Week 11: Lab

Module 5: Graphs

Collaboration level 0 (no restrictions). Open notes.

1. Briefly describe and analyze algorithms for the following problems. In each problem you are given an undirected graph $G = (V, E)$ as an adjacency list.
   - Is G connected?
   - Given two vertices, are they in the same CC?
   - How many CCs are in G?
   - Given two vertices u, v, is there a path between them? How do you find such a path?
   - Given two vertices u, v, describe how you can find the shortest path between them.
   - Does G contain a cycle?
   - Is G a *tree* (a tree is an undirected graph that's connected and has no cycles)?
2. Briefly describe and analyze algorithms for the following problems. In each problem you are given a directed graph $G = (V, E)$ as an adjacency list.
   - Find all vertices reachable from a given vertex u.
   - Given a vertex u, compute all vertices v such that u is reachable from v.
   - Given two vertices u, v, is there a (directed) path from u to v? If yes, how do you find such a path?
   - Does G have a cycle, or is it acyclic? (a graph is called acyclic if it does not have a cycle).
3. All-pair connectivity: Given an undirected graph, you want to be able to answer queries of the form: are u, v connected?
   (a) Describe an algorithm to answer such a query. How long does it take?
   (b) Now assume the graph is large and doe snot change, and you have a lot of such queries to answer. You have the idea to pre-process the graph and store some information, so that you can then answer connectivity queries in $O(1)$ time.
   What sort of pre-processing can you do? Describe the space and time complexity of the pre-processing.

   Note: The same question can be asked about all-pair reachability and all-pair shortest paths, and will be in the homework.
4. The *square* of a digraph $G = (V, E)$ is the graph $G^2 = (V, E^2)$ such that $(u, w) \in E^2$ if and only if for some vertex $v \in V$, both $(u, v) \in E$ and $(v, w) \in E$. That is, G^2 contains an edge from u to w whenever G contains a path with exactly two edges from u to w. Describe and analyze an efficient algorithm for computing G^2 from G.

5. In a directed graph, two vertices u and v are said to be in the same *strongly connected component (SCC)* if u can reach v and v can reach u.

   (a) Describe a linear time algorithm for computing the *strong component* containing a given vertex v.

   (b) On the basis of that algorithm, describe a simple algorithm for computing the strong components of a directed graph G.

   (c) Describe an algorithm which, given a directed graph G and two arbitrary vertices u, v, determines whether u and v are in the same SCC.

Additional Problems

1. **The Kevin Bacon game:** One of the classic application of graphs is to find the degree of separation between two individuals in a social network. We'll discuss this in terms of the Kevin Bacon game. Most of you have heard about Kevin Bacon. He is a known actor who appeared in a lot of movies. We assign every Hollywood actor a Kevin Bacon number (BN) as follows: Bacon himself has BN=0; any actor (except Bacon himself) who has been in the same movie as Kevin Bacon has a BN= 1; every actor who does not have a BN of 0 or 1, and has been in a movie with an actor who has a BN of 1, gets a BN=2; and so forth.

   Example: Meryl Streep has BN=1, because she appeared in *The river Wild* with Kevin Bacon. Nicole Kidman's number is 2 because she did not appear in any movie with Kevin Bacon, she was in *Days of Thunder* with Tom Cruise, and Cruise appeared in *A few good men* with Kevin Bacon.

   Given an actor/actress name, the simplest version of the game is to find a sequence of movies alternating with actors connecting that actor to Kevin Bacon. For example: a movie buff might know that Tom Hanks was in *Joe versus the volcano* with Lloyd Bridges, who was in *High noon* with Grace Kelly, who was in *Dial M for murder* with Patrick Allen, who was in *The eagle has landed* with Donald Sutherland, who was in *Animal house* with Kevin Bacon. Based on this, Tom Hanks is at distance 5 from Kevin Bacon. But this is *not* Hanks' BN: Hanks has BN=1, because he was in *Apollo 13* with Kevin Bacon.

   Model this problem as a graph problem and describe algorithmically how you would solve the Kevin Bacon Game, by answering the questions below:

   (a) What are the vertices and edges?

   (b) Assume we get as input a file `movies.txt` from the Internet Movie Database. This file consists of lines, each line contains a movie title, followed by all actors who played in that movie. Describe how you go about building the graph corresponding to this file.

   (c) Given an actor, you want to find the sequence of movies /actors that connect him/her to Kevin Bacon. Describe how you would do this.