

Problem-Statement

Let an object have a reflectance given by r . Then the complex-distribution of the object-field is given by $g \sim CN(0, D(r))$ [1]. The problem statement is given

$$y = Ag + w$$

we would like to estimate r .

1 Case I: Linear measurements ($A = I$)

1.1 Forward-model:

Let the reflectance of the object that is imaged be given by $r \in [0, 1]$. Under the assumed model of the object-field given by $g \sim CN(0, D(r))$, and the additive white-gaussian noise model we have that the measured complex-signal is given as

$$y = g + w.$$

A single-realization of this is given in Figure with std. deviations of the noise at $\sigma_w = \{0.001, 0.1\}$.

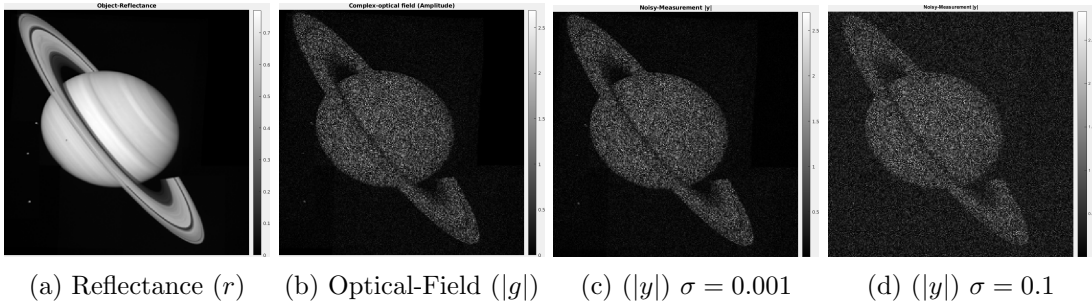


Figure 1: An example-realization of the forward-model $y = g + w$. Two realizations of the noisy observations are shown at std. deviations of $\{0.001, 0.1\}$.

1.2 ML-estimate

The likelihood of the measurements y is given as $p(y|r) \sim CN(0, D(r) + \sigma_w^2 I)$. Applying the maximum-likelihood estimator to the above problem we have

$$\begin{aligned}
 \hat{r} &= \arg \max \log p(y|r) \\
 &= \arg \max \frac{-1}{2} \left(\log ((2\pi)^N |D(r) + \sigma_w^2 I|) + y^H (D(r) + \sigma_w^2 I)^{-1} y \right) \\
 &= \arg \max \frac{-1}{2} \left(\sum_{i=1}^N \log (2\pi(r_i + \sigma_w^2)) + \frac{|y_i|^2}{r_i + \sigma_w^2} \right) \\
 \implies \hat{r}_i &= \arg \max \frac{-1}{2} \left(\log (2\pi(r_i + \sigma_w^2)) + \frac{|y_i|^2}{r_i + \sigma_w^2} \right) \\
 &= \arg \min \log (2\pi(r_i + \sigma_w^2)) + \frac{|y_i|^2}{r_i + \sigma_w^2}.
 \end{aligned}$$

Taking the differential of the above equation and equating to zero we get the ML estimate

$$\hat{r}_i = |y_i|^2 - \sigma_w^2.$$

The reconstructions for varying degrees of noise is shown in Figure 3 (a-d).

