Approximate Proximity Problems in High Dimensions via Locality-Sensitive Hashing

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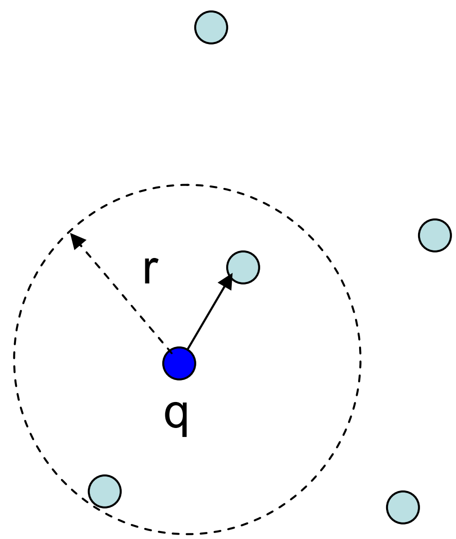
Closest Pair

- We have seen several algorithms for the closest pair problem over n points in R^d
  - $d=2$: $O(n \log n)$
  - $d>2$: $d^{O(d)} n$ (or $O(dn^2)$)
- The exponential dependence on d, a.k.a. "curse of dimensionality", is a very common phenomenon
  - See talk by Nina Amenta, 4pm, Kiva/Star
- Next few lectures: how to get around this problem

Nearest Neighbor

- For a set P of n points in R^d
  - Nearest Neighbor: for any query q, returns a point $p \in P$ minimizing $||p-q||$
  - r-Near Neighbor: for any query q, returns a point $p \in P$ s.t. $||p-q|| \leq r$ (if it exists)

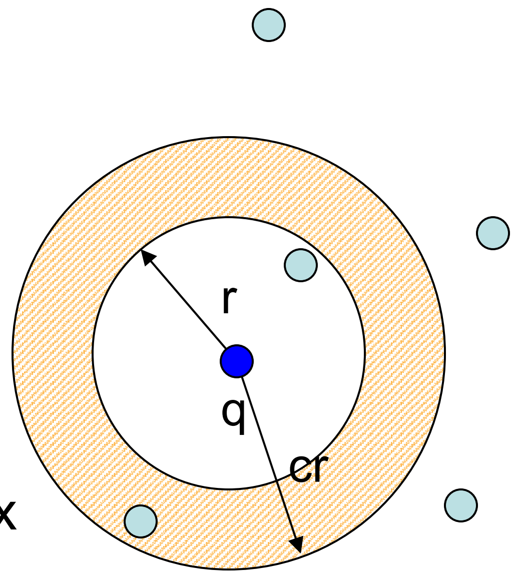


- If we have data structure with
  - Construction time T(n)
  - Query time Q(n)

  then we can solve r-Close Pair in time $[\ T(n) + n\ Q(n)\ ] \log n$
- Unfortunately, no algorithm with small $T(n)$ and $Q(n)$ is known

Approximate Near Neighbor

- c-Approximate r-Near Neighbor: build data structure which, for any query q:
  - If there is a point $p \in P$, $||p-q|| \leq r$
  - it returns $p' \in P$, $||p-q|| \leq cr$



- Reductions:
  - c-Approx r-Close Pair
  - c-Approx Nearest Neighbor reduces to c-Approx Near Neighbor

    (log overhead)
  - One can enumerate all approx near neighbors

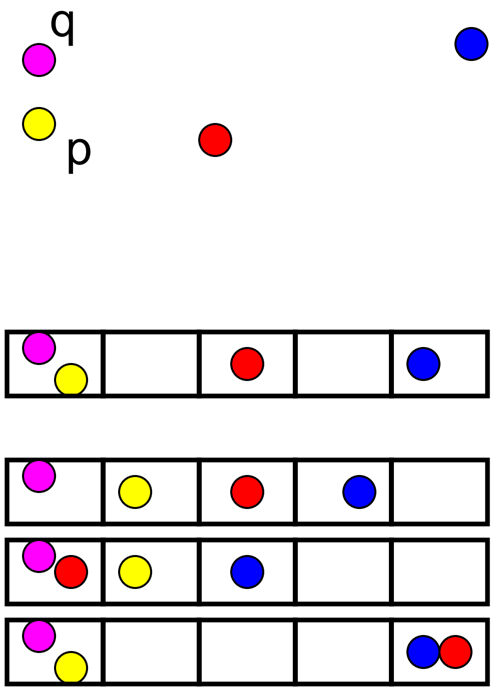
  → can solve exact near neighbor problem
  - Other apps: c-approximate Minimum Spanning Tree, clustering, etc.

