II: Solving equations

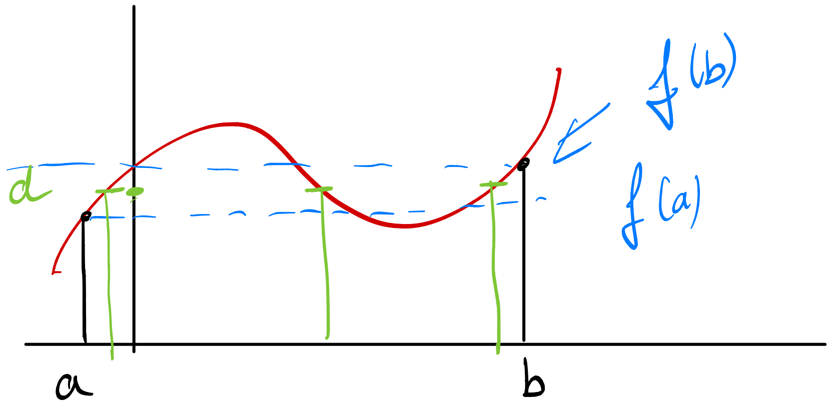
given x, find x st $f(x) = 0$

IVT: let f be continuous on $[a,b]$

then for any $d \in [f(a), f(b)]$

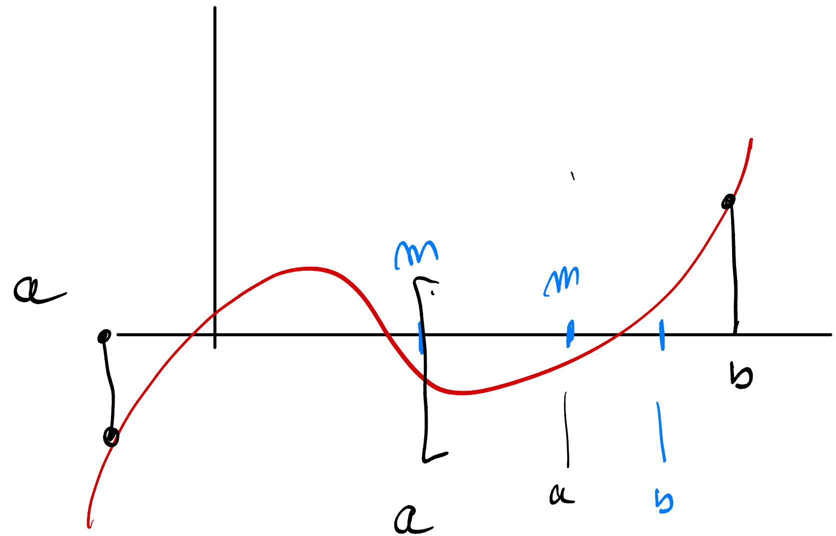
(or $[f(b), f(a)]$ if $f(b) < f(a)$)

there exists c s.t. $f(c) = d$



Corollary: If f is continuous on $[a,b]$, $f(a) \cdot f(b) < 0$ then there exists at least one c in (a,b) s.t $f(c) = 0$

Bisection algorithm.



Algorithm: given f, a, b st $f(a)\,f(b) < 0$

while [some convergence test] :

$$m = \frac{a+b}{2}$$

if $f(a) f(m) < 0$:

$b = m$

else if $f(a) f(m) > 0$:

