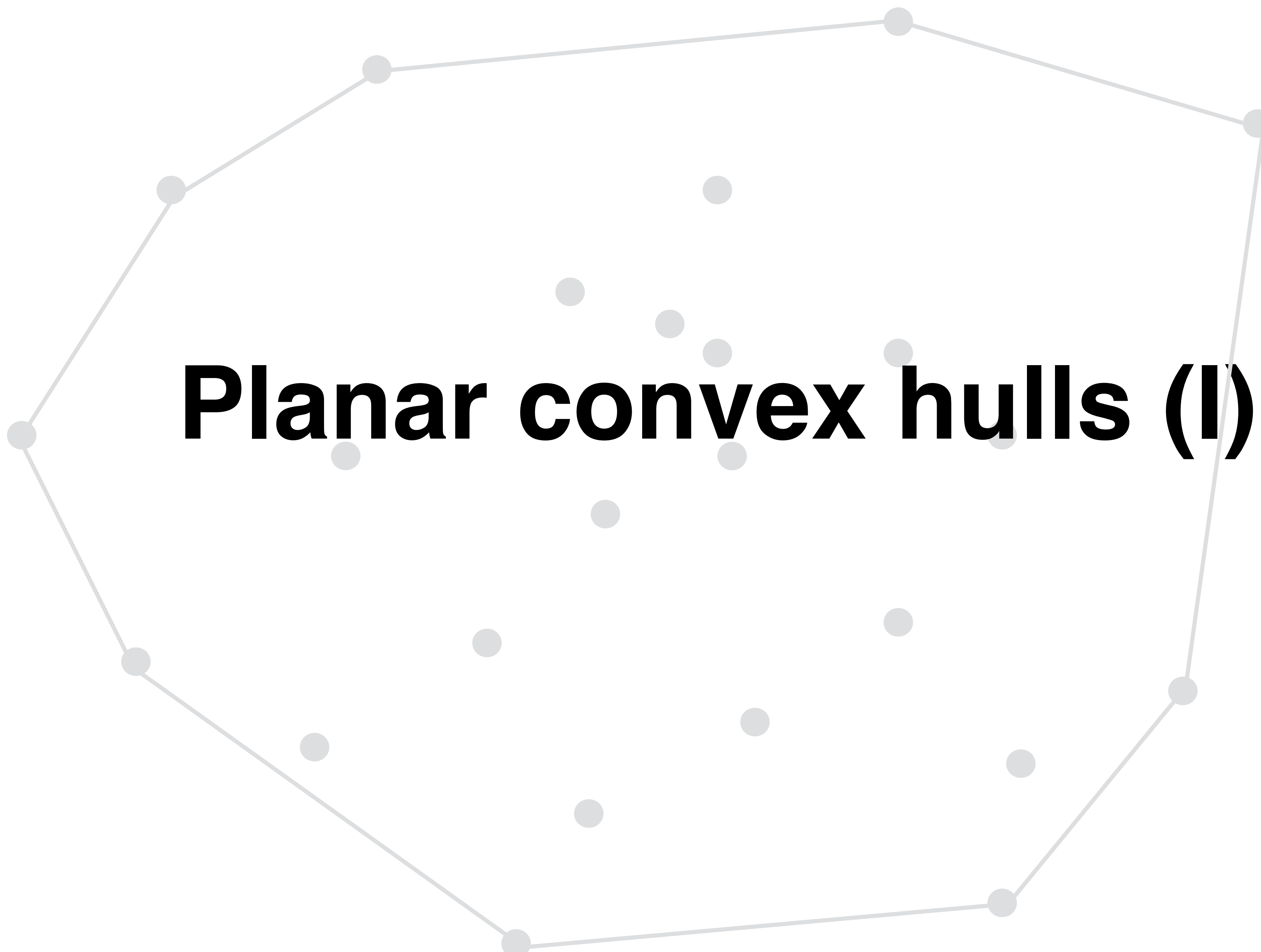


Planar convex hulls (I)

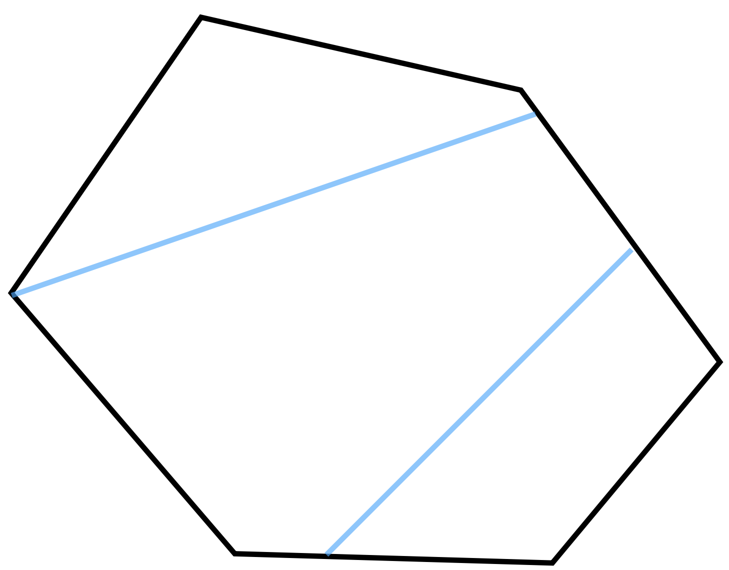
Computational Geometry [csci 3250]
Laura Toma
Bowdoin College

Algorithms for computing the convex hull

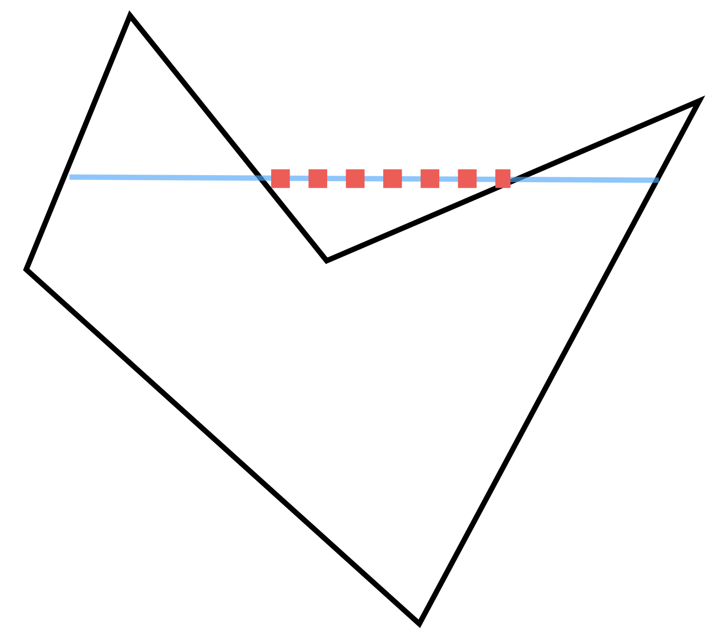
- Last time
  - Brute force
  - Gift wrapping
- Today
  - Quickhull
  - Graham scan
  - Andrew’s monotone chain
- Next time
  - incremental hull
  - divide-and -conquer hull
  - lower bound

Convexity

A polygon P is **convex** if for any p, q in P, the segment pq lies entirely in P.



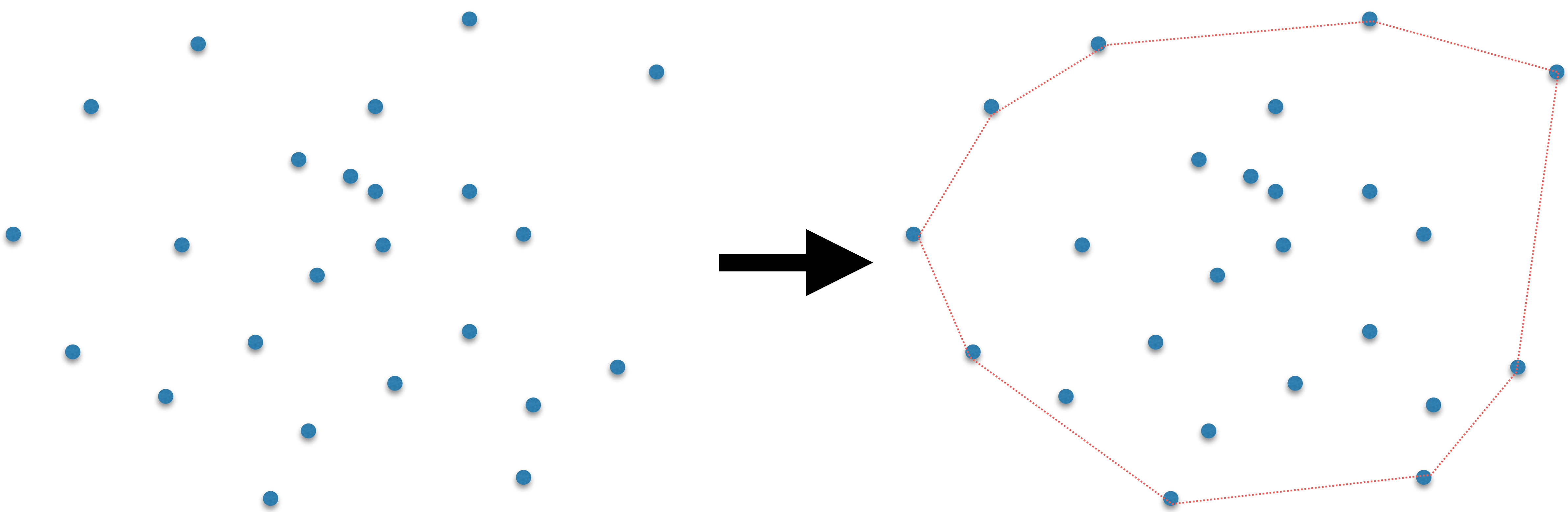
convex



non-convex

Compute the Convex Hull

Given a set P of points in 2D, describe an algorithm to compute their convex hull



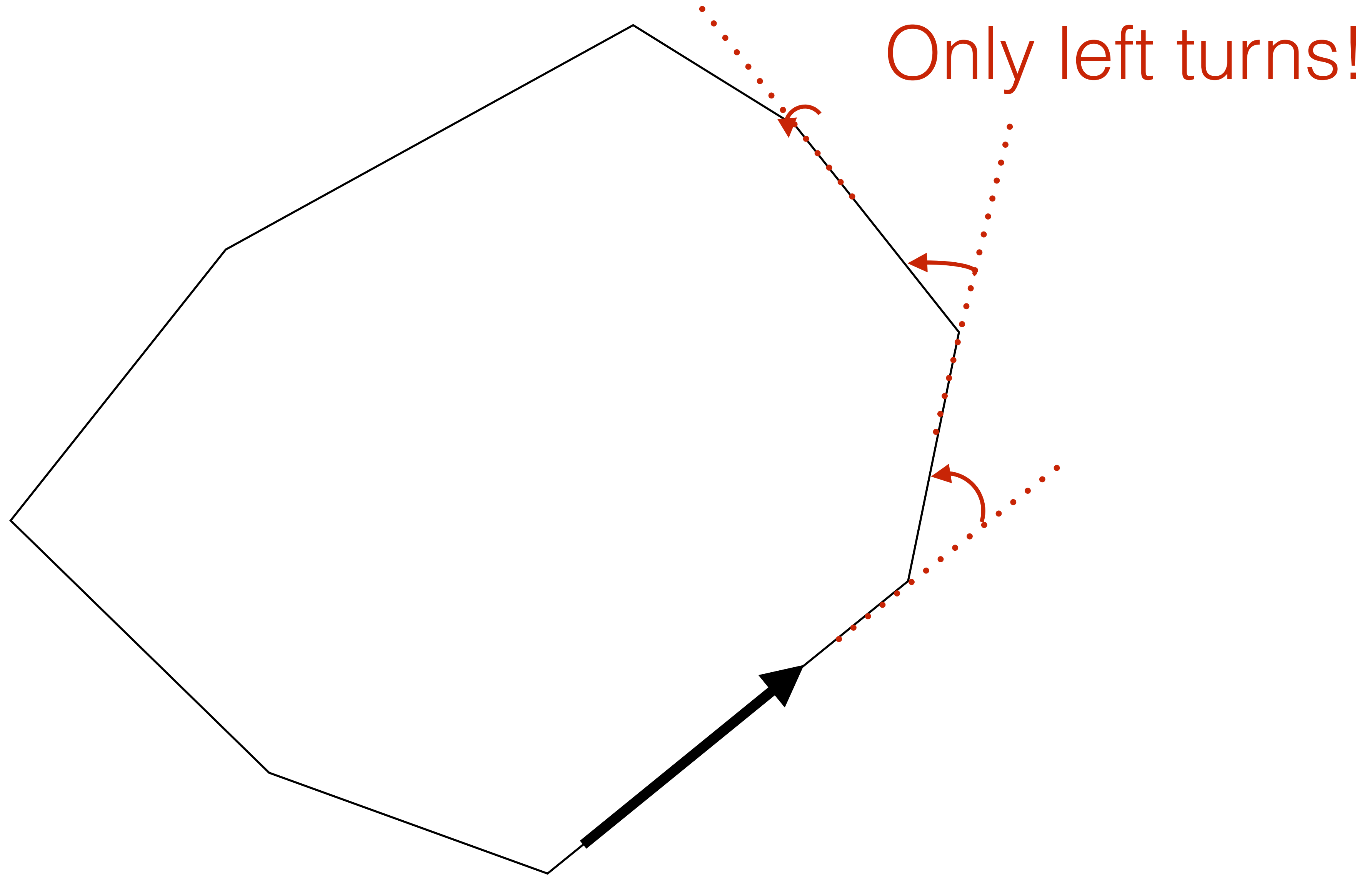
Input:
array P of points (in 2D)

Output:
array/list of points on the CH (in boundary order)

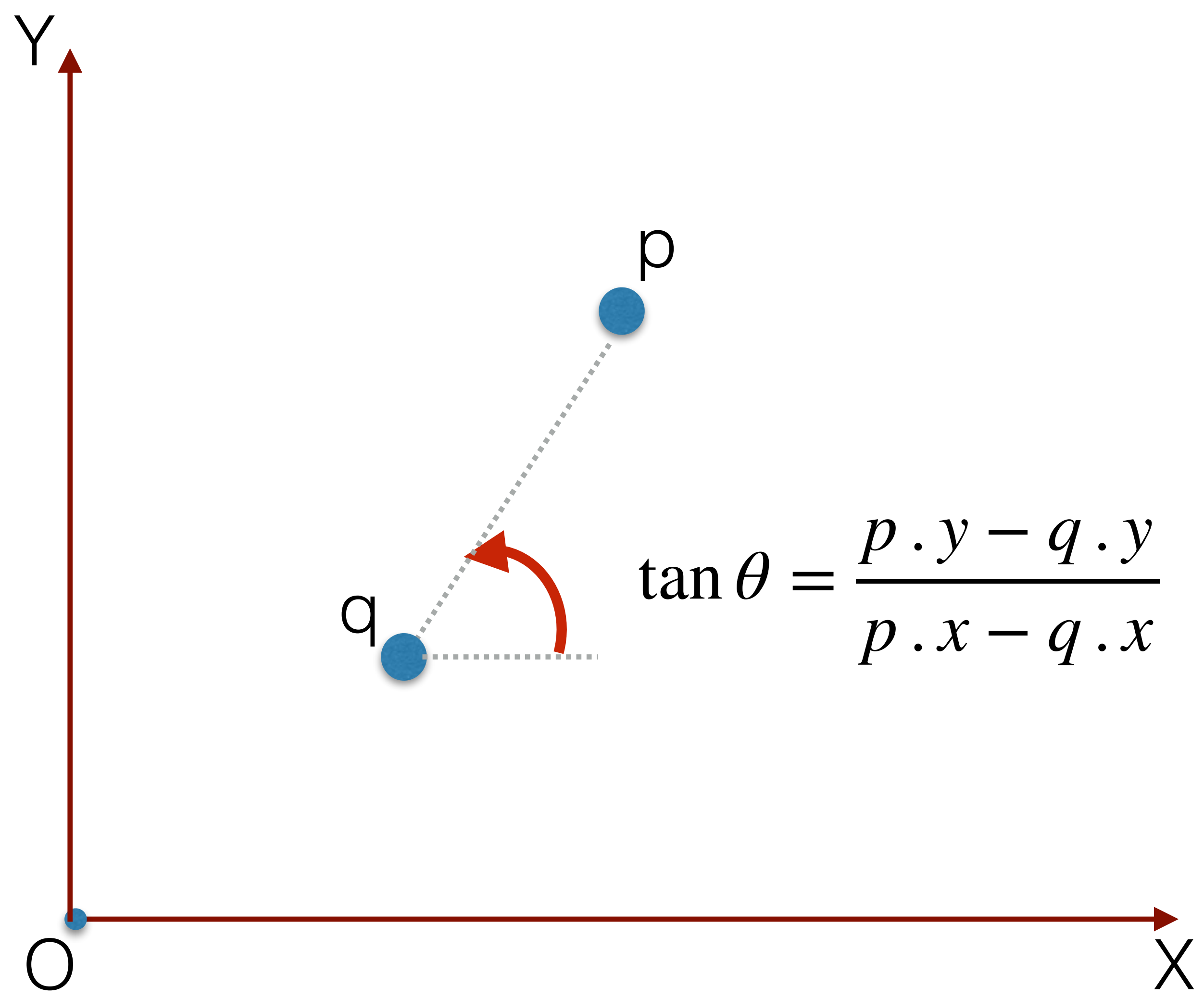
Convex hull properties

- Walking counter-clockwise on the boundary of the CH you make only left turns
- Consider a point p inside the CH. Then the points on the boundary of the CH are encountered in sorted radial order around p
- CH consists of extreme points and edges
    - point is extreme $\iff$ it is on the CH
    - (p_i, p_j) form an edge on the CH $\iff$ edge (p_i, p_j) is extreme
    - point p is interior $\iff$ p not on the CH

Walk ccw along the boundary of a convex polygon

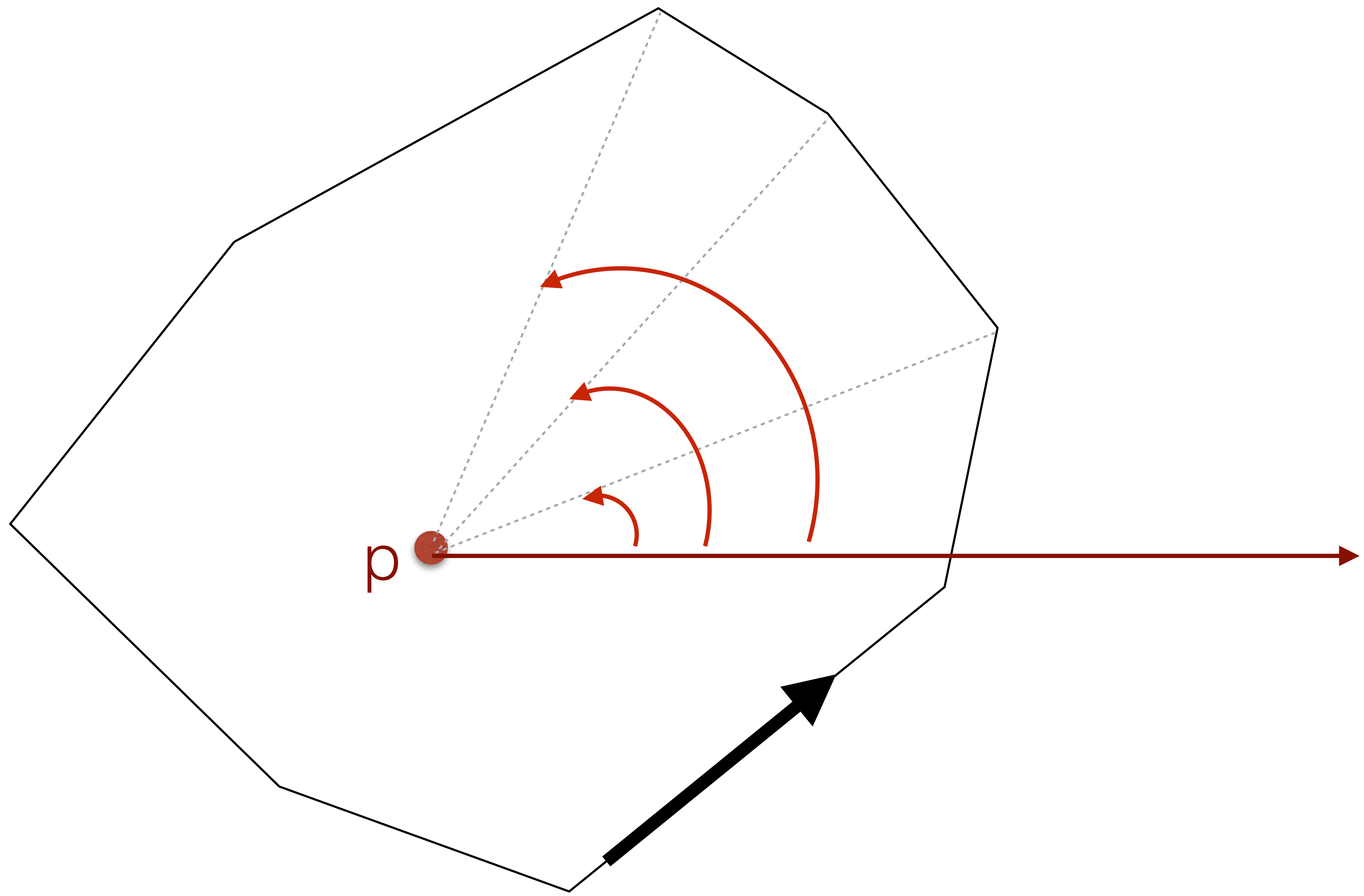


The radial angle of p with-respect-to q



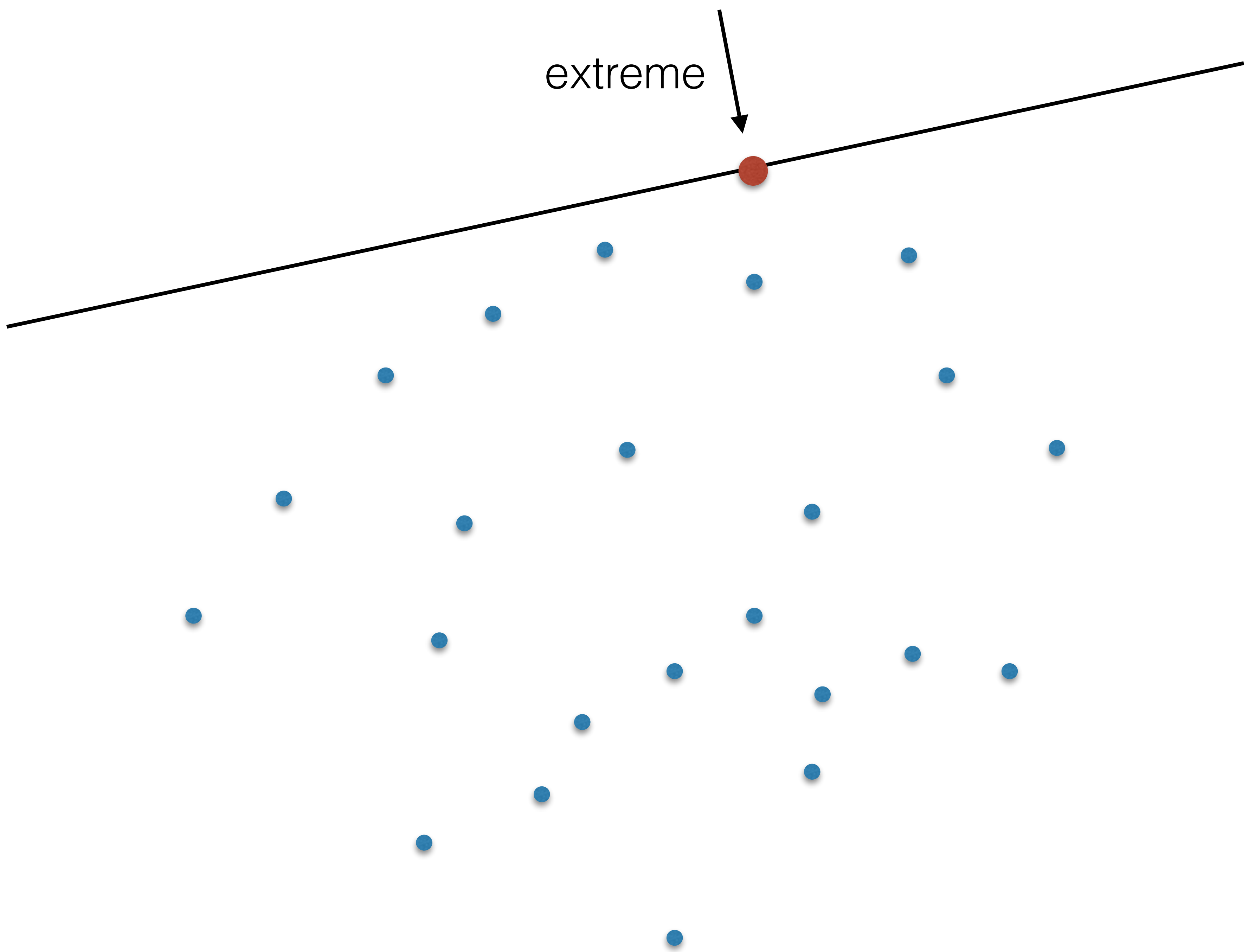
$$\tan\theta = \frac{p.y - q.y}{p.x - q.x}$$

