Solution Comprehensive Exam.

Q.1 a) Since there are no non zero solutions dimension of the Kernel is 0 [1 Mark]

Hence by Rank Nullity Theorem, Rank(A) = No of columns of A = 3 [1 Mark]

b) If W_1 and W_2 are subspaces of V, consider

$W = W_1 \cup W_2$

Let x and y be in W and c in F.

We need $cx + y$ to be in W

↳ only possibility is to have $W_1 \subset W_2$ or $W_2 \subset W_1$.

[1 Mark]

To counter it otherwise, consider examples

$W_1 = \{(a, 0) \mid a \text{ in } R\}$ $W_2 = \{(0, b) \mid b \text{ in } R\}$.

Both W_1 & W_2 are subspaces of R^2. However $W_1 \cup W_2$ is not a subspace (1 Mark).

c) What kind of conic section is represented by

$4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$.

$$A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$ Eigenvalues of A are 16, 1 [1 Mark]

Transformation to principal axes reduces to

$16y_1^2 + y_2^2 = 16$ or $y_1^2 + 16y_2^2 = 16$.

This is an ELLIPSE (1 Mark)

1d) Apply four steps of the power method, using scaling using initial guess [1, 1] to $\begin{pmatrix} 7 & -3 \\ -3 & 1 \end{pmatrix}$

0.5 Marks to each of these steps.

Iteration	q
1	1
2	8.2
3	8.24260355
4	8.242640655.

Q 2 a) Prove that if graphs G and H are isomorphic, then their complements $\overline{G}$ and $\overline{H}$ are also isomorphic

Given :- G & H are isomorphic simple graphs

$G = (V_1, E_1)$ $H = (V_2, E_2)$ be isomorphic

There exists one to one & onto function $f : V_1 \to V_2$ such that a and b are adjacent in G if and only if f(a) and f(b) are adjacent in H.

$\overline{G} = (V_1, \overline{E}_1)$ $\overline{H} = (V_1, \overline{E}_2)$ complementary graphs contain same set of vertices while contain edges not contained in original graphs. (1 Mark)

If c and d are adjacent in $\overline{G}$

c and d are not adjacent in G. (by definition of complementary graph)

f(c) and f(d) can Then also not be adjacent in H

f(c) and f(d) are adjacent in $\overline{H}$ (1 Mark)

Hence $\overline{G}$ and $\overline{H}$ are also isomorphic (with the same function as isomorphic relation between G & H)

Q 2 b) Prove / disprove :- There exists a connected simple graph with 6 vertices & 5 edges such that G has Euler circuit.

$|V| = 6$ $|E| = 5$ If G has Euler circuit, G must be connected, all vertices must have Even degree

Sum of degree of all vertices must be atleast 12.

(as degree of each vertex must be atleast 2) because degree of vertex 0 would lead to G to be disconnected. [1 Mark]

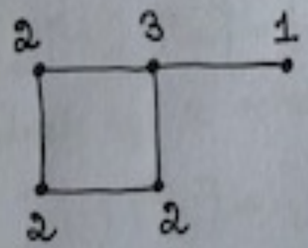
by Handshaking Thm we must have

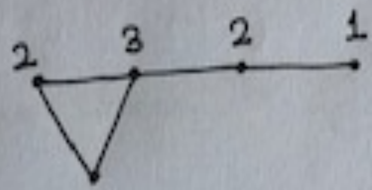
$\sum_{v \in V} \deg(v) = 2|E| = 10$ a contradiction

[1 Mark]

$\boxed{\text{disproved}}$

Q 2 c) Draw two non isomorphic 5 vertex, 5 edge simple graphs with the same degree sequence





drawing simple graphs with same degree sequence – 1 Mark

Proving they are Non isomorphic – 1 Mark.

Q 2 d) Write a pseudocode to decide whether a graph is bipartite based on the coloring theorem.

