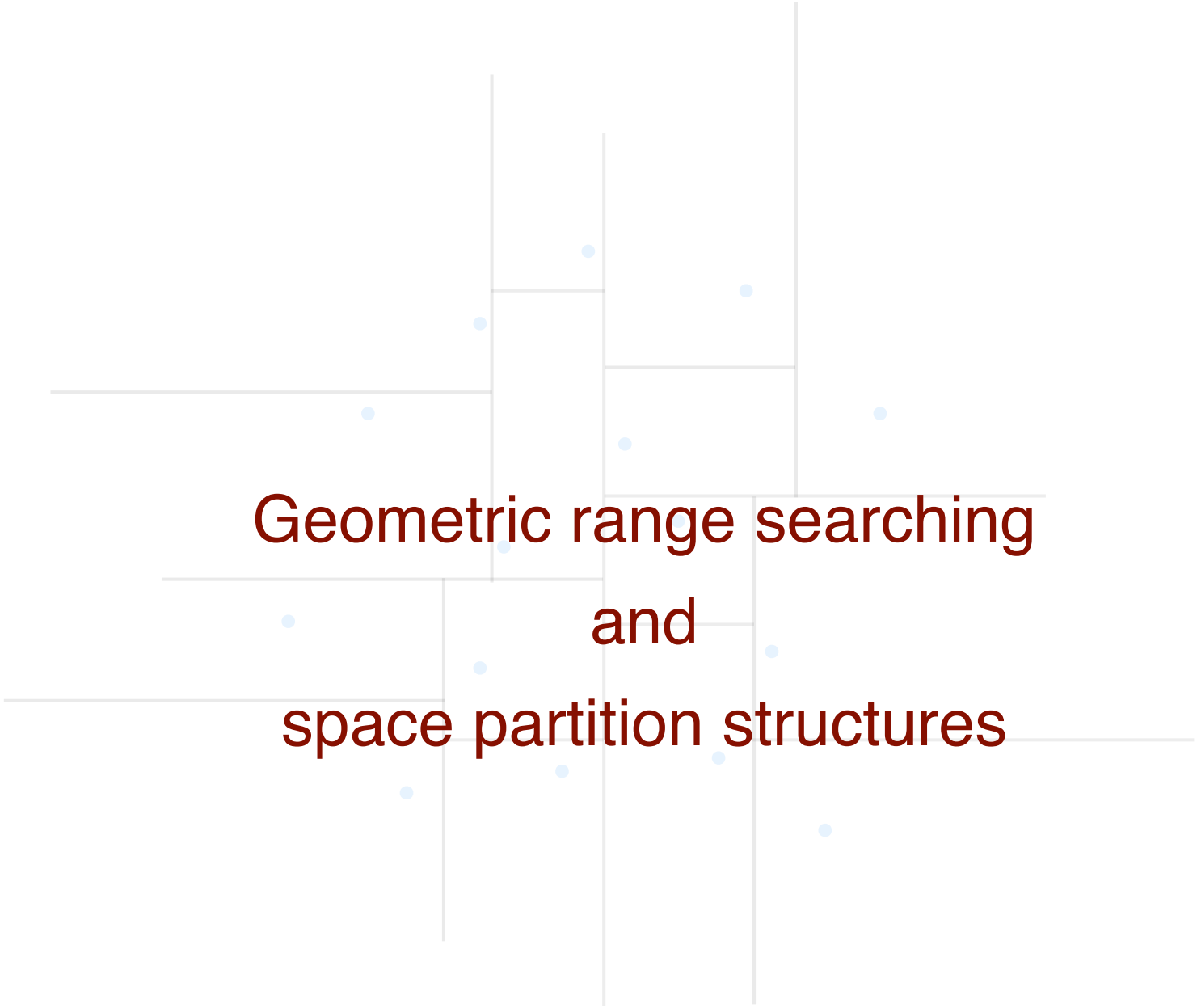


Geometric range searching and space partition structures

Computational Geometry [csci 3250]
Laura Toma
Bowdoin College

Where we are

"Global" problems

- closest pair
- convex hull
- intersections
- ..

Geometric search problems

- range searching
- nearest neighbor
- k-nearest neighbor
- find all roads within 1km of current location
- ..

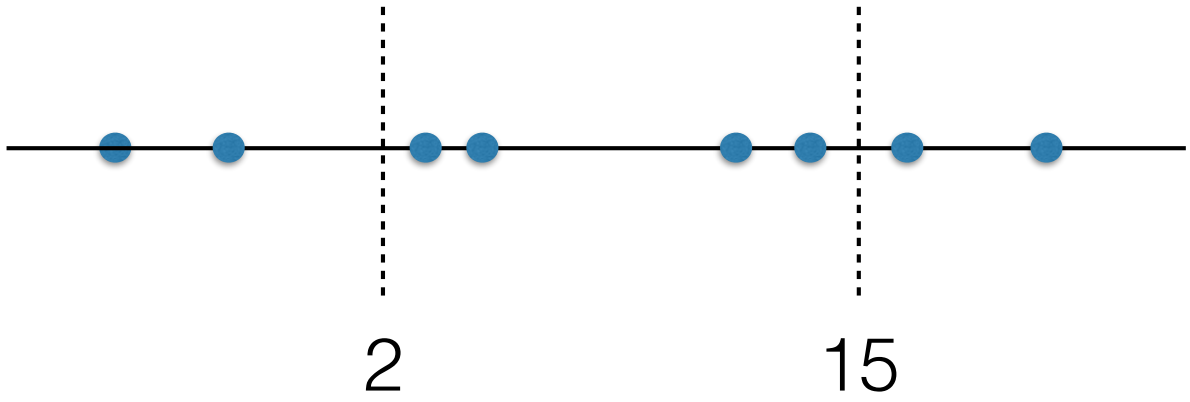
Techniques

- divide-and-conquer
- incremental
- space decomposition
- plane sweep
- …

next

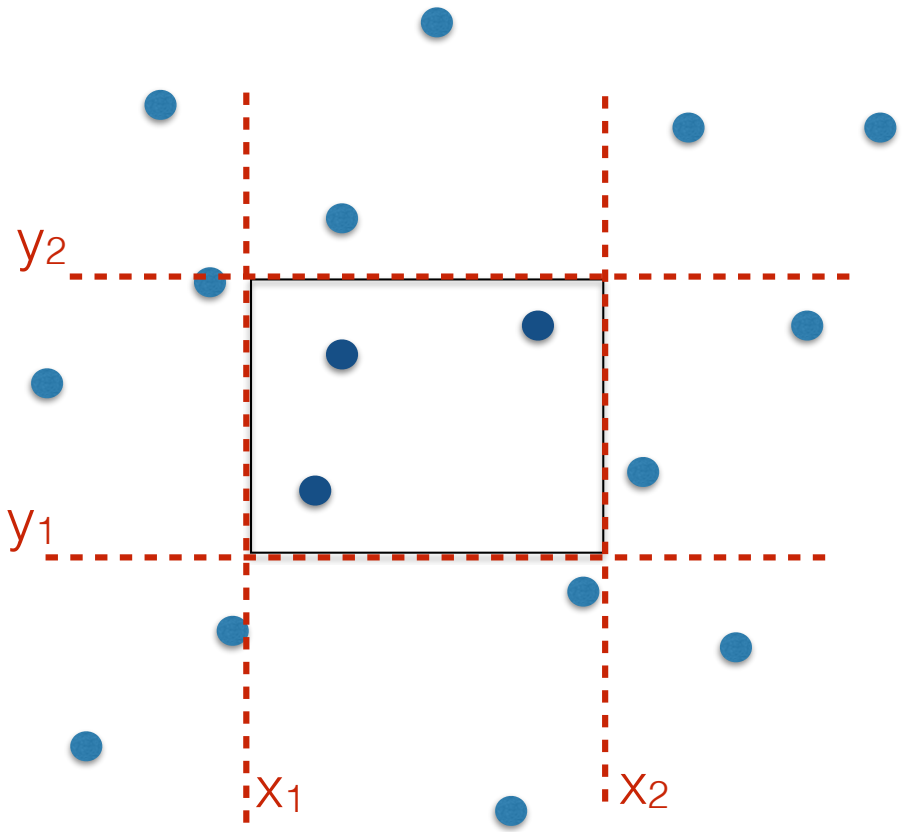
1D Range searching

Given a set of n points on the real line and an arbitrary interval [a,b], find all points in [a,b]

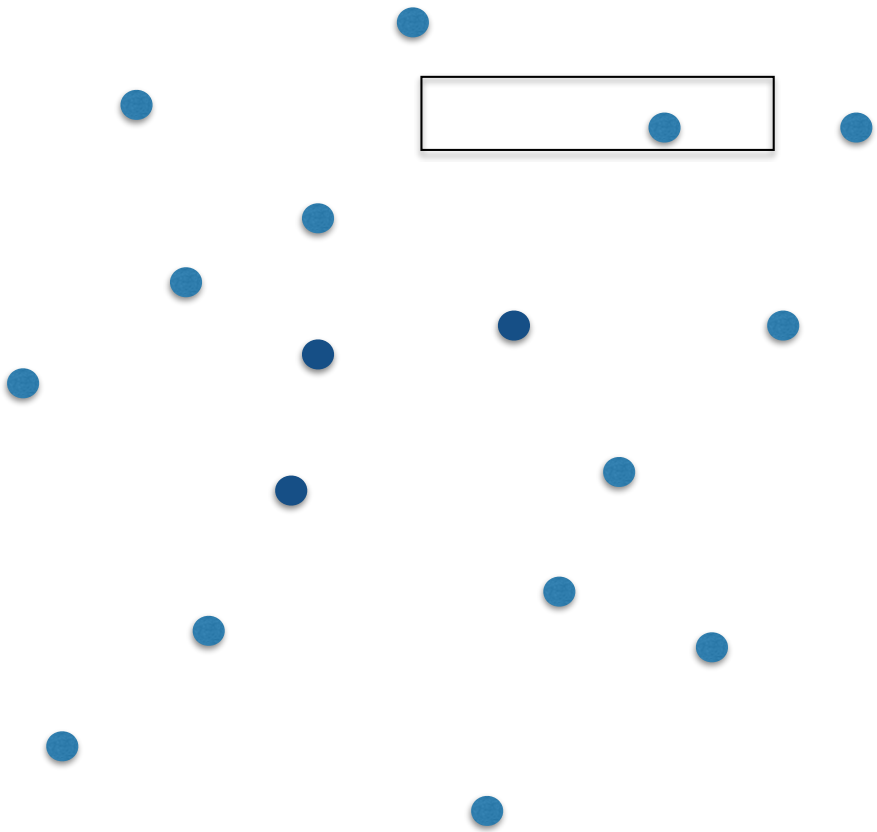


2D Range searching

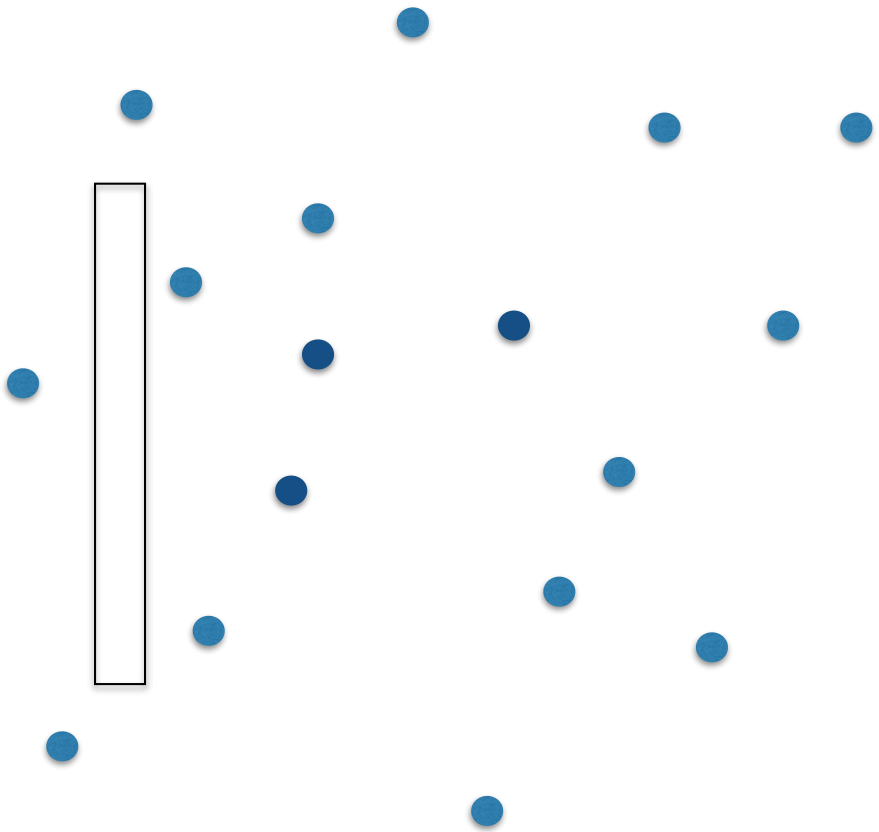
Given a set of n points in 2D and an arbitrary range $[x_1, x_2] \times [y_1, y_2]$, find all points in this range



2D Range searching



2D Range searching

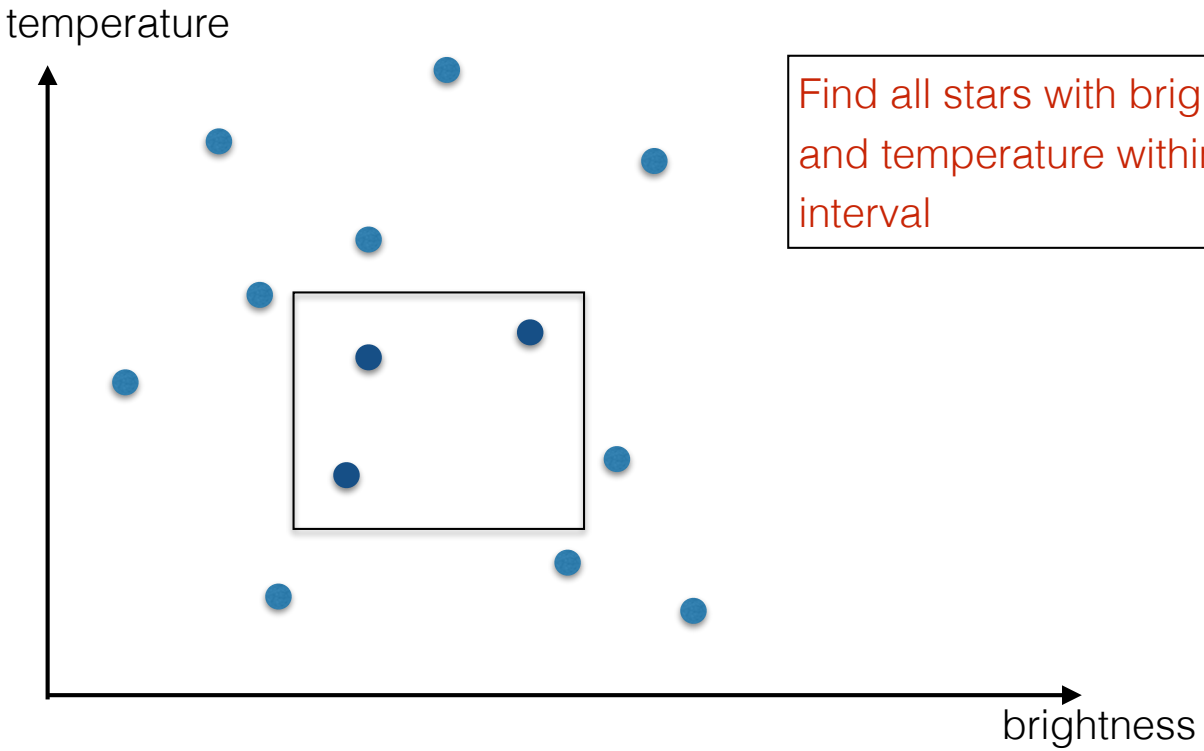


Why range searching?

Searching is a fundamental operation. This is the multi-dimensional version of the “report all points in this interval”

Interestingly, it comes up in settings that are not geometrical

e.g. Database of stars. A star = (brightness, temperature,……)

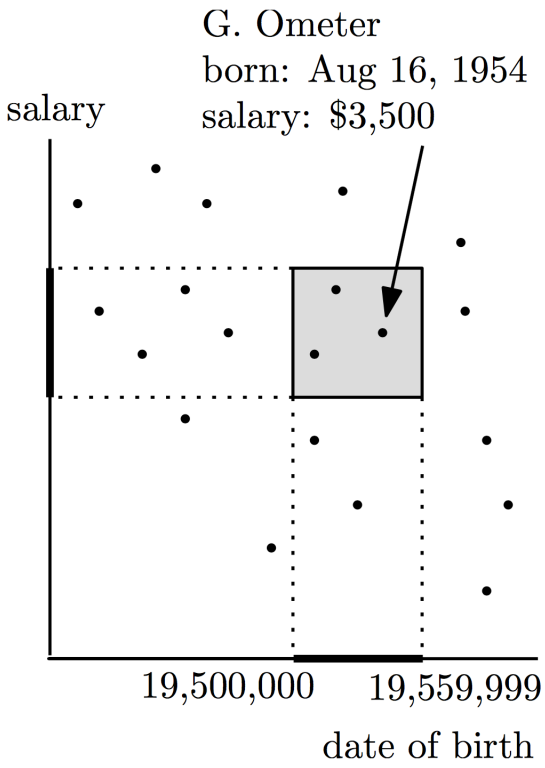


Find all stars with brightness and temperature within a given interval

Why range searching?

e.g. Database of employees. An employee = (age, salary,……)

A database query may ask for all employees with age between a_1 and a_2, and salary between s_1 and s_2

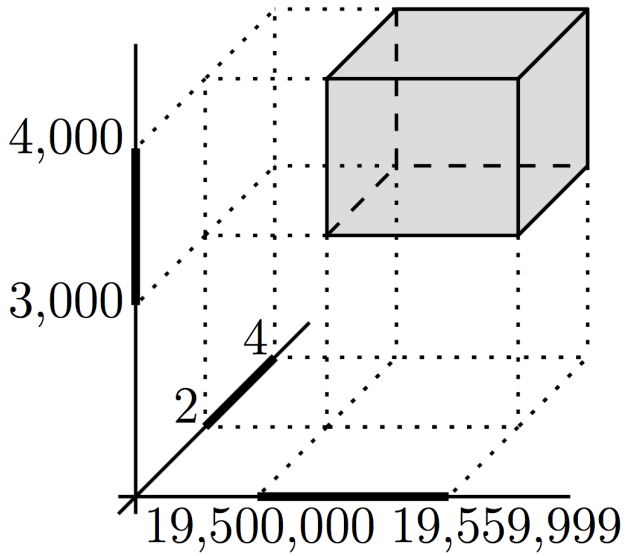


screenshot from Mark van Kreveld slides at http://www.cs.uu.nl/docs/vakken/ga/slides5a.pdf

Why range searching?

3d-range searching, etc

Example of a 3-dimensional (orthogonal) range query: children in $[2, 4]$, salary in $[3000, 4000]$, date of birth in $[19,500,000\ ,\ 19,559,999]$

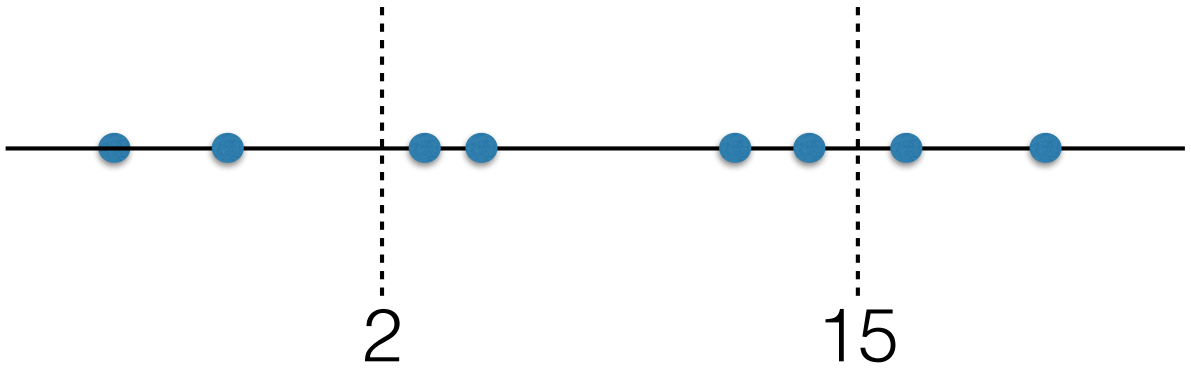


screenshot from Mark van Kreveld slides at http://www.cs.uu.nl/docs/vakken/ga/slides5a.pdf

To see what we can expect in 2D, let's look at 1D range searching

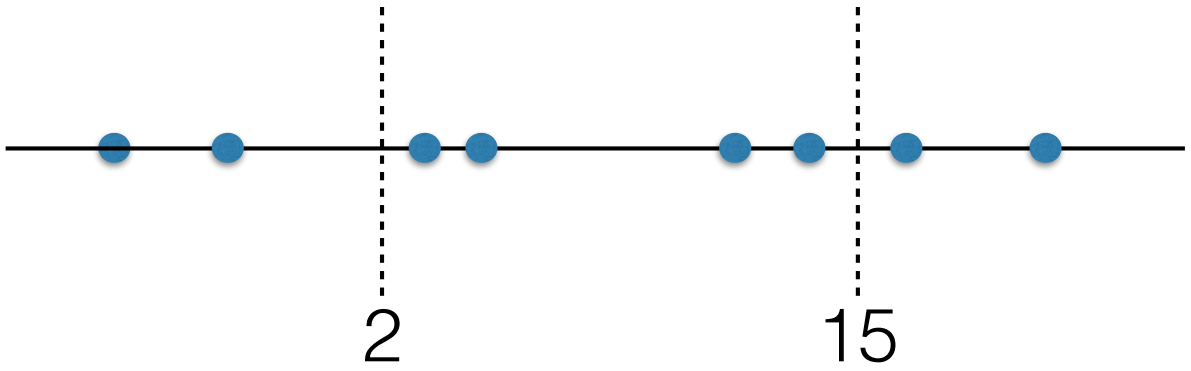
1D Range searching

Given a set of n points on the real line and an interval [a,b], find all points in [a,b]



find all values in [2,15]

1D Range searching

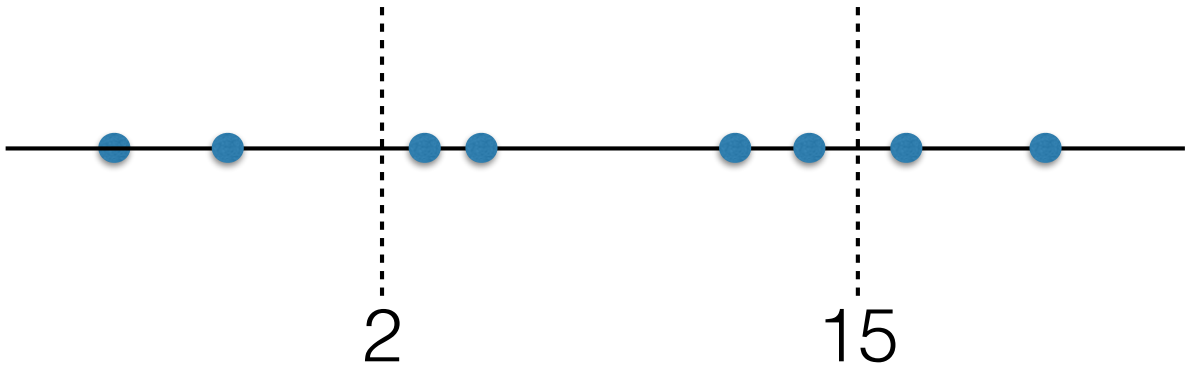


find all values in [2,15]

Assume first that the points are **fixed**, i.e. don't change.

What can we do?

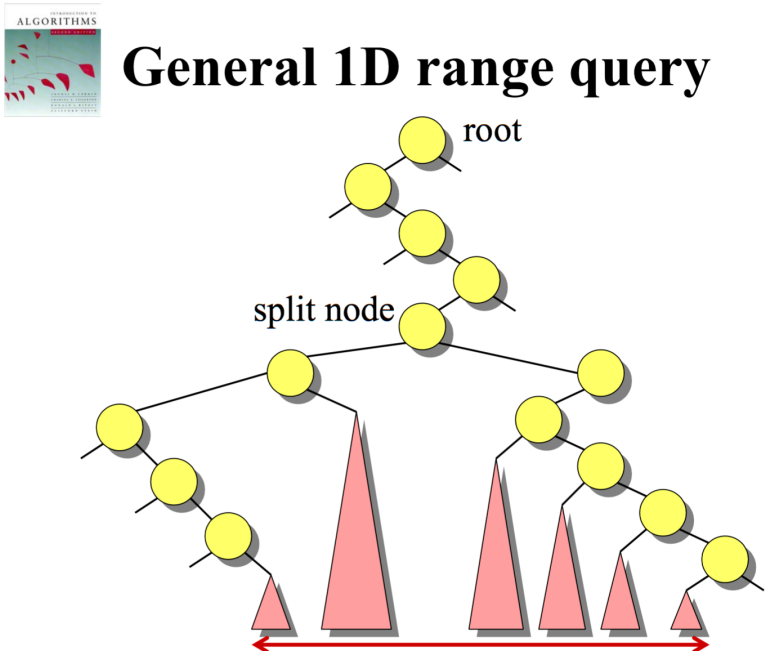
1D Range searching



Assume now that the points are **dynamic**, i.e. in addition to range queries, we want to be able to **insert** and **delete** points.

1D Range searching

- A set of n points in 1D can be pre-processed into a BBST such that:
  - Build: $O(n \lg n)$
  - Space: $\Theta(n)$
  - Range queries: $O(\lg n + k)$
  - Dynamic: points can be inserted/deleted in $O(\lg n)$



The k points in the range sit in O(lg n) subtrees

1D

- A set of n points in 1D can be pre-processed into a BBST such that:
  - Build: $O(n \lg n)$
  - Space: $\Theta(n)$
  - Range queries: $O(\lg n + k)$
  - Dynamic: points can be inserted/deleted in $O(\lg n)$

2D

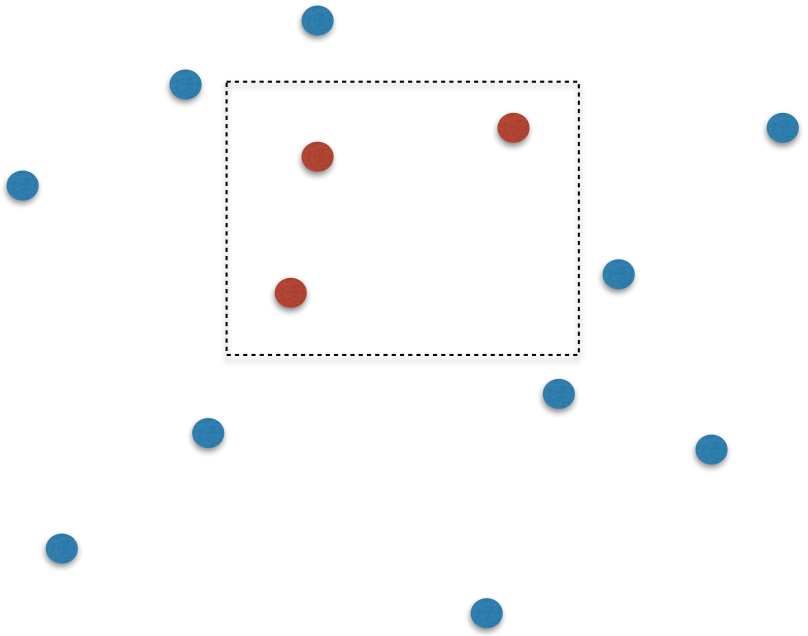
- A set of n points in 2D can be pre-precessed in a ??2d-BBST??? such that
  - Build: $O(n \lg n)$
  - Space: $\Theta(n)$
  - Range queries: $O(\lg^2 n + k)$

These bounds would be nice

But how?

2D Naive Approach

- n: size of the input (number of points)
- k: size of output (number of points inside range)



Points are static or dynamic?