Today

- A c-Approx r-Near Neighbor:
  - Preprocessing: $dn^{1+1/c}$
  - Query time: $dn^{1/c}$
  - …for binary vectors
- Sketch how to improve $1/c$ to $1/c^2+\delta$

Locality-Sensitive Hashing

- Idea: construct hash functions $g: R^d \rightarrow U$ such that for any points p,q:
  - If $||p-q|| \leq r$, then $Pr[g(p)=g(q)]$ is ~~“high”~~ “not-so-small”
  - If $||p-q|| > cr$, then $Pr[g(p)=g(q)]$ is “small”
- Then we can solve the problem by hashing



LSH [Indyk-Motwani'98]

- A family H of functions $h: R^d \rightarrow U$ is called (P_1,P_2,r,cr)-sensitive, if for any p,q:
  - if $||p-q|| <r$ then $\Pr[h(p)=h(q)] > P_1$
  - if $||p-q|| >cr$ then $\Pr[h(p)=h(q)] < P_2$
- Example: Hamming distance
  - LSH functions: $h(p)=p_i$, i.e., the i-th bit of p
  - Probabilities: $\Pr[h(p)=h(q)] = 1-D(p,q)/d$

p=10010010
q=11010110

LSH Algorithm

- We use functions of the form

  $$g(p) = \langle h_1(p), h_2(p), \ldots, h_k(p) \rangle$$

- Preprocessing:
  - Select $g_1 \ldots g_L$
  - For all $p \in P$, hash p to buckets $g_1(p) \ldots g_L(p)$

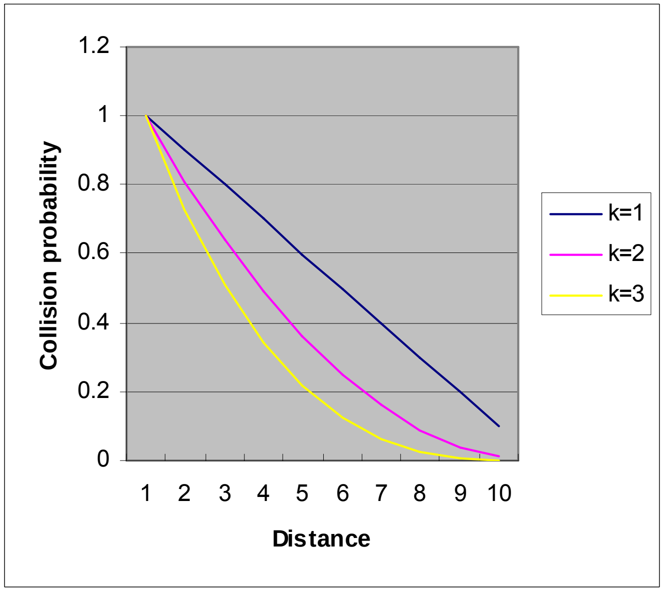
- Query:
  - Retrieve the points from buckets $g_1(q)$, $g_2(q)$, … , until
    - Either the points from all L buckets have been retrieved, or
    - Total number of points retrieved exceeds $3L$
  - Answer the query based on the retrieved points
  - Total time: $O(dL)$

