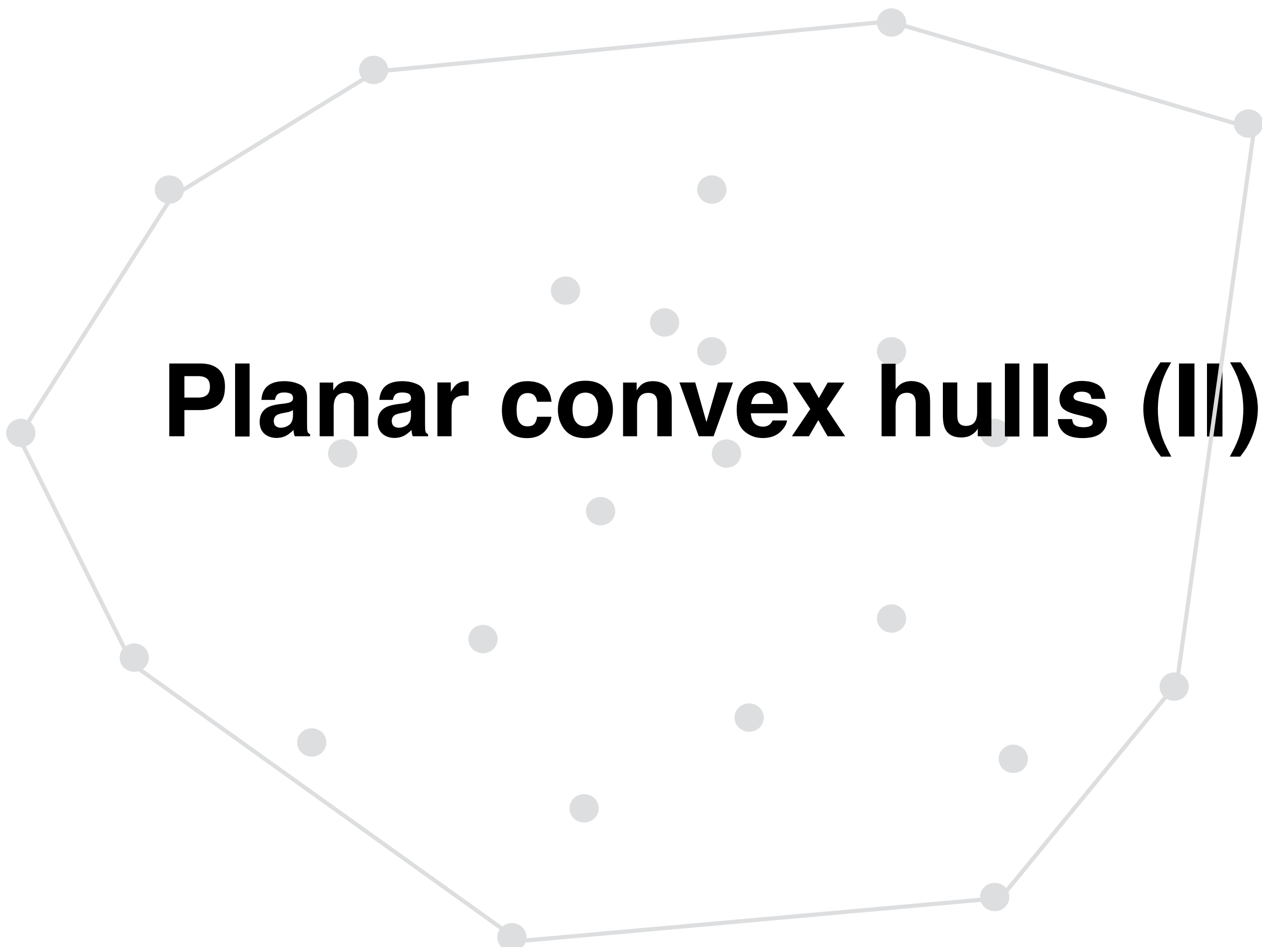
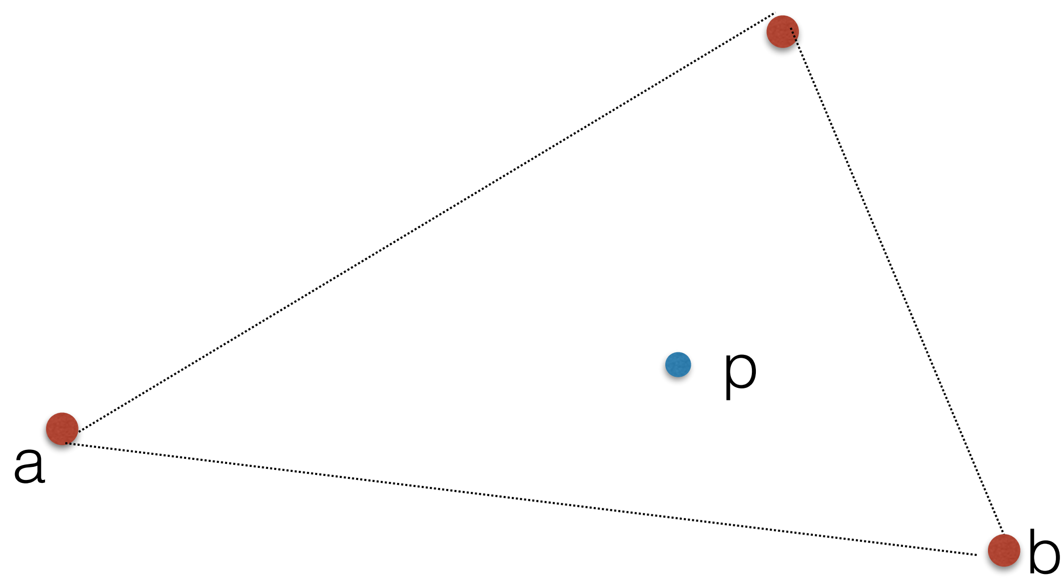
Planar convex hulls (II)



Computational Geometry [csci 3250]
Laura Toma
Bowdoin College

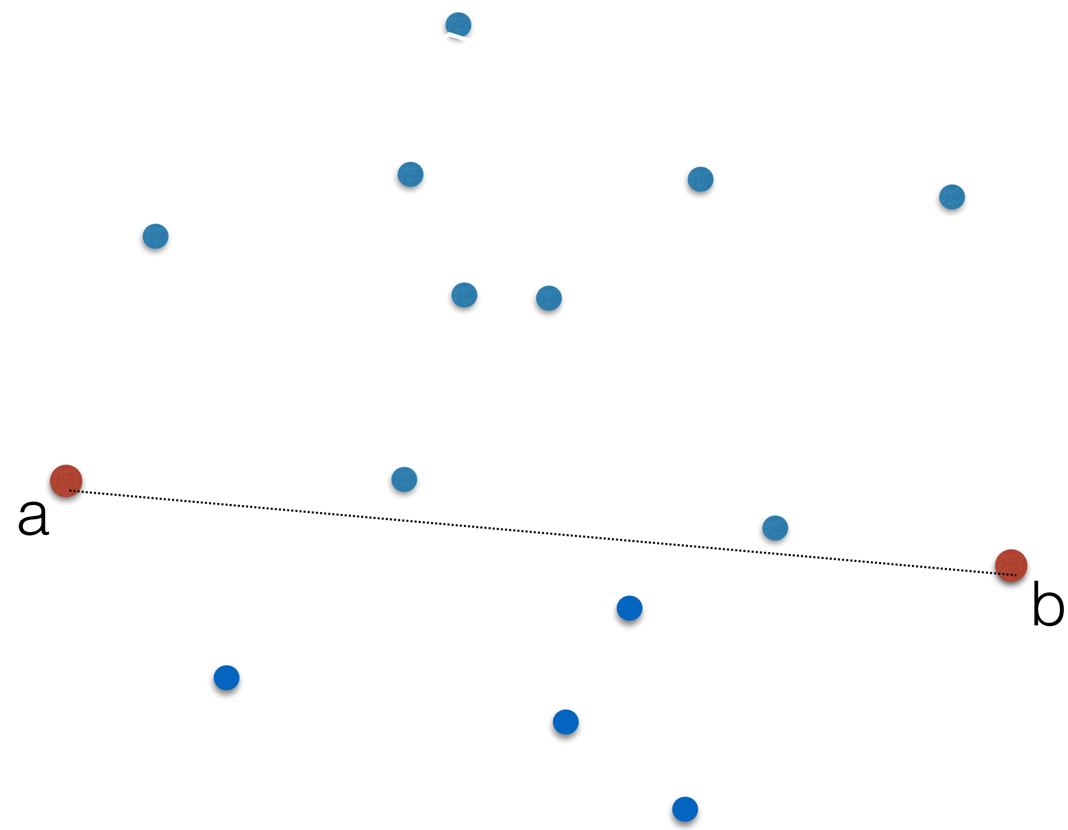
Classwork: Given a point p and a triangle a, b, c

```
//return true if p is inside (or on) abc, and false otherwise
bool isInside (p, a, b, c)
```



Andrew's Monotone Chain Algorithm

- Alternative to Graham's scan, **faster in practice**
- Idea: Find upper hull and lower hulls separately



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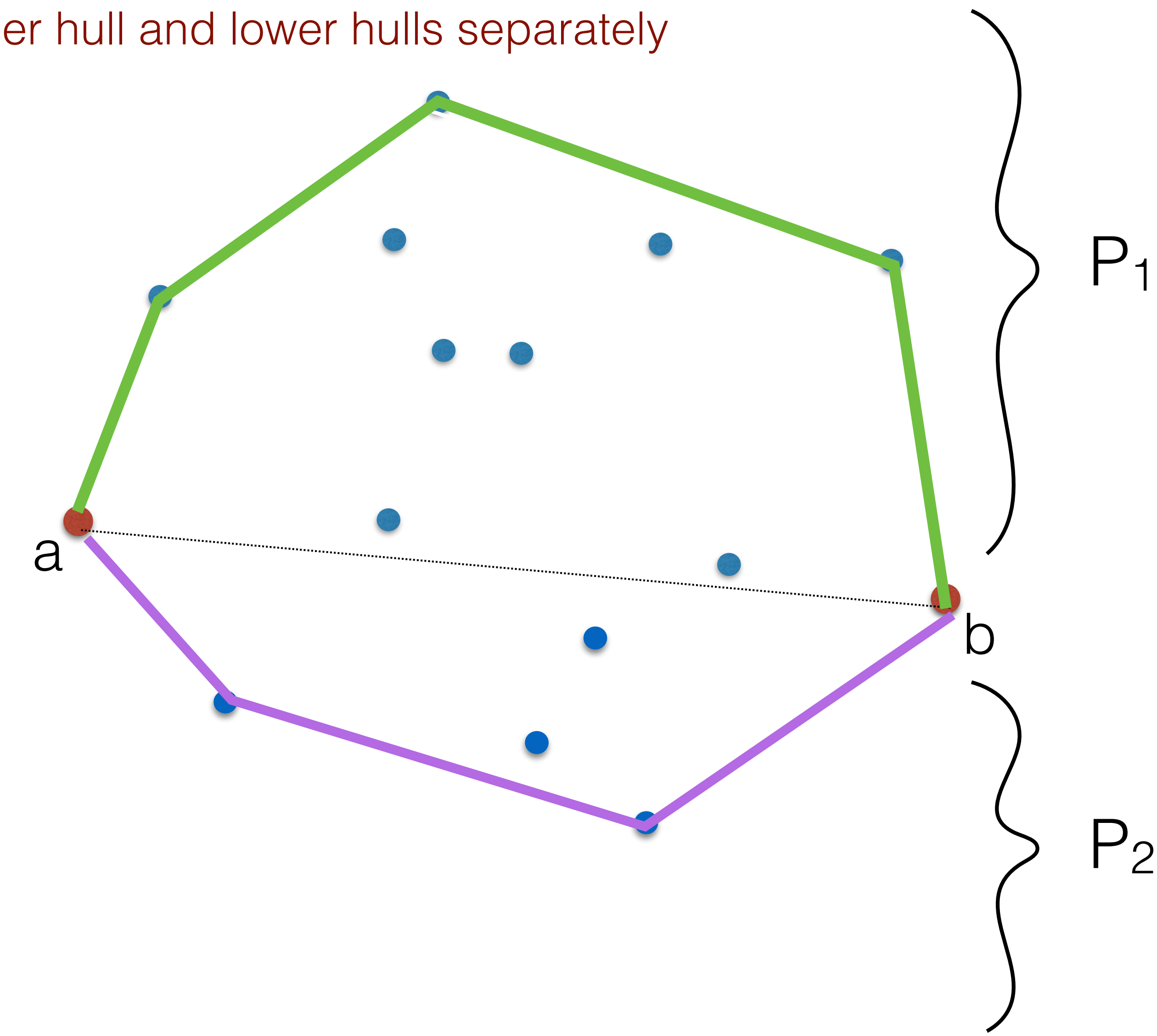
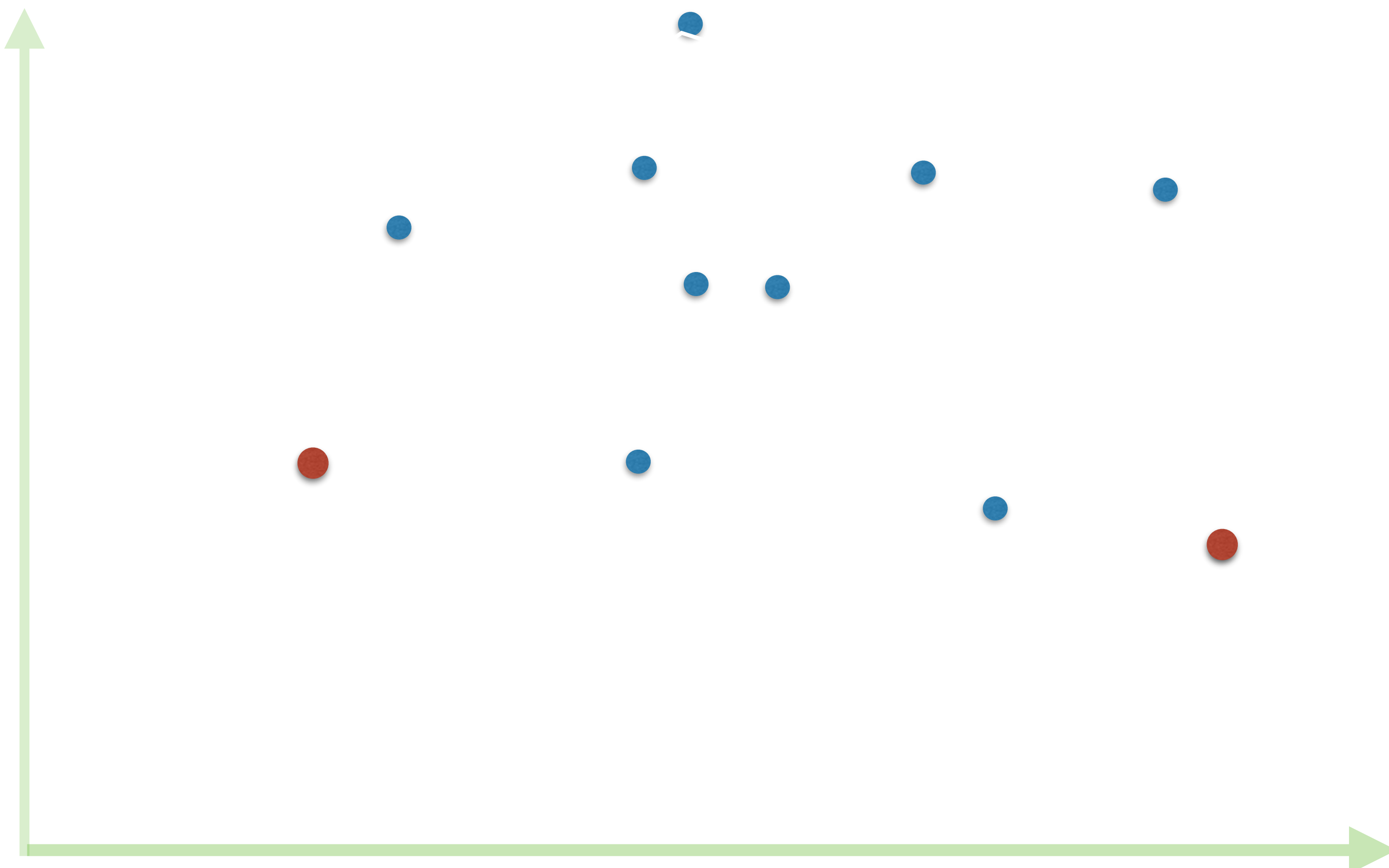


- Order these points in (x,y) lexicographic order



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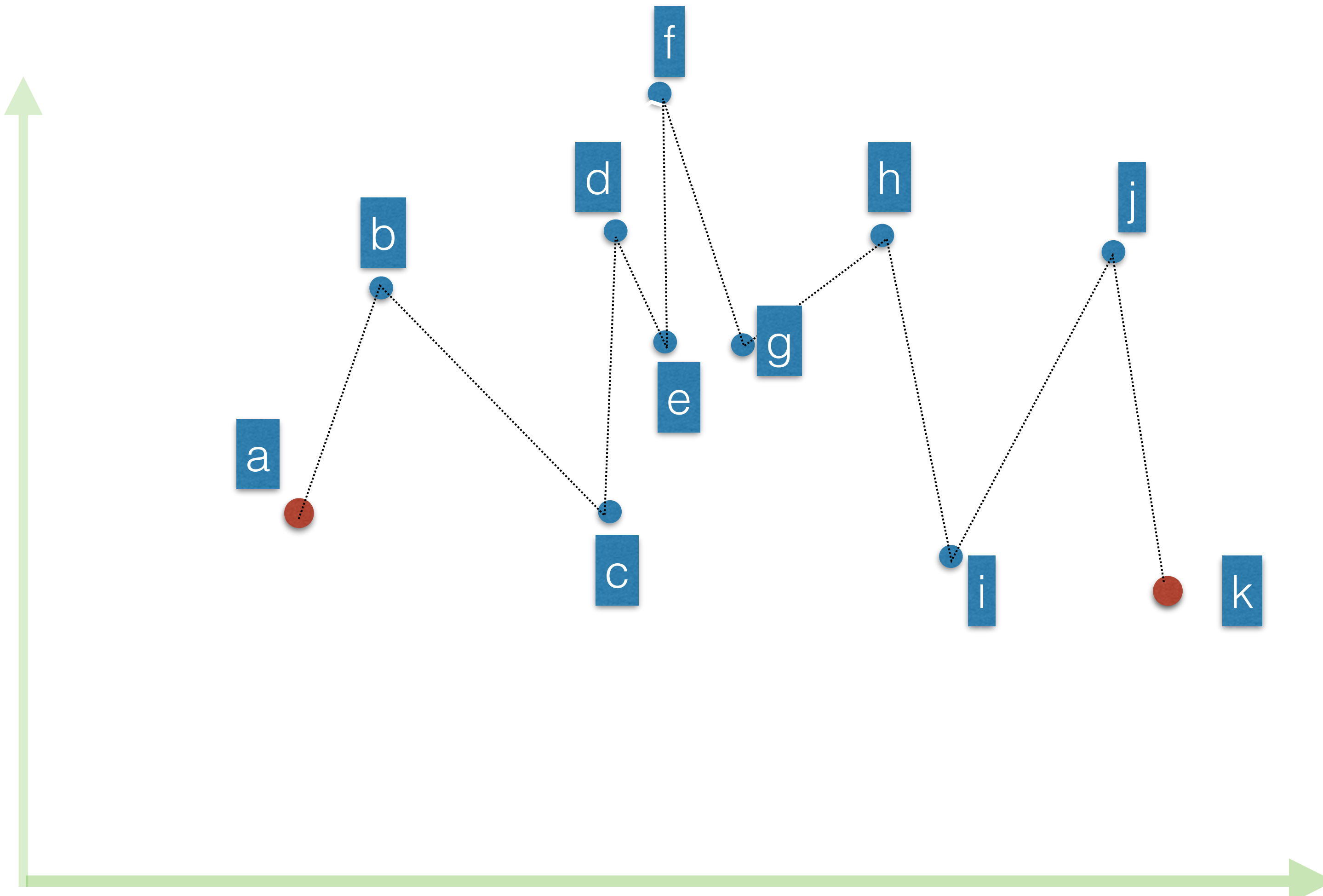
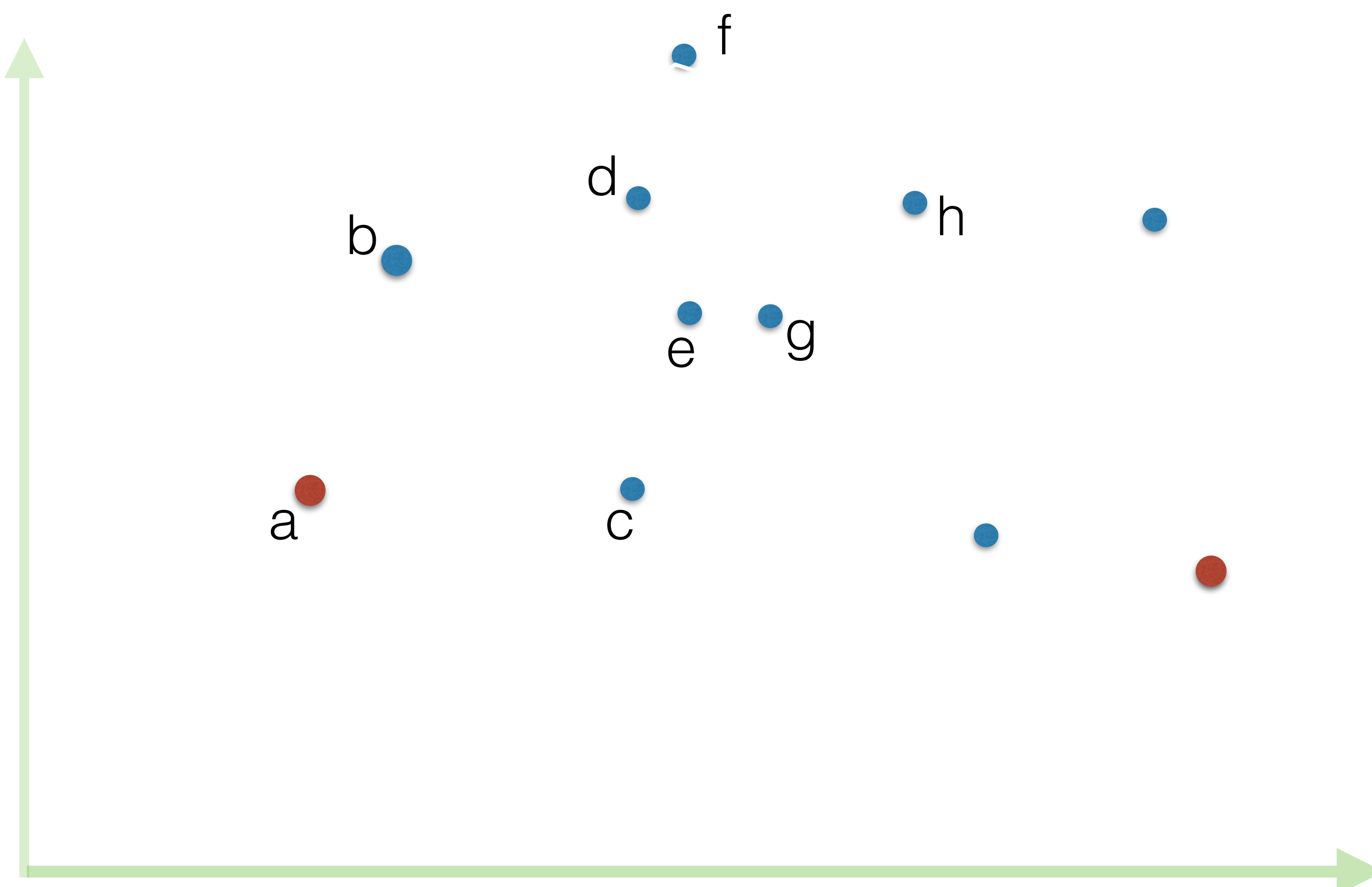


Finding the upper hull of P1

- Traverse points in (x,y) order and build the upper hull, like in Graham scan



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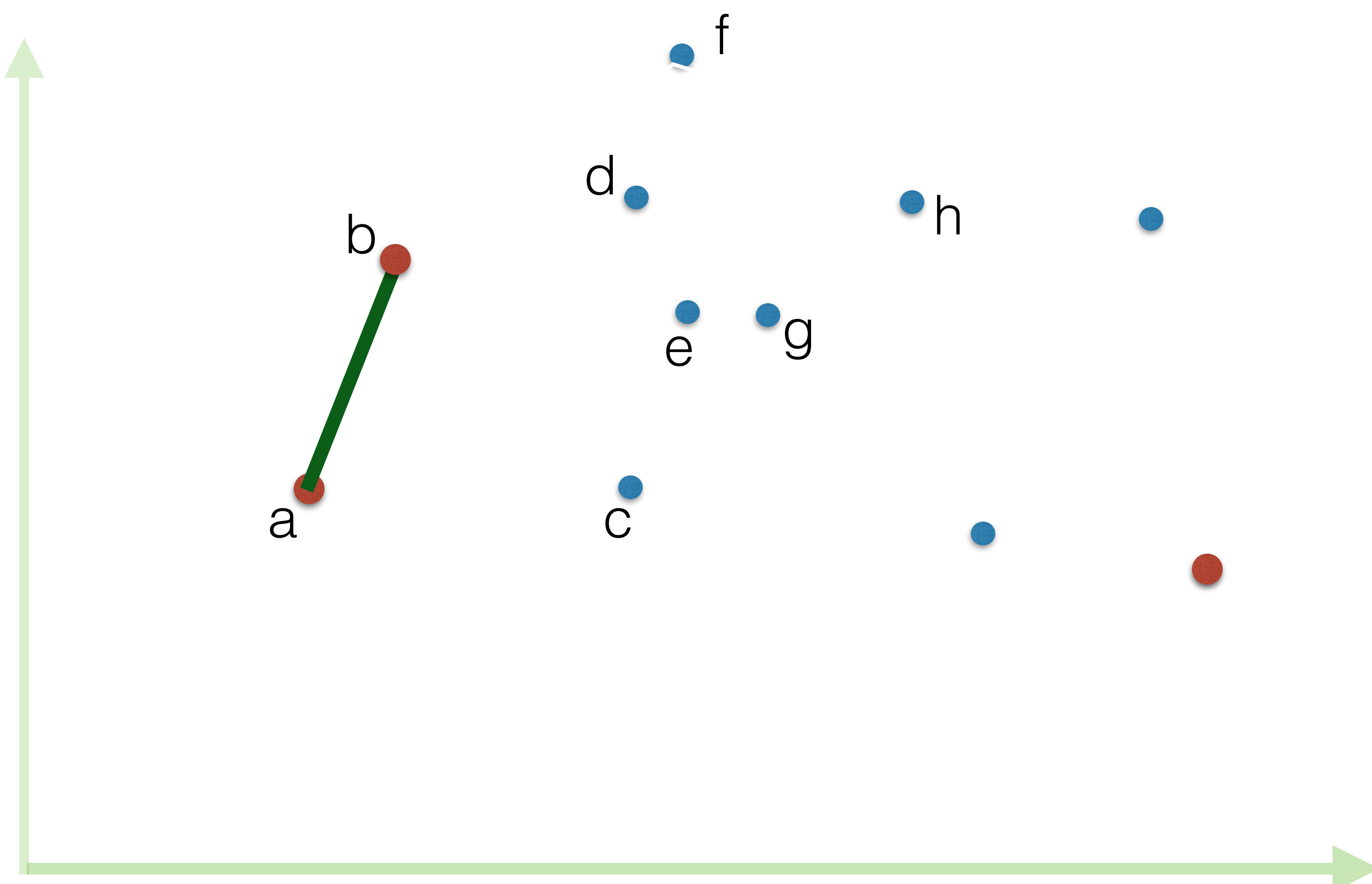