We'll assume static (it's hard enough)

The naive approach: just traverse and check in O(n)

Analysis:

Build: none

Space: none

2d range query: $O(n)$ ⟵ We want $O(\lg^2 n + k)$

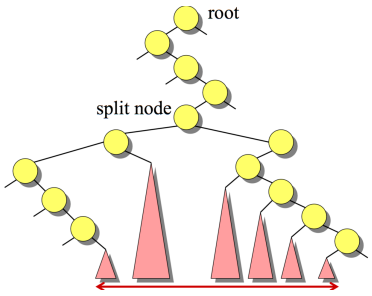
query $[x_1, x_2] \times [y_1, y_2]$

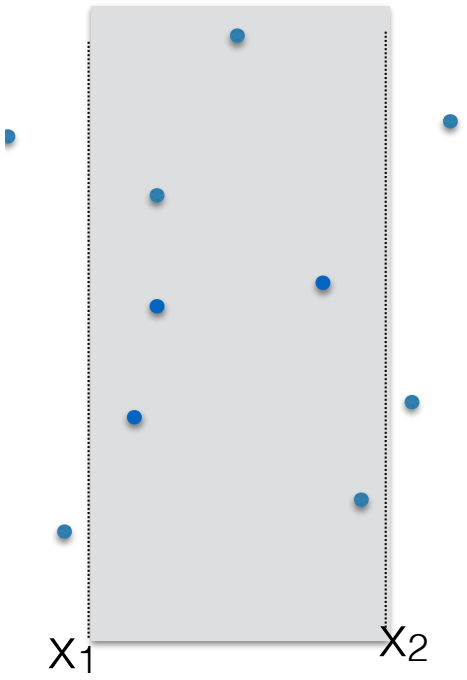
How about this:

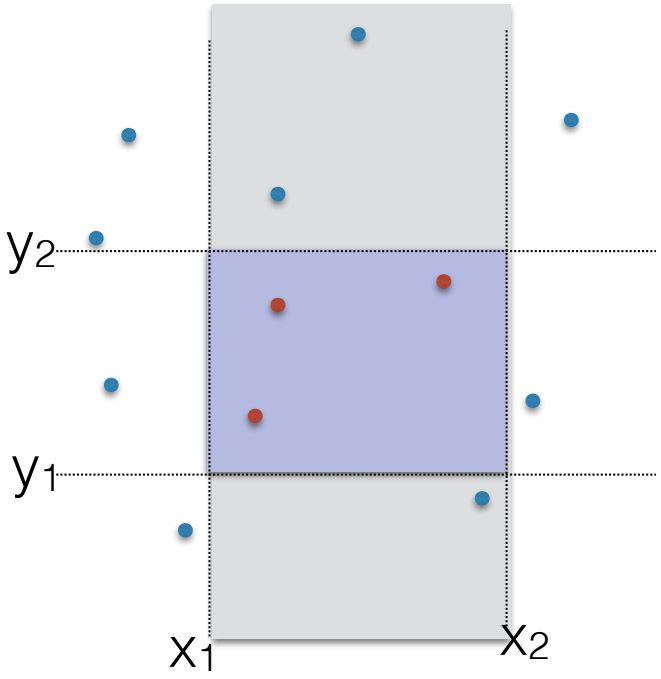
1. Find all points with the x-coords in $[x_1, x_2]$

2. Out of these points, find all points with the y-coord in $[y_1, y_2]$

BST in (x,y) order

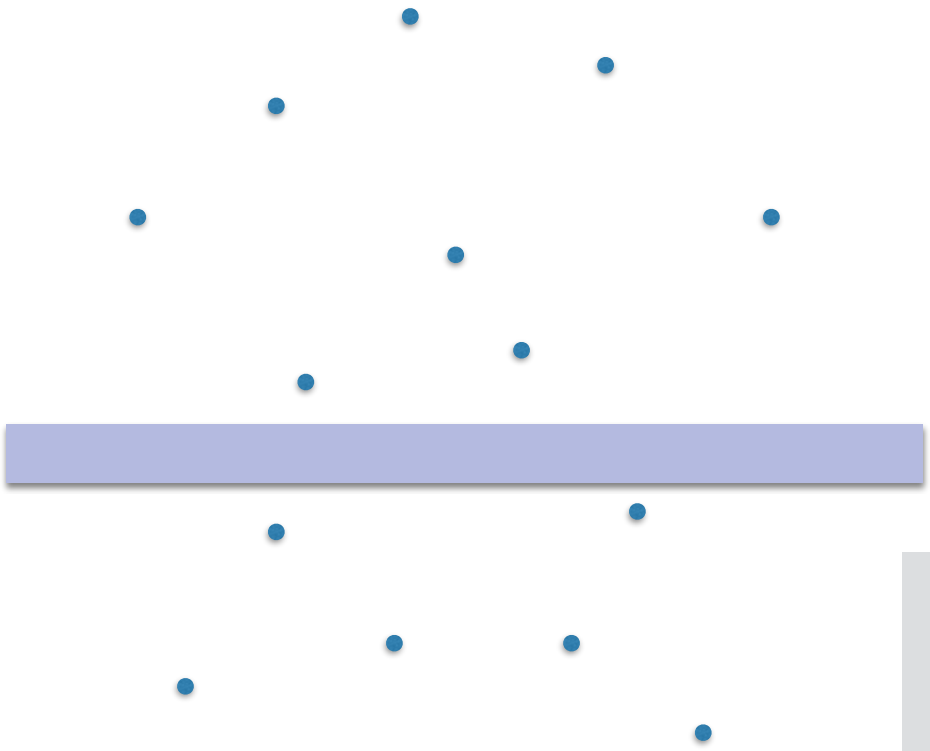






- $O(\lg n + k')$ to find the points in the vertical strip, and then $O(k')$ to traverse and find the points in $[y_1, y_2]$

- The problem is that the nb. of points in $[x_1, x_2]$ can be large, and the nb. of points in $[x_1, x_2] \times [y_1, y_2]$ may be small
- Worst case: $k' = n, k = 0$



to find the points in this range takes $O(\lg n + n) = O(n)$ time

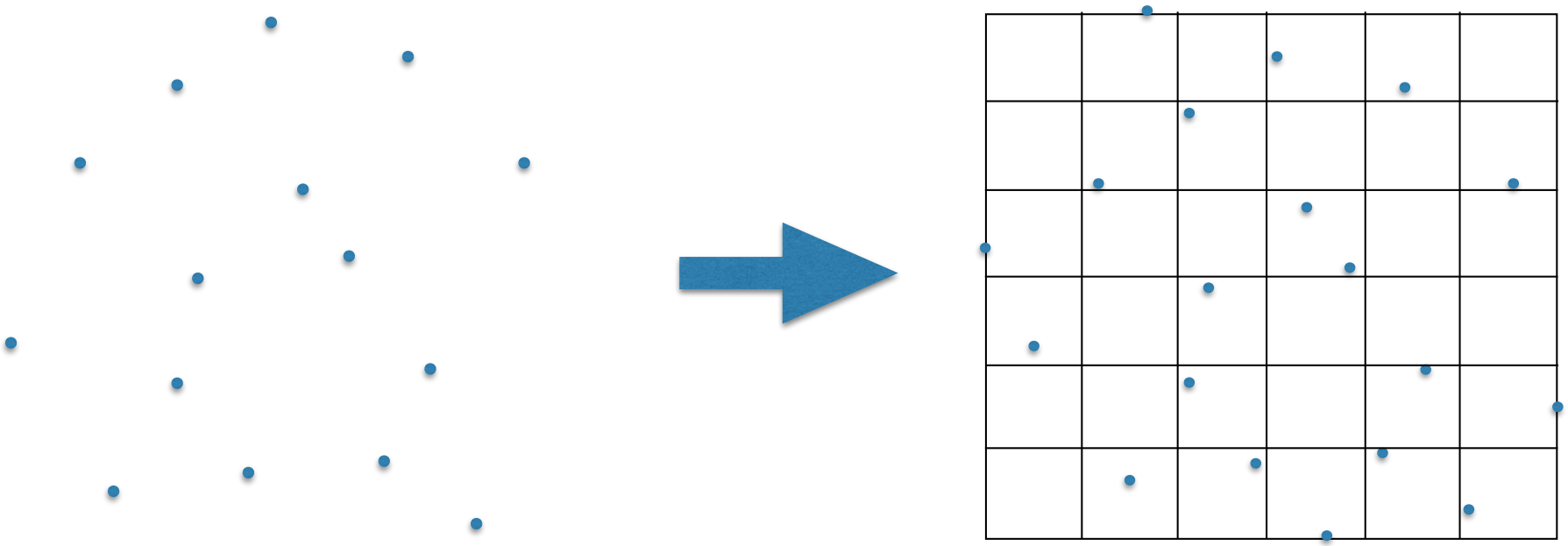
We'll partition the space, store it in a data structure and use it to speed up searching

Space decomposition methods

- the grid heuristic
- kd-trees
- range-trees

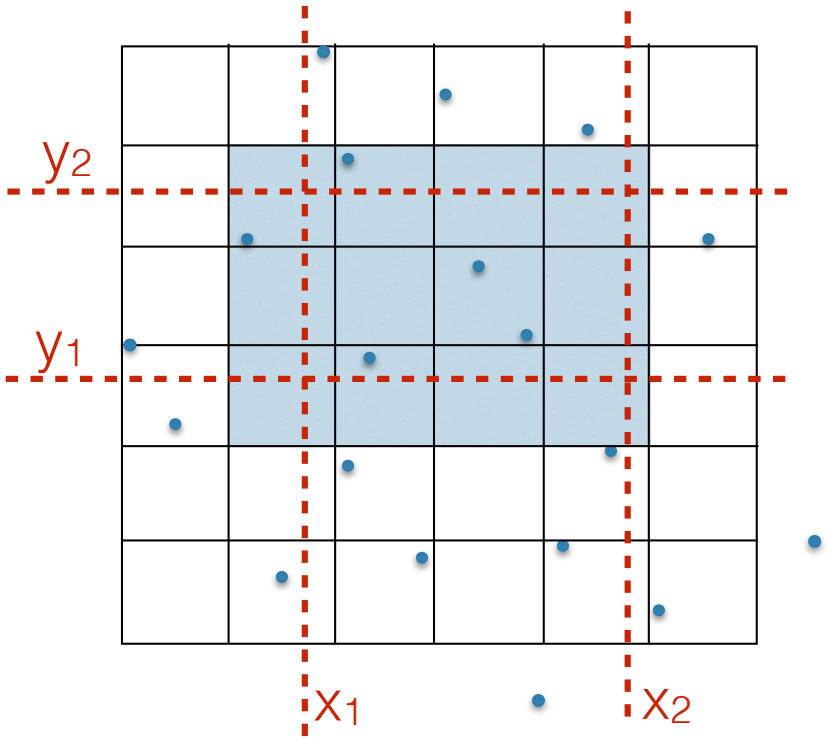
The grid heuristic

The simplest space decomposition is a grid



- Build: O(n)
- Space: O(n)

2D range searches with a grid



- 2d range queries: traverse all cells that intersect the range
- Exact bound depends on how many points are in the cells
- Choose grid size $m = O(\sqrt{n})$ and hope for $O(1)$ points per cell. In this case, a range query takes $O(k)$
- Worst case is bad: points are not uniformly distributed, a range query could take $O(n)$ even if no points are reported

The grid method

+ Simple to implement

+ Perform well if points are uniformly distributed

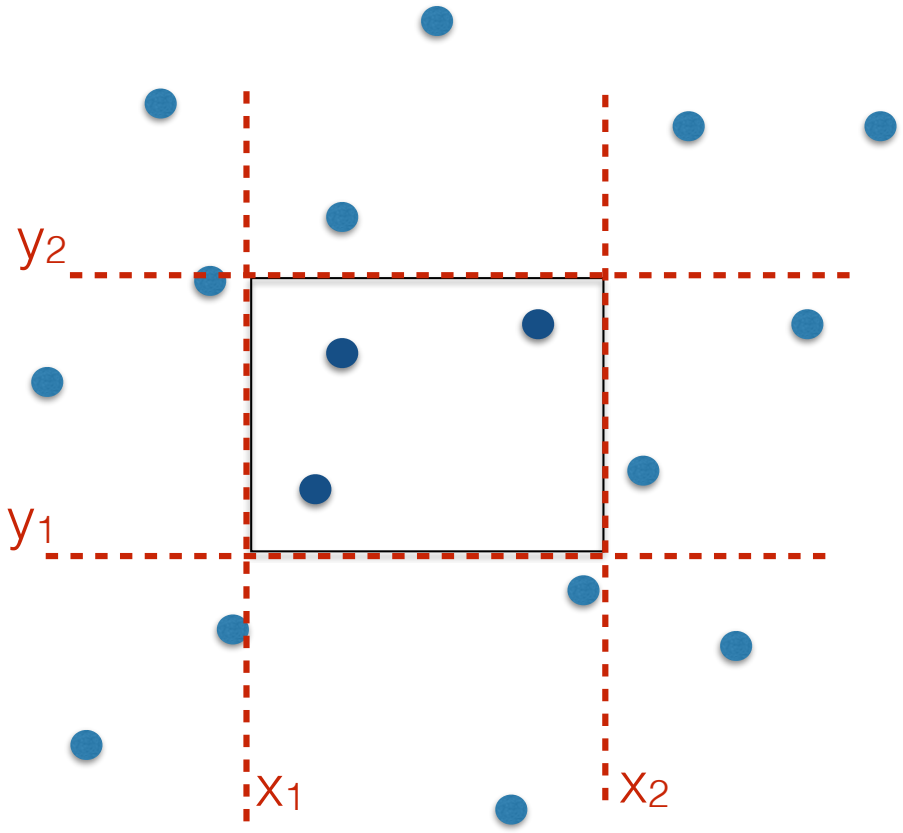
+ Can be used for many other problems besides range searching (e.g. closest pair, neighbor queries)

- Gridding is an heuristic. No guarantee on bounds.

2D Range searching

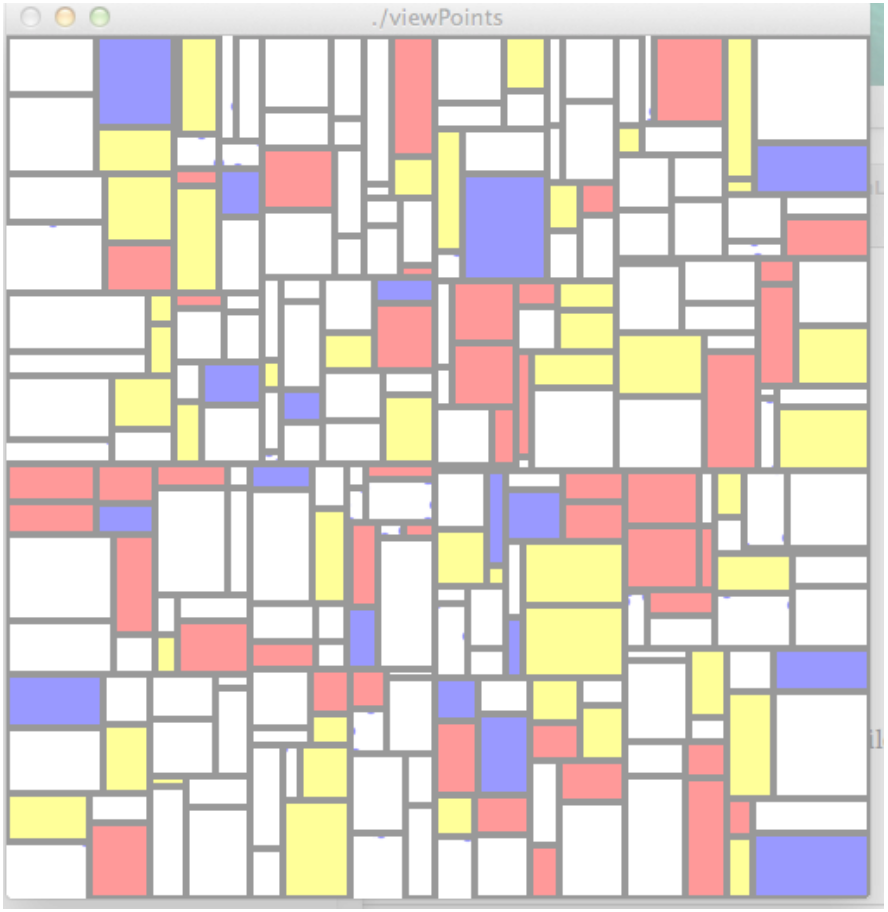
Given a set of n points in 2D and an arbitrary range $[x_1, x_2] \times [y_1, y_2]$, find all points in this range

Build a structure to answer this efficiently



kd-trees

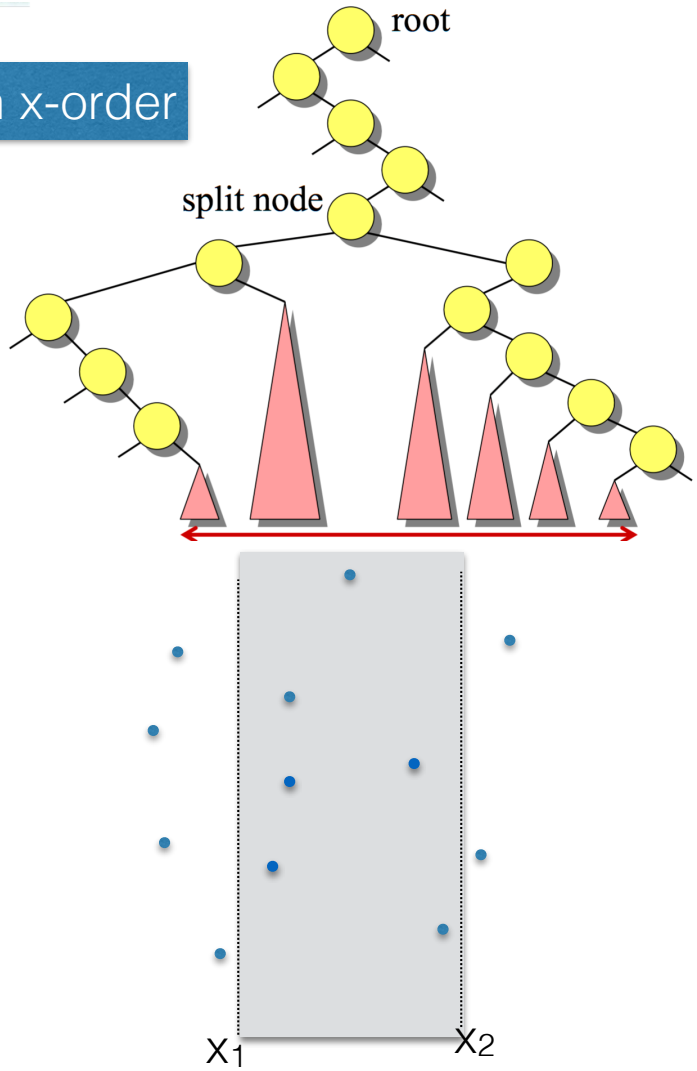
k-dimensional search trees



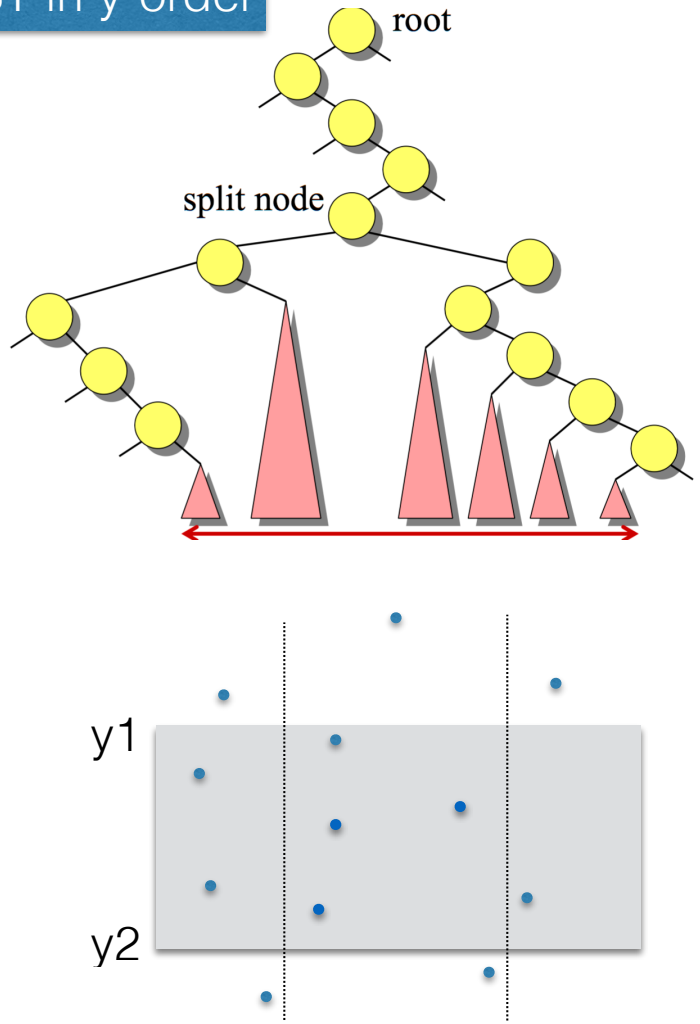
A BST is a 1D structure. If it is ordered by x, it can answer x-range queries. If it is ordered by y, it can answer y-range queries.

We'll extend it to be a 2D structure and symmetrical on x and y.