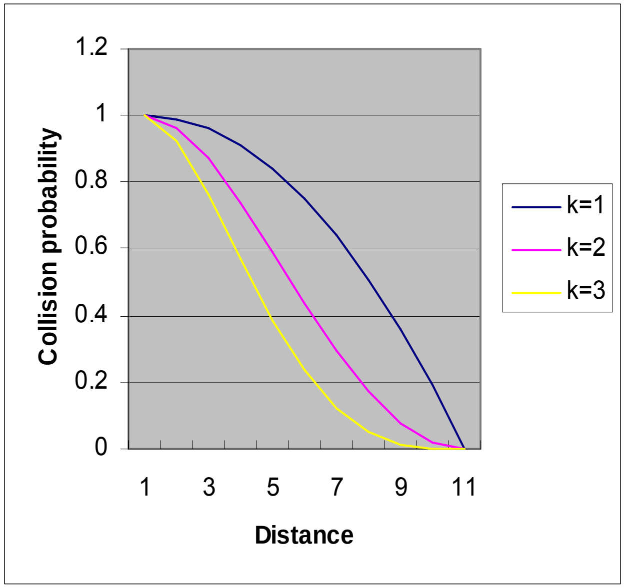
Analysis

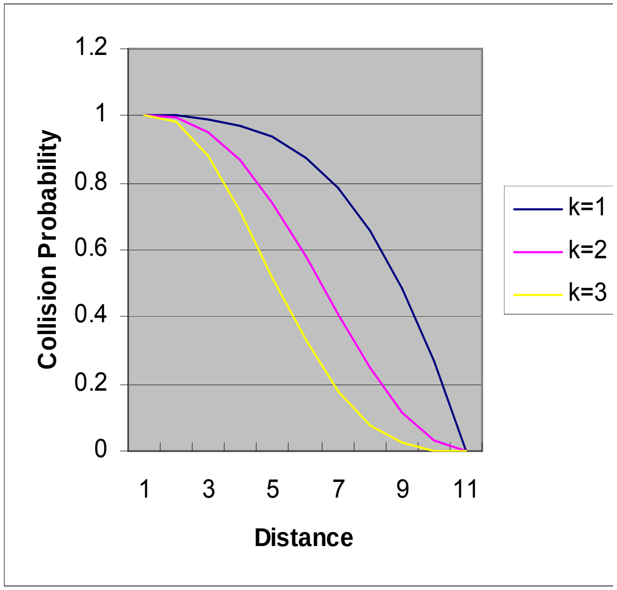
- Lemma1: LSH solves c-approximate NN with:
  - Number of hash fun: $L=n^{\rho}$, $\rho=\log(1/P1)/\log(1/P2)$
  - Constant success probability per query q
- Lemma 2: for LSH functions as seen earlier we have $\rho=1/c$

Proof of Lemma 1 by picture

- Points in $\{0,1\}^d$
- Collision prob. for k=1..3, L=1..3 (recall: L=#indices, k=#h’s)
- Distance ranges from 0 to d=10







Proof

- Define:
  - p: a point such that $||p-q|| \le r$
  - $FAR(q)=\{ p' \in P: ||p'-q|| > c\, r \}$
  - $B_i(q)=\{ p' \in P: g_i(p')=g_i(q) \}$
- Will show that both events occur with >0 probability:
  - E_1: $g_i(p)=g_i(q)$ for some $i=1\ldots L$
  - E_2: $\Sigma_i |B_i(q) \cap FAR(q)| < 3L$

Proof ctd.

- Set $k=\log_{1/P2} n$
- For $p' \in FAR(q)$,

$$Pr[g_i(p')=g_i(q)] \leq P_2^{k} =1/n$$

- $E[\ |B_i(q) \cap FAR(q)|\ ] \leq 1$
- $E[\Sigma_i\ |B_i(q) \cap FAR(q)|\ ] \leq L$
- $Pr[\Sigma_i\ |B_i(q) \cap FAR(q)| \geq 3L\ ] \leq 1/3$

Proof, ctd.

- $\Pr[\ g_i(p)=g_i(q)\ ] \geq 1/P_1{}^k = 1/n^\rho = 1/L$
- $\Pr[\ g_i(p) \neq g_i(q),\ i=1..L] \leq (1-1/L)^L \leq 1/e$

Proof, end

- $\Pr[E_1 \text{ not true}]+\Pr[E_2 \text{ not true}]$
  $\leq 1/3+1/e = 0.7012.$
- $\Pr[E_1 \cap E_2] \geq 1-(1/3+1/e) \approx 0.3$

Proof of Lemma 2

- Statement: for
  - $P1=1-r/d$
  - $P2=1-cr/d$

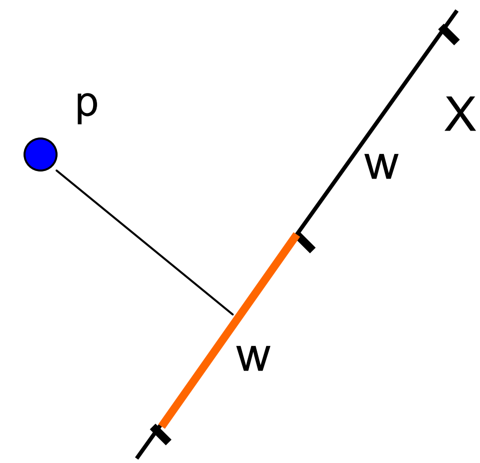
  we have $\rho=\log(P1)/\log(P2) \leq 1/c$
- Proof:
  - Need $P1^c \geq P2$
  - But $(1-x)^c \geq (1-cx)$ for any $1>x>0,\ c>1$

For the curious…

Projection-based LSH

[Datar-Immorlica-Indyk-Mirrokni'04]

