Correction de la Feuille N° 10

Exercice 1. *Echantillonneur de Gibbs*

1. On a

$$\begin{aligned}
\pi(\theta|\mathbf{x}) &\propto \pi(\theta,\mathbf{x}) \\
&\propto \pi(\mathbf{x}|\theta)\pi(\theta) \quad \text{(règle de Bayes)} \\
&= \prod_{i=1}^{n} \pi(x_i|\theta) \times \pi(\theta) \quad \text{(par indépendance)} \\
&= \prod_{i=1}^{n} \left(\sum_{k=1}^{K} p_k f_{\mathcal{N}(m_k,s_k^2)}(x_i) \right) \times \pi(\theta) \quad \text{(densité de mélange).}
\end{aligned}$$

2. Pour la loi a posteriori de $\mathbf{Z}$ sachant que $\mathbf{X}=\mathbf{x}$ et $\Theta=\theta$, on a

$$\begin{aligned}
\pi(\mathbf{z}|\mathbf{x},\theta) &\propto \pi(\mathbf{z},\mathbf{x},\theta) \\
&\propto \pi(\mathbf{x}|\mathbf{z},\theta)\pi(\mathbf{z}|\theta) \quad \text{(règle de Bayes)} \\
&= \prod_{i=1}^{n} \pi(x_i|\mathbf{z},\theta) \times \prod_{i=1}^{n} \pi(z_i|\theta) \\
&= \prod_{i=1}^{n} \pi(x_i|z_i,\theta)\pi(z_i|\theta) \\
&= \prod_{i=1}^{n} p_{z_i} f_{\mathcal{N}(m_k,s_k^2)}(x_i).
\end{aligned}$$

Pour la loi a posteriori de Θ sachant que $\mathbf{X}=\mathbf{x}$ et $\mathbf{Z}=\mathbf{z}$, on a

$$\begin{aligned}
\pi(\theta|\mathbf{z},\mathbf{x}) &\propto \pi(\mathbf{x}|\mathbf{z},\theta)\pi(\mathbf{z}|\theta)\pi(\theta) \\
&= \pi(\mathbf{x}|\mathbf{z},\mathbf{m},\mathbf{s}^2)\pi(\mathbf{z}|\mathbf{p})\pi(\mathbf{m},\mathbf{s}^2)\pi(\mathbf{p}) \\
&= \underbrace{\pi(\mathbf{x},\mathbf{z},\mathbf{m},\mathbf{s}^2)}_{\propto \pi(\mathbf{m},\mathbf{s}^2|\mathbf{x},\mathbf{z})} \quad \underbrace{\frac{\pi(\mathbf{m},\mathbf{s}^2)}{\pi(\mathbf{z},\mathbf{m},\mathbf{s}^2)}}_{=1/\pi(\mathbf{z}|\mathbf{m},\mathbf{s}^2)=1/\pi(\mathbf{z})} \quad \underbrace{\frac{\pi(\mathbf{z},\mathbf{p})\pi(\mathbf{p})}{\pi(\mathbf{p})}}_{\propto \pi(\mathbf{p}|\mathbf{z})} \\
&\propto \pi(\mathbf{m},\mathbf{s}^2|\mathbf{x},\mathbf{z})\pi(\mathbf{p}|\mathbf{z}).
\end{aligned}$$

Or, d'une part, on a

$$\begin{aligned}
\pi(\mathbf{p}|\mathbf{z}) &\propto \pi(\mathbf{z}|\mathbf{p})\pi(\mathbf{p}) \\
&\propto \prod_{i=1}^{n} p_{z_i} \times \prod_{k=1}^{K} p_k^{\gamma_k-1} \\
&= \prod_{k=1}^{K} p_k^{n_k+\gamma_k-1} \quad \text{où} \quad n_k = \#\{i : z_i = k\},
\end{aligned}$$

ce qui implique que

$$\mathbf{P}|\mathbf{z} \sim \mathcal{D}(\gamma_1 + n_1, \ldots, \gamma_K + n_k).$$