For any point p inside, the points on the boundary are in radial order around p



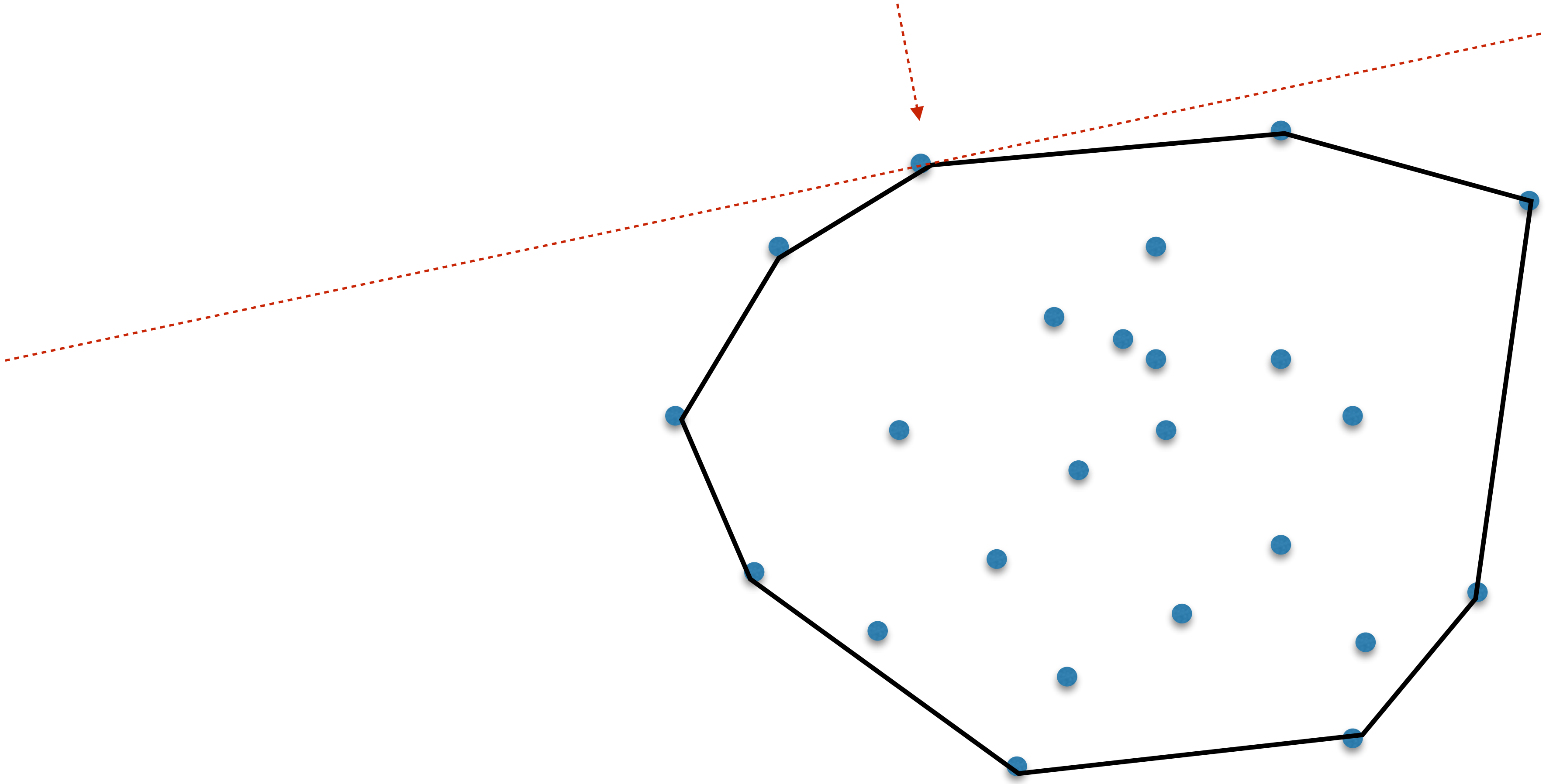
Extreme points

- A point p is called **extreme** if there exists a line l through p, such that all the other points of P are on the same side of l (and not on l)



A point is on the CH <==> it is extreme

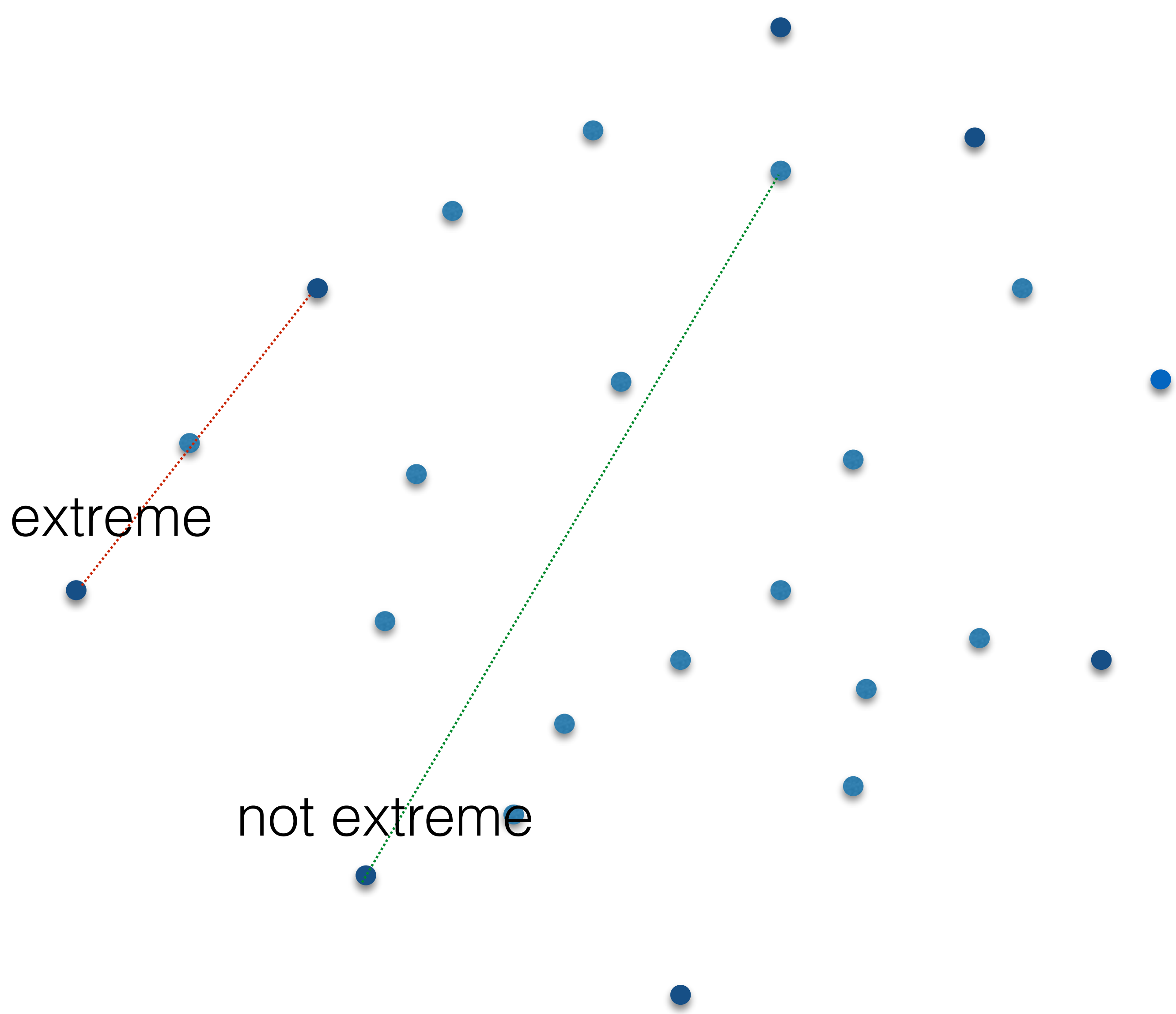
All points on the CH are extreme



All extreme points are on the CH

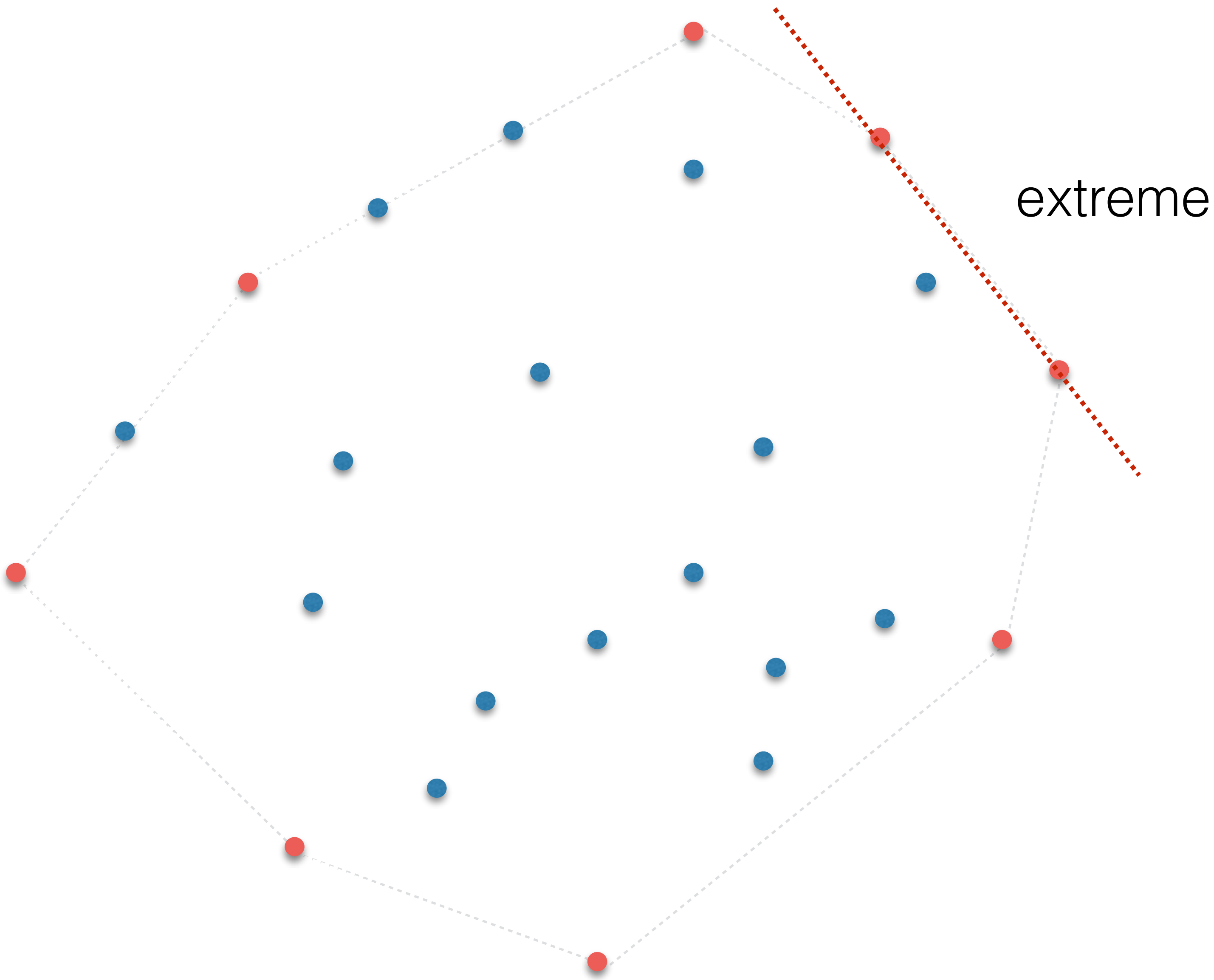
Extreme edges

- An edge (p_i, p_j) is **extreme** if all the other points of P are on one side of it (or on)



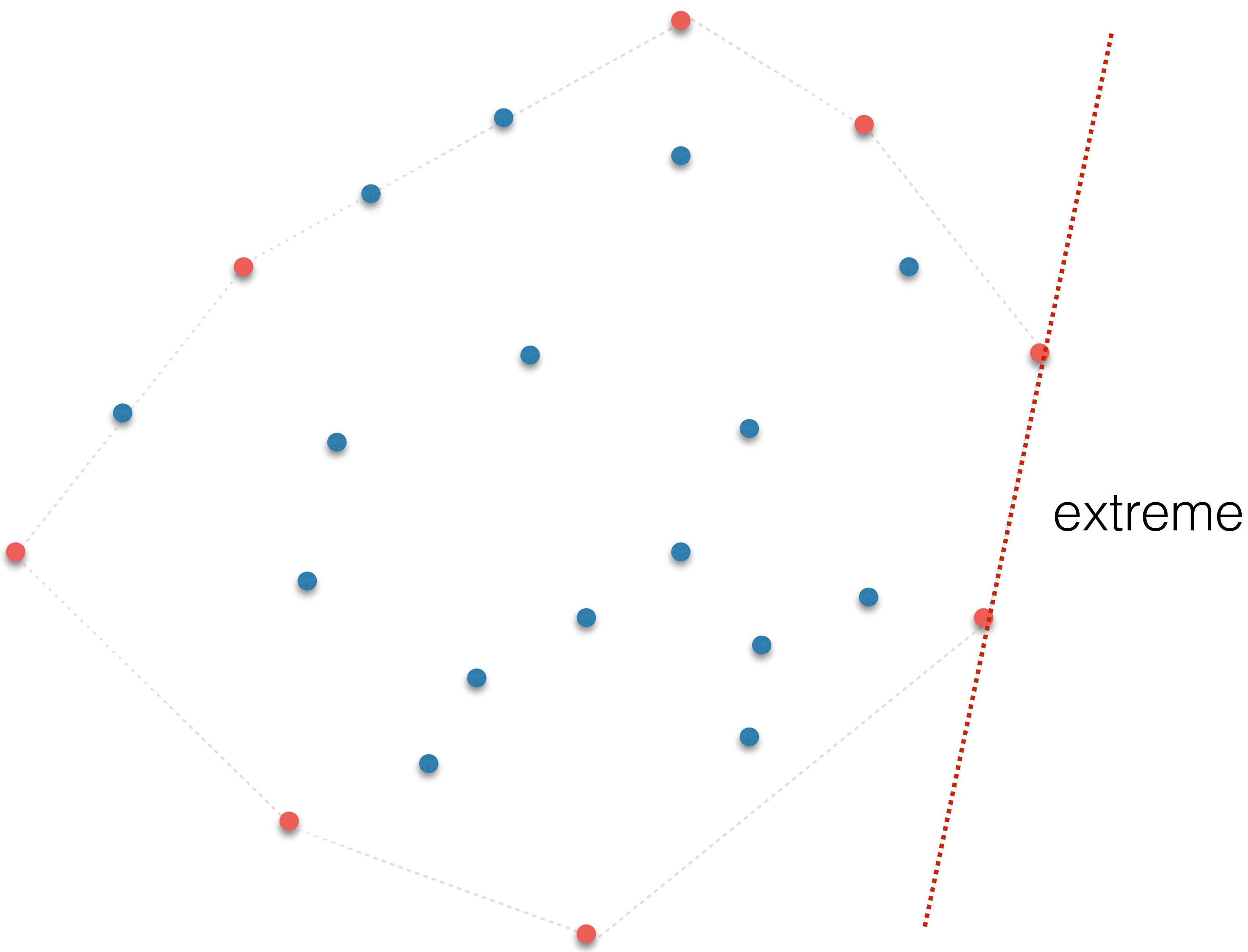
Extreme edges

- An edge (p_i, p_j) is **extreme** if all the other points of P are on one side of it (or on)



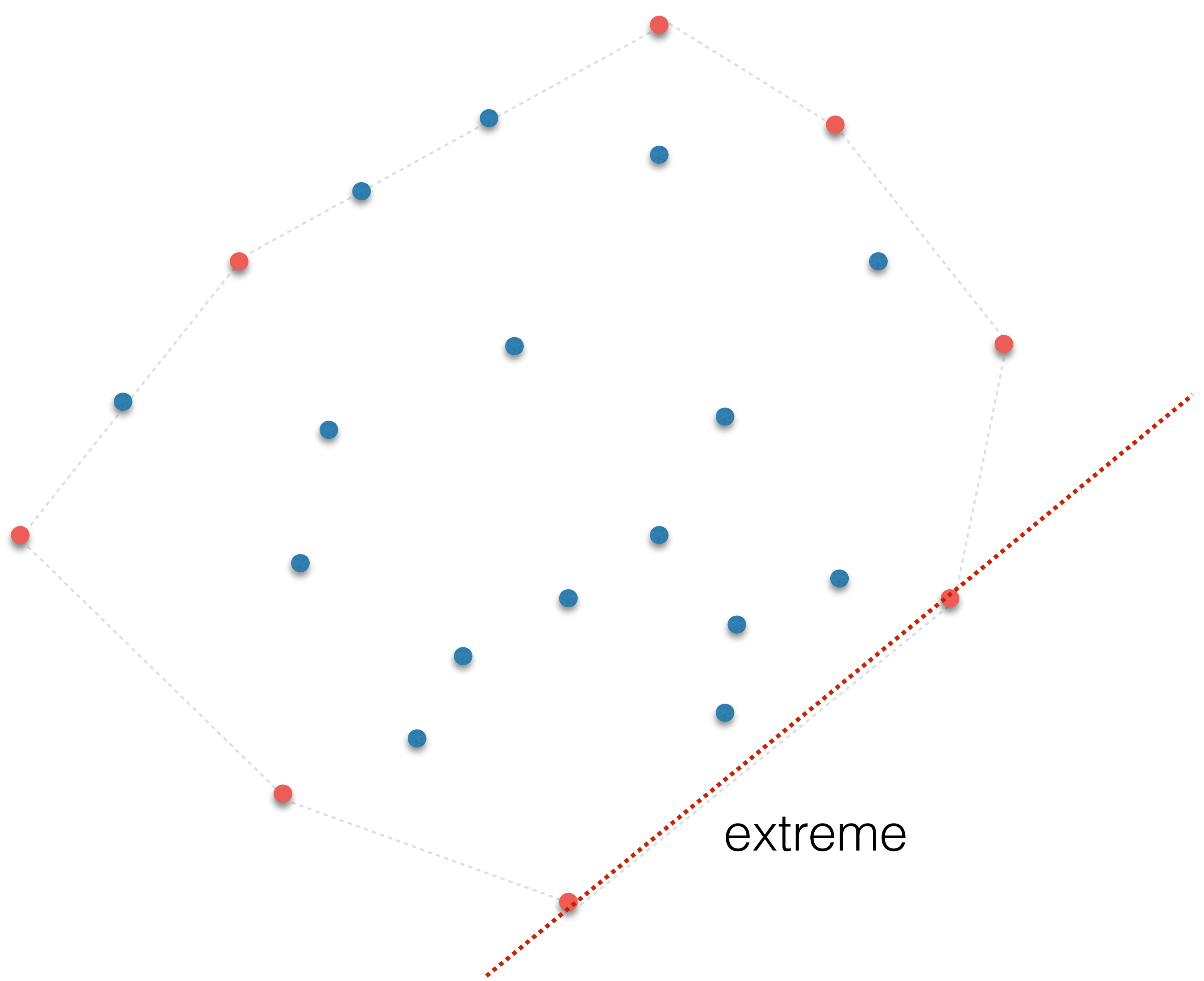
Extreme edges

- An edge (p_i, p_j) is **extreme** if all the other points of P are on one side of it (or on)



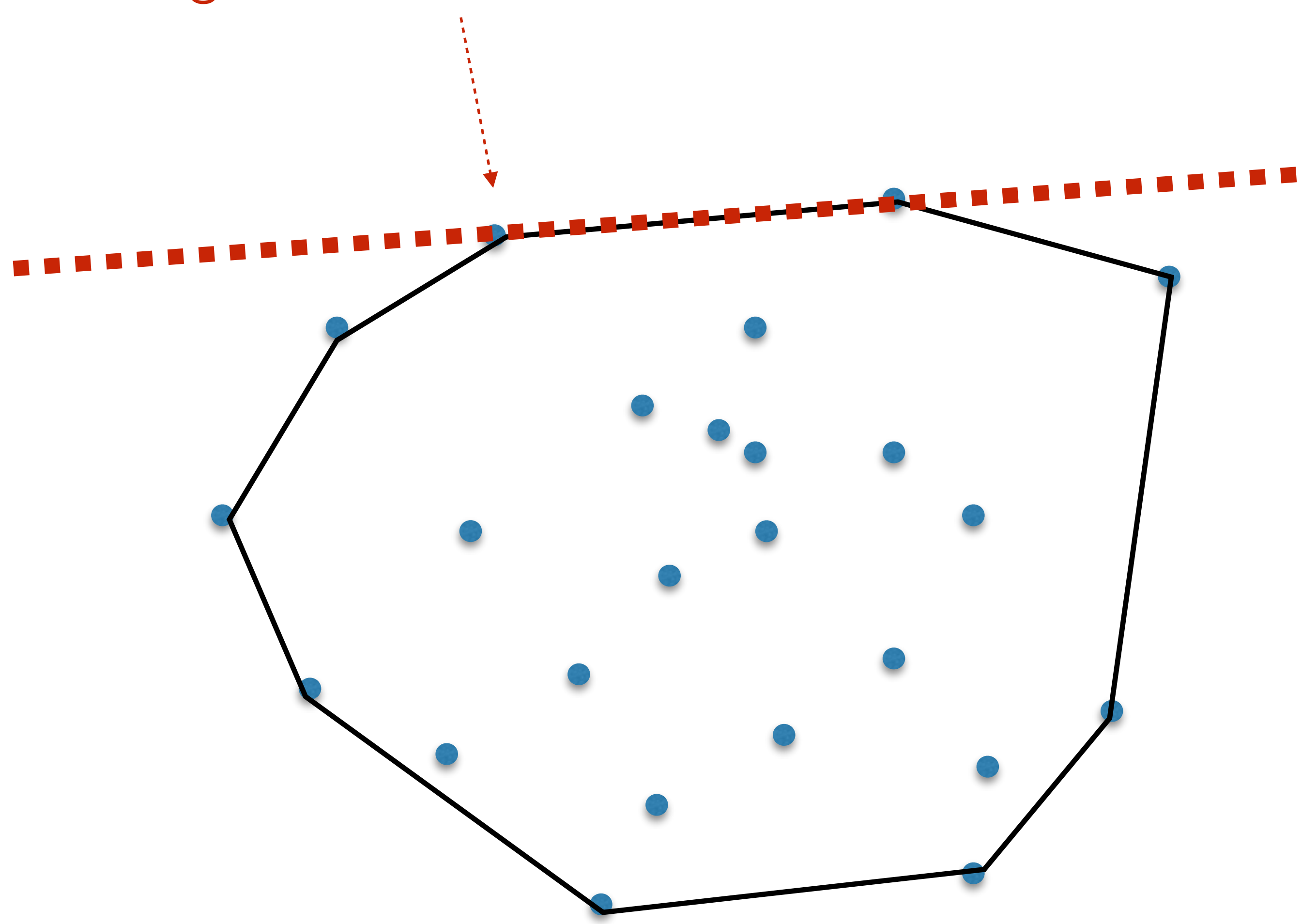
Extreme edges

- An edge (p_i, p_j) is extreme if all the other points of P are on one side of it (or on)