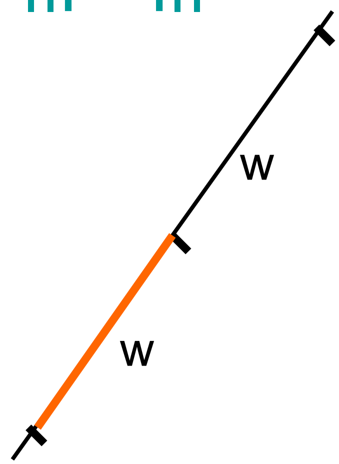
- Define $h_{X,b}(p)=\lfloor (p*X+b)/w \rfloor$:
  - $w \approx r$
  - $X=(X_1 \ldots X_d)$, where X_i is chosen from:
    - Gaussian distribution (for l_2 norm)
  - b is a scalar



Analysis

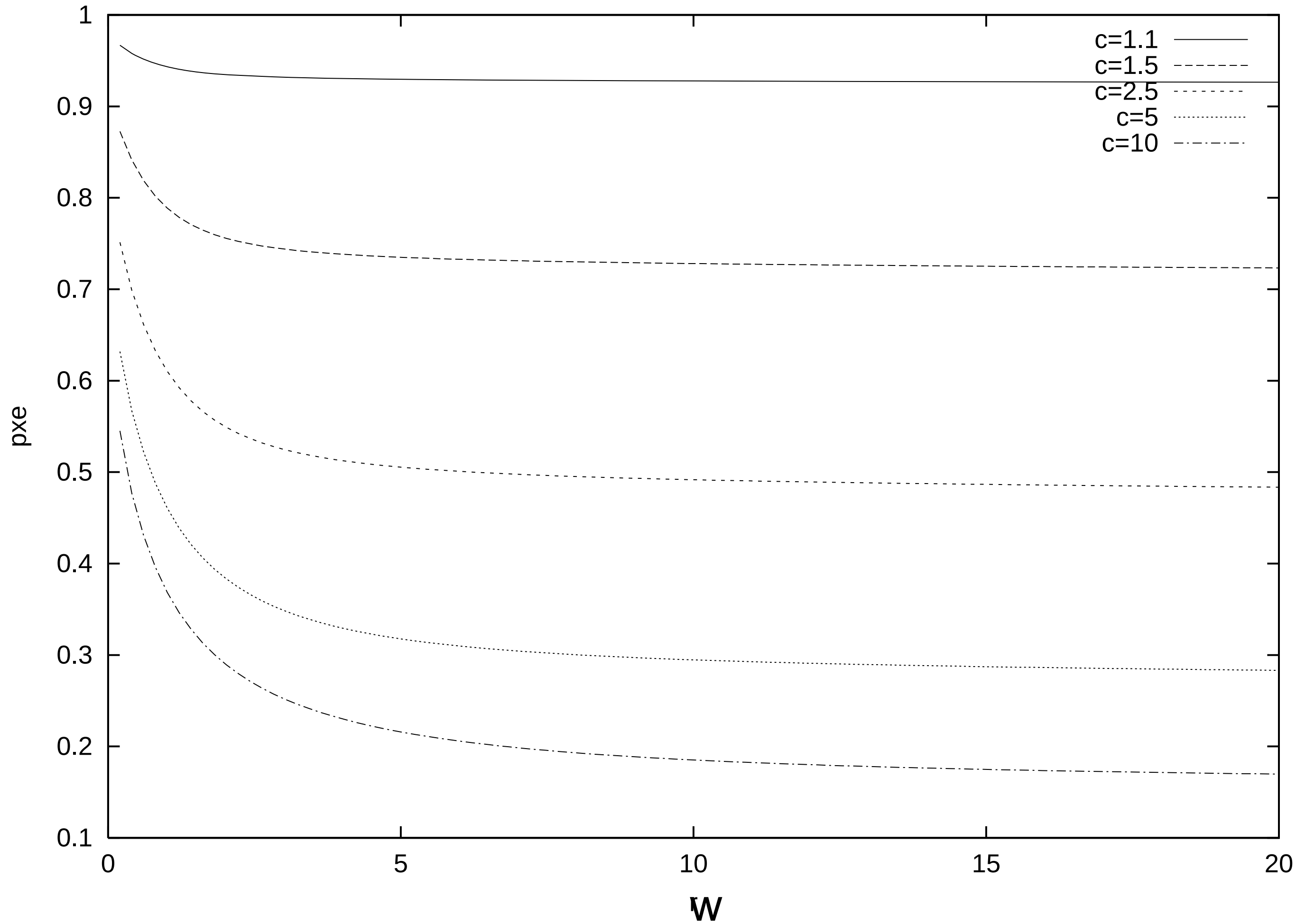
- Need to:
  - Compute $Pr[h(p)=h(q)]$ as a function of $||p-q||$ and w; this defines P_1 and P_2
  - For each c choose w that minimizes