D'autre part, on a

$$\begin{aligned}
\pi(\mathbf{m}, \mathbf{s}^2|\mathbf{x}, \mathbf{z}) \propto \pi(\mathbf{x}|\mathbf{z}, \mathbf{m}, \mathbf{s}^2) &\underbrace{\pi(\mathbf{z}|\mathbf{m}, \mathbf{s}^2)}_{=\pi(\mathbf{z})\propto 1} \pi(\mathbf{m}|\mathbf{s}^2)\pi(\mathbf{s}^2) \\
&\propto \prod_{i=1}^{n} \frac{1}{s_{z_i}} \exp\left\{-\frac{1}{2s_{z_i}^2}(x_i - m_{z_i})^2\right\} \times \prod_{k=1}^{K} \frac{1}{s_k} \exp\left\{-\frac{\lambda_k}{2s_k^2}(m_k - \alpha_k)^2\right\} \times \\
&\quad \prod_{k=1}^{K} (s_k^2)^{-\lambda_k/2-1} \exp\left\{-\frac{\beta_k}{2s_k^2}\right\} \\
&= \prod_{k=1}^{K} (s_k^2)^{-n_k/2-\lambda_k/2-3/2} \exp\left\{-\frac{\beta_k}{2s_k^2}\right\} \times \\
&\quad \prod_{k=1}^{K} \exp\left\{-\frac{1}{2s_k^2}\left[\sum_{i:z_i=k} (x_i - m_k)^2 + \lambda_k(m_k - \alpha_k)^2\right]\right\}.
\end{aligned}$$

Or,

$$\begin{aligned}
\sum_{i:z_i=k} (x_i - m_k)^2 + \lambda_k(m_k - \alpha_k)^2 &= \sum_{i:z_i=k} x_i^2 - 2m_k \sum_{i:z_i=k} x_i + n_k m_k^2 + \lambda_k m_k^2 - 2\lambda_k m_k \alpha_k + \lambda_k \alpha_k^2 \\
&= (n_k + \lambda_k)m_k^2 - 2m_k(\sum_{i:z_i=k} x_i + \lambda_k\alpha_k) + \sum_{i:z_i=k} x_i^2 + \lambda_k\alpha_k^2 \\
&= (n_k + \lambda_k)\left(m_k - \underbrace{\frac{\sum_{i:z_i=k} x_i + \lambda_k\alpha_k}{n_k + \lambda_k}}_{=:\tau_k}\right)^2 + \sum_{i:z_i=k} x_i^2 + \lambda_k\alpha_k^2 - \frac{(\sum_{i:z_i=k} x_i + \lambda_k\alpha_k)^2}{n_k + \lambda_k}.
\end{aligned}$$

D'où

$$\begin{aligned}
\pi(\mathbf{m}, \mathbf{s}^2|\mathbf{x}, \mathbf{z}) &\propto \prod_{k=1}^{K} (s_k^2)^{-(n_k+\lambda_k)/2-1-1/2} \exp\left\{-\frac{1}{2s_k^2}\left(\underbrace{\beta_k + \sum_{i:z_i=k} x_i^2 + \lambda_k\alpha_k^2 - \frac{(\sum_{i:z_i=k} x_i + \lambda_k\alpha_k)^2}{n_k + \lambda_k}}_{=:\rho_k}\right)\right\} \times \\
&\quad \prod_{k=1}^{K} f_{\mathcal{N}(\tau_k, s_k^2/(n_k+\lambda_k))}(m_k) \frac{s_k}{\sqrt{n_k + \lambda_k}} \\
&\propto \prod_{k=1}^{K} f_{\mathcal{IG}((n_k+\lambda_k)/2, \rho_k/2)}(s_k^2) \prod_{k=1}^{K} f_{\mathcal{N}(\tau_k, s_k^2/(n_k+\lambda_k))}(m_k).
\end{aligned}$$