An edge is on the CH <==> it is extreme

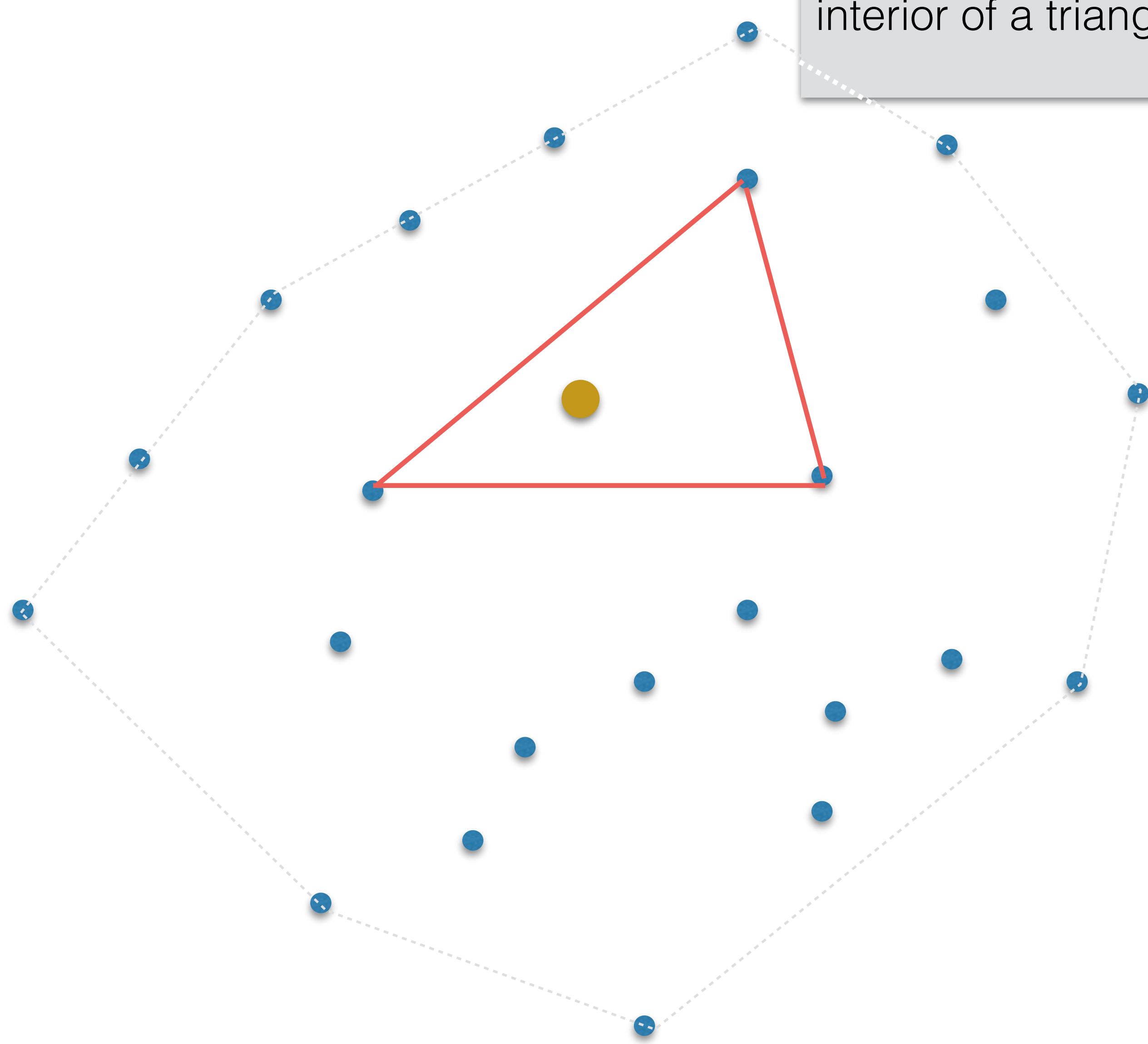
All edges on the CH are extreme



All extreme edges are on the CH

Interior points

- p interior <==> p **not** on the CH



A point p is called interior if p is contained in the interior of a triangle formed by three other points of P

Algorithms

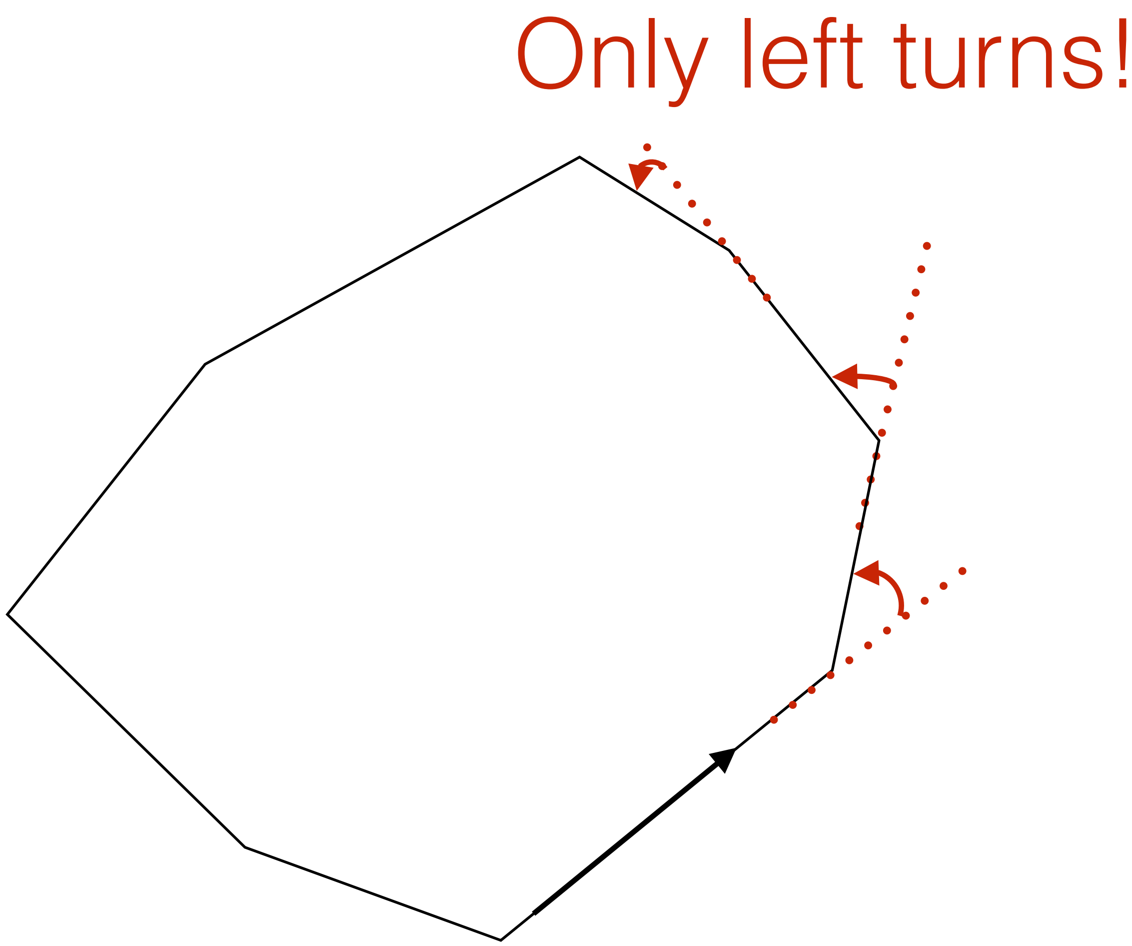
- Brute force: $O(n^3)$
- Gift wrapping: $O(kn)$
    - output-size sensitive: $O(n)$ best case, $O(n^2)$ worst case
    - ✦ by Chand and Kapur [1970]. Extends to 3D and to arbitrary dimensions; for many years was the primary algorithm for higher dimensions
- Next
    - Graham scan
    - Quickhull

Algorithm:

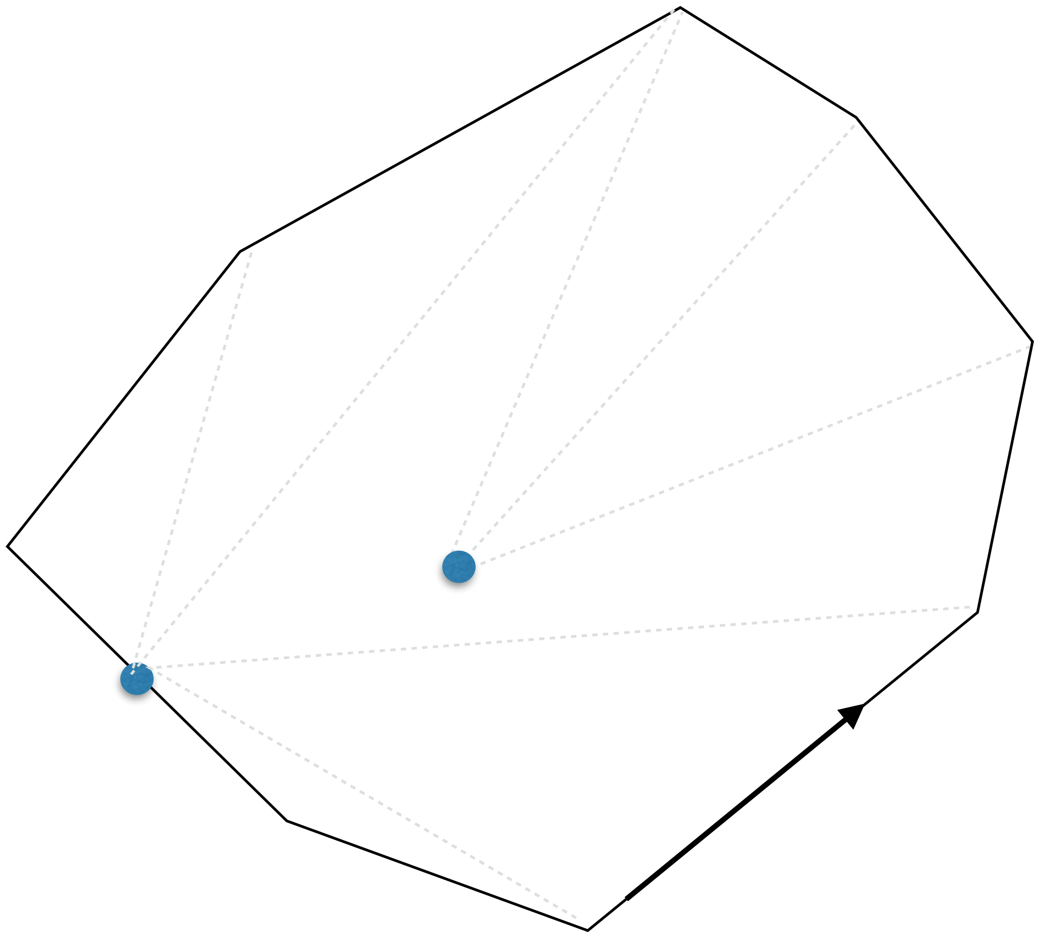
Graham scan

- In late 60s an application at Bell Labs required the hull of 10,000 points, for which a quadratic algorithm was too slow
- Graham developed an algorithm which runs in O(n lg n)
    - It runs in one sort plus a linear pass!!
    - Simple, intuitive, elegant and practical

Ideas



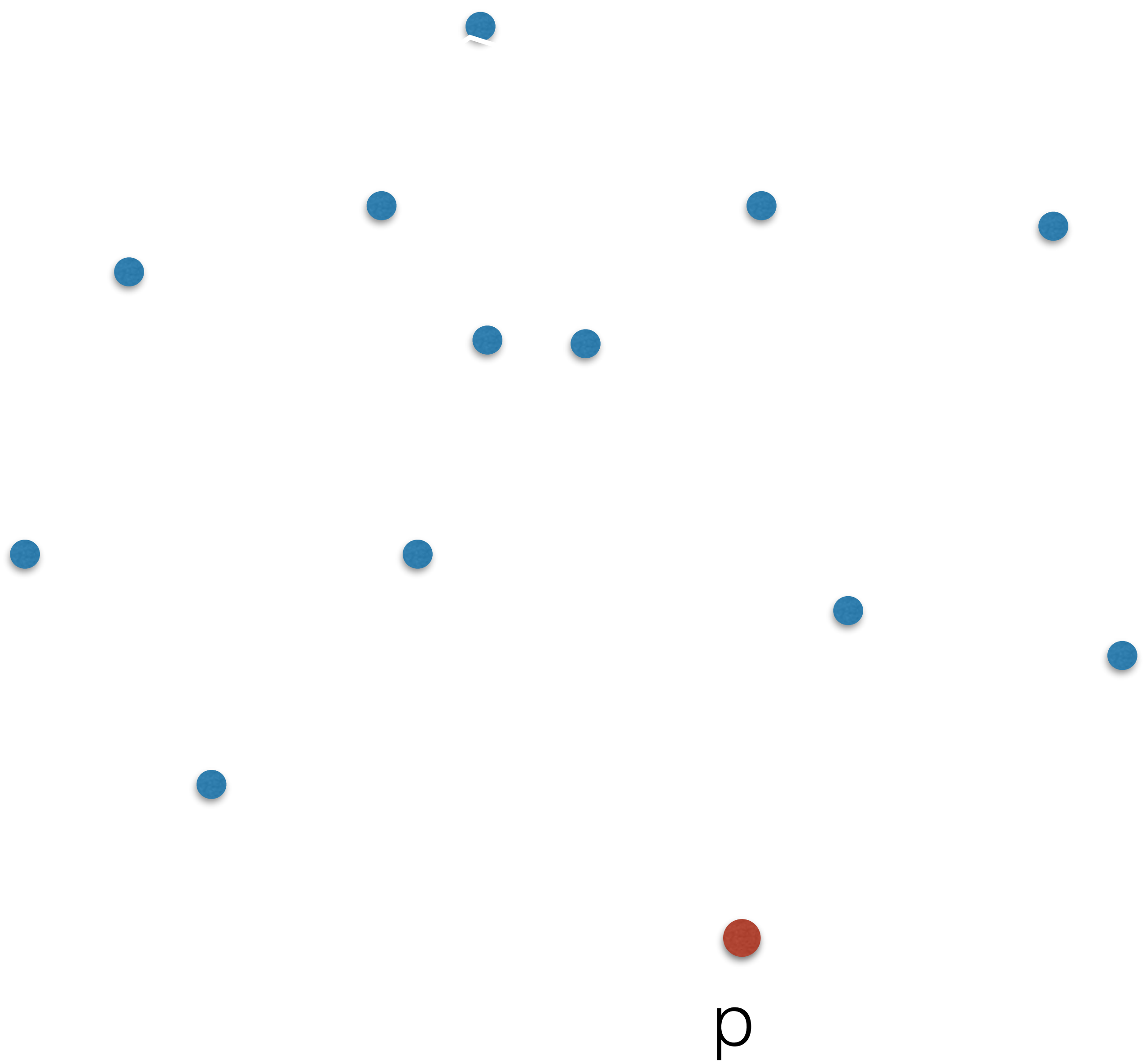
Walk ccw along the boundary of a convex polygon



For any point p inside or on, the points on the boundary are in radial order around p

Graham scan

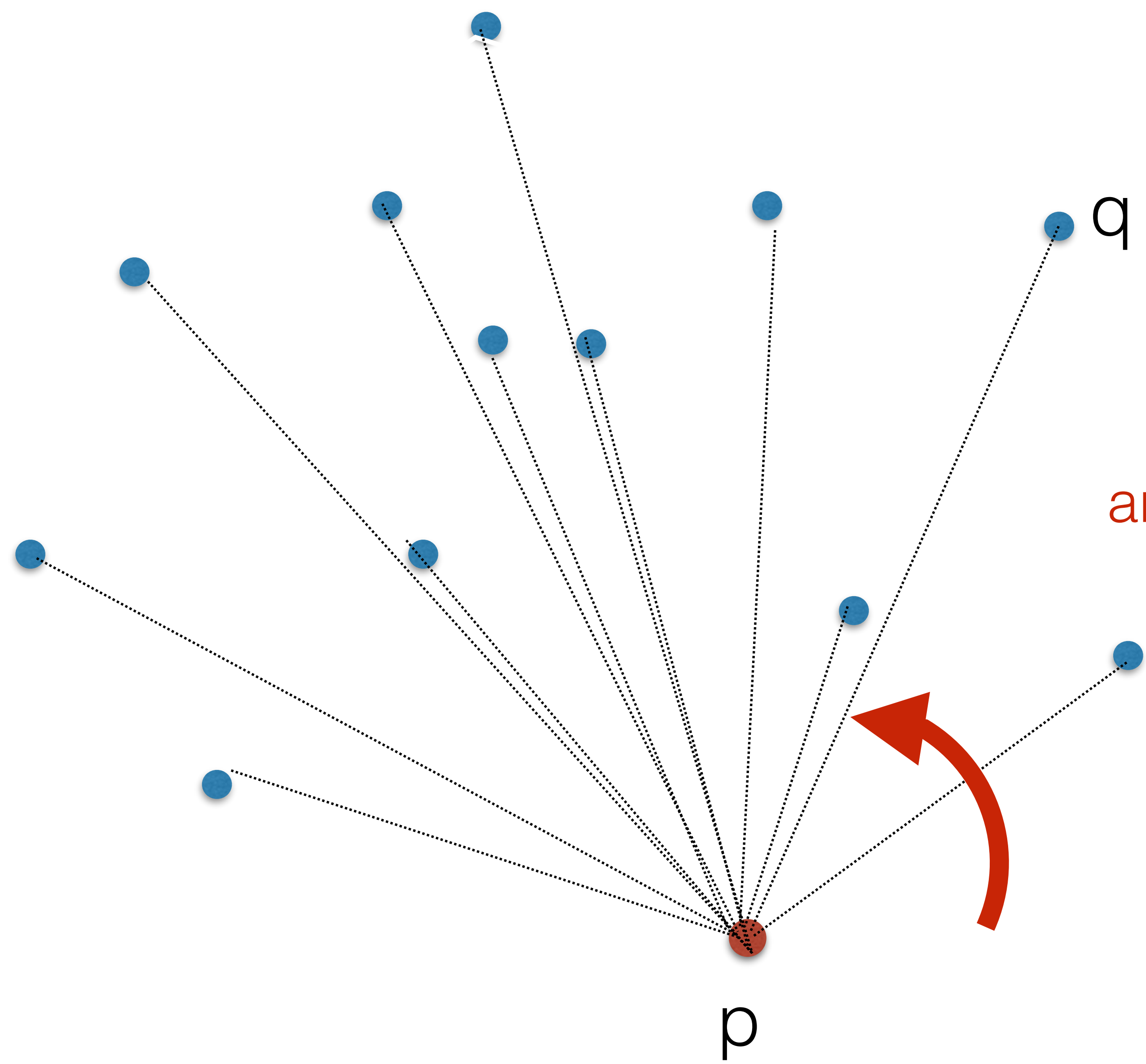
- Idea: start from a point p on the hull (e.g. lowest point)



Graham scan

- Idea: start from a point p on the hull (e.g. lowest point)

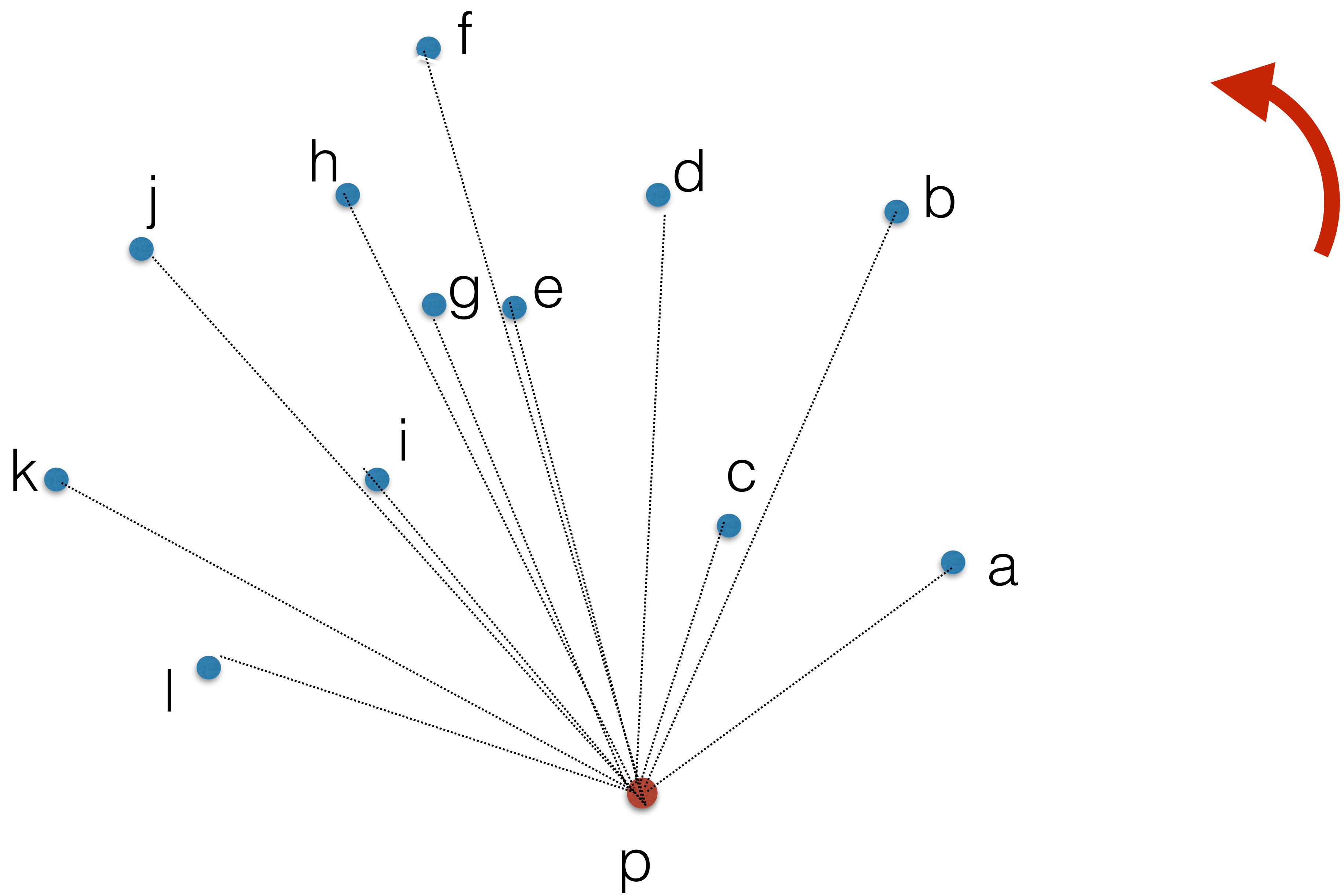
  order all points by their ccw angle wrt p



