LOW DEGREE POINTS ON CURVES IN TORIC SURFACES

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Abstract. In this paper we investigate the minimum density degree of certain nice curves C embedded in split, smooth toric surfaces, building on the work of [SV22]. We prove that ample curves C in a smooth toric surface X with minimal density degree not equal to their gonality must correspond to one of a finite number of possible Newton Polygons P. We calculate polygons for some specific surfaces such as Hirzebruch surfaces $\mathbb{F}_r$.

1. Introduction

A curve C over a number field K is nice if it is smooth, projective, and geometrically integral. Given such a curve, a fundamental question in arithmetic geometry is if the curve has an infinite number of rational points. By Falting's Theorem, if $g(C) \geq 2$, then the K-points in C are finite [Fal83]. To further study higher genus curves, we may want to extend our question to a larger class of closed points. One such way of enlarging this class is by taking points defined in an extension of bounded degree over K. For a given $d > 1$, a degree d point is a closed point p whose residue field is a degree d extension of K.

A recent question of interest is how large a degree d is needed before the degree d points are dense (equivalently infinite). This is known as the minimum density degree, or $\min \delta(C/K)$. This invariant is strongly related to the gonality $\mathrm{gon}_K C$, or the minimum d such that there is a degree d map $C \to \mathbb{P}^1_K$. By considering the fibers of this map and applying the Hilbert Irreducibility theorem, one finds that $\mathrm{gon}_K C \leq \min \delta(C/K)$ [VV25].

This inequality may be strict, but its failure will be described by rational points in the image $W_d C$ of the Abel-Jacobi map $\mathrm{Sym}^d C \to \mathrm{Pic}^d C$. By application of the Mordell-Lang conjecture (proven by Faltings), if this image has infinite rational points, it must contain a translate of a positive rank abelian subvariety of $\mathrm{Pic}^0 C$. While the gonality of a curve C is relatively easy to understand, the abelian varieties inside $W_d C$ are more subtle. Therefore, much research in this direction concerns circumstances in which $\min \delta(C/K) = \mathrm{gon}_K C$. For instance, in 1994, Debarre and Klassen proved such an equality for smooth curves in $\mathbb{P}^2_K$ of degree $d \geq 8$ [DK94]. In 2022, Smith and Vogt applied a similar approach for smooth curves in $\mathbb{P}^1_K \times \mathbb{P}^1_K$, demonstrating that if a curve is of bidegree $(d_1, d_2) \neq (2,2)$ or $(3,3)$, then the equality holds [SV22].

In [SV22], they develop techniques that can be used more generally for curves in surfaces S with $h^0(S, \mathcal{O}_S) = 0$, but the list of exceptions for this general result is not explicit. Both $\mathbb{P}^2_K$ and $\mathbb{P}^1_K \times \mathbb{P}^1_K$ are examples of smooth, split toric surfaces. A split toric surface S is a compactification of the split torus $(K^*)^2$ such that the action of the torus on itself extends to an action on all of S. We generalize these results across all smooth, split toric surfaces as follows:

Theorem 1.1. *Let C be a smooth curve in an ample class on a smooth, split toric surface X_Σ, with associated Newton-polygon P. Then, if the lattice width $\mathrm{lw}(P) \leq \frac{2}{9}\mathrm{Vol}(P)$, one has that $\min(\delta(C/K)) = \mathrm{gon}_K C$. In particular, if $\min(\delta(K/C)) < \mathrm{gon}_K C$, then $\mathrm{Vol}(P) < 54$.*

Hence, as one ranges over all smooth, split toric surfaces, there are a finite number of ample classes in which a curve can satisfy $\min(\delta(C/K)) < \mathrm{gon}_K C$.

As a result, we are able to prove a novel result about Hirzebruch surfaces $\mathbb{F}_n$, generalizing the results of Smith-Vogt on $\mathbb{P}^1 \times \mathbb{P}^1$. If we think of the Hirzebruch surface $\mathbb{F}_n$ as the projectivization

$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^1$, then the nef cone will be generated by G and F, where F is a fiber of projection onto $\mathbb{P}^1$ and G is a section corresponding to an inclusion of vector bundles $\mathcal{O} \hookrightarrow \mathcal{O}(n) \oplus \mathcal{O}$. More precisely, $[G]$ and $[F]$ are a basis of the nef cone such that G has self-intersection 0 and G has self-intersection n. In this language, we have the following result:

Theorem 1.2. Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$ be the Hirzebruch surface over a number field K, with $n \geq 0$, and let $C \hookrightarrow \mathbb{F}_n$ be a curve in an ample class $C \sim aF + bG$. Then $\mathrm{gon}_K C = \min(\delta(C/K))$ if:

$$(n, a, b) \notin \{(0,2,2), (0,3,3), (1,1,2), (1,1,3), (1,1,4), (1,1,5), (1,1,6)\}$$

Removing the case $(1,2,4)$ requires some additional effort, and is ruled out in an appendix by Isabel Vogt. In the case that $n = 0$, this is just Smith-Vogt's result for $\mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, we will demonstrate that the five counterexamples of the form $(1,1,b)$ are essentially not new, and are just detecting failures of smooth plane curves after blowing up at a toric point.

1.1. **Structure of the Paper.** We begin with discussion of background in Section 2. This will start with an exposition of the motivating question of low-degree points on curves in 2.1. This will establish the two sides of our proof, a "gonality side" and an "abelian variety side." Subsection 2.2 will introduce common notation for the discussion of toric surfaces, including introduction of some results and examples to be used later. This will be followed in 2.2 with a discussion relating the gonality of curves in toric surfaces to their associated Newton polygons.

In Section 3 we will prove Theorem 1.1. This will be followed by a few immediate corollaries, including an application to certain nodal plane curves. In Section 4, these results will be applied specifically to the Hirzebruch surfaces, proving Theorem 1.2.

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2. Background

2.1. **Low Degree Points on Curves.** In this subsection we will recount basic results about low degree points on curves, mostly following the survey [VV25]. We will finish by summarizing some results in [SV22], which will be our main tool of proving nonexistence of AV-parametrized points.

A foundational question is algebraic geometry is whether or not an algebraic variety has dense rational points over a given number field. In the case of nice curves, this question is answered by Falting's Theorem:

Theorem 2.1. [Fal83] Let C be a nice curve of genus $g(C) \geq 2$ defined over a number field K. Then $C(K)$ is finite.

Because the answer to this question is negative for all higher genus curves, one may wish to expand the set of points they are considering. One such way to do this is by considering closed points of a bounded degree. Given a (scheme theoretic) closed point p in a variety X, the degree of p is defined to be the degree of the residue field at p, or $[\kappa(p) : K]$.

Definition 2.1. Let X be a variety over a number field K. The *density degree set* $\delta(X/K)$ is the set of $d > 0$ such that the degree d points in X are Zariski dense. The *minimum density degree* $\min(\delta(X/K))$ is the smallest such degree.

We will be primarily interested in the case that X is a nice curve C, so from now on this will be assumed. One way to study low degree points is with the symmetric product:

$$\mathrm{Sym}^d_C := (\underbrace{C \times \cdots \times C}_{d \text{ times}})/S_d.$$

Where S_d acts by permuting the copies of C. When C is a nice curve over K, the symmetric product will itself be smooth d-dimensional variety over K. Given a degree d-point p in C, we can associate a K-rational point in Sym^d_C. One can first think of p as a size d orbit of $\mathrm{Gal}(\overline{K}/K)$ on $C(\overline{K})$. We can then consider the $d!$-possible orderings of this orbit in $(C \times \cdots \times C)(\overline{K})$, which will be permuted by the action of D_K, and thus give rise to a single point in $(C \times \cdots \times C)/S^n(\overline{K})$, fixed by $\mathrm{Gal}(\overline{K}/K)$, and thus rational.

This argument can be expanded more broadly to say that the K-points of Sym^d_C are in bijection with the degree d effective divisors of C [VV25]. With this in mind, we can translate the minimum density degree of C to a fact about Sym^d_C as follows:

Lemma 2.1.1. Let C be a nice curve over K. Then

$$\min(\delta(C/K)) = \min\{d : (\mathrm{Sym}^d_C)(K) \text{ is infinite}\}$$

Proof. Let $d = \min(\delta(C/K))$, then $(\mathrm{Sym}^d_C)(K)$ will have one point for every degree d point in C, which are infinite. Thus, $(\mathrm{Sym}^d_C)(K)$ is infinite.

Conversely, let $d = \min\{d : (\mathrm{Sym}^d_C)(K) \text{ is infinite}\}$. Then there are an infinite number of degree d divisors D, which may all be written as sums $\sum p_i$, where $\sum \deg p_i = d$. Because there are only a finite number of points with degree $< d$, there can only be a finite number of such sums where all $\deg p_i$ are strictly less than d. Thus, the infiniteness of degree d divisors must arise from infiniteness of degree d points, or sums that consist of a single point. □

A canonical way to study Sym^d_C (and thus the minimum density degree) is via the degree d Abel-Jacobi map:

$$\Phi^d : \mathrm{Sym}^d_C \to \mathrm{Pic}^d_C$$

This map can be thought of as mapping degree d effective divisors to their associated vector bundle. We write W^d_C for the image of Φ^d_C under this map. In effect, there are two ways that Sym^d_C can have infinite K-points: Either there are an infinite number of K-points in the image W^d_C, or there are an infinite number in the fiber of some K-point in Pic^d_C[1]. We start by considering the first case, related to the image of Φ^d. Assuming Pic^d_C contains a rational point (as otherwise we'd have trivially that Sym^d_C has none either), there is an isomorphism $\mathrm{Pic}^d_C \cong \mathrm{Pic}^0_C$ defined via subtraction of this point. Under this isomorphism, we can exploit the abelian structure of the Jacobian, and apply the Mordell-Lang Conjecture:

Theorem 2.2. Let $W \subseteq P$ be a closed subvariety of an abelian variety over a number field K. Then:

$$W(K) = \bigcup_{i=1}^{N} (w_i + A_i(K))$$

Where $w_i \in W(K)$ and A_i are abelian subvarieties $A_i \subseteq P$ such that $w_i + A_i \subseteq W$.