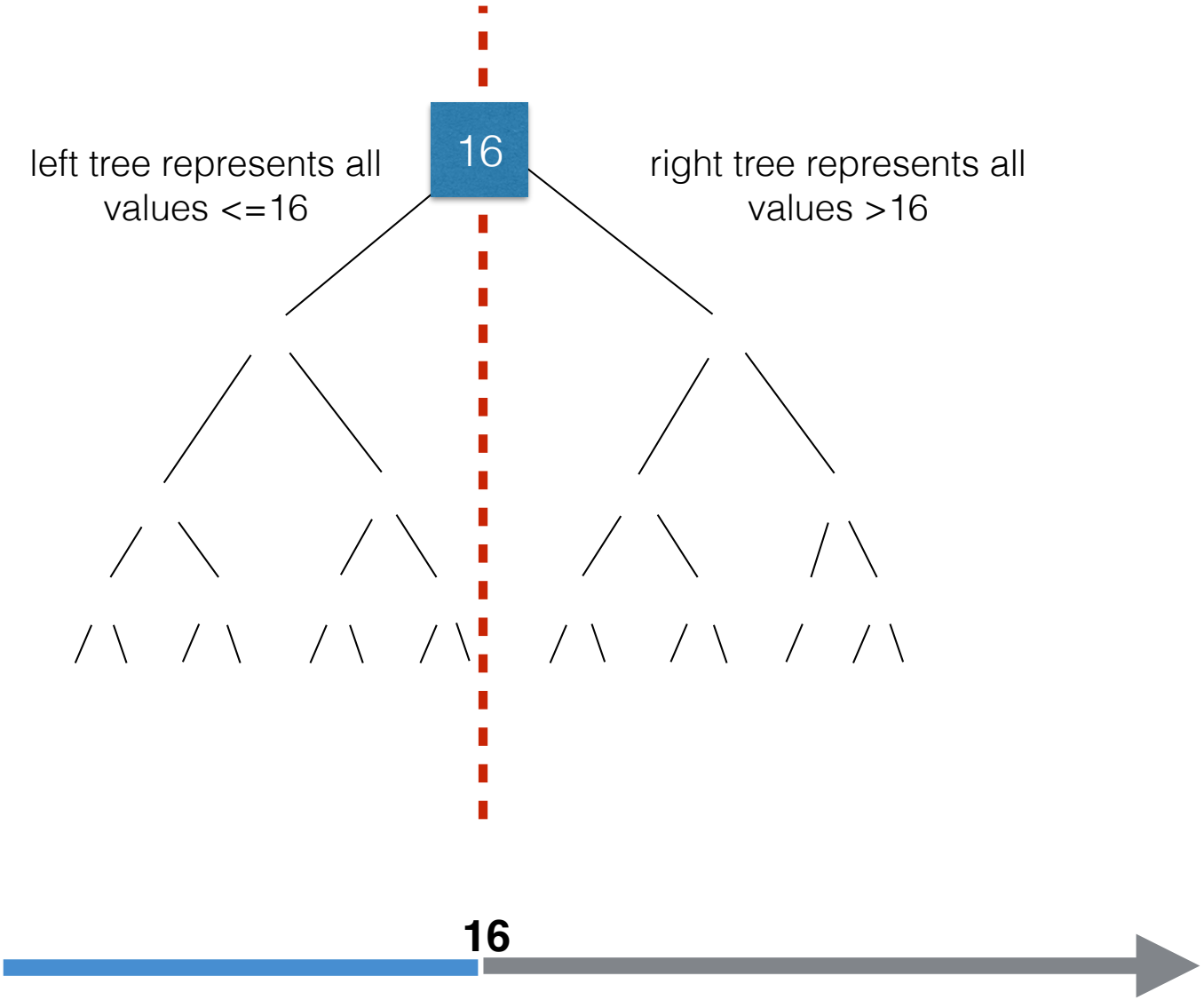
BST in x-order



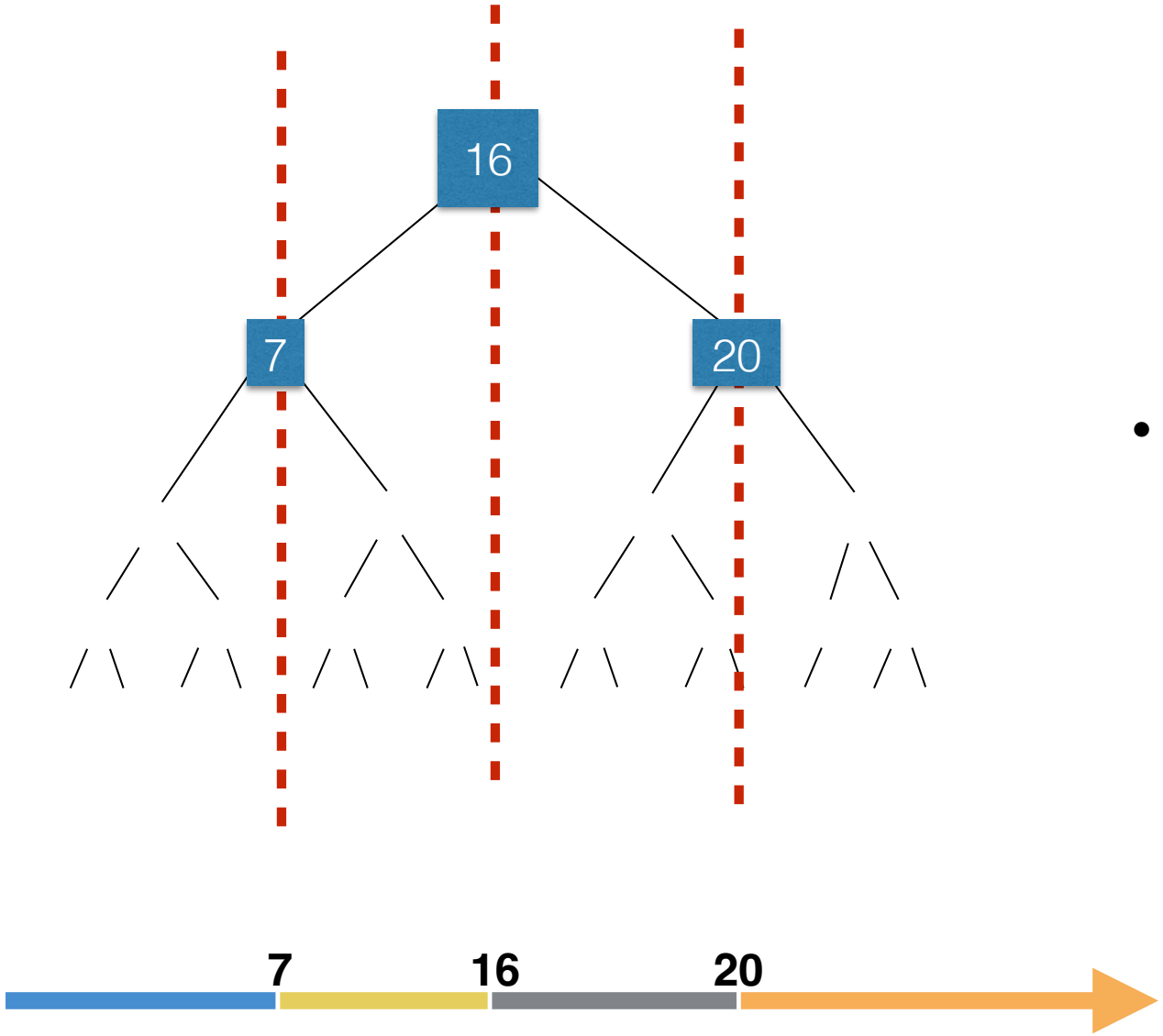
BST in y-order



A BST creates an implicit space partition

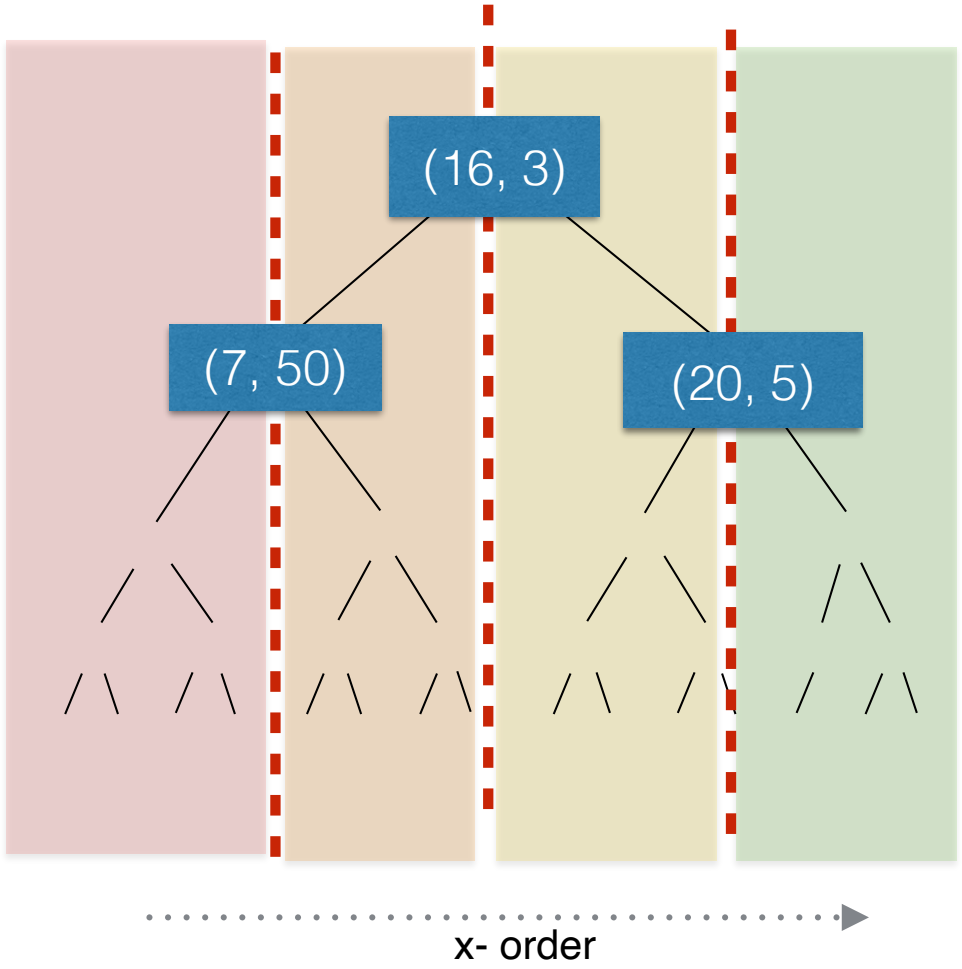


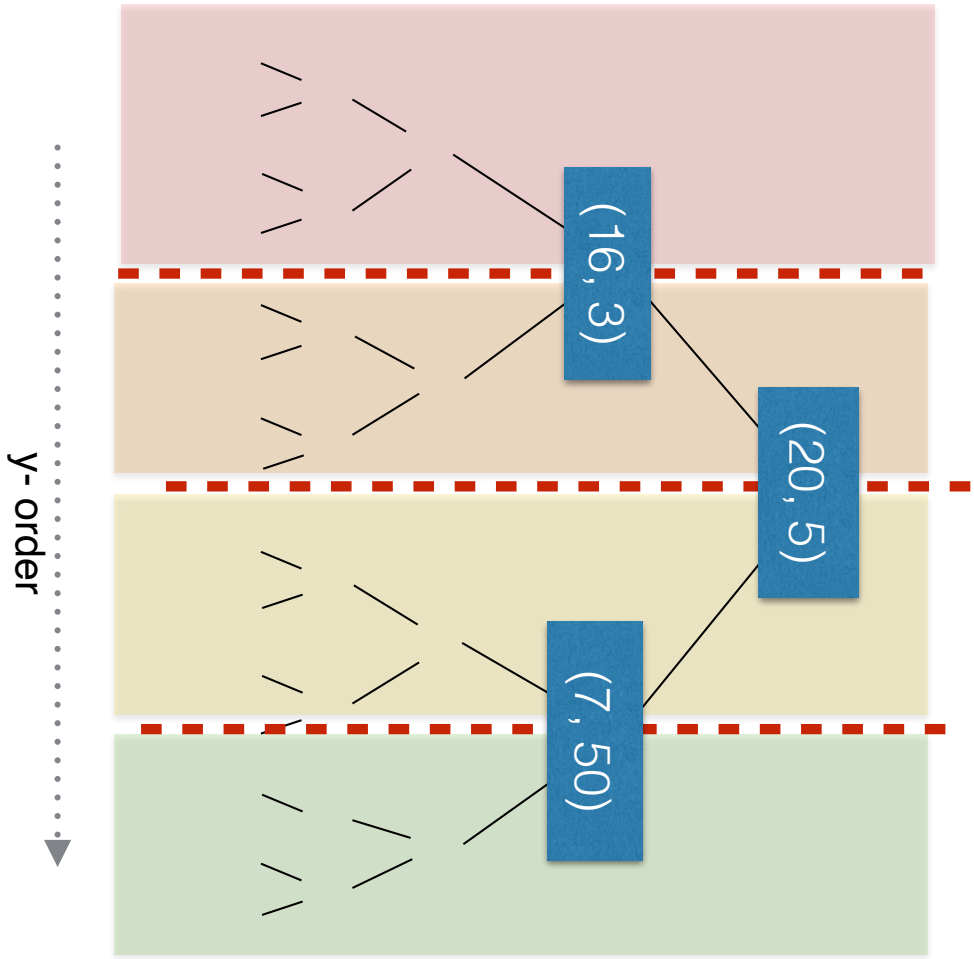
A BST creates an implicit space partition



- To search for a value we need to find the region of space that would contain this value

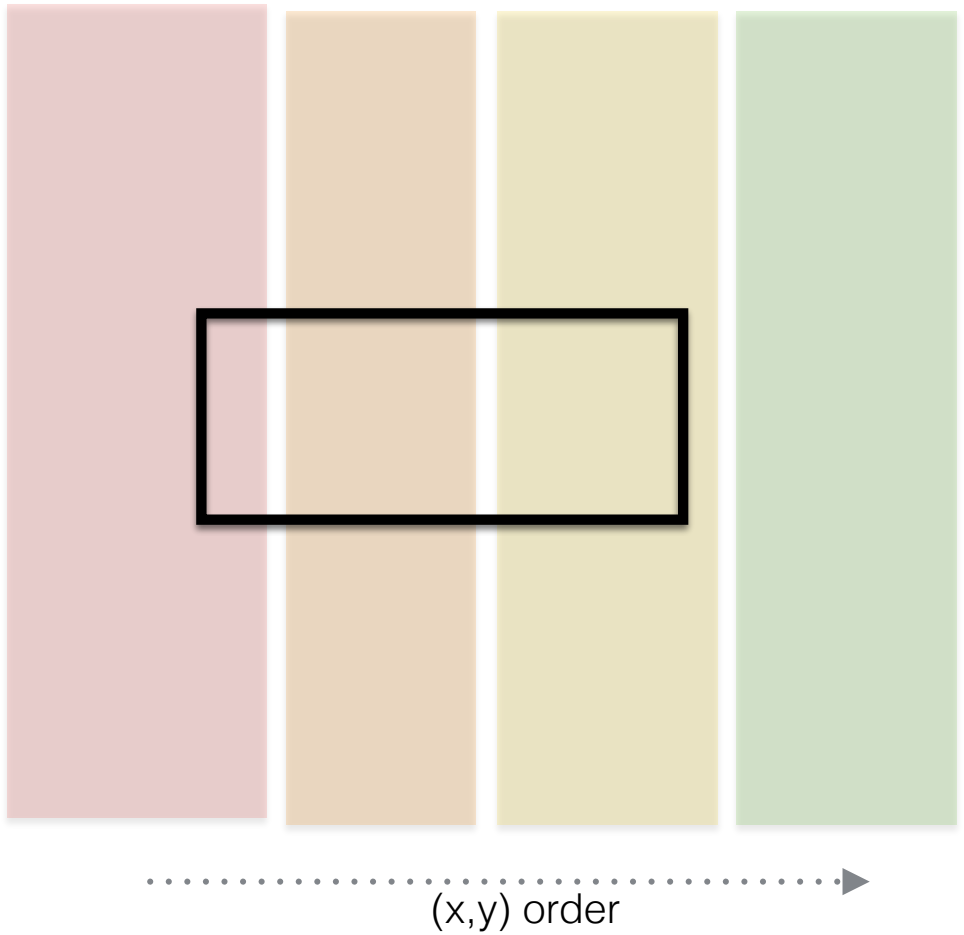
Using a BST on 2d points





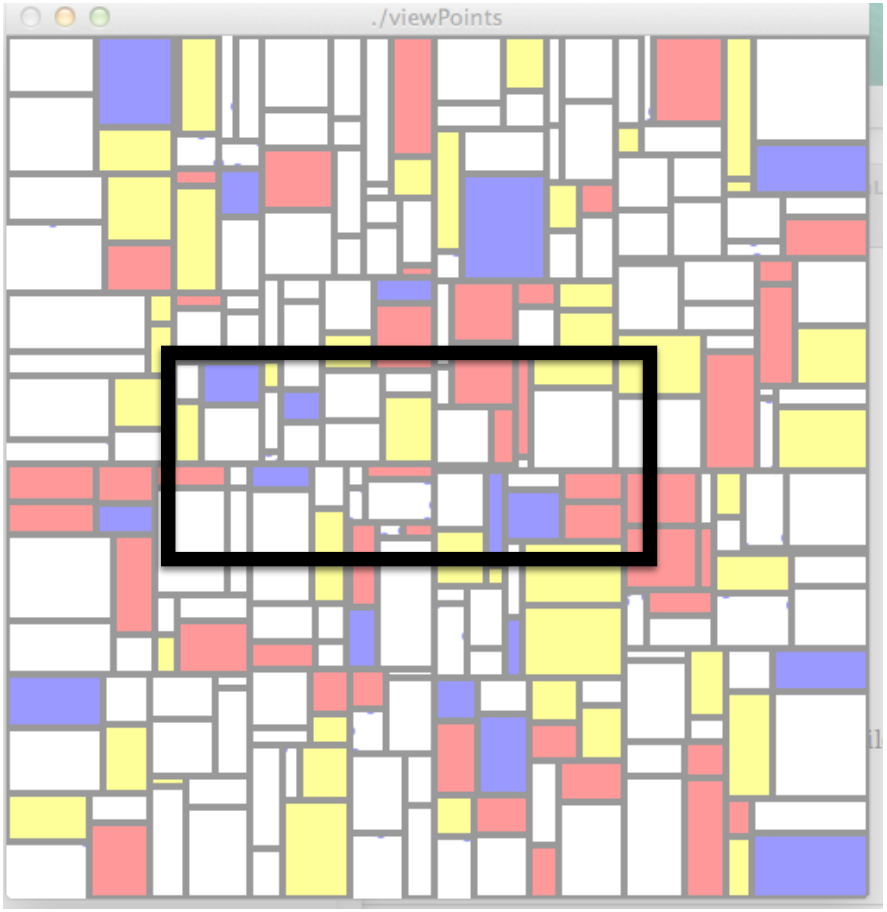
Partitions the space in vertical/horizontal stripes

Not a good partition for range-searching!



We have to search all vertical strips that intersect the range, which could have a lot of points outside the range.

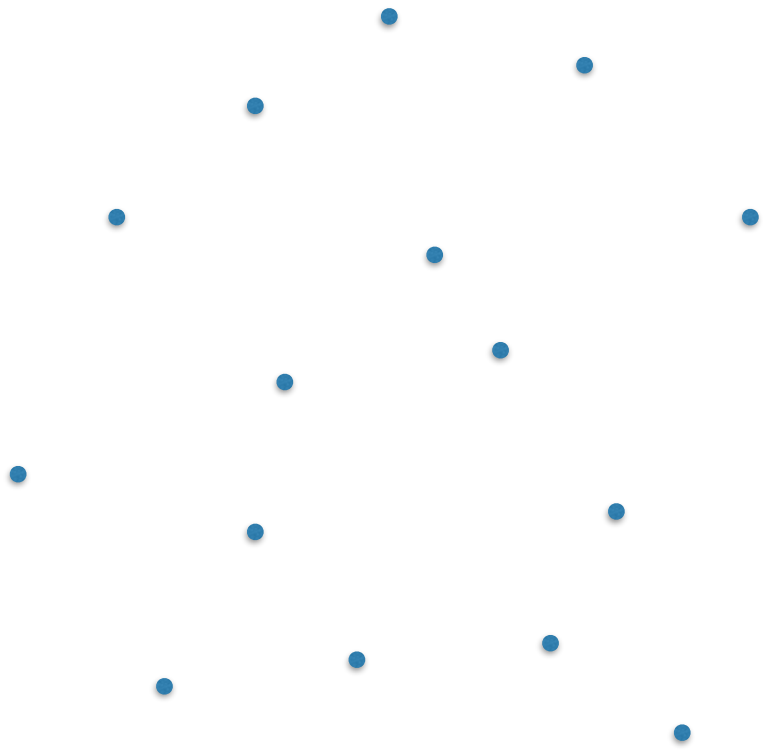
The 2d-search tree



Space partition of a 2d-search tree

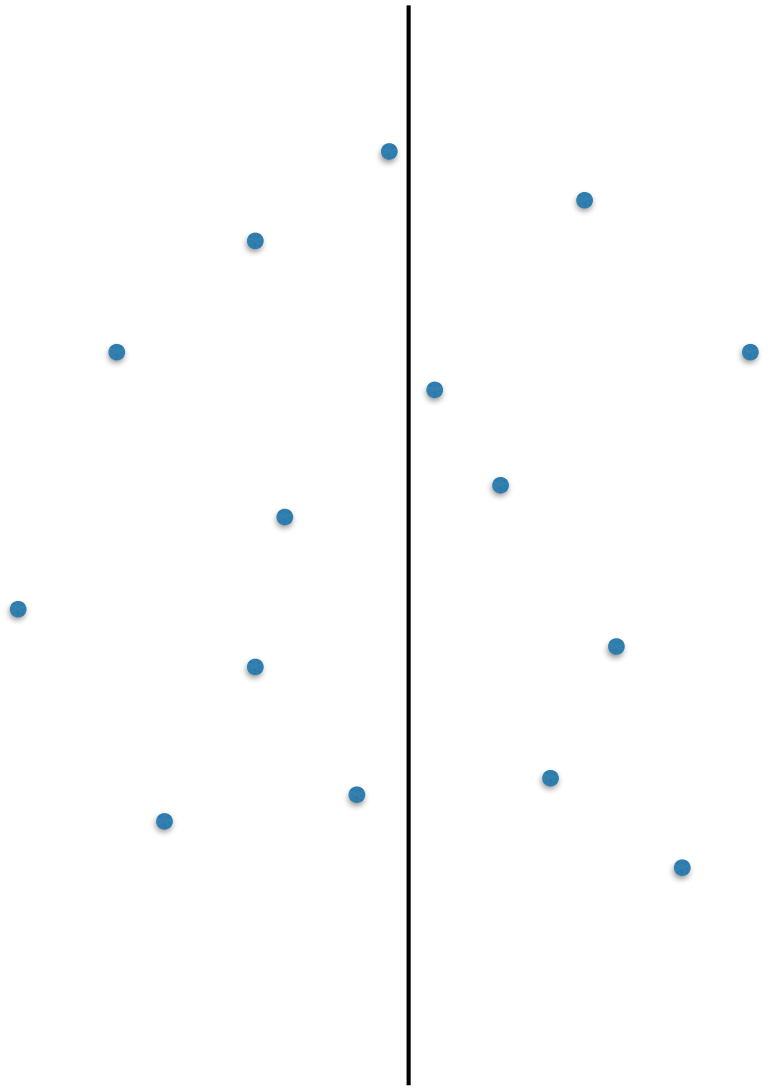
- **The idea:** recursively subdivide the plane by vertical and horizontal cut lines which alternate
- Cut lines are chosen to split the points in half ==> logarithmic height)

The 2d-search tree



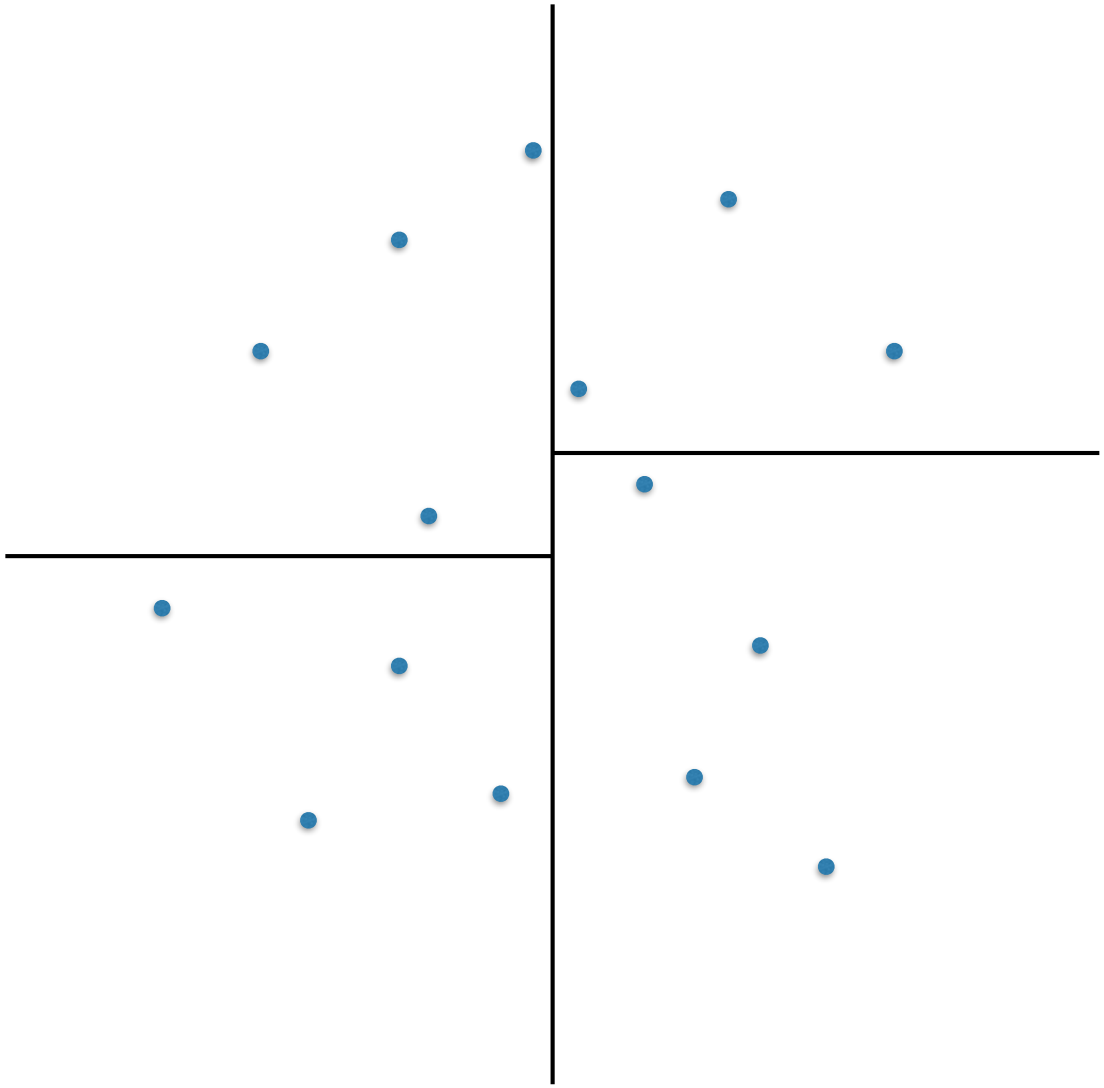
The 2d-search tree

split points in two halves with a **vertical** line

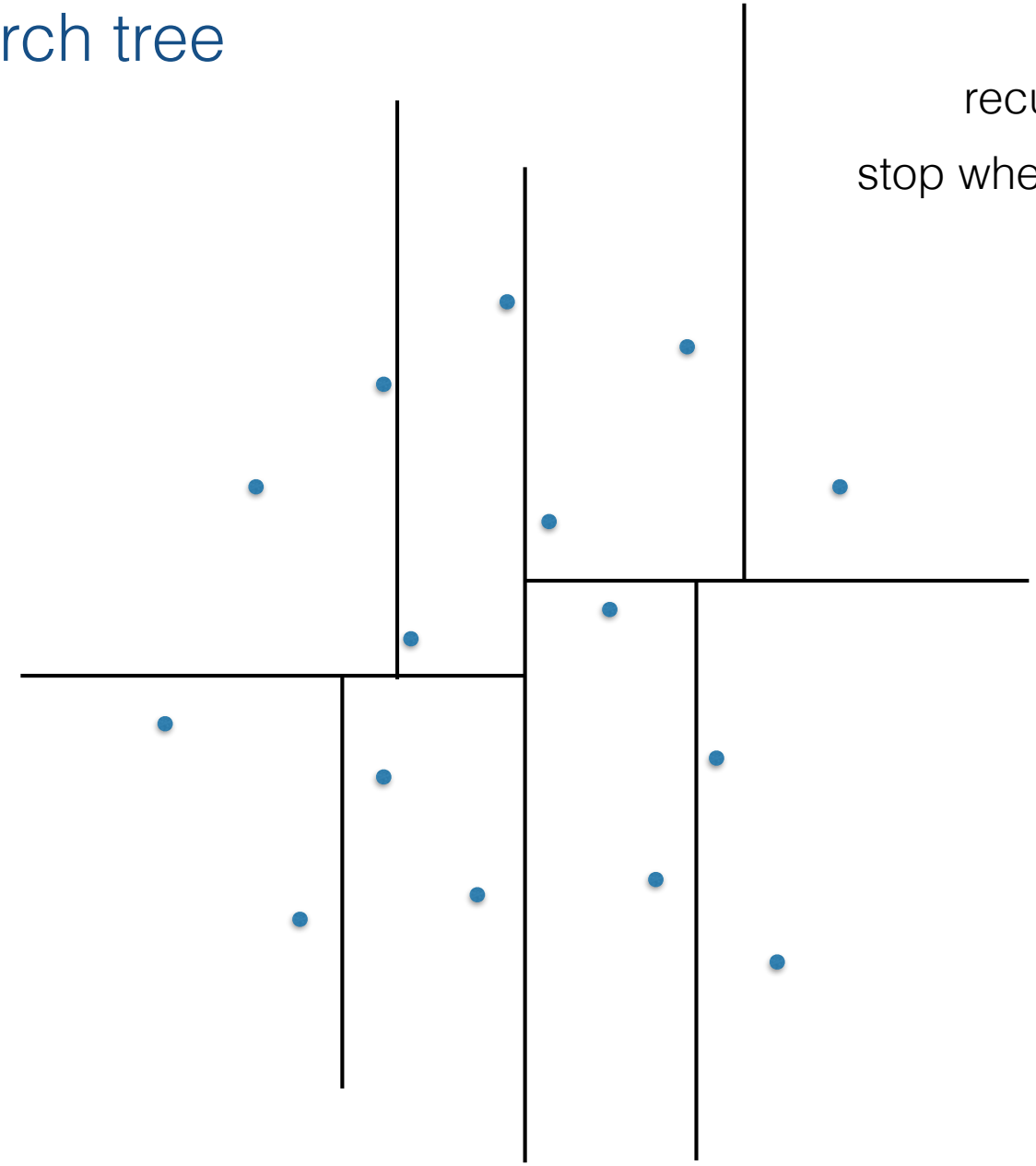


The 2d-search tree

split each side into half with a **horizontal** line



The 2d-search tree



recurse

stop when a region has 1 point

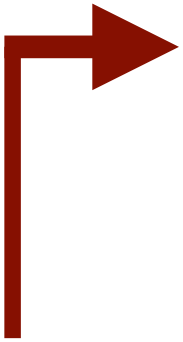
The 2d-search tree

stop when a region has 1 point

The 2d-search tree

Couple of variants based on how exactly to choose the splitting line

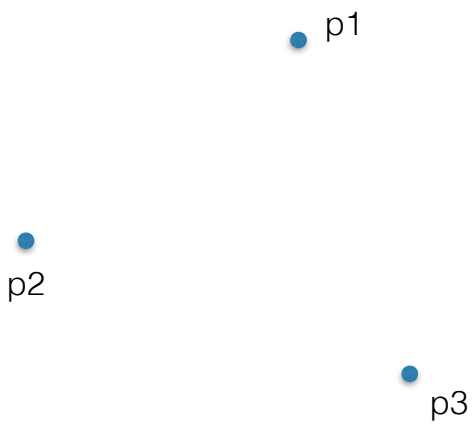
- Choose the cut line so that it falls in between the points. Internal nodes store lines, and points are only in leaves.
- Chose the cut line so that it goes through the median point, and store the median in the internal node.
- Choose the cut line so it goes through the median point. Internal nodes store lines, and points are only in leaves.



This is the standard choice and simplifies the details

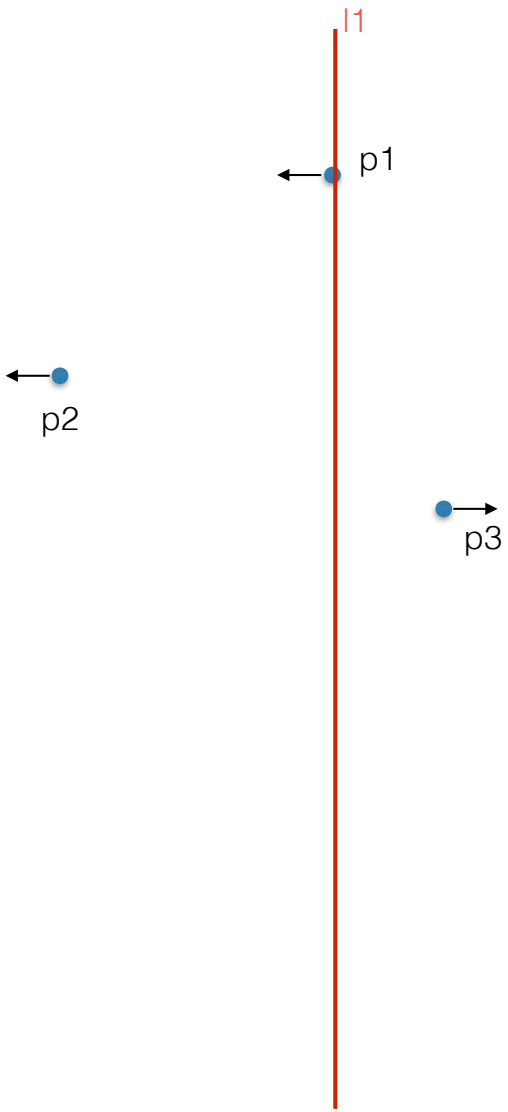
The 2d-search tree

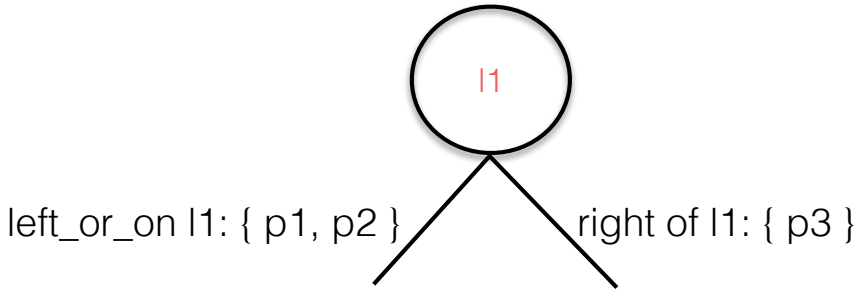
Include the median point to the **first** side, consistently.