Bipartite graph is simple graph whose vertices can be partitioned into two sets V_1 & V_2 such that no edges among vertices of V_1 & no edges among vertices of V_2

```
procedure bipartite(v1, v2 ... vn vertices with n >= 1;
                    E: set of edges)
 w1 := Red
 for i: 2 to n
   wi := Unassigned
 b := True                                  (1 Mark)
 while b = True
   for i := 1 to n
     ai := 0
   for j := 1 to n
    for k := 1 to n
      if wj = wk and j ≠ k then b := False
      else if wk = Unassigned and wj = Red and j ≠ k then
         wk := Blue
         ai := 1
      else if wk = Unassigned and wj = Blue and j ≠ k then
         wk := Red
         ai := 1                            (1 Mark)
   if ai = 0 then
     c = 0
     m := 2
     while c = 0
       if wm = Unassigned then
         wm := Red
         c := 1
       else m := m + 1
return b
```

Q 3 a) Prove that $\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n^2}{4} \right\rfloor$ for all $n \in N$

Case 1 n is even (1 Mark)

$\hookrightarrow$ n is divisible by 2, $n/2$ is an integer

$\lfloor n/2 \rfloor \lceil n/2 \rceil = \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$

Product of 2 integers is an integer. So $\frac{n^2}{4}$ is an integer

$\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n^2}{4} \right\rfloor$

Case 2 n is odd. $\Rightarrow (n-1) \& (n+1)$ are even (1 Mark)

$n-1 < n < n+1$

$\frac{n-1}{2}$ and $\frac{n+1}{2}$ are integers

$\frac{n-1}{2} < \frac{n}{2} < \frac{n+1}{2}$ $\lfloor n/2 \rfloor = \frac{n-1}{2}$

$\frac{n+1}{2} \rightarrow$ Smallest integer $\geq n/2$ $\lceil n/2 \rceil = \frac{n+1}{2}$

$\frac{n-1}{2} \rightarrow$ largest integer $\leq n/2$

$\lfloor n/2 \rfloor \lceil n/2 \rceil = \frac{(n-1)}{2} \frac{(n+1)}{2} = \frac{n^2-1}{4} = \frac{n^2}{4} - \frac{1}{4}$

$\boxed{\lfloor n/2 \rfloor \lceil n/2 \rceil = \left\lfloor \frac{n^2}{4} \right\rfloor}$

Can also prove Case 2 $n = 2k+1$

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$\frac{n^2}{4} = \frac{4k^2+4k+1}{4} = k^2 + k + \frac{1}{4}$

$\left\lfloor \frac{n^2}{4} \right\rfloor = k^2 + k = \frac{n^2}{4} - \frac{1}{4}$

$\left\lfloor \frac{n^2}{4} \right\rfloor = \lfloor n/2 \rfloor \lceil n/2 \rceil$

Q3b) Determine whether the Symmetric difference operation is ASSOCIATIVE

If A, B, C are sets

$A \oplus (B \oplus C) = (A \oplus B) \oplus C$ (2 Marks)

A	B	C	(½M) $B \oplus C$	(½M) $A \oplus B$	(½M) $A \oplus (B \oplus C)$	(½M) $(A \oplus B) \oplus C$
0	0	0	0	0	0	0
0	0	1	1	0	1	1
0	1	0	1	1	1	1
0	1	1	0	1	0	0
1	0	0	0	1	1	1
1	0	1	1	1	0	0
1	1	0	1	0	0	0
1	1	1	0	0	1	1

$A \oplus B$ = All elements in A or B, but not in both A and B.

Hence $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ is proved from Membership Table (last two columns have same values)

Q3c) If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

Given :- $g: A \to B$ and $f: B \to C$

f & $f \circ g$ are one to one.

To prove :- g is one to one.

Assume $g(a) = g(b)$ —— ①

Taking function of each side of eqn ①

$f(g(a)) = f(g(b))$ (1 Mark)

Use definition of composition

$(f \circ g)(a) = (f \circ g)(b)$

Since $f \circ g$ is one to one $a = b$. By definition of one to one

Thus $g(a) = g(b) \Rightarrow a = b$. (1 Mark)

we have proved that g is one to one $\boxed{g \text{ is one to one}}$

Q3 d) Draw the graph of the function $f(x) = \left\lceil \left\lfloor x - \frac{1}{2} \right\rfloor + \frac{1}{2} \right\rceil$ in the domain $[-3, 3]$

$f(-2.5) = \lceil \lfloor -2.5 - \frac{1}{2} \rfloor + \frac{1}{2} \rceil$ (1 Mark)