Because the rational points of W^d_C may be decomposed in this way, it follows that W^d_C can contain infinite rational points only if it contains some positive rank abelian subvariety. Thus, we can think of the first situation as being detecting the existence of certain abelian varieties.

We now turn our attention to the second situation. Given some point $p \in \mathrm{Pic}^d_C(K)$, p will correspond to a degree d line bundle $\mathcal{L}$, and the fiber of p under the Abel-Jacobi map will retrieve the complete linear system of this $\mathcal{L}$, or $\mathbb{P}H^0(\mathcal{L})$. It follows that this fiber will have infinite K-points if and only if $\mathcal{L}$ has more than 2 global sections, which corresponds to $\mathcal{L}$ inducing a degree d map $C \to \mathbb{P}^1$ (potentially of lower degree after removing the base locus).

Conversely, given a map $\pi : C \to \mathbb{P}^1$ of degree d, the Hilbert Irreducibility Theorem implies that there is a Zariski-dense set of K-points $x \in \mathbb{P}^1$ such that the preimage of x in C is a degree d-point [VV25]. In this sense we are mapping C onto a one-dimensional linear subspace of $\mathbb{P}H^0(\mathcal{L})$, where

[1] This explanation, requiring infinitude of rational points to come either from the fibers or the image is intentionally imprecise. See [VV25] for a more rigorous treatment.

$\mathcal{L}$ is the line bundle inducing π. The upshot here is that positive-dimensional fibers over K-points in Pic_C^d are really detecting the existence of degree d or lower maps into $\mathbb{P}^1$. This discussion leads us to the following definitions:

Definition 2.2. Let C be a nice curve over a number field K. Let $x \in C$ be a degree d closed point.

We say x is $\mathbb{P}^1$-*parametrized* if there exists a morphism $\pi : C \to \mathbb{P}^1$ with $\deg \pi = \deg x$ and $\pi(x) \in \mathbb{P}^1(K)$. Otherwise it is $\mathbb{P}^1$-*isolated*.

We say x is *AV-parametrized* if there exists a positive rank abelian subvariety $A \subseteq \mathrm{Pic}_C^d$ such that $[x] + A \subseteq W_C^d$. Otherwise it is *AV-isolated*.

In general, we say x is *parametrized* if it is either $\mathbb{P}^1$-parametrized or AV-parametrized. Otherwise we say x is *isolated*.

As we imprecisely argued before, infinite degree d-points ought to arise either in $\mathbb{P}^1$-parametrized families or AV-parametrized families, corresponding either to positive dimensional fibers or to positive-rank abelian subvarieties of W_C^d. This argument can be made precise to say not only that there are a finite number of isolated degree d points, but that there are a finite number of isolated points across all degrees:

Theorem 2.3. [Bou+19] Let C be a nice curve over a number field.

(1) There are finitely many isolated points on C.

(2) There are infinitely many degree d points if and only if there exists a degree d parametrized point.

By the Hilbert Irreducibility theorem, existence of a degree d $\mathbb{P}^1$-parametrized point is equivalent to existence of a degree d map $C \to \mathbb{P}^1$, so the minimum degree containing a degree d parametrized point is just $\mathrm{gon}_K C$. As we will demonstrate in Subsection 2.3, calculating gonality is often very simple. On the other hand, detecting positive-rank abelian varieties inside W_C^d is often extremely difficult, and so most techniques for calculating $\mathrm{gon}_K C$ focus on proving when such abelian subvarieties cannot exist. One such result is the following:

Theorem 2.4. [SV22] Let S be a nice surface with $h^1(S, \mathcal{O}_S) = 0$, and let C be a smooth, ample curve on S. Then for $e < C^2/9$, the locus $W_e C$ contains no positive-dimensional abelian varieties.

Before continuing, it is worth making some comments about what fields we'll be considering in this discussion. Much of the literature on toric surfaces treats exclusively toric surfaces over $\mathbb{C}$. For a field that is not algebraically closed, such as number fields, there is a richer theory of toric varieties where the Galois group $\mathrm{Gal}(\overline{K}/K)$ is allowed to have a nontrivial relationship with the combinatorial structure of the surface. However, if we require that all toric surfaces under consideration are *split*, (in other words that they contain $(K^*)^2$), one retrieves a theory isomorphic to that over the complex numbers. For more on nonsplit toric surfaces, see [Eli+14] or the expository article [McF22].

On the "abelian variety side," our results will primarly rely on Theorem 2.4, and thus only relate the the intersection theory of the surface, which may be computed over the complex numbers. On the "gonality" side of the equation, the computation becomes slightly more complicated, necessitating the split hypothesis included in Theorem 1.1.

Before leaving this section, we note some the following results helpful results about nice curves:

Lemma 2.4.1. [KV25] Suppose that $\min(\delta(C/K)) = 2$. Then C emits a degree 2 cover of $\mathbb{P}^1$ or an elliptic curve.

Lemma 2.4.2. Suppose C emits a degree 3 map to the projective line $\mathbb{P}^1$ and a degree 2 map to an elliptic curve. Then $g(C) \leq 4$.

Proof. We consider the product map $C \to \mathbb{P}^1 \times E$. Because (2) and (3) are coprime, this map will be birational onto its image C'. We can calculate the arithmetic genus of C' via the adjunction formula:

$$p_a(C') = 1 + \frac{1}{2}(C^2 + K_{\mathbb{P}^1\times E} \cdot C) = 1 + \frac{1}{2}(12 - 6) = 4$$

Here we are using the fact that $\mathcal{O}_E(C) \cong \mathcal{O}_E(3e) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2)$ for some closed point $e \in E$, and that $K_{\mathbb{P}^1\times E} = \mathcal{O}_E \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)$. Because C covers C' birationally, it follows that $g(C) \leq p_a(C') = 4$. □

2.2. **Curves in Toric Surfaces.** In this subsection, we will establish notation and fundamental results for discussion of toric surfaces, and specific curves inside of them. Our notation for toric surfaces, lattice polygons, and normal fans will follow [CLS11], whereas notation for curves will follow [CC17].

Throughout the paper, we will use M and N to denote the character lattice and lattice of one-parameter subgroups of a 2-dimensional torus T_N, and use inner product notation to denote the dual relation $\langle\cdot,\cdot\rangle : M \times N \to \mathbb{Z}$. Because we are interested exclusively in toric surfaces, we will assume that M and N both have rank 2, and will fix isomorphisms $M \cong \mathbb{Z}^2$ and $N \cong \mathbb{Z}^2$ such that two bases are dual. Under this isomorphism, we may unambiguously write $T_N = \operatorname{Spec}\mathbb{C}[x^\pm, y^\pm]$, where x and y are characters corresponding $(1,0)$ and $(0,1)$. Under our identification, an element $(a,b) \in M$ corresponds to the character $\operatorname{Spec}\mathbb{C}[x,y] \to \mathbb{C}^*$ defined $(x,y) \mapsto x^a y^b$, and elements $(c,d) \in \mathbb{Z}^2 \cong N$ correspond two one-parameter subgroups $z \mapsto (z^c, z^b)$ (We maintain the notation for M and N to differentiate the two dual lattices, although we will treat both as $\mathbb{Z}^2$).

The notation $M_\mathbb{R}$ and $N_\mathbb{R}$ will be used to denote extensions of scalars $M \otimes_\mathbb{Z} \mathbb{R}$ and $N \otimes_\mathbb{Z} \mathbb{R}$, respectively. Given a convex lattice polygon $P \subseteq M_\mathbb{R}$ or a fan Σ in N, we write $T_N \subseteq X_P$ or $T_N \subseteq X_\Sigma$ for the associated toric surfaces, respectively (in the polygon case, this toric surface can be obtained by first taking the interior fan Σ_P and then the toric surface associated with it).

When discussing a smooth irreducible curve $C \in X_P$, we are usually interested in the case that C is not contained in the toric boundary (the boundary itself is a union of rational curves). When this is the case, there is a dense subset of C contained in T_N. We can write $T_N = \operatorname{Spec}\mathbb{C}[x^\pm, y^\pm]$, where x and y are characters forming a basis for M. Because C is irreducible, it follows that $C \cap T_N$ is cut out by some Laurent polynomial $f \in \operatorname{Spec}\mathbb{C}[x^\pm, y^\pm]$. Because only a finite number of points in C are outside T_N, discussing smooth curves C in toric surfaces can be translated entirely into discussing the properties of nicely behaved Laurent polynomials $f \in \mathbb{C}[x^\pm, y^\pm]$. There are a few possible candidates for this nice behavior, but we will use the notion of Newton-polygon nondegeneracy, which will will proceed to define.

We define the *Newton polygon* P of a Laurent polynomial $f \in \mathbb{C}[x^\pm, y^\pm]$ is the convex hull of the support of f in M (Note that in this case the lattice M is just monomials $\{x^i y^j\}_{(i,j)\in\mathbb{Z}^2}$). Given a face $\tau \subseteq P$ of the Newton polygon, we will write $f_\tau \subseteq P$ for the polynomial containing all the monomials of f supported on τ. See Example 2.5.1 for an example.

Definition 2.3. Given a Laurent polynomial $f \in \mathbb{C}[x^\pm, y^\pm]$ with associated Newton polygon P, we say that f is *P-non-degenerate* if for every face $\tau \subseteq P$ (including P itself)

$$\frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = f = 0$$

has no solutions in T_N.