The 2d-search tree

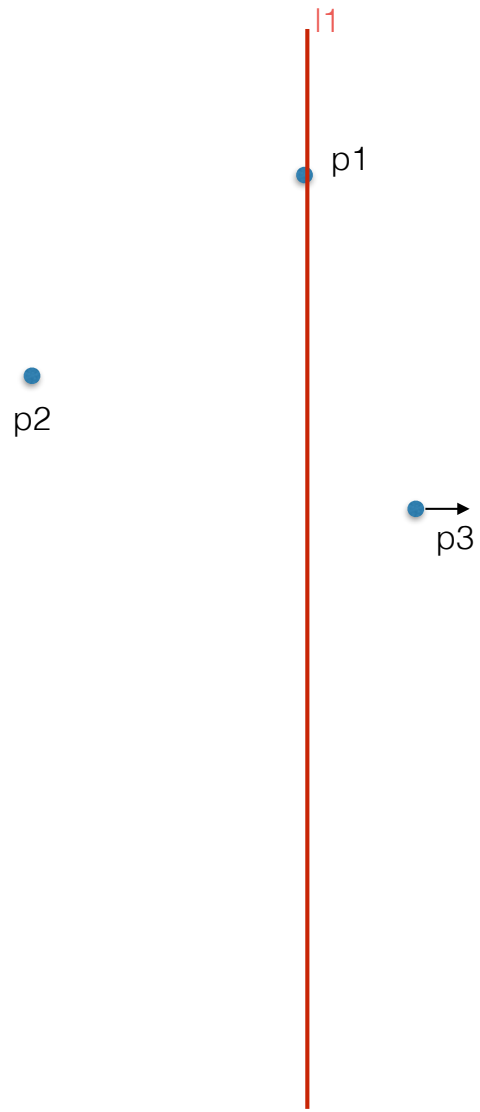
split with vertical line through x-median
include median in left child

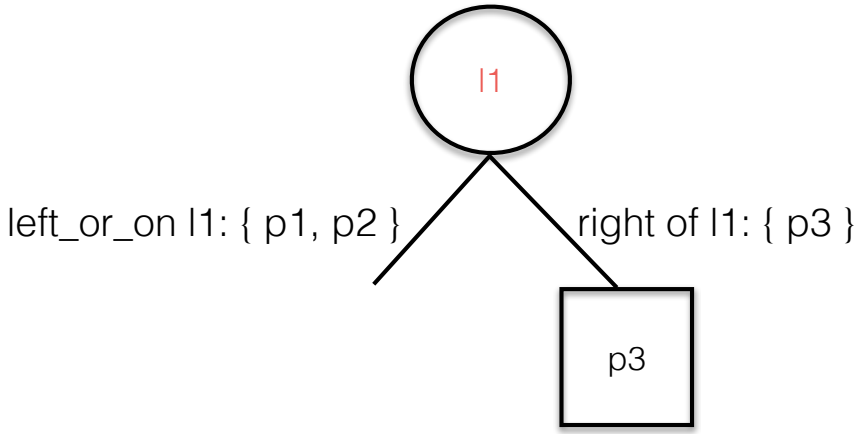




The 2d-search tree

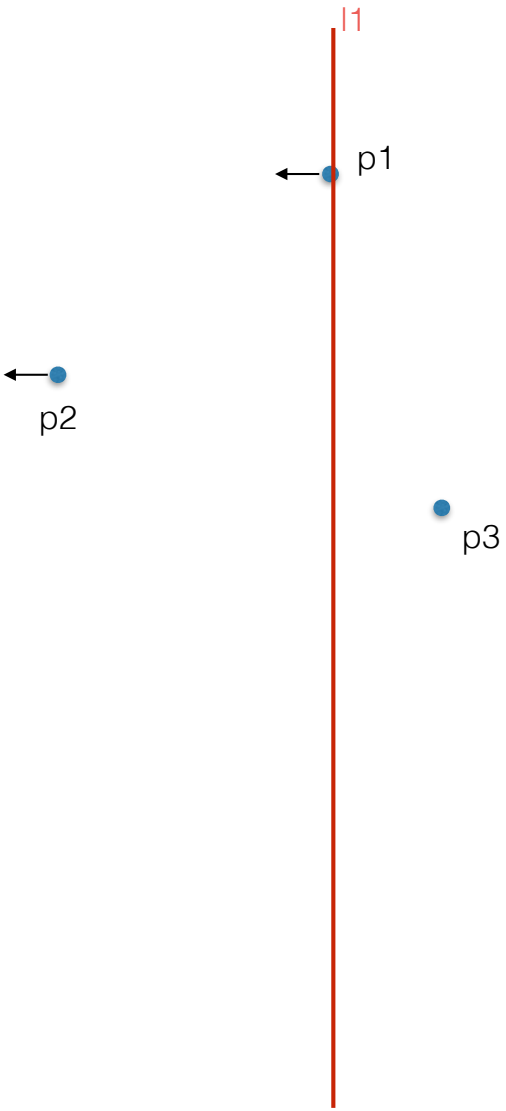
right of l1: p3 => leaf

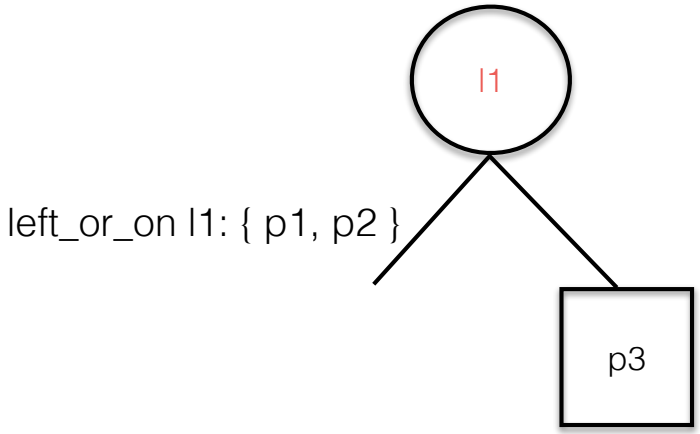




The 2d-search tree

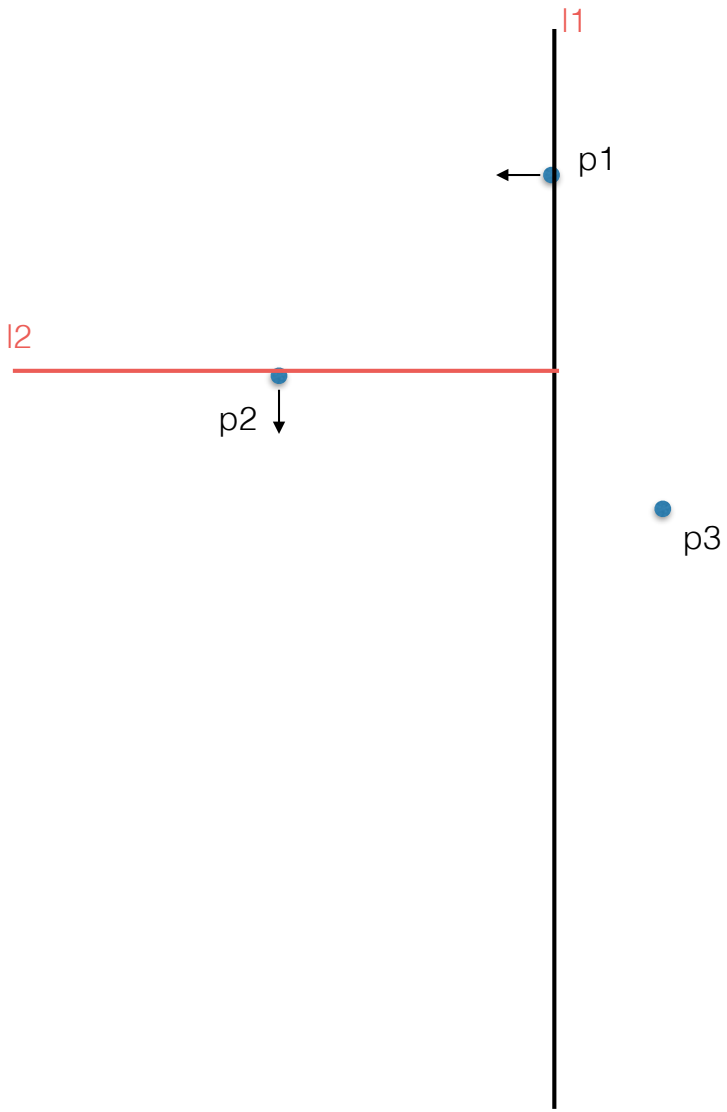
left_on l1: p1,p2 => recurse

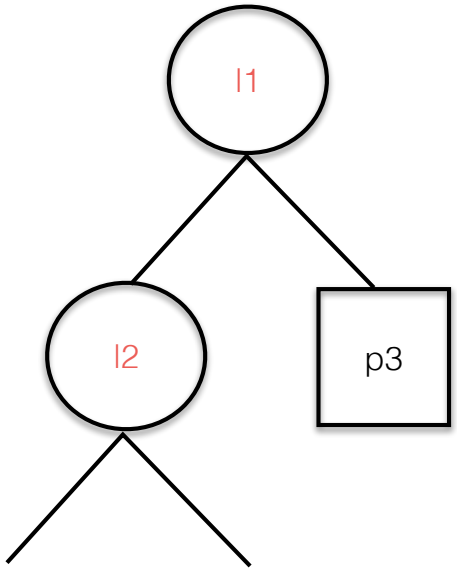




The 2d-search tree

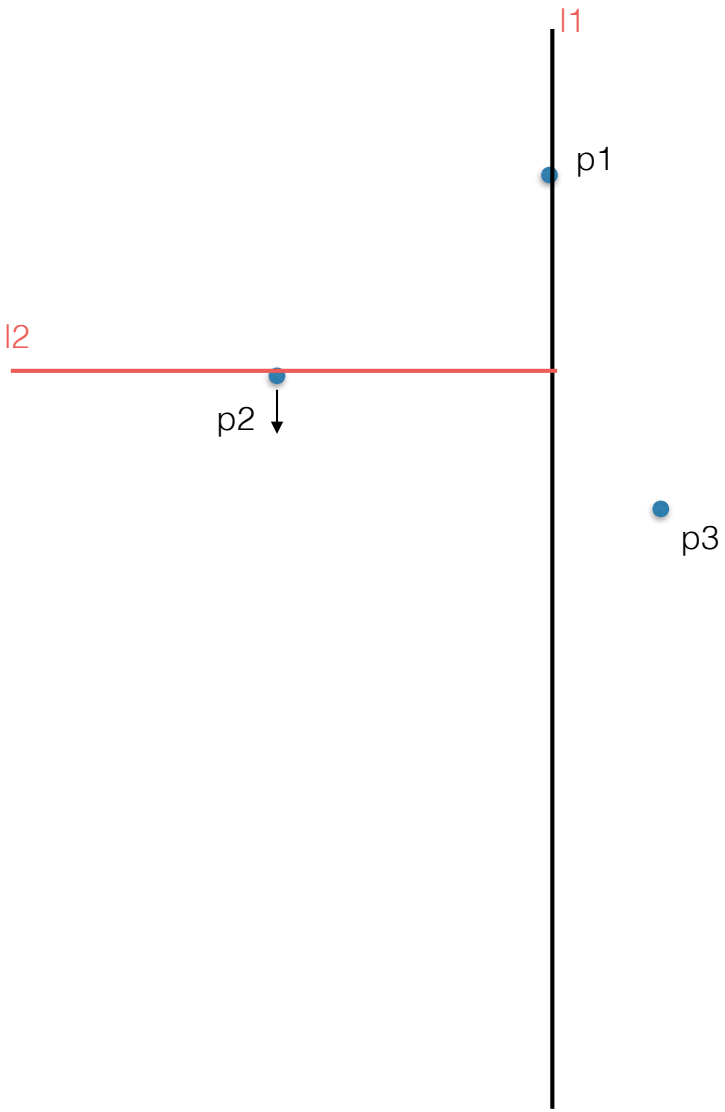
split with horizontal line through y-median
include median in left child

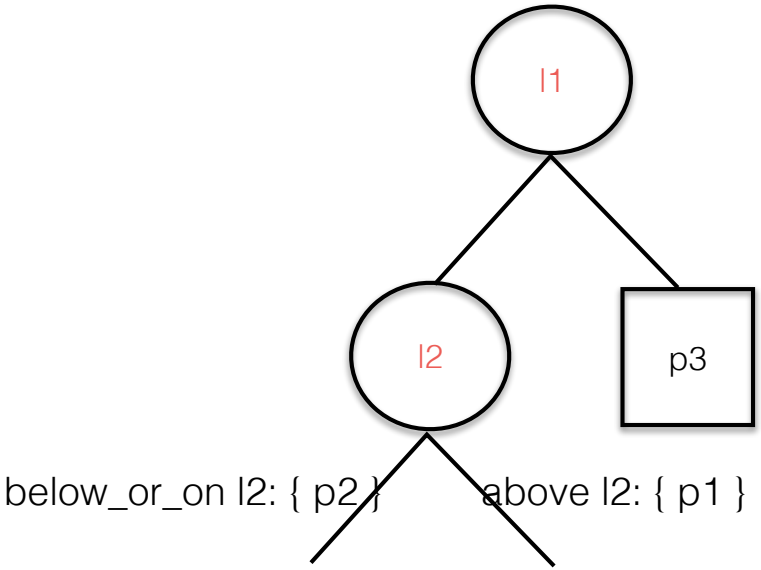




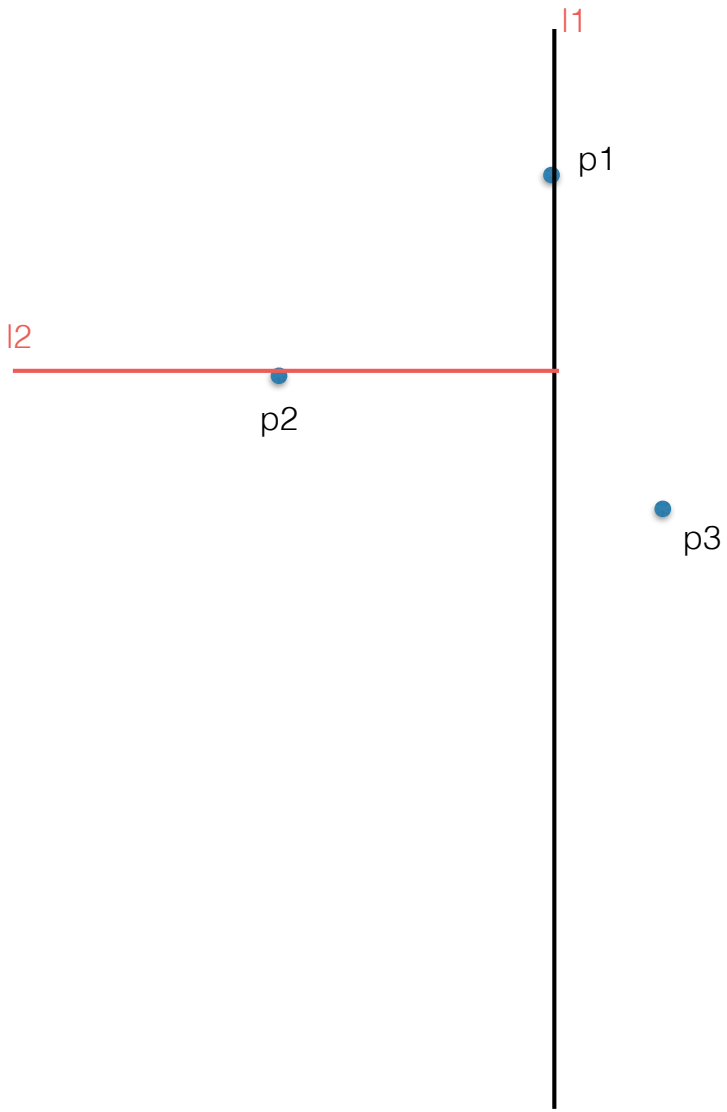
The 2d-search tree

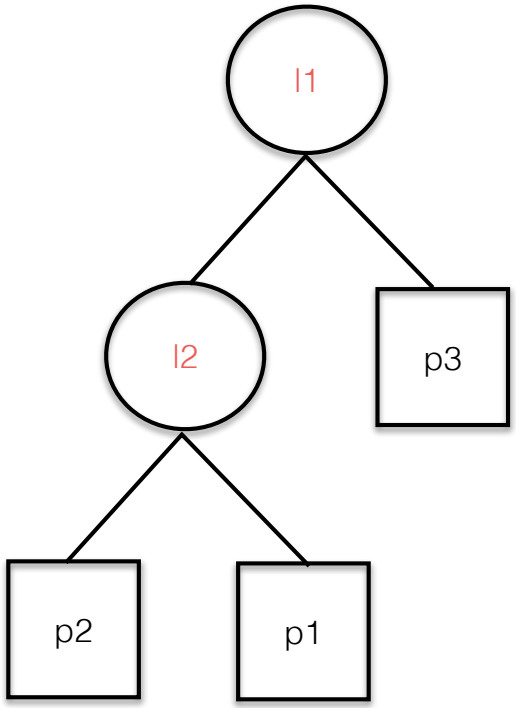
split with horizontal line through y-median
include median in left child



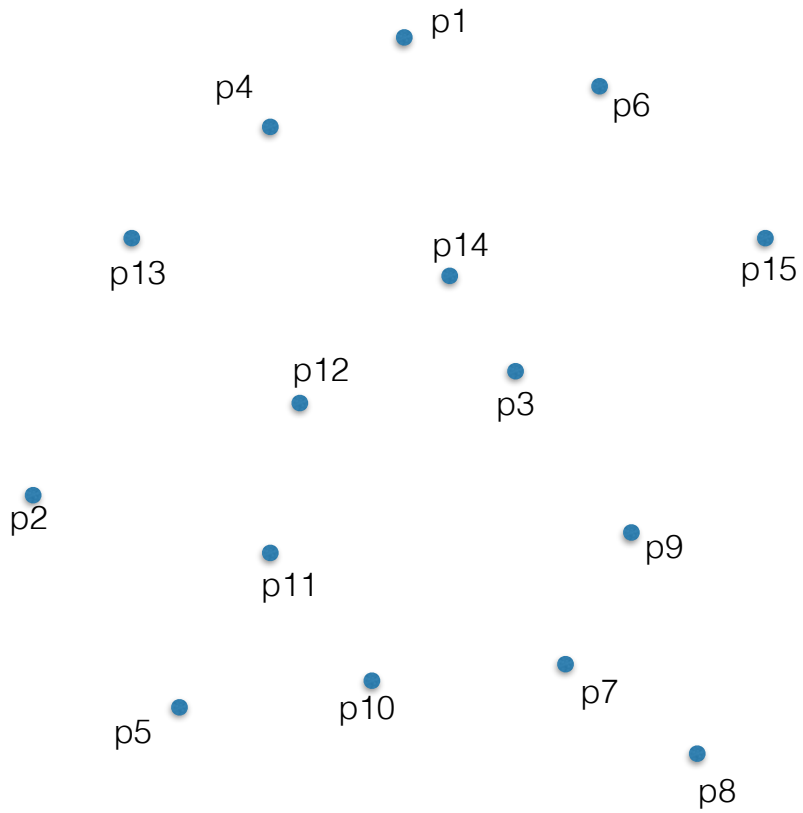


The 2d-search tree



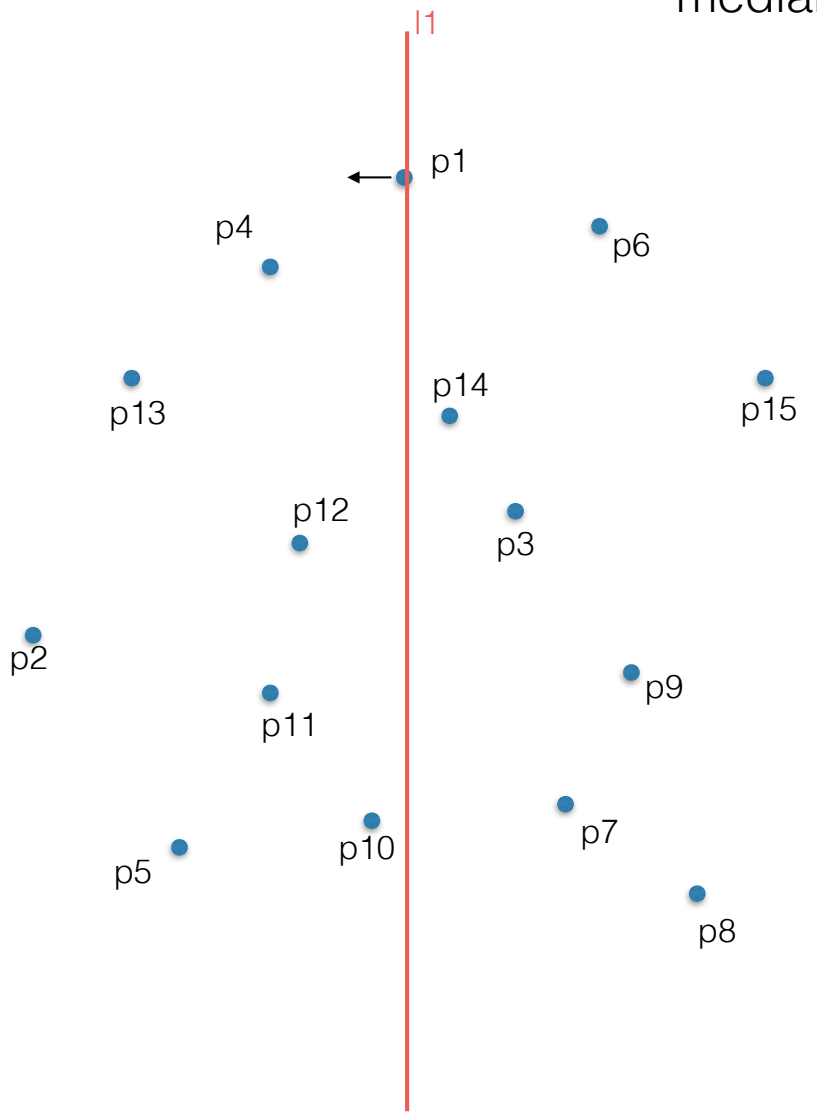


A bigger example



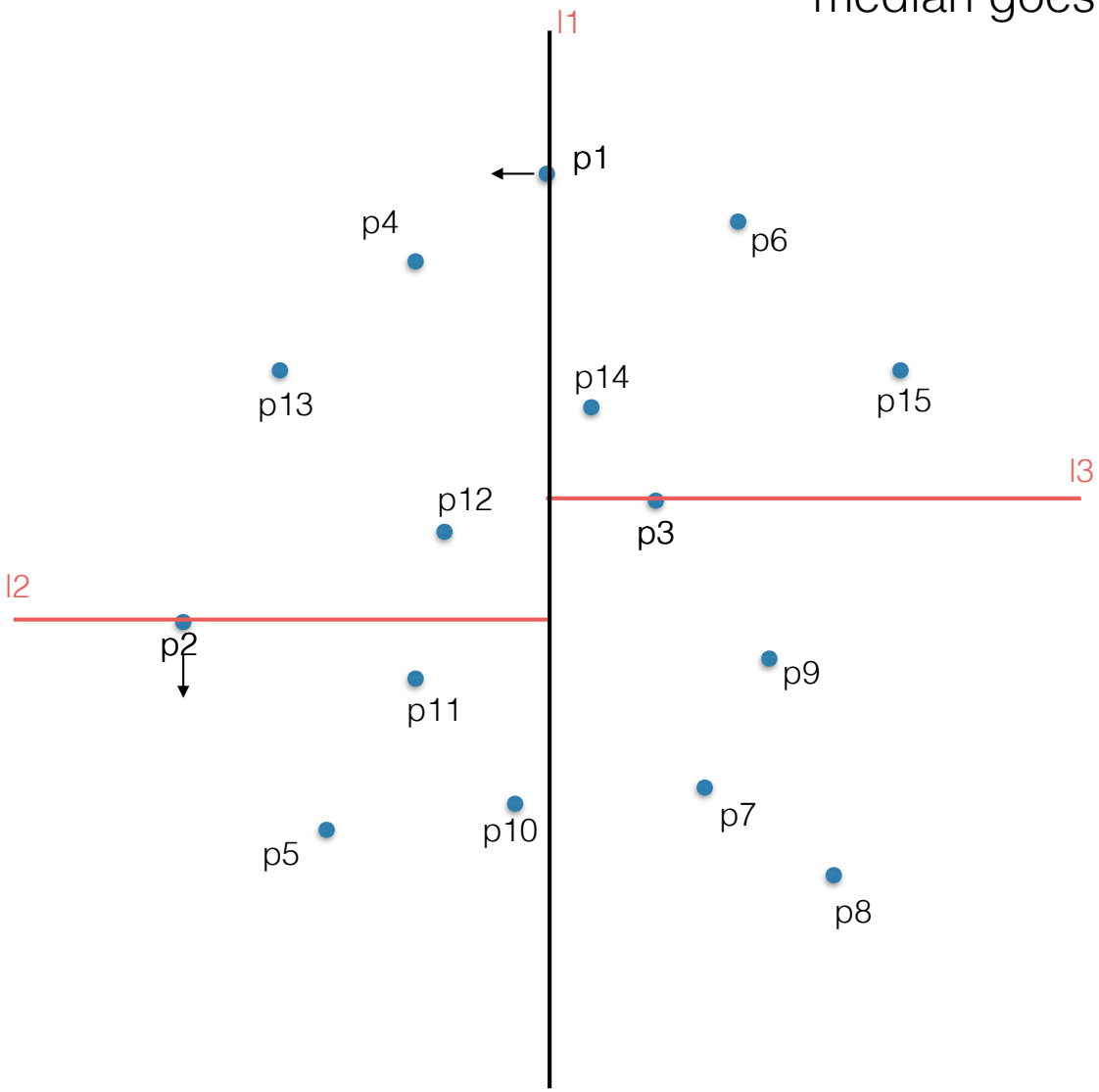
A bigger example

split with vertical line through x-median
median goes to the left side

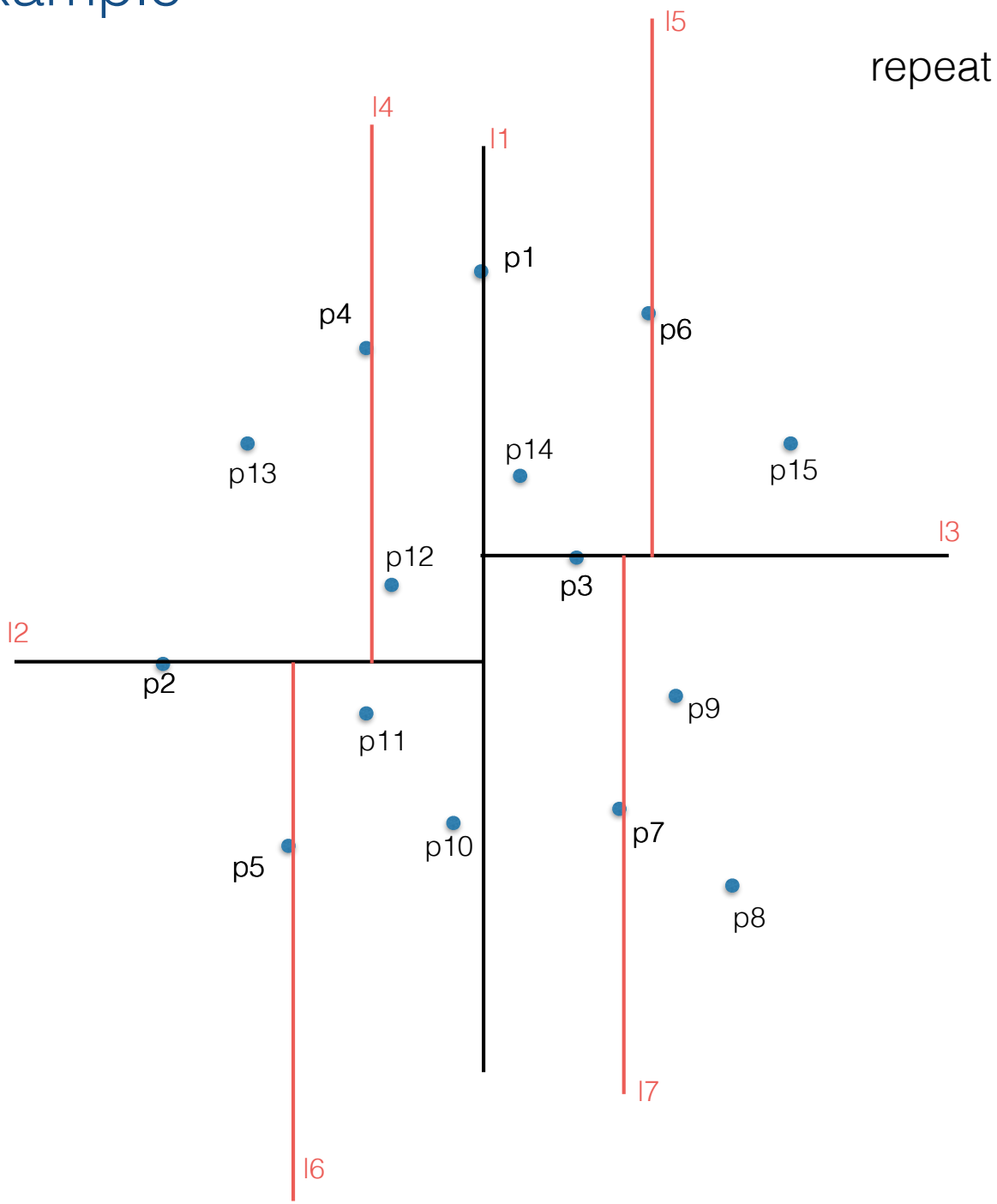


A bigger example

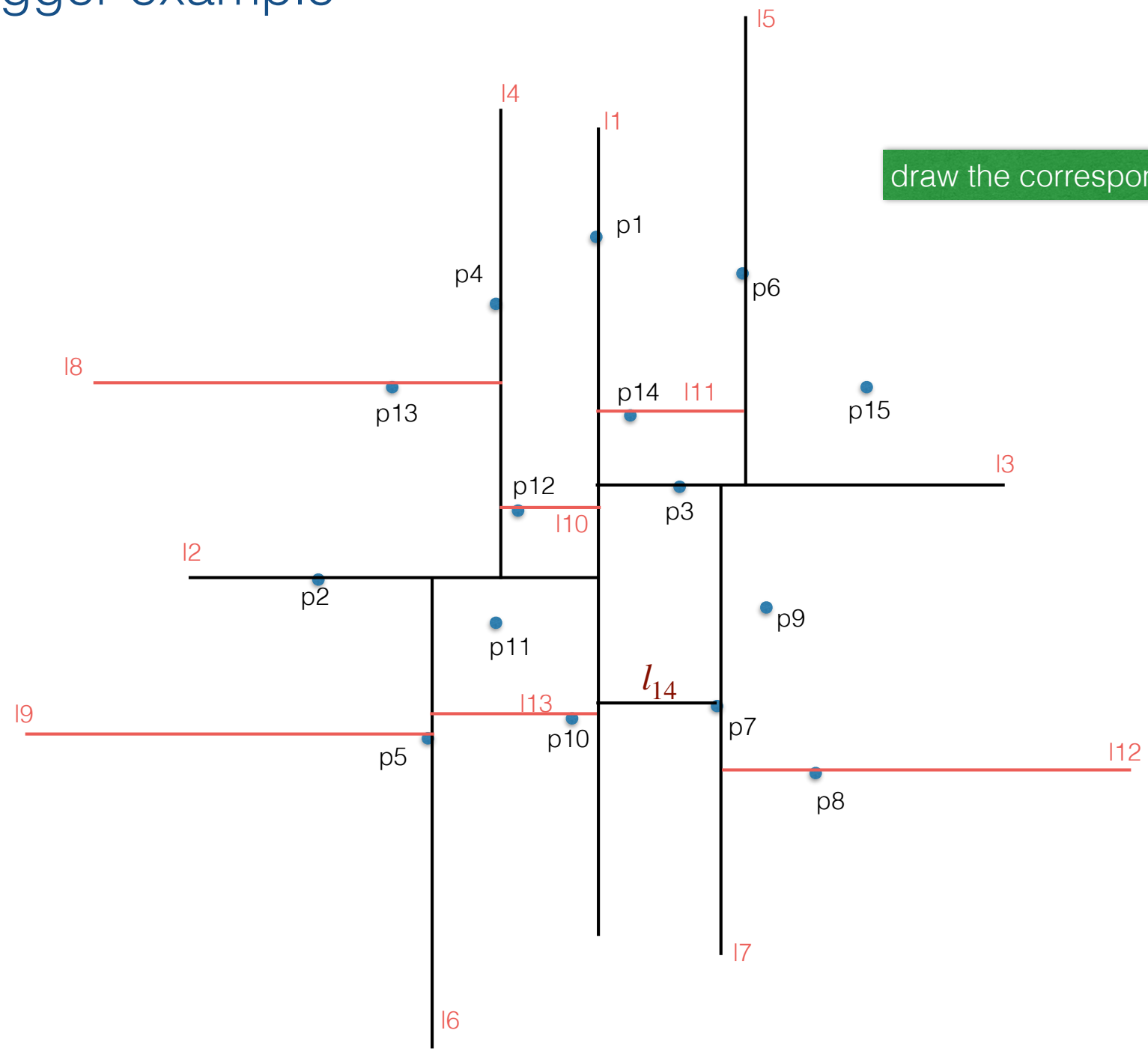
split each side with horizontal line through y-median
median goes to the left side



A bigger example



A bigger example



Given a set of points P:

How do we build their kd-tree?

