Minimum Enclosing Ball

- The objective function is convex.

The traditional approach of the solution contains multiple parts:

- Convex hull-like optimizations in order to better the optimization process afterwards.
- Starting point: we start at an initial point $x^{(0)} \in \mathbb{R}^d$.
  - Although every point in the input is a 2-approximation to the MEB center, it might be better to start at a less trivial point.
  - Mean and median might be as bad as any other point.
  - The starting point $x^{(0)} = \begin{pmatrix} \vdots \\ \frac{\max_{p \in P} p_i + \min_{p \in P} p_i}{2} \\ \vdots \end{pmatrix}$ looks promising (didn't prove anything yet).
- Moving towards the central path, via Newton's method, TRM, etc.
- It may be possible to reduce MEB to medoid: $\min \left\|\alpha\right\|_\infty \mapsto \min \left\|\alpha\right\|_1$.

Pseudo-code:

- Input: Points $p_1, \ldots, p_n \in \mathbb{R}^d$, accuracy ε.

1. $x^{(0)} \leftarrow p_1$.
2. for $1 \leq i \leq k$, $k = \log^{\mathcal{O}(1)} \frac{1}{\varepsilon}$.
   (a) Use 1st order methods (e.g. GD) to derive a point x', such that $f\left(x'\right) \leq f\left(x^{(i-1)}\right)$.
   (b) Use 2nd order methods to derive $x^{(i)}$ from x'.
3. return $x^{(k)}$.

(It is equivalent to use $\left\|\cdot\right\|_2^2$)

$$f\left(x\right) = \max_{1 \leq i \leq n} \left\|x - p_i\right\|_2^2$$

Computing ∇f:

$$\frac{\partial}{\partial x_i} f\left(x\right) = \lim_{p \to \infty} \frac{\partial}{\partial x_i} \left\| \begin{bmatrix} | \\ \left\|x - p_j\right\|_2^2 \\ | \end{bmatrix} \right\|_p$$

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\left\|\begin{bmatrix} | \\ \|x-p_j\|_2^2 \\ | \end{bmatrix}\right\|_p &= \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\|x-p_j\|_2^{2p}\right)^{1/p} \\
&= \frac{1}{p}\left(\sum_{j=1}^{n}\|x-p_j\|_2^{2p}\right)^{1/p-1}\frac{\partial}{\partial x_i}\sum_{j=1}^{n}\|x-p_j\|_2^{2p}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\sum_{j=1}^{n}\|x-p_j\|_2^{2p} &= \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\sum_{k=1}^{d}(x_k-p_{j,k})^2\right)^{p} \\
&= p\left(\sum_{j=1}^{n}\sum_{k=1}^{d}(x_k-p_{j,k})^2\right)^{p-1}\frac{\partial}{\partial x_i}\sum_{j=1}^{n}(x_i-p_{j,i})^2 \\
&= 2p\left(\sum_{j=1}^{n}\sum_{k=1}^{d}(x_k-p_{j,k})^2\right)^{p-1}\sum_{j=1}^{n}(x_i-p_{j,i})
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \frac{\partial}{\partial x_i}\left\|\begin{bmatrix} | \\ \|x-p_j\|_2^2 \\ | \end{bmatrix}\right\|_p &= \frac{1}{p}\left(\sum_{j=1}^{n}\|x-p_j\|_2^{2p}\right)^{1/p-1}\cdot 2p\left(\sum_{j=1}^{n}\sum_{k=1}^{d}(x_k-p_{j,k})^2\right)^{p-1}\sum_{j=1}^{n}(x_i-p_{j,i}) \\
&= 2\left(\sum_{j=1}^{n}\|x-p_j\|_2^{2p}\right)^{1/p-1}\left(\sum_{j=1}^{n}\sum_{k=1}^{d}(x_k-p_{j,k})^2\right)^{p-1}\sum_{j=1}^{n}(x_i-p_{j,i})
\end{aligned}
$$

hence, the gradient can be computed using this limit (probably there's a simpler approch)

$$
\lim_{p\to\infty}\frac{\partial}{\partial x_i}\left\|\begin{bmatrix} | \\ \|x-p_j\|_2^2 \\ | \end{bmatrix}\right\|_p = \lim_{p\to\infty} 2\left(\sum_{j=1}^{n}\|x-p_j\|_2^{2p}\right)^{1/p-1}\left(\sum_{j=1}^{n}\sum_{k=1}^{d}(x_k-p_{j,k})^2\right)^{p-1}\sum_{j=1}^{n}(x_i-p_{j,i})
$$