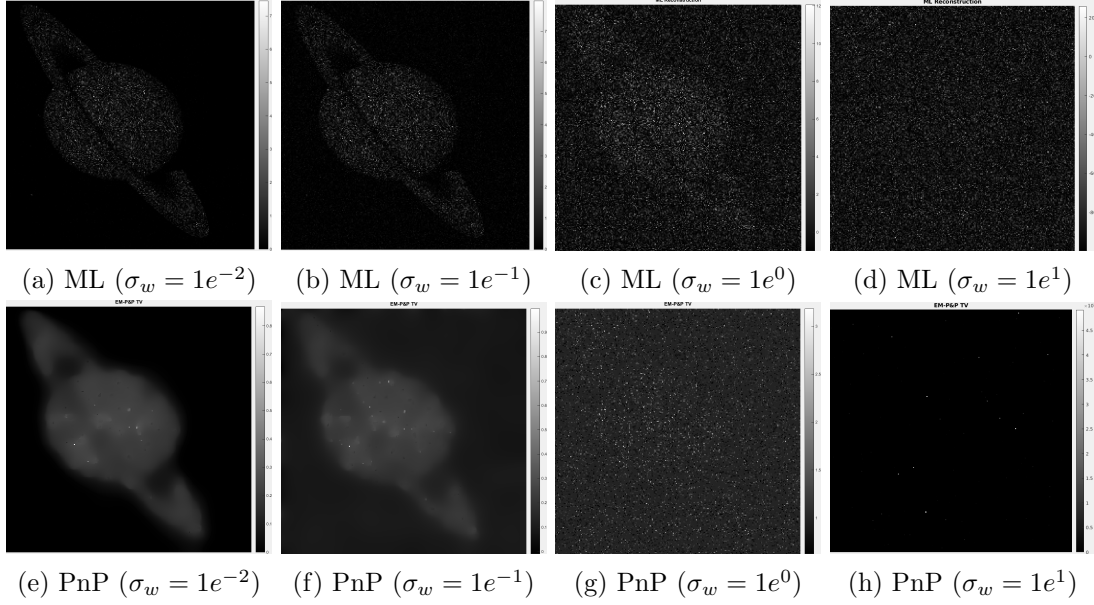


Figure 2: An example-realization for different inversion-models. The inversions are shown for different std. deviations of $\{0.01, 0.1, 1, 10\}$.

1.3 Plug and play (PnP) estimate (MAP)

The MAP estimate for the above problem is given as

$$\begin{aligned} \hat{r} &= \arg \min -\log p(r|y) \\ &= \arg \min -\log p(y|r) - \log p(r). \end{aligned} \quad (1)$$

The PnP algorithm [2] decouples the above problem into one where we can apply the forward and prior models in the form of denoisers. Specifically, the PnP algorithm operates by splitting the unknown variable r into two variables r, v and converts the unconstrained optimization problem in (1) into a constrained optimization problem

$$\hat{r} = \arg \min_{r=v} \ell(r) + \beta s(v) \quad (2)$$

where $\beta s(v) = p(v)$ is the prior term and $\ell(r)$ enforces the measurement constraints on r . Adopting an ADMM approach to solve the above problem results in an algorithm containing two primary operations:

(1) Inversion operator

$$F(\tilde{r}, \sigma_\lambda) = \arg \min_r \left\{ \ell(r) + \frac{1}{2\sigma_\lambda^2} \|r - \tilde{r}\|^2 \right\},$$

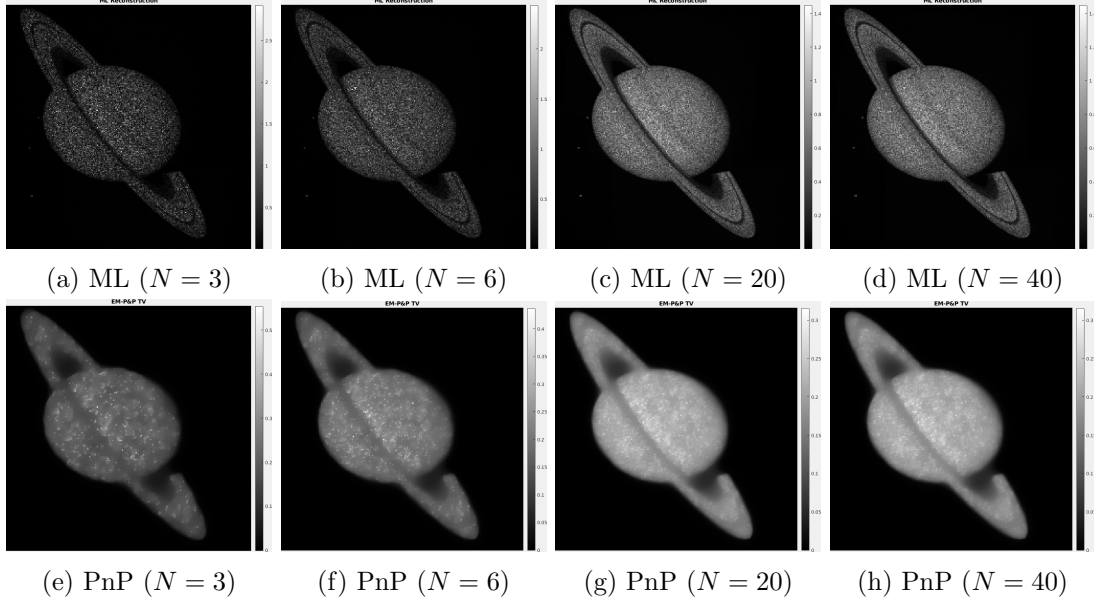


Figure 3: Multi-image plug and play algorithm for: For the image with $\sigma_w = 1e^{-3}$.