$= \lceil -3 + 1/2 \rceil = -2$

$f(-1.5) = \lceil \lfloor -1.5 - 1/2 \rfloor + 1/2 \rceil = \lceil -2 + 1/2 \rceil$

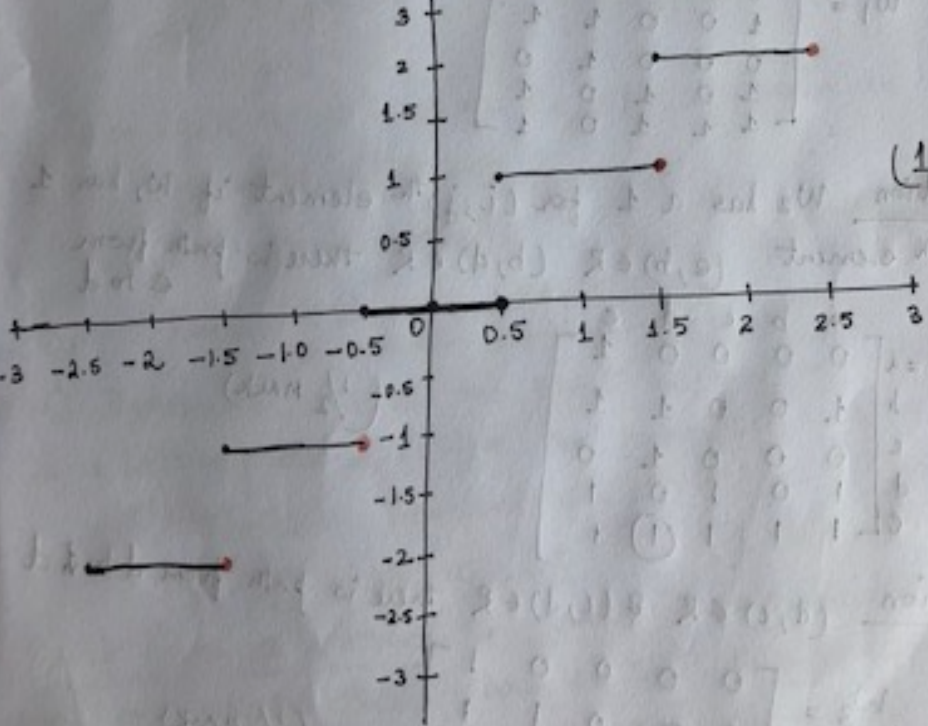
$= -1$

$f(-0.5) = \lceil \lfloor -0.5 - 1/2 \rfloor + 1/2 \rceil = \lceil -1 + 1/2 \rceil = 0$

$f(0.5) = \lceil \lfloor 0.5 - 1/2 \rfloor + 1/2 \rceil = \lceil 0 + 1/2 \rceil = 1$

$f(1.5) = \lceil \lfloor 1.5 - 1/2 \rfloor + 1/2 \rceil = \lceil 1 + 1/2 \rceil = 2$

$f(2.5) = \lceil \lfloor 2.5 - 1/2 \rfloor + 1/2 \rceil = \lceil 2 + 1/2 \rceil = 3$



(1 Mark)

Red points - not included in graph of $\lceil \lfloor x - 1/2 \rfloor + 1/2 \rceil$

pts are $(-2.5, -2)$, $(-1.5, -1)$, $(-0.5, 0)$, $(0.5, 1)$, $(1.5, 2)$, $(2.5, 3)$

Q4 a) Find the transitive closure of relation given by $\{(a,e),(b,a),(b,d),(c,d),(d,a),(d,c),(e,a),(e,b),$
$(e,c),(e,e)\}$ on $\{a,b,c,d,e\}$ using Warshall's algorithm

Matrix that represents relation R

$$W_0 = M_R = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

1st iteration W_1 has 1 for (i,j)th element if W_0 has a 1 for $(i,j)^{th}$ element / if there is a path from a_i to a_j with $a_1 = a$ as its only interior vertex

$(b,a) \in R$ & $(a,e) \in R$ there is path from b to e

$(d,a) \in R$ & $(a,e) \in R$ there is path from d to e.