Example 1. Consider the polynomial $f = x^3 + 3x + xy + y^2 + 1$. We adopt the notation of writing $x^i y^j$ as $(i,j) \in \mathbb{Z}^2$ (identifying M with $\mathbb{Z}^2$). Then the Newton polygon will be $P = \operatorname{conv}((0,0),(3,0),(0,2))$. In this case the polygon has eight nonempty faces: three corresponding to individual points, three corresponding to edges of the triangle, and one corresponding to all of P. We will call the horizontal face τ_1, the vertical face τ_2, and the diagonal face τ_3. We call the three points

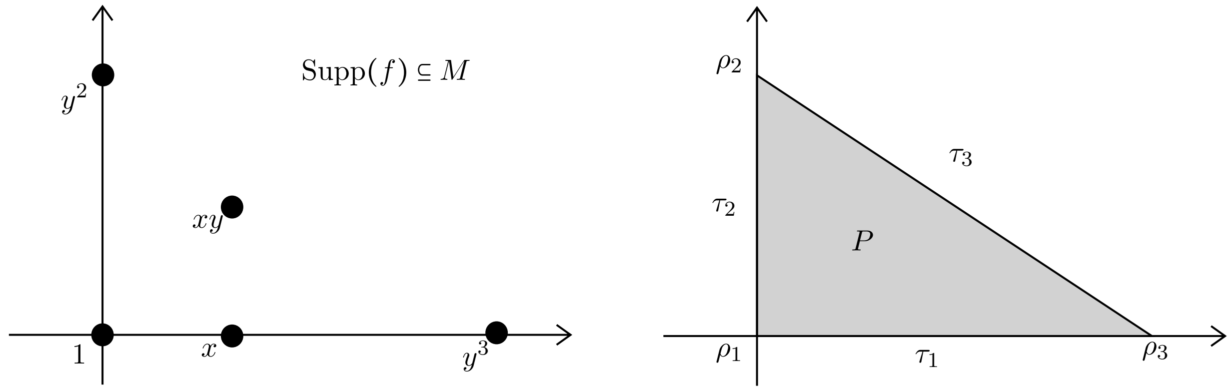

FIGURE 1. Support of the polygon f in Example 1

ρ_1, ρ_2, ρ_3 clockwise from the origin. This labelling is pictured on the right half of Figure 1. The corresponding polynomials will be as follows:

$$f_P = x^3 + 3x + xy + y^2 + 1$$

$$\begin{aligned} f_{\tau_1} &= x^2 + 3x + 1 & f_{\rho_1} &= 1 \\ f_{\tau_2} &= y^2 + 1 & f_{\rho_2} &= x^3 \\ f_{\tau_3} &= x^3 + y^2 & f_{\rho_3} &= y^2 \end{aligned}$$

From here one can check by hand that all such polynomials have the property that they and their derivatives do not vanish anywhere on the torus.

It will become clear through the remainder of this subsection why P-nondegeneracy is a suitable choice. We note briefly that, while it may appear to be a lot of hypotheses, one can prove that a dense subset of polynomials f supported on a given polygon P will be P-nondegenerate [CC12]. Recall now that we can think of the toric surface X_P as the closure of the embedding

$$\phi_P : T_N \hookrightarrow \mathbb{P}^{\#(P \cap M)-1} \qquad (x, y) \mapsto (x^i y^j)_{(i,j) \in P \cap M}$$

[CLS11]. If $f = \sum_{(i,j) \in P \cap M} a_{i,j} x^i y^j$, then the scheme-theoretic image of $V(f)$ under this map will be the intersection of X_P with the hypersurface H, where $H = \sum_{(i,j) \in P \cap M} a_{i,j} X_{i,j}$, where $X_{i,j}$ is the homogenous coordinate associated (i, j). We may now motivate the following lemma:

Lemma 2.4.3. [CC12]. Let $f \in \mathbb{C}[x^{\pm}, y^{\pm}]$ be a P-nondegenerate Laurent polynomial. Then the closure of $V(f) \subseteq T_N \subseteq X_P$ is a smooth curve whose associated divisor is Cartier and very ample.

Conversely, any smooth curve $C \subseteq X_P$ meeting toric divisors transversely and not meeting toric points will restrict to the vanishing of some P-nondegenerate Laurent polynomial on $T_N = \operatorname{Spec} k[x^{\pm}, y^{\pm}]$.

In the first part of the statement, because the image in X_P is smooth and birational to $V(f)$, it will be our curve of interest C_f. The upshot here is that the Newton polygon P, through its associated toric surface, provides a natural setting to understand the geometry of C_f in its ambient space. We now discuss some facts about divisors on toric surfaces. Throughout this discussion, we will assume that Σ is a complete (lattice?) fan in $N_{\mathbb{R}}$. We will write $\{\sigma_1, \cdots, \sigma_n\} = \Sigma(1)$ for the one-dimensional cones in Σ. Each σ_i will contain a unique generating lattice point, which we'll name $\mu_i \in N$. To each σ_i, the orbit-cone correspondence establishes an associated one-dimensional, lattice invariant subspace, which we name D_σ. We refer to these as the *prime toric divisors*. With this notation set, we collect the following useful lemmas:

Lemma 2.4.4. [CLS11] Every Weil divisor D on X_Σ is linearly equivalent to a sum of prime toric divisors.

We won't prove this fact, but in our case of interest, it is more or less trivial: given a curve $C_f \subseteq \Sigma_P$, one can consider $C_f - \mathrm{div}(f) \sim C_f$, which will have support outside the torus and thus can be represented by toric divisors. To a torus-invariant divisor $D = \sum_{\sigma\in\Sigma(1)} a_i D_i$, we associate the polygon $P_D = \bigcap_{i\in\Sigma(1)} H_i \subseteq M_\mathbb{R}$, where H_i is the half-plane defined $H_i = \{m \in M_\mathbb{R} : \langle m, \mu_i\rangle \geq -a_i\}$.

There is a similar procedure in the opposite direction: given a polygon P, there is a correspondence between facets F and one-dimensional fans σ of the interior fan of P (Thinking of M as $N^\wedge$, one can think of this one-dimensional fan as the internal orthogonal direction to F). If we write $\mu_F \in N$ for the primitive vector in this direction, we can express P as

$$P = \{m \in M_\mathbb{R} | \text{for all } F \text{ a facet of } P, \langle m, \mu_F\rangle \geq -a_F\}$$

for some integers a_F. This description suggests that we should consider the toric divisor $\sum_{f\in\Sigma(1)} a_F D_F$ on X_P. For more on this procedure, see [CLS11] or Example 2.

Theorem 2.5. [CLS11] The maps $P \mapsto (X_P, D_P)$ and $(X_\Sigma, D) \mapsto P_D$ establish a bijection between the sets:

$$\{P \subseteq M_\mathbb{R} | P \text{ is a full-dimensional lattice polytope}\}$$

and

$$\{(X_\Sigma, D) | \Sigma \text{ a complete fan in } N_\mathbb{R}, D \text{ a torus-invariant ample divisor on } X_\Sigma\}.$$

This description of ample divisors extends nicely to our questions about curves specifically:

Lemma 2.5.1 (in Castryck, look earlier)**.** Let $f \in K[x^\pm, y^\pm]$ be a Laurent polynomial nondegenerate with respect to its Newton Polygon P. Let $D \sim C_f$ be a torus invariant divisor linearly equivalent to C_f. Then, up to translation, $P_D = P$.

The translation ambiguity amounts to different choices of linearly equivalent divisor D. We finish with a way to translate intersections on X_Σ to geometry of associated Newton polygons:

Theorem 2.6. [CC12] Let D_1, D_2 be a nef (or ample) torus invariant divisors on a smooth toric surface X_Σ. Then:

$$D_1 \cdot D_2 = \mathrm{Area}(P_{D_1} + P_{D_2}) - \mathrm{Area}(P_{D_1}) - \mathrm{Area}(P_{D_1}).$$

Where addition of polygons is the Minkowski sum. In particular, if the Newton polygon of f is smooth, then

$$C_f^2 = \mathrm{Area}(P + P) - \mathrm{Area}(P) - \mathrm{Area}(P) = 2\,\mathrm{Area}(P).$$

Lastly, we can relate the genus of C_f to the Newton polygon with the following classical result:

Theorem 2.7. (Baker's Theorem) [CC17] Let $f \in \mathbb{C}[x^\pm, y^\pm]$ be a Laurent polynomial nondegenerate with respect to its Newton Polygon P_f. Then the genus of C_f is equal to the number of internal lattice points of P_f.

Example 2. Consider the polynomial $f = -x^2 + y^2 + x^4 + 2y^4 = 0$. Again identifying $M = \mathbb{Z}^2$ with usual xy-axes, it follows that $P_f = \mathrm{Conv}((2,0), (4,0), (0,2), (0,4))$. We claim, but will not prove, that f is P_f-nondegenerate. Going clockwise from the vertical edge, we name the edges F_1, F_2, F_3, and F_4. The associated internal normals (which will in turn be generators for the one dimensional cones in the internal fan Σ_{P_f}) will be

$$\mu_{F_1} = (1,0), \qquad \mu_{F_2} = (-1,-1), \qquad \mu_{F_3} = (0,1), \qquad \mu_{F_3} = (1,1).$$

This is visualized in Figure 2. This fan is canonically the fan associated with the projective plane blown up at the origin $\mathrm{Bl}_{(0,0)}\,\mathbb{P}^2_\mathbb{C}$. If we use D_i to refer to the irreducible toric divisor associated to μ_{F_i}, then under this description, D_4 will be the exceptional divisor, D_2 will be any line not