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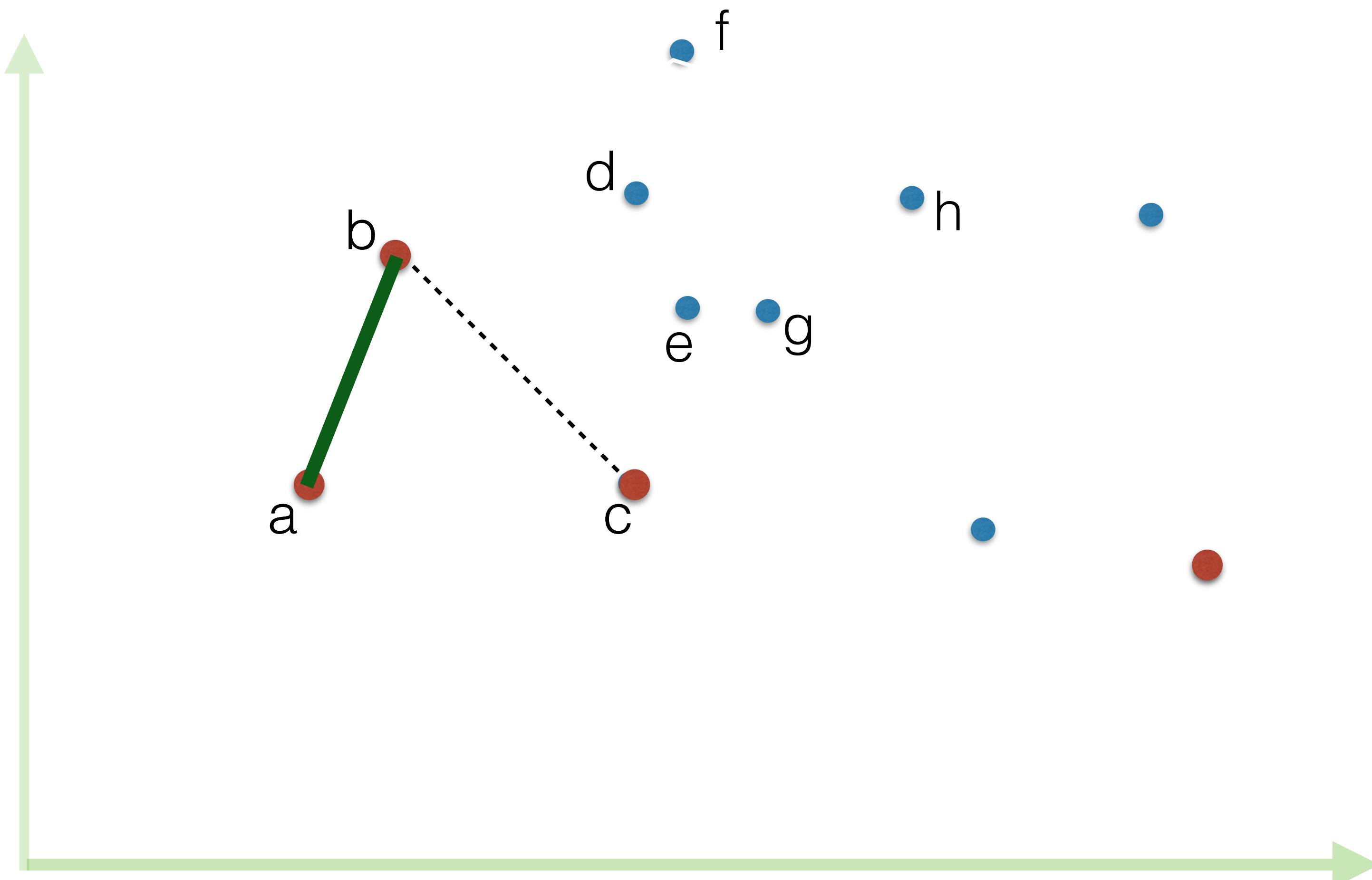

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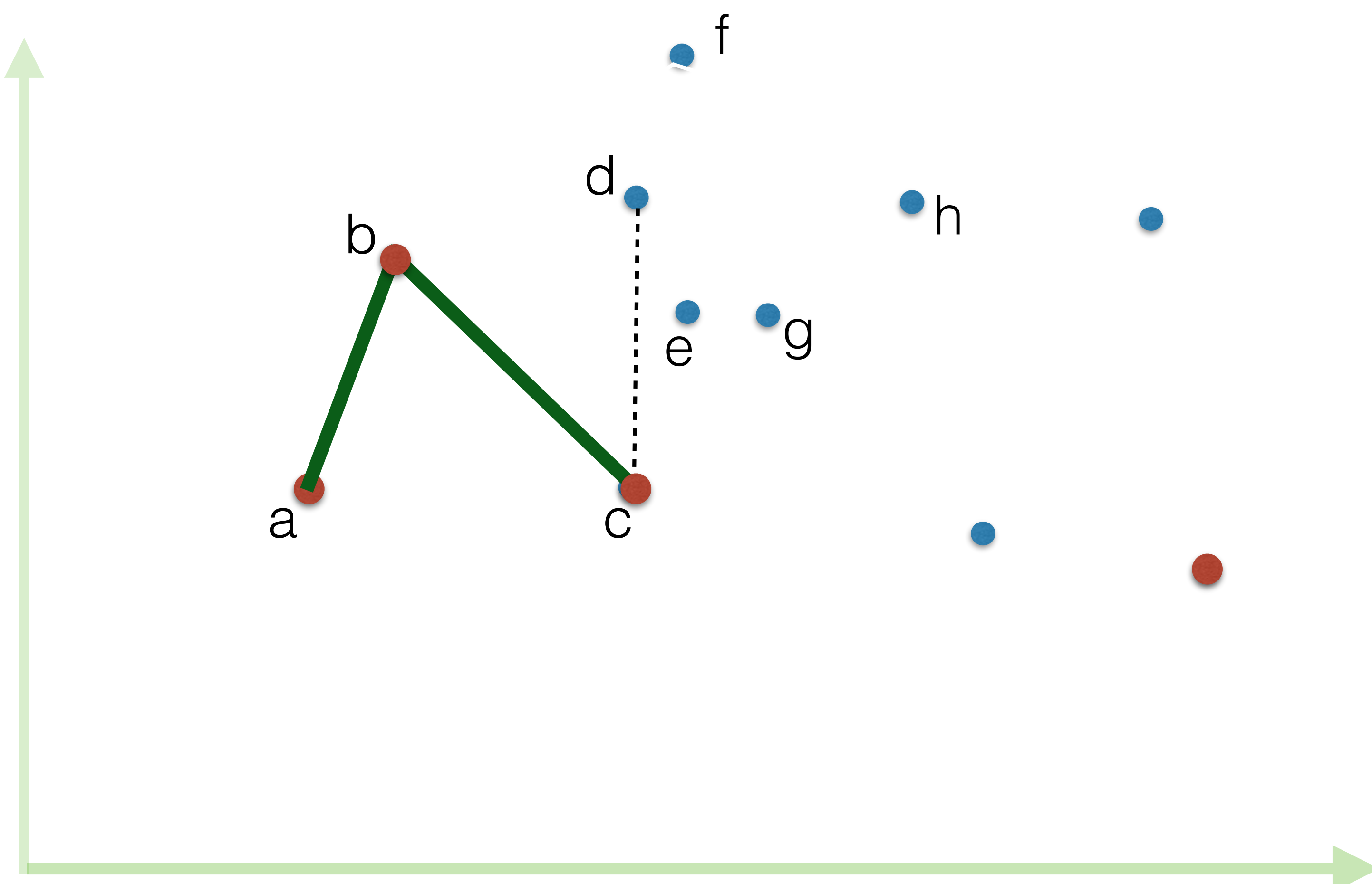

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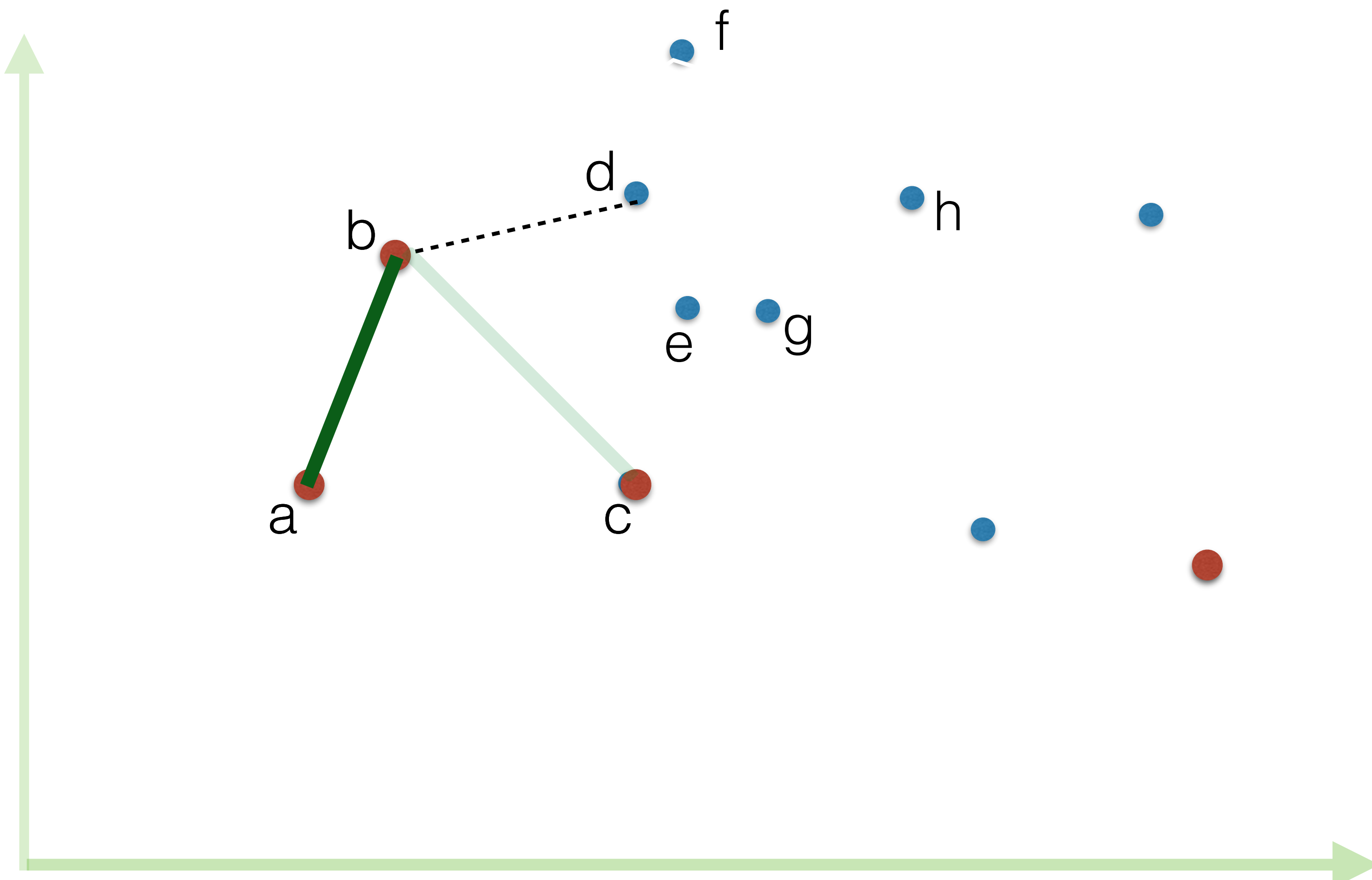

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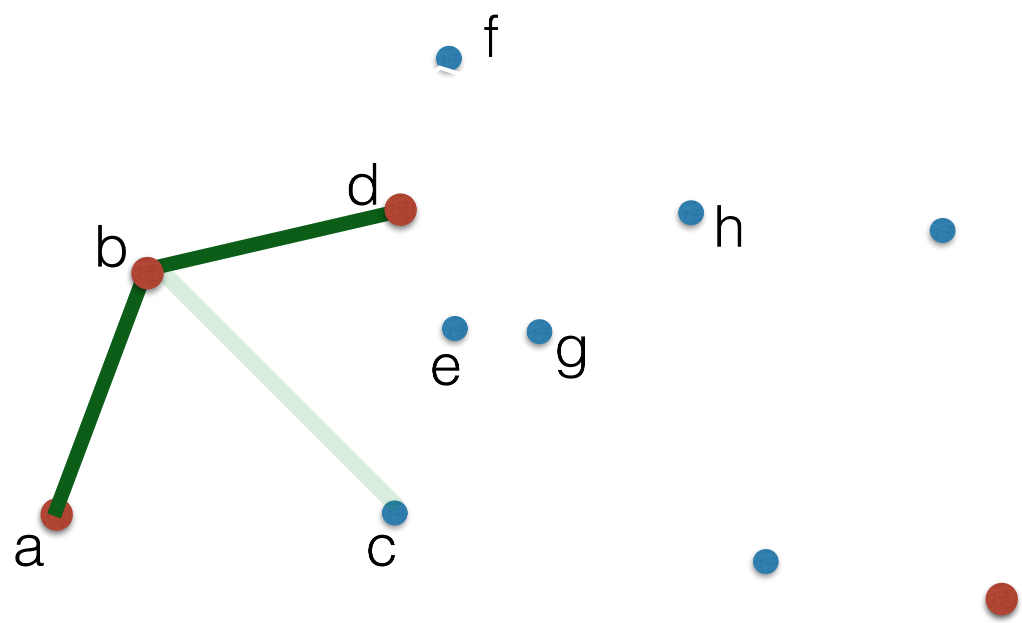

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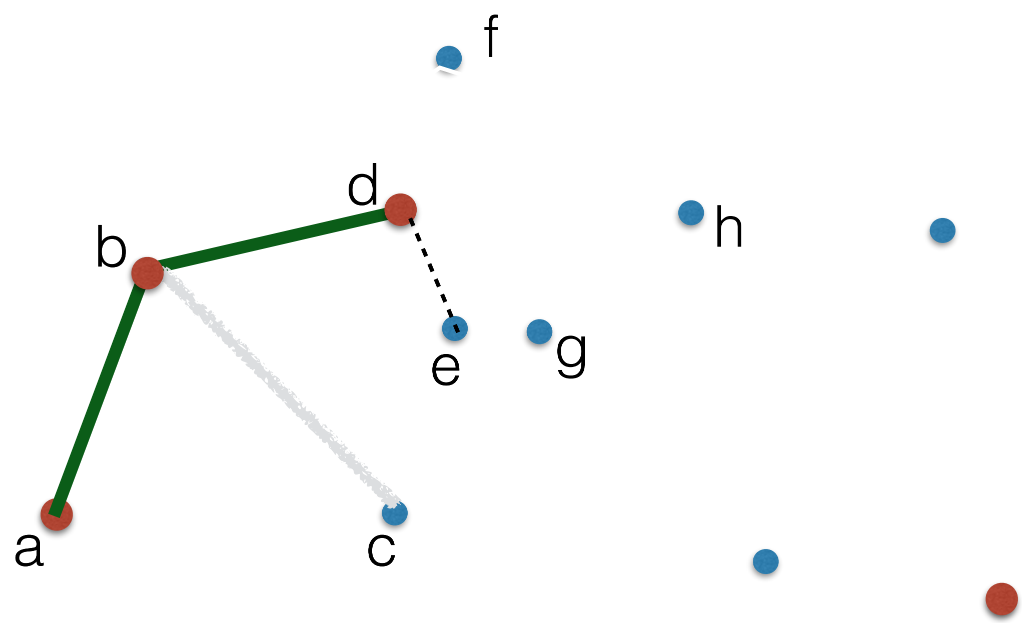

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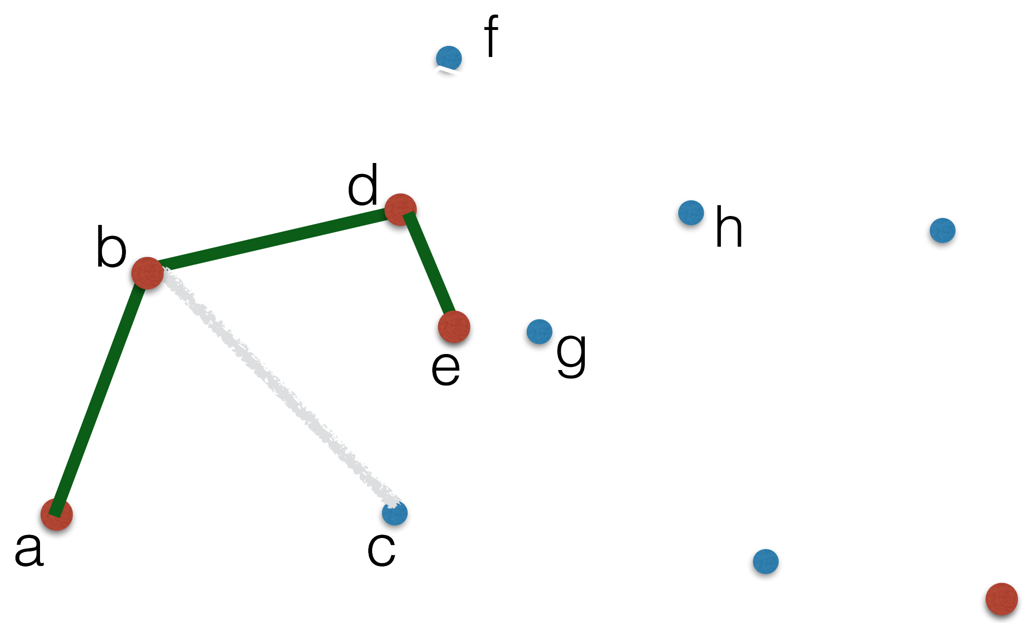

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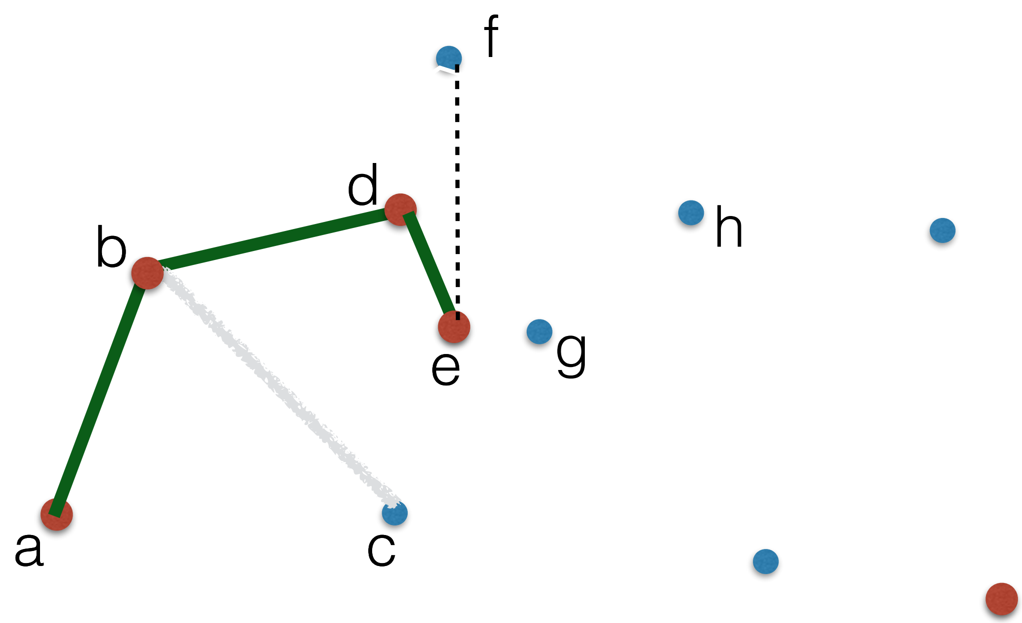

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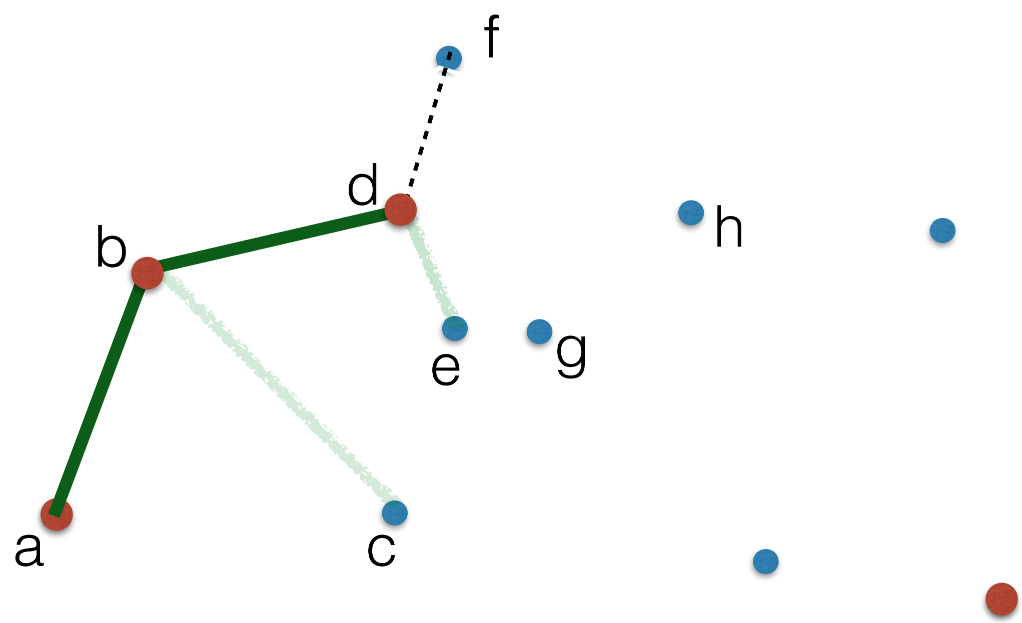

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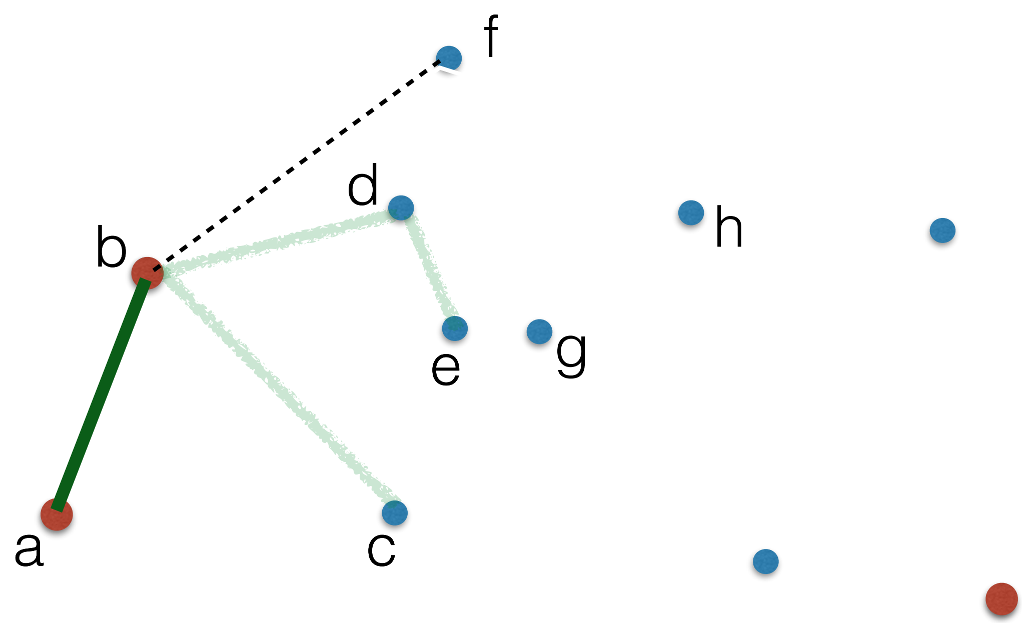

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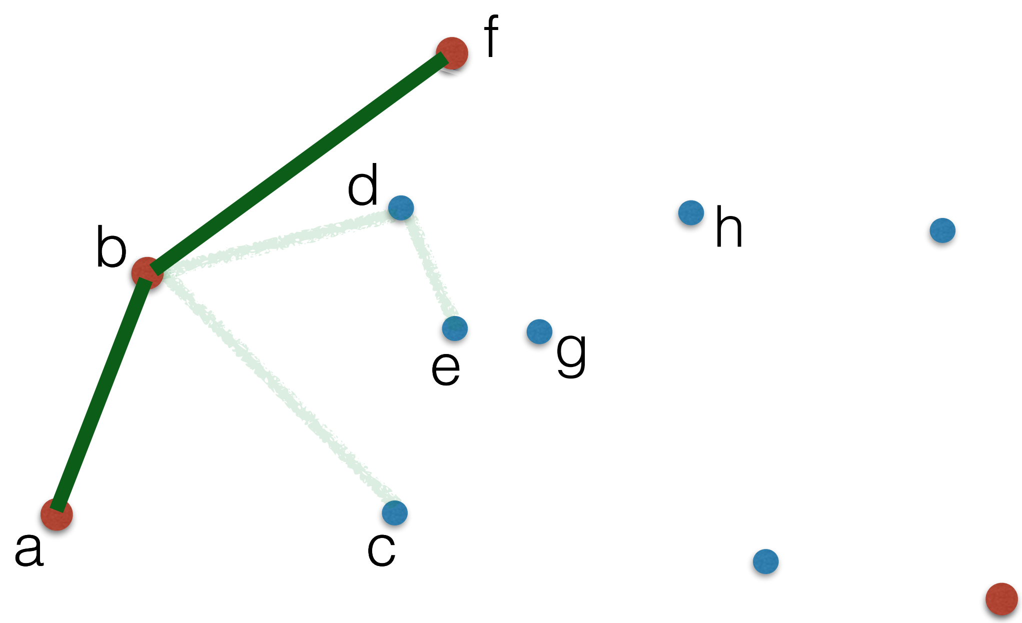