$$\text{angle}_q = \tan^{-1}\frac{q.y - p.y}{q.x - p.x}$$

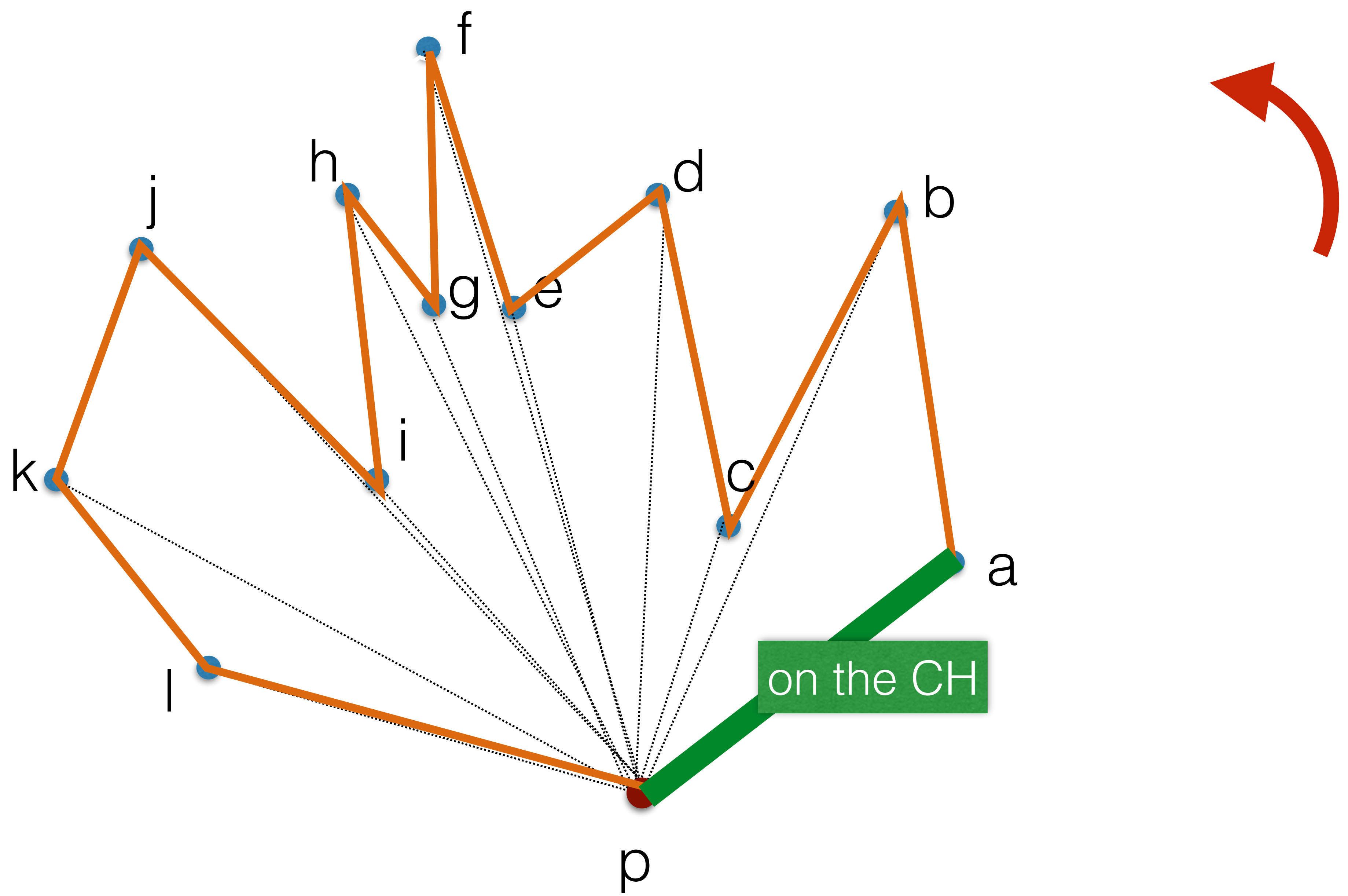
Graham scan

- Idea: traverse the points in this order a, b, c, d, e, f, g, …

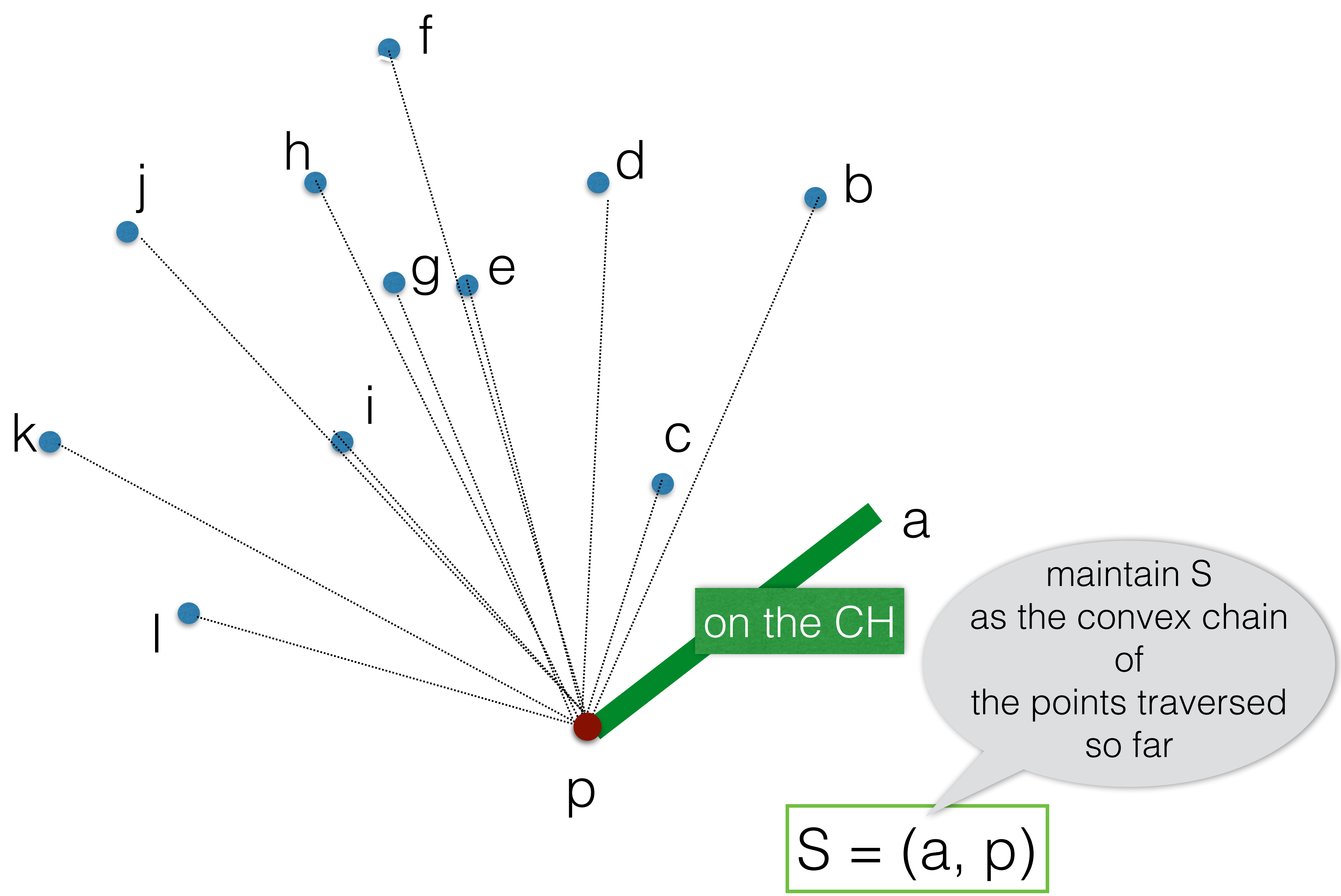


Graham scan

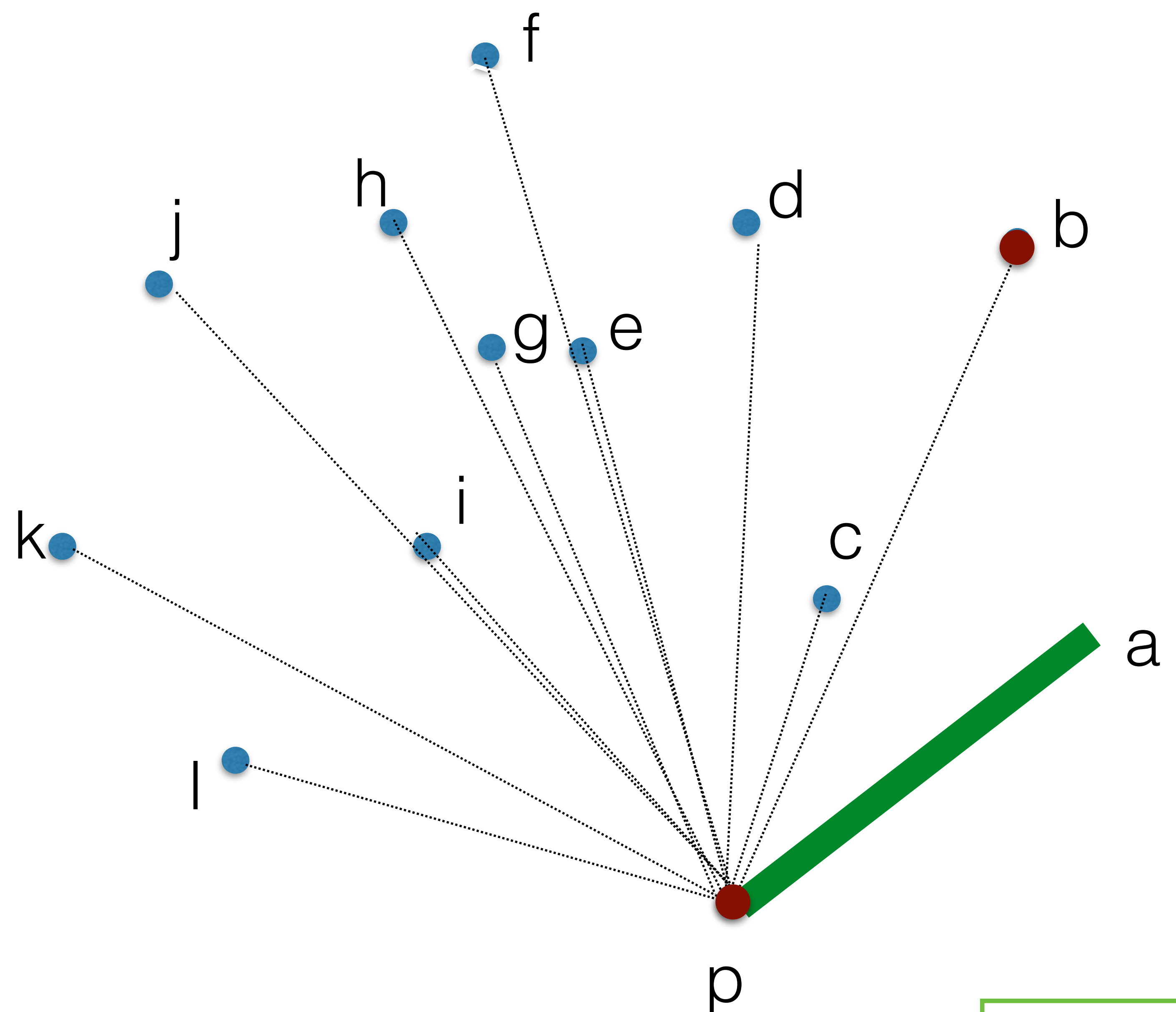
- Idea: traverse the points in this order a, b, c, d, e, f, g, …, making it convex



Graham scan


f
h
d
b
j
g
e
i
k
c
a
on the CH
l
p
maintain S as the convex chain of the points traversed so far
S = (a, p)

Next point b: what do we do with it?



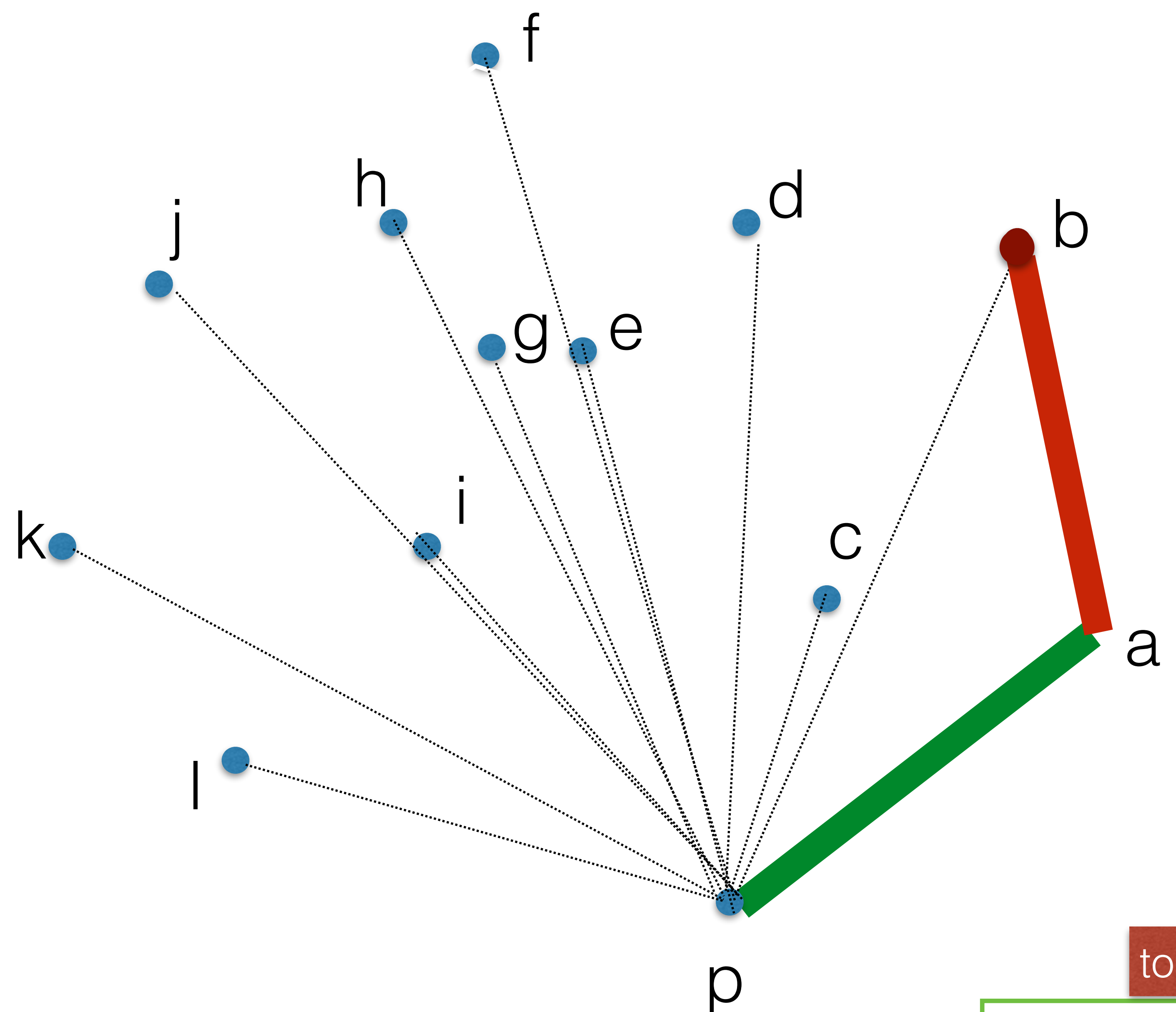
$S = (a, p)$

is b left of pa?

Next point b: what do we do with it?

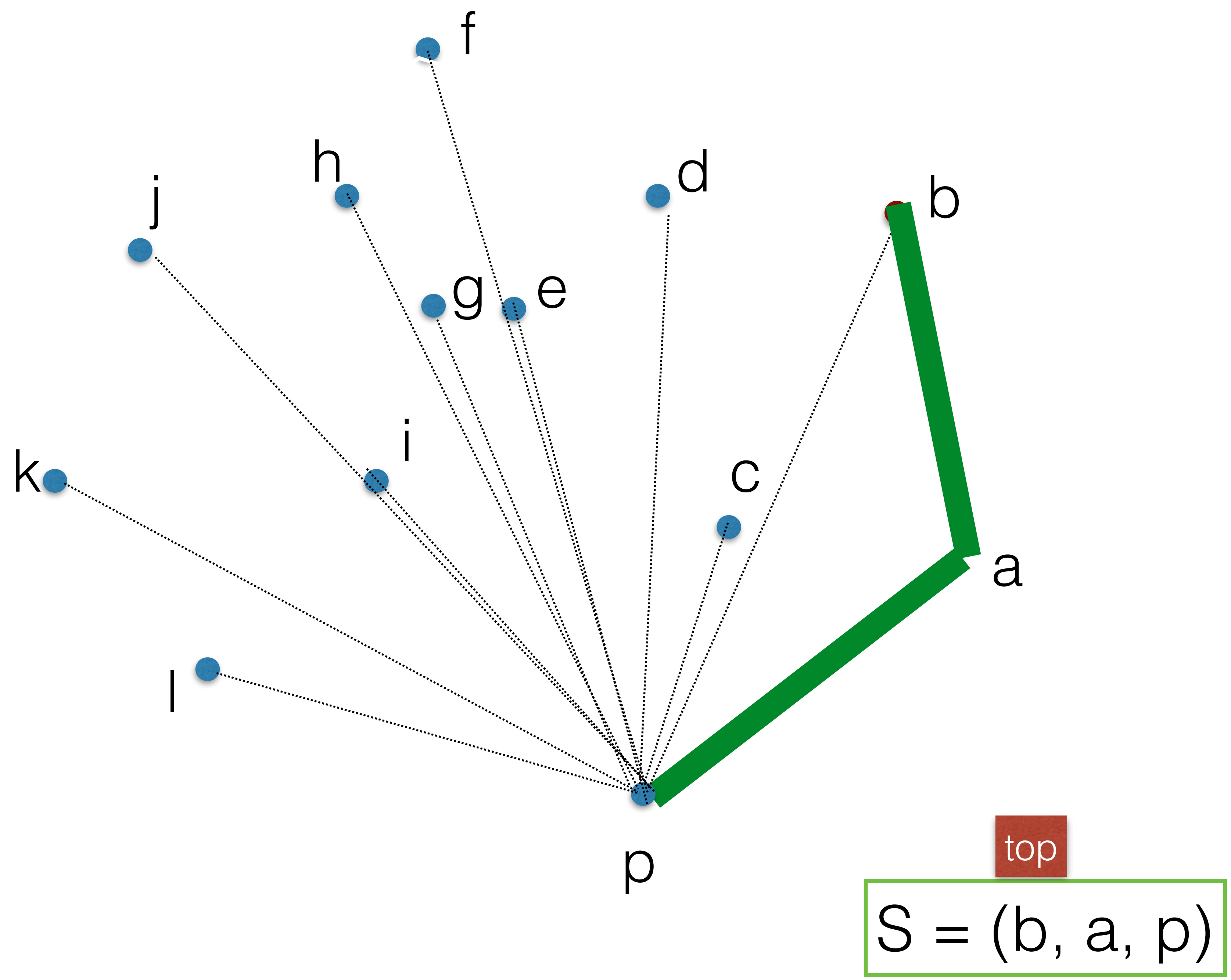
Is b + (a, p) convex? YES!

push b to S



top

S = (a, p)



f
h
d
j
b
g
e
i
k
c
a
l
p
top
S = (b, a, p)

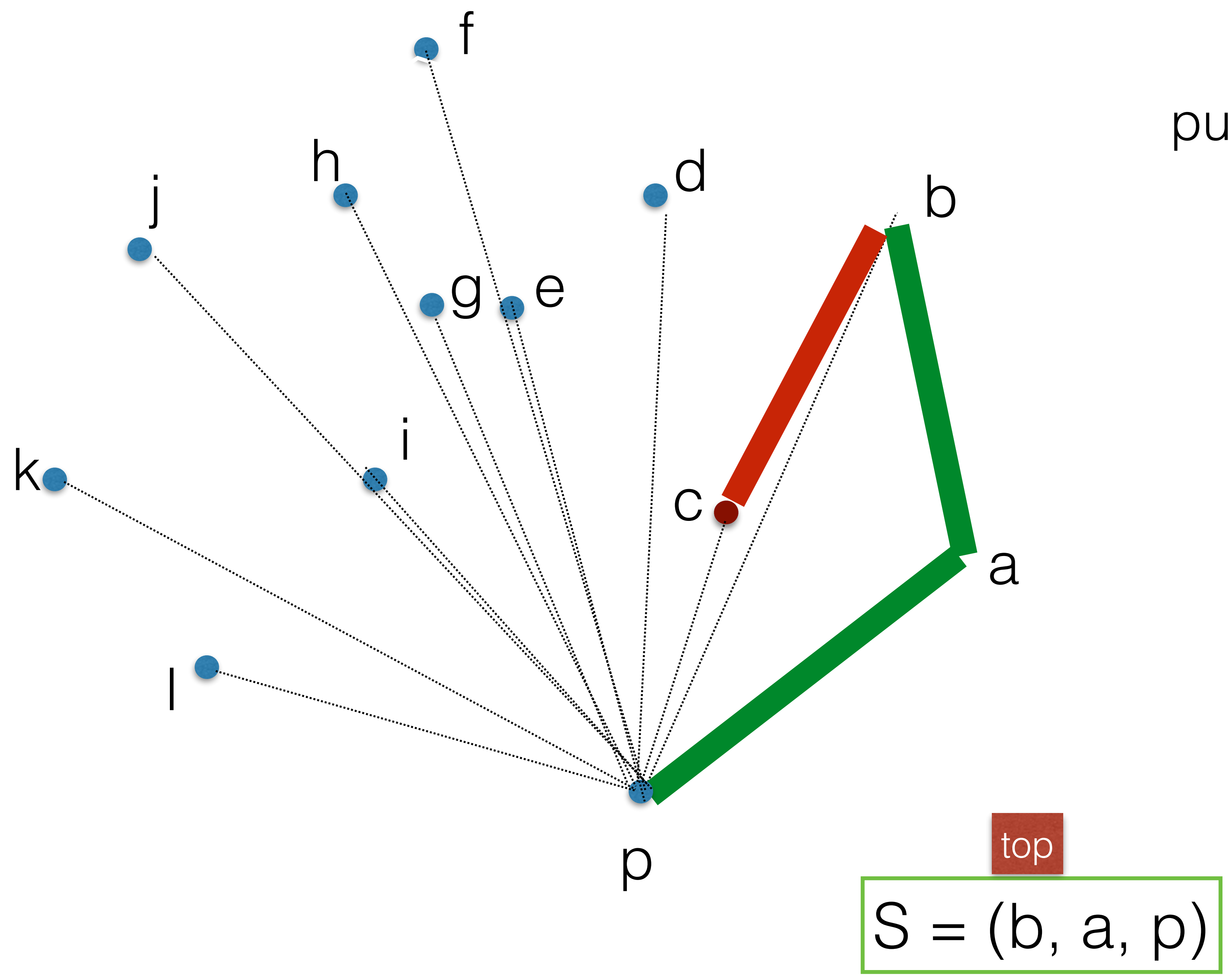
Next point c:

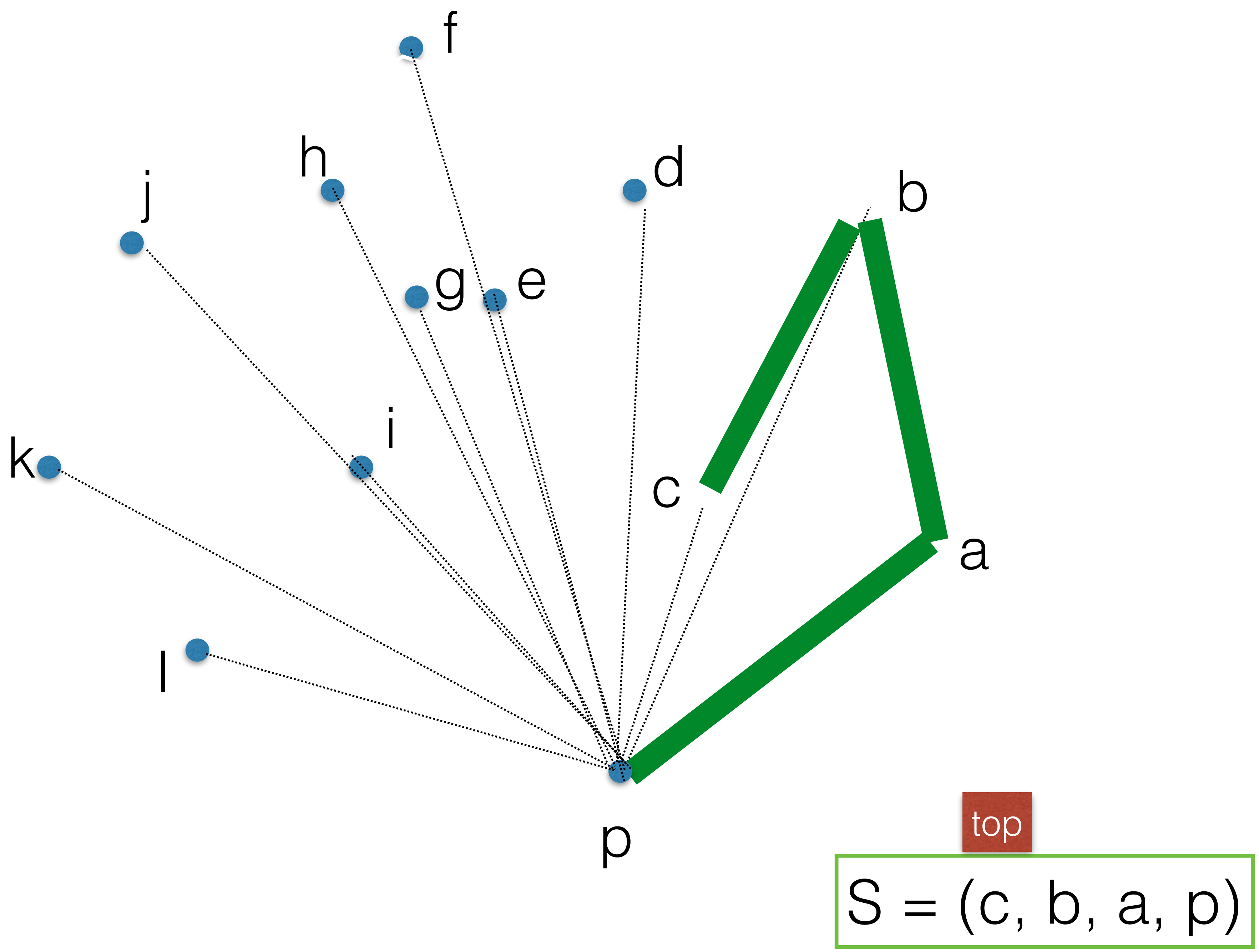
Is c + S convex?

is c left of ab?