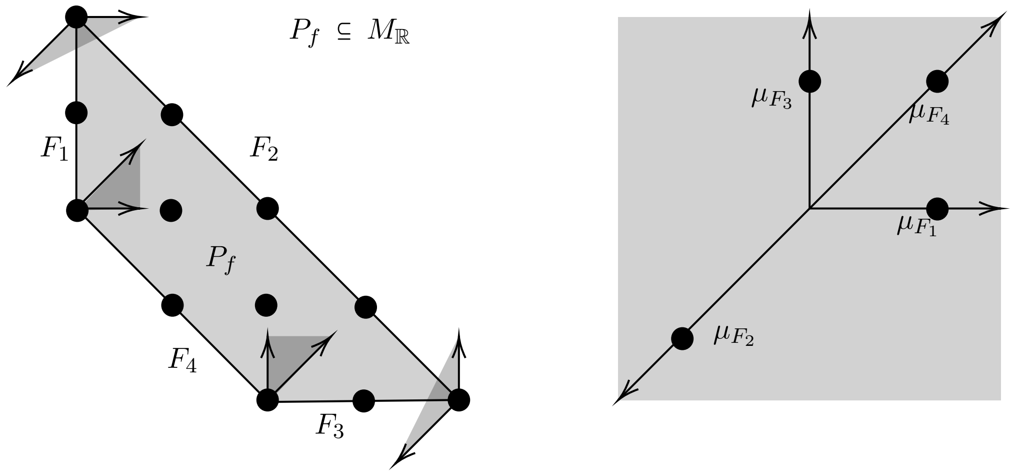

FIGURE 2. Newton polygon P_f of polynomial f appearing in Example 2 on the left, along with the associated internal fan on the right, assembled from dual cones to the vertices of P_f.

crossing the origin, and $D_1 \simeq D_3$ will be two lines through the origin. More on this identification can be found in [CLS11]. It is classical that these classes will generate $\operatorname{Pic} X_{P_f} = \operatorname{Pic} \operatorname{Bl}_{(0,0)} \mathbb{P}^2$, with relations $D_1 = D_3 = D_2 - D_4$, and intersection information

$$D_4^2 = -1, \qquad D_2 \cdot D_4 = 0, \qquad D_2^2 = 1,$$

(from which we can deduce intersections involving the two other divisors). We will now calculate C_f^2 in two ways. For the first method, because C_f is ample in X_{P_f}, Theorem 2.6 tells us that $C_f^2 = 2\operatorname{Area}(P_f) = 12$.

For the second, we would like to compute an explicit torus-invariant divisor D that is linearly equivalent to C_f, and use the intersection information above to complete our calculation. We begin by computing the divisor P_D associated with the Newton polygon. To each face F_i we can associate a half-plane $H_i = \{m \in M_{\mathbb{R}} | \langle m, \mu_{F_i} \rangle \geq -a_i\}$ that consists of those points on the interior side of F_i. For instance, points on the interior side of F_1 are just points to the right of the x-axis, which reflects the fact that:

$$H_1 = \{m \in M_{\mathbb{R}} | \langle m, (0,1) \rangle \geq 0\} \implies a_1 = 0.$$

We similarly find that $a_3 = 0$. We observe that H_4 is those points $m = (x, y)$ such that $x + y \geq 2$, and H_2 is those points (x, y) such that $x + y \leq 4$. We can rewrite these as:

$$H_4 = \{m \in M_{\mathbb{R}} | \langle m, (1,1) \rangle \geq 2\} \qquad \text{and} \qquad H_2 = \{m \in M_{\mathbb{R}} | \langle m, (-1,-1) \rangle \geq -4\}.$$

It follows that $a_4 = -2$ and $a_2 = 4$, and so $D_{P_f} = 4D_2 - 2D_4$. By Theorem 2.5 and Lemma 2.5.1, it follows that $D_{P_f} \sim C_f$, and so we can compute

$$C_f^2 = D_{P_f}^2 = (4D_2 - 2D_4)^2 = 16D_2^2 - 8D_2 \cdot D_4 + 4D_4^2 = 16 - 4 = 12.$$

Matching our previous computation. Lasty, we'll verify Baker's Theorem, or Theorem 2.7, which suggests that the genus of C_f should be equal to the number of interior lattice points of P_f. Since the only interior points are $(2,1)$ and $(1,2)$, this would suggest $g(C_f) = 2$.

It is well known that the canonical divisor on $\operatorname{Bl}_{(0,0)} \mathbb{P}^2$ is $K_X = D_4 - 3D_2$, since D_2 is a line not through the origin and D_4 is the exceptional divisor. It follows from the adjunction formula that:

$$\begin{aligned} 2g - 2 &= (K_X + C) \cdot C = (D_4 - 3D_2 + 4D_2 - 2D_4) \cdot (4D_2 - 2D_4) = (D_2 - D_4) \cdot (4D_2 - 2D_4) \\ &= 4D_2^2 + 2D_4^2 = 4 - 2 = 2 \end{aligned}$$

It follows that $g = 2$, as predicted.

2.3. **Gonality and the Newton Polygon.** Throughout this section, $f \in K[x^{\pm}, y^{\pm}]$ is an irreducible Laurent polynomial, $V(f)$ is the curve it defines in $(K^*)^2$, P_f is its Newton polygon, and C_f is the closure of $V(f)$ in X_{P_f}. In this subsection we will recount the main results of [CC12] and [CC17] to relate the gonality of a curve C_f to an invariant of the Newton Polygon called the lattice width. We will not attempt to prove these results, but will briefly motivate them. The results in these papers are primarily concerned with the complex case, but we will mention where they can or cannot be extended to other fields, in particular number fields for our purposes. Recall that the gonality of a curve C is the lowest d such that there is a degree d map $C \to \mathbb{P}^1$. Note that our lattice M can be thought of as the set of characters, or maps $T_N \to K^*$ preserving the group structure. If we think of K^* as sitting inside $\mathbb{P}^1$, then this restricts to a map $V(f) \to \mathbb{P}^1$. Because $V(f)$ is a dense subset of C_f, by the curve to projective theorem we can extend to a map $C_f \to \mathbb{P}^1$. The upshot here is a map to the projective line associated to every lattice element M. As it turns out, these combinatorial maps often completely explain the gonality of the curve C.

Before proceeding, we note that in terms of detecting gonal maps, we only need to consider primitive lattice elements. This is because if we have some character $(a, b) \in M$, and some other character $(da, db) \in M$, we can factor the associated maps $(da, db) : T_N \to \mathbb{C}^*$ into $T_N \to \mathbb{C}^* \to \mathbb{C}^*$, where the first map takes that associated to (a, b) and the second map is $z \mapsto z^d$. Therefore, the degree of (da, db) will be d times the degree of (a, b). This motivates the following definitions:

Definition 2.4. A *lattice direction* $(a, b) \in \mathbb{Z}^2$ is an element such that $\gcd(a, b) = 1$.

The *width* of a nonempty lattice polygon $P \subseteq \mathbb{R}^2$ *with respect to a lattice direction* $v = (a, b)$ is:

$$w(P, v) := \max_{(c,d)\in P\cap\mathbb{Z}^2}(ad - bc) - \min_{(c,d)\in P\cap\mathbb{Z}^2}(ad - bc)$$

The *lattice width* $\mathrm{lw}(P)$ of a polygon P is $\min_v w(P, v)$, where the minimum is taken over all lattice directions.

The width in a certain lattice direction can be thought of as the number of lines parallel to that lattice direction passing through the polygon. For instance, if the lattice direction is $v = (0, 1)$, the width $w(P, v)$ would be the number of horizontal lattice lines passing through the polygon, and therefore just the height of the polygon in the y direction. Intuitively, the lattice width is how much P can be "flattened" by applying unimodular transformations. As it turns out, there is a strong relationship between the width in a lattice direction (a, b) and the degree of the map $(x, y) \to x^n y^m$ on $V(f)$.

Example 3. Suppose we have some Laurent polynomial $f(x, y) \in K[x^{\pm}, y^{\pm}]$ that can be written in the form $f(x, y) = \sum_{n=n_{\min}}^{n_{\max}} y^n g_n(x)$ Where $n_{\min}$ is the minimum degree of y showing up in $f(x, y)$, $n_{\max}$ is the maximum such degree, and each $g_n(x) \in K[x^{\pm}]$. Consider now the map $\Phi : (K^*)^2 \supseteq V(f) \to K^*$ defined $(x, y) \mapsto x$. Because $g_{n_{\min}}$ and $g_{n_{\max}}$ are nonzero, for a general point $x_0 \in K^*$ one has that $g_{n_{\min}}(x_0), g_{n_{\max}}(x_0) \neq 0$. It follows that the fiber over x_0 will consist of solutions $y \in K^*$ to the equation $\sum_{n=n_{\min}}^{n_{\max}} y^n g_n(x_0) = 0$. Because y is nonzero, we can freely divide by $y^{n_{\min}}$, and rewrite the equation:

$$\sum_{n=0}^{n_{\max}-n_{\min}} y^{n-n_{\min}} g_{n-n_{\min}}(x_0) = 0$$

Because neither the constant or top term vanishes, it follows that this equation will have $n_{\min} - n_{\max}$ solutions with multiplicity in K^*. Because this is true over a general point x_0 in the target, it follows that Φ has degree $n_{\min} - n_{\max}$. We note that Φ is the map induced by the lattice direction $v = (1, 0)$, and that:

$$w(P_f, v) = \max_{(c,d)\in P_f\cap\mathbb{Z}^2}(1d - 0c) - \min_{(c,d)\in P_f\cap\mathbb{Z}^2}(1d - 0c) = \max_{(c,d)\in P_f\cap\mathbb{Z}^2}(d) - \min_{(c,d)\in P_f\cap\mathbb{Z}^2}(d)$$

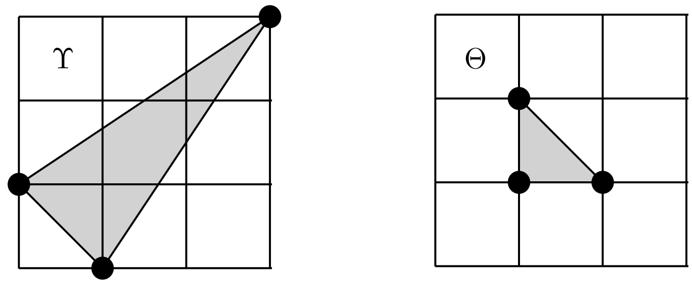

FIGURE 3. The lattice polygons Υ and Θ

Because elements $(c,d) \in P_f \cap \mathbb{Z}^2$ correspond to monomials $x^c y^d$ appearing in $f(x,y)$, it follows that the maximum and minimum values of d are just $n_{\max}$ and $n_{\min}$, respectively, and so $w(P_f, v) = \deg \Phi$.