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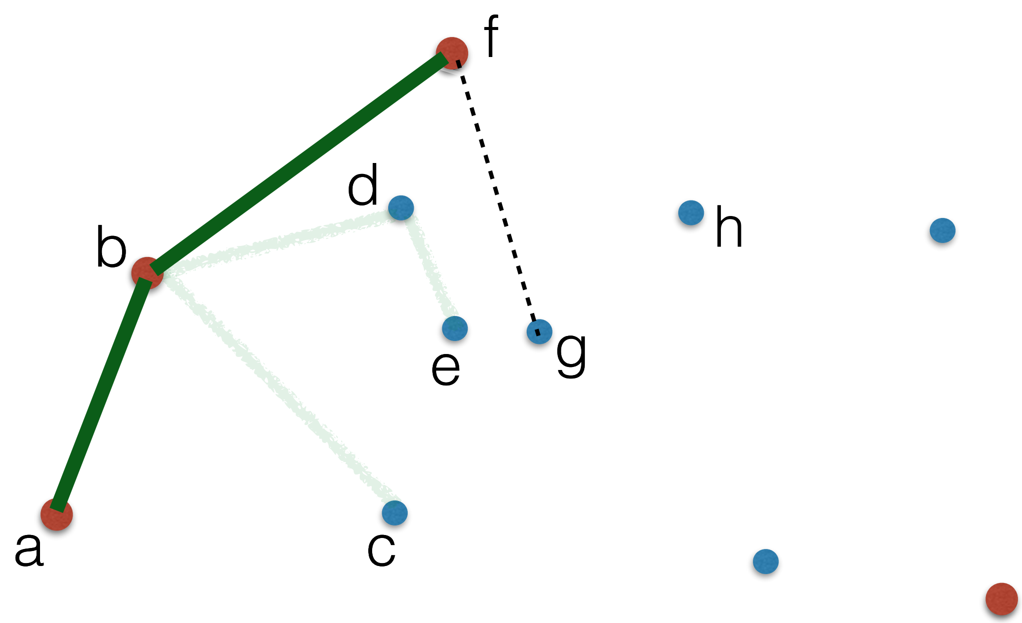

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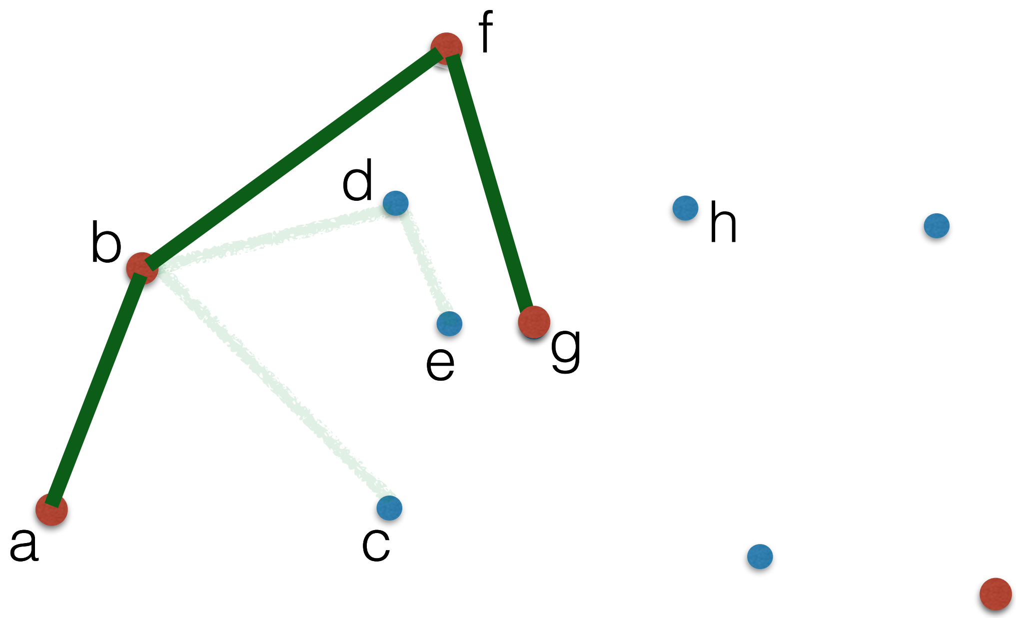

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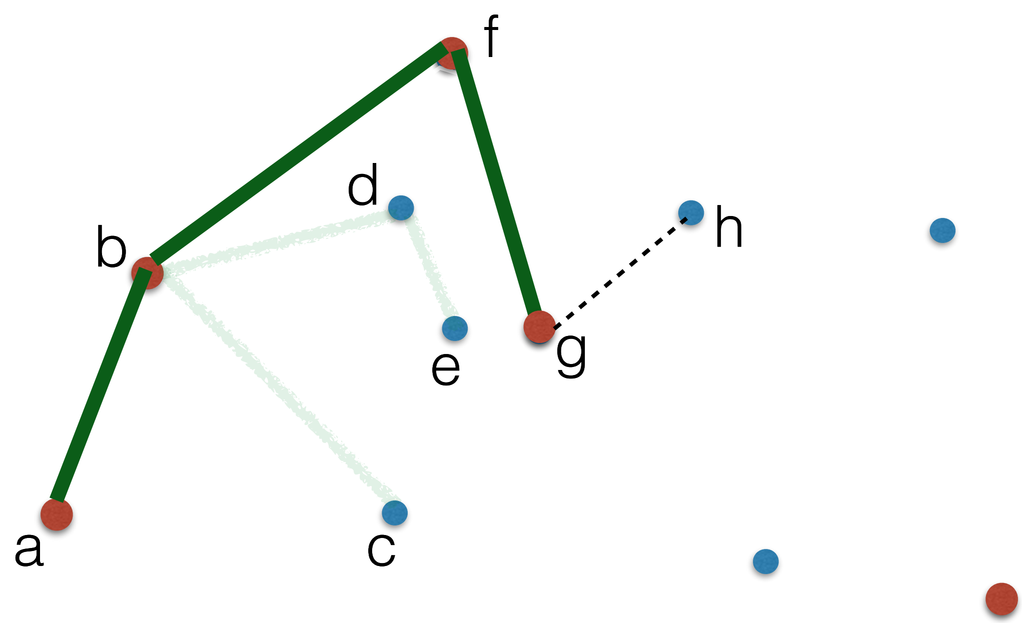

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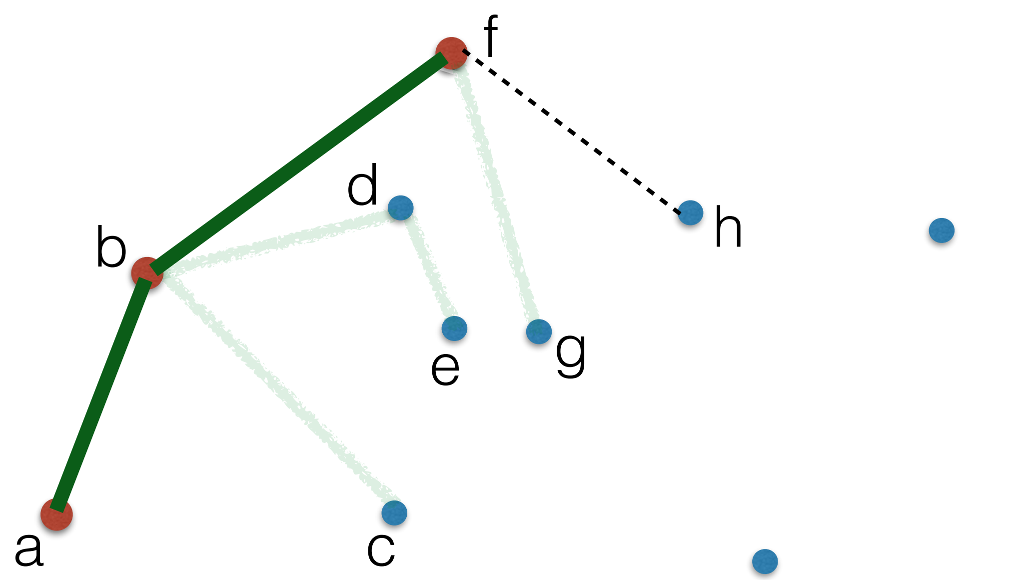

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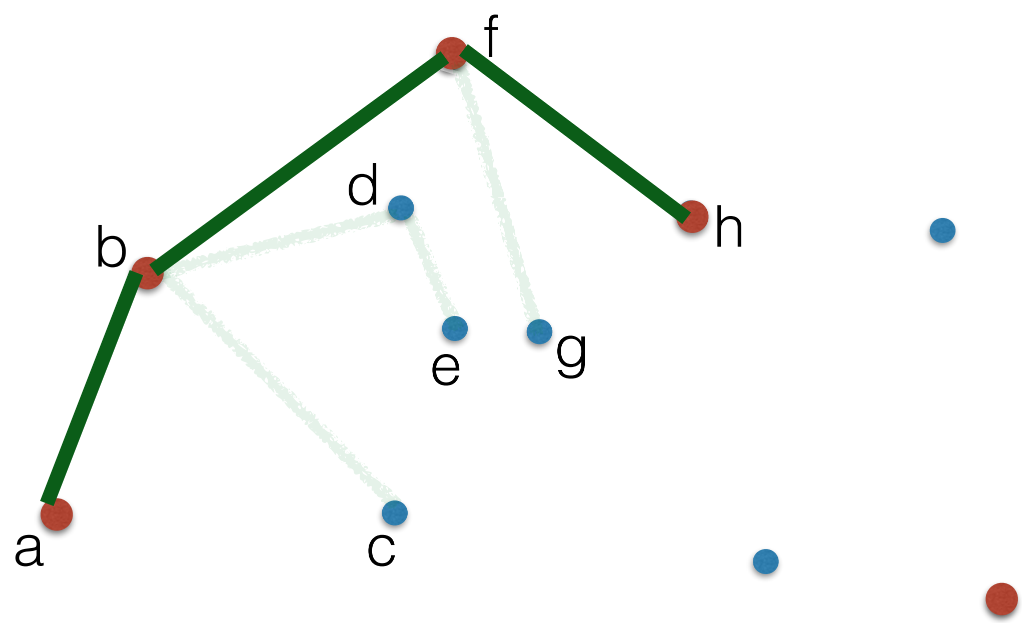


and so on..

Andrew's Monotone Chain Algorithm

- Alternative to Graham's scan
- Same running time
- Sorting lexicographically is faster than sorting radially => faster in practice

Convex hull: summary

Naive	$O(n^3)$
Gift wrapping	$O(nh)$
Quickhull	$O(n^2)$
Graham scan	$O(n \lg n)$
Andrew monotone chain	$O(n \lg n)$

Can we do better?

Lower bound

What is a lower bound?

- Given an algorithm A, its **worst-case running time** is the **largest** running time on any input of size n

  $T_A(n) = \max_{|P|=n} \{ T(n) \mid T(n)$ is the running time of A on input P$\}$

- A lower bound $L(n)$ for a problem is a lower bound on the worst-case running time of any algorithm that solves that problem

  $T_A(n) = \Omega(L(n))$, for all algorithms A that solve the problem

- We could say that Convex hull has a lower bound $L(n) = \Omega(1)$ (trivial). We could also say that $L(n) = \Omega(n)$, also trivial.
- We want larger lower bounds (and lower upper bounds!)
- When the best-known worst-case $T(n)$ of an algorithm, matches the bests-known lower bound for that problem, the problem is considered "solved". The algorithm that matches the lower bound is optimal!

Proving lower bounds

- Lower bounds depend on the machine model.
  - The standard model is the decision tree (comparison) model
- Prove directly
  - Theorem: Any sorting algorithm that uses only comparisons uses at least $\Omega(n \lg n)$ comparisons in the worst case.
  - Proof: We saw this in Algorithms..
- Or via **reduction** from a problem known to have a lower bound
  - aka: $n \lg n < A$ and $A < B \Longrightarrow n \lg n < B$

Lower bounds by reduction

- We know that $\Omega(n \lg n) \leq$ Sorting
- If we could show that ConvexHull is at least as hard as Sorting

Sorting $\leq$ Convex hull

This would imply that ConvexHull is $\Omega(n \lg n)$

How do we show Sorting $\leq$ Convex hull ?

- We'll show that we can use ConvexHull to Sort:
  - Let P be a set of values that need to be sorted. We'll show that there exists some instance of the CH problem that sorts P, and we can build this instance in O(n) time

sortViaCH (array P of n real values)

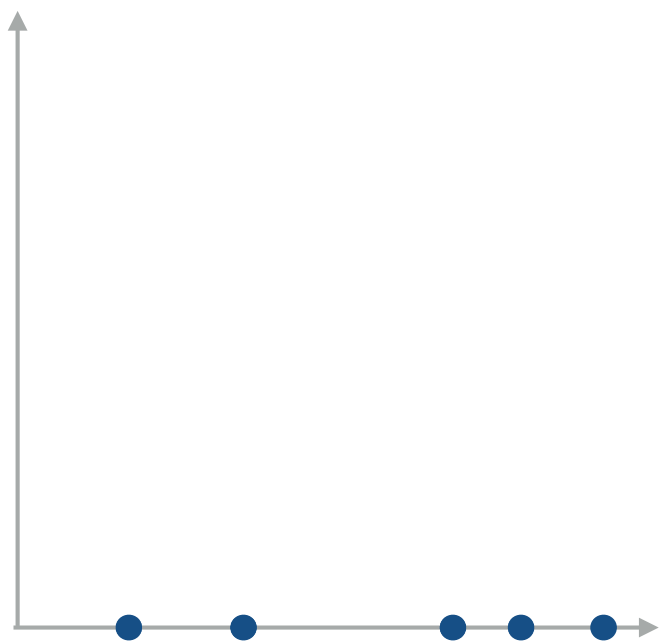
- create a set P' of points from P ← O(n)
- findConvexHull(P')
- use the convex hull to infer sorted order of P ← O(n)

Running time: O(n) + O(findConvexHull))

- If we could find the CH faster than $\Theta(n \lg n)$ in the worst case, we could use it to sort faster than $\Theta(n \lg n)$ in the worst case, which is impossible!

Sorting via ConvexHull

- Let P: set of values x_1, x_2, …x_n. to sort



Our goal is to find an instance of a convex hull problem that sorts our numbers.

Sorting via ConvexHull

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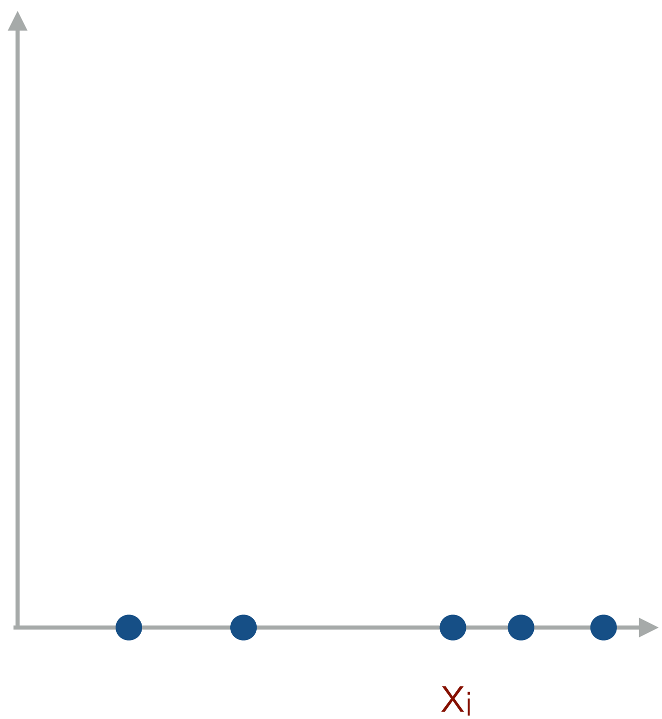
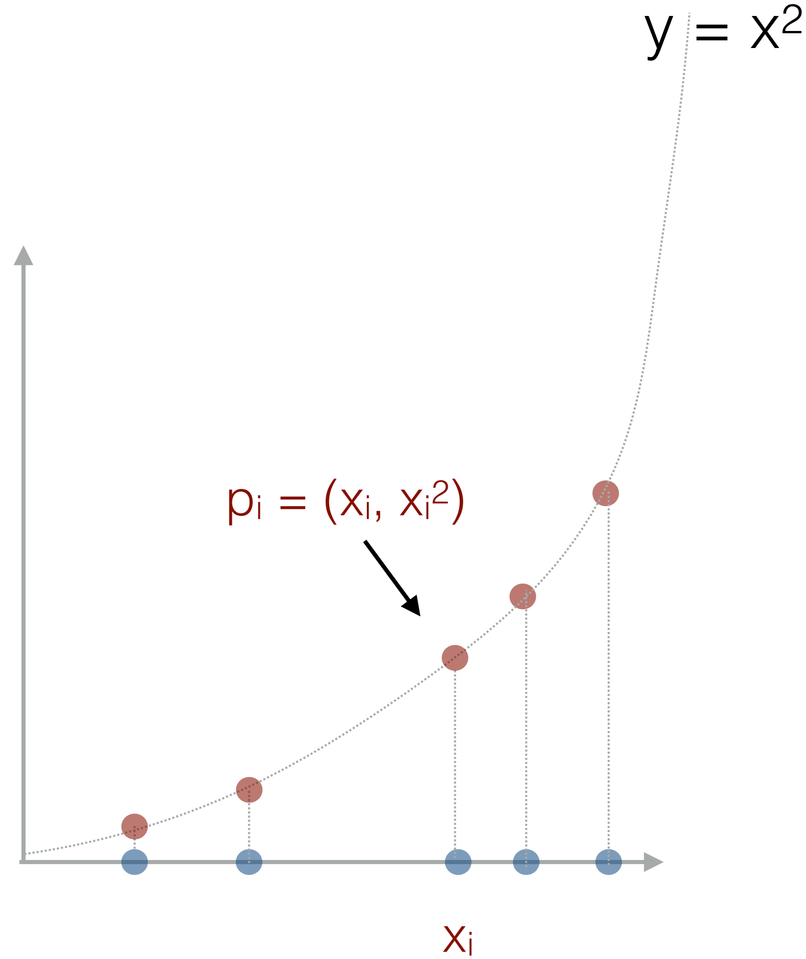


- Let P': set points { $p_i = (x_i, x_i^2)$}



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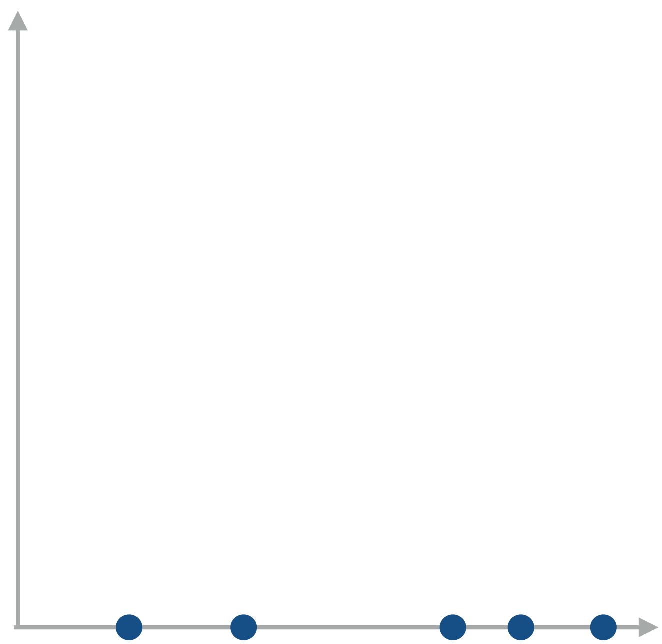
- Let P: set of values $x_1, x_2, \ldots x_n$. to sort



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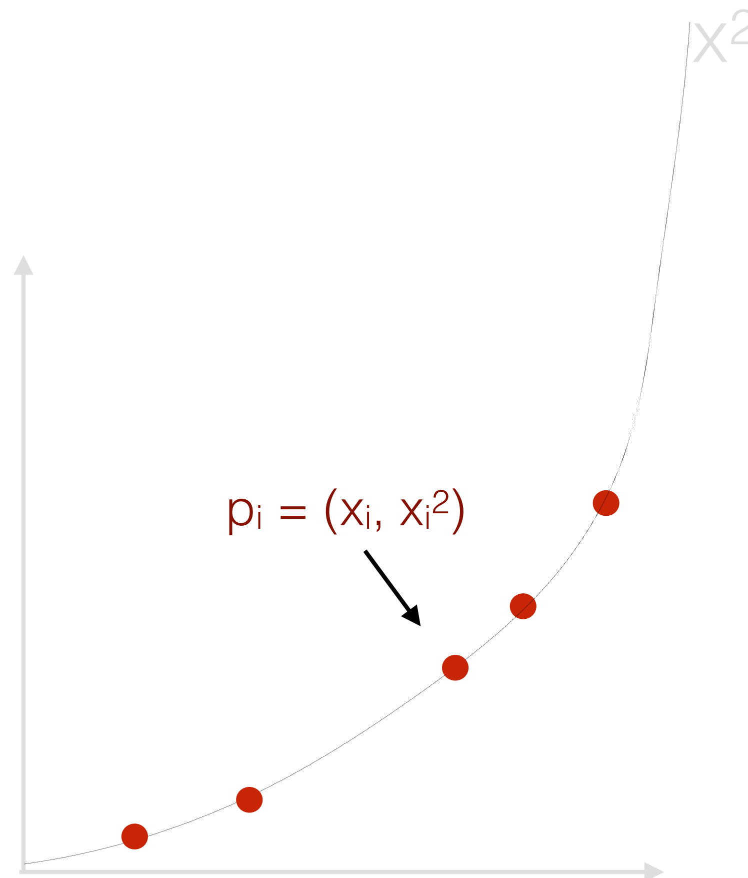

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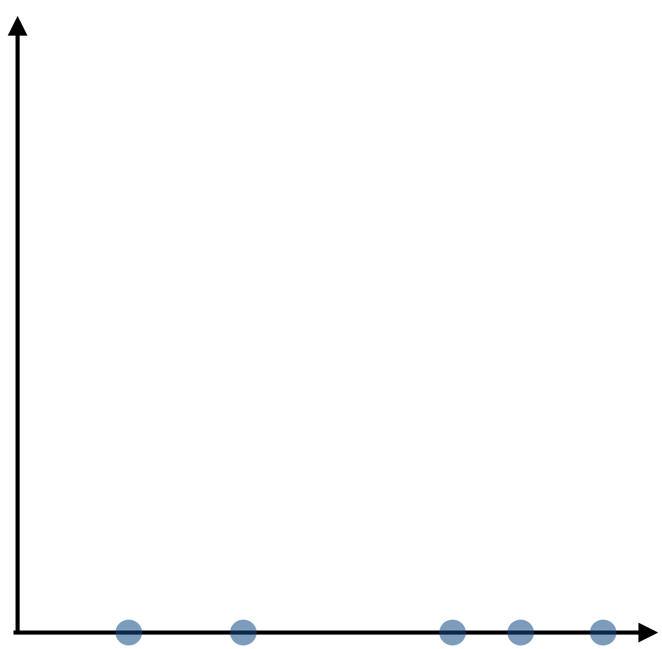
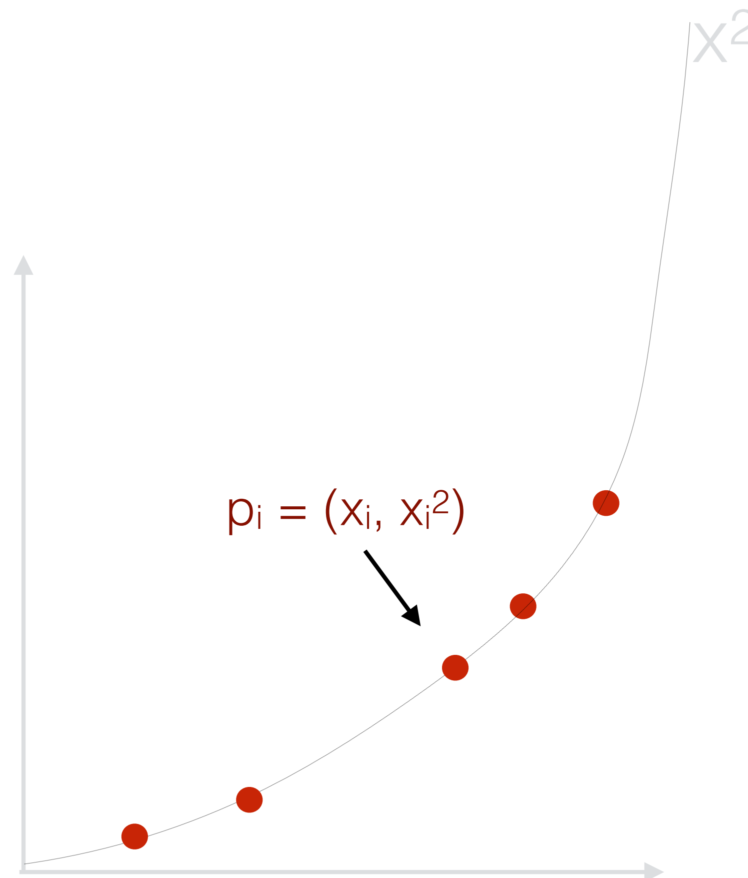


- Let P’: set points { $p_i = (x_i, x_i^2)$}
- Run CH(P’) to find their convex hull