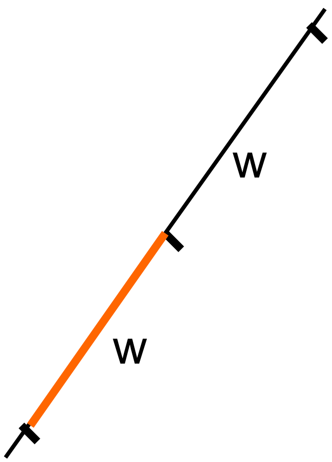
$$\rho=\log_{1/P2}(1/P_1)$$



- Method:
  - For l_2: computational
  - For general l_s: analytic

$\rho(w)$ for various c's: l_1





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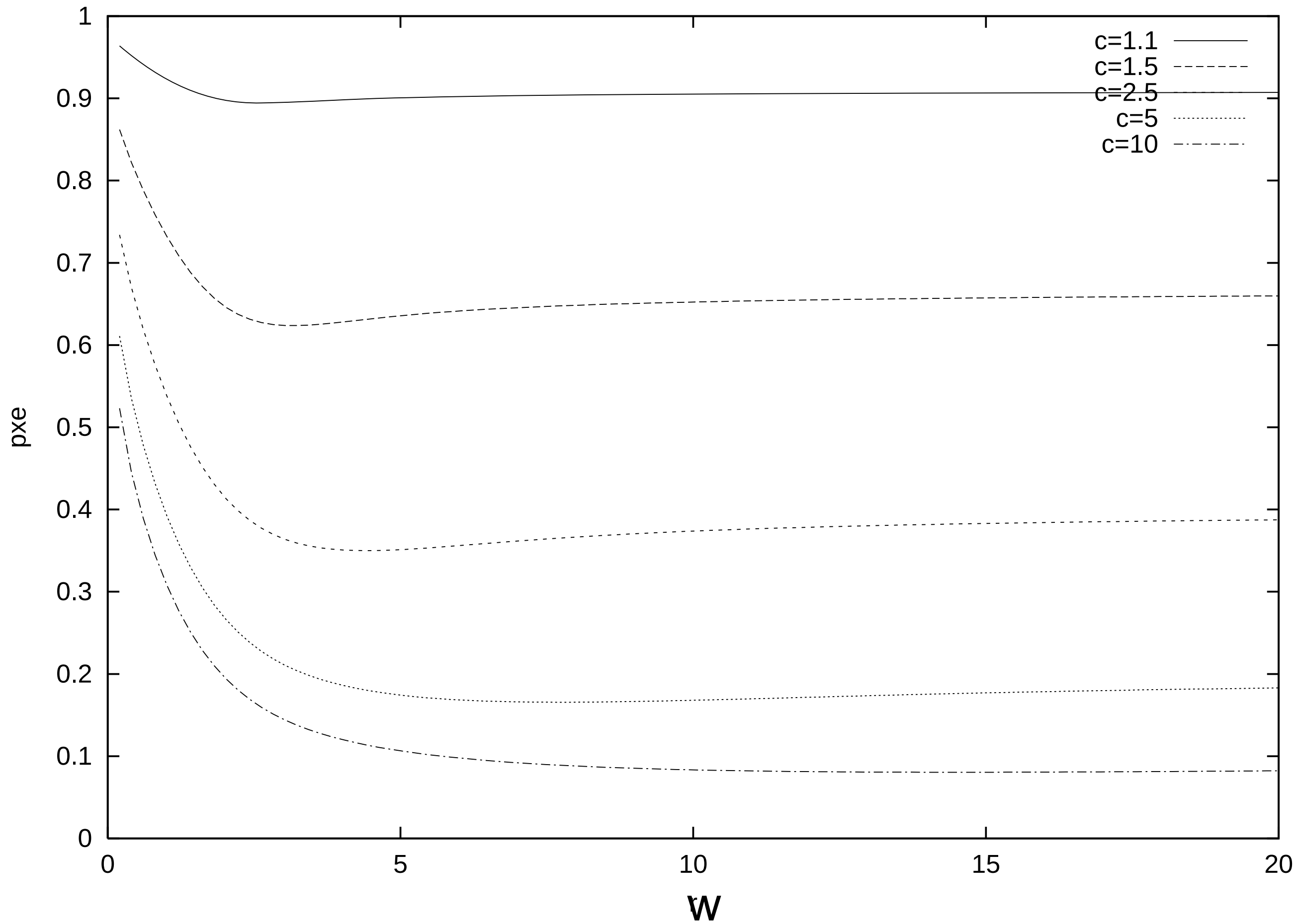
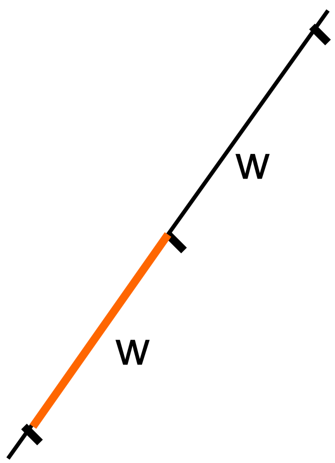
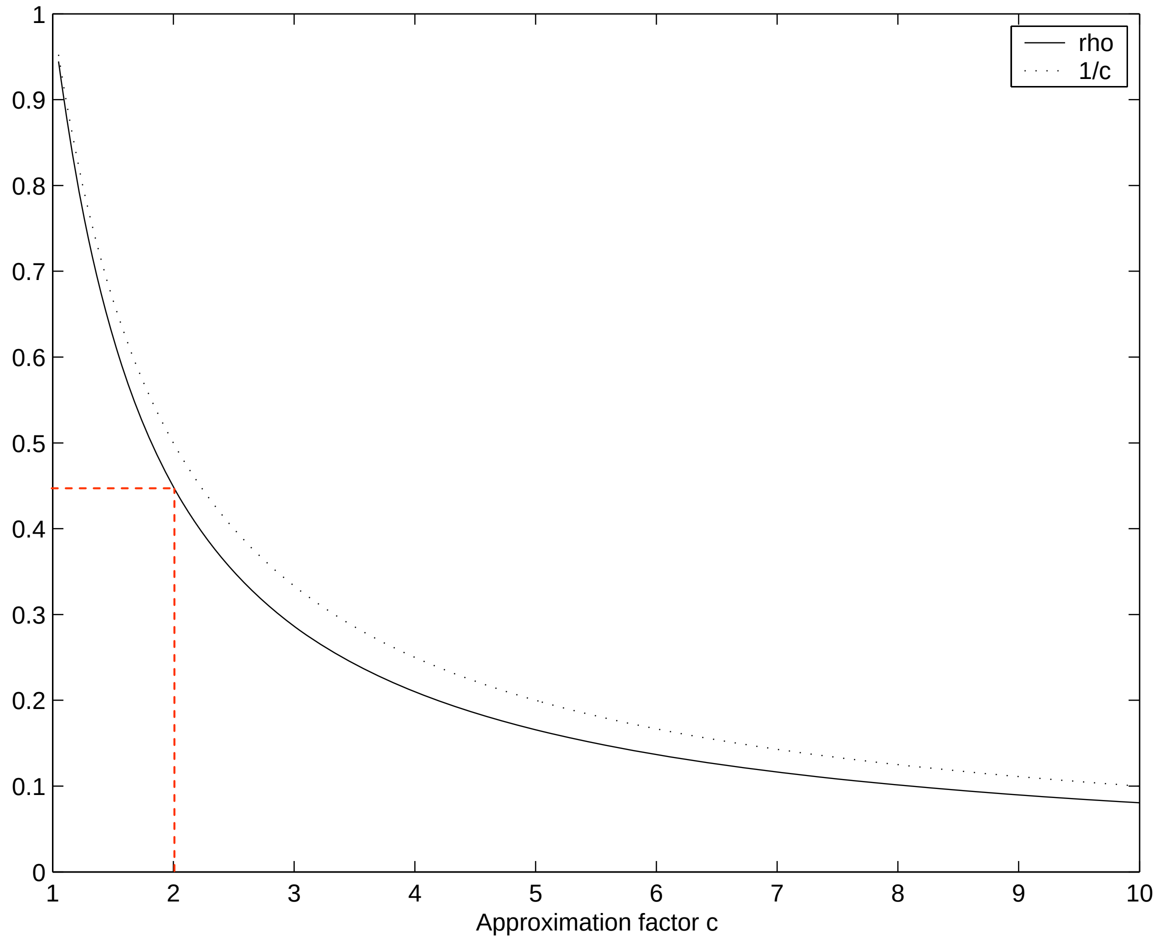
$\rho(w)$ for various c's: l_2