YES!

push c to S



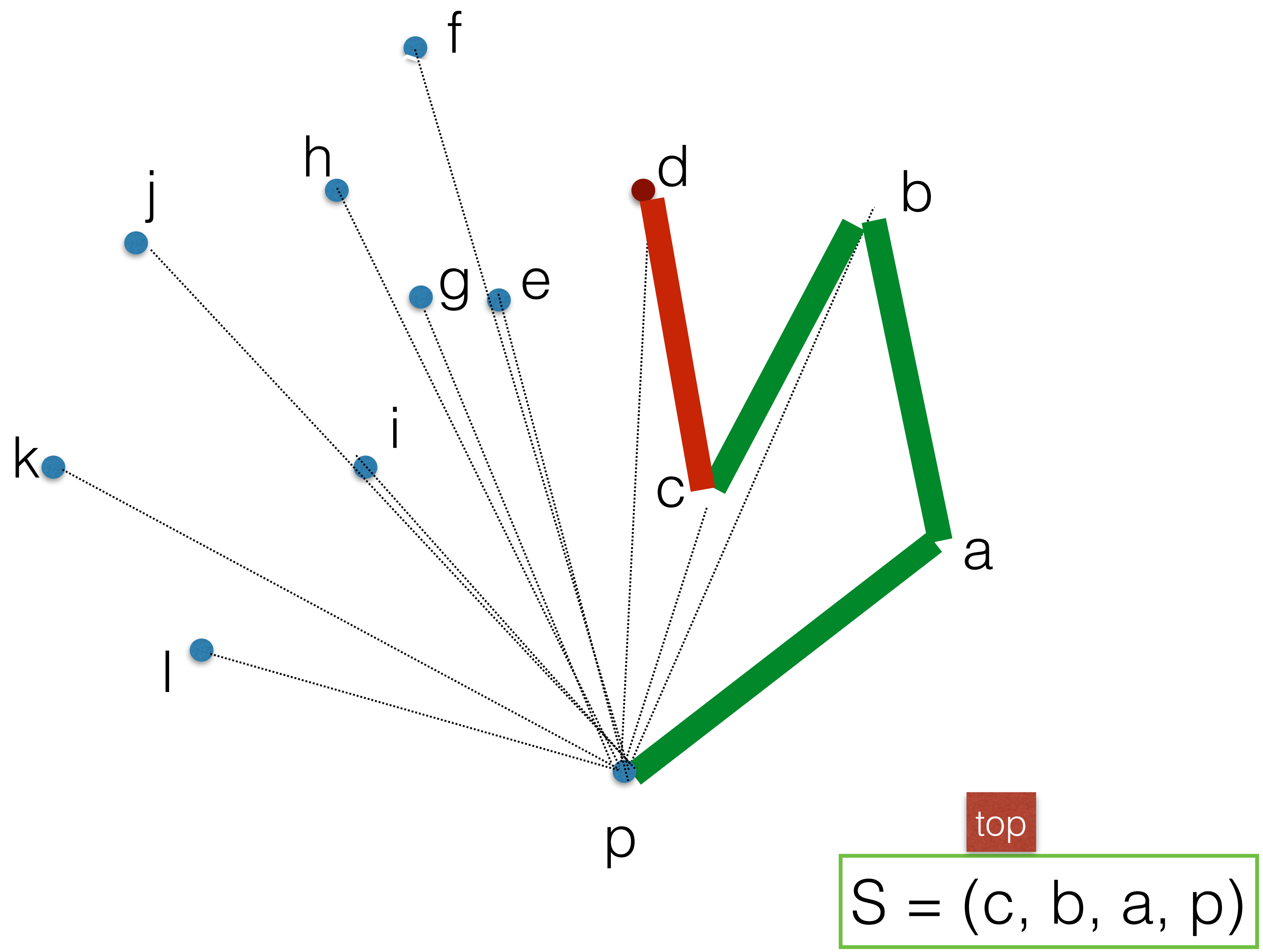

f
h
d
b
j
g
e
i
k
c
a
l
p
top
S = (c, b, a, p)

is d left of bc?

Is d + S convex?

NO!

Next point d:



is d left of bc?

Next point d:

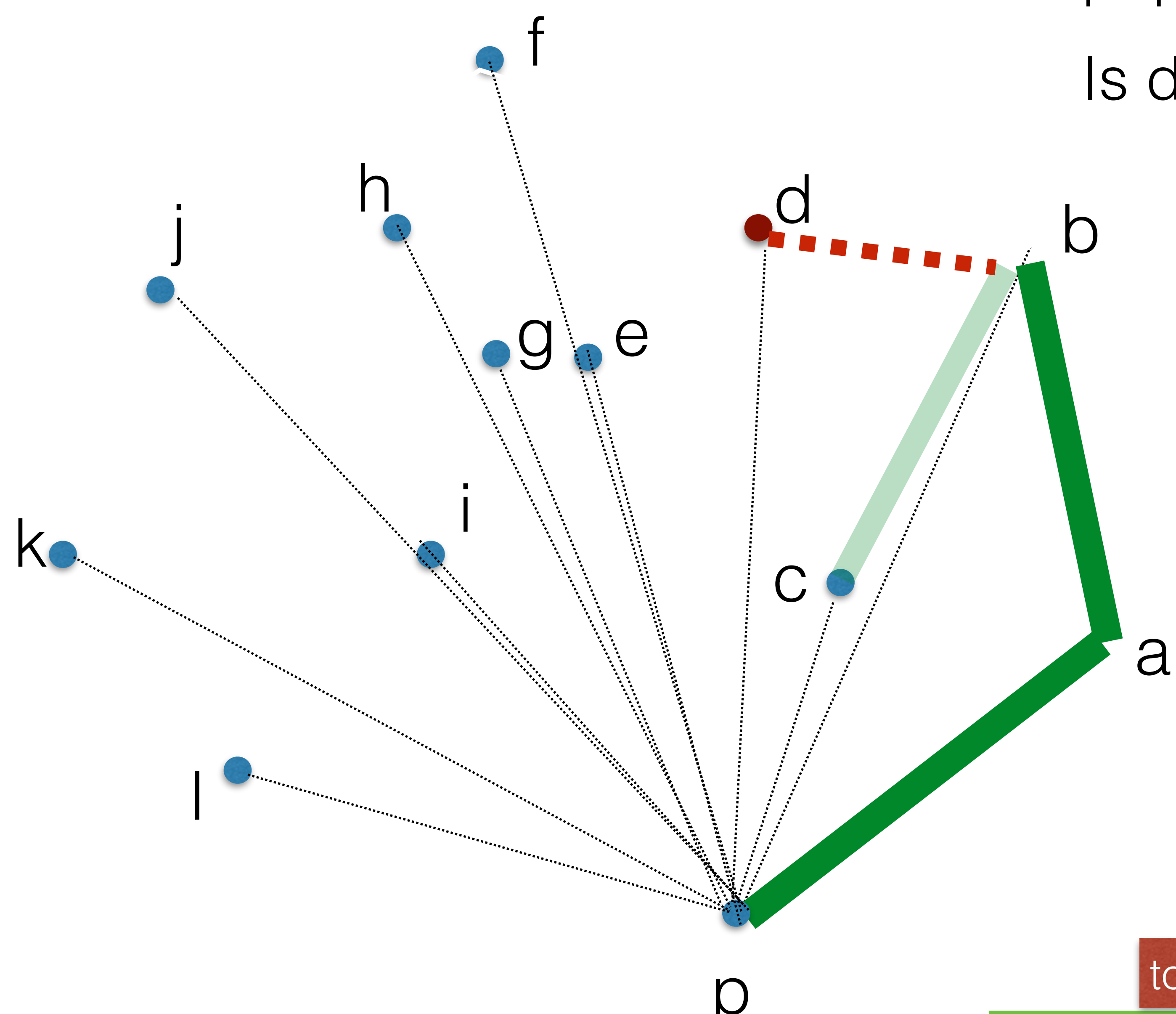
Is d + S convex? NO!

pop c

Is d + S convex?

YES!

push d to S



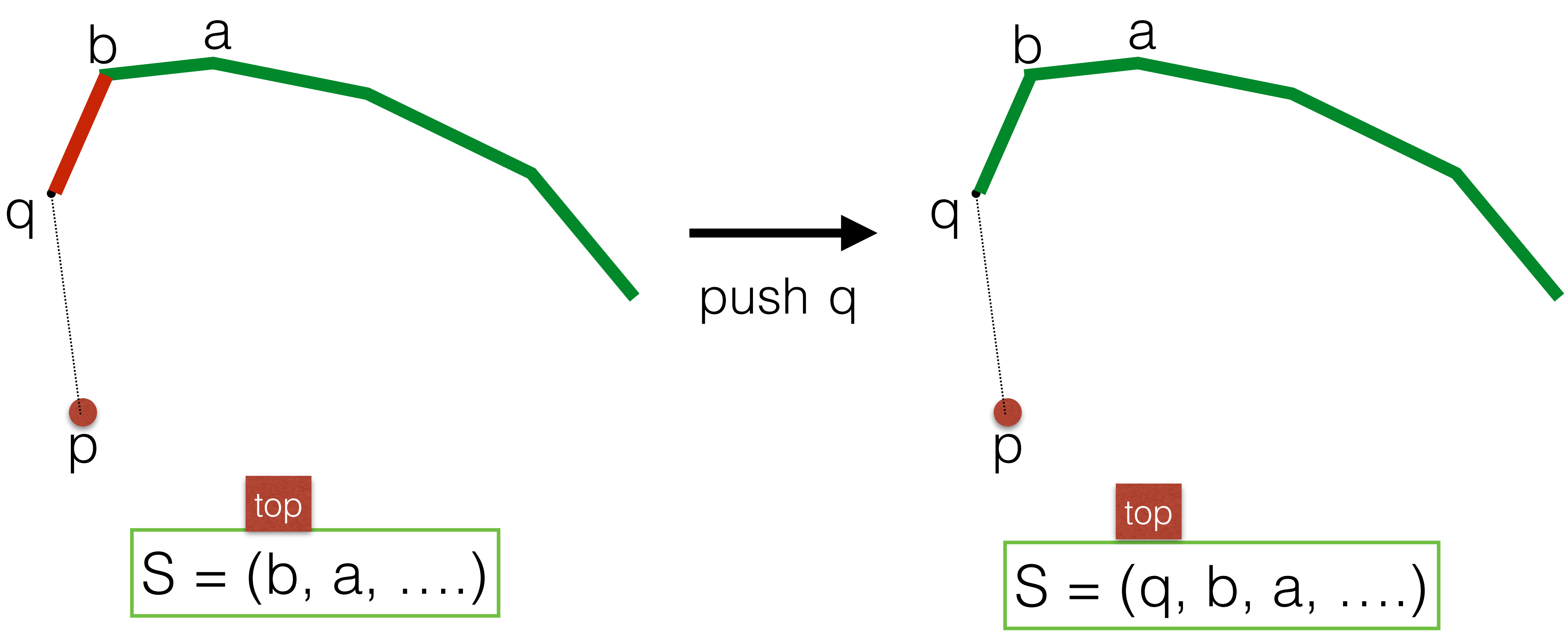
top

S = (d, ~~c~~, b, a, p)

In general

b = top(S), a = second(S)

Next point q: • if q is left of a, b: push(q, S)

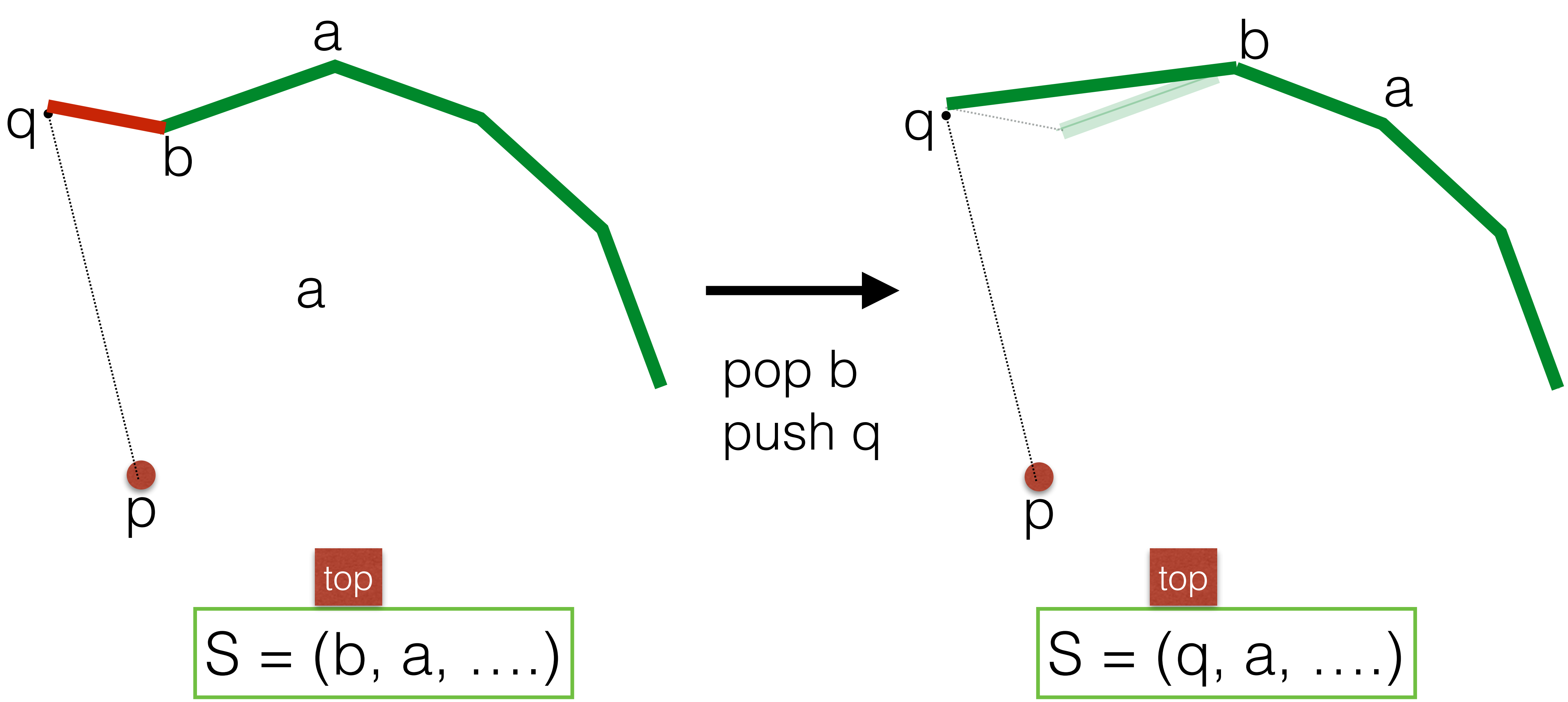


In general

b = top(S), a = second(S)

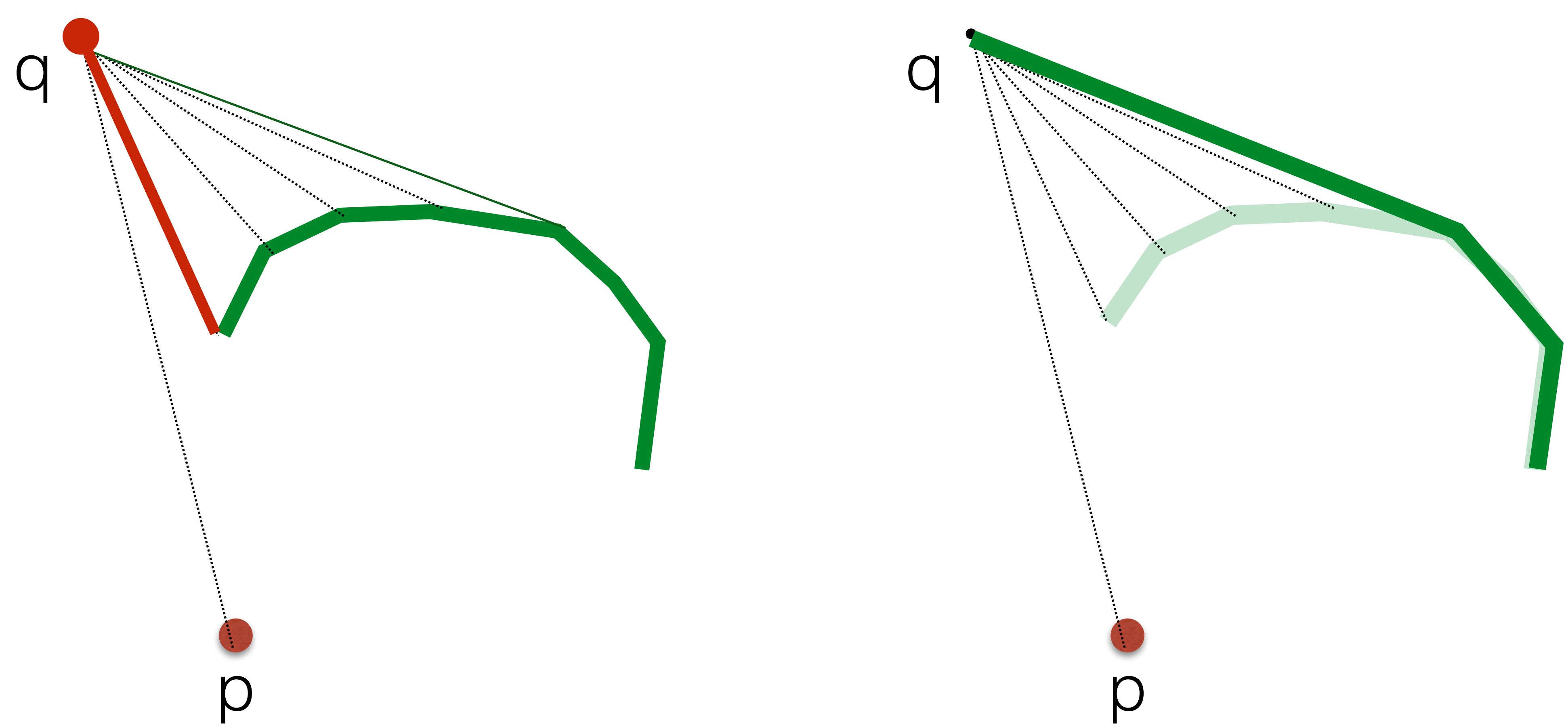
Next point q:

- while q is right of a, b:
    - pop (S)
    - b = top(S), a = second(S)
- push(q, S)



In general

A vertex can trigger more than one pop



Graham scan

- Find lowest point p_0
- Sort all other points ccw around p_0

call them p_1, p_2, p_3, …p_{n-1} in this order

- Initialize stack $S = (p_2, p_1)$
- for $i = 3$ to $n - 1$ do
  - if p_i is left of (second(S),first(S)):
    - push p_i on S
  - else
    - while p_i is right of (second(S), first(S))
      - pop S
    - push p_i on S

note that we are ignoring some details, such as collinear points

Graham scan

- Find lowest point p_0
- Sort all other points ccw around p_0 ← call them p_1, p_2, p_3, …p_{n-1} in this order
- Initialize stack $S = (p_2, p_1)$
- for $i = 3$ to $n - 1$ do
    - if p_i is left of (second(S),first(S)):
        - push p_i on S
    - else
    - while p_i is right of (second(S), first(S))
        - pop S
    - push p_i on S

ANALYSIS

- Find lowest point p_0
- Sort all other points ccw around p_0
- Initialize stack S = (p_2, p_1)
- for i=3 to n-1 do
    - while p_i is right of (second(S), first(S))
        - pop(S)
    - push p_i on S

$O(n \lg n)$

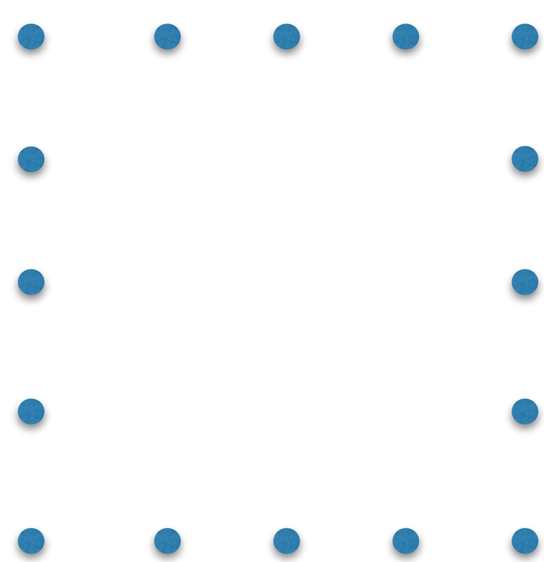
We can show that this takes $O(n)$

Graham-scan runs in $O(n \lg n) + O(n)$

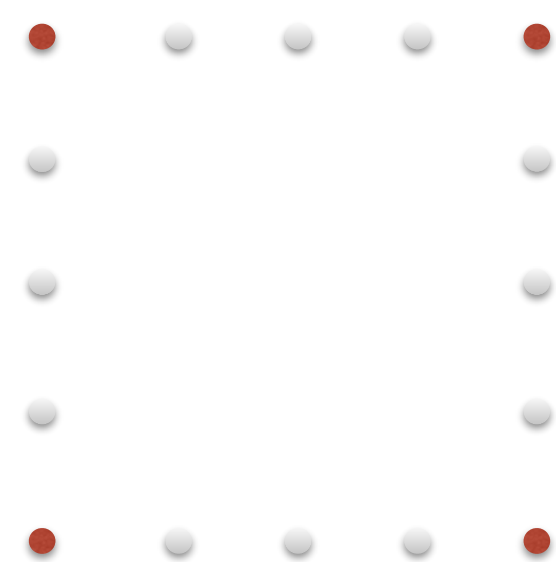
One point can trigger many pops. But we can only pop points that were previously pushed.
Every point is pushed once and popped at most once.