The 2d-search tree

Algorithm BUILDKDTREE($P, depth$)

1. **if** P contains only one point
2. **then return** a leaf storing this point
3. **else if** $depth$ is even
4. **then** Split P with a vertical line ℓ through the median x-coordinate into P_1 (left of or on ℓ) and P_2 (right of ℓ)
5. **else** Split P with a horizontal line ℓ through the median y-coordinate into P_1 (below or on ℓ) and P_2 (above ℓ)
6. $v_{\text{left}} \leftarrow$ BUILDKDTREE($P_1, depth+1$)
7. $v_{\text{right}} \leftarrow$ BUILDKDTREE($P_2, depth+1$)
8. Create a node v storing ℓ, make v_{left} the left child of v, and make v_{right} the right child of v.
9. **return** v

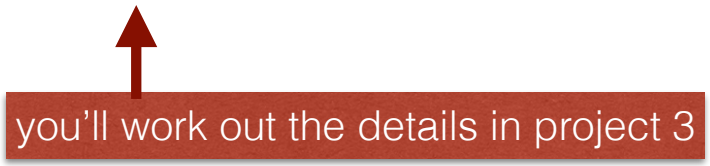
Analysis: Let T(n) be the time to build a kdtree of n points.

Then T(n) = 2T(n/2) + O(n) , which solves to O(n lg n).

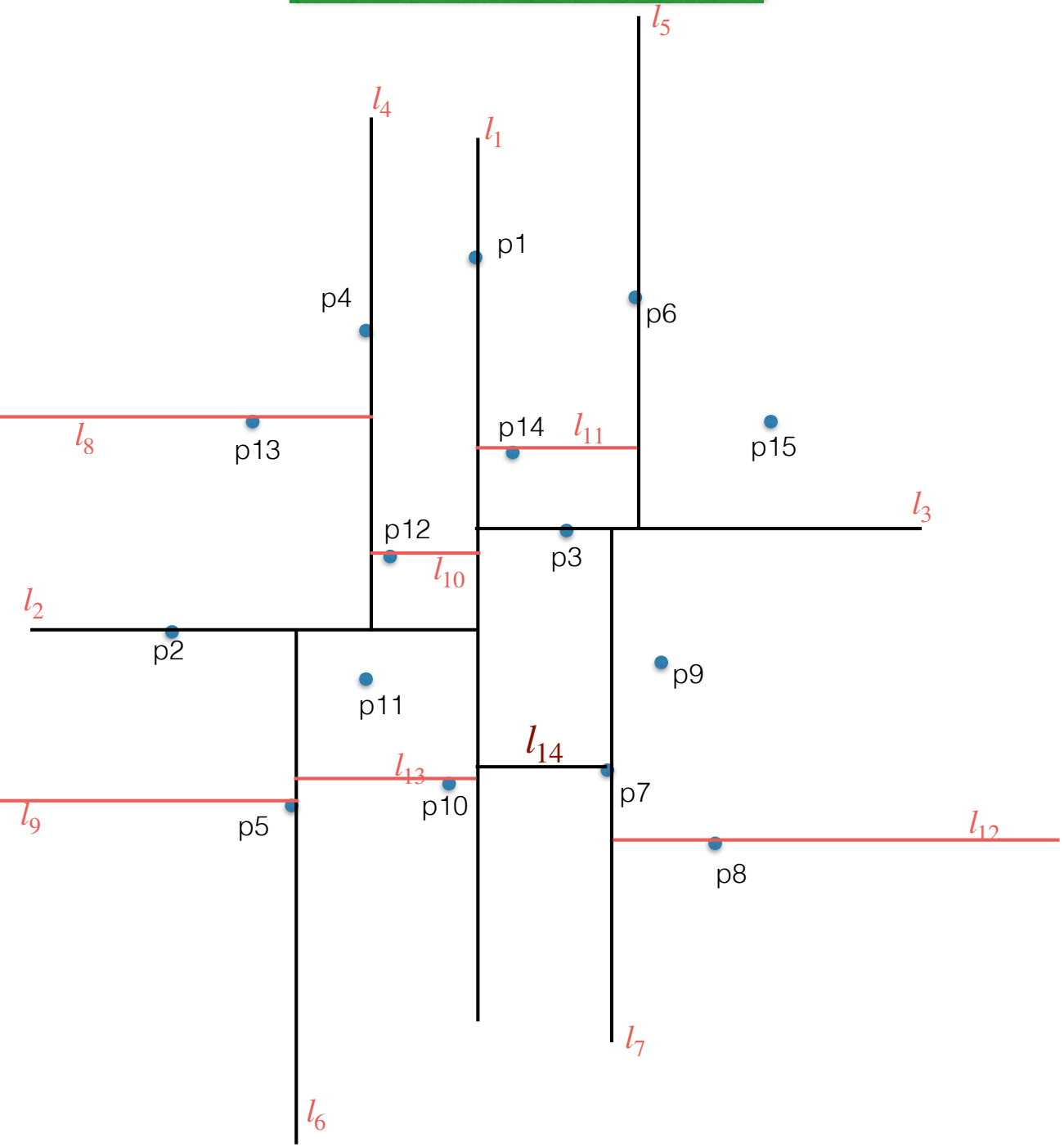
The 2d-search tree

Theorem: A 2d-tree for a set of n points in the plane can be built in $\Theta(n \lg n)$ time and uses $\Theta(n)$ space.

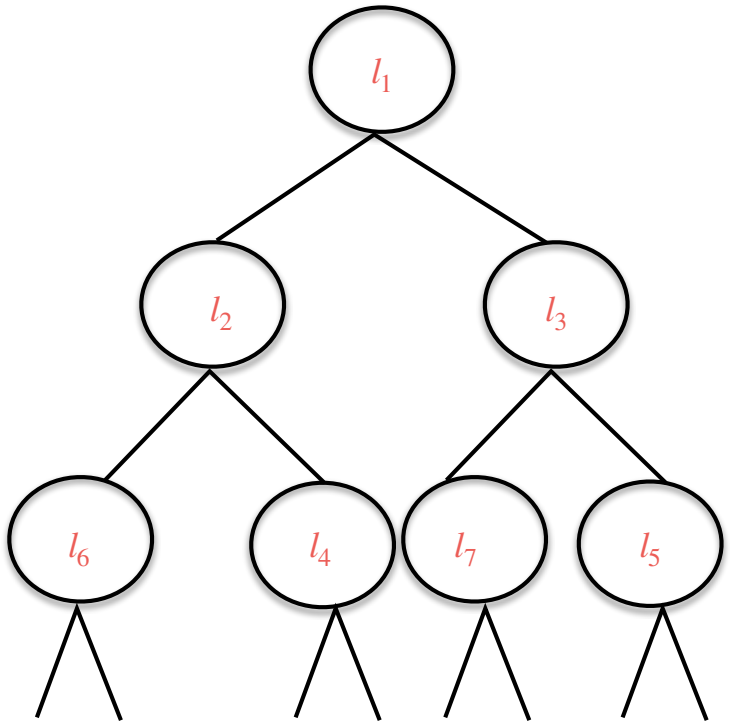
- Practical notes
  - The O(n) median finding algorithm is not practical. Either use a randomized median finding (QuickSelect); or, better,
  - Avoid needing to find a median by pre-sorting the points
    - sort P by x- coord and, separately by y-coord **only once** at the beginning, before building the tree, and pass them as parameters
    - *BuildKDtree (P-sorted-by-x, P-sorted-by-y, depth)*



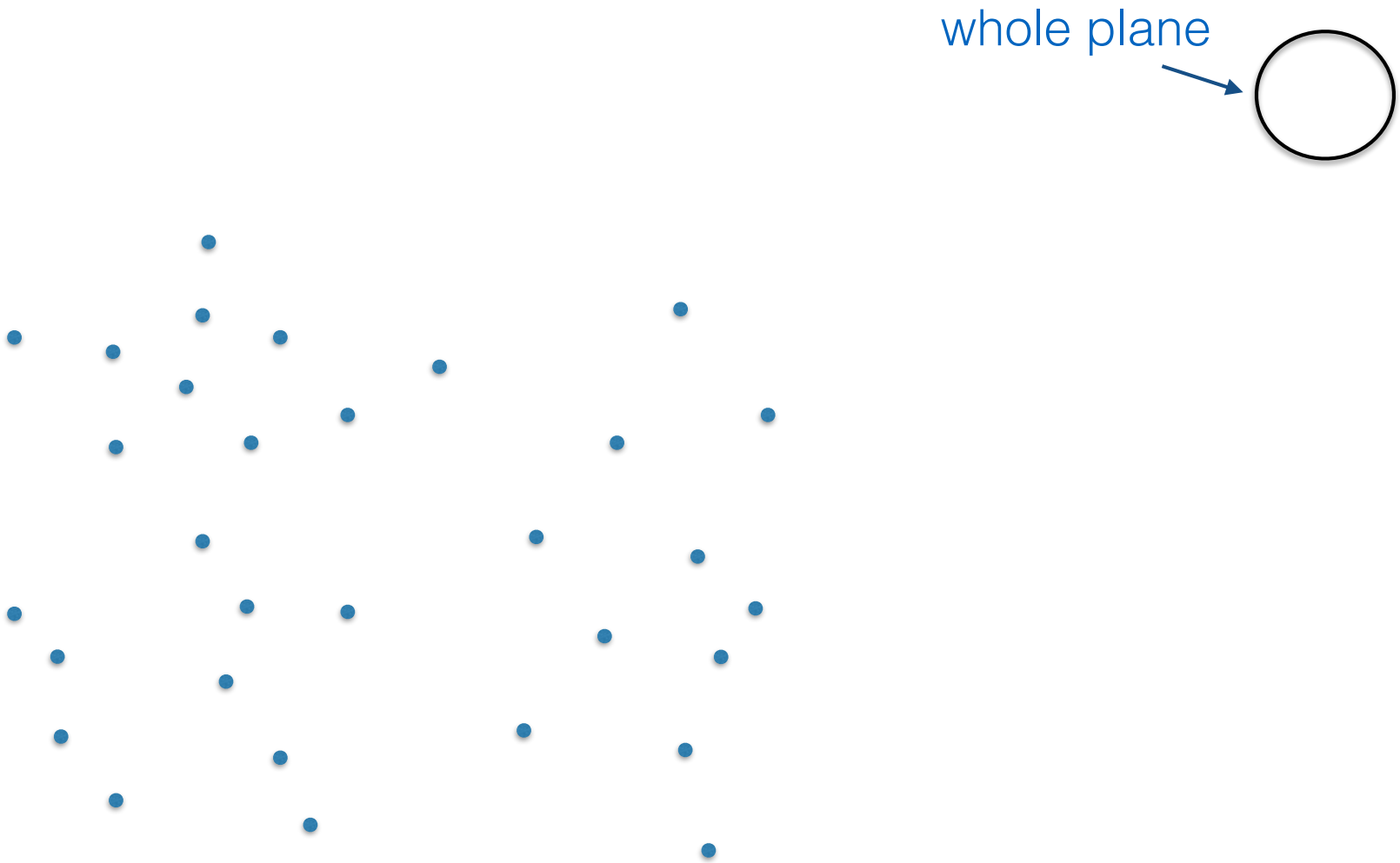
space partition corresponding to the 2d-tree



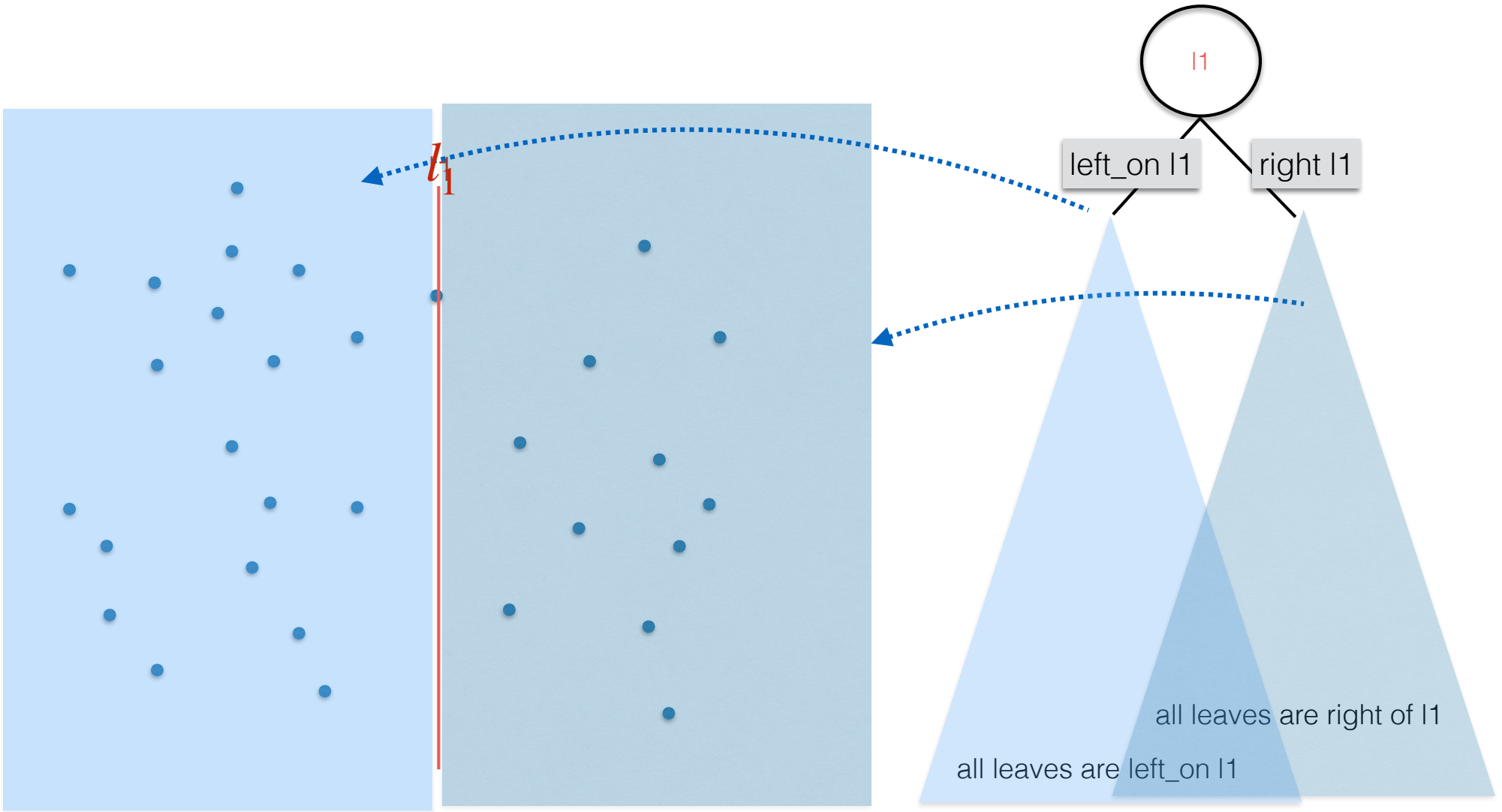
the corresponding 2d-tree



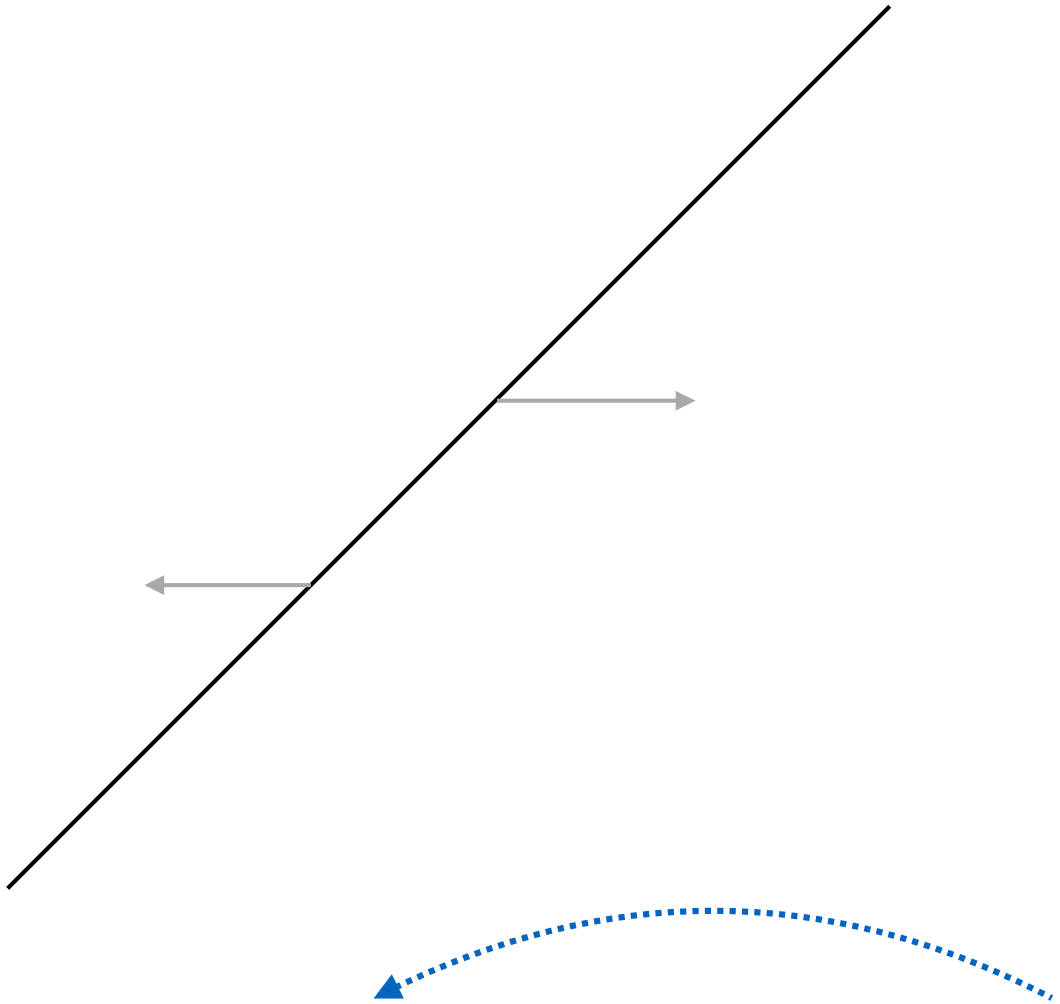
. . . . . .



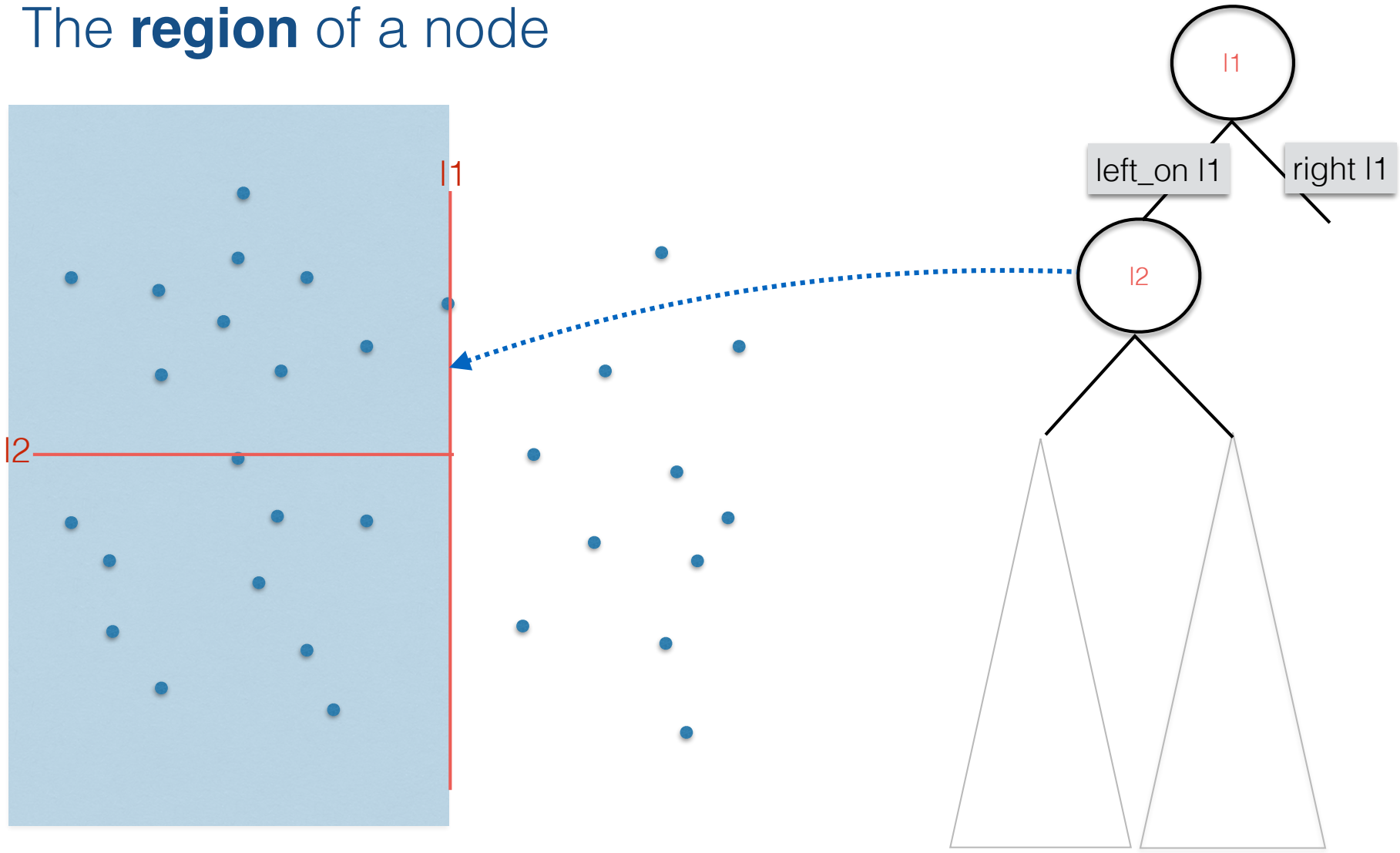
Each node in the tree corresponds to a region in the plane.


l_1
l1
left_on l1
right l1
all leaves are right of l1
all leaves are left_on l1

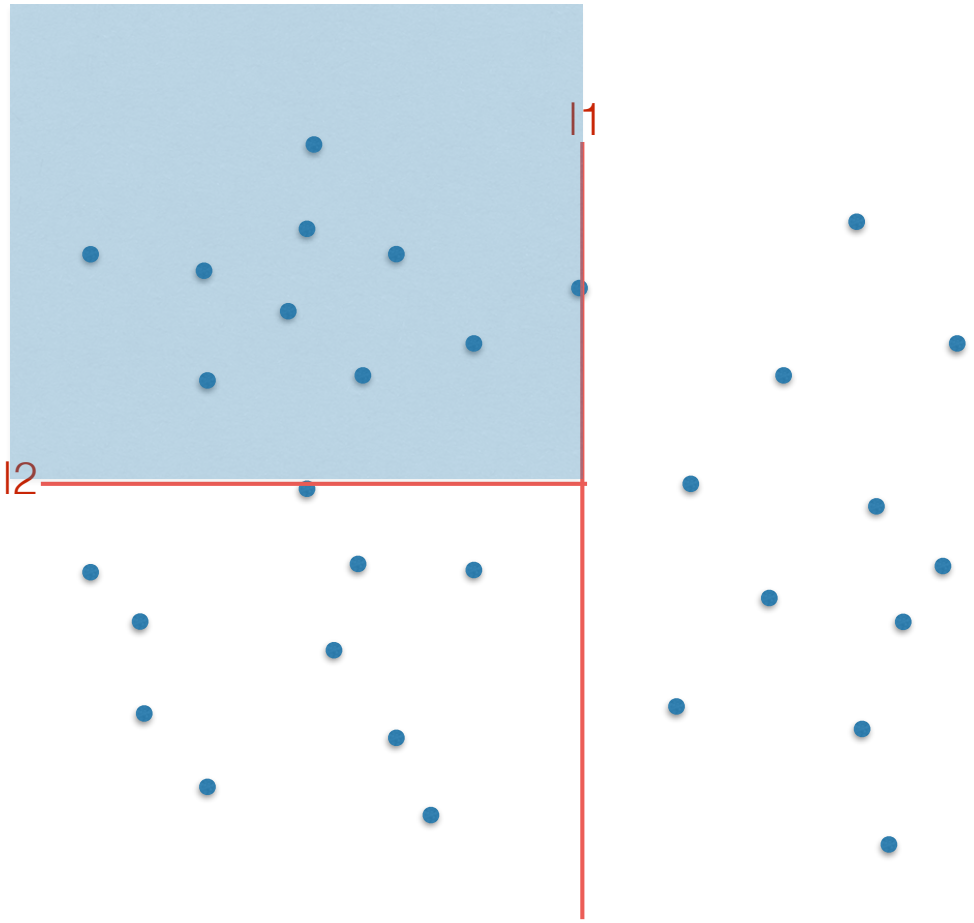
A line in the plane defines two **half-planes**

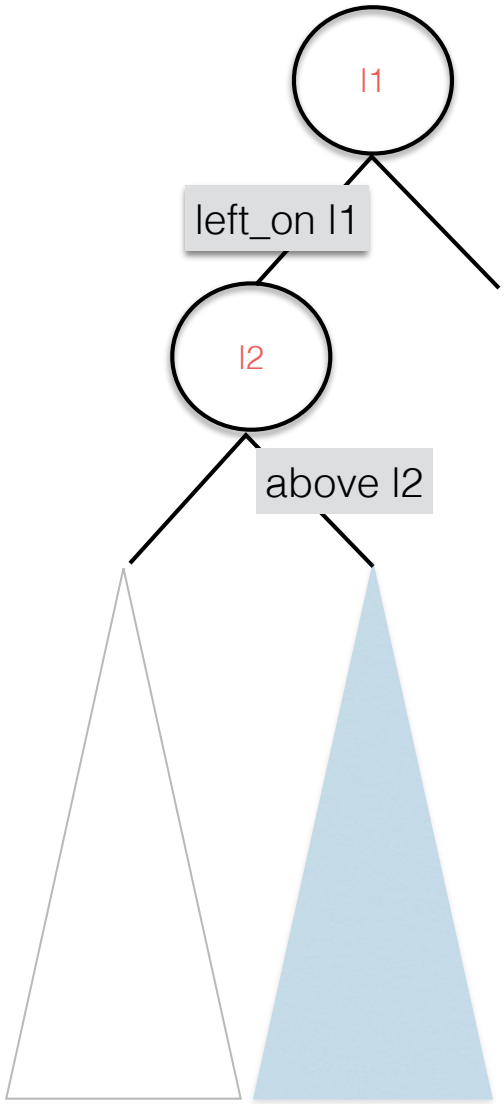


The **region** of a node

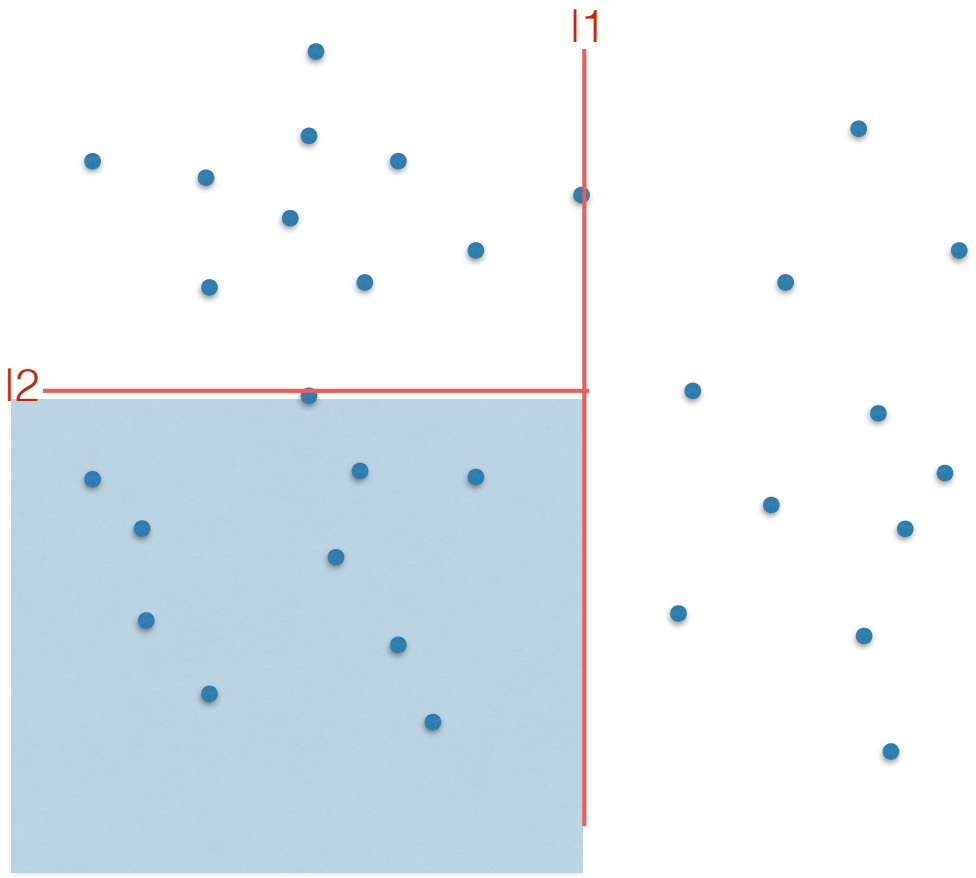


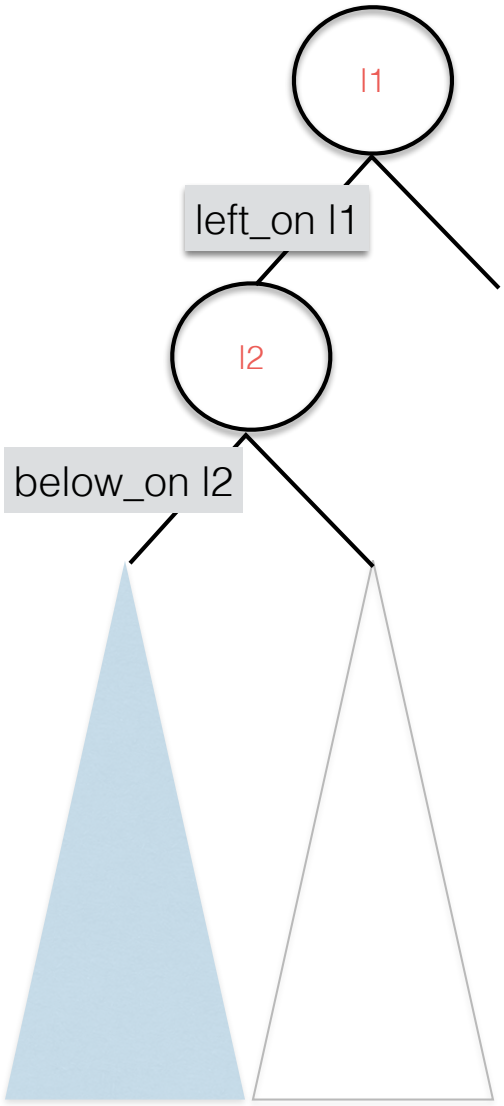
The **region** of a node



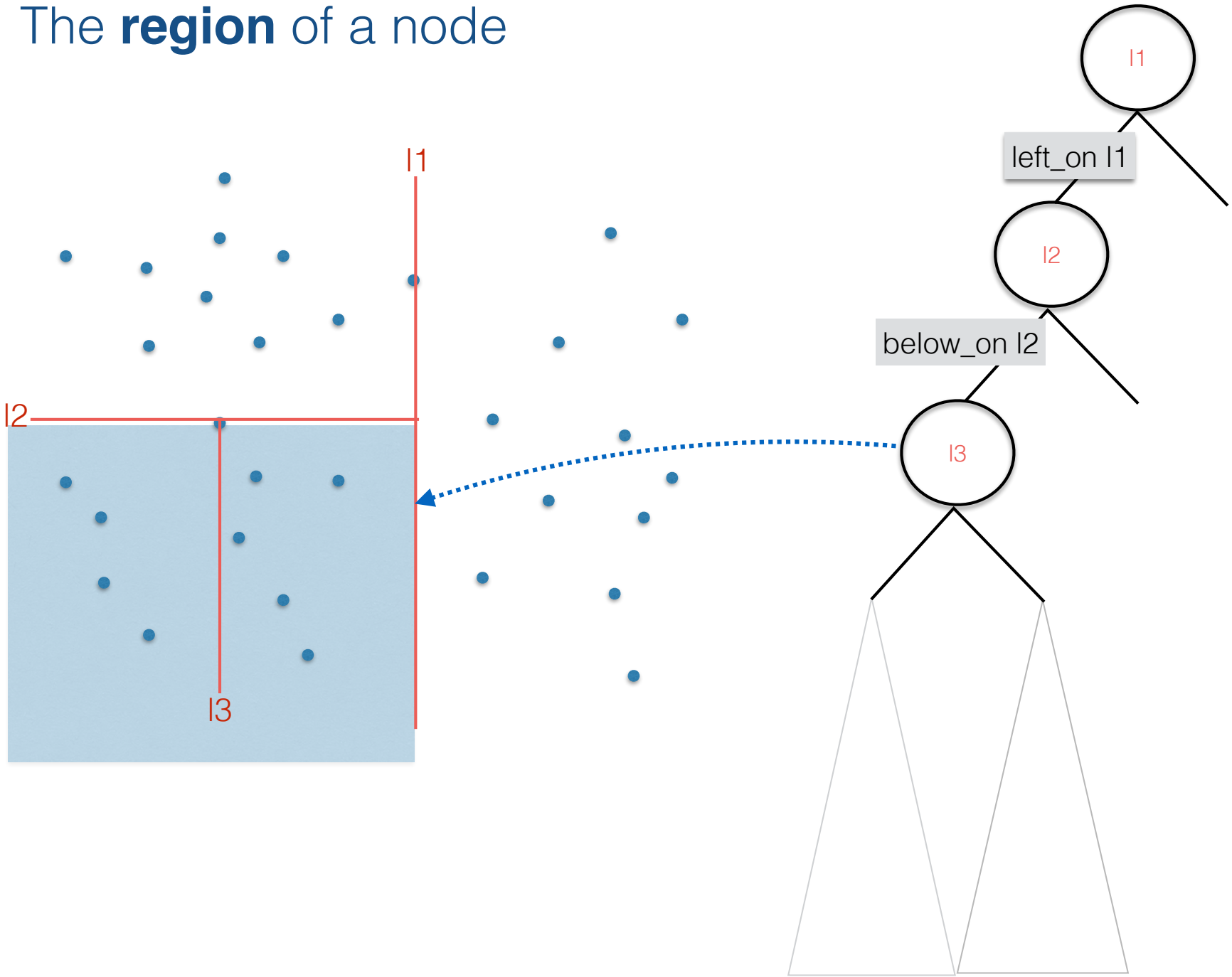


The **region** of a node

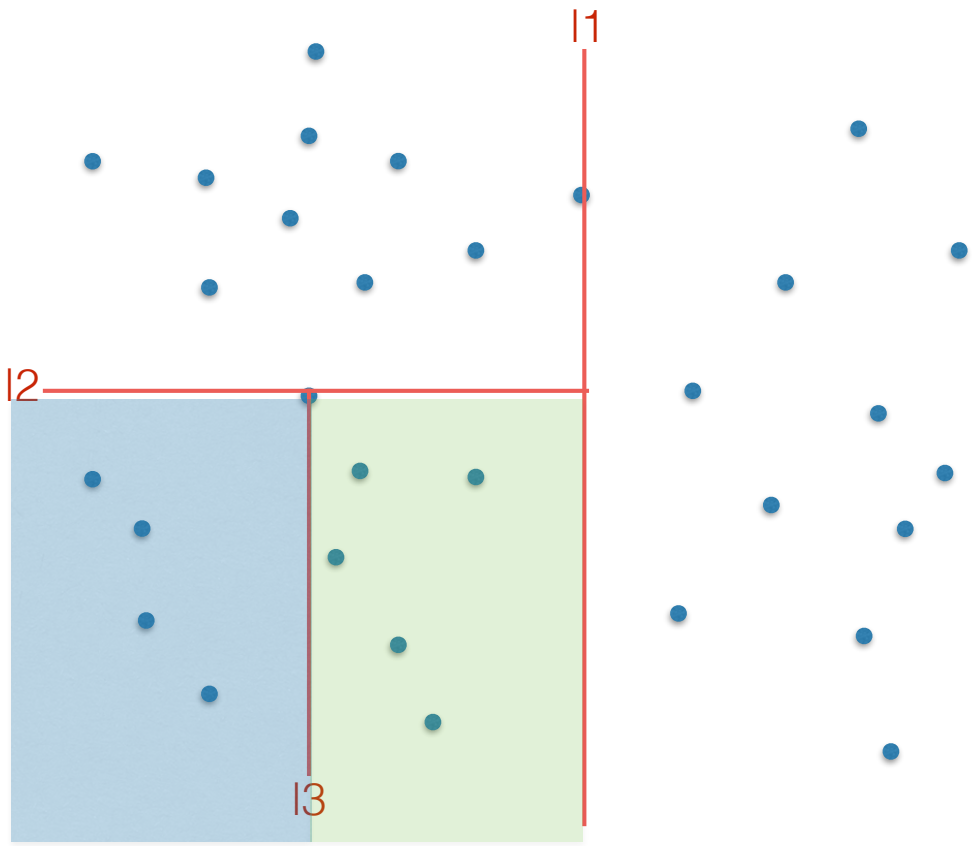


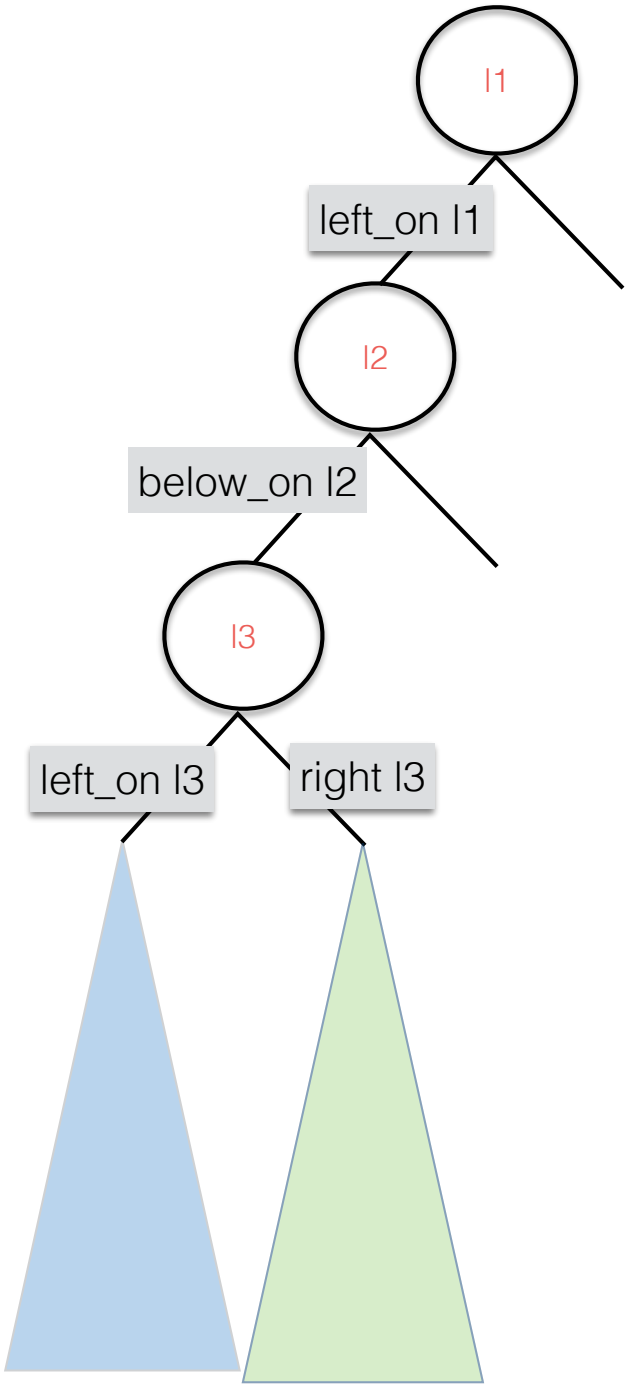


The **region** of a node

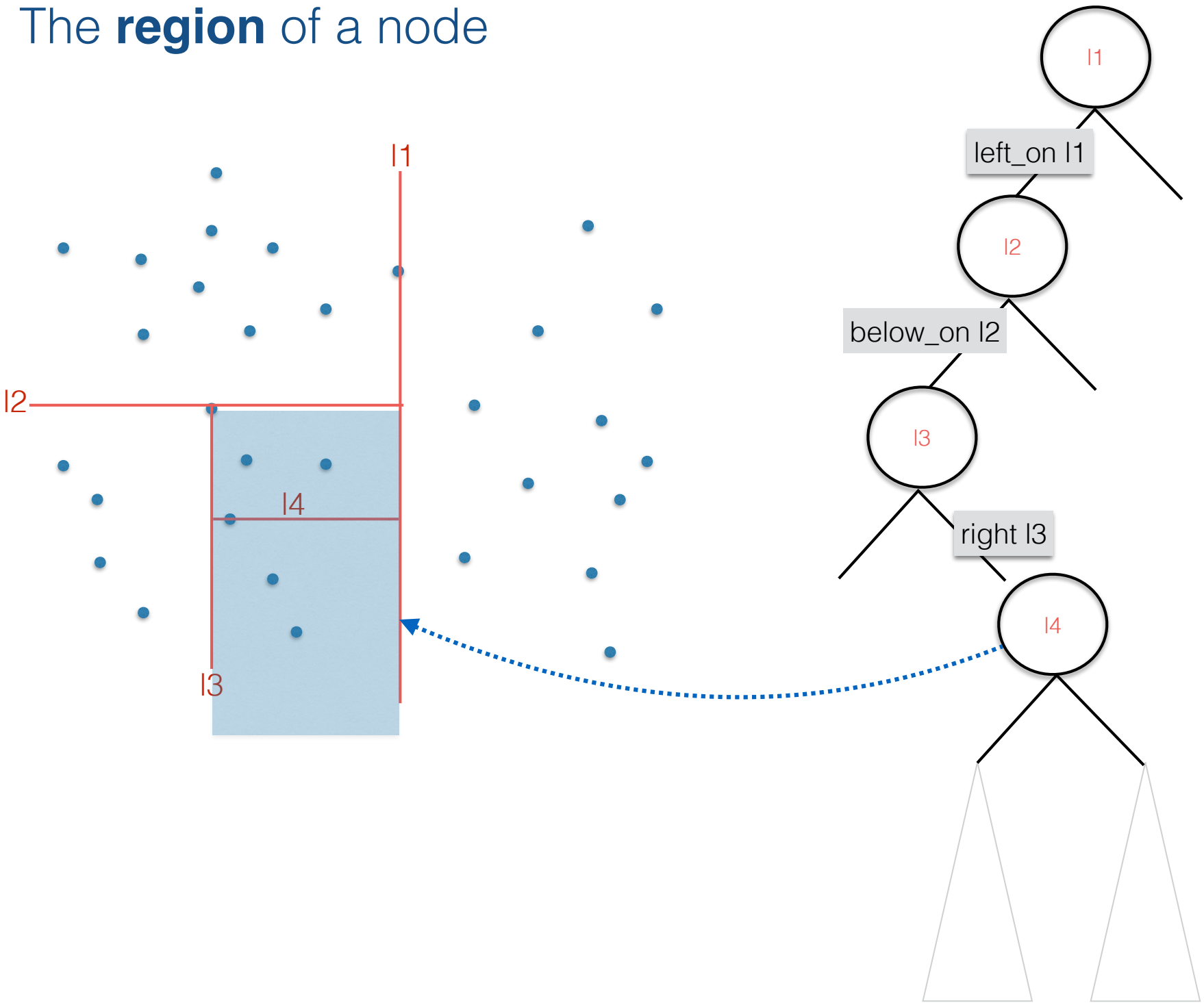


The **region** of a node

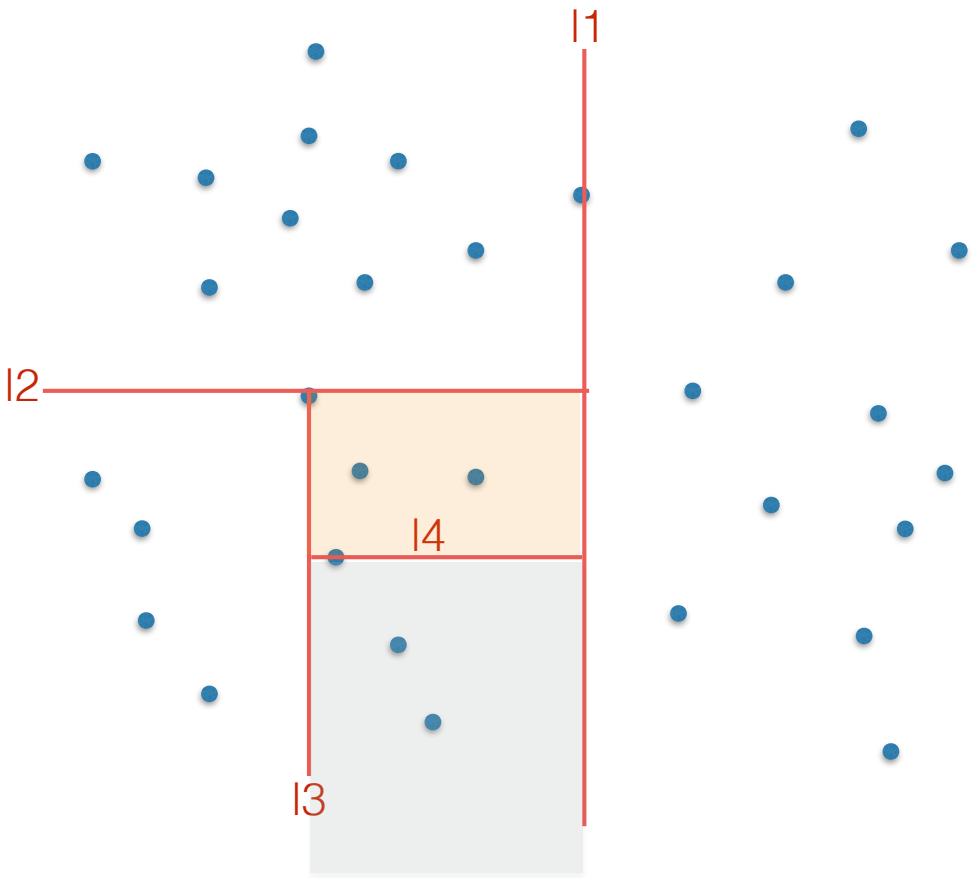


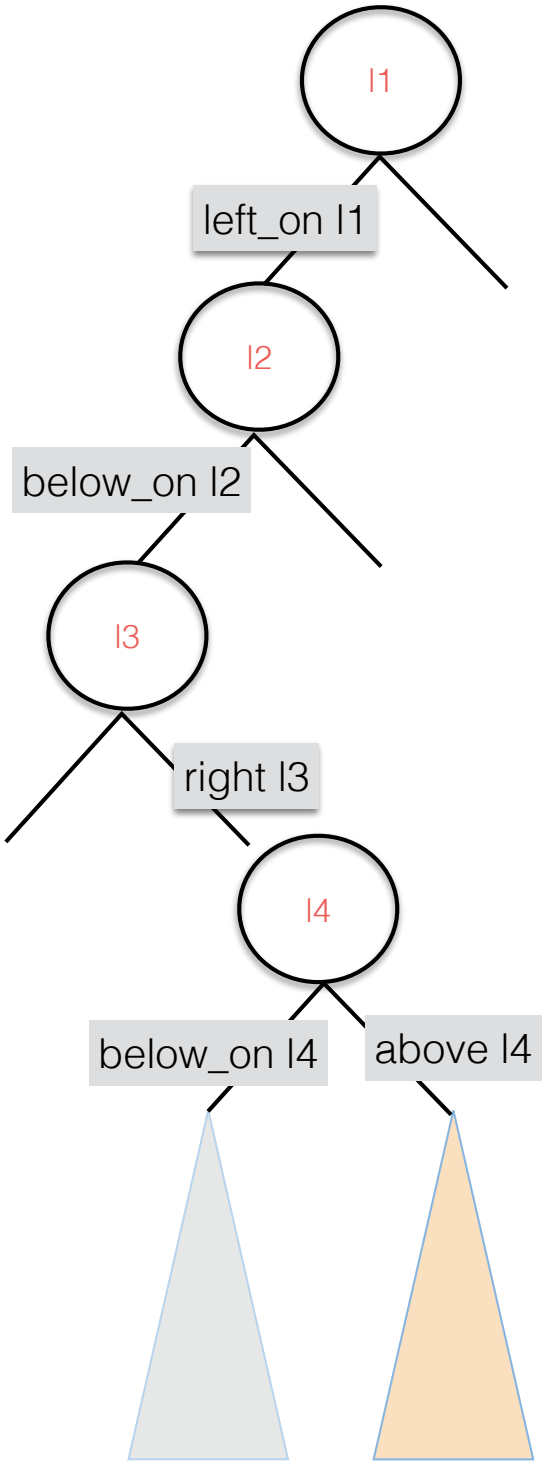


The **region** of a node

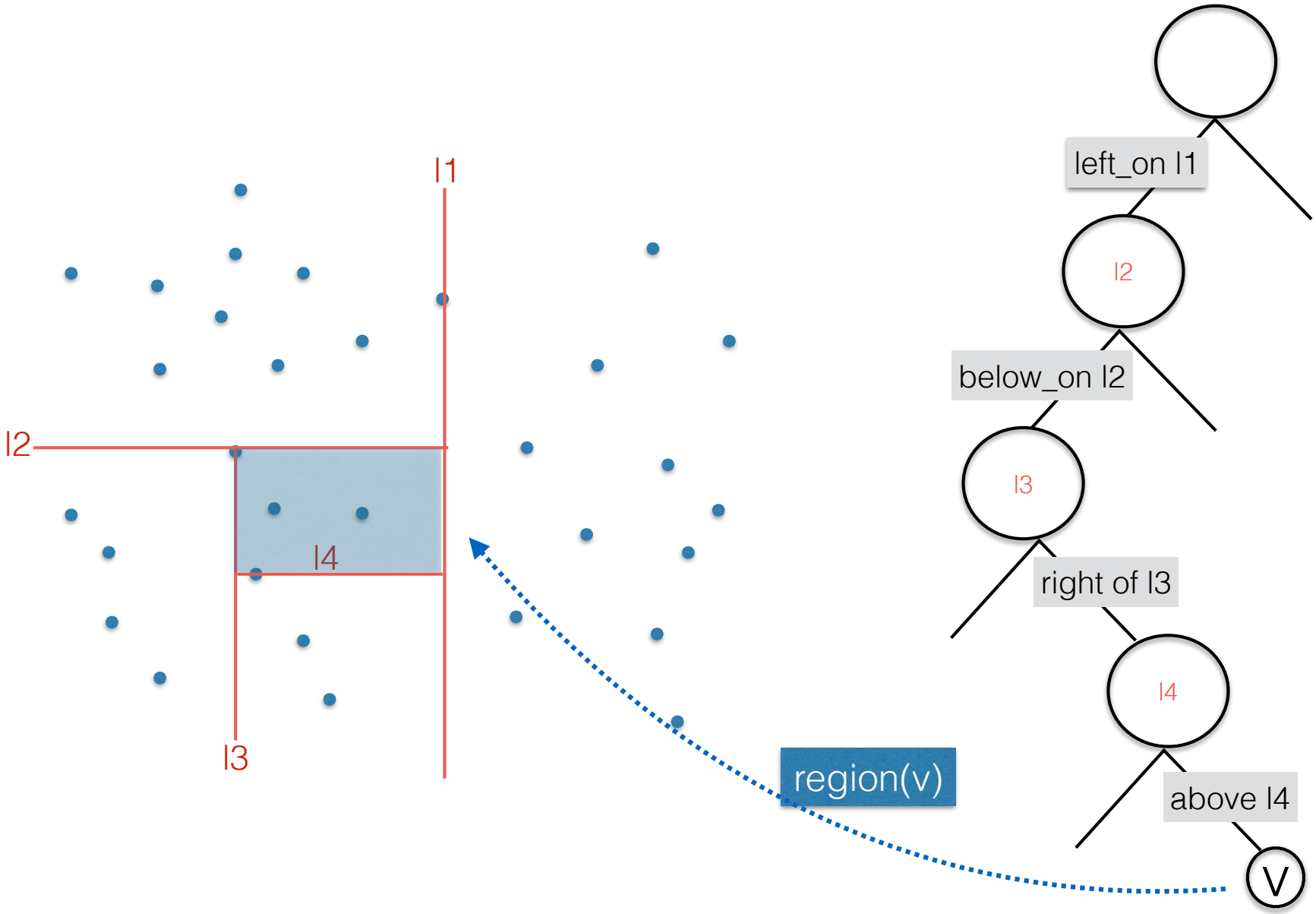


The **region** of a node

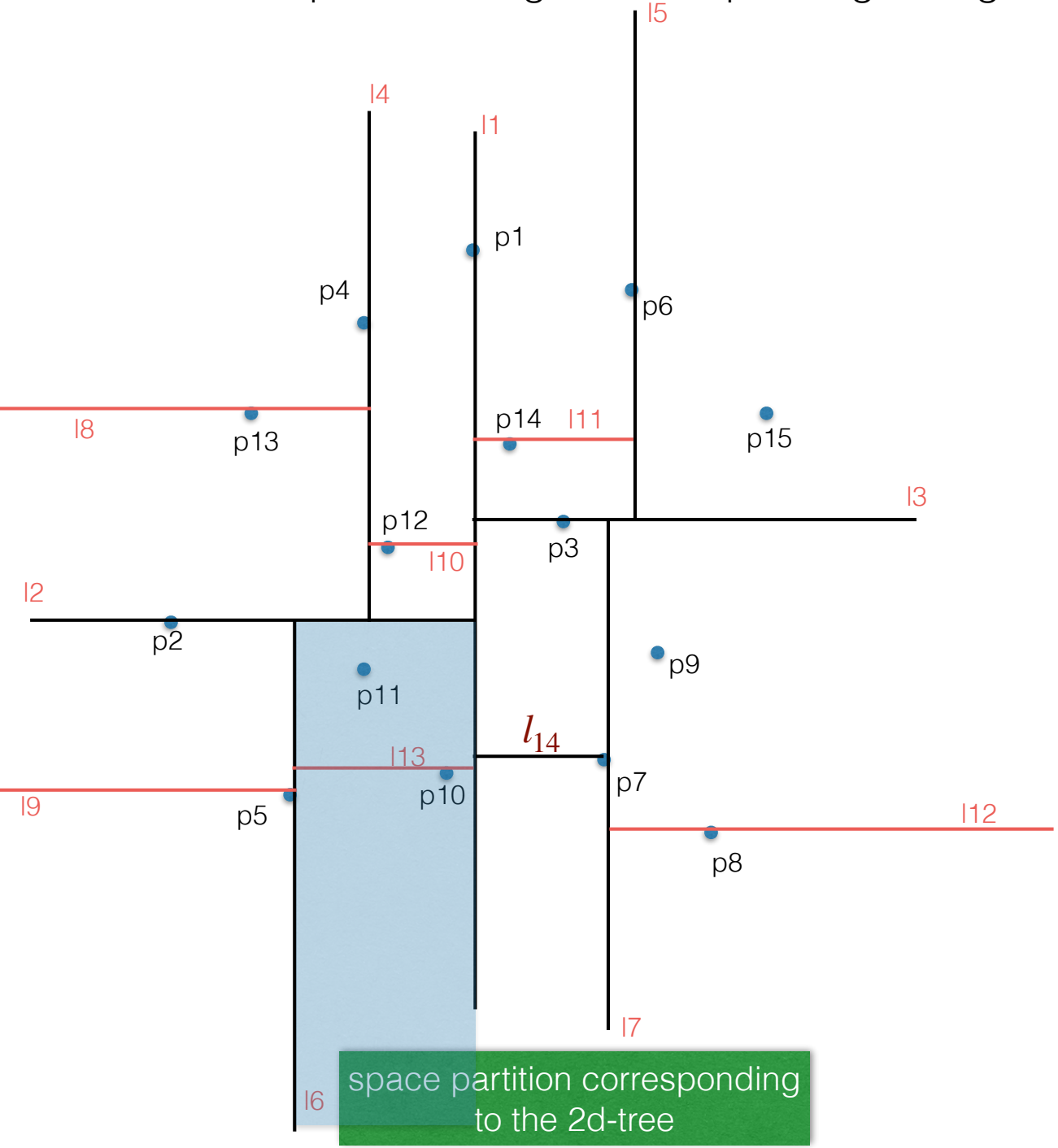




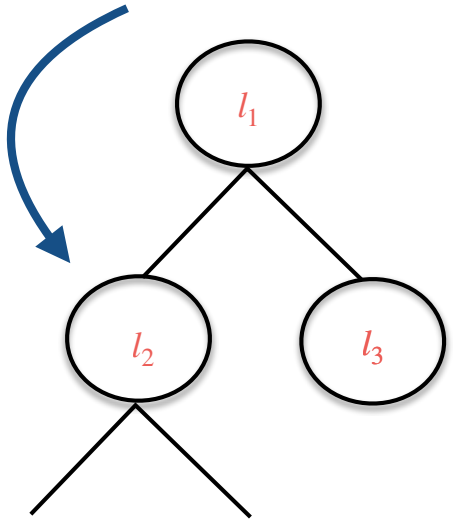
Each node in the tree corresponds to a region in the plane, which is the intersection of the half-planes of all it's ancestors.



Exercise: express the region corresponding to, e.g., second grandchild of this node

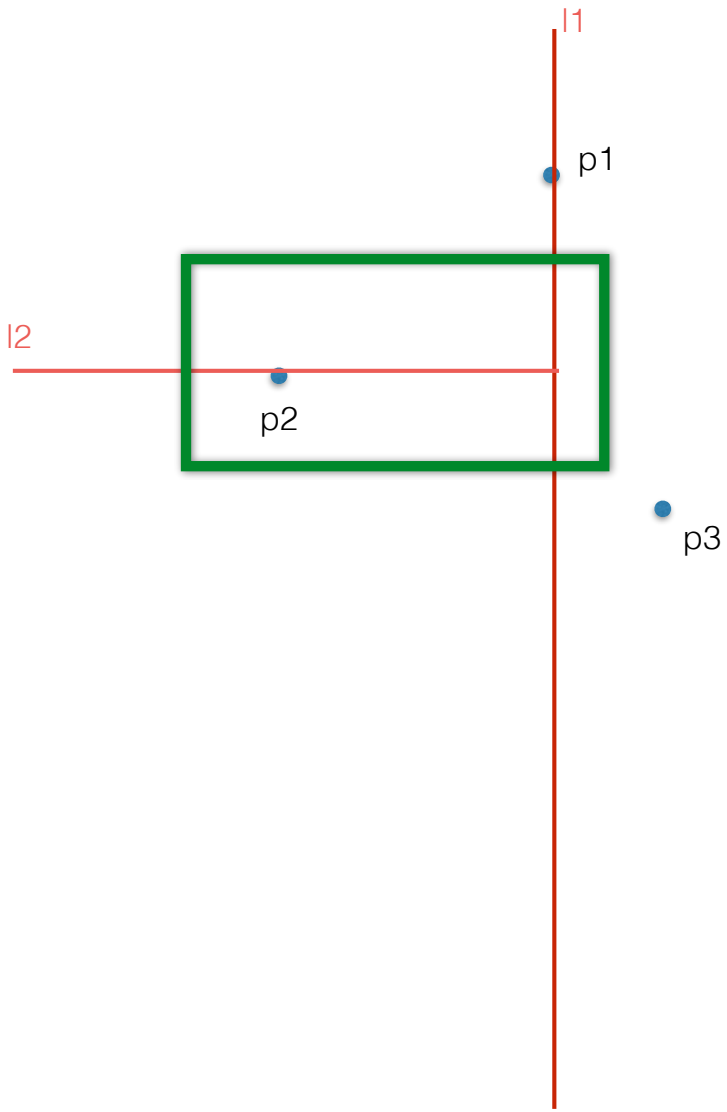


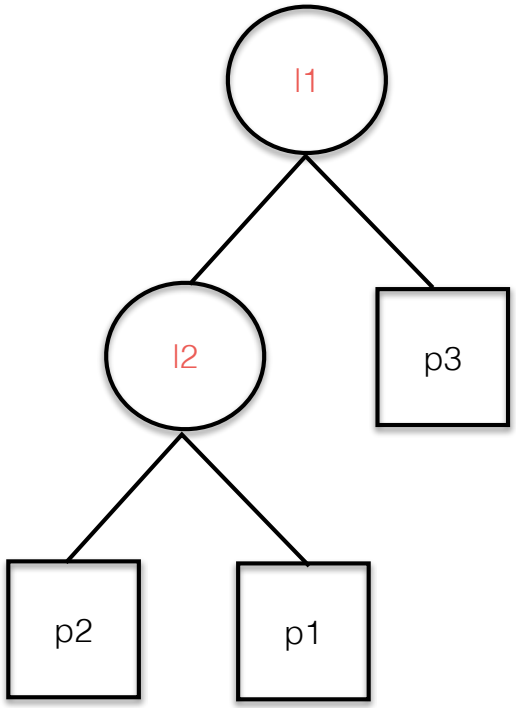
space partition corresponding to the 2d-tree

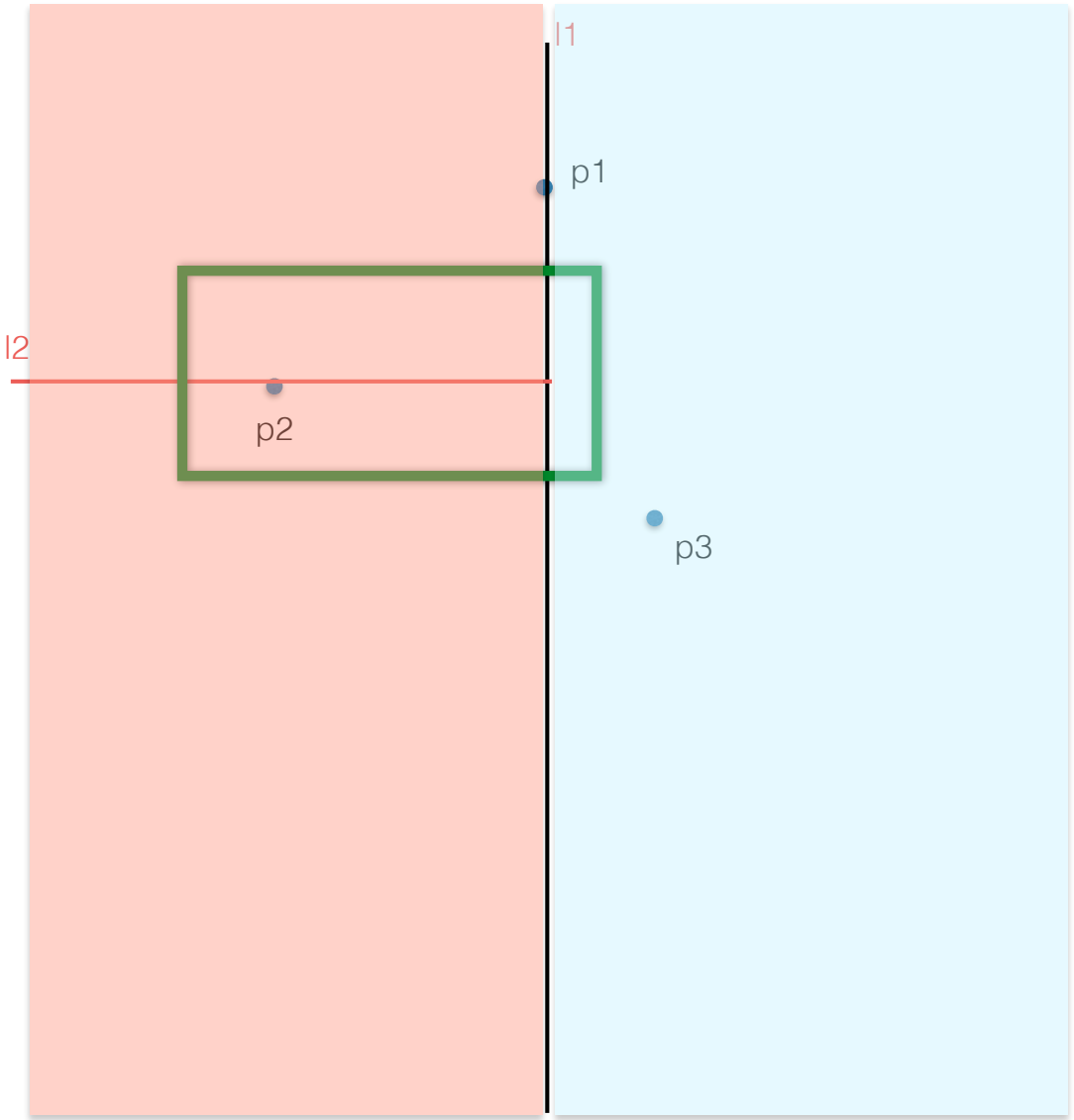


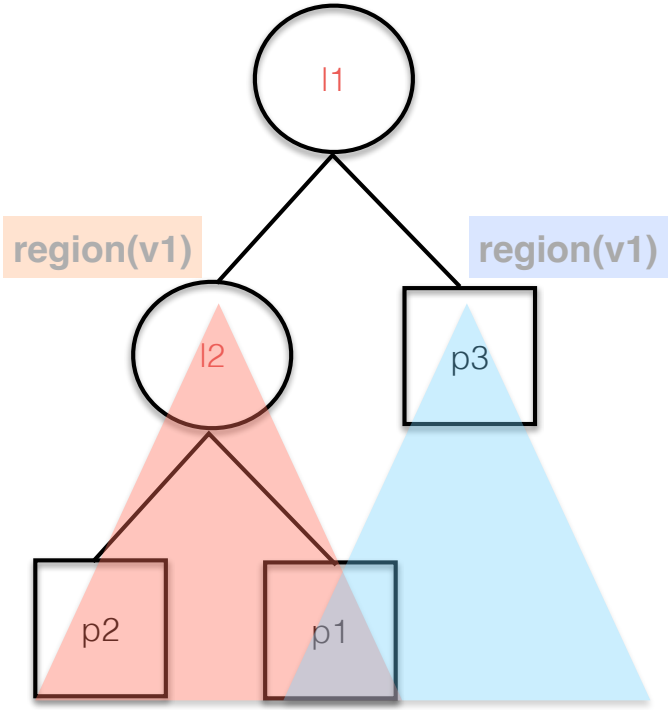
the corresponding 2d-tree

How do we answer range queries on kd-trees ?

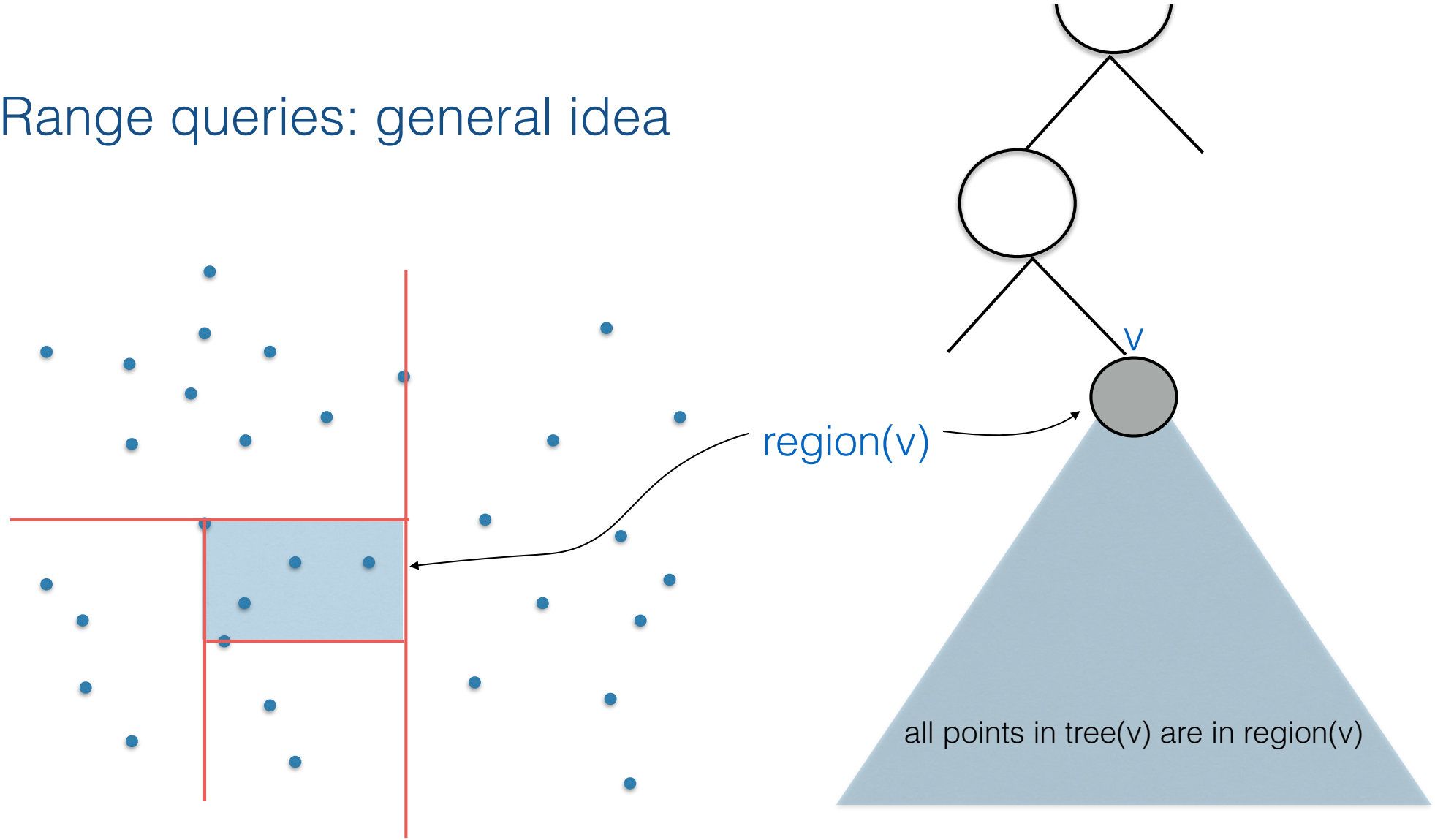


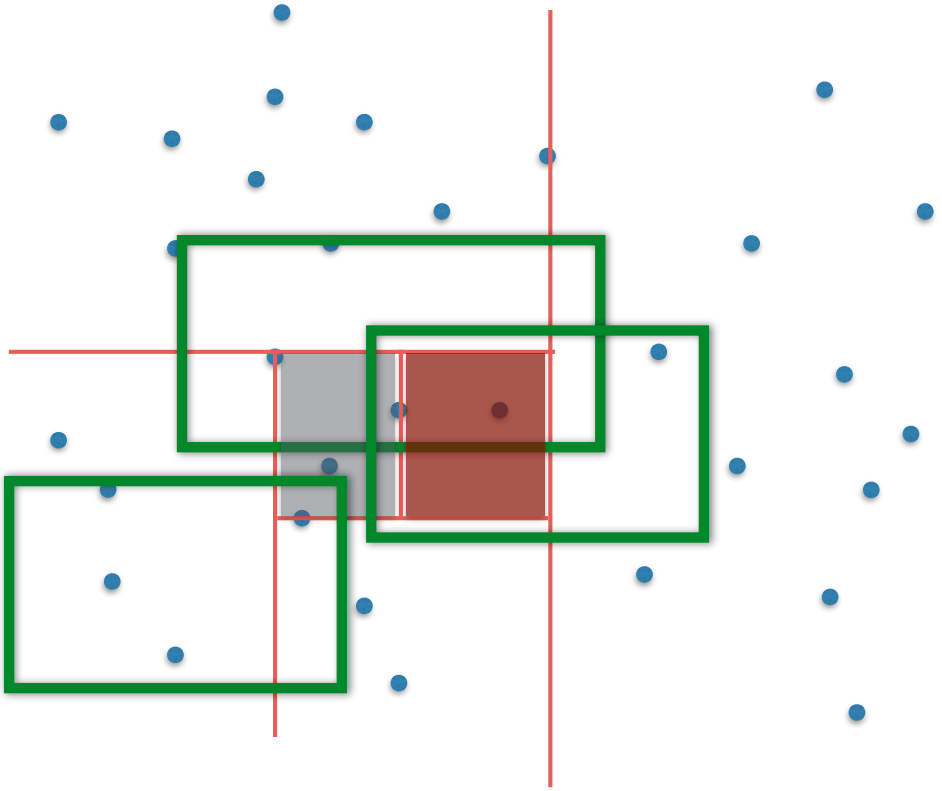


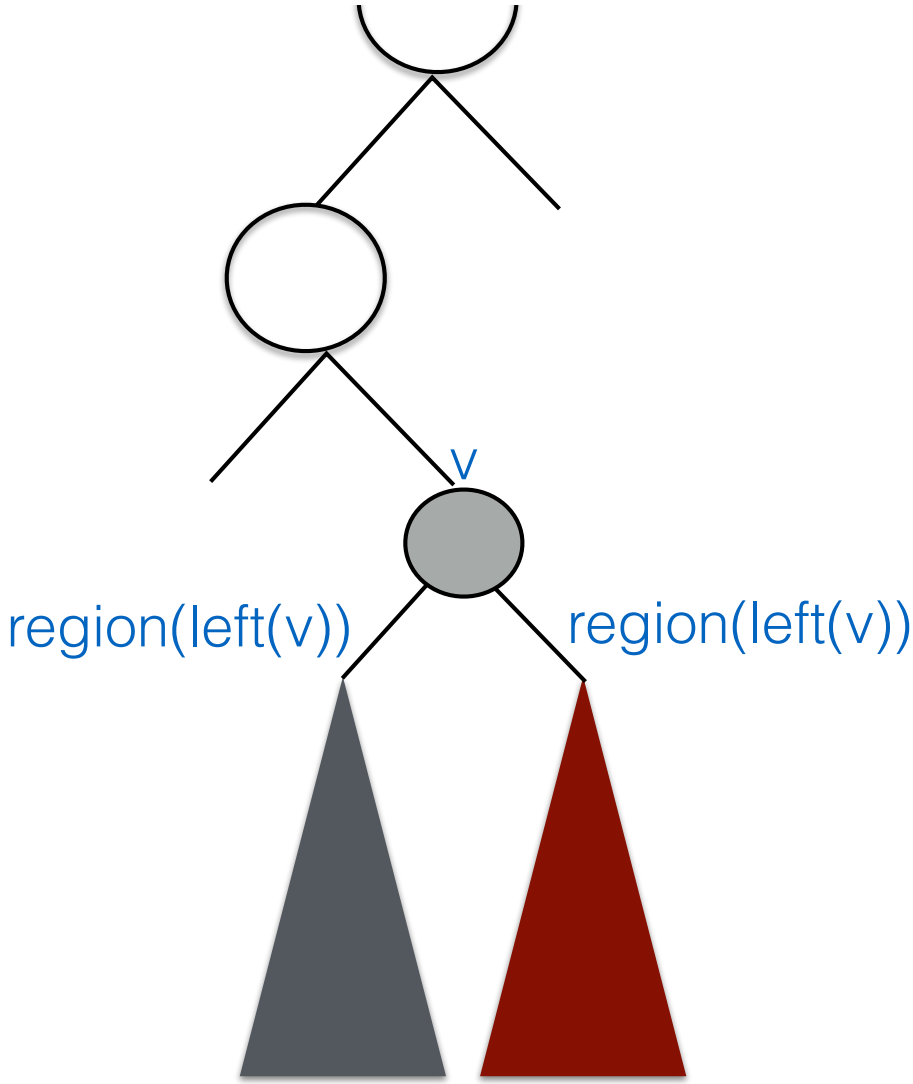

l1
p1
l2
p2
p3


l1
region(v1)
region(v1)
l2
p3
p2
p1

Range queries: general idea







Case 1:range intersects both children

Case 2: range intersects only one child

Case 3: child completely contained in range

Algorithm SEARCHKDTREE(v, R)
Input. The root of (a subtree of) a kd-tree, and a range R
Output. All points at leaves below v that lie in the range.

```
1.  if v is a leaf
2.     then Report the point stored at v if it lies in R
3.     else  if region(lc(v)) is fully contained in R
4.              then REPORTSUBTREE(lc(v))
5.              else  if region(lc(v)) intersects R
6.                       then SEARCHKDTREE(lc(v), R)
7.           if region(rc(v)) is fully contained in R
8.              then REPORTSUBTREE(rc(v))
9.              else  if region(rc(v)) intersects R
10.                      then SEARCHKDTREE(rc(v), R)
```

screenshot from Mark van Kreveld slides at http://www.cs.uu.nl/docs/vakken/ga/slides5a.pdf

Analysis:

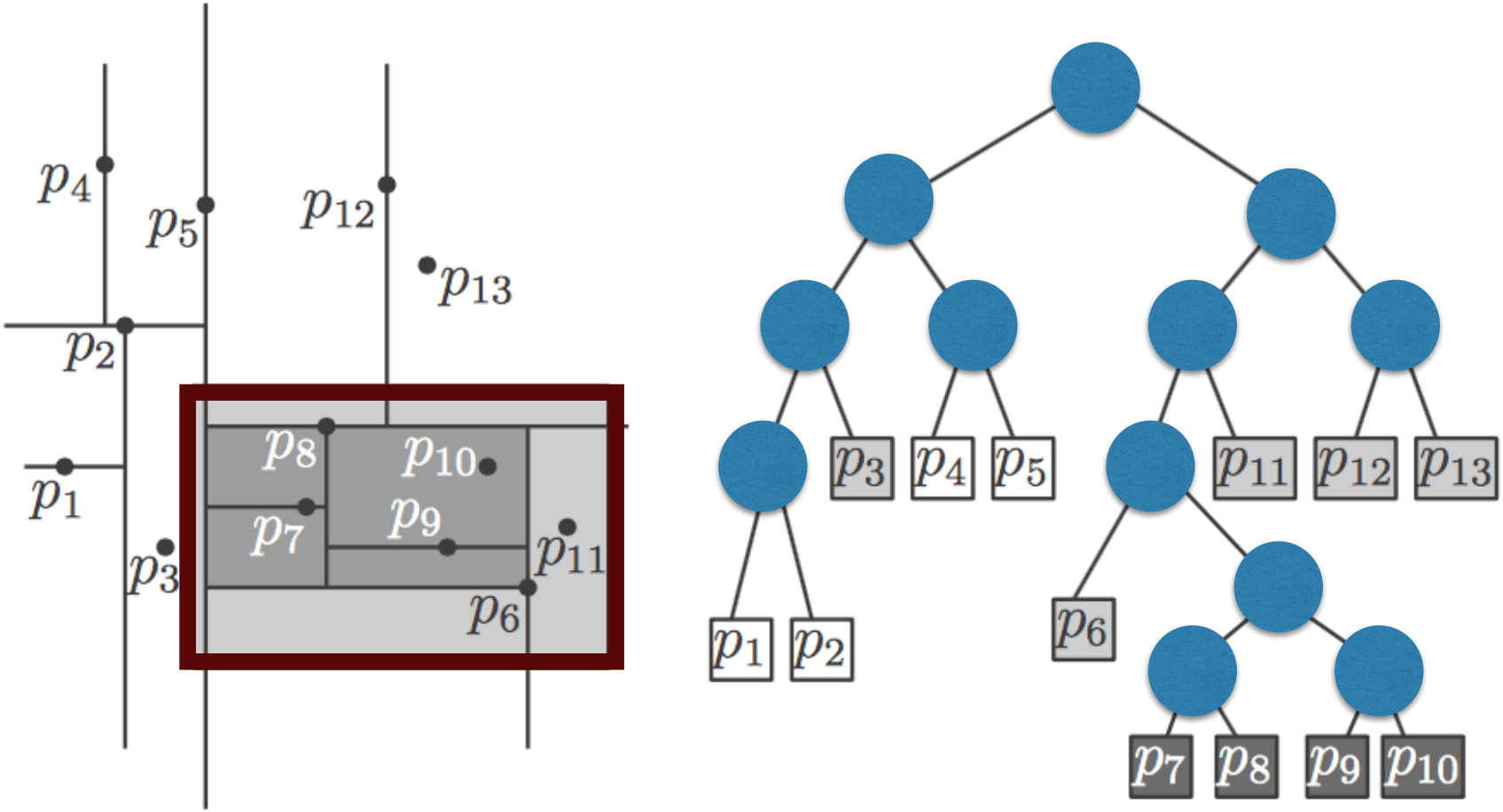
What's the running time of a range search with a kd-tree?

The running time of a range-search with a kd-tree

- For each node we can visit and recurse on **one** or **both** of its children
- The nb. of points in a child is half the nb. of points in the parent
- If at every node v we only recurse on one child =>
  $T(n) = T(n/2) + O(1)$ which solves to $O(\lg n)$
- If at every node v we recurse on both children:
  $T(n) = 2T(n/2) + O(1)$ which solves to $O(n)$
- Here we visit the children intersected by the range, which can be one or both

- So what's the running time of a search? $O(\lg n)$? or $O(n)$?

To analyze the time to answer a range query we'll look at the nodes visited in the tree



screenshot from Mark van Kreveld slides at http://www.cs.uu.nl/docs/vakken/ga/slides5a.pdf

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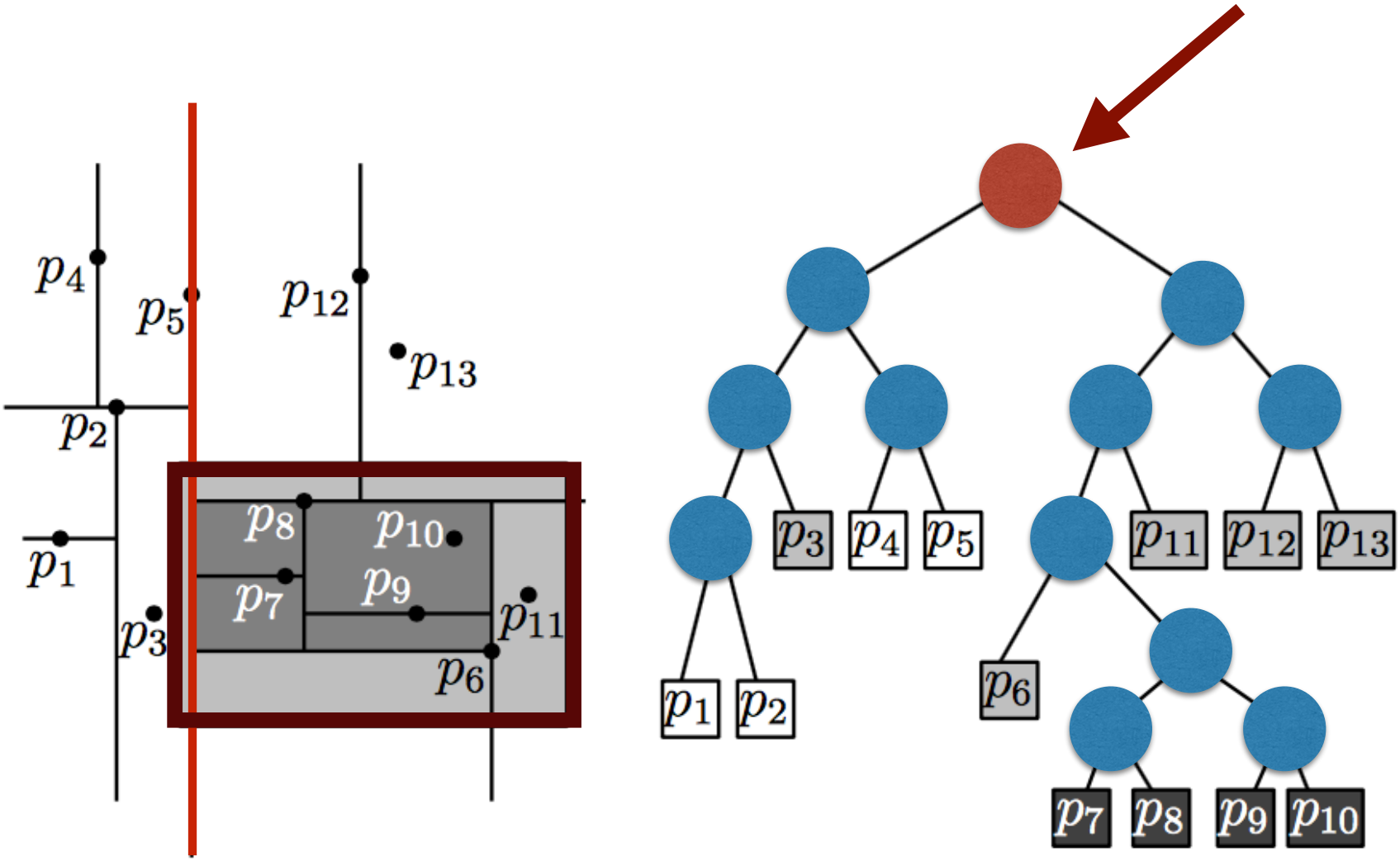
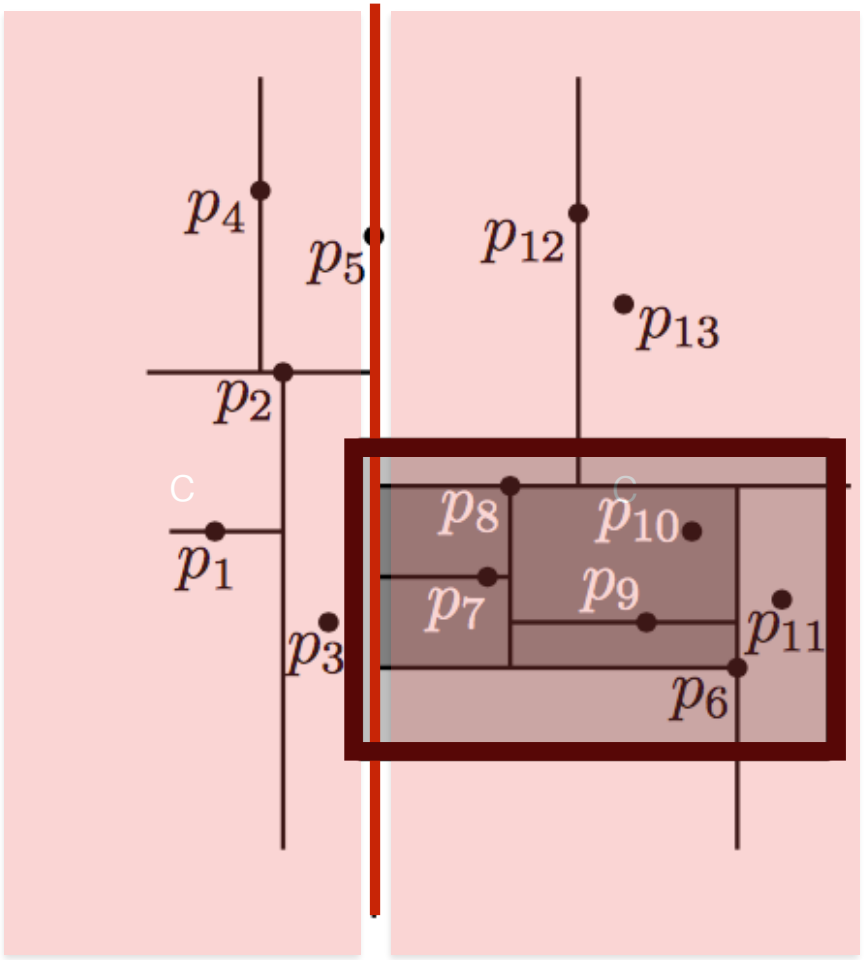
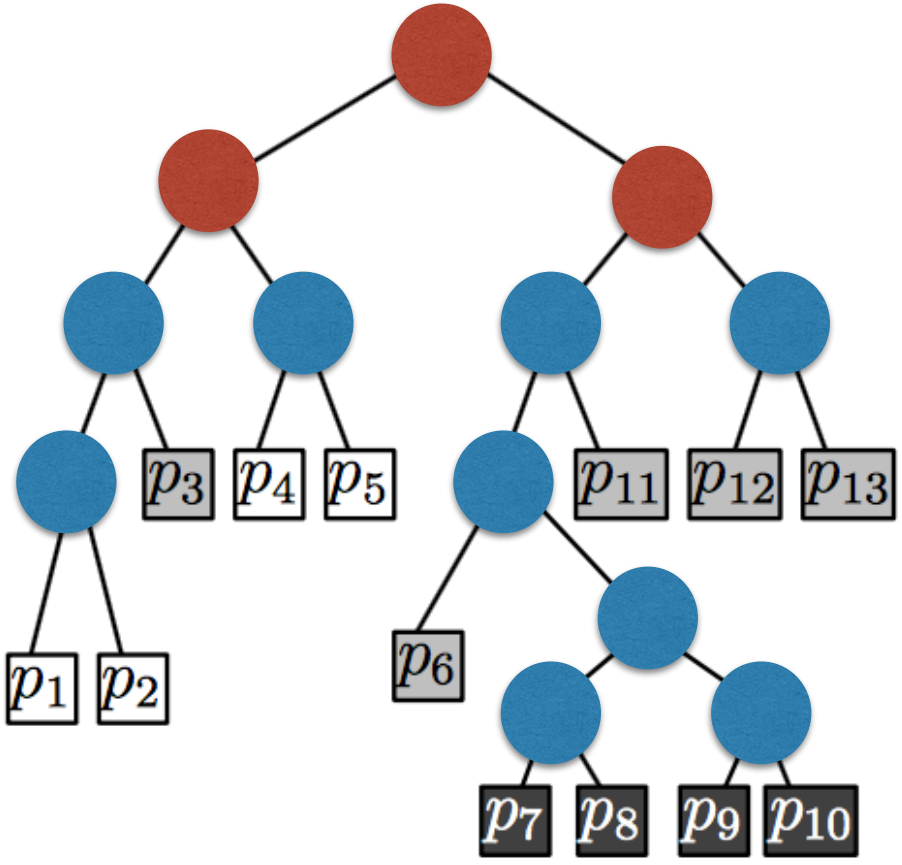


screenshot from Mark van Kreveld slides at http://www.cs.uu.nl/docs/vakken/ga/slides5a.pdf

To analyze the time to answer a range query we'll look at the nodes visited in the tree





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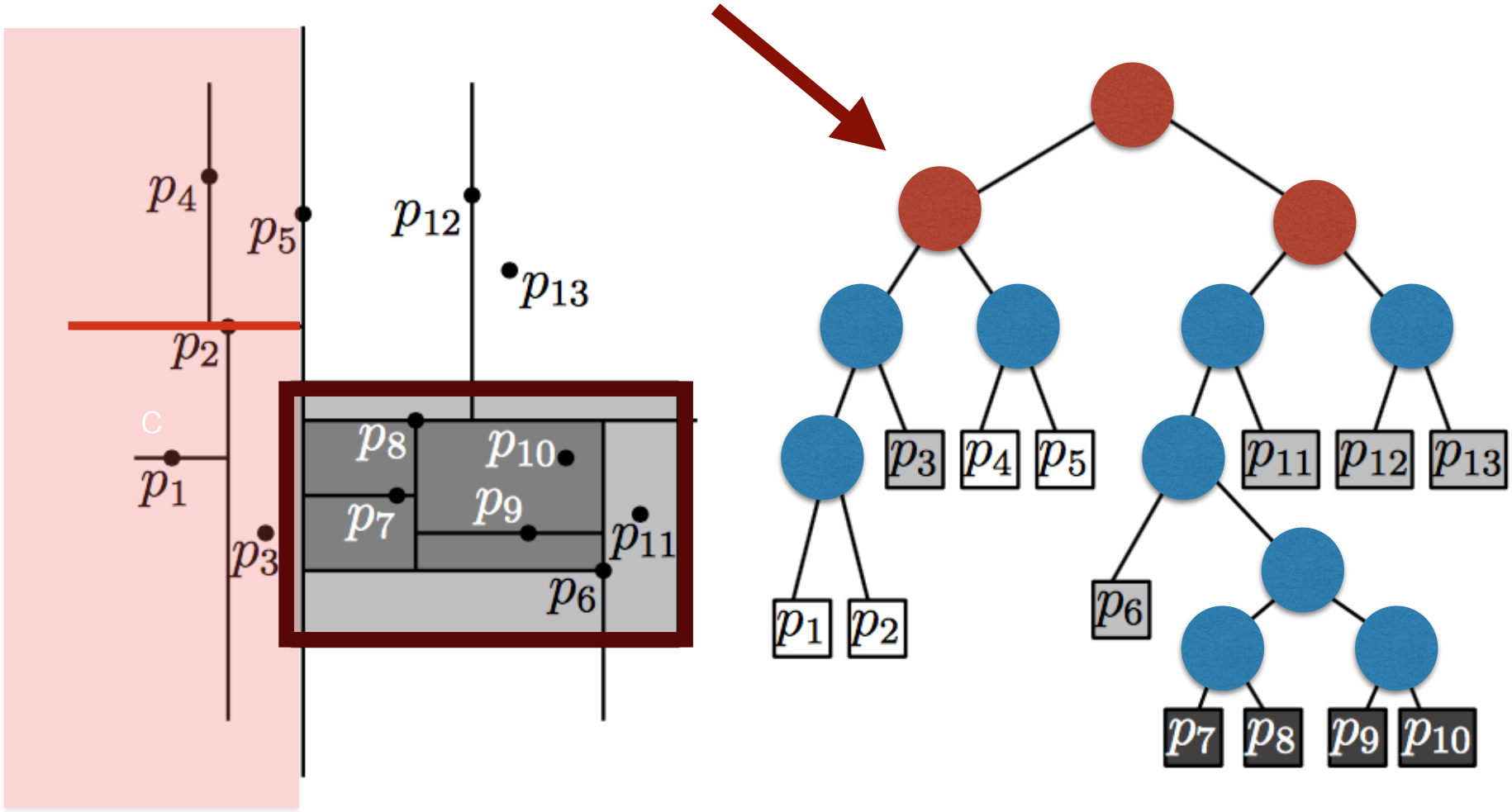
To analyze the time to answer a range query we'll look at the nodes visited in the tree



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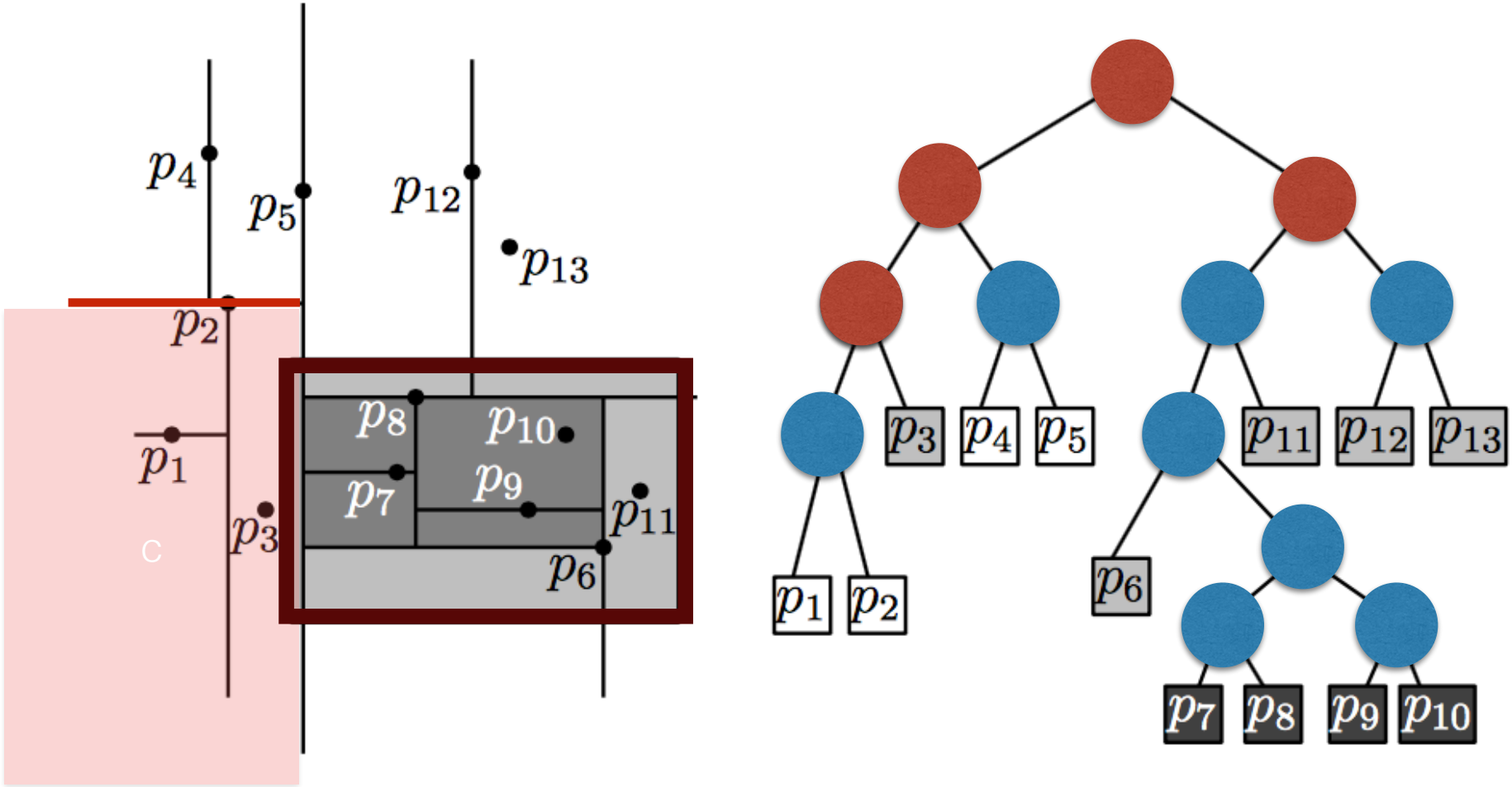

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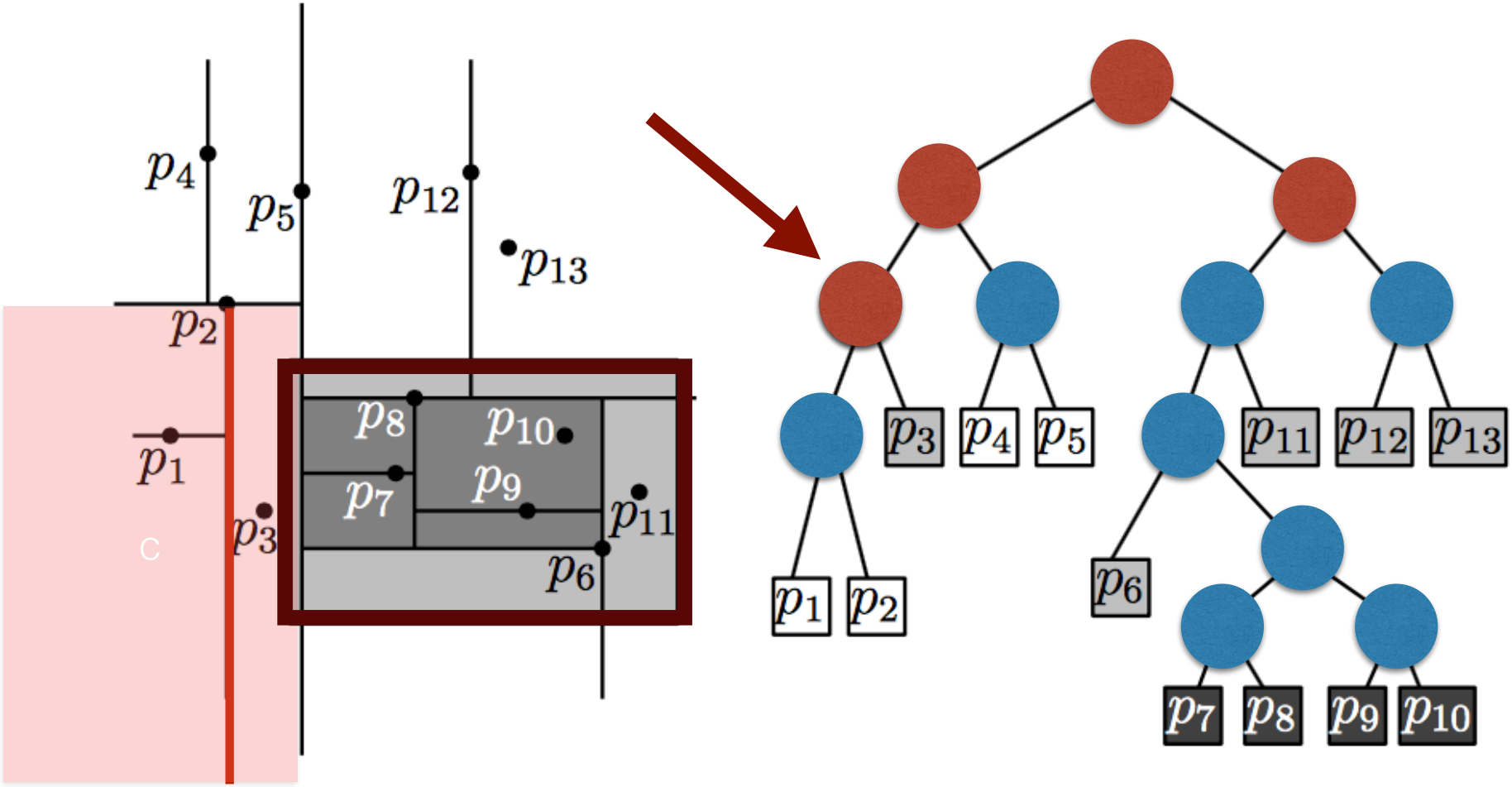

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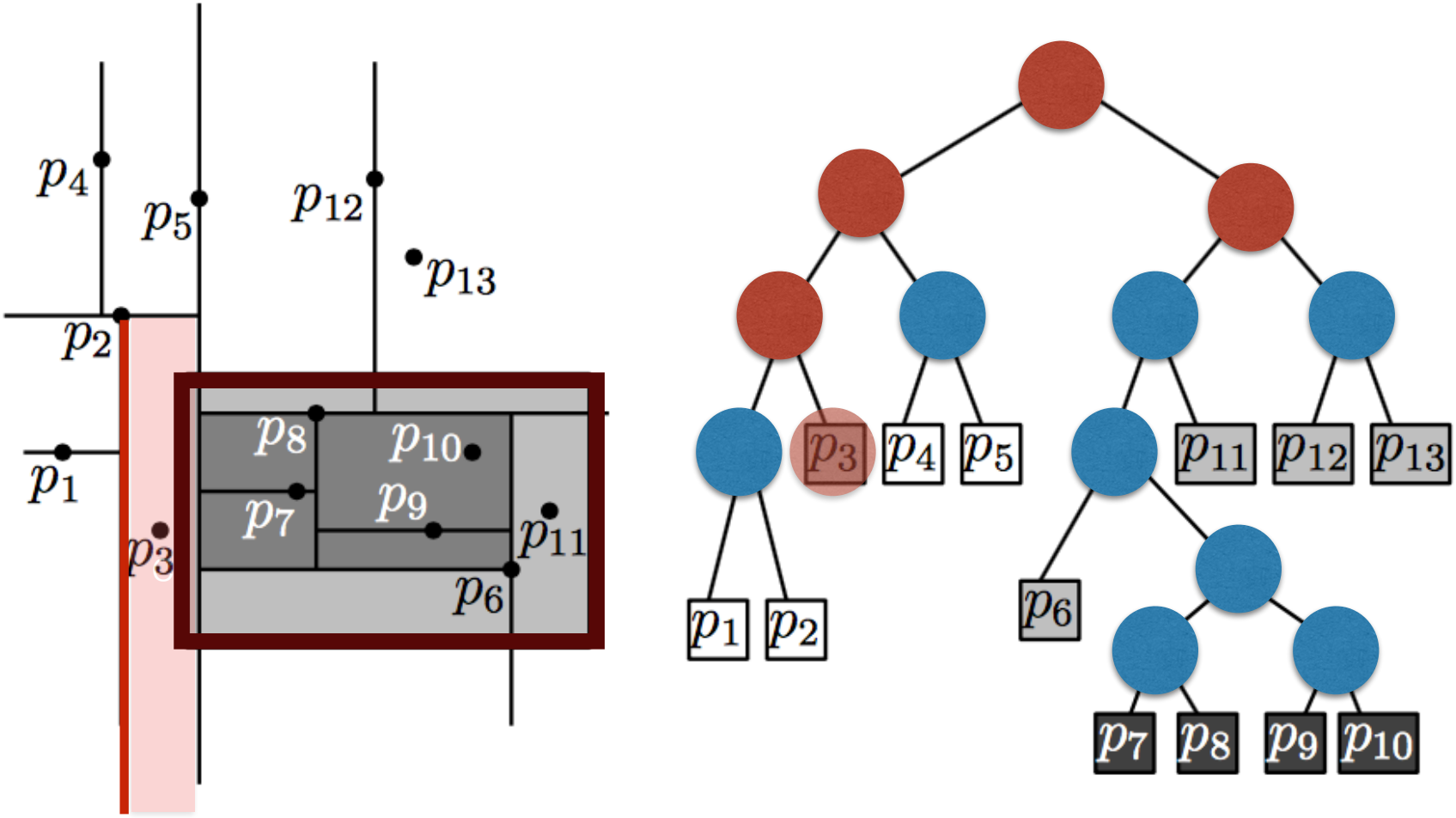

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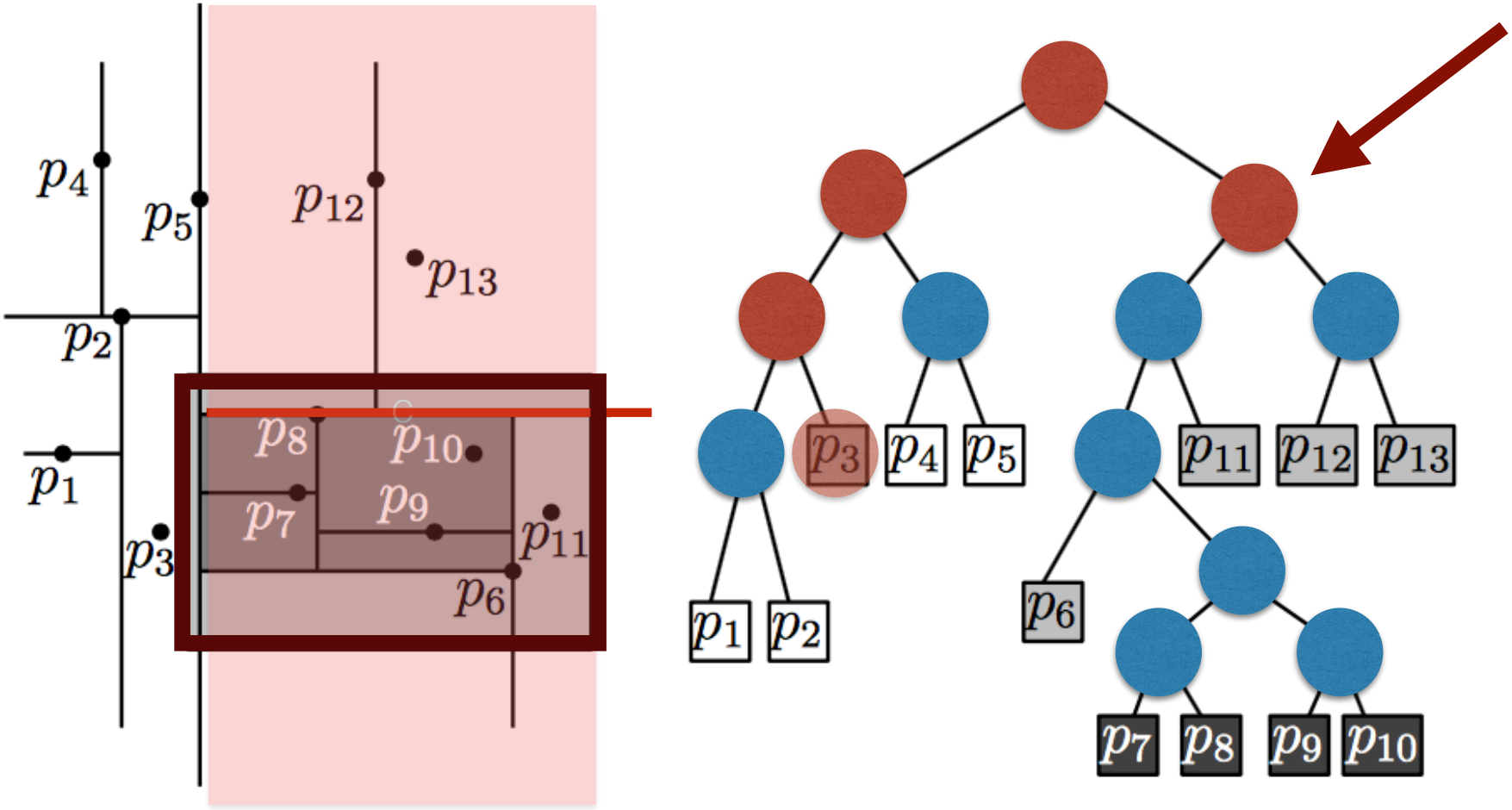

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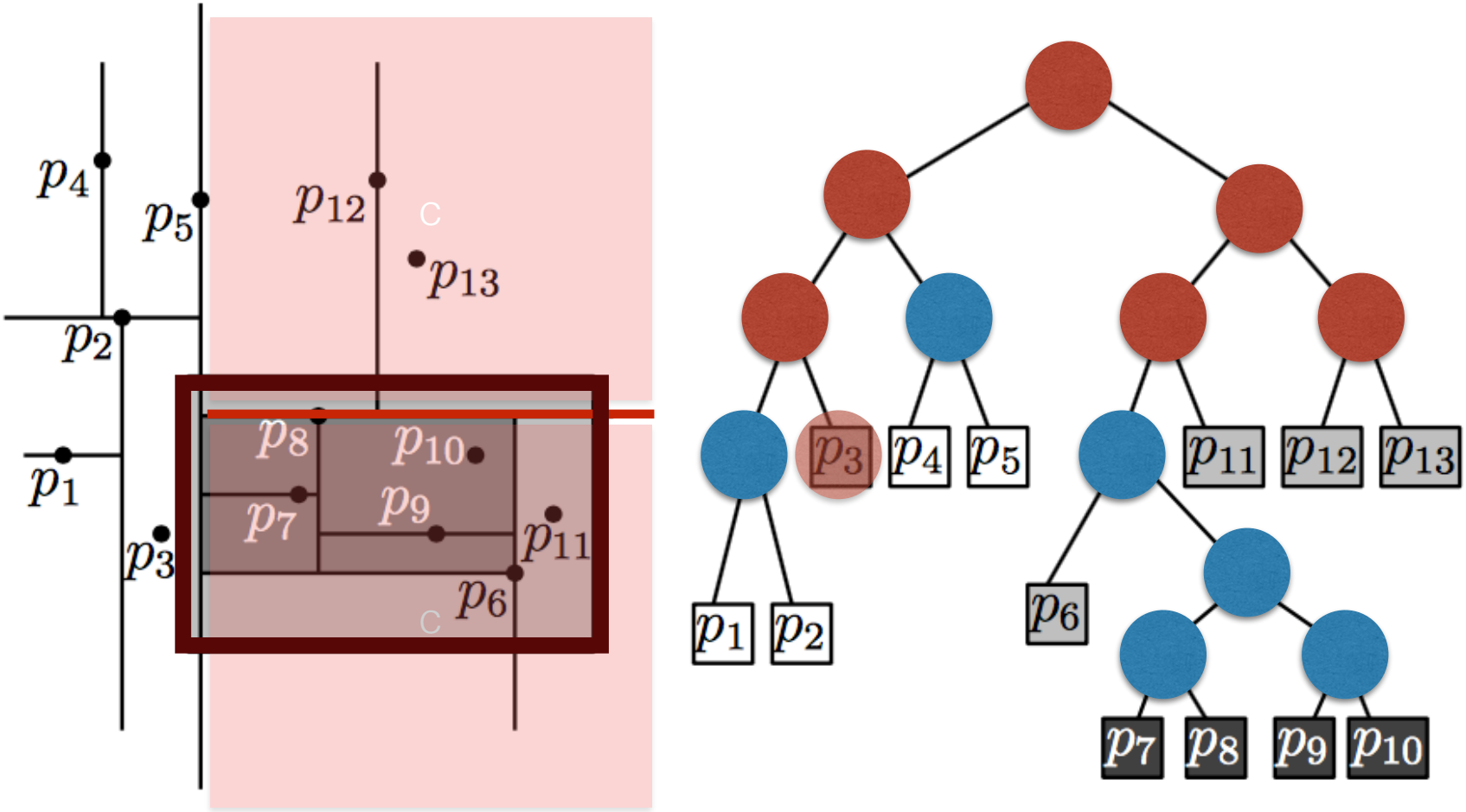

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