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

2nd iteration W_2 has a 1 for $(i,j)^{th}$ element if W_1 has 1 for $(i,j)^{th}$ element $(e,b) \in R$ $(b,d) \in R$ there is path from e to d

$$W_2 = \begin{array}{c|c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(½ MARK)

3rd iteration $(d,c) \in R$ & $(c,d) \in R$ there is path from d to d

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(½ mark)

4th iteration

$$W_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$ (1/2 Mark)

5th iteration

$$W_5 = M_{R^*} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$ (1/2 Mark)

Q4b) Suppose that R_1 & R_2 are Reflexive relations on a set A, then show that $R_1 \oplus R_2$ is Irreflexive

Let $a \in A$

Since R_1 & R_2 are Reflexive

$(a,a) \in R_1$ (1 Mark)

$(a,a) \in R_2$

Since (a,a) is in R_1 and in R_2, (a,a) cannot be an element in the symmetric difference. (1 Mark)

$(a,a) \notin R_1 \oplus R_2$.

hence $R_1 \oplus R_2$ is IRREFLEXIVE

Q4c) Determine whether the poset defined by $(P(S), \supseteq)$ is a lattice, where P(S) is the power set of S.

$S = P(A)$ Powerset of a set A.

$R = \{(a,b) \mid a \supseteq b\}$ (1 Mark)

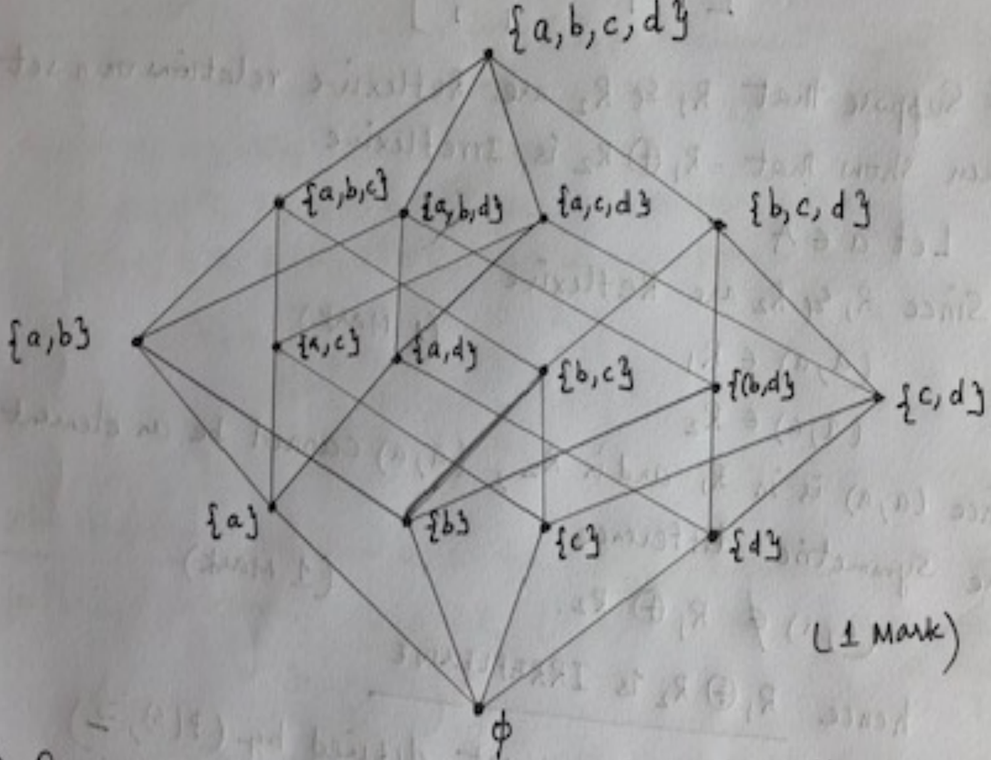
Poset forms a Lattice as greatest lower bound (GLB) of any two elements $B \in Z$ and $C \in Z$ is their intersection and the least upper bound (LUB) is their union (1 Mark)

$\boxed{GLB = B \cap C, \; LUB = B \cup C}$

Q 4 d) Draw the Hasse diagram for inclusion on the set $P(S)$ where $S = \{a, b, c, d\}$

$P(S)$ represent the set of all subsets of S

$P(S) = \{ \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$ (1 Mark)



(1 Mark)