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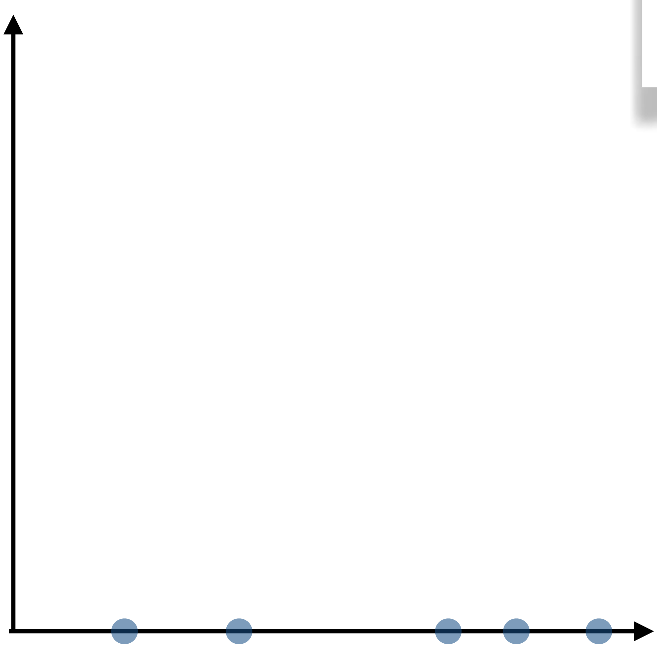
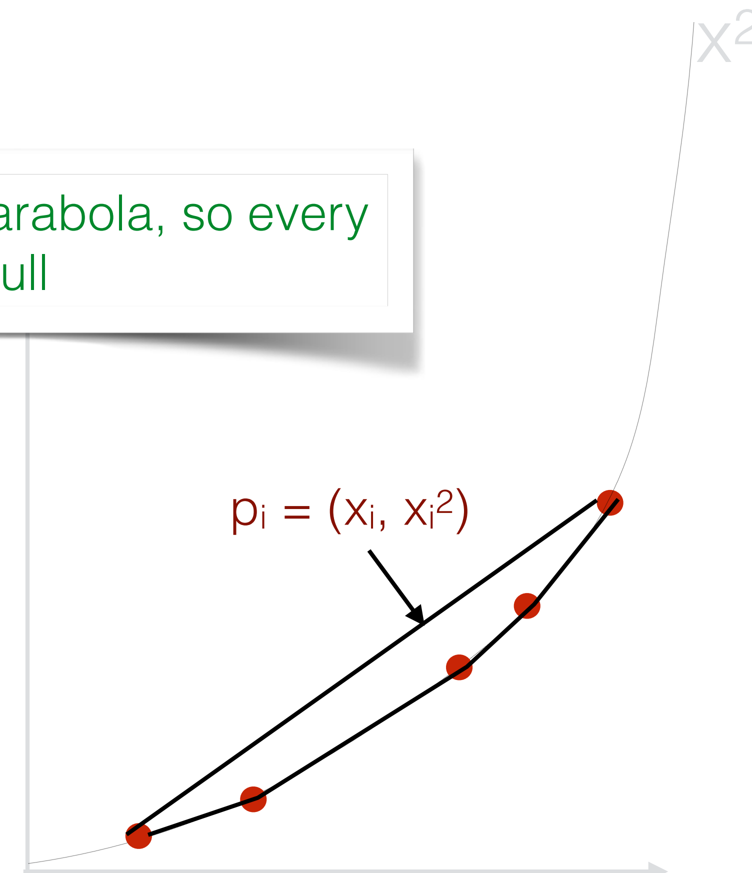


- Let P': set points $\{ p_i = (x_i, x_i^2) \}$
- Run CH(P') to find their convex hull

- They fall on a parabola, so every point is on the hull



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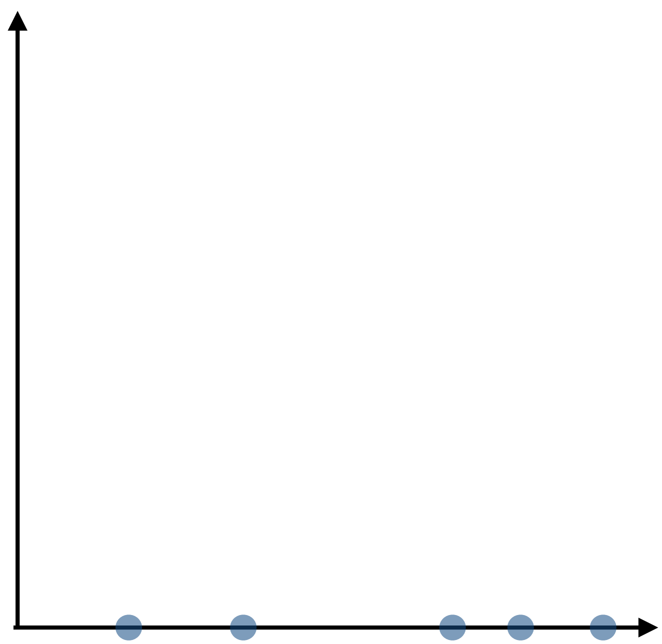
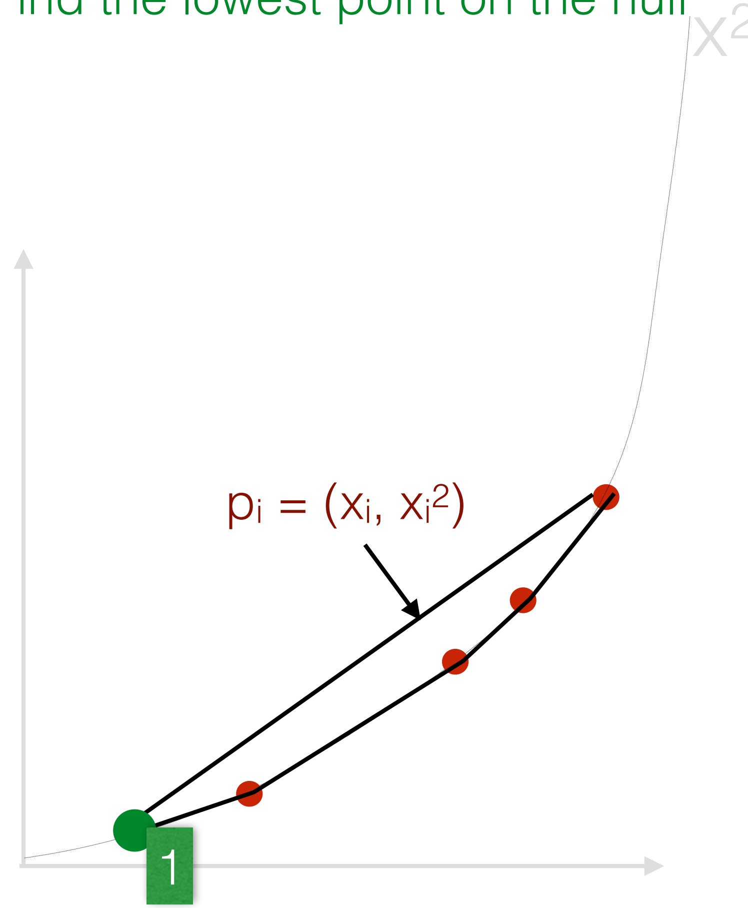
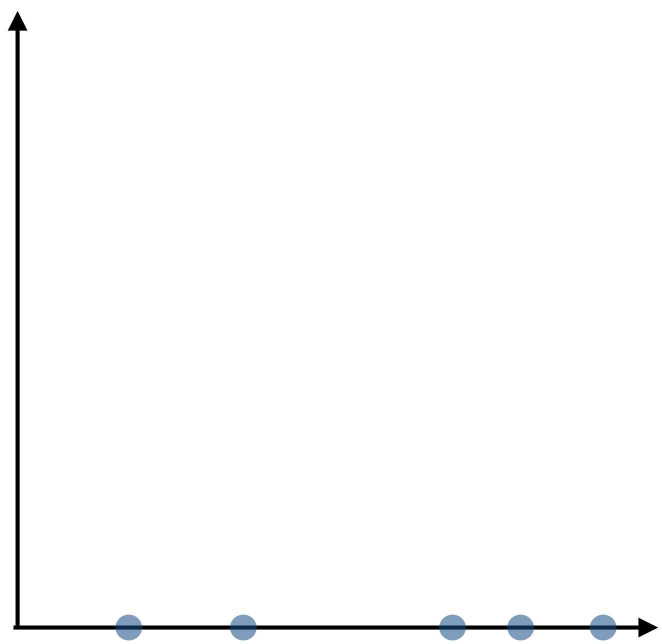


- Let P': set points $\{ p_i = (x_i, x_i^2)\}$
- Run CH(P') to find their convex hull
- Find the lowest point on the hull

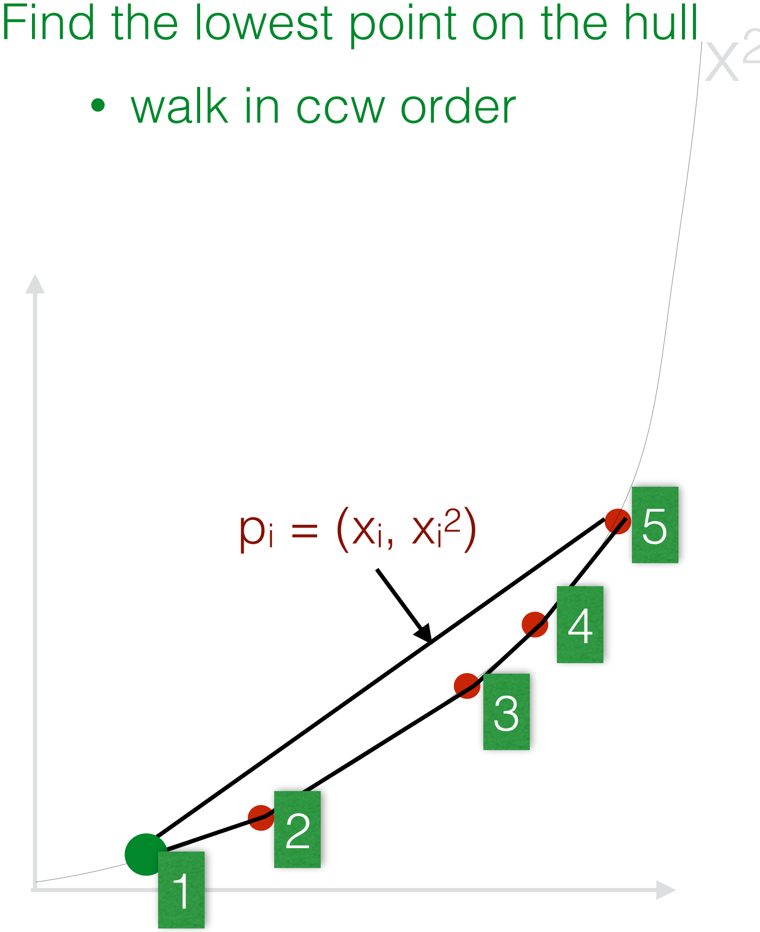


Sorting via ConvexHull

- Let P: set of values $x_1, x_2, \ldots x_n$ to sort



- Let P’: set points $\{ p_i = (x_i, x_i^2)\}$
- Run CH(P’) to find their convex hull
- Find the lowest point on the hull
  - walk in ccw order



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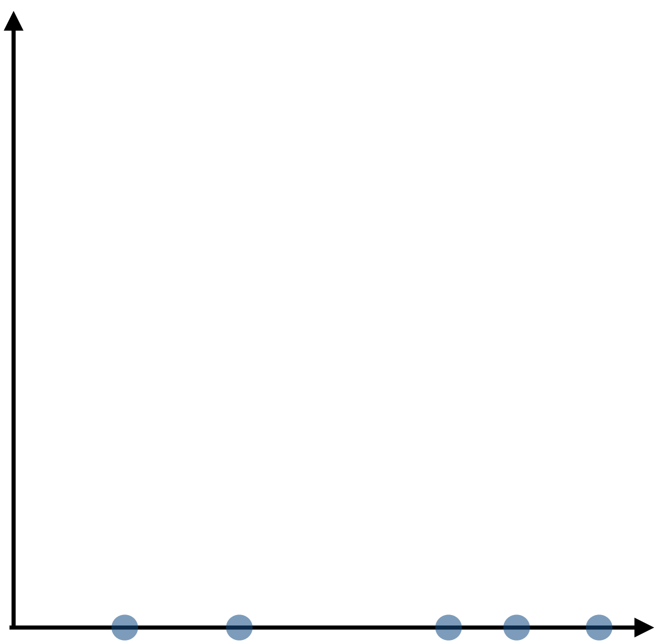

- Let P’: set points { $p_i = (x_i, x_i^2)$}
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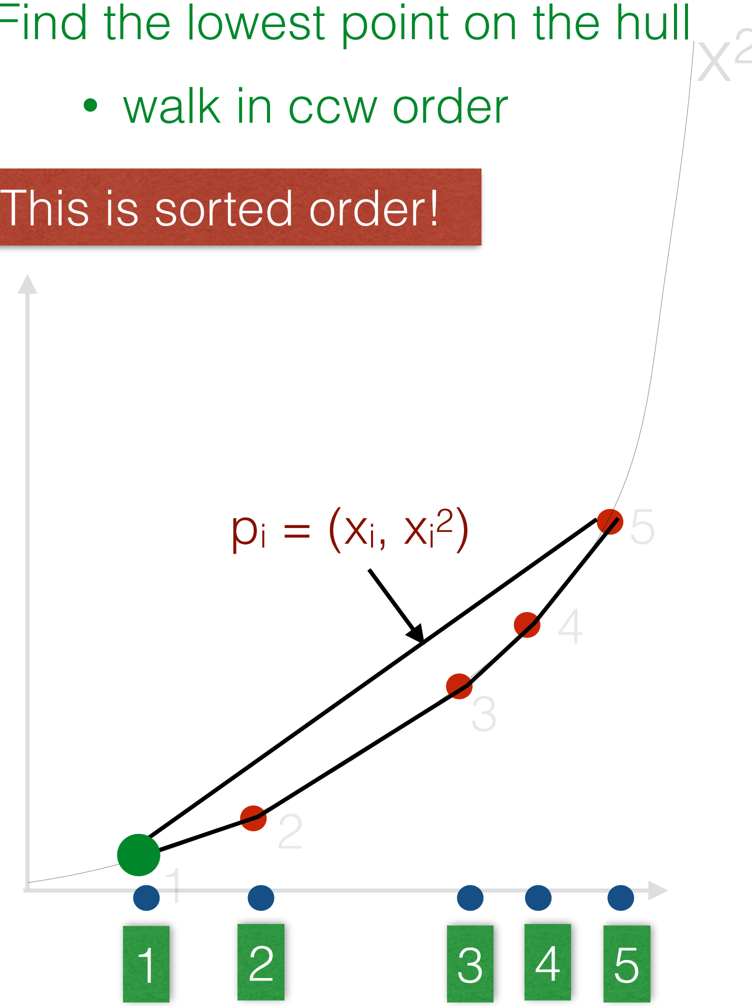
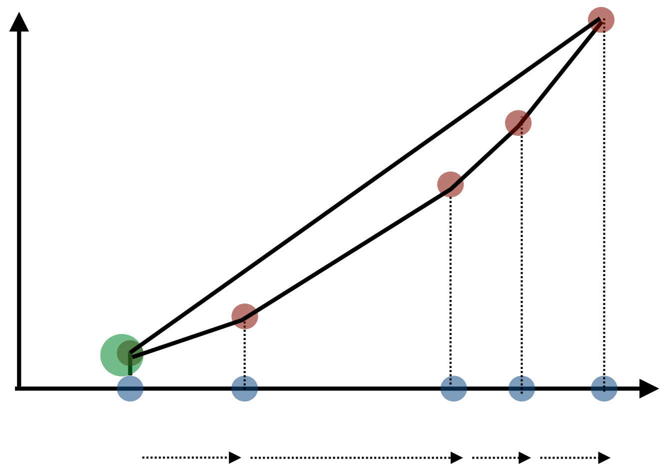
- Find the lowest point on the hull
  - walk in ccw order

This is sorted order!



Sorting via ConvexHull

- Input: set of points $x_1, x_2, \ldots x_n$
  - Create a set of 2D points (x_i, x_i^2).
  - Run the CH algorithm to construct their convex hull.
  - Find the lowest point on the hull, and walk from in ccw order. This is sorted order!



Analysis: runs in O(CH(n)) + O(n)

- This shows that sorting runs in O(CH) + O(n)
  - CH is an upper bound for sorting, or Sorting $\leq$ ConvexHull
- If we could find the CH faster than $\Theta(n \lg n)$, we could use it to sort faster than $\Theta(n \lg n)$, which is impossible!

Summary

sorting $\xrightarrow{O(n)}$ convex hull

reduces to
(or: solves via)

sorting is $\Omega(n \lg n)$

CH must be $\Omega(n \lg n)$

Sorting reduces to CH

- What we actually proved is that
  - Any CH algorithm **that produces the boundary in order** must take Omega (n lg n) in the worst case.
- If we did not want the boundary in order, can the CH be constructed faster?
  - It was an open problem for a while
  - Finally, it was established quite recently that a convex hull algorithm, even if it does not produce the boundary in order, still needs $\Omega(n \lg n)$ in the worst case

- Yes, Graham scan is the ultimate CH algorithm but…
    - not output sensitive
    - does not extend to 3D

- The (re)search continues

An incremental algorithm for CH

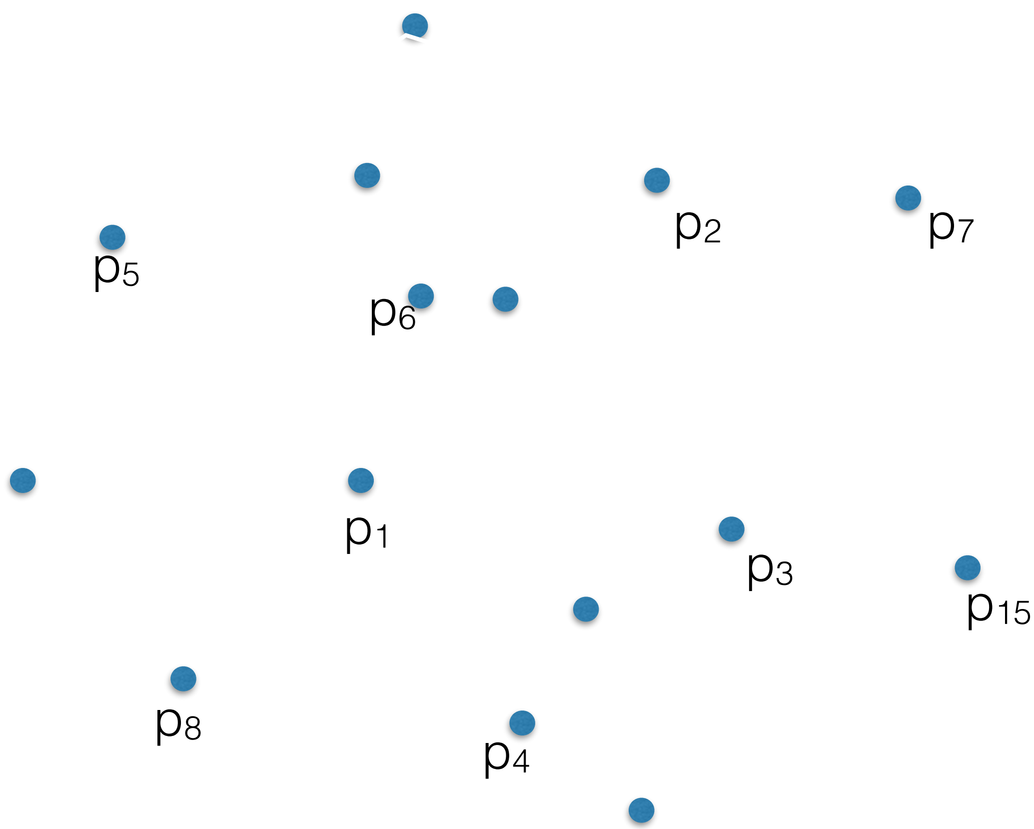
Incremental algorithms

- Idea: Traverse the points one at a time and solve the problem for the points seen so far

- Incremental Algorithm
    - initialize solution S
    - for i=1 to n
        - //S represents solution of $p_1 \ldots\ldots p_{i-1}$
        - update S to represent solution of $p_1 \ldots\ldots p_{i-1}$ p_i

Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



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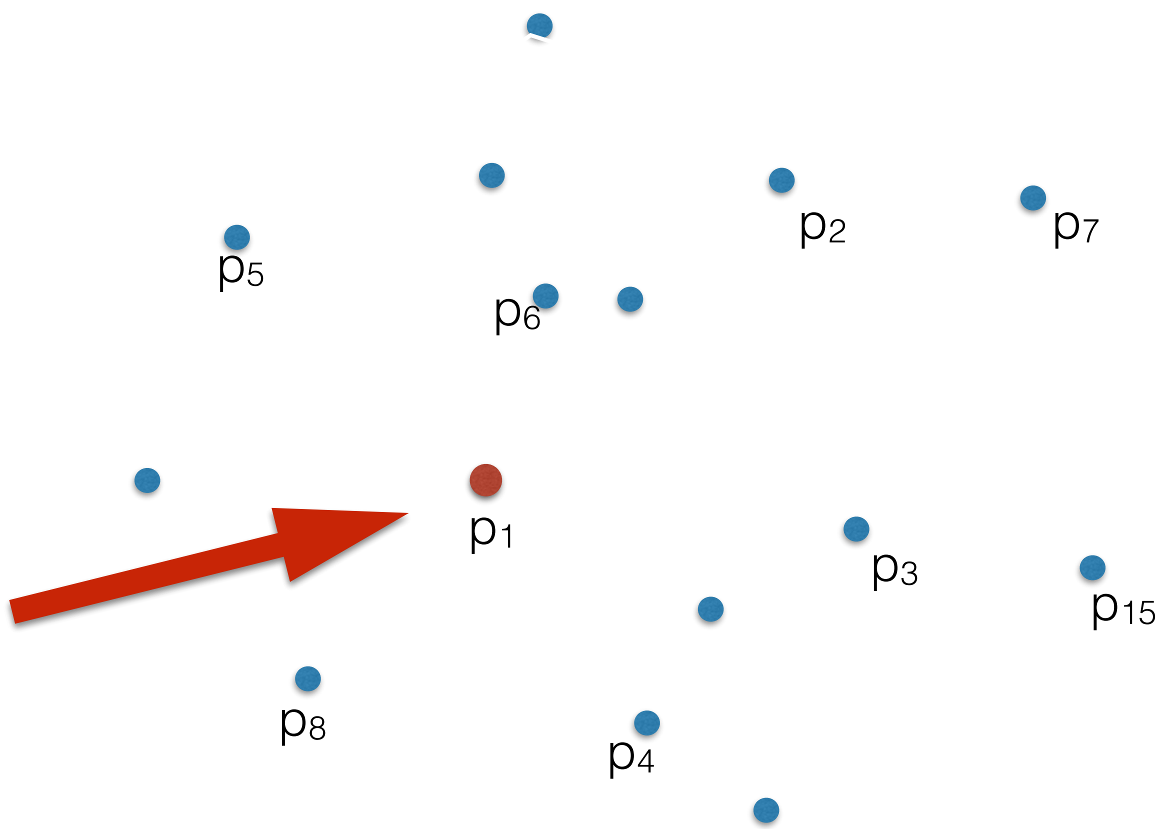

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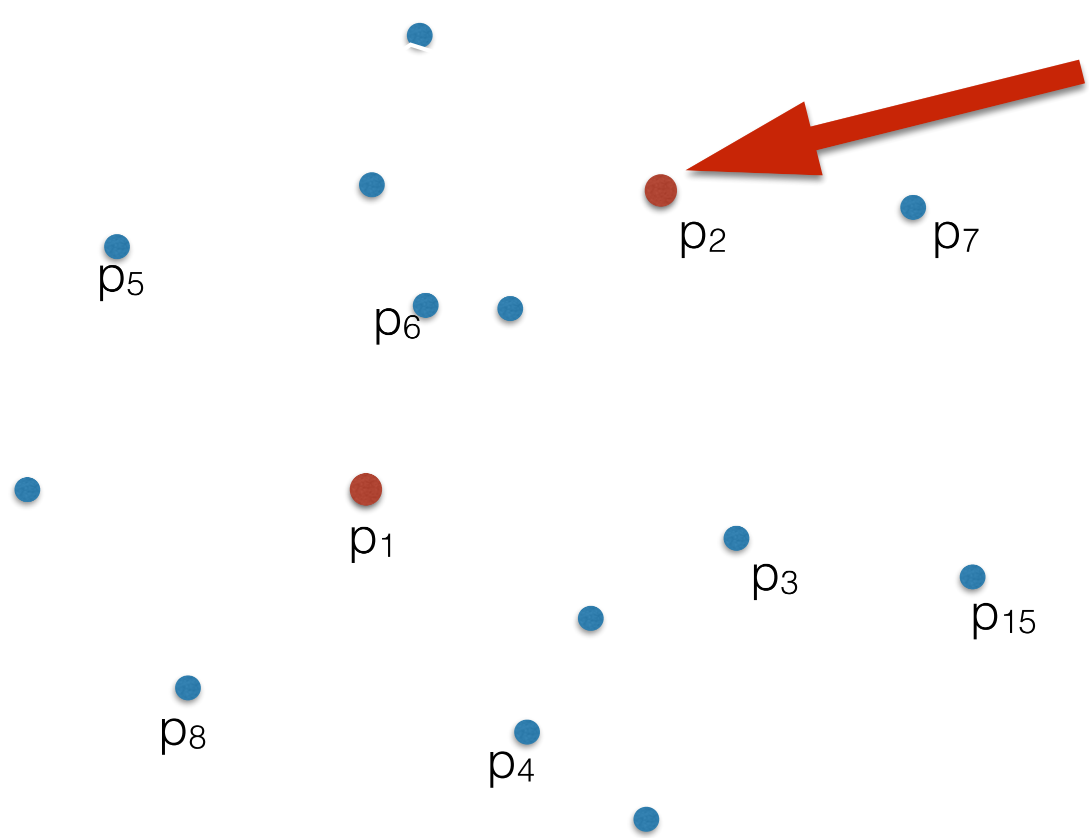

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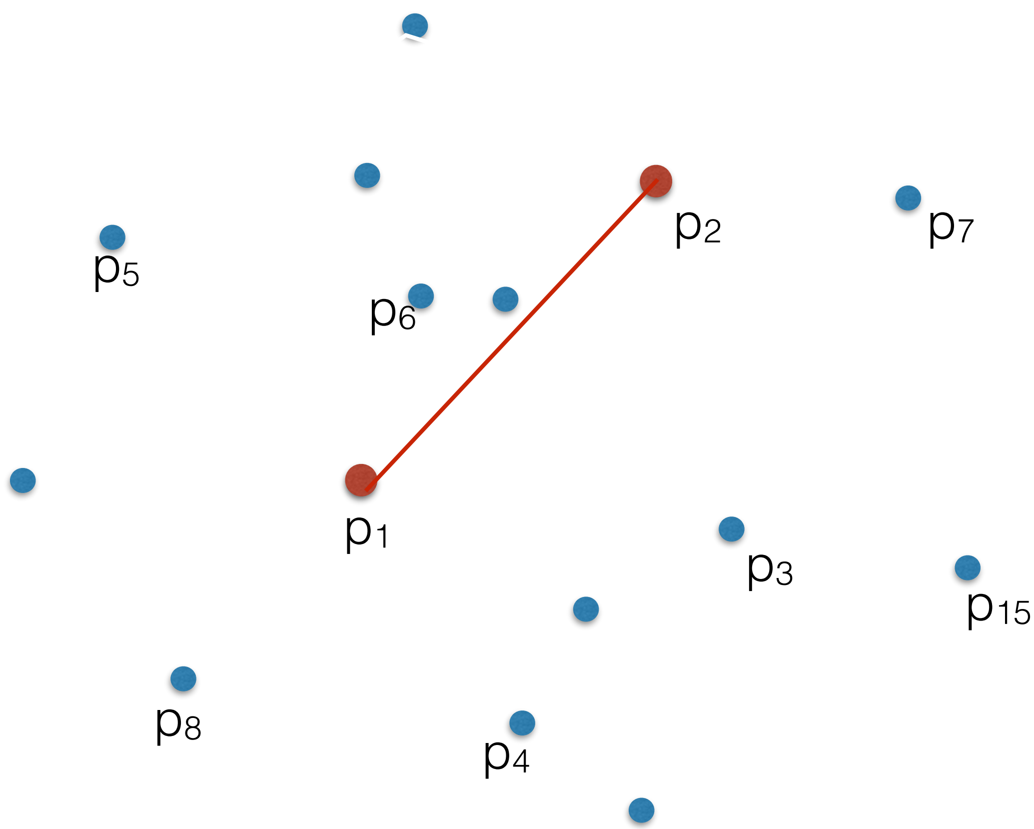