$a = m$

else

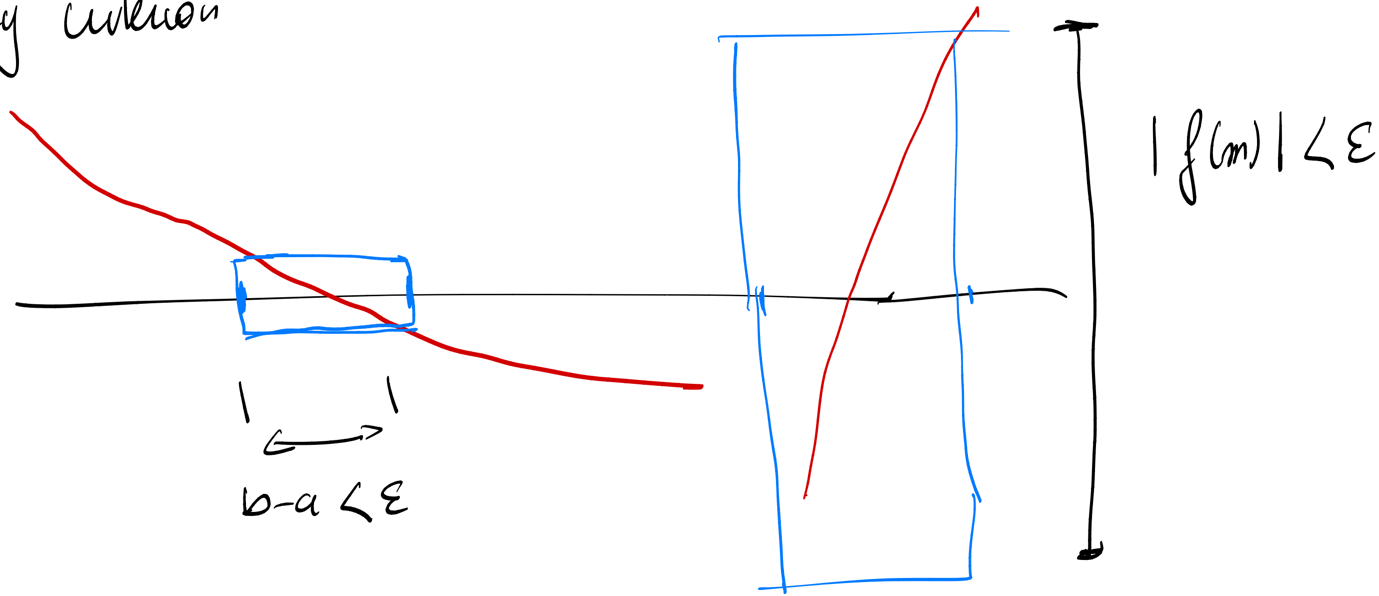
$a = m$

$b = m$

$|b-a| \leq tol_1$ or $|f(m)| \leq tol_2$

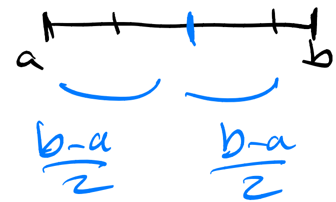
\# iteration $\leq$ nmax

Stopping criterion



step 0 $\quad (a, b) \to$ length is $|b-a|$

step 1



step 2 $\quad$ length $\quad \frac{b-a}{2^2} \quad 2^{-2}(b-a)$

step n: $\quad 2^{-n}(b-a)$

given $\varepsilon > 0$

$$2^{-n}(b-a) < \varepsilon$$

$$\ln\left(2^{-n}(b-a)\right) < \ln \varepsilon$$

$$\ln(b-a) + \ln(2^{-n}) \leq \ln \varepsilon$$

$$\ln(b-a) - n \ln 2 \leq \ln \varepsilon$$

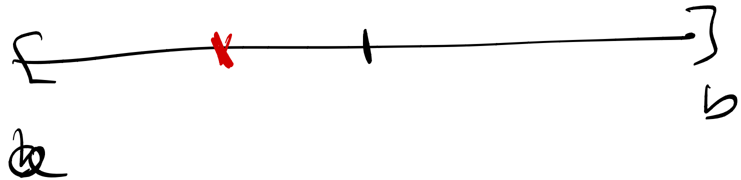
$$-n \ln 2 \leq \ln \varepsilon - \ln(b-a)$$

$$-n \leq \ln \varepsilon/(b-a) / \ln 2$$

$$n \geq \frac{\ln\left(\frac{\varepsilon}{b-a}\right)}{\ln 2}$$

$$\boxed{n \geq \log_2\left(\frac{\varepsilon}{b-a}\right)}$$

$$m = \frac{a+b}{2} \quad \left(= a + \frac{b-a}{2}\right)$$



$$m = a + \frac{b-a}{3}$$

modified bisection: set $m = \frac{b-a}{3}$ instead of $\frac{b-a}{2}$

suppose $b-a = 1$, after $2n$ iterations:

Std bisection $|b-a| = \left(\frac{1}{2}\right)^{2n} = \left(\frac{1}{4}\right)^{n} = \left(\frac{2}{8}\right)^{n}$

mod bisection $|b-a| = \left(\frac{1}{3}\right)^{n} \left(\frac{2}{3}\right)^{n}$

$$= \left(\frac{2}{9}\right)^{n} < \left(\frac{2}{8}\right)^{n}$$