This equality holds in general:

Theorem 2.8. [CC12] Let $f \in \mathbb{C}[x^{\pm}, y^{\pm}]$ be a Laurent polynomial with Newton polygon Δ, and let $v \in \mathbb{Z}^2 = M$ be a lattice direction. Then:
(1) Then the induced map $\Phi : V(f) \to \mathbb{C}^*$ has degree $w(P_f, v)$.
(2) If C_f is the closure of $V(f)$ in X_{P_f}, $\mathrm{gon}_{\mathbb{C}}(C_f) \leq \mathrm{lw}(P_f)$.

We note that the second statement is an immediate corollary of the first, since maps into $\mathbb{C}^*$ may be extended to maps into $\mathbb{P}^1_{\mathbb{C}}$, and the gonality is defined to be the minimum degree of all such maps. The first statement may be reduced to the case in Example 3, as any lattice direction v may be sent to $(1,0)$ via a unimodular transformation. This transformation would induce an automorphism on the torus T_N, and the same computation may be made under this automorphism. While this theorem is proven over the complex numbers, it can be extended freely to any number field:

Corollary 2.8.1. Let K be a number field and let $f \in K[x^{\pm}, y^{\pm}]$ be a Laurent polynomial with Newton polygon Δ, and let $v \in \mathbb{Z}^2 = M$ be a primitive lattice direction. Then:
(1) Then the induced map $\Phi : V(f) \to K^*$ has degree $w(P_f, v)$.
(2) If C_f is the closure of $V(f)$ in X_{P_f}, $\mathrm{gon}_K(C_f) \leq \mathrm{lw}(P_f)$.

Proof. (1) The map Φ associated to a lattice element (a,b) is of the form $(x,y) \mapsto x^a y^b$, and thus can be defined over any field. We can freely calculate degree of this map over the complex numbers, where Theorem 2.8 implies that $\deg \Phi = w(P_f, v)$ over the complex numbers, and thus over K as well. (2) Follows immediately by considering a lattice direction with minimal width. $\square$

As it turns out, in a large number of cases, this inequality can be strengthened to an equality. First, we adopt the notation $P^{(1)}$ to represent the internal lattice polygon of a polygon P, or the convex hull of the points in the interior of P. We also fix the names Υ and Θ for the two polygons in Figure 3 (Note that in [CC17], Θ is referred to as Σ. We rename it here to avoid confusion with our notation for lattice fans). We can then state the following theorem (Corollary 6.2 in [CC17]):

Theorem 2.9. [CC17] Let $f \in K[x^{\pm}, y^{\pm}]$ be non-degenerate with respect to its Newton polygon P_f. Then $\mathrm{gon}_{\mathbb{C}} C_f$ equals $\mathrm{lw}(P_f^{(1)}) + 2$ unless $P_f^{(1)} \cong \Upsilon$, in which case it equals 3.

Here $\cong$ indicates equality up to equivalence by translation and unimodular transformation. Because we will mostly be looking at smooth lattice polygons, and Υ is not smooth, it follows that, in the P-nondegenerate case, we will mostly be able to compute the gonality directly. Furthermore, we can very simply categorize the lattice width of $P_f^{(1)}$:

Lemma 2.9.1. Let P be a two dimensional lattice polygon. Then $\mathrm{lw}(P^{(1)}) = \mathrm{lw}(P) - 2$ unless $P \cong d\Theta$ for some $d \geq 2$, in which case $\mathrm{lw}(P^{(1)}) = \mathrm{lw}(P) - 3$.

In toric terms, this corresponds to when P represents an ample class on the the plane $\mathbb{P}^2$. This can be translated to the fact that smooth plane curves of degree d with a rational point are gonality at most $d-1$, as projection from a rational point will define a map of this degree. However, this map is *not* in general toric, or corresponding to some element of the character lattice $(x, y) \mapsto x^n y^m$. The above theorem suggests that the existence of this map is somehow unique to $\mathbb{P}^2$, and does not occur in any other toric surface. Before finishing this section, we note the two following facts about lattice width, which will be useful for some of our examples later on:

Lemma 2.9.2. [FM74] Let P be a lattice polygon. Then $\mathrm{lw}(P)^2 \leq \frac{8}{3}\mathrm{Vol}(P)$.

Lemma 2.9.3. Let P be a lattice polygon, let v be a lattice direction, and let $m \in \mathbb{Z}^2$ be a lattice point. Suppose that P contains the line of points $\{m, m+v, m+2v, \cdots m+nv\}$, where $n \geq 0$. Then for all other lattice directions $u \neq v$, $w(P, u) \geq n$.

Proof. The theorem is preserved under unimodular transformation, so we may assume that $u = (0,1)$ and $v = (a, b)$, where $a \geq 0$. It follows that:

$$w(P, v) = \max_{(c,d) \in P \cap \mathbb{Z}^2}(c) - \min_{(c,d)}(c).$$

By considering the points m and $m + nv$, one finds that the maximum will $\geq m_1 + av$, and the minimum will be $\leq m_1$. Therefore, $w(P, v) \geq na \geq n$. □

3. Proof of Main Theorem

In this section we will prove Theorem 1.1 and some of its immediate corollaries:

Proof. (Of Theorem 1.1) First, observe that C cannot be contained in the toric boundary, as in this case it would need to be equal to one of the toric divisors D, which is necessarily non-ample. Therefore an open subset of C is contained in the torus. Because C is split, this torus is of the form $\operatorname{Spec} K[x^{\pm}, y^{\pm}]$, and so C can be realized as the vanishing of some Laurent polynomial $f \in \operatorname{Spec} K[x^{\pm}, y^{\pm}]$. The Newton polygon of this Laurent Polynomial will in fact be the polygon P associated to C because of the correspondence established in Theorem 2.5 and Lemma 2.5.1. in this situation, we can apply Corollary 2.8.1, which says that $\mathrm{gon}_K C \leq \mathrm{lw}(P)$.

On the abelian variety side, Theorem 2.4 tells us that, for every $d \leq C^2/9$, there are no positive-dimensional abelian varieties contained inside W_C^d, which in particular means that there are no AV-Parametrized points of such degree d. By Theorem 2.6, we can rewrite:

$$\frac{C^2}{9} = \frac{2}{9}\mathrm{Vol}(P).$$

Because $\mathrm{gon}_K \leq \mathrm{lw}(P) \leq \frac{2}{9}\mathrm{Vol}(P)$, for any $e < \mathrm{gon}_K C$, there cannot be any AV-parametrized points of degree e. Because the minimum density degree is the minimum of the smallest degree containing a parametrized point, and no degree lower than the gonality contains a parametrized point, it follows that $\mathrm{gon}_K C = \min(\delta(C/K))$ as desired.

For the bounded volume statement, it will suffice to show that $\mathrm{lw}(P) > \frac{2}{9}\mathrm{Vol}(P) \implies \mathrm{Vol}(P) < 54$. By Lemma 2.9.2, for any lattice polygon we have that $\mathrm{lw}(P)^2 \leq \frac{8}{3}\mathrm{Vol}(P)$. Squaring our first equation, we find:

$$\frac{4}{81}\mathrm{Vol}(P)^2 < \mathrm{lw}(P)^2 \leq \frac{8}{3}\mathrm{Vol}(P)$$

Dividing by $\operatorname{Vol} P$ on either side and simplifying, one finds that $\operatorname{Vol}(P) < 54$.

For the final statement of the theorem, we note that isomorphism classes of ample classes on toric surfaces are in one-to-one correspondence with lattice polygons up to unimodular equivalence. Therefore, to show there are a finite number of counterexamples, it will suffice to show that there are a finite number of smooth lattice polygons having $\operatorname{Vol}(P) < 54$ up to unimodular equivalence. This is a special case of a Theorem of Lagarius and Ziegler [LZ91], that there are a finite number of lattice polytopes of any dimension with bounded volume. We are done. □

In the case that the curve is P-nondegenerate, we can strengthen our result to say that the gonality is in fact the lattice width:

Corollary 3.0.1. Let C be a smooth curve in an ample class on a smooth, split toric surface X_Σ, with associated Newton polygon P, where:
(i) $P \not\cong d\Theta$ for $d \geq 2$,
(ii) $P^{(1)} \not\cong \Upsilon$,
(iii) C intersects toric divisors transversely, and
(iv) C does not meet any toric points of X_P.
Then $\min(\delta(C/K)) = \operatorname{gon}_K(C) = \operatorname{lw}(P)$.

Proof. Theorem 1.1 implies that $\operatorname{gon}_K(C) = \min(\delta(C/K))$. The third and fourth requirement are equivalent to C being of the form C_f for some P-non-degenerate laurent polynomial f. Theorem 2.9, along with the fact that $P^{(1)} \not\cong \Upsilon$ implies that $\operatorname{gon}_K C = \operatorname{lw}(P^{(1)}) + 2$. Because $P^{(1)} \not\cong d\Theta$, it follows from Lemma 2.9.1that $\operatorname{lw}(P^{(1)}) = \operatorname{lw}(P) - 2$. Putting these three equations together, we get the desired equalities. □

Because there are a finite number of Newton polygons of bounded weight up to unimodular equivalence, it follows that, classified by their associated polygons, there are a finite number of counterexamples when we range across all split, smooth toric surfaces X_Σ. For an immediate application of Theorem 1.1, we can consider certain well-behaved nodal plane curves.

