Problem-Statement

Let an object have a reflectance given by r. Then the complex-distribution of the object-field is given by $g \sim CN(0, D(r))$ [1]. The problem statement is given

$$y = Ag + w$$

we would like to estimate r.

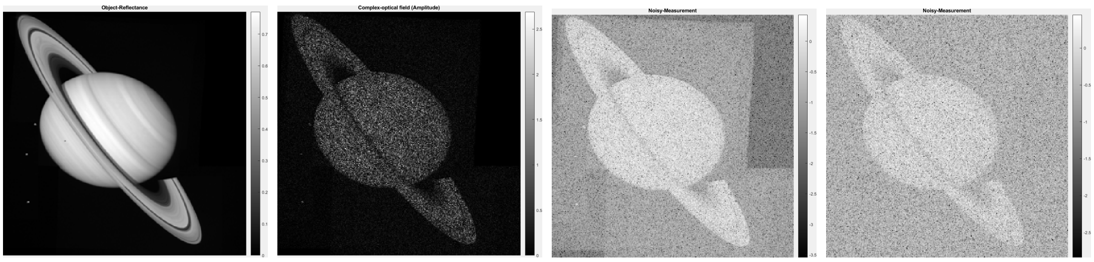
1 Case I: Linear measurements $(A = I)$

1.1 Forward-model:

Let the reflectance of the object that is imaged be given by $r \in [0, 1]$. Under the assumed model of the object-filed given by $g \sim CN(0, D(r))$, and the additive white-gaussian noise model we have that the measured complex-signal is given as

$$y = g + w.$$

A single-realization of this is given in Figure with std. deviations of the noise at $\sigma_w = \{0.001, 0.1\}$.



(a) Reflectance (r) (b) Optical-Field ($|g|$) (c) ($|y|$) $\sigma = 0.001$ (d) ($|y|$) $\sigma = 0.1$

Figure 1: An example-realization of the forward-model $y = g + w$. Two realizations of the noisy observations are shown at std. deviations of $\{0.001, 0.1\}$.

1.2 ML-estimate

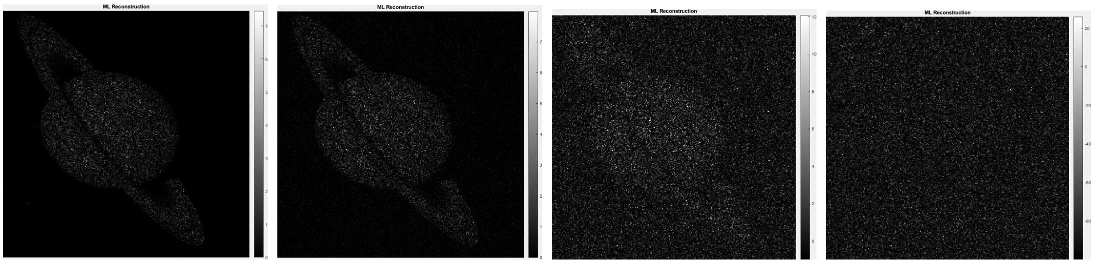
The likelihood of the measurements y is given as $p(y|r) \sim CN(0, D(r) + \sigma_w^2 I)$. Applying the maximum-likelihood estimator to the above problem we have

$$\begin{aligned}
\hat{r} &= \arg\max p(y|r) \\
&= \arg\max \frac{-1}{2}\left(\log\left((2\pi)^N |D(r) + \sigma_w^2 I|\right) + y^H (D(r) + \sigma_w^2 I)^{-1} y \right) \\
&= \arg\max \frac{-1}{2}\left(\sum_{i=1}^{N} \log\left(2\pi(r_i + \sigma_w^2)\right) + \frac{|y_i|^2}{r_i + \sigma_w^2} \right) \\
\implies \hat{r}_i &= \arg\max \frac{-1}{2}\left(\log\left(2\pi(r_i + \sigma_w^2)\right) + \frac{|y_i|^2}{r_i + \sigma_w^2} \right) \\
&= \arg\min \log\left(2\pi(r_i + \sigma_w^2)\right) + \frac{|y_i|^2}{r_i + \sigma_w^2}.
\end{aligned}$$

Taking the differential of the above equation and equating to zero we get the ML estimate

$$\hat{r}_i = |y_i|^2 - \sigma_w^2.$$

The reconstructions for varying degrees of noise is shown in Figure 2 (a-d).



(a) ML ($\sigma = 1e^{-2}$) (b) ML ($\sigma = 1e^{-1}$) (c) ML ($\sigma = 1e^{0}$) (d) ML ($\sigma = 1e^{1}$)

Figure 2: An example-realization for different inversion-models. The inversions are shown for different std. deviations of $\{0.01, 0.1, 1, 10\}$.

1.3 Plug and play (PnP) MAP estimate

The MAP estiamate for the above problem is given as

$$\begin{aligned} \hat{r} &= \arg\max \log p(r|y) \\ &= \arg\max \log p(y|r) + \log p(r). \end{aligned} \tag{1}$$

The PnP algorithm [2] decouples the above problem into one where we can apply the forward and prior models in the form of denoisers. Specifically, the PnP algorithm operates by splitting the unknown variable r into two variables r, v and converts the unconstrained optimization problem in (1) into a constrained optimization problem

$$\hat{r} = \arg\max_{r=v} \log p(y|r) + \beta s(v) \tag{2}$$

where $\beta s(r) = p(r)$ is the prior term and β is the tuning parameter. Adopting an ADMM approach to solve the above problem results in an algorithm containing two primary operations:(1) an inversion operator

$$F(\tilde{r}, \sigma_\lambda) = \arg\min_r \left\{ \ell(r) + \frac{1}{2\sigma_\lambda^2}\|r - \tilde{r}\|^2 \right\},$$

where $\tilde{r} = v - u$ and the denoising operator is given as

$$H(\tilde{v}, \sigma_n) = \arg\min_v \left\{ \frac{1}{2\sigma_n^2}\|v - \tilde{v}\|^2 + s(v) \right\}$$

where $\sigma_n^2 = \beta\sigma_\lambda^2$ and $\tilde{v} = r + u$. Using the above notation the PnP algorithm is given as

References

[1] Casey J. Pellizzari et al. *Optically coherent image formation and denoising using a plug and play inversion framework in OSA.*

[2] S. V. Venkatakrishnan et al. *Plug-and-play priors for model based reconstruction in GLOBALSIP.*