

0 – 1 Knapsack summary

- The problem: We are given a knapsack of capacity W and a set of n items; an each item i , with $1 \leq i \leq n$, is worth $v[i]$ and has weight $w[i]$ pounds. Assume that weights $w[i]$ and the total weight W are integers. The goal is to fill the knapsack so that the value of all items in the knapsack is maximized.
- Choice of subproblem: Let $optknapsack(k, w)$ return the maximal value obtainable when filling a knapsack of capacity w using items among items 1 through k . To solve our problem we call $optknapsack(n, W)$.
- Recursive definition of $optknapsack(k, w)$:

```
OPTKNAPSACK( $k, w$ )
1 // returns the max value to pack a knapsack of capacity  $w$  using items 1 through  $k$ .
2 if ( $w == 0$ ): return 0
3 if ( $k \leq 0$ ): return 0
4 IF ( $weight[k] \leq w$ ):  $with = value[k] + OPTKNAPSACK(k - 1, w - weight[k])$ 
5 ELSE:  $with = 0$ 
6  $without = OPTKNAPSACK(k - 1, w)$ 
7 RETURN  $\max \{ with, without \}$ 
```

- Correctness: tries both possibilities for an item (take or not) and recurses on the rest (which is correct bec. of optimal substructure).
- Dynamic programming solution, recursive (top-down) with memoization: We create a table $table[1..n][1..W]$, where $table[i][w]$ will store the result of $optknapsack(i, w)$. We initialize all entries in the table as 0. To solve the problem, we call $optknapsackDP(n, W)$.

```
OPTKNAPSACKDP( $k, w$ )
1 // global variable  $table[1..n][1..W]$  initialized to 0. Also global  $v[1..n]$  and  $w[1..n]$ .
2 // returns the max value to pack a knapsack of capacity  $w$  using items 1 through  $k$ .
3 if ( $w == 0$ ): return 0
4 if ( $k \leq 0$ ): return 0
5 IF ( $table[k][w] \neq 0$ ): RETURN  $table[k][w]$ 
6 IF ( $w[k] \leq w$ ):  $with = v[k] + OPTKNAPSACKDP(k - 1, w - w[k])$ 
7 ELSE:  $with = 0$ 
8  $without = OPTKNAPSACKDP(k - 1, w)$ 
9  $table[k][w] = \max \{ with, without \}$ 
10 RETURN  $table[k][w]$ 
```

Running time: $O(n \cdot W)$

- Dynamic programming, iterative (bottom-up):

```

OPTKNAPSACKDP_ITERATIVE()
1  create table[0..n][0..W] and initialize all entries to 0
2  for  $k = 1; k \leq n; k++$ 
3      for  $w = 1; w \leq W; w++$ 
4          with =  $v[k] + \text{table}[k-1][w-w[k]]$ 
5          without =  $\text{table}[k-1][w]$ 
6           $\text{table}[k][w] = \max \{ \text{with}, \text{without} \}$ 
7  RETURN  $\text{table}[n][W]$ 

```

Running time: $O(n \cdot W)$

- Computing full solution:

Input: The table $\text{table}[1..n][1..W]$ as computed above, where $\text{table}[i][x]$ stores the max value to pack a knapsack of weight x using items $1..i$.

Output: the set of items corresponding to $\text{table}[n][W]$

$i = n, w = W$

while ($i > 0$) do:

 //is the value $\text{table}[i][w]$ achieved by including item i or not?

 if $\text{table}[i][w] == v[i] + \text{table}[i-1][w-w[i]]$:

 output item i

$w = w - w[i], i = i - 1$

 else: $i = i - 1$

Running time: $\Theta(n + W)$, with no extra space