Example 4. Let $C \subseteq \mathbb{P}^2_K$ be a degree $d \geq 5$ curve in $\mathbb{P}^2_K$ whose singular locus consists of three non-collinear simple nodes. After applying a linear transformation to $\mathbb{P}^2_K$, we may assume that these three points are in the coordinate directions $[1:0:0], [0:1:0]$, and $[0:0:1]$, which we'll name p_1, p_2, p_3. It follows that the strict transform of C in $\operatorname{Bl}_{p_1,p_2,p_3}\mathbb{P}^2_K$ will be a smooth curve. We'll call this strict transform C' and the blow-up map $\beta : \operatorname{Bl}_{p_1,p_2,p_3}\mathbb{P}^2_K \to \mathbb{P}^2_K$. However these are also the three toric fixed points of $\mathbb{P}^2$. If we let Σ be the standard lattice fan for $\mathbb{P}^2$, namely that generated by $\{(0,1),(1,0),(-1,-1)\}$, then the two toric points correspond to the three two dimensional fans:

$$\operatorname{Cone}((0,1),(1,0)), \operatorname{Cone}((1,0),(-1,-1)), \operatorname{Cone}((-1,-1),(0,1)).$$

These cones can be seen in the fan on the left of Figure 4. Blowing up all three points corresponds geometrically to taking the star subdivision of each cone. In this circumstance, this just means rewriting $\operatorname{Cone}(a,b) = \operatorname{Cone}(a, a+b), \operatorname{Cone}(a+b, a)$. The result is that $\operatorname{Bl}_{p_1,p_2,p_3}\mathbb{P}^1_K$ will be the toric variety associated to the complete fan Σ' whose edges have ray generators

$$\mu_1 = (0,1), \quad \mu_2 = (1,1), \quad \mu_3 = (1,0), \quad \mu_4 = (0,-1), \quad \mu_5 = (-1,-1), \quad \mu_6 = (-1,0).$$

We will use these names to refer to the ray generators in both Σ and Σ'. The blowup map $\beta : X_{\Sigma'} \to X_\Sigma$ corresponds to to the inclusion of $\Sigma \subseteq \Sigma'$. This is all visualized in Figure 4 We see now that the strict transform of $\tilde{C}$ of C may be written as: $\tilde{C} = \beta^{-1}C - 2E_1 - 2E_2 - 2E_3$, where E_i is the exceptional divisor associated to p_i.

Note that we can rewrite E_1, E_2, and E_3 as D_2, D_4, and D_6, as the exceptional divisors at the blown-up points will be precisely the toric divisors associated to the three new rays. Because C is

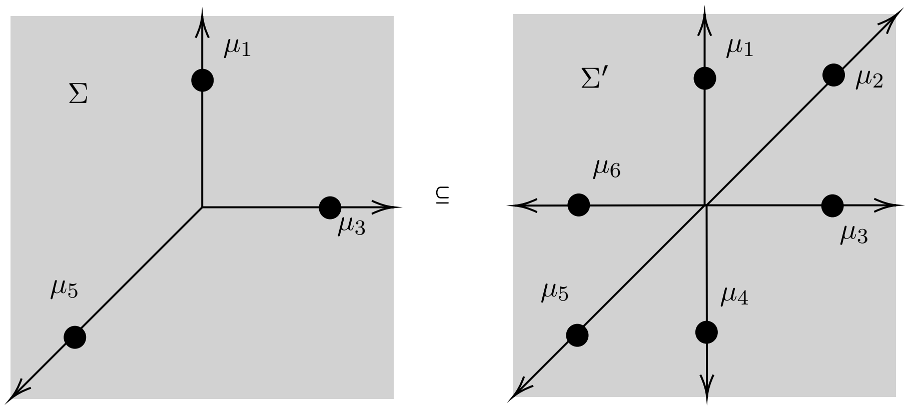

FIGURE 4. The lattice fan Σ for $\mathbb{P}^2$ sitting within the lattice fan Σ' for $\mathrm{Bl}_{p_1,p_2,p_3}\mathbb{P}^2$.

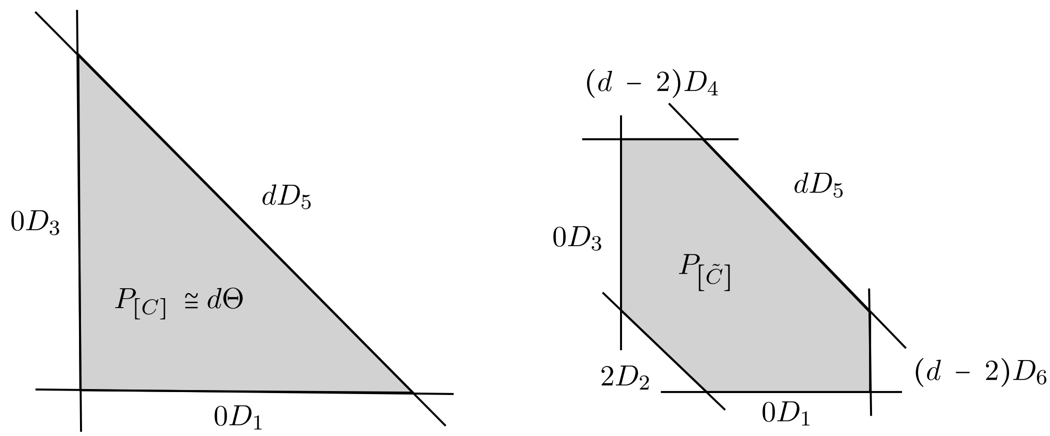

FIGURE 5. The polygons $P_{[C]}$ and $P_{[\tilde{C}]}$ associated with C inside X_Σ and $\tilde{C}$ inside $X_{\Sigma'}$, respectively. The labels on the defining half-plane indicate which of the prime toric divisors it is associated to.

degree d, inside $X_\sigma \cong \mathbb{P}^2$ we have that $C \cong dD_5$ on C (since all toric divisors are $\mathcal{O}_{\mathbb{P}^2}(1)$s). Because D_5 intersects the points p_2 and p_3, it follows that

$$\beta^{-1}C \sim d\beta^{-1}D_2 = dD_5 + dE_1 + dE_2 = dD_5 + dD_4 + dD_6.$$

It follows that we can express the divisor class of $\tilde{C}$ in $X_{\Sigma'}$ as:

$$\tilde{C} \sim dD_5 + (d-2)D_4 + (d-2)dD_6 - 2D_2.$$

We can plug this expression in to find the corresponding lattice polygon $P_{\tilde{C}}$. This will be a hexagon with points $(2,0), (0,2), (0,d-2), (2,d-2), (d-2,2), (d-2,0)$. In other words, it will be the polygon $d\Theta$ associated to degree d-divisors in $\mathbb{P}^2 \cong \mathrm{Bl}_0\,\mathbb{P}^2$ with the corners "chipped off" corresponding to subtracting the two copies of each exceptional divisors to be removed. This operation is visualized in Figure 5.

Because $d > 5$, none of the six sides will be nondegenerate, and therefore $\tilde{C}$ is ample. It is a straightforward calculation to find that $\mathrm{Vol}(P_{[\tilde{C}]}) = d^2/2 - 6$. The width of $P_{[\tilde{C}]}$ in either lattice

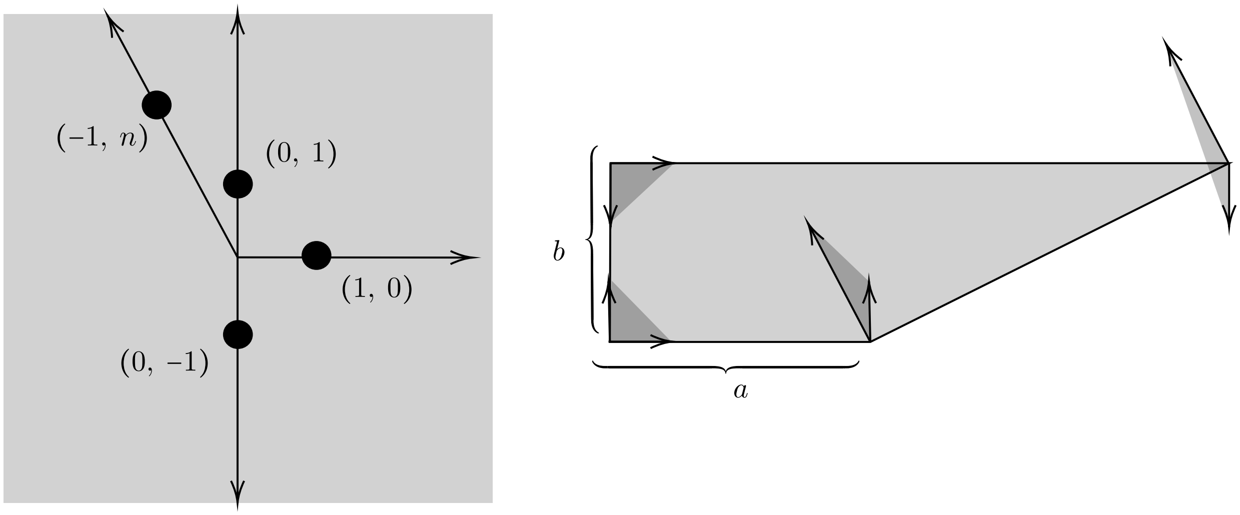

FIGURE 6. The lattice fan Σ_n realizing the Hirzebruch surface (left) and an example of a trapezoid realizing this fan, corresponding to an ample class in $X_{\Sigma_n} = \mathbb{F}_n$.

direction will be $d-2$, so $\mathrm{lw}(P_{[\tilde{C}]}) \leq d-2$. Because the line of lattice points $\{(0,2),(1,2),\cdots,(d-2,2)\}$ is inside $P_{[\tilde{C}]}$, it follows from Lemma 2.9.3 that $\mathrm{lw}(P_{[\tilde{C}]}) = d-2$. We can now check directly for which values of d we can apply Theorem 1.1:

$$d-2 = \mathrm{lw}(P_{[\tilde{C}]}) \leq \frac{2}{9}\,\mathrm{Vol}(P_{[\tilde{C}]}) = \frac{d^2}{9} - \frac{4}{3}.$$

One can check quickly that this is satisfied whenever $d \geq 9$. Because C and $\tilde{C}$ are birational, they have the same minimum density degree. We've thus proven the following corollary after applying Theorem 1.1:

Corollary 3.0.2. Let C be a geometrically integral plane curve of degree $d \geq 9$ over a number field K whose singular local consists of three noncollinear simple nodes, and let $\tilde{C}$ be its normalization. Then $\min(\delta(C/K)) = \mathrm{gon}_K\,\tilde{C}$.

