Problem-Statement

Let an object have a reflectance given by r. Then the complex-distribution of the object-field is given by $g \sim CN(0, D(r))$ [1]. The problem statement is given

$$y = Ag + w$$

we would like to estimate r.

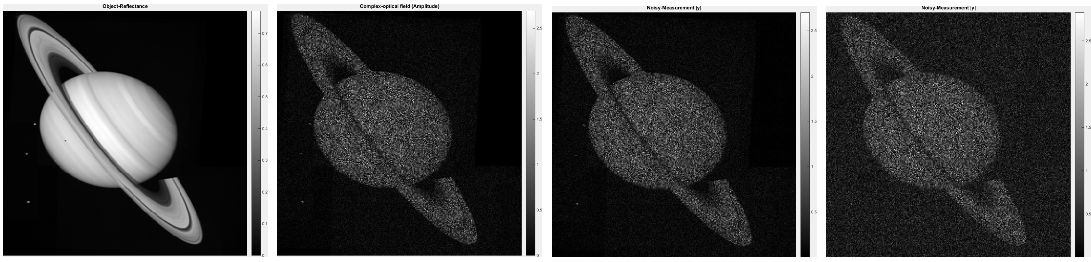
1 Case I: Linear measurements $(A = I)$

1.1 Forward-model:

Let the reflectance of the object that is imaged be given by $r \in [0, 1]$. Under the assumed model of the object-filed given by $g \sim CN(0, D(r))$, and the additive white-gaussian noise model we have that the measured complex-signal is given as

$$y = g + w.$$

A single-realization of this is given in Figure with std. deviations of the noise at $\sigma_w = \{0.001, 0.1\}$.



(a) Reflectance (r) (b) Optical-Field ($|g|$) (c) ($|y|$) $\sigma = 0.001$ (d) ($|y|$) $\sigma = 0.1$

Figure 1: An example-realization of the forward-model $y = g + w$. Two realizations of the noisy observations are shown at std. deviations of $\{0.001, 0.1\}$.

1.2 ML-estimate

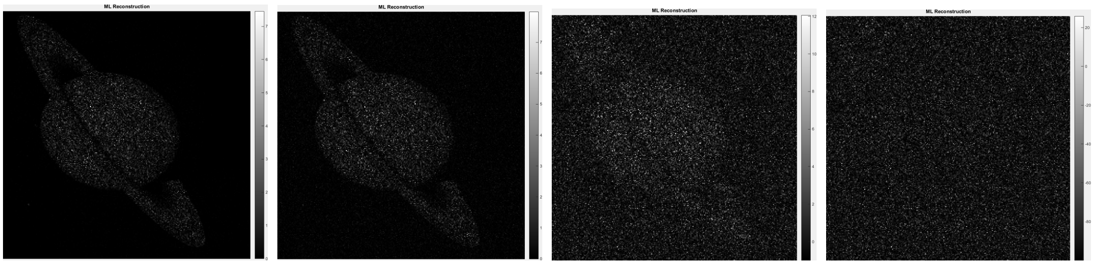
The likelihood of the measurements y is given as $p(y|r) \sim CN(0, D(r) + \sigma_w^2 I)$. Applying the maximum-likelihood estimator to the above problem we have

$$\begin{aligned}
\hat{r} &= \arg\max \log p(y|r) \\
&= \arg\max \frac{-1}{2}\left(\log\left((2\pi)^N |D(r) + \sigma_w^2 I|\right) + y^H (D(r) + \sigma_w^2 I)^{-1} y \right) \\
&= \arg\max \frac{-1}{2}\left(\sum_{i=1}^{N} \log\left(2\pi(r_i + \sigma_w^2)\right) + \frac{|y_i|^2}{r_i + \sigma_w^2} \right) \\
\implies \hat{r}_i &= \arg\max \frac{-1}{2}\left(\log\left(2\pi(r_i + \sigma_w^2)\right) + \frac{|y_i|^2}{r_i + \sigma_w^2} \right) \\
&= \arg\min \log\left(2\pi(r_i + \sigma_w^2)\right) + \frac{|y_i|^2}{r_i + \sigma_w^2}.
\end{aligned}$$

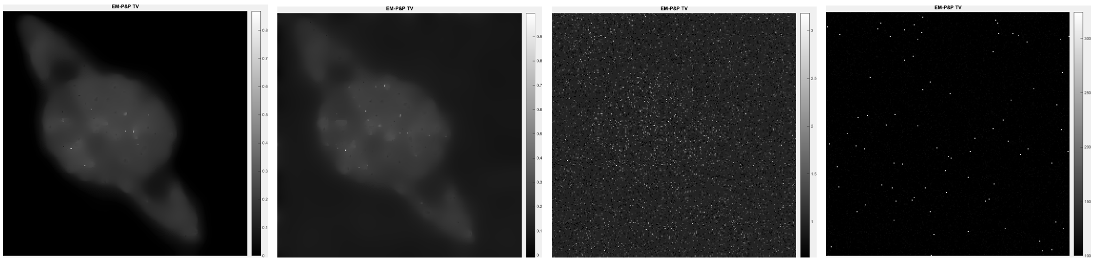
Taking the differential of the above equation and equating to zero we get the ML estimate

$$\hat{r}_i = |y_i|^2 - \sigma_w^2.$$

The reconstructions for varying degrees of noise is shown in Figure 2 (a-d).