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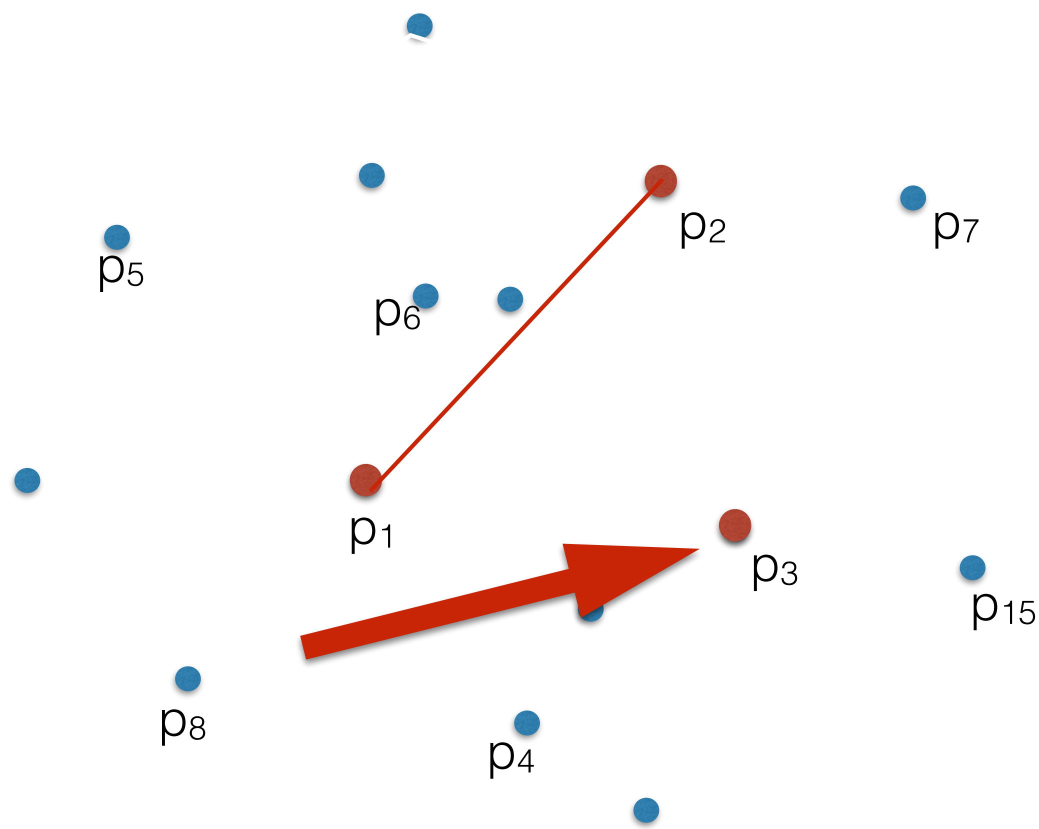

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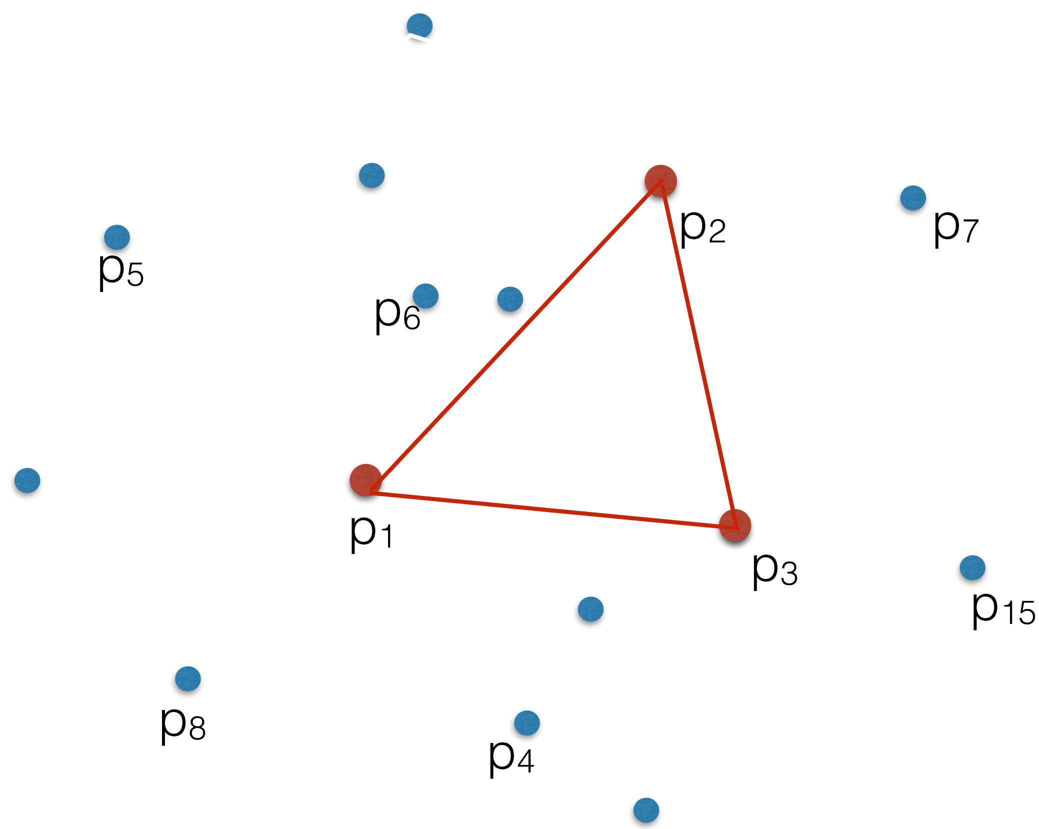

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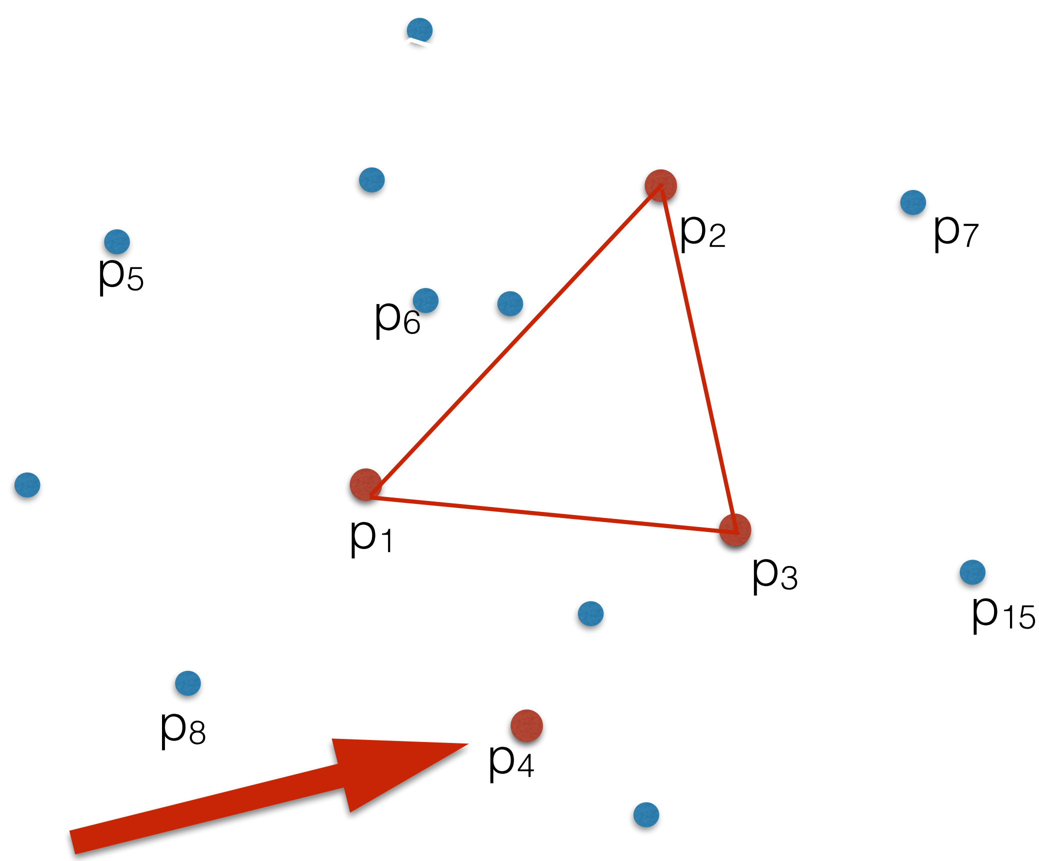

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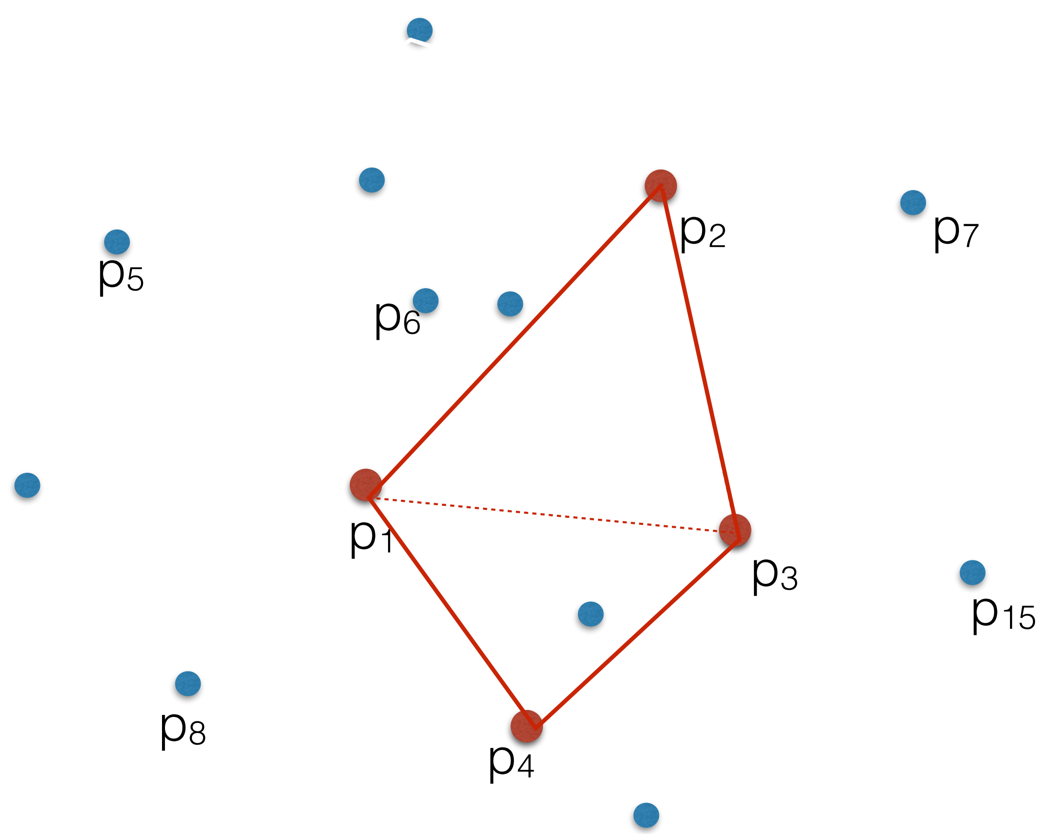

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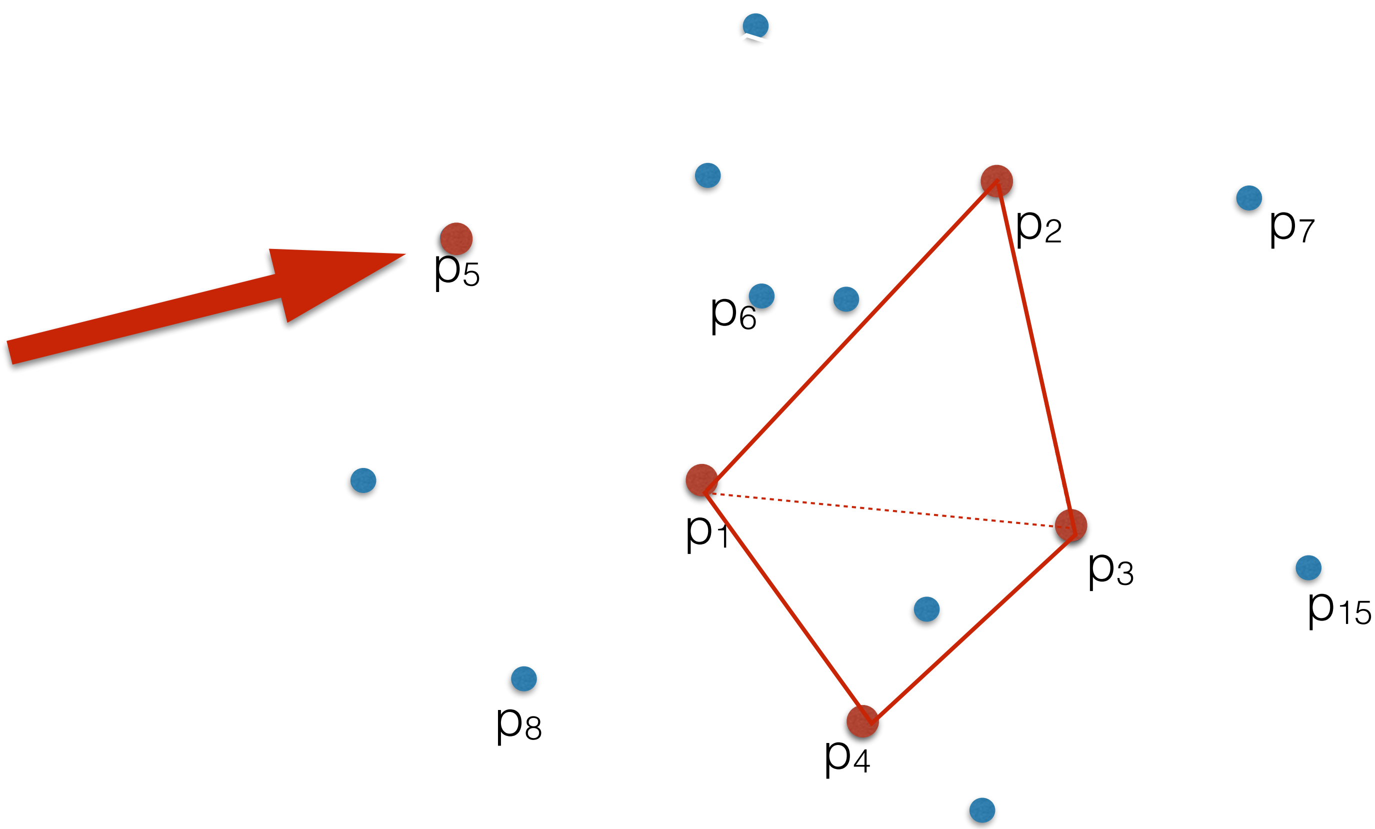

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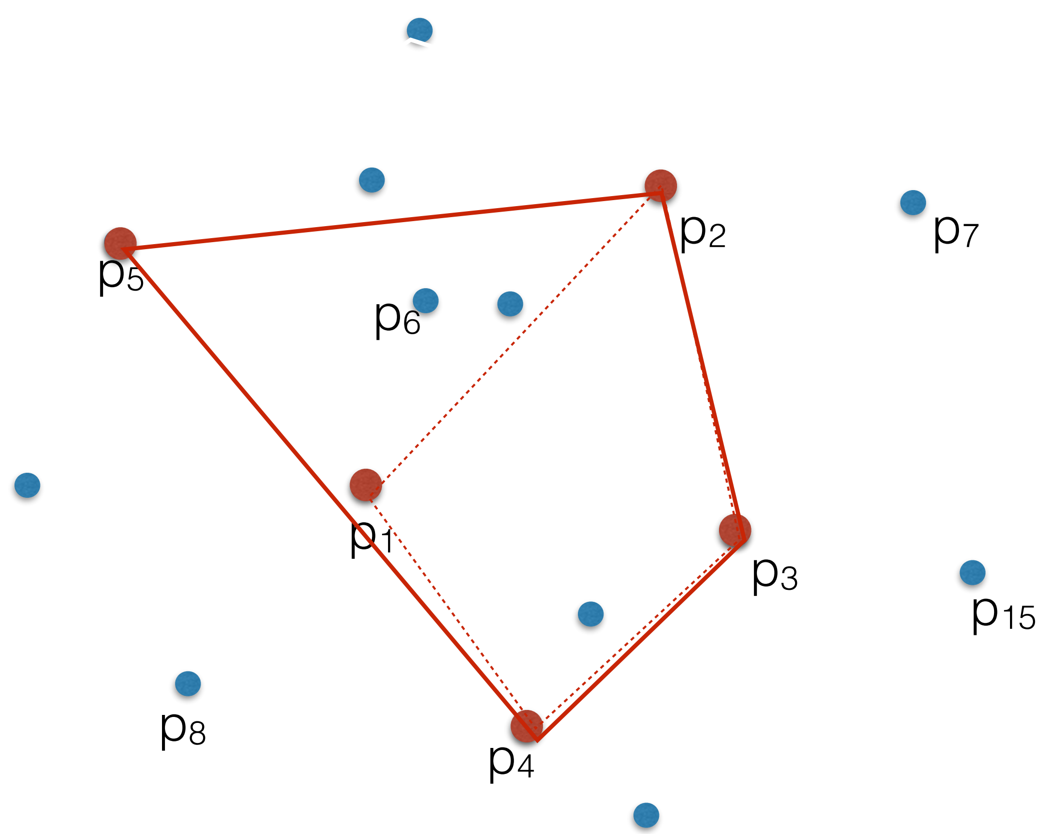

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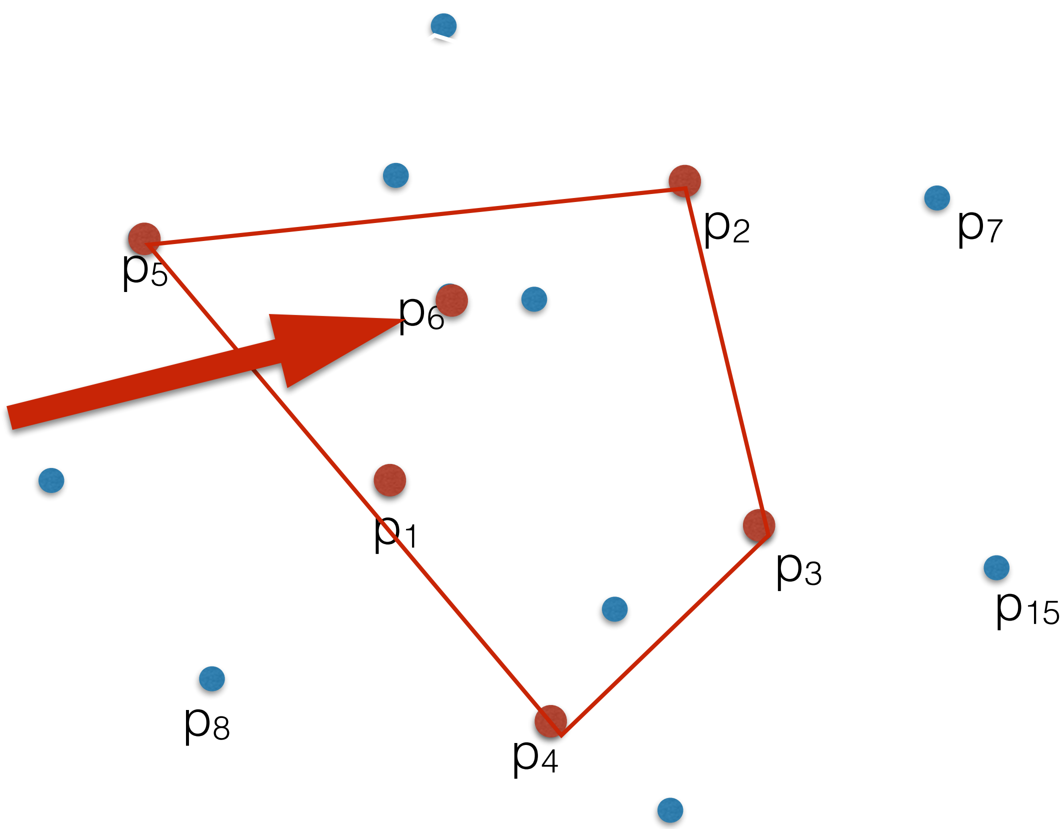

Incremental algo for CH

- CH = {}
- for i=1 to n
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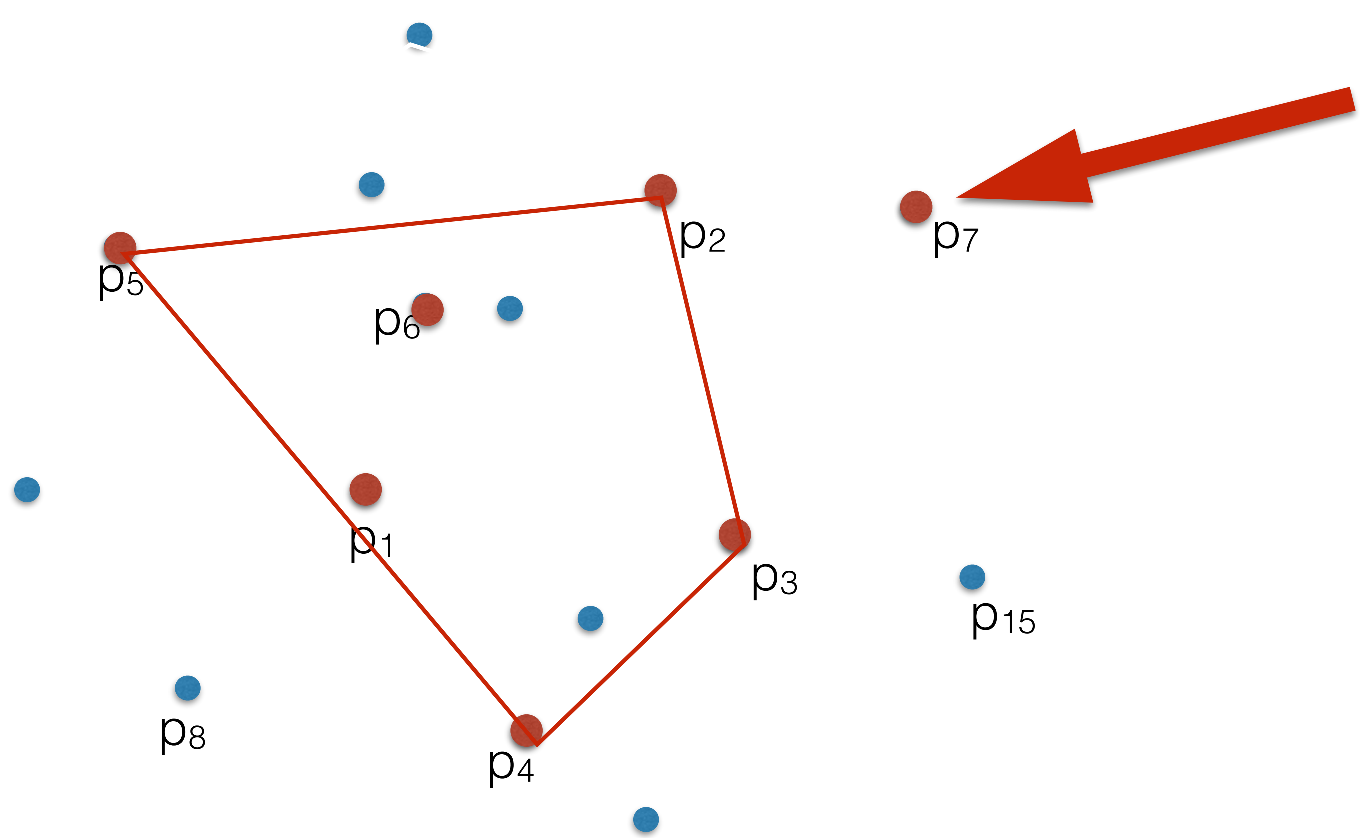

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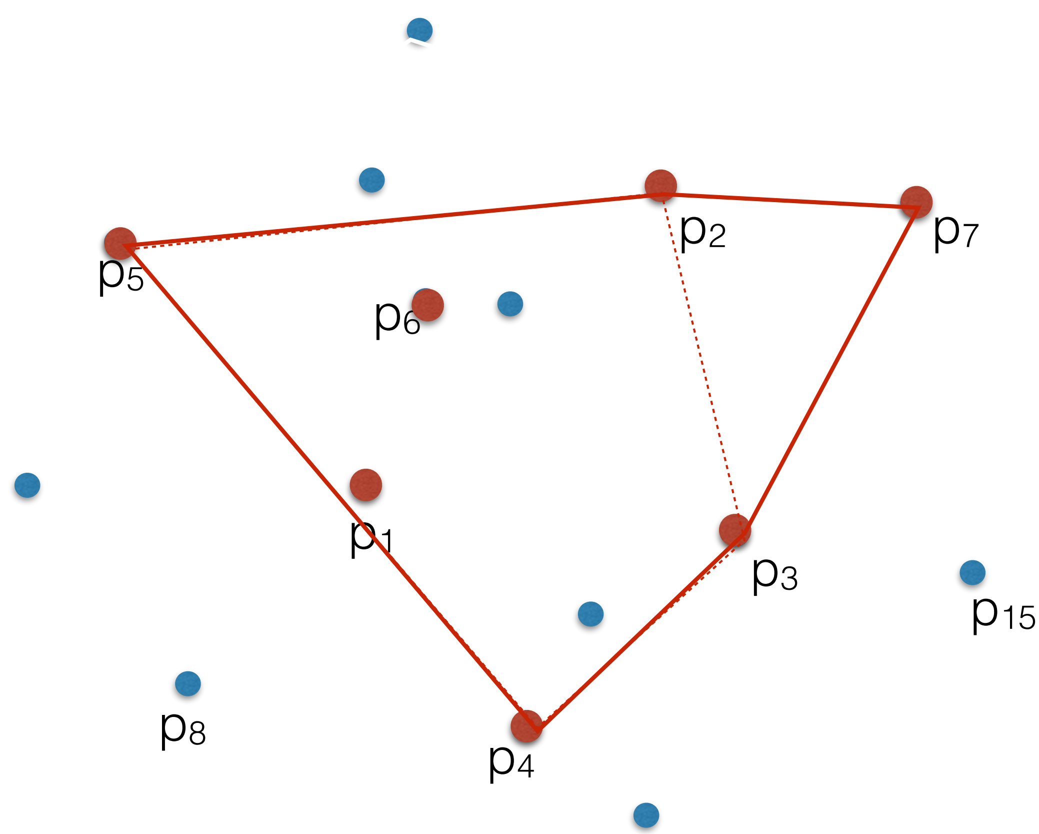