Project 2

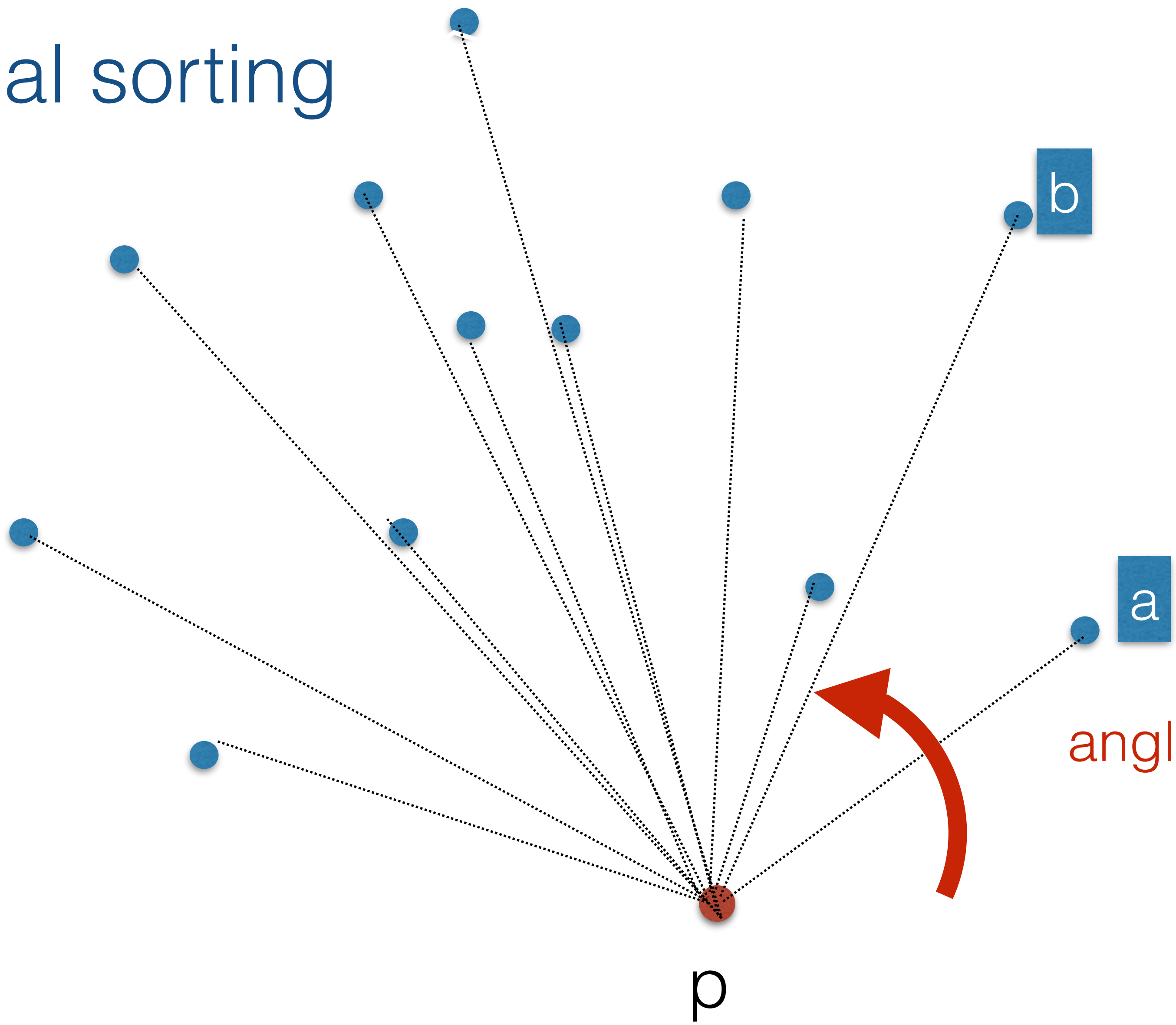
- You'll implement this!
- Along the way you'll get to figure out what situations can cause problems and how to handle them



This is what we want



Note on radial sorting

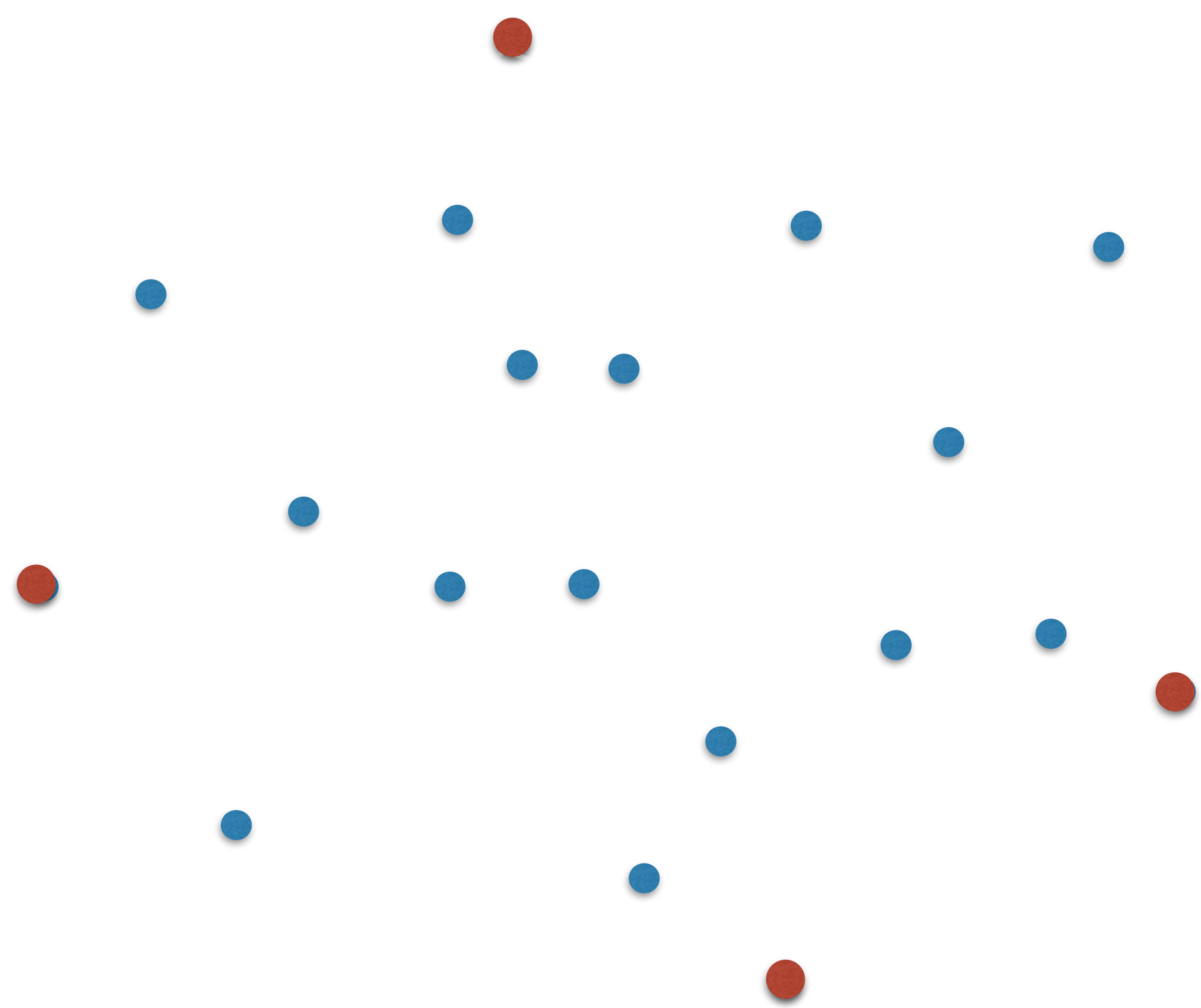


$$\text{angle}_q = \tan^{-1} \frac{q.y - p.y}{q.x - p.x}$$

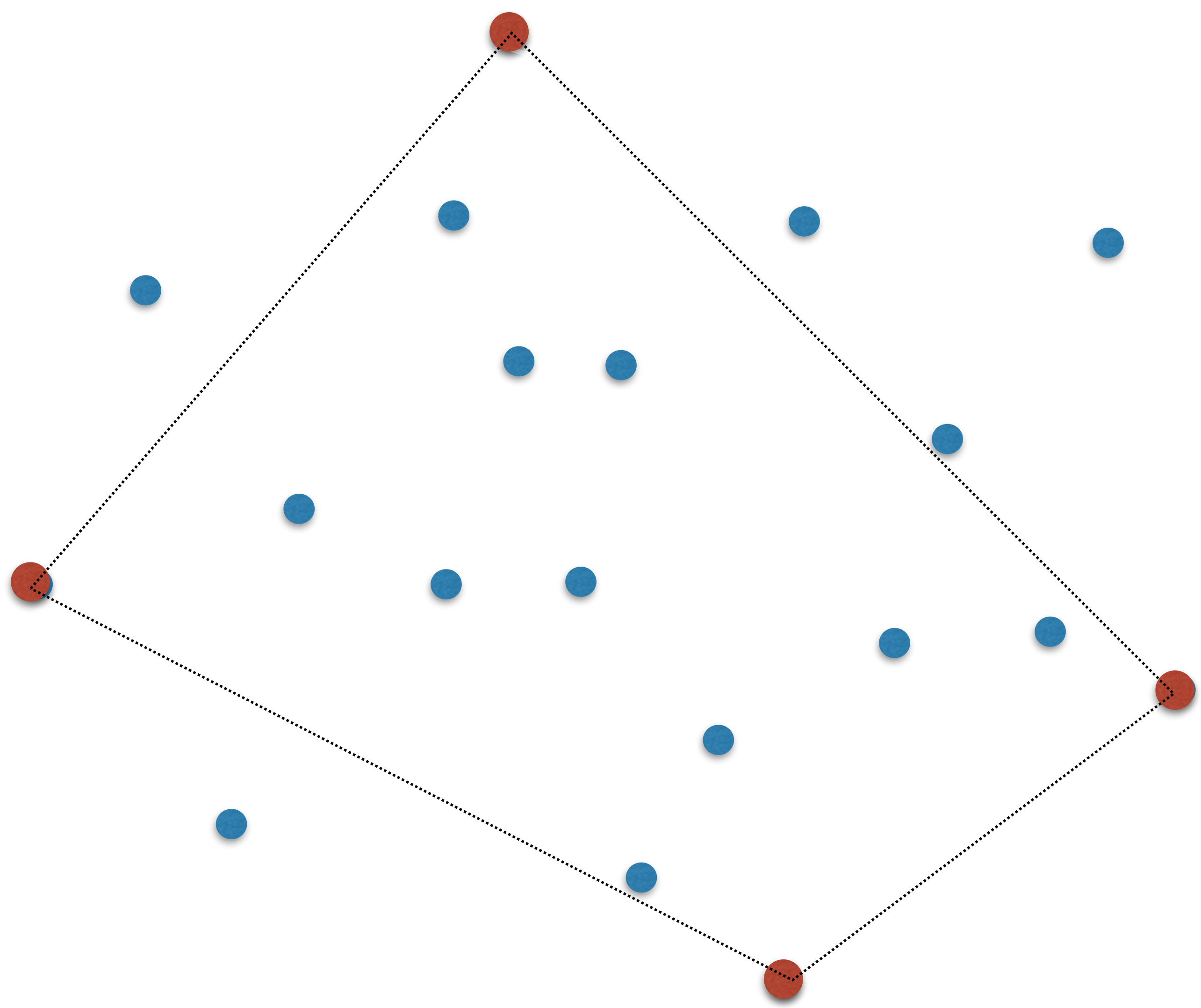
- point a comes before point b in radial order around p if a is right of pb
- rightOf() is to radial sort what < is to sort

And final note..