where $\tilde{r} = v - u$. The operator F is the proximal mapping of $\ell(r)$ and is equivalent to a MAP estimate of r with a Gaussian prior having a distribution $p(r) \sim N(\tilde{r}, \sigma_\lambda^2 I)$. Specifically when $\ell(r) = -\log p(y|r)$, the inversion operator is given as

$$\begin{aligned}
F(\tilde{r}_i, \sigma_\lambda) &= \arg \min_{r_i} \left\{ \log(r_i + \sigma_w^2) + \frac{|y_i|^2}{r_i + \sigma_w^2} + \frac{1}{2\sigma_\lambda^2} (r_i - \tilde{r}_i)^2 \right\}, \\
\Rightarrow \frac{1}{r_i + \sigma_w^2} - \frac{|y_i|^2}{(r_i + \sigma_w^2)^2} + \frac{1}{\sigma_\lambda^2} (r_i - \tilde{r}_i) &= 0, \\
\Rightarrow \sigma_\lambda^2 (r_i + \sigma_w^2) - |y_i|^2 + (r_i - \tilde{r}_i)(r_i + \sigma_w^2)^2 &= 0, \\
\Rightarrow r_i^3 + (-\tilde{r}_i + 2\sigma_w^2)r_i^2 + (-2\tilde{r}_i\sigma_w^2 + \sigma_w^4 + \sigma_\lambda^2)r_i + (\sigma_\lambda^2\sigma_w^2 - \tilde{r}_i\sigma_w^4) &= 0.
\end{aligned}$$

Thus the solution of the inversion-operator is simply the root of the cubic-polynomial. We are constrained that the root of the cubic-polynomial is real.

(2) Denoising operator

$$H(\tilde{v}, \sigma_n) = \arg \min_v \left\{ \frac{1}{2\sigma_n^2} \|v - \tilde{v}\|^2 + s(v) \right\}$$

where $\sigma_n^2 = \beta\sigma_\lambda^2$ and $\tilde{v} = r + u$. The denoising operator H is a proximal mapping of $s(v)$. Mathematically, it is equivalent to a Gaussian denoising operation. Thus, we can replace H with a denoiser which removes noise with a variance σ_n^2 . We use the following denoising-operators to use for Gaussian-denoising: (1) TV, (2) BM3D denoiser. The complete PnP algorithm is given in Algorithm 1.

1.3.1 Parameter-tuning

In the following analysis we assume that we know the true-value of σ_w . The parameters to be tuned for the problem are σ_λ, σ_n ; which are essentially the tuning parameters for each of the proximal-operators (inversion and denoising).

Algorithm 1 Plug and Play $(y, \sigma_w, \sigma_\lambda, \sigma_n)$

```
1:  $v = y, u = 0$ 
2: Repeat
3: {
4:  $\tilde{r} = v - u$ 
5:  $r = F(y, \tilde{r}, \sigma_w, \sigma_\lambda)$     Inversion-Operator
6:  $\tilde{v} = r + u$ 
7:  $v = H(\tilde{v}, \sigma_n)$     Denoising-Operator
8:  $u = u + (r - v)$ 
9: }
```

1.4 Regularization by denoising

Recently there has been an alternative approach instead of the ADMM or Plug-and-Play (P^3) approach: regularization by denoising [3]. The key idea of the paper is the regularization proposed in the paper:

$$\rho(x) = \frac{1}{2}x^T[x - f(x)],$$

where $f(x)$ is the denoised version of x . The intuition behind this regularization is:

- Penalty introduced is the inner-product between the candidate image x and its denoising residual.

A small value for $\rho(x)$ implies that

- Residual is very small, image x is at the fixed point or $x \approx f(x)$.
- Cross-correlation between the image x and its residual is very small or alternatively resembles white-noise or contains no image-like features.

1.5 Properties of the denoiser $f(x)$

The RED paper assumes that a denoiser can be assumed to be a psuedo-linear filter of the form

$$f(x) = W(x)x.$$

The following properties are further assumed on the denoiser:

1.6 Solving the above regularized optimization problem

A general inverse-problem is of the form

$$\hat{x} = \arg \min_x E(x) \triangleq \ell(x, y) + \frac{\lambda}{2}x^T[x - f(x)].$$

The gradient of the above expression is of the form

$$\nabla_x E(x) = \nabla_x \ell(y, x) + \frac{\lambda}{2} \nabla_x \{x^T(x - f(x))\}$$

References

- [1] Casey J. Pellizzari et al. *Optically coherent image formation and denoising using a plug and play inversion framework in OSA.*
- [2] S. V. Venkatakrishnan et al. *Plug-and-play priors for model based reconstruction in GLOBALSIP.*
- [3] Y. Romano et al. *The little engine that could: Regularization by denoising.*