$\rho(c)$ for l_2

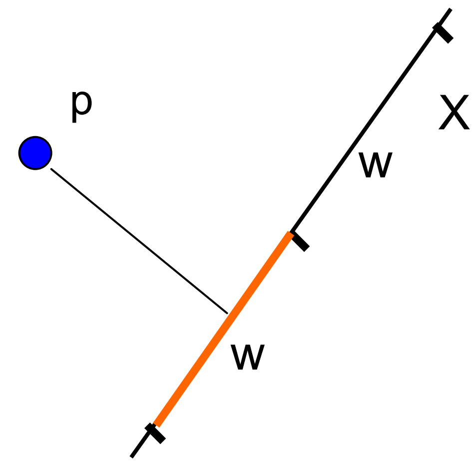


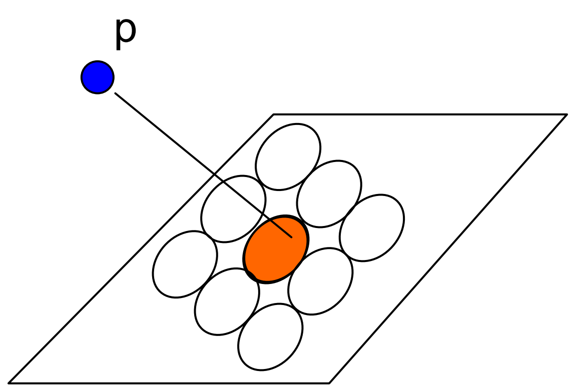
rho
1/c
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
8
9
10
Approximation factor c

New LSH scheme

[Andoni-Indyk'06]

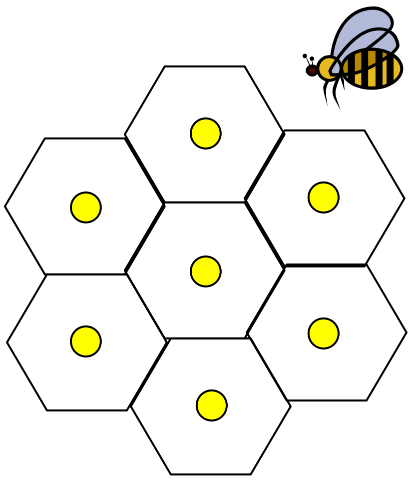
- Instead of projecting onto R^1, project onto R^t, for constant t
- Intervals → lattice of balls
  - Can hit empty space, so hash until a ball is hit
- Analysis:
  - $\rho = 1/c^2 + O(\log t / t^{1/2})$
  - Time to hash is $t^{O(t)}$
  - Total query time: $dn^{1/c^2+o(1)}$
- [Motwani-Naor-Panigrahy'06]: LSH in l_2 must have $\rho \geq 0.45/c^2$





New LSH scheme, ctd.

- How does it work in practice ?
- The time $t^{O(t)} d n^{1/c^2+f(t)}$ is not very practical
  - Need $t \approx 30$ to see some improvement
- Idea: a different decomposition of R^t
  - Replace random balls by Voronoi diagram of a lattice
  - For specific lattices, finding a cell containing a point can be very fast →fast hashing



Leech Lattice LSH

- Use Leech lattice in R^{24} , $t=24$
  - Largest kissing number in 24D: 196560
  - Conjectured largest packing density in 24D
  - 24 is 42 in reverse…
- Very fast (bounded) decoder: about 519 operations [Amrani-Beery'94]
- Performance of that decoder for $c=2$:
  - $1/c^2$ 0.25
  - $1/c$ 0.50
  - Leech LSH, any dimension: $\rho \approx 0.36$
  - Leech LSH, 24D (no projection): $\rho \approx 0.26$

Experiments

Experiments (with '04 version)

- E^2LSH: Exact Euclidean LSH (with Alex Andoni)
  - Near Neighbor
  - User sets r and P = probability of NOT reporting a point within distance r (=10%)
  - Program finds parameters k,L,w so that:
    - Probability of failure is at most P
    - Expected query time is minimized
- Nearest neighbor: set radius (radiae) to accommodate 90% queries (results for 98% are similar)
  - 1 radius: 90%
  - 2 radiae: 40%, 90%
  - 3 radiae: 40%, 65%, 90%
  - 4 radiae: 25%, 50%, 75%, 90%

Data sets

- MNIST OCR data, normalized (LeCun)
  - d=784
  - n=60,000
- Corel_hist
  - d=64
  - n=20,000
- Corel_uci
  - d=64
  - n=68,040
- Aerial data (Manjunath)
  - d=60
  - n=275,476