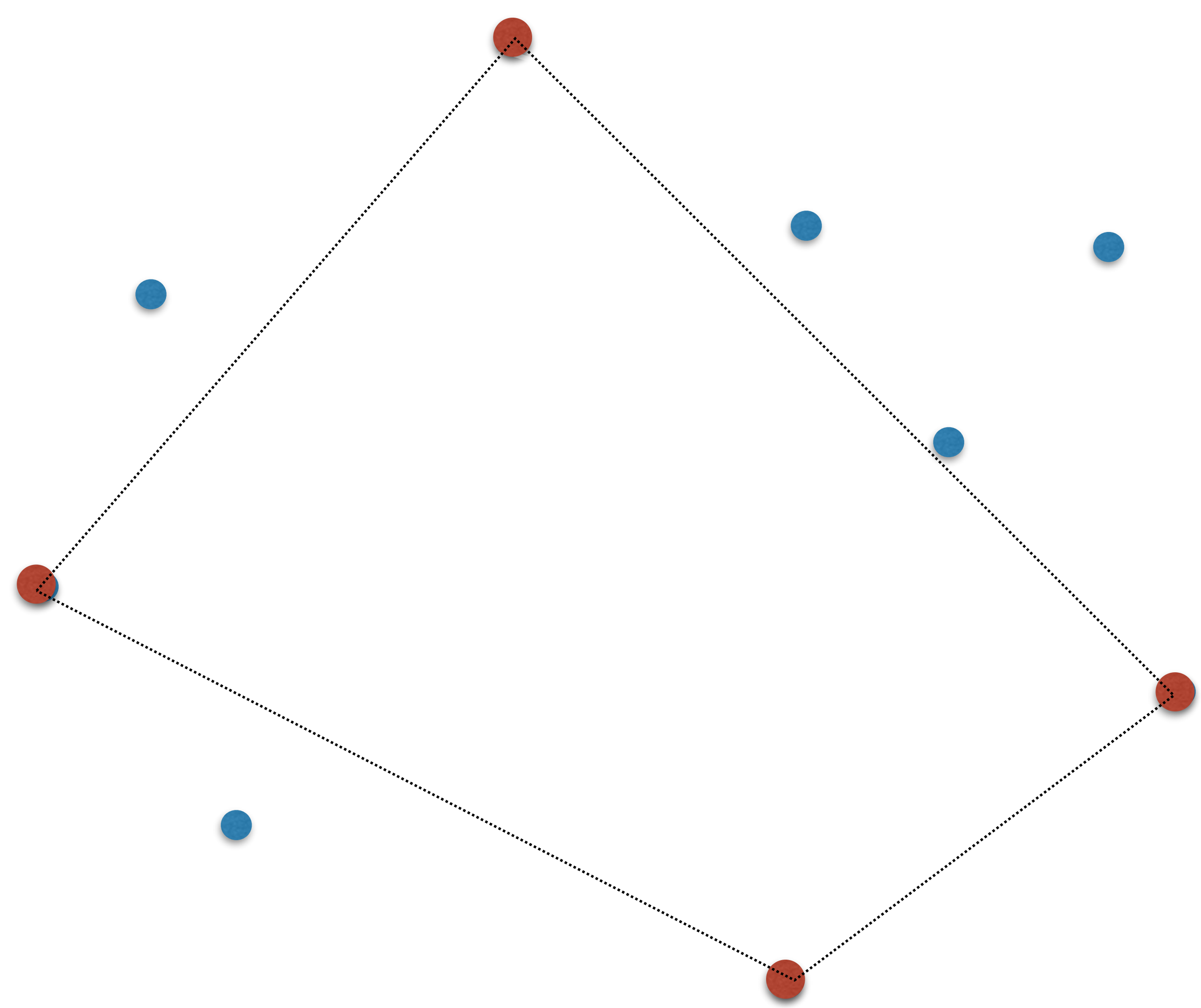
It is possible to speed up Graham scan



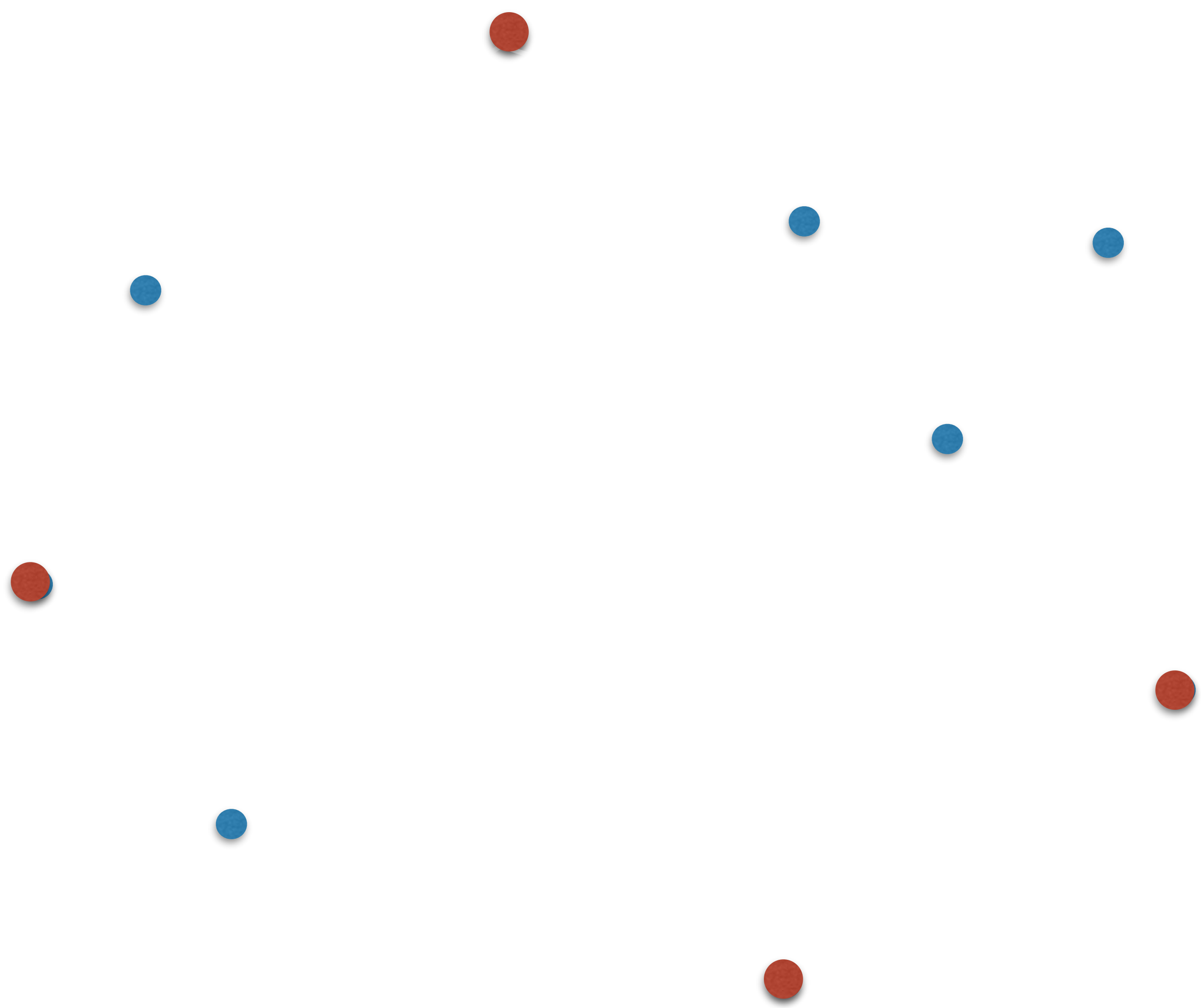
It is possible to speed up Graham scan



It is possible to speed up Graham scan

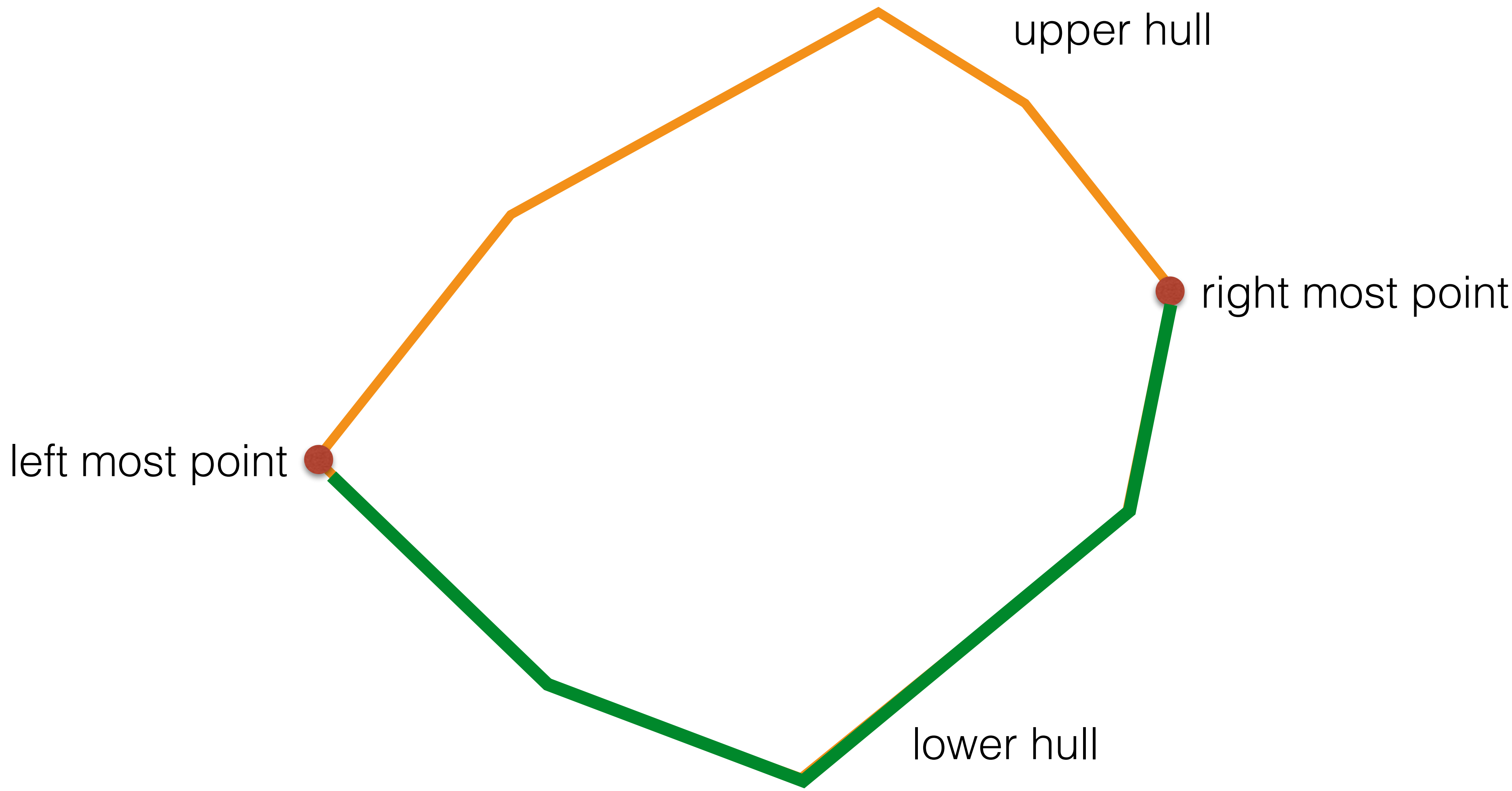


It is possible to speed up Graham scan



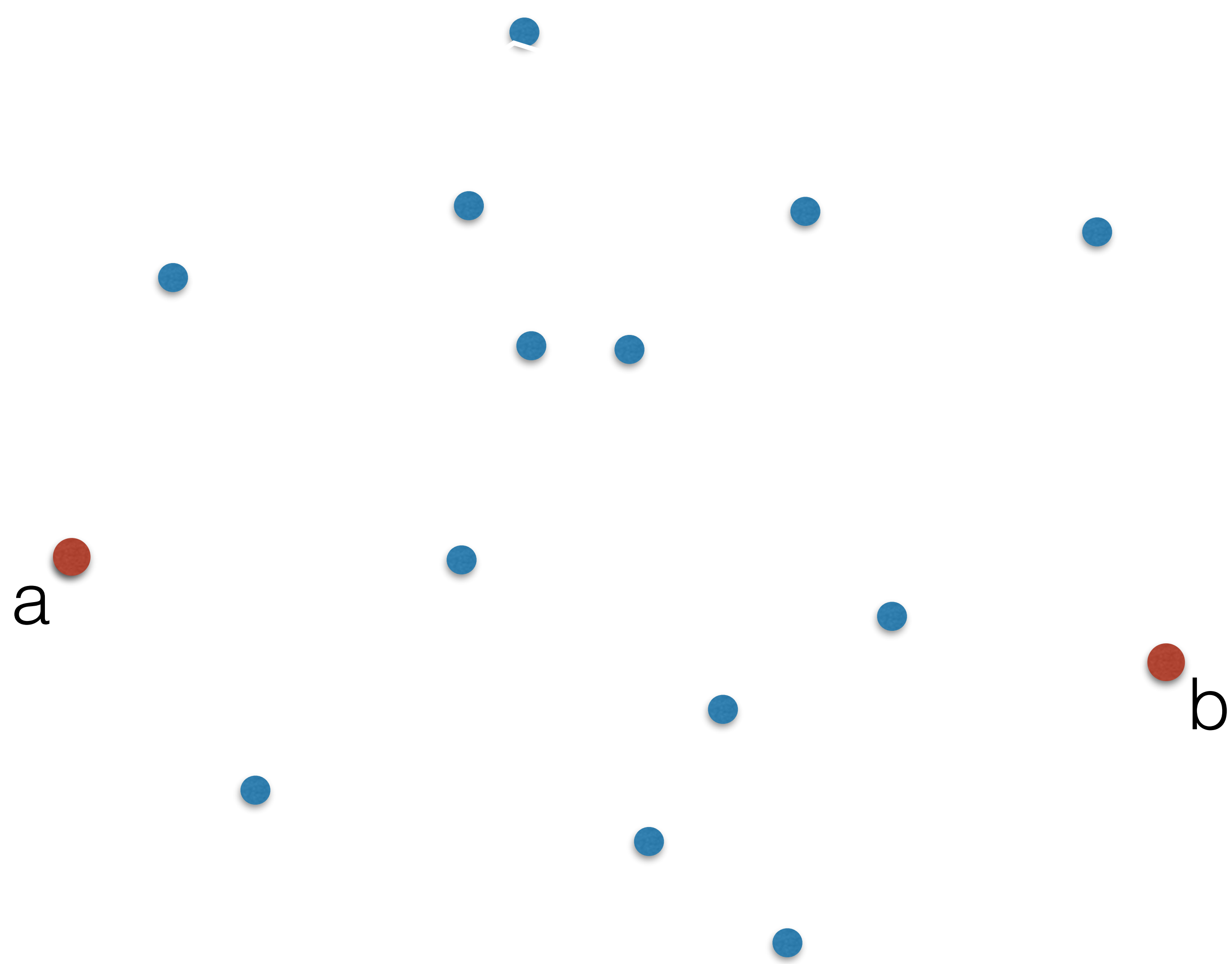
Algorithm: Quickhull

Convex polygons: Properties



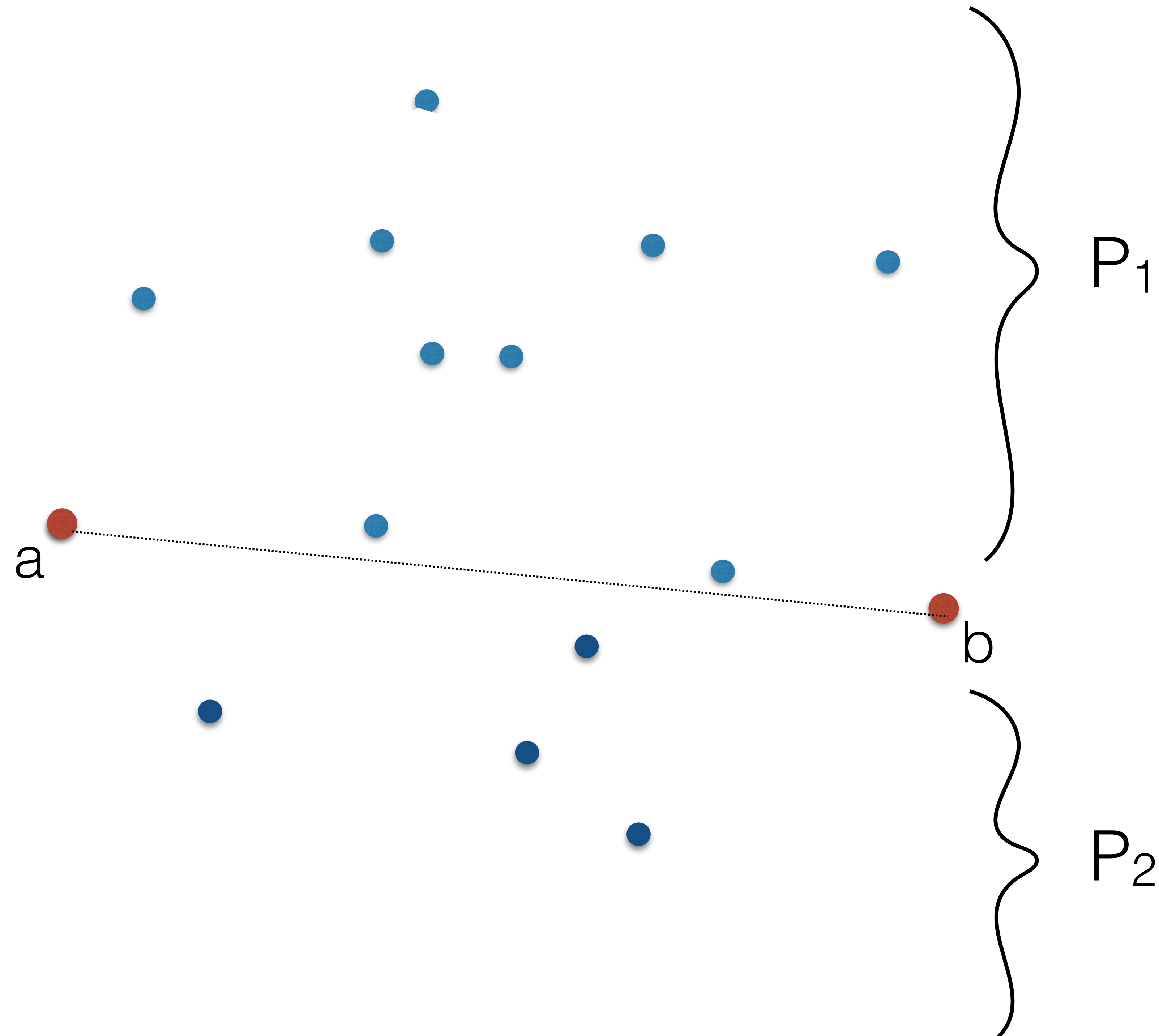
Quickhull (late 1970s)

- Similar to Quicksort
- Idea: start with 2 extreme points



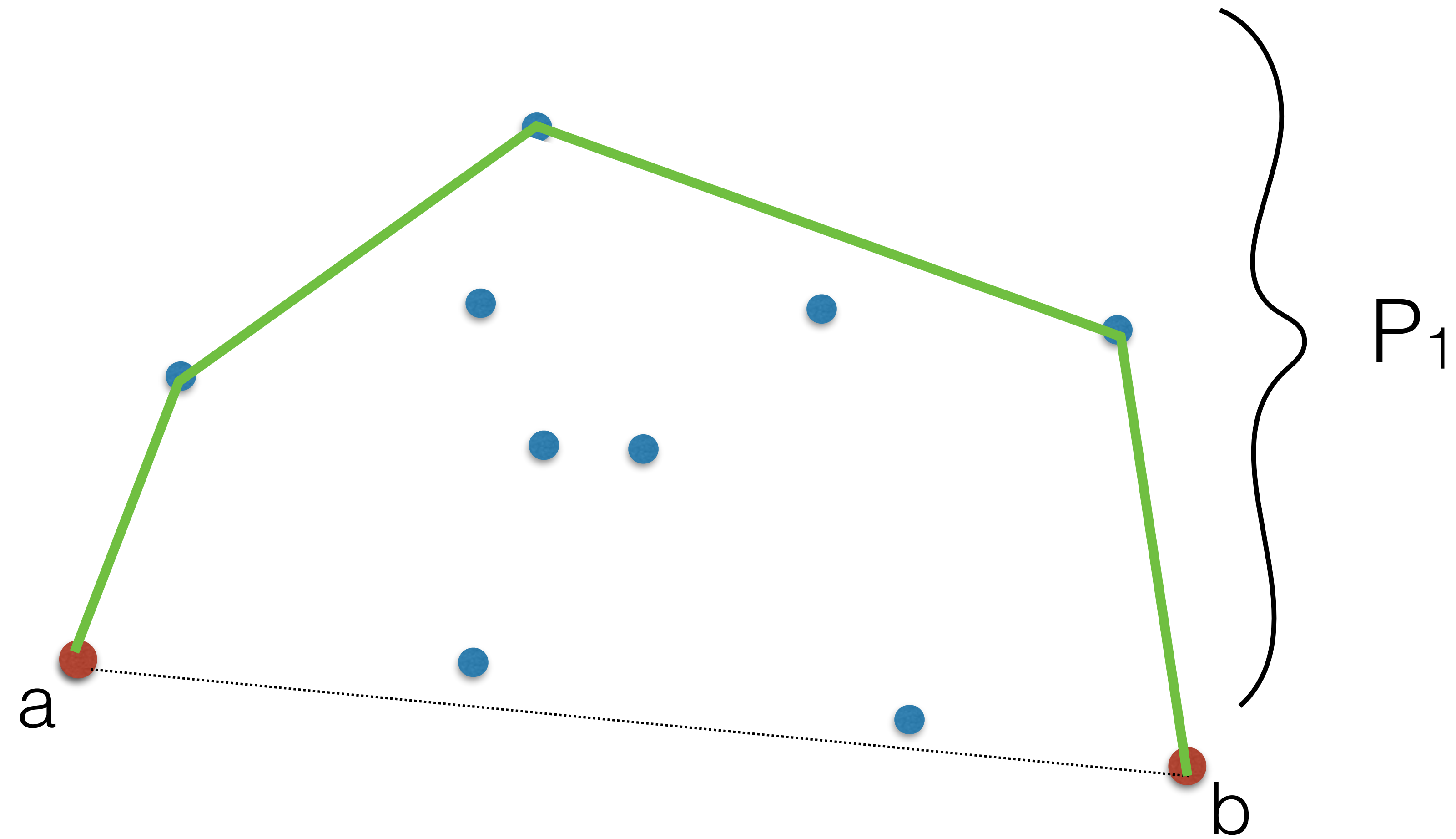
Quickhull

- CH = CH of P_1 (upper hull) + CH of P_2 (lower hull)



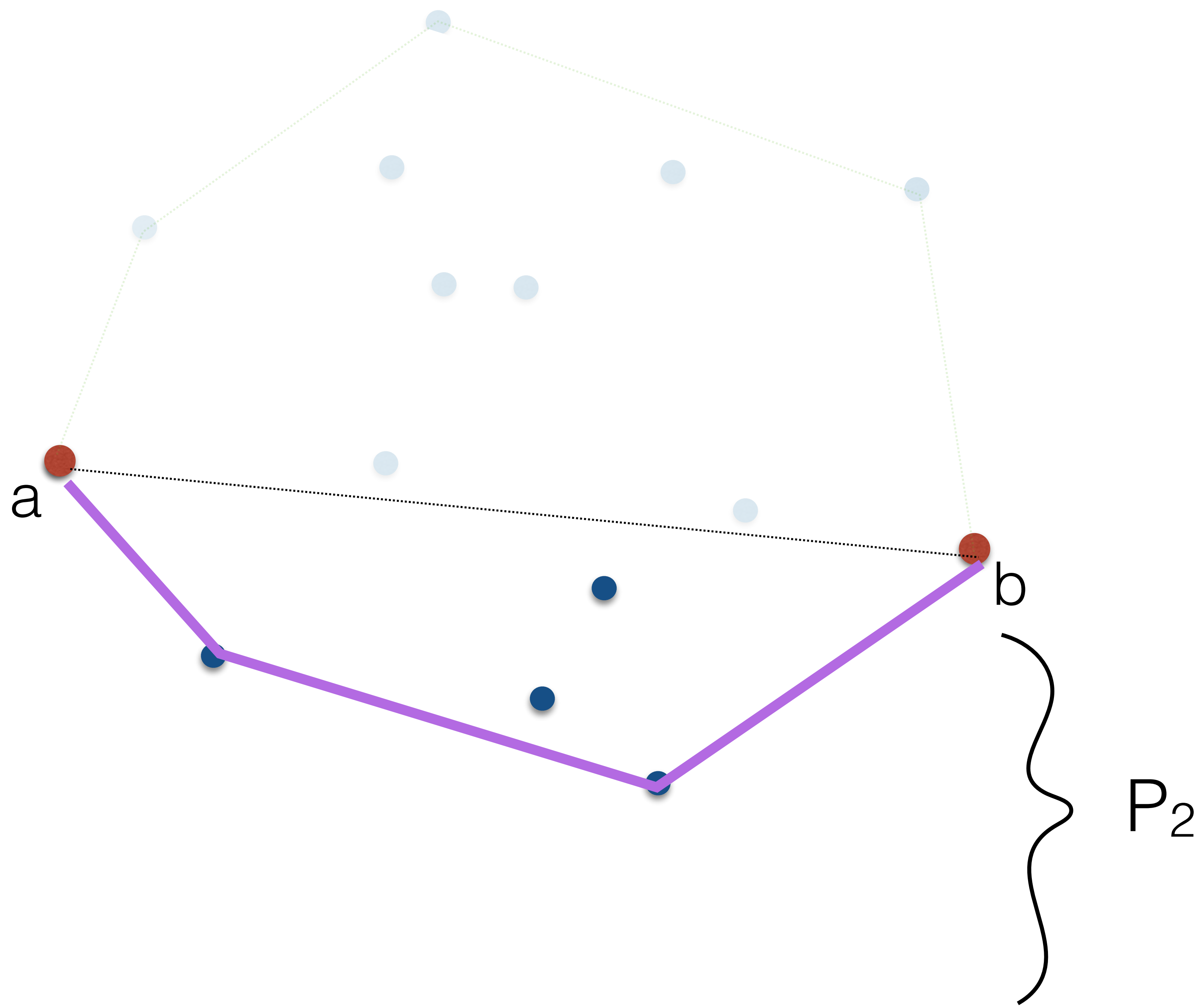
Quickhull

- CH = CH of P_1 (upper hull) + CH of P_2 (lower hull)



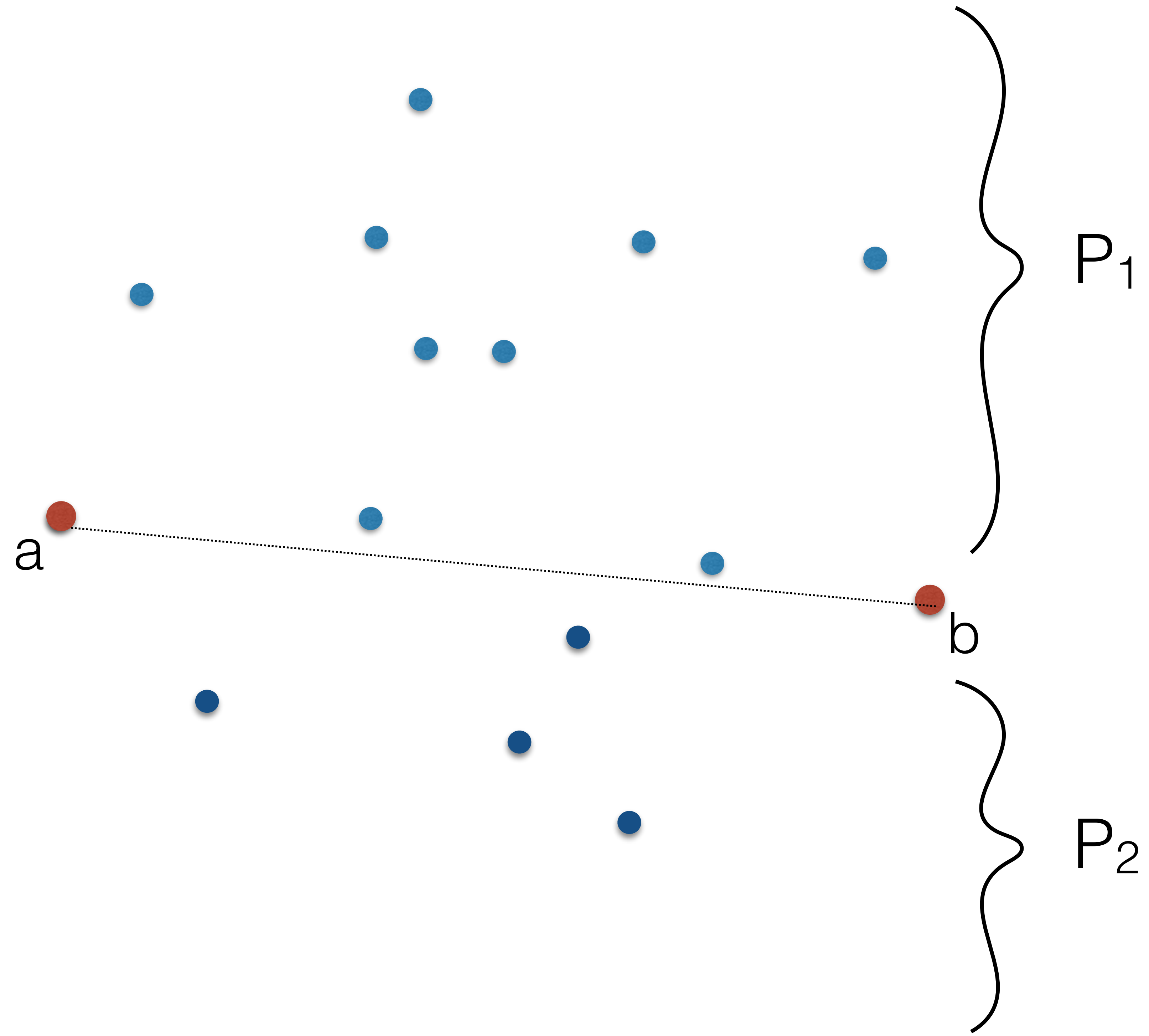
Quickhull

- CH = CH of P_1 (upper hull) + CH of P_2 (lower hull)