4. Application: Hirzebruch Surfaces

In this section we will apply our main theorem to attain bounds on ample classes on Hirzebruch surfaces, similar those known on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. We recall to start that the Hirzebruch surface $\mathbb{F}_n$ is determined as a toric surface by the lattice fan Σ_n with unit rays $\{(0,1),(1,0),(0,-1),(-1,n)\}$ [CLS11]. Throughout this section, we will assume that $n \geq 1$, as otherwise we return the $\mathbb{P}^1 \times \mathbb{P}^1$ case. The lattice polygons giving rise to Σ_n will look like trapezoids with a straight left side and a right side slanted with a $\frac{1}{n}$ slope. This fan and an example trapezoid are visualized in Figure 6. By Theorem 2.5, these trapezoids will correspond to ample classes on $\mathbb{F}_n = X_{\Sigma_n}$. This trapezoid is completely described by the length of its bottom and left sides, which can be any nonnegative pair. It follows that we can parametrize ample classes with these lengths $(a,b) \in \mathbb{Z}_+ \times \mathbb{Z}_+$.

For the remainder of this section, P will refer to the trapezoid corresponding to such a pair (a,b). We observe that:

$$\mathrm{Vol}(P) = ab + \frac{nb^2}{2} \qquad \text{and} \qquad \mathrm{lw}(P) = b$$

The volume is a straightforward observation. The lattice width of b is clearly obtained by the lattice direction $(1,0)$. To show that this is optimal, observe that in any other lattice direction, we can consider the line of points $(0,b),(1,b),\cdots,(a+bn,n)$. For any lattice direction $u \neq (1,0)$, Lemma 2.9.3 implies that $w(P,u) \geq a+bn$, and so $\mathrm{lw}(P) = b$.

Remark 1. Classically, the Hirzebruch surface is obtained by projectivizing $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$ to create a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. In this context, the nef cone (and Picard group) is generated by F and G, where F is a fiber of $\mathbb{F}_n \to \mathbb{P}^1$, and G is the section corresponding to an inclusion $\mathcal{O} \hookrightarrow \mathcal{O}(n) \oplus \mathcal{O}$. We've implicitly shown that the (degenerate) trapezoids corresponding to $(1,0)$ and $(0,1)$ generate the nef cone. By comparing self-intersection, it follows that (a,b) corresponds to $(a,b) = aF + bG$.

Before proceeding to the proof of Theorem 1.2, we begin by interpreting some of exceptional cases listed therein:

Remark 2. We note that 5 of the 7 cases enumerated in Theorem 1.2 are of the form $(n,a,b) = (1,1,b)$. We consider this case in more detail now. The first thing to observe is that $\mathbb{F}_1$ is just $\mathrm{Bl}_0 \mathbb{P}^2$. We can retrieve the standard lattice fan for $\mathrm{Bl}_0 \mathbb{P}^2$ by applying the sheer $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to the lattice of one parameter subgroups N. This results in applying the transpose to the dual lattice M, and thus to the defining trapezoid. The result of this transformation will be a trapezoid with vertices $(1,0), (0,1), (0,d)$, and $(d,0)$. This process is visualized in Figure 7. Using our notation from Example 2, it follows that the class of our curve C will be $(b+1)D_2 - D_4$, where D_4 is the exceptional divisor and D_2 is the line through the origin. It follows that $C \cdot D_4 = 1$, and so C intersects the exceptional divisor transversally. Therefore we can project C down to get a smooth curve of degree $b+1$ in $\mathbb{P}^2$. Therefore the curves corresponding to counterexamples $(1,1,b)$ are actually just smooth plane curves of degree $b+1$, examples already studied by Debarre and Klassen in [DK94]. In other words, there are no *new* counterexamples.

We may now proceed to our proof of Theorem 1.2:

Proof. (of Theorem 1.2) The $n = 0$ case is covered in [SV22], so we will omit it here, and return to assuming $n \geq 1$. From our prior discussion we can associate with $aF + bG$ a polygon P. For our discussion, it will be occasionally useful to have coordinates, so we assume that the bottom left corner of P is at the origin, such that $P = \mathrm{conv}((0,0),(0,a),(0,b),(nb+a,b))$. By Theorem 1.1, for any (a,b,n) emiting a counterexample we'd have:

$$b = \mathrm{lw}(P) > \frac{2}{9}\mathrm{Vol}(P) = \frac{2}{9}ab + \frac{nb^2}{2} \iff 9 > 2a + nb.$$

Thus, we are looking for solutions to $9 > 2a + nb$.

Case 1: $b = 1$. There are no interior lattice points, so $g(C) = 0$, and thus we have equality $\mathrm{gon}_K C = \min \delta(C/K)$ trivially.

Case 2: $b = 2$, $a > 1$ or $n > 1$. In this case the curve is hyperelliptic, so we wish to show it cannot have dense rational points. By Falting's Theorem (Theorem 2.1) it will suffice to show that $g(C) > 1$, i.e. P has more than one internal lattice point. We observe that $(1,1),(1,2) \in \mathrm{int}\,\mathrm{conv}((0,0),(0,2),(0,2n+a)) \subseteq \mathrm{int}\, P$, as $2n + a > 2$. Thus, the genus must be greater than 1, and we are done.

Case 3: $n \geq 2$. We already know that, when $n > 1$, we must have $b \geq 3$. Therefore we are looking for solutions to $9 > 2a + nb$. One can see relatively easily that the only solution subject to these constraints is the minimum in all three variables, or $(n,a,b) = (2,1,3)$. However $b = 3$ would imply the curve emits a map to $\mathbb{P}^1$ of degree at most 3. There are two cases to consider. The only way that $\mathrm{gon}_K C > \min(\delta(C/K))$ is if $\mathrm{gon}_K C = 3$ and $\min(\delta(C/K)) = 2$. By Lemma 2.4.2, this can only be the case of C emits a degree 2 covering of an elliptic curve E. In this case Lemma 2.4.2 implies that $g(C) < 4$, but the associated polygon has 6 internal lattice points: $(1,1),(1,2),(2,1),(2,2),(2,3)$, and $(2,4)$, so this cannot be.

Case 4: $n = 1$, $a > 1$, and $(n,a,b) \neq (1,2,4)$. By case 2, we may assume that $b \geq 3$. A quick search finds that the only solutions are $(n,a,b) = (1,2,3)$ or $(1,2,4)$. For the former case, we can make the same argument as in Case 3. Because $b = 3$, the curve is trigonal. But if there are quadratic

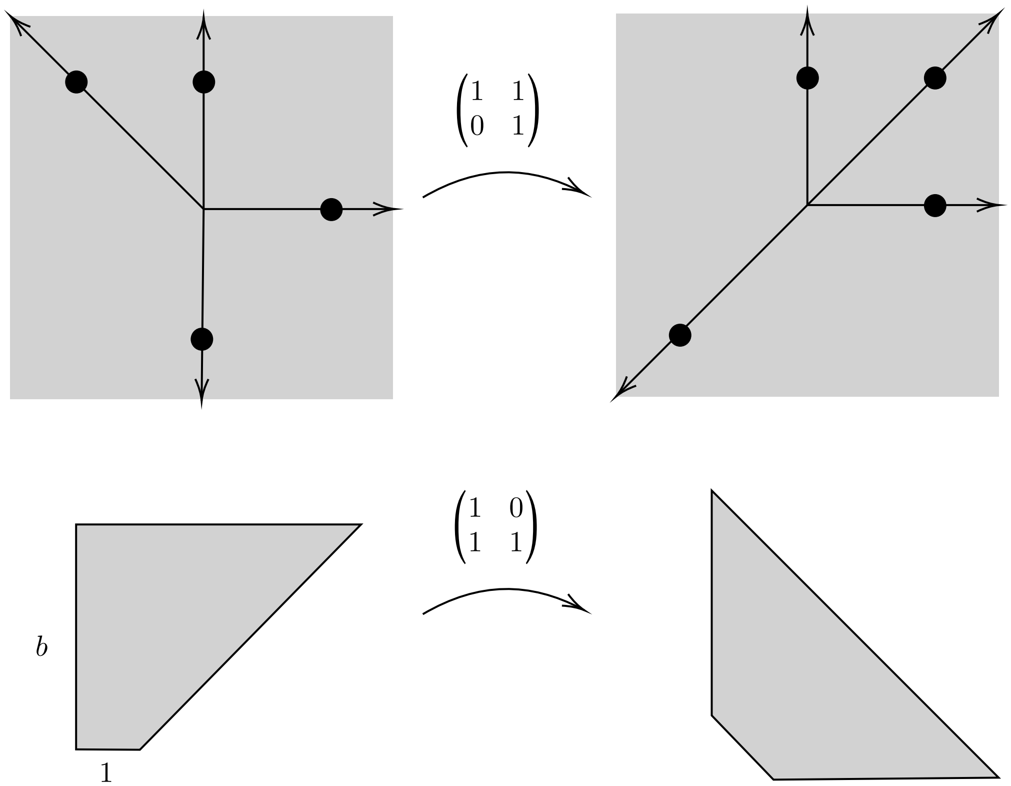

FIGURE 7. Isomorphism of the standard fans associated to $\mathbb{F}_1$ and $\mathrm{Bl}_0 \mathbb{P}^2$
.

AV-parametrized points, it would need to emit a degree 2-map to E. However P has five interior lattice points $(1,1), (2,1), (2,1), (2,2), (3,2)$, so this cannot be. The case of $(1,2,4)$ will postponed to Case 6.

Case 5: $n = a = 1$. In this case we get a bound on b via $9 > 2a + nb = 2 + b \implies b < 7$.