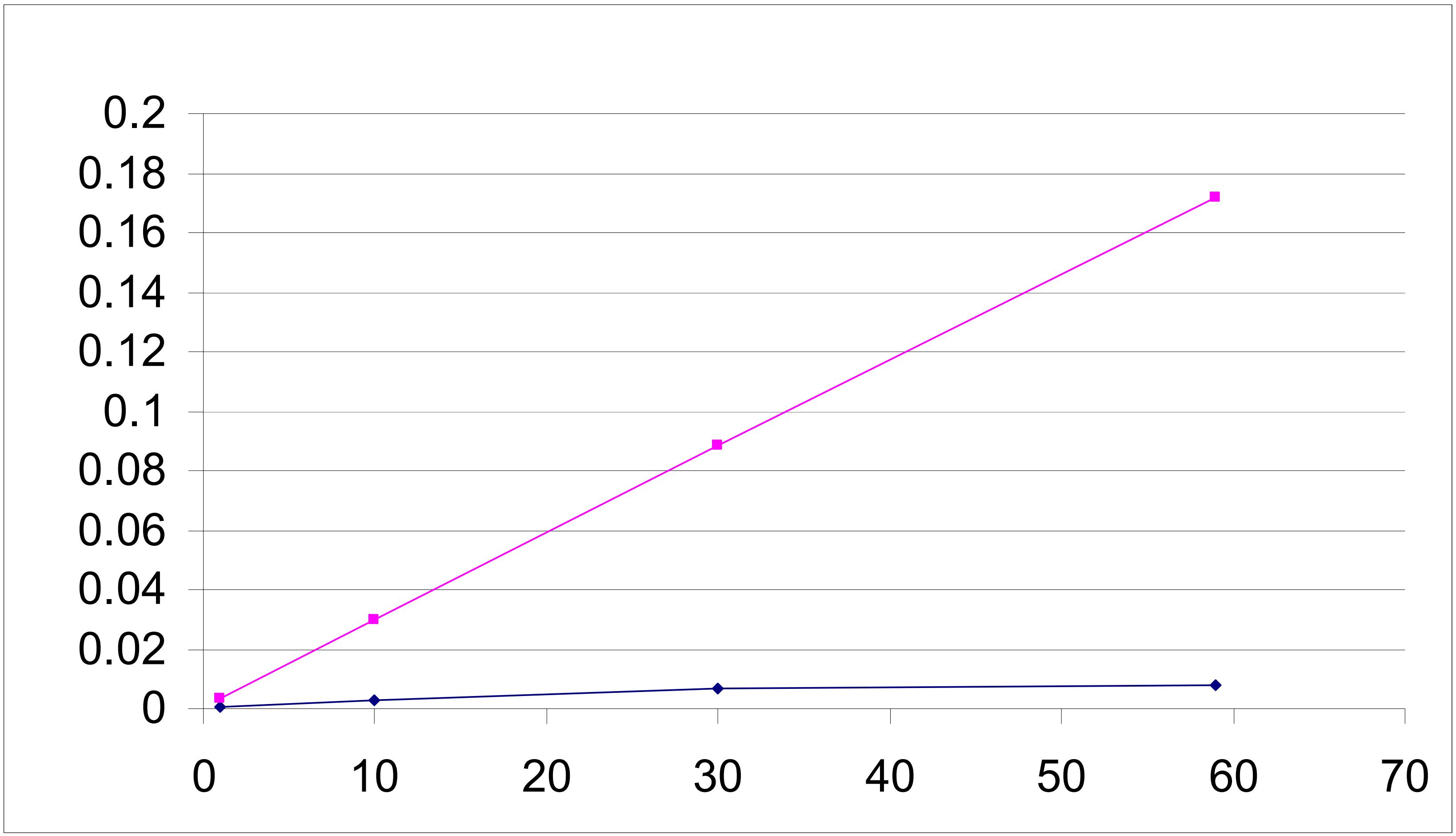
Other NN packages

- ANN (by Arya & Mount):
  - Based on kd-tree
  - Supports exact and approximate NN
- Metric trees (by Moore et al):
  - Splits along arbitrary directions (not just x,y,..)
  - Further optimizations

Running times

	MNIST	Speedup	Corel_hist	Speedup	Corel_uci	Speedup	Aerial	Speedup
E2LSH-1	0.00960							
E2LSH-2	0.00851		0.00024		0.00070		0.07400	
E2LSH-3			0.00018		0.00055		0.00833	
E2LSH-4							0.00668	
ANN	0.25300	29.72274	0.00018	1.011236	0.00274	4.954792	0.00741	1.109281
MT	0.20900	24.55357	0.00130	7.303371	0.00650	11.75407	0.01700	2.54491

LSH vs kd-tree (MNIST)



Caveats

- For ANN (MNIST), setting ε=1000% results in:
  - Query time comparable to LSH
  - Correct NN in about 65% cases, small error otherwise
- However, no guarantees
- LSH eats much more space (for optimal performance):
  - LSH: 1.2 GB
  - Kd-tree: 360 MB

Conclusions

- Locality-Sensitive Hashing
  - Very good option for near neighbor
  - Worth trying for nearest neighbor
- E^{2}LSH [DIIM'04] available – check my web page for more info