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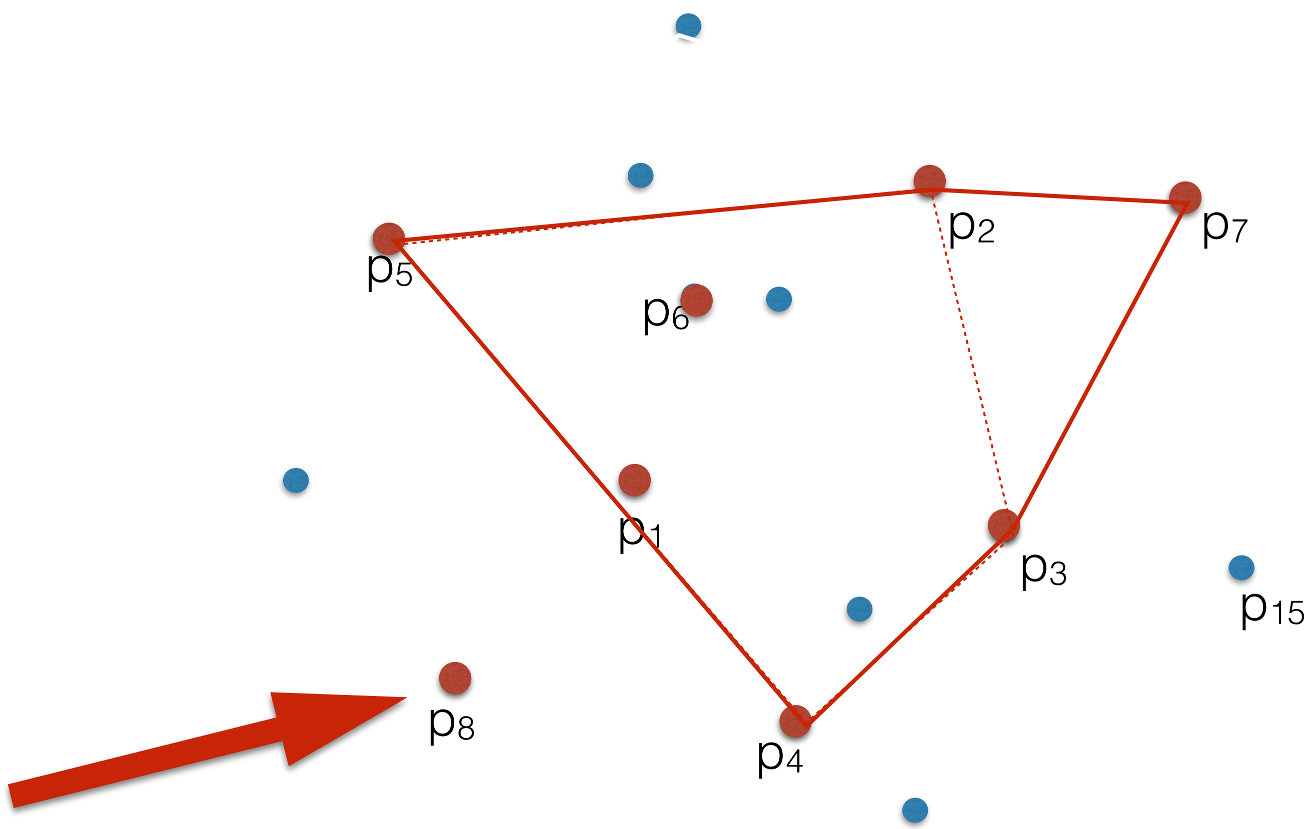

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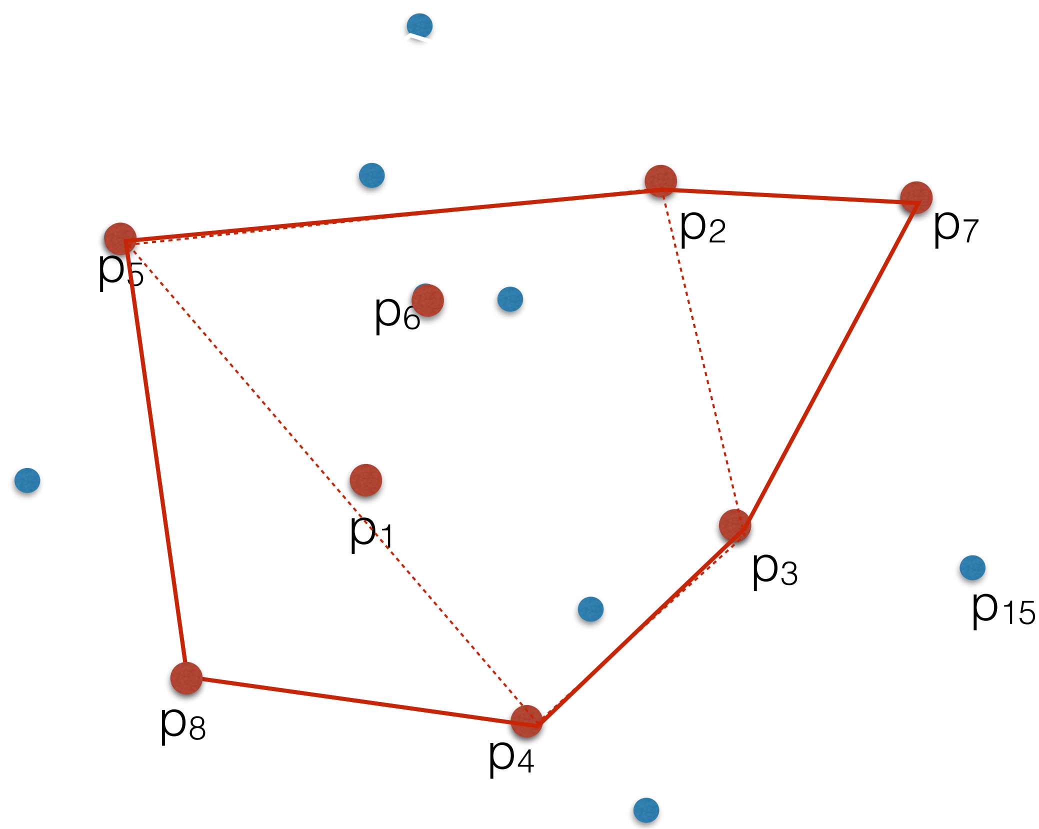

Incremental algo for CH

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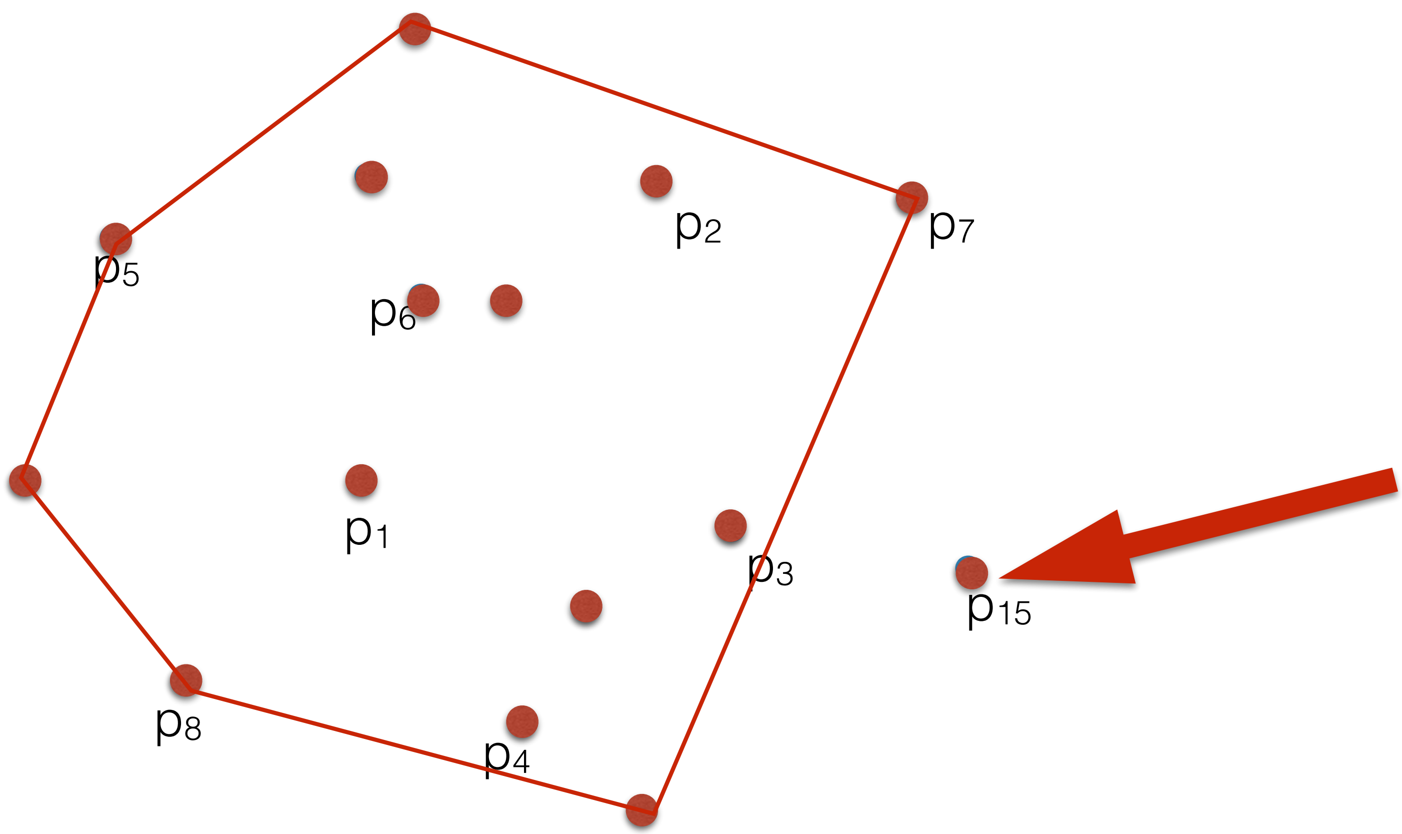
and so on



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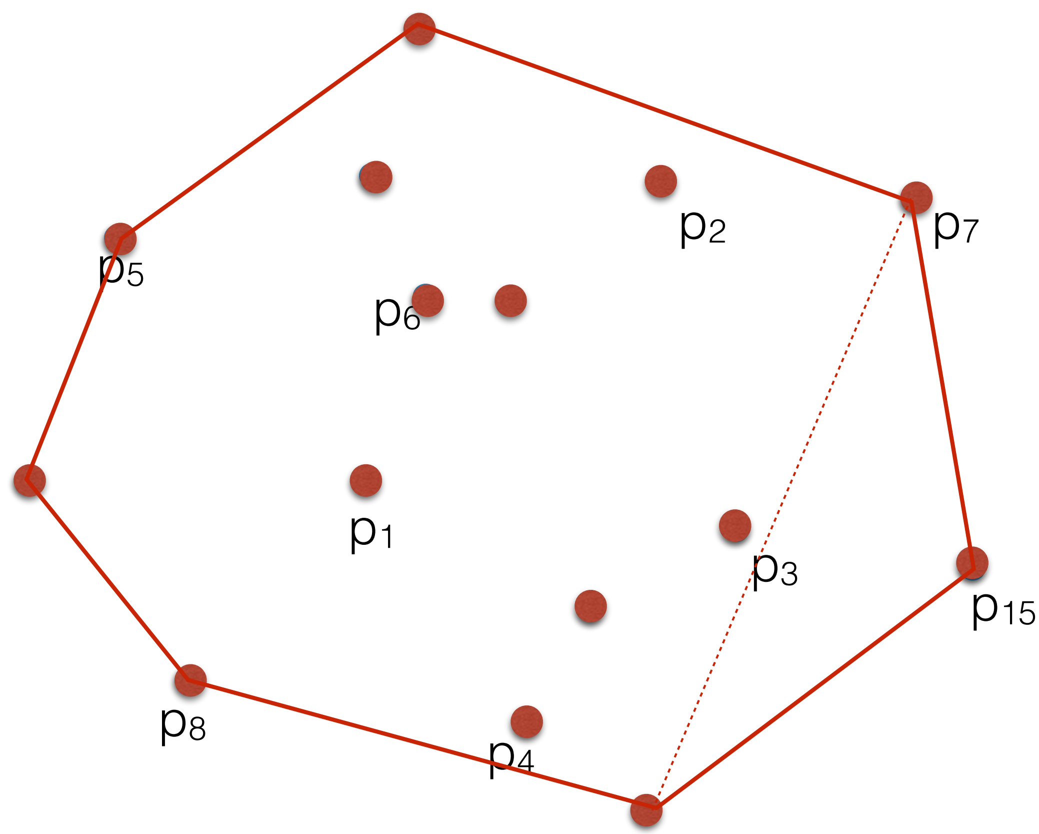

Incremental algo for CH

- CH = {}
- for i=1 to n
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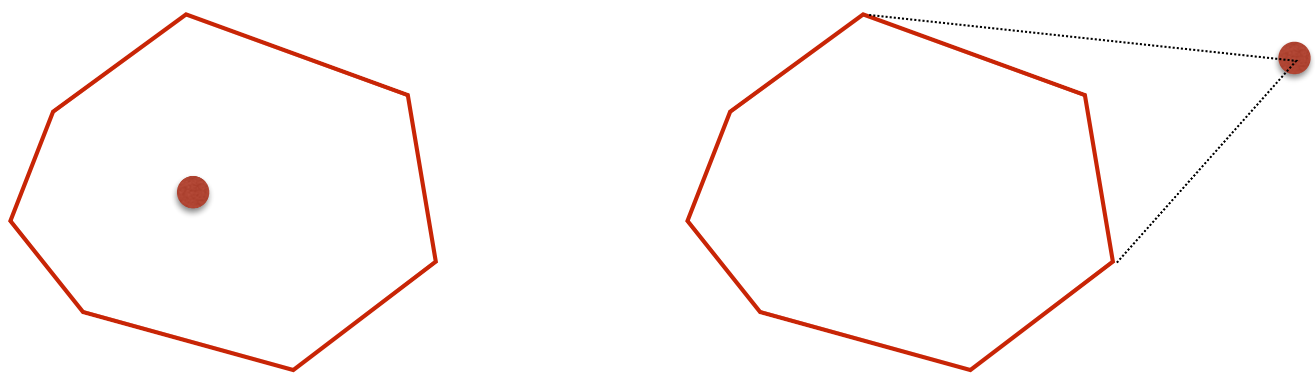
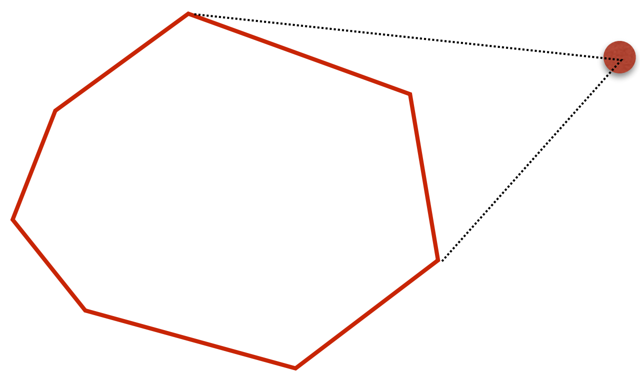
- The basic operation is adding a point to a convex polygon
    - CASE 1: p is in polygon
    - CASE 2: p outside polygon



Incremental algo for CH

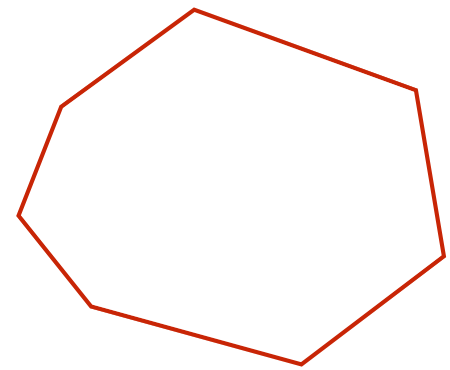
- Issues to solve
  - What's a good representation for a (convex) polygon?
  - We need a point-in-convex-polygon test
  - How to handle CASE 2 ?



Representing a polygon

A polygon is represented as a list of vertices in boundary order.

(the convention is counter-clockwise order)

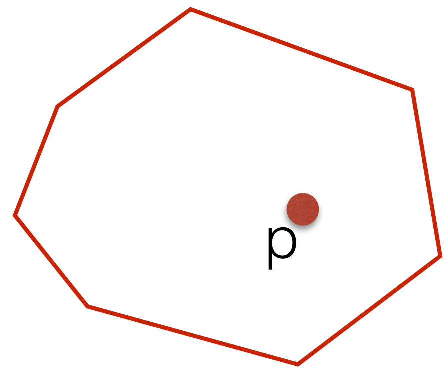


```
typedef struct _polygon{
    int k; //number of vertices
    Point* vertices; //the vertices, ccw in boundary order
 } Polygon;

 or

Vector<Point>          //note: the vertices, ccw in boundary order
```

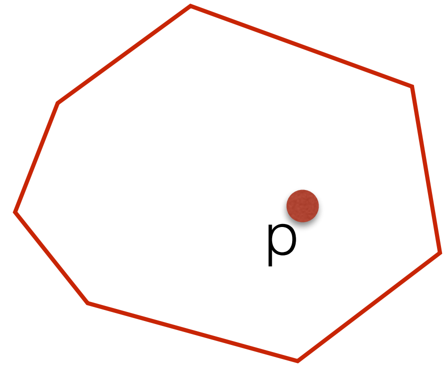
Point in convex polygon




```
//return TRUE iff p on the boundary or inside H; H is convex a polygon
bool point_in_polygon(point p, polygon H)
```

What has to be true in order for p to be inside?

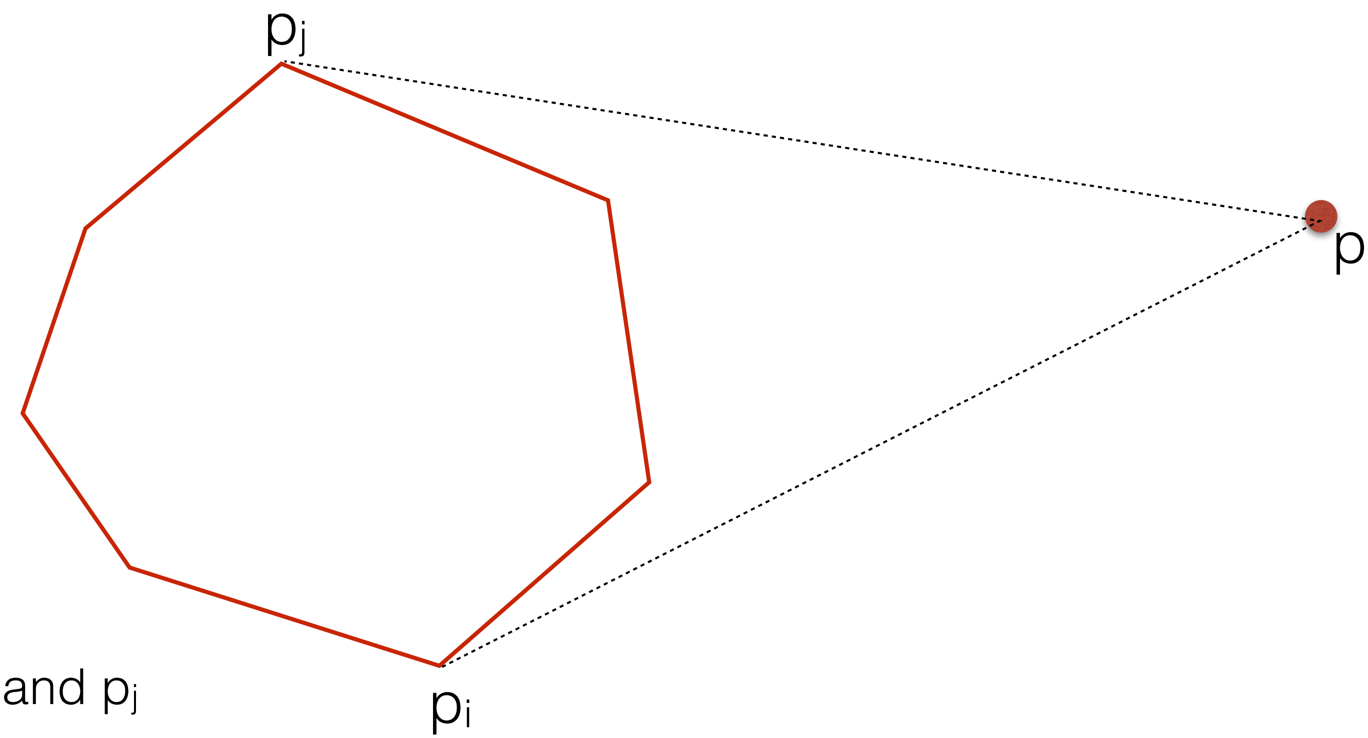
Point in convex polygon




```
//return TRUE iff p on the boundary or inside H; H is convex a polygon
bool point_in_convex_polygon(point p, polygon H)
  //p is inside if and only if it is on or to the left of all edges, oriented ccw
  //note:  this is NOT true for a non-convex polygon — can you show a
  //counter-example?
```

Analysis: O(k) where k is the size of the polygon

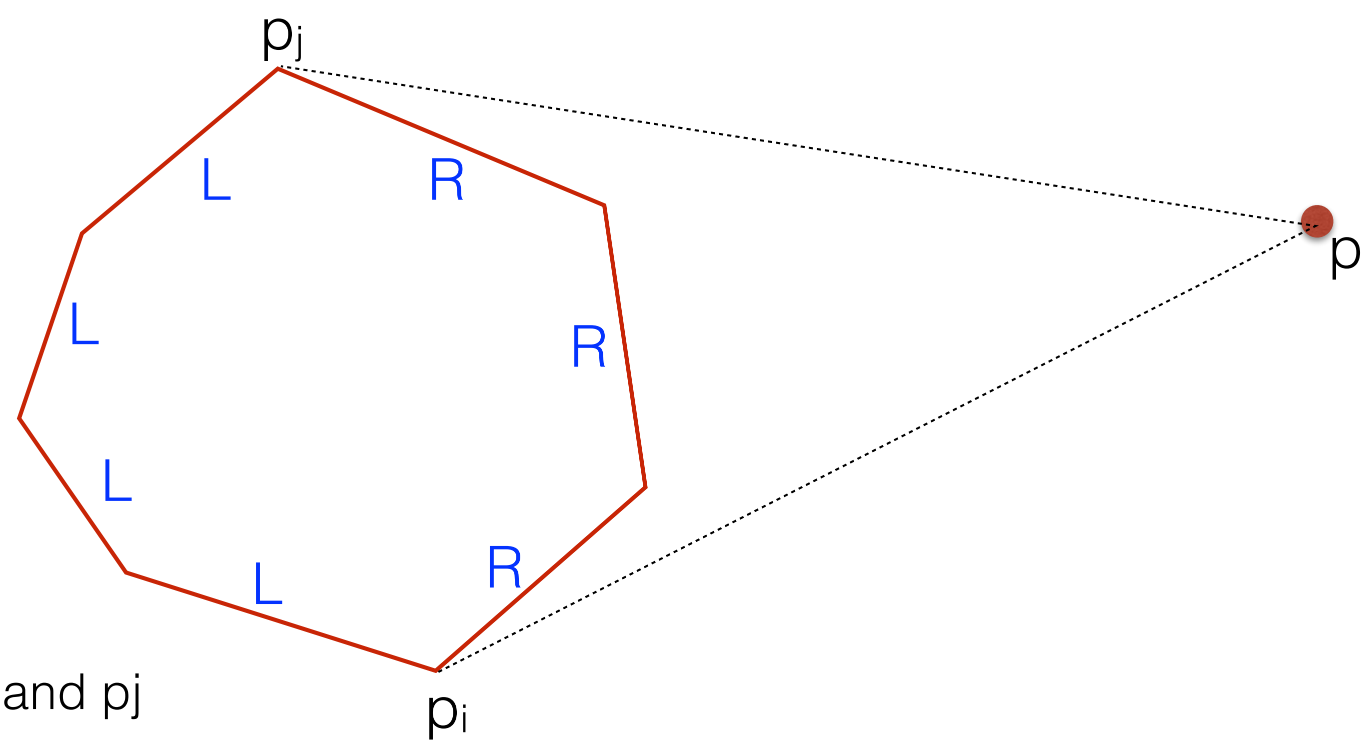
Case 2:



We want to find p_i and p_j

Hint: Check the orientation of p wrt the edges of the polygon.