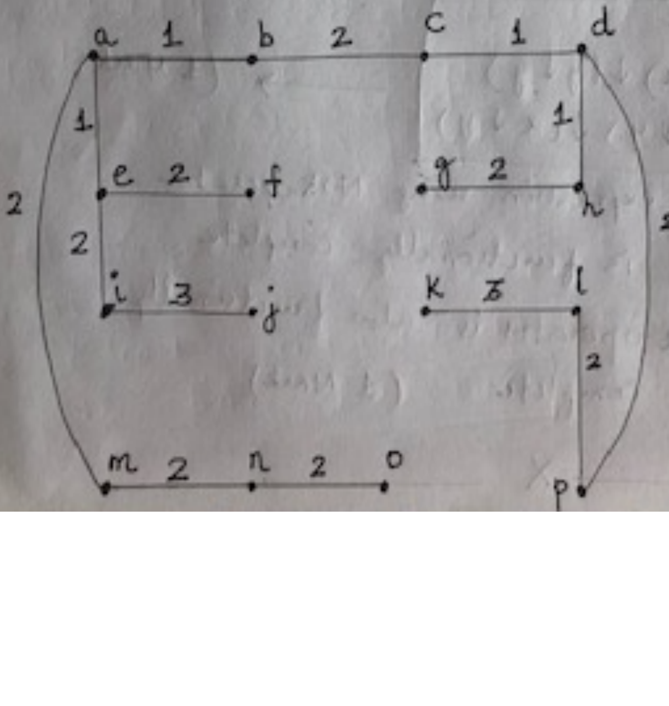
1) Draw a dot for each element in $P(S)$, label it
2) Place all sets with same No of elements in same row
3) Sets with least number placed at bottom & sets with most number of elements are at the top.
4) Add straight line between dots if one of set is subset of other set.

Q5 a) Use Kruskal's algorithm to find minimum spanning tree for weighted graph.

Edges	Weight
(a,b)	1
(a,e)	1
(c,d)	1
(d,h)	1
(a,m)	2
(b,c)	2
(d,p)	2
(e,f)	2
(e,i)	2
(g,h)	2
(l,p)	2
(i,j)	3
(k,l)	3
Total.	24

Table
↳ 2 Marks
selection of edges along with weight.

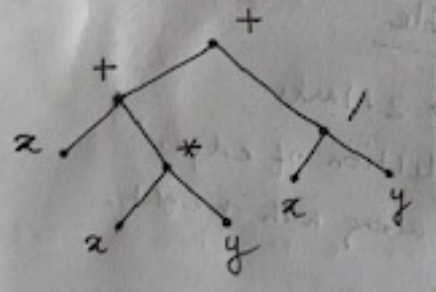
— (1 Mark)



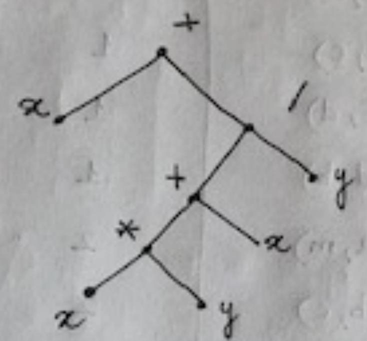
→ (1 Mark)

Q 5 b) Represent the expressions
$(x+xy)+(x|y)$ and $x+((xy+x)/y)$ using binary trees.

$(x+xy)+(x|y)$ ½ Mark



$x+((xy+x)/y)$ ½ Mark.



Prefix notation :- (½ Mark)

$++x*xy/$
(first expression)

Second expression
$+x/+*xyxy$

Postfix Notation (½ Mark)

First expression
$xxy*+xy$

Second expression
$xxy*x+y/+$

5 c) Show that NOR operator $\downarrow$ is functionally complete.

All Boolean functions are represented by $\cdot, +, ^{-}$ operators
$\{\cdot, +, -\}$ is functionally complete.

$$\boxed{\begin{aligned} \bar{x} &= x \downarrow x \\ xy &= (x\downarrow x)\downarrow(y \downarrow y) \\ x+y &= (x\downarrow y)\downarrow(x\downarrow y) \end{aligned}} \longrightarrow (1\ \text{Mark})$$

We can write any expression using NOR operator.
Since $\{\cdot, +, ^{-}\}$ is functionally complete
$\{\downarrow\}$ NOR operator is also functionally complete (1 Mark)

—— X ——