Case 6: $(n, a, b) = (1, 2, 4)$. As explained in 2, The Hirzebruch Surface $\mathbb{F}_1$ is isomorphic to the plane $\mathbb{P}^2$ blown up at the origin. Under the isomorphism in Figure 7, the trapezoid associated to $(a, b) = (2, 4)$ will be the convex hull of $(2,0), (6,0), (0,6), (0,2)$. It follows that an equation for C in the torus can be written with positive degree monomials of total degree ≥ 2 and ≤ 6. Therefore, C is birational to a degree 6 plane curve with a node at the origin. We can count interior lattice points of this polygon to find that $g(C) = 9$. With this information, then Theorem A.1 in Appendix A implies that $\min(\delta(C/K)) = \mathrm{gon}_K C = 4$, and so we're done.

This completes our case reduction, and we are done. $\square$

Appendix A. Minimum density degree of a genus 9 curve with a plane model of degree 6

by Isabel Vogt

The goal of this appendix is to compute the minimum density degree of a nice curve of genus 9 over a number field that admits a plane model of degree 6 (which necessarily has a single node/cusp.). Since the gonality is 4, we can utilize classification theorems of Harris–Silverman [Sil91] and Abramovich–Harris [Har91] to rule out small elements of the density degree set $\delta(C/k)$.

Theorem A.1. Let k be a number field and let C be a smooth curve of genus 9 with a (birational) plane model of degree 6. Then $\min \delta(C/k) = \mathrm{gon}(C/k) = 4$.

Proof. We first show that $\mathrm{gon}(C/k) = 4$. By the adjunction formula, the plane model of C has a single singularity p with δ-invariant 1, i.e., a node or cusp. Projection from this point gives a tetragonal map. The fact that C does not also admit a map of degree 3 to $\mathbb{P}^1$ follows from the adjunction fromula on $\mathbb{P}^1 \times \mathbb{P}^1$. To prove that $\mathrm{gon}(C/k) = 4$, it suffices to show that a hyperelliptic curve of genus 9 does not have a birationally very ample line bundle of degree 6. By Clifford's theorem, the only basepoint free linear series of degree at most $2g-2$ on a hyperelliptic curve are symmetric powers of the hyperelliptic linear series. Consequently a linear series of degree at most $2g-2$ is never birationally very ample, and so a hyperelliptic curve of genus 9 cannot have a plane model of degree 6.

If $\min \delta(C/k) < \mathrm{gon}(C/k)$ then $\min \delta(C/k) = 2, 3)$, by Faltings' Theorem. We rule out each of these possibilities in turn. If $\min \delta(C/k) = 2$, then by [Sil91], C has a map of degree 2 to a curve of genus at most 1. We have already ruled out the case that C is hyperelliptic, so it suffices to show that C cannot have a degree 2 map $\alpha\colon C \to E$, for a genus 1 curve E. Let L be the line bundle of degree 6 on C which is the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$ from the plane model. The pushforward $\alpha_* L$ is a vector bundle of rank 2 and degree $\chi(E, \alpha_* L) = \chi(C, L) = -2$. Since $3 = h^0(C, L) = h^0(E, \alpha_* L)$, the bundle $\alpha_* L$ must be the direct sum of a line bundle L_3 of degree 3 and a line bundle L_{-8} of degree -8. The complete linear series $|L|\colon C \to \mathbb{P}^2$ factors through the map of the projective bundle $\mathbb{P}(\alpha_* L)^\vee \dashrightarrow \mathbb{P}^2$ by the complete linear series of $\mathcal{O}_{\mathbb{P}(\alpha_* L)^\vee}(1)$ (see [Vog, Section 2] for the case of a cover of $\mathbb{P}^1$, though the arguments go through mutatis mutandis in the case of a cover of a higher genus curve). Since $h^0(\mathbb{P}^1, \alpha_* L) = h^0(\mathbb{P}^1, L_3)$, the map $\mathbb{P}(\alpha_* L)^\vee \dashrightarrow \mathbb{P}^2$ factors through the projection $\mathbb{P}(\alpha_* L)^\vee \dashrightarrow \mathbb{P}(\alpha_* L)^\vee / L_3^\vee \simeq E$. Thus the complete linear system of L yields a map $C \to \mathbb{P}^2$ that factors through E, and so is not a birational model of C. Hence C cannot be a double cover of an elliptic curve.

Now assume that $\min \delta(C/k) = 3$. By [Har91] (see also [KV25, Theorem 1.2(2)]), C is a triple cover of a curve of genus at most 1; since C has gonality 4, it must be a triple cover $\alpha\colon C \to E$ of a genus 1 curve E. Consider the surface $E \times \mathbb{P}^1$ and let F_1 be the numerical class of a fiber of the first projection π_1 to E and let F_2 be the numerical class of a fiber of the second projection π_2 to $\mathbb{P}^1$. Recall that $\mathrm{Pic}(E \times \mathbb{P}^1) \simeq \mathrm{Pic}(E) \times \mathrm{Pic}(\mathbb{P}^1)$: every line bundle on $E \times \mathbb{P}^1$ is the tensor product of the pullback of a line bundle from E and from $\mathbb{P}^1$. Since $\mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$, we will slightly abuse notation and write F_2 for the linear equivalence class of a fiber of the second projection as well. A line bundle in numerical class $aF_1 + bF_2$ is the pullback of a line bundle of degree a from E twisted by bF_2. The curve C maps to $E \times \mathbb{P}^1$ by the produt of α and the tetragonal map and the image has class $4F_1 + 3F_2$. This map must be birational onto its image since 3 and 4 are relatively prime. Since $[K_{E \times \mathbb{P}^1}] = -2F_2$, the adjunction formula yields that the arithmetic genus of a curve in class $4F_1 + 3F_2$ is 9, so C in fact embeds in $E \times \mathbb{P}^1$. The pullback H of $\mathcal{O}_{\mathbb{P}^1}(1)$ under the tetragonal map is the restriction of F_2 to C. Using the adjunction formula, the Serre dual $K_C \otimes H^\vee$ is the restriction of a line bundle M_{4F_1} in numerical class $4F_1$ to C.

Again, we will let L denote the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ under the plane model. Since the tetragonal map is given by projection from p in the plane model of C, we have $L \simeq H(p_1 + p_2)$, where p_1, p_2 are the two (possibly nondistinct) points of C above the singularity p. In order to have

$h^0(C, H(p_1+p_2)) = 3 = h^0(C,H)+1$, the divisor p_1+p_2 must impose only 1 condition on the Serre dual linear system $|K_C \otimes H^\vee|$. In other words, the linear system $|K_C \otimes H^\vee|$ must contract the divisor p_1+p_2. Since $K_C \otimes H^\vee \simeq M_{4F_1}|_C$, restriction to C yields an exact sequence

$$0 \to \mathcal{O}_{E\times\mathbb{P}^1}(-3F_2) \to M_{4F_1} \coloneqq \mathcal{O}_{E\times\mathbb{P}^1}(C-3F_2) \to K_C \otimes H^\vee \to 0.$$

Since $-3F_2$ is not effective, we have an inclusion $H^0(E\times\mathbb{P}^1, M_{4F_1}) \hookrightarrow H^0(C, K_C \otimes H^\vee)$, inducing a projection

$$\mathbb{P}H^0(C, K_C \otimes H^\vee)^\vee \dashrightarrow \mathbb{P}H^0(E\times\mathbb{P}^1, M_{4F_1})^\vee.$$

The image of C under this composition is contained in the image of $E\times\mathbb{P}^1$ under the complete linear system $|M_{4F_1}|$. Thus if p_1+p_2 is contracted by the complete linear system $|K_C\otimes H^\vee|$, these points must also be contracted by $|M_{4F_2}|$. By the description of the Picard group of $E\times\mathbb{P}^1$, there exists a line bundle L_4 of degree 4 on E such that $M_{4F_1} \simeq \pi_1^* L_4 \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}$. By the Künneth formula, $H^0(E\times\mathbb{P}^1, M_{4F_1}) \simeq \pi_1^*H^0(E, L_4)\otimes\pi_2^*H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \simeq \pi_1^*H^0(E,L_4)$. The complete linear system $|M_{4F_1}|$, is the composition of projection π_1 with the embedding of E by the (very ample) complete linear system $|L_4|$. We conclude that the divisor p_1+p_2 must be contained in a fiber F of π_1. In particular, for $q = F\cap C \smallsetminus p_1+p_2$, we have $L \simeq \mathcal{O}_{E\times\mathbb{P}^1}(F+F_2)|_C(-q)$. If $h^0(C,L) = 3$, then, in particular, $h^0(C, \mathcal{O}_{E\times\mathbb{P}^1}(F+F_2)|_C) \geq 3$.

Consider the exact sequence for restriction to C:

$$0 \to \mathcal{O}_{E\times\mathbb{P}^1}(F+F_2-C) \to \mathcal{O}_{E\times\mathbb{P}^1}(F+F_2) \to \mathcal{O}_{E\times\mathbb{P}^1}(F+F_2)|_C \to 0.$$

By the Künneth formula, $h^0(E\times\mathbb{P}^1, O_{E\times\mathbb{P}^1}(F+F_2)) = 2$. The line bundle $\mathcal{O}_{E\times\mathbb{P}^1}(F+F_2-C)$ has numerical class $-3F_1-2F_2$, so there exists a line bundle L_{-3} of degree -3 on E for which $\mathcal{O}_{E\times\mathbb{P}^1}(F+F_2-C) \simeq \pi_1^*L_{-3}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(-2)$. Again, by the Künneth formula,

$$H^1(E\times\mathbb{P}^1, \pi_1^*L_{-3}\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(-2)) \simeq H^0(E, L_{-3})\otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \oplus H^1(E, L_{-3})\otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0,$$

since $h^0(E, L_{-3}) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$. We conclude that $(h^0(C, \mathcal{O}_{E\times\mathbb{P}^1}(F+F_2)|_C) = 2$, which is a contradiction. $\square$

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