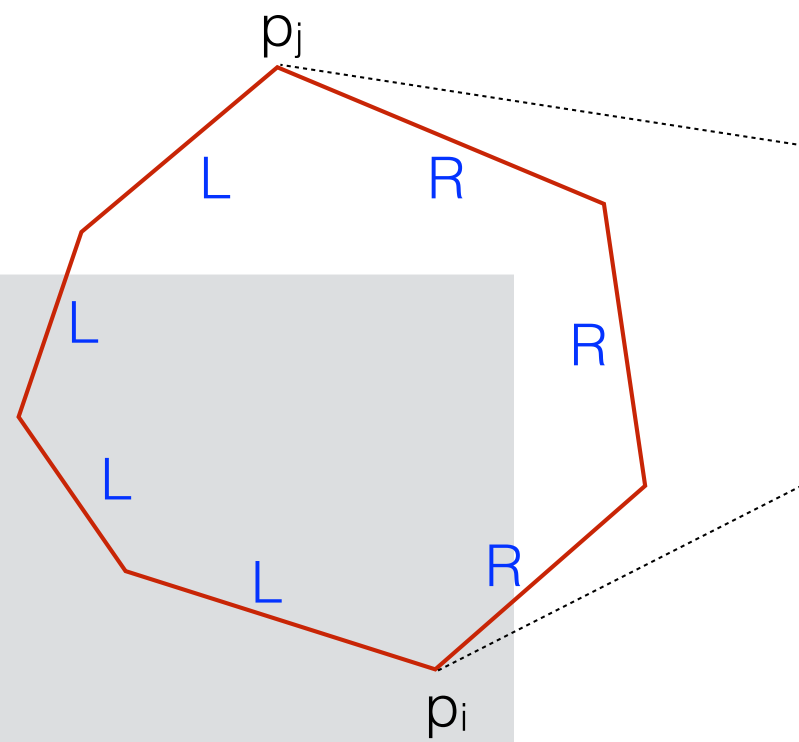
Case 2:



We want to find pi and pj

Hint: Check the orientation of p wrt the edges of the polygon.

Finding tangent points



Input: point p outside H

polygon H = $[p_0, p_1, \ldots, p_{k-1}]$ convex

- for i=0 to k-1 do
  - prev = ((i == 0)? k-1: i-1);
  - next = (i==k-1)? 0; k+1);
  - if XOR (p is left-or-on (p_{prev}, p_i), p is left-or-on(p_i, p_{next}))
    - then: p_i is a tangent point

Putting it all together

Incremental CH

- H = [p1, p2, p3]
- for i=4 to n do
  - //add p_i to H
  - if point_in_polygon(pi, H)
    - //do nothing
  - else
    - find p_k the tangent point where orientation changes from L to R
    - find p_j the tangent point where orientation changes from R to L
    - cut out the part from p_k to p_j in H (note: pk not necessarily before pj in the vertex array of H. view H as wrapping around)

Incremental CH

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Simulate the algorithm on a couple of examples. Think how p_i could come before p_j in H or the other way around.

Analysis:

Incremental CH

- H = [p1, p2, p3]
- for i=4 to n do
  - //add p_i to H
  - if point_in_polygon(pi, H) ← O(i)
    - //do nothing
  - else
    - find p_k the tangent point where orientation changes from L to R
    - find p_j the tangent point where orientation changes from R to L ← O(i)
    - cut out the part from p_k to p_j in H (note: pk not necessarily before pj in the vertex array of H. view H as wrapping around)

Analysis: $\sum_i O(i) = \Theta(n^2)$

Incremental CH

- Improvement: pre-sort the points by their x-coordinates and add them in this order. What happens?

Incremental CH

- Improvement: pre-sort the points by their x-coordinates and add them in this order. What happens?
  - point p_i is to the right of p_{i-1}, so it will be outside CH{p_1, p_2, …, pi-1}
  - No need to check!

- pre-sort the points by their x-coordinates. Let H = [p1, p2, p3]
- for i=4 to n do
  - ~~//add p_i to H~~
  - ~~if point_in_polygon(pi, H)~~
    - ~~//do nothing~~
  - ~~else~~
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Incremental CH

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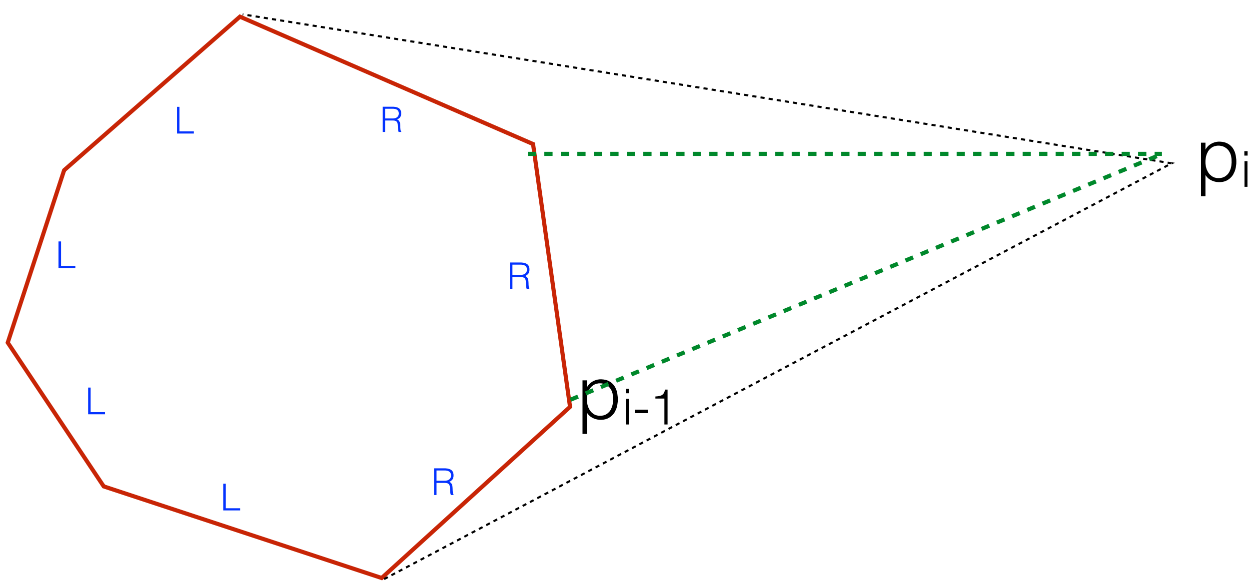
  - cut out the part from p_k to p_j in H

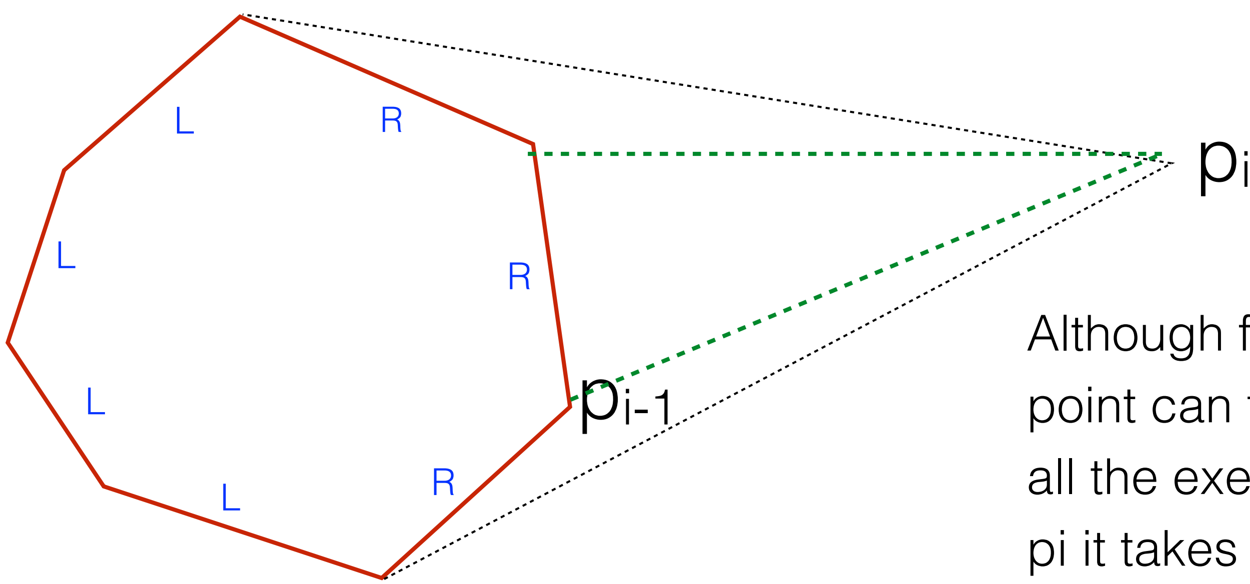
How do we make this run in O(n) once sorted?

Incremental CH



Finding tangent points of pi to the hull H of {p1, p2, …, pi-1}

- find vertex p_{i-1} on H
- v = p_{i-1}
- while point pi lies to the right of (v, succ(v)): v = succ(v)
- v is the upper tangent point
- find lower tangent point analogously



Although finding a tangent point can take O(n), over all the executions over all pi it takes O(n)

Theorem: Incremental CH (in 2D) runs in $O(n \lg n)$ to sort the points followed by $O(n)$ to construct the convex hull.

A divide-and-conquer algorithm for CH

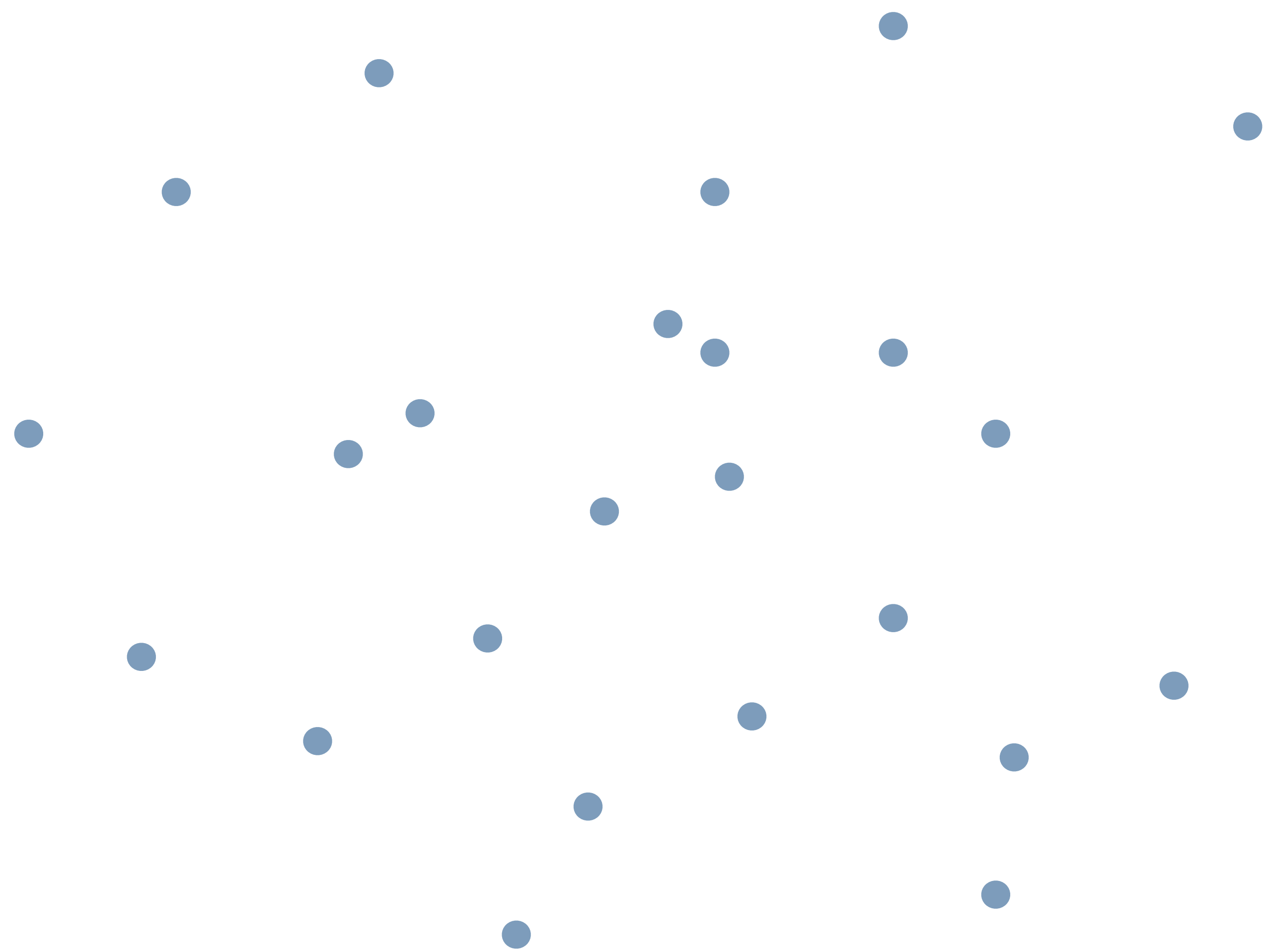
Divide-and-conquer

```
DC(input P)
  if P is small, solve and return
  else
    //divide
    divide input P into two halves, P1 and P2
    //recurse
    result1 = DC(P1)
    result2 = DC(P2)
    //merge
    do_something_to_figure_out_result_for_P
    return result
```

Analysis: $T(n) = 2T(n/2) + O(\text{merge phase})$

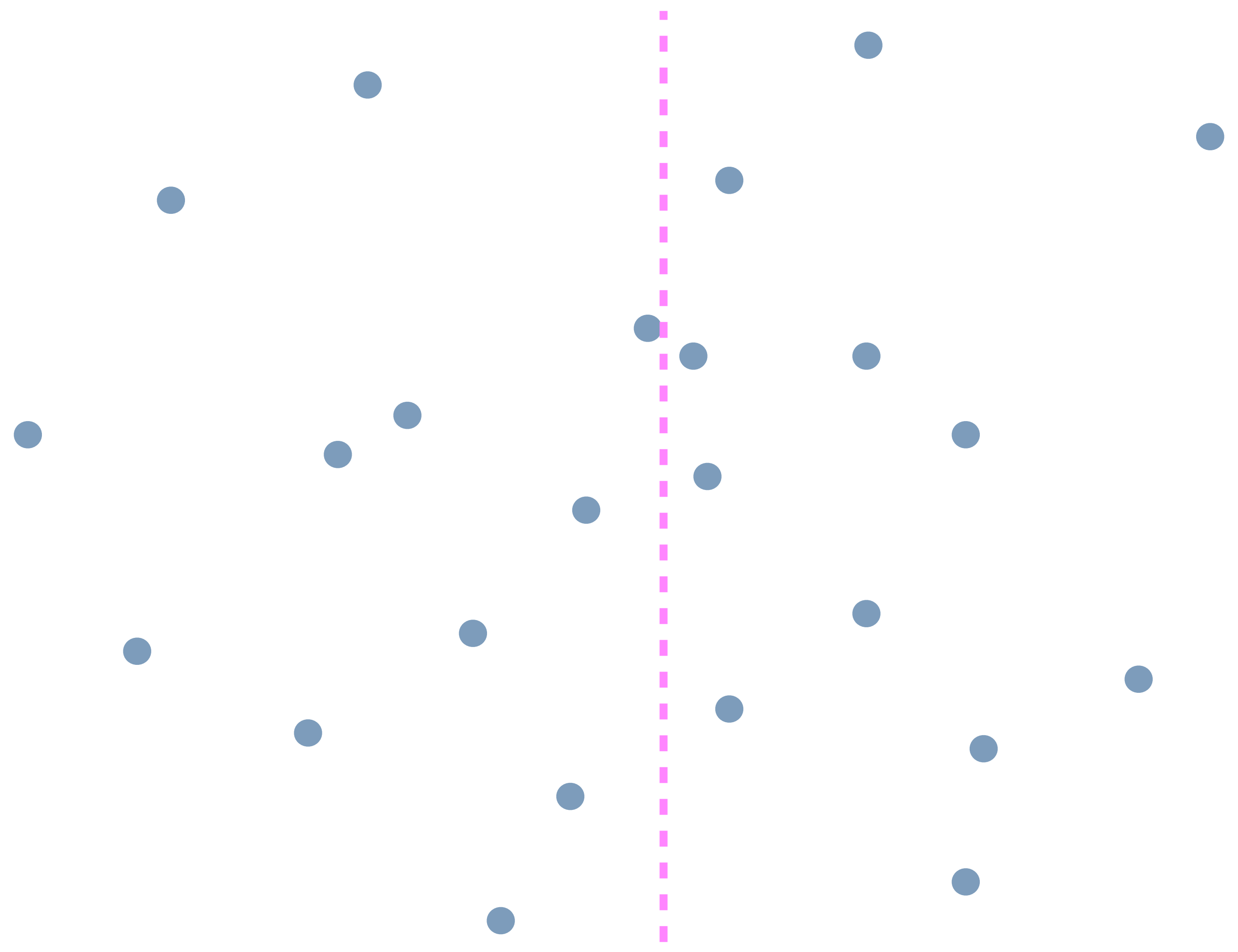
- if merge phase is **O(n)**: $T(n) = 2T(n/2) + O(n)$ => $O(n \lg n)$

CH via divide-and-conquer



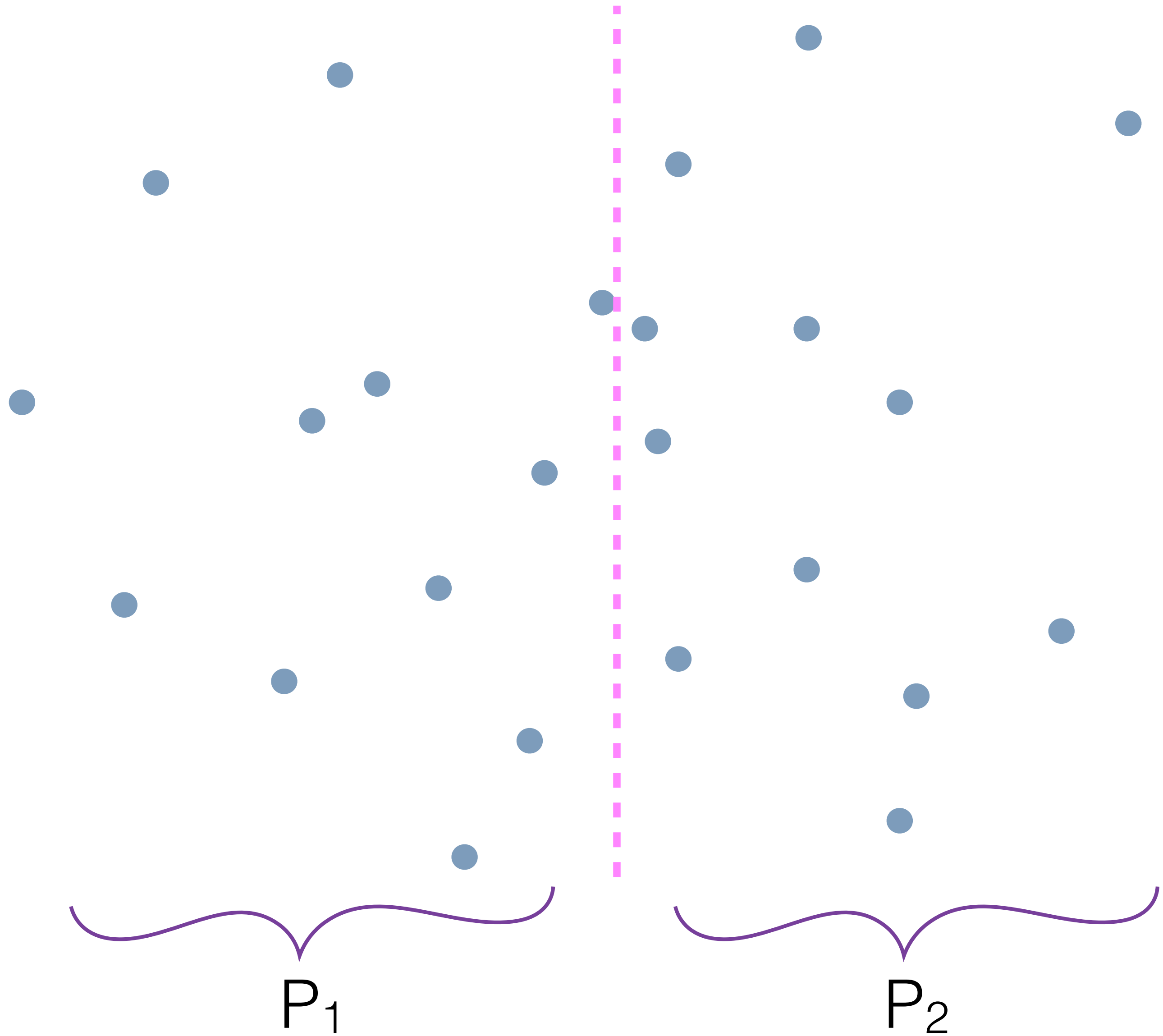
CH via divide-and-conquer

- find vertical line that splits P in half



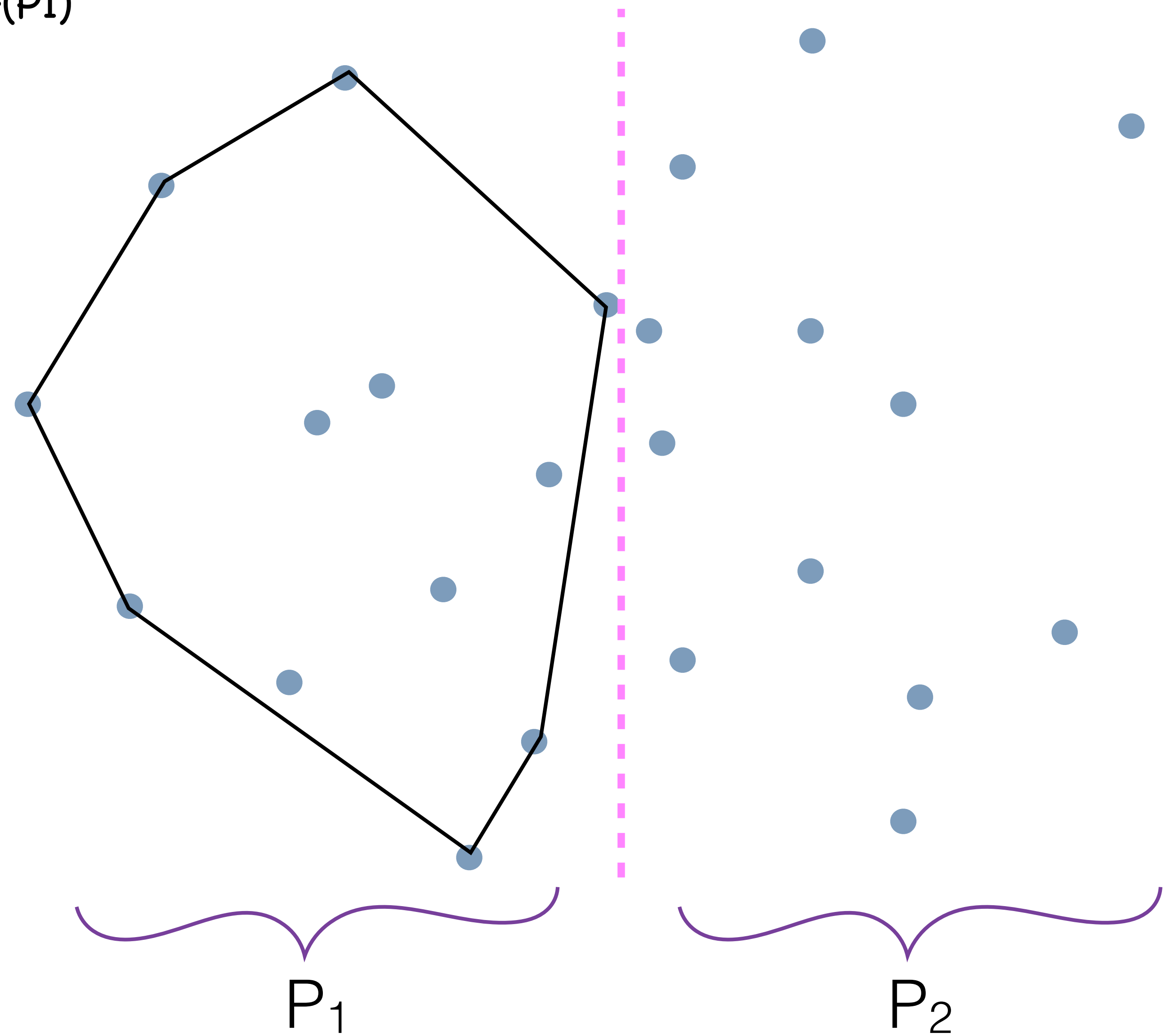
CH via divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line



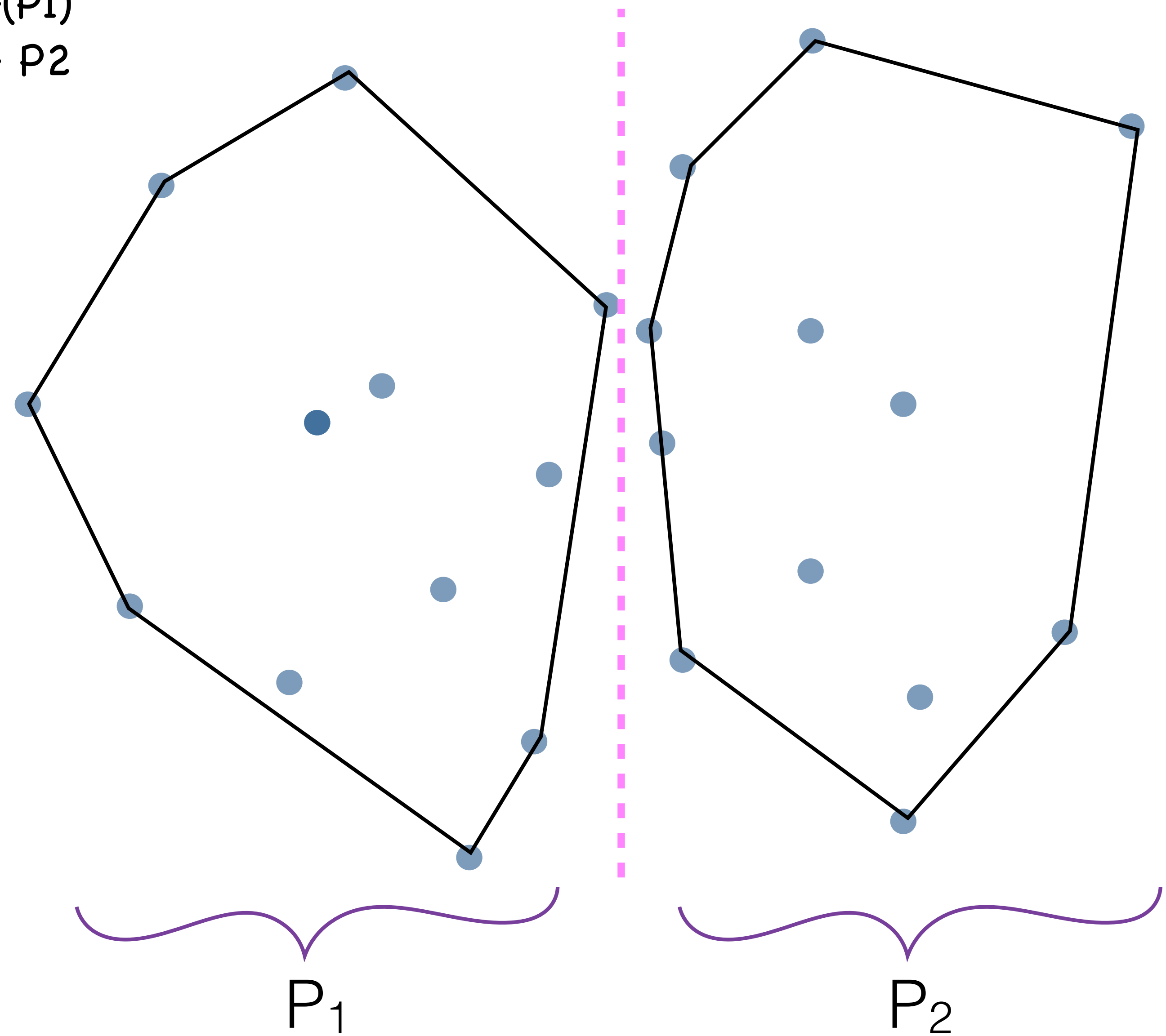
CH via divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find CH(P1)