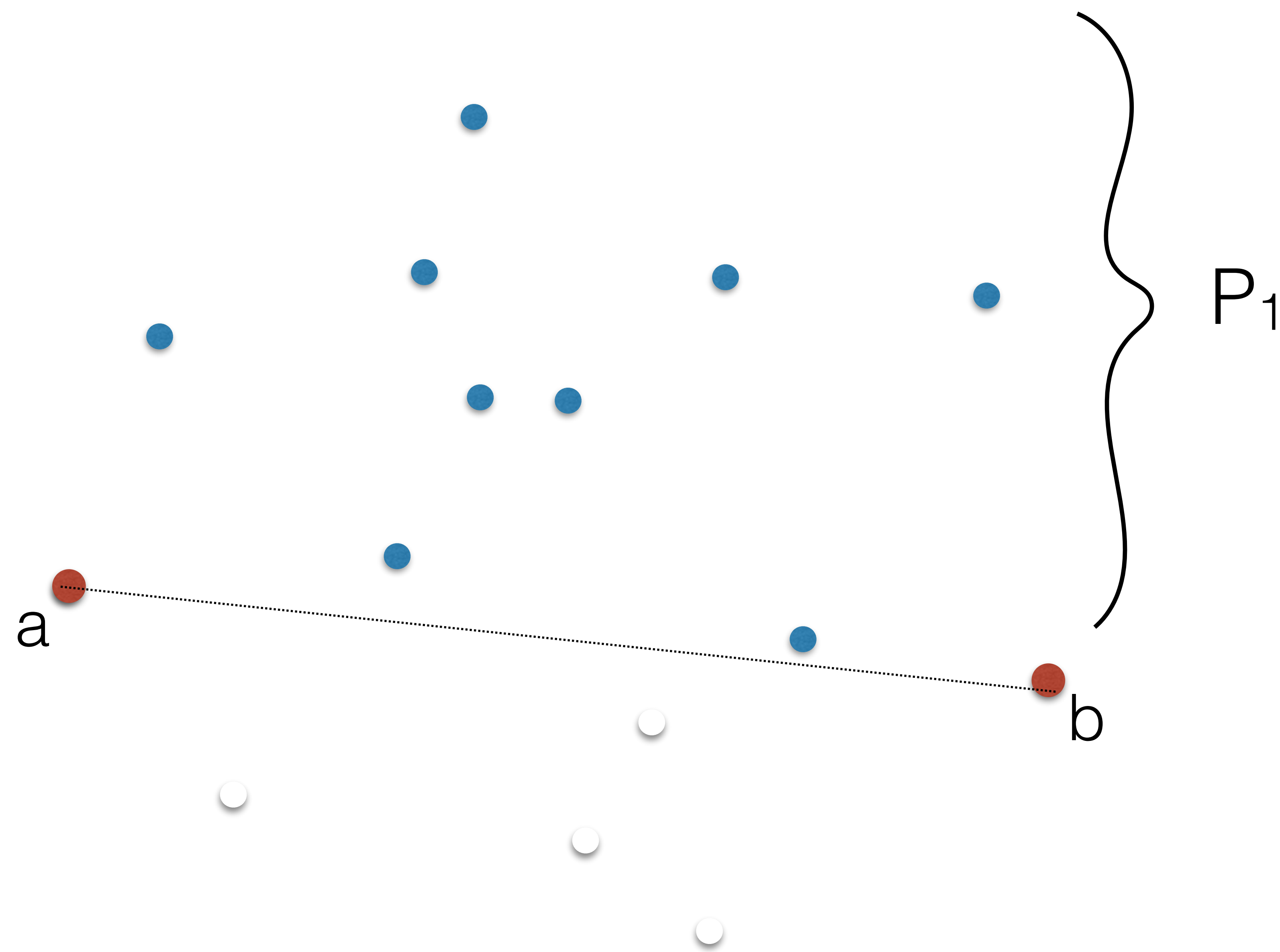
Quickhull

- We'll find the CH(P_1) and CH(P_2) separately



Quickhull

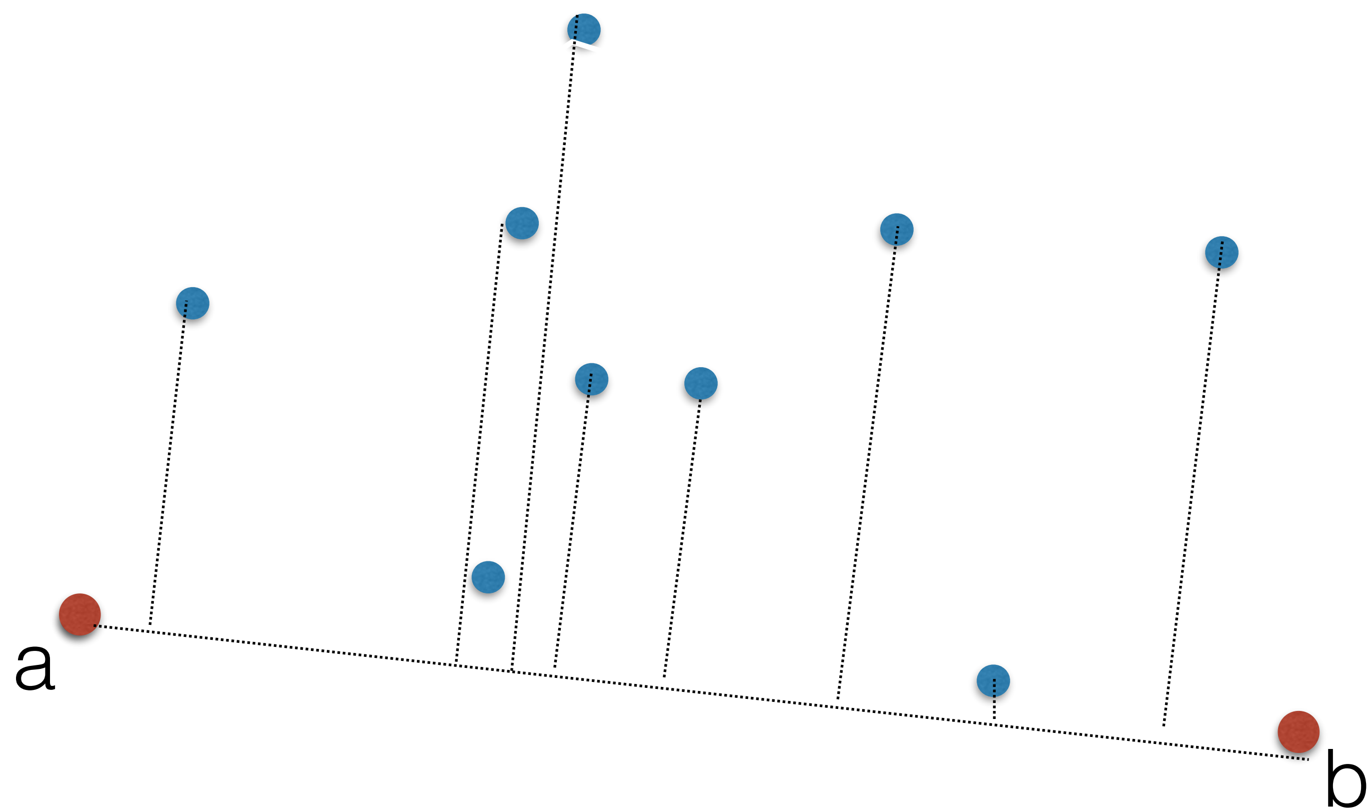
- First let's focus on P1



Quickhull

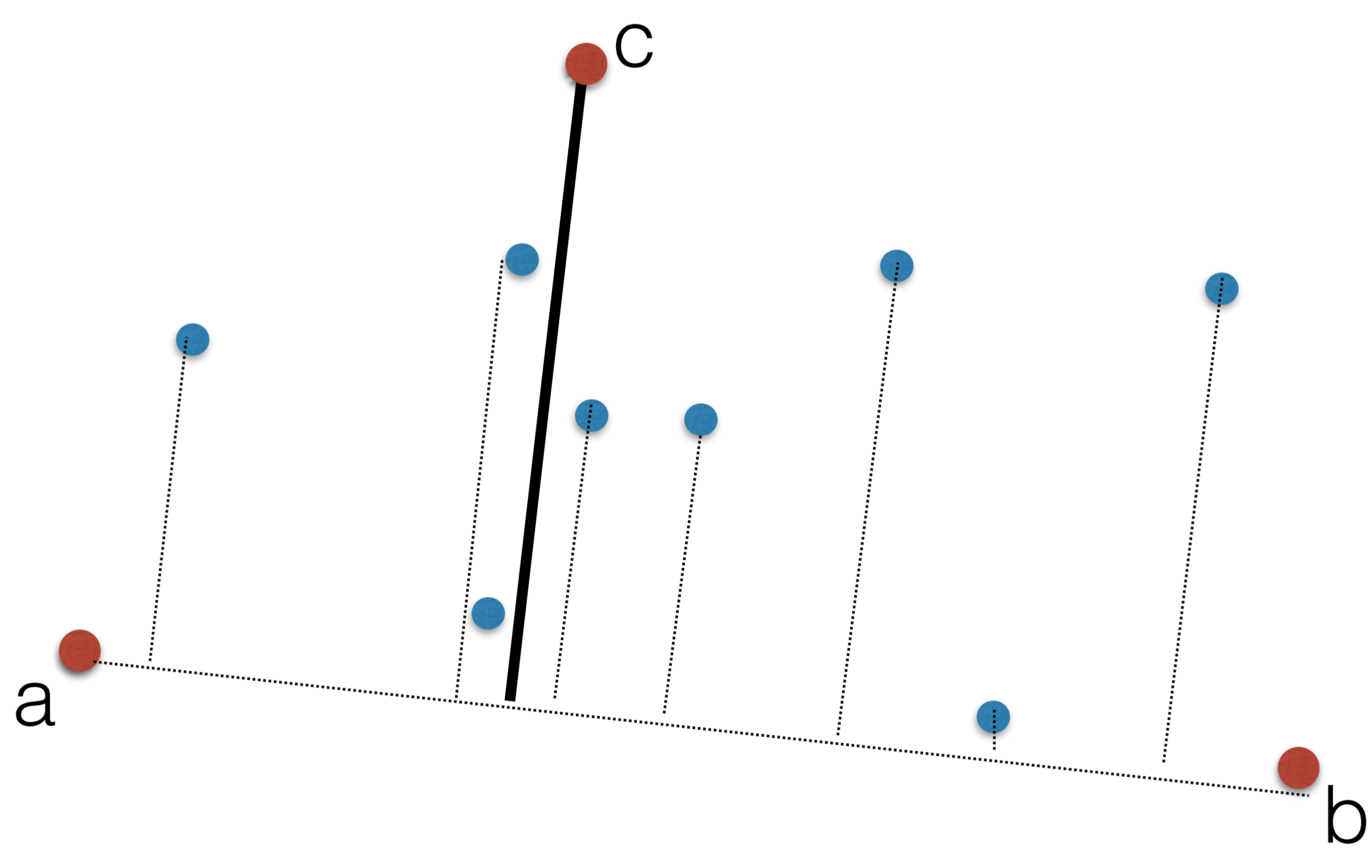
- For all points p in P1: compute dist(p, ab)

assume no collinear points (for now)



Quickhull

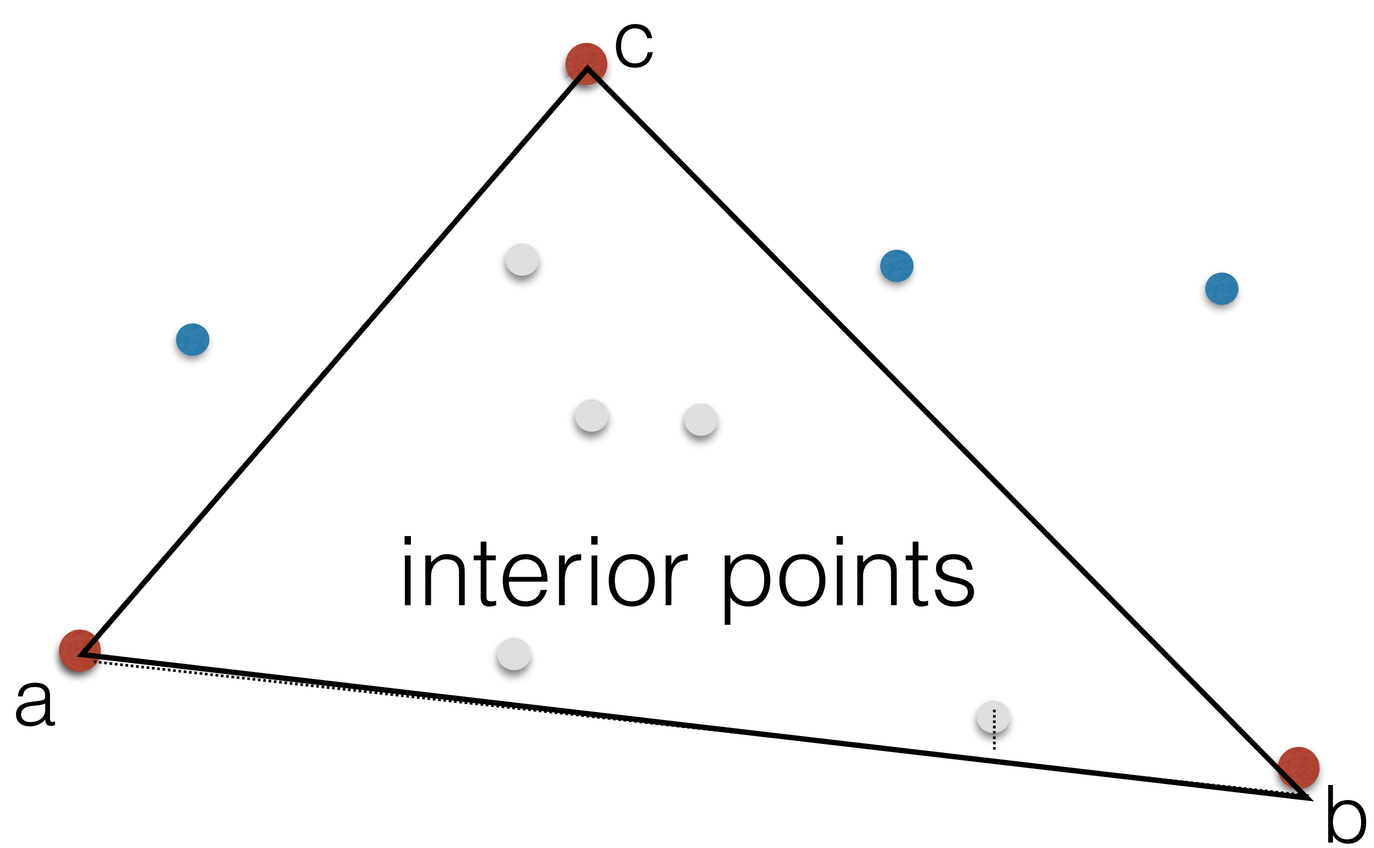
- For all points p in P1: compute dist(p, ab)
- Find the point c with largest distance (i.e. furthest away from ab)



- Claim: Point c must be an extreme point and thus on the CH of P1. (Why?)

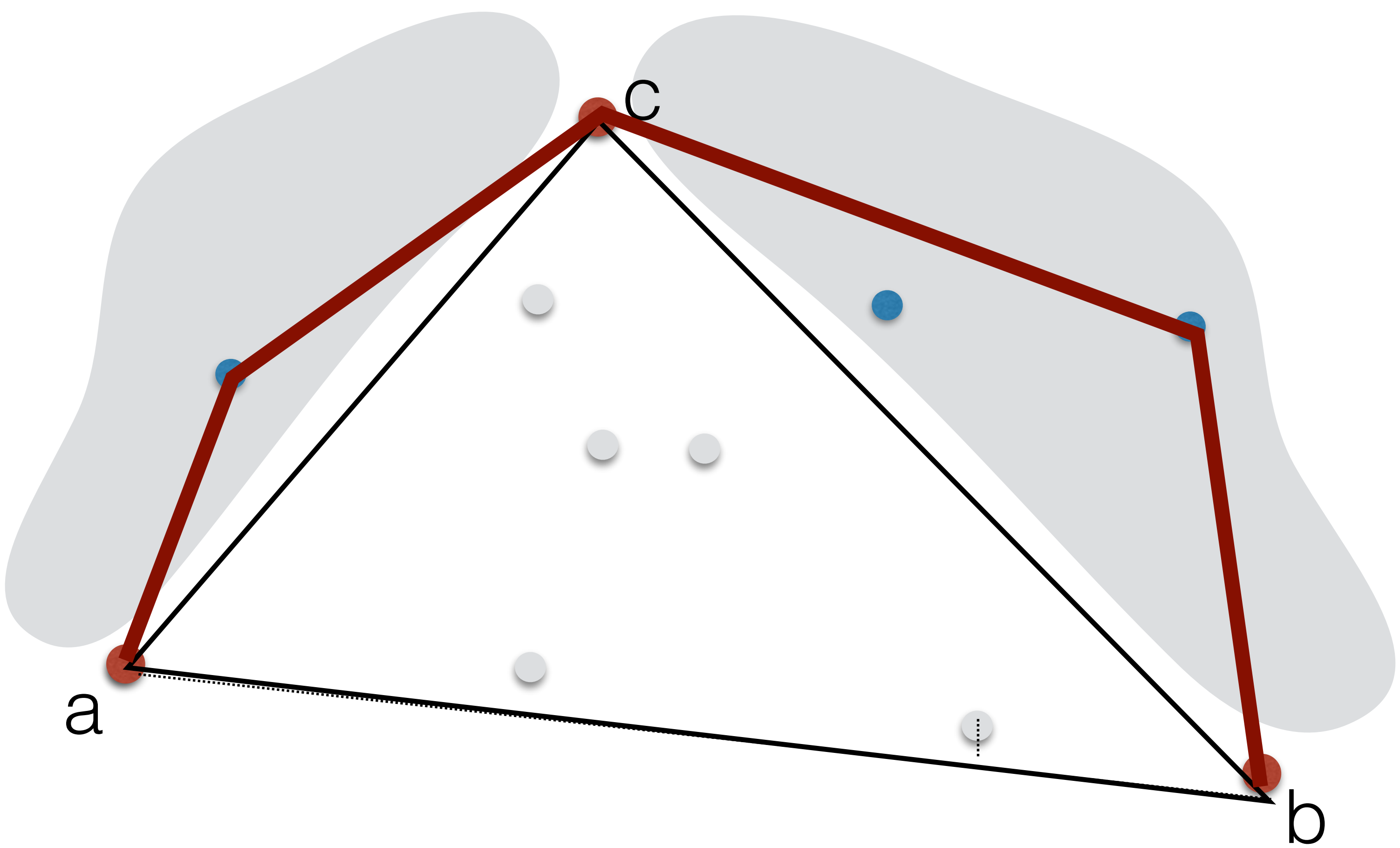
Quickhull

- Discard all points inside triangle abc



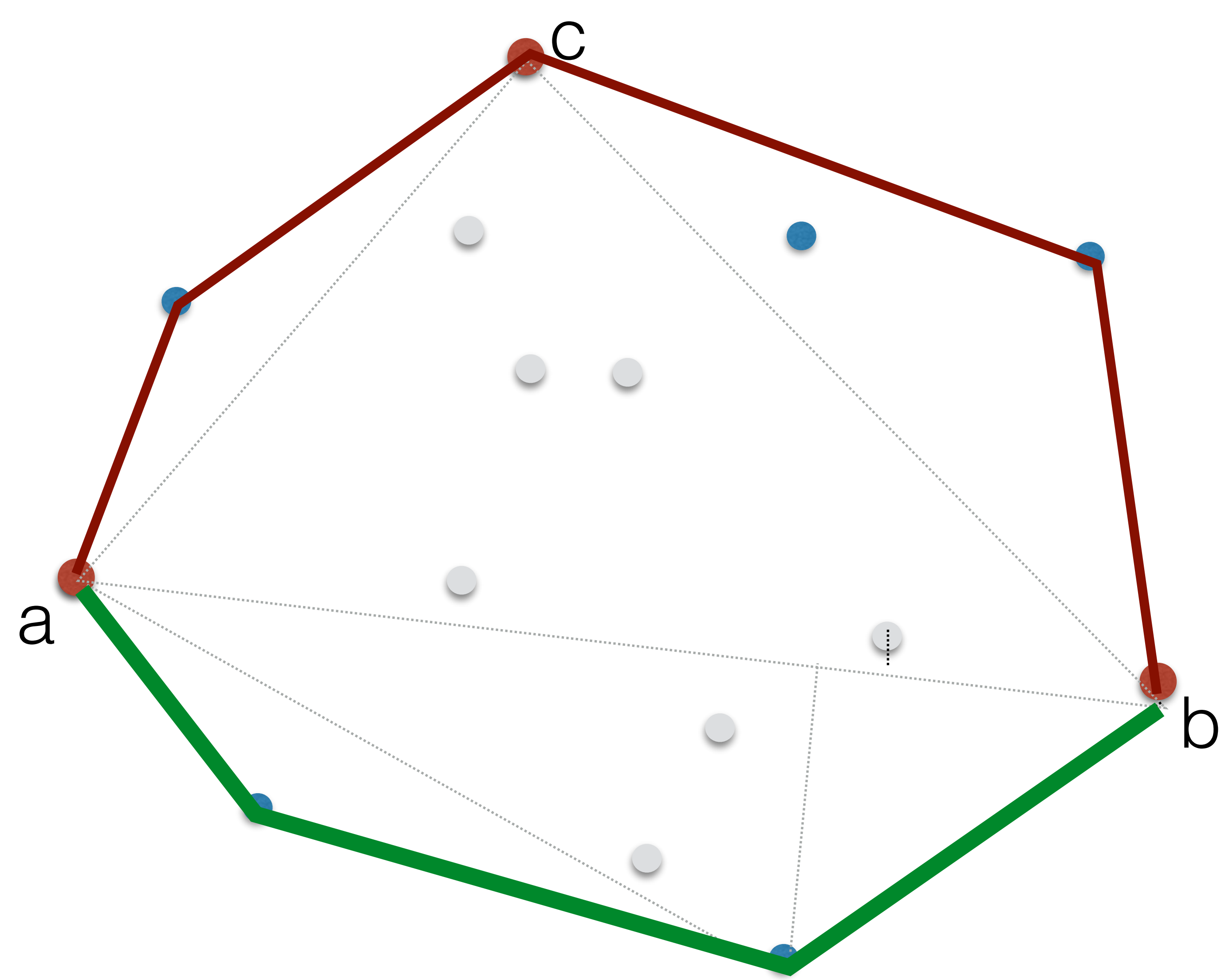
Quickhull

- **Recurse** on the points left of ac and right of bc

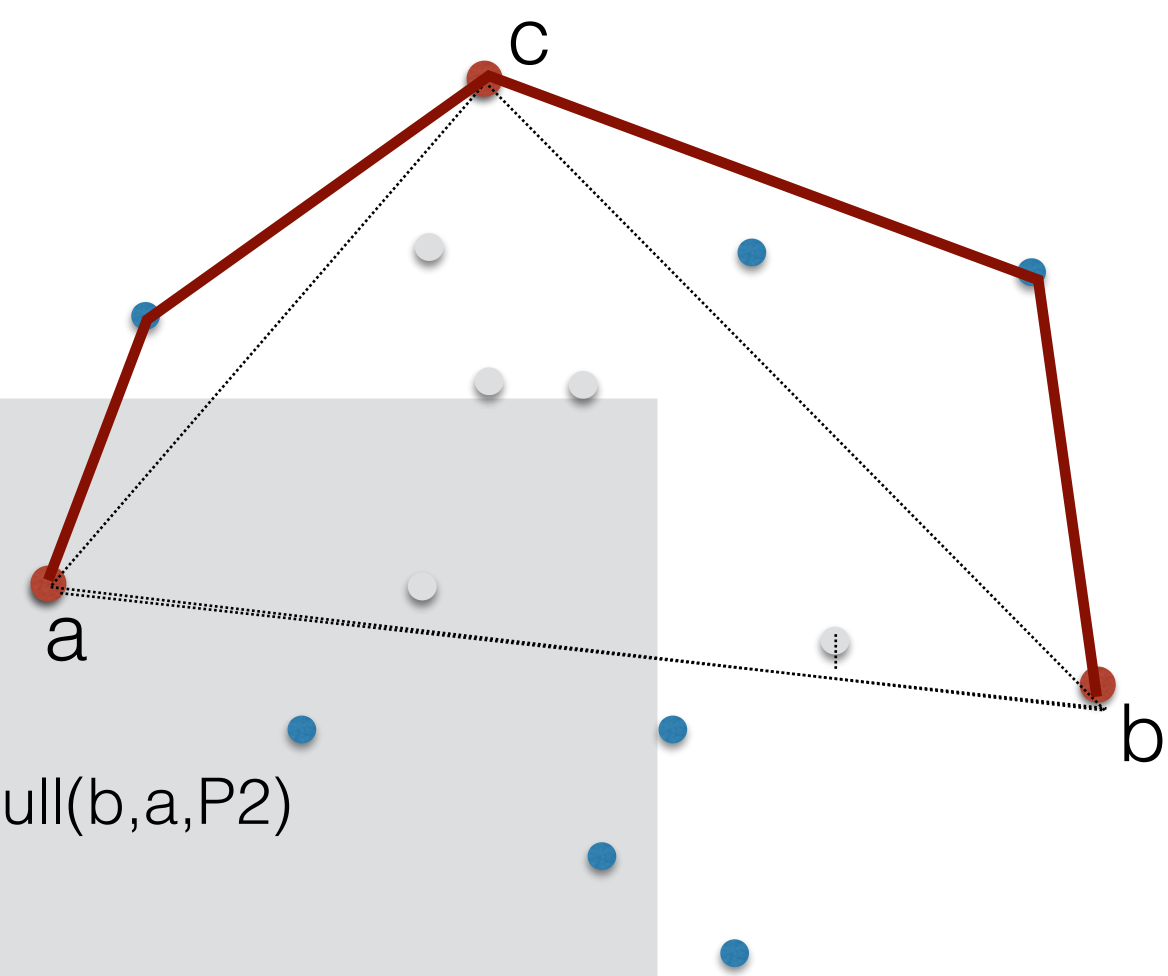


Quickhull

- Compute CH of P_2 similarly



Quickhull



- **Quickhull (P)**
  - find a, b
  - partition P into P1, P2
  - return a + Quickhull(a,b, P1) + b + Quickhull(b,a,P2)

- **Quickhull(a, b, P)**

  //P is a set of points all on the left of ab. return the upper hull of P

  - if P empty => return {}
  - for each point p in P: compute its distance to ab
  - let c = point with max distance
  - let P1 = points to the left of ac
  - let P2 = points to the left of cb
  - return Quickhull(a, c, P1) + c + Quickhull(c, b, P2)

Classwork

- Simulate Quickhull on an arbitrary small set of points (assume no collinear-ities)
- Analysis:
    - Write a recurrence relation for its running time
    - What/when is the worst case running time ?
    - What/when is the best case running time ?
- Argue that Quickhull's average complexity is O(n) on points that are uniformly distributed

Summary

- Brute force: $O(n^3)$
- Gift wrapping: $O(kn)$
    - output-size sensitive: $O(n)$ best case, $O(n^2)$ worst case
    - ✦ by Chand and Kapur [1970]. Extends to 3D and to arbitrary dimensions; for many years was the primary algorithm for higher dimensions
- Graham scan: $O(n \lg n)$, but
    - not output-sensitive
    - does not transfer to 3d
- Quickhull: $O(n^2)$

- Next time
    - incremental, divide-and-conquer
    - $\Omega(n \lg n)$ lower bound