(a) ML ($\sigma_w = 1e^{-2}$) (b) ML ($\sigma_w = 1e^{-1}$) (c) ML ($\sigma_w = 1e^{0}$) (d) ML ($\sigma_w = 1e^{1}$)



(e) PnP ($\sigma_w = 1e^{-2}$) (f) PnP ($\sigma_w = 1e^{-1}$) (g) PnP ($\sigma_w = 1e^{0}$) (h) PnP ($\sigma_w = 1e^{1}$)

Figure 2: An example-realization for different inversion-models. The inversions are shown for different std. deviations of $\{0.01, 0.1, 1, 10\}$.

1.3 Plug and play (PnP) estimate (MAP)

The MAP estiamate for the above problem is given as

$$\begin{aligned} \hat{r} &= \arg\min -\log p(r|y) \\ &= \arg\min -\log p(y|r) - \log p(r). \end{aligned} \tag{1}$$

The PnP algorithm [2] decouples the above problem into one where we can apply the forward and prior models in the form of denoisers. Specifically, the PnP algorithm operates by splitting the unknown variable r into two variables r, v and converts the unconstrained optimization problem in (1) into a constrained optimization problem

$$\hat{r} = \underset{r=v}{\arg\min}\ \ell(r) + \beta s(v) \tag{2}$$

where $\beta s(v) = p(v)$ is the prior term and $\ell(r)$ enforces the meausrement constraints on r. Adopting an ADMM approach to solve the above problem results in an algorithm containing two primary operations:

(1) **Inversion operator**

$$F(\tilde{r}, \sigma_\lambda) = \arg\min_r \left\{ \ell(r) + \frac{1}{2\sigma_\lambda^2} \|r - \tilde{r}\|^2 \right\},$$

where $\tilde{r} = v - u$. The operator F is the proximal mapping of $\ell(r)$ and is equivalent to a MAP estimate of r with a Gaussian prior having a distribution $p(r) \sim N(\tilde{r}, \sigma_\lambda^2 I)$.
Specifically when $\ell(r) = -\log p(y|r)$, the inversion operator is given as

$$\begin{aligned}
F(\tilde{r}_i, \sigma_\lambda) &= \arg\min_{r_i} \left\{ \log(r_i + \sigma_w^2) + \frac{|y_i|^2}{r_i + \sigma_w^2} + \frac{1}{2\sigma_\lambda^2}(r_i - \tilde{r}_i)^2 \right\}, \\
&\implies \frac{1}{r_i + \sigma_w^2} - \frac{|y_i|^2}{(r_i + \sigma_w^2)^2} + \frac{1}{\sigma_\lambda^2}(r_i - \tilde{r}_i) = 0, \\
&\implies \sigma_\lambda^2(r_i + \sigma_w^2) - |y_i|^2 + (r_i - \tilde{r}_i)(r_i + \sigma_w^2)^2 = 0, \\
&\implies r_i^3 + (-\tilde{r}_i + 2\sigma_w^2) r_i^2 + (-2\tilde{r}_i\sigma_w^2 + \sigma_w^4 + \sigma_\lambda^2) r_i + (\sigma_\lambda^2\sigma_w^2 - \tilde{r}_i\sigma_w^4) = 0.
\end{aligned}$$

Thus the solution of the inversion-operator is simply the root of the cubic-polynomial. We are constrained that the root of the cubic-polynomial is real.

(2) **Denoising operator**

$$H(\tilde{v}, \sigma_n) = \arg\min_{v} \left\{ \frac{1}{2\sigma_n^2} \|v - \tilde{v}\|^2 + s(v) \right\}$$

where $\sigma_n^2 = \beta\sigma_\lambda^2$ and $\tilde{v} = r + u$. The denosing operator H is a proximal mapping of $s(v)$. Mathematically, it is equivalent to a Gaussian denosing operation. Thus, we can replace H with a denoiser which removes noise with a variance σ_n^2. We use the following denoising-operators to use for Gaussian-denosing: (1) TV, (2) BM3D denoiser. The complete PnP algorithm is given in Algorithm 1.

Algorithm 1 Plug and Play $(y, \sigma_w, \sigma_\lambda, \sigma_n)$

```
1: v = y, u = 0
2: Repeat
3: {
4: r̃ = v − u
5: r = F(y, r̃, σ_w, σ_λ)    Inversion-Operator
6: ṽ = r + u
7: v = H(ṽ, σ_n)    Denoising-Operator
8: u = u + (r − v)
9: }
```

1.3.1 Parameter-tuning

In the following analysis we assume that we know the true-value of σ_w. The parameters to be tuned for the problem are σ_λ, σ_n; which are essentially the tuning parameters for each of the proximal-operators (inversion and denoising).

References

[1] Casey J. Pellizzari et al. *Optically coherent image formation and denoising using a plug and play inversion framework in OSA.*

[2] S. V. Venkatakrishnan et al. *Plug-and-play priors for model based reconstruction in GLOBALSIP.*