CH via divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find CH(P1)
- recursively find CH P2



CH via divide-and-conquer

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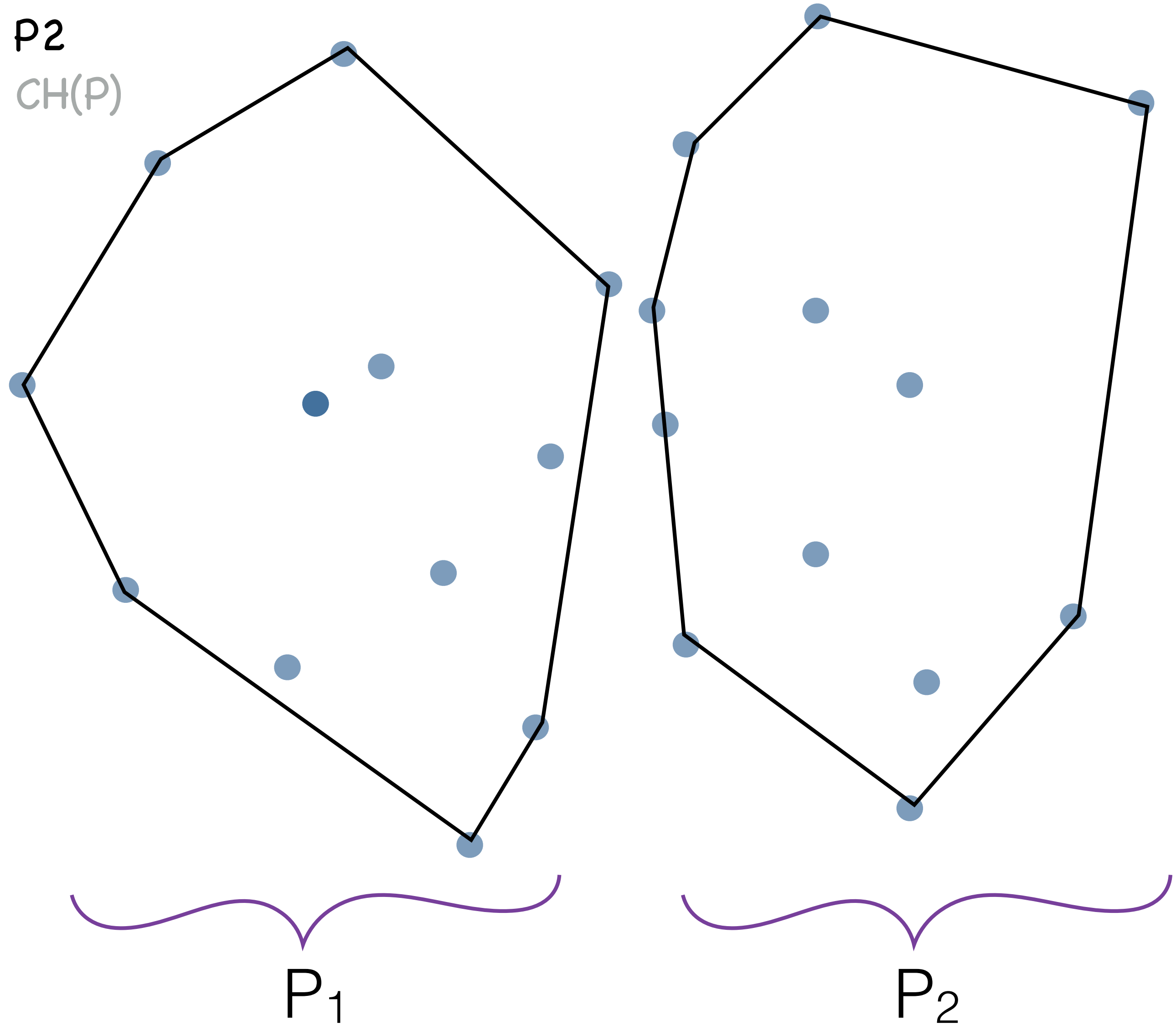
- recursively find CH P2

//now get somehow CH(P)



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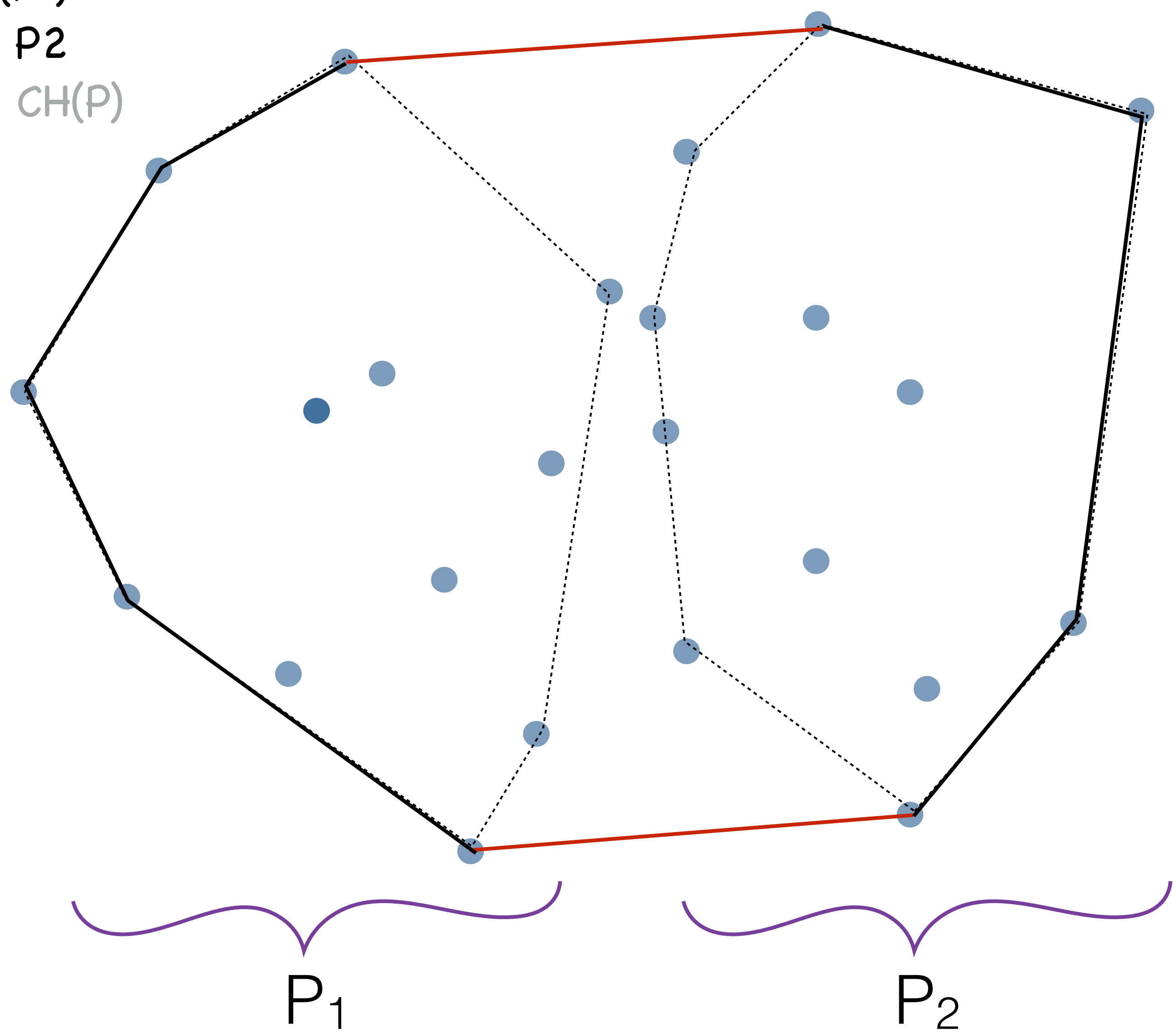
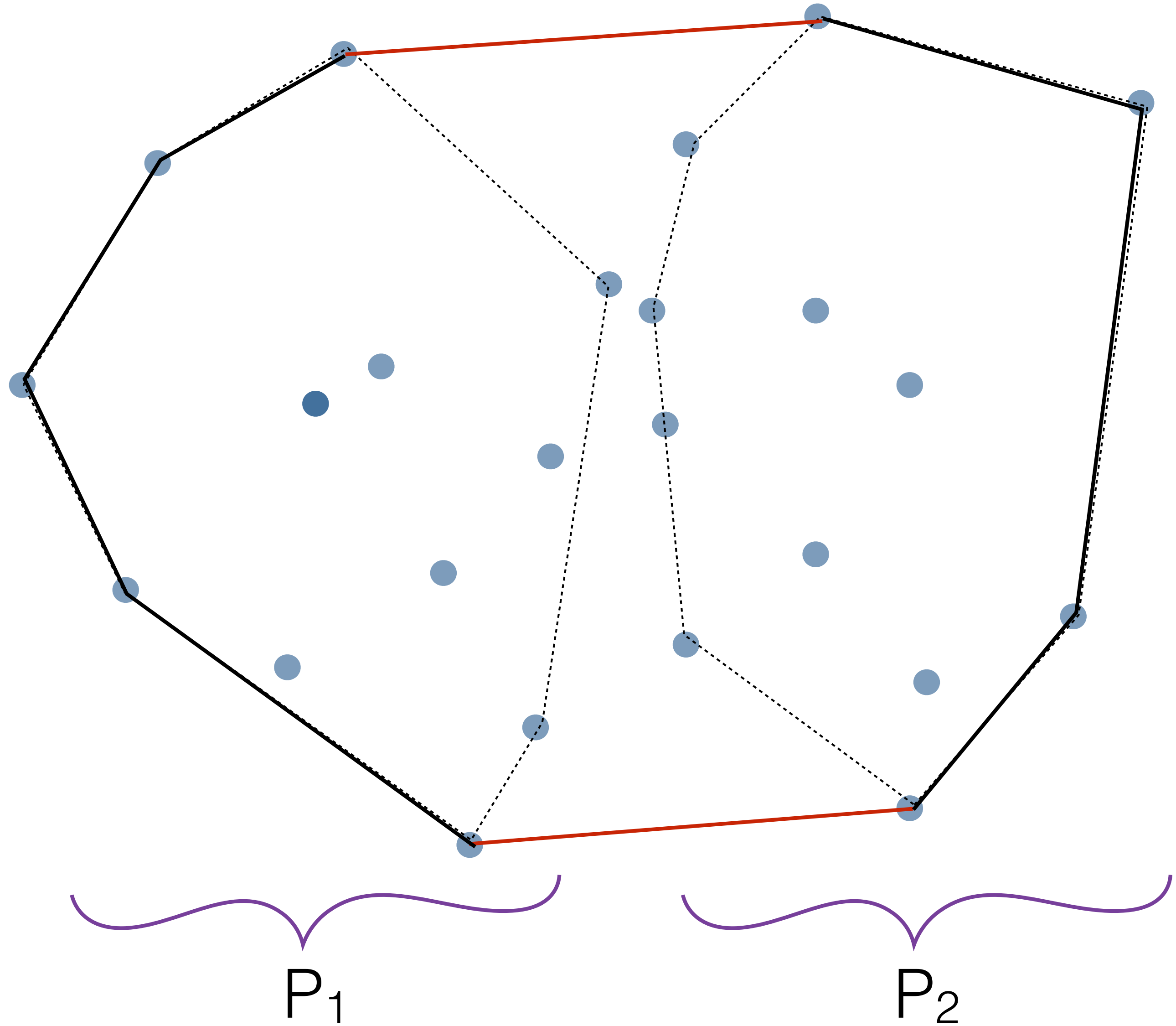


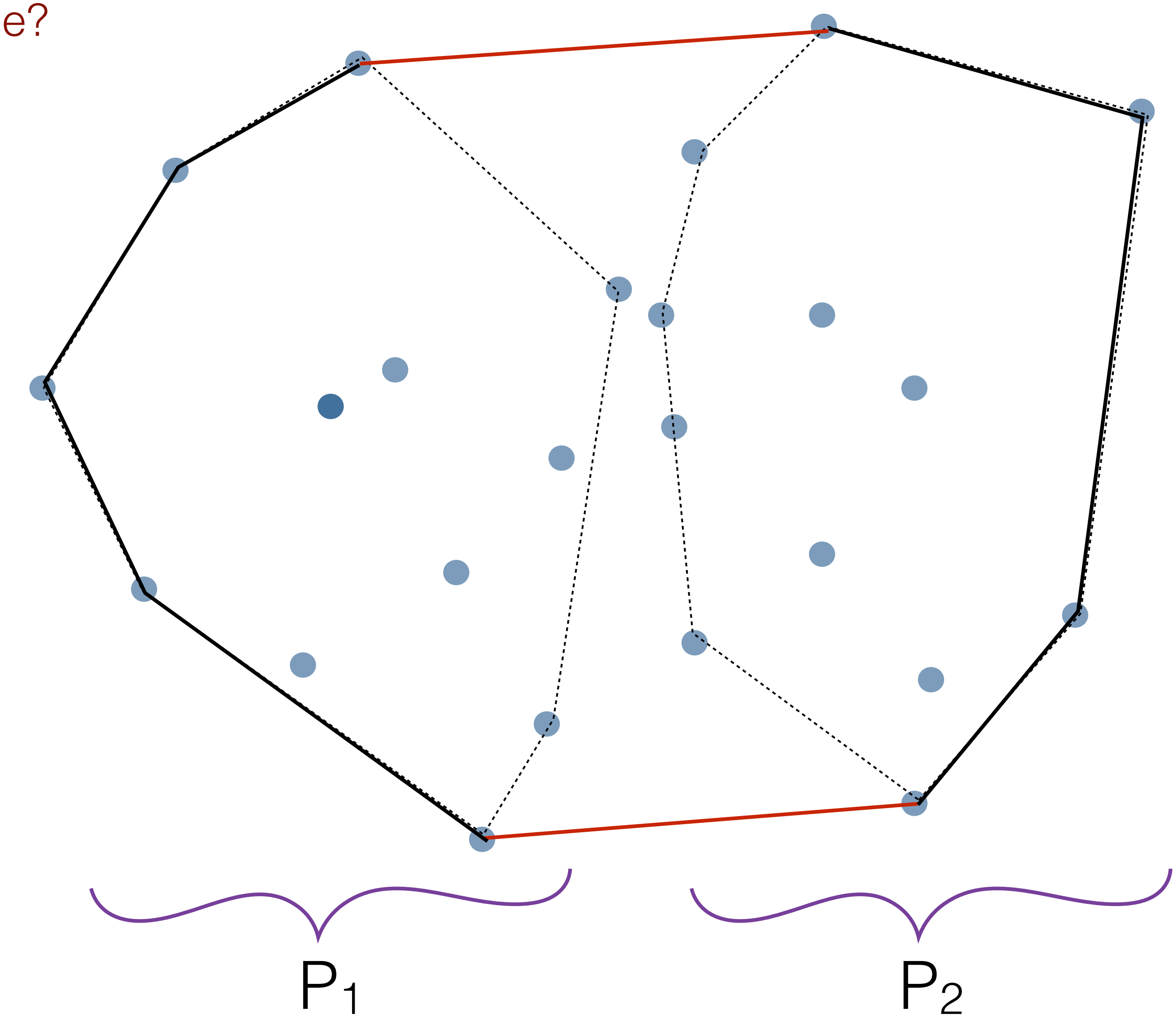
Merging two hulls..in linear time

- Need to find the two “tangents” (bridges?)



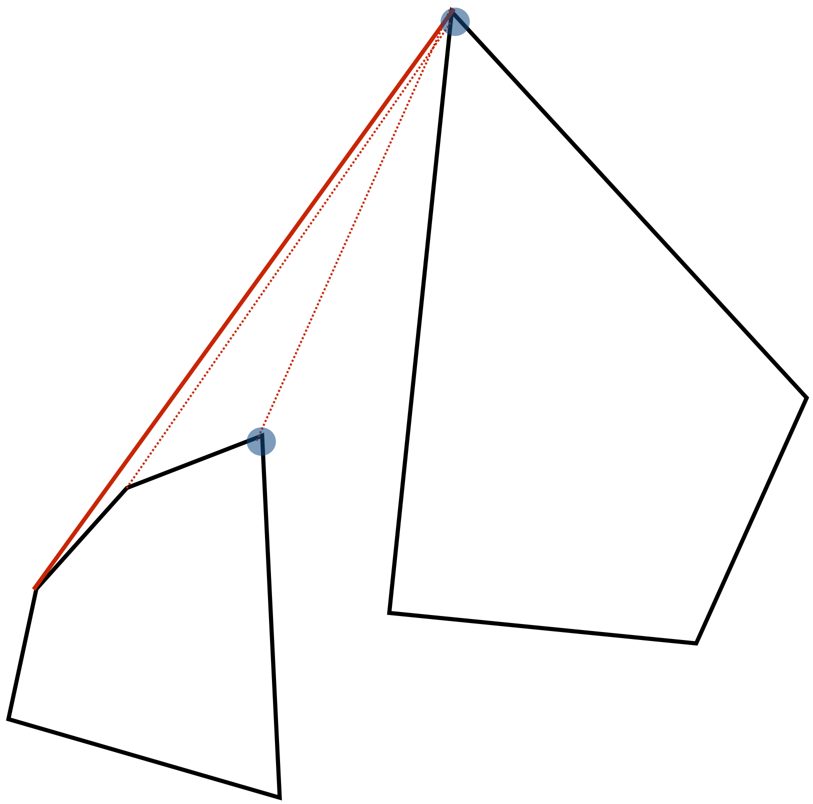
Merging two hulls..in linear time

- Here it looks like the upper tangent is between the **top** points in P_1 and P_2
- Is that always true?

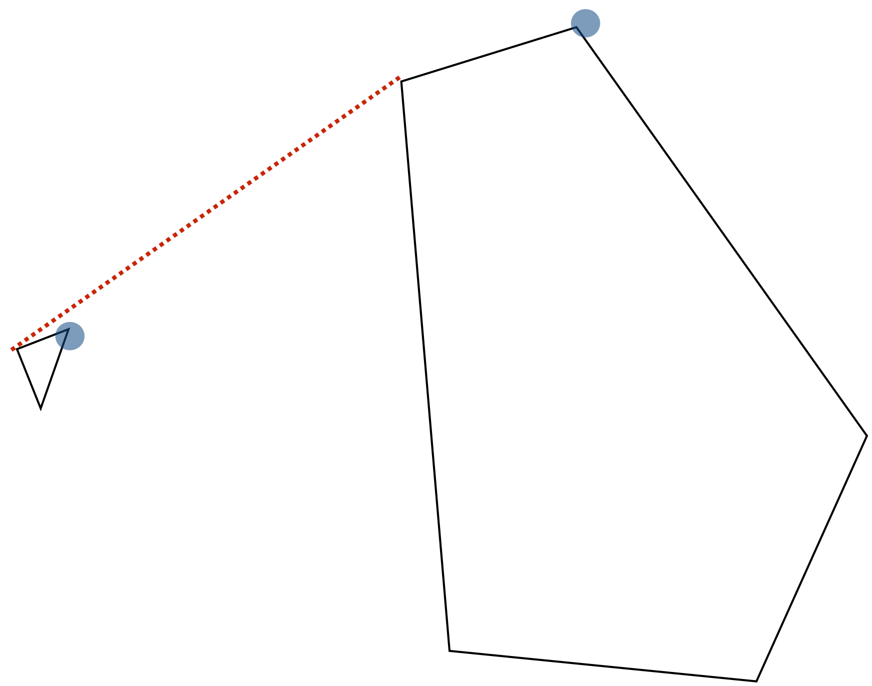


- Is the upper tangent guaranteed to connect the **top** points in P_1 and P_2 ?

  Not necessarily…



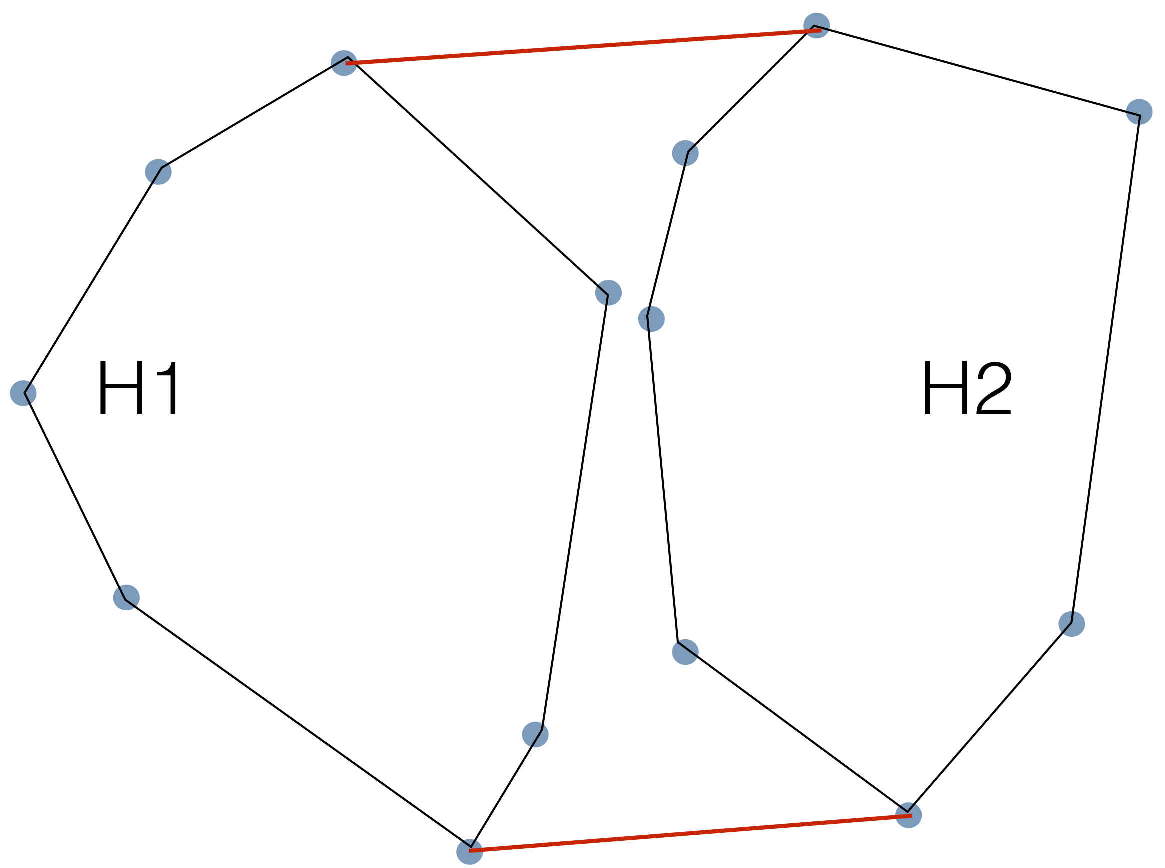
The top-most point overall is on the CH, but not necessarily on the upper tangent



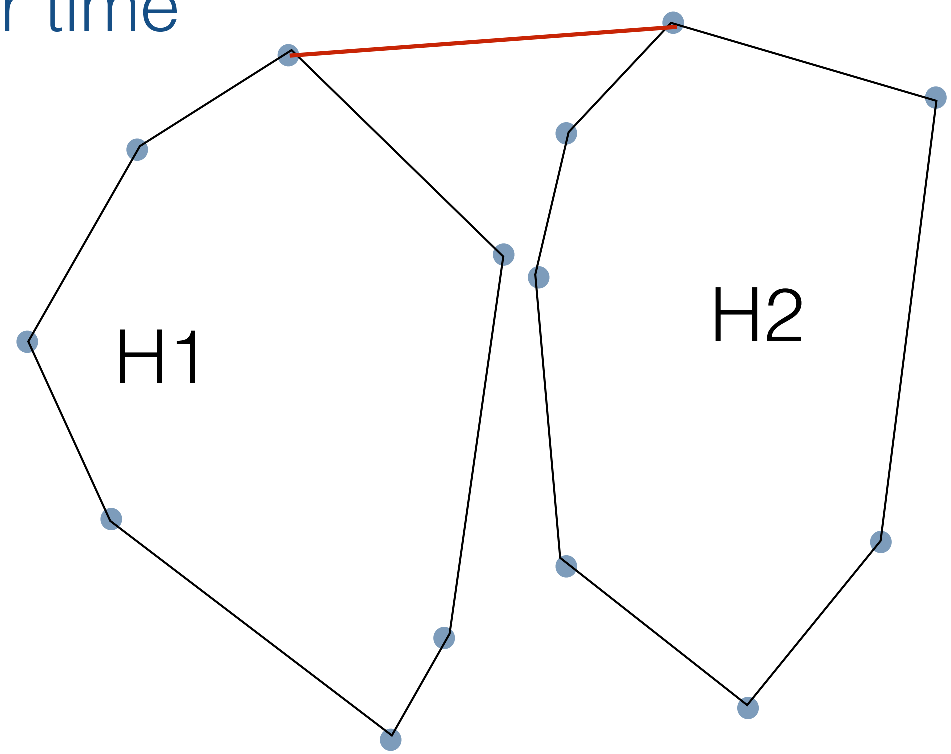
Merging two hulls..in linear time

- Naive algorithm: try all segments (a,b) with a in H_1 and b in H_2

  Too slow. => $O(n^2)$ merge, $O(n^2 \lg n)$ CH algorithm



Merging two hulls..in linear time



- To find the upper bridge:
  - let P1, P2 = set of points to the left/right of line
  - start with a = right most point of P1, b = left most point of P2
  - while one of succ(a) and pred(b) lies above line ab do:
    - if succ(a) lies above ab then set a = succ(a)
    - else : set b = pred(b)
  - return ab as the upper bridge

Theorem: D&C CH (in 2D) takes $O(n \lg n)$

CH via divide-and-conquer

- Yet another illustration of divide-and-conquer paradigm!
- Runs in $O(n \lg n)$
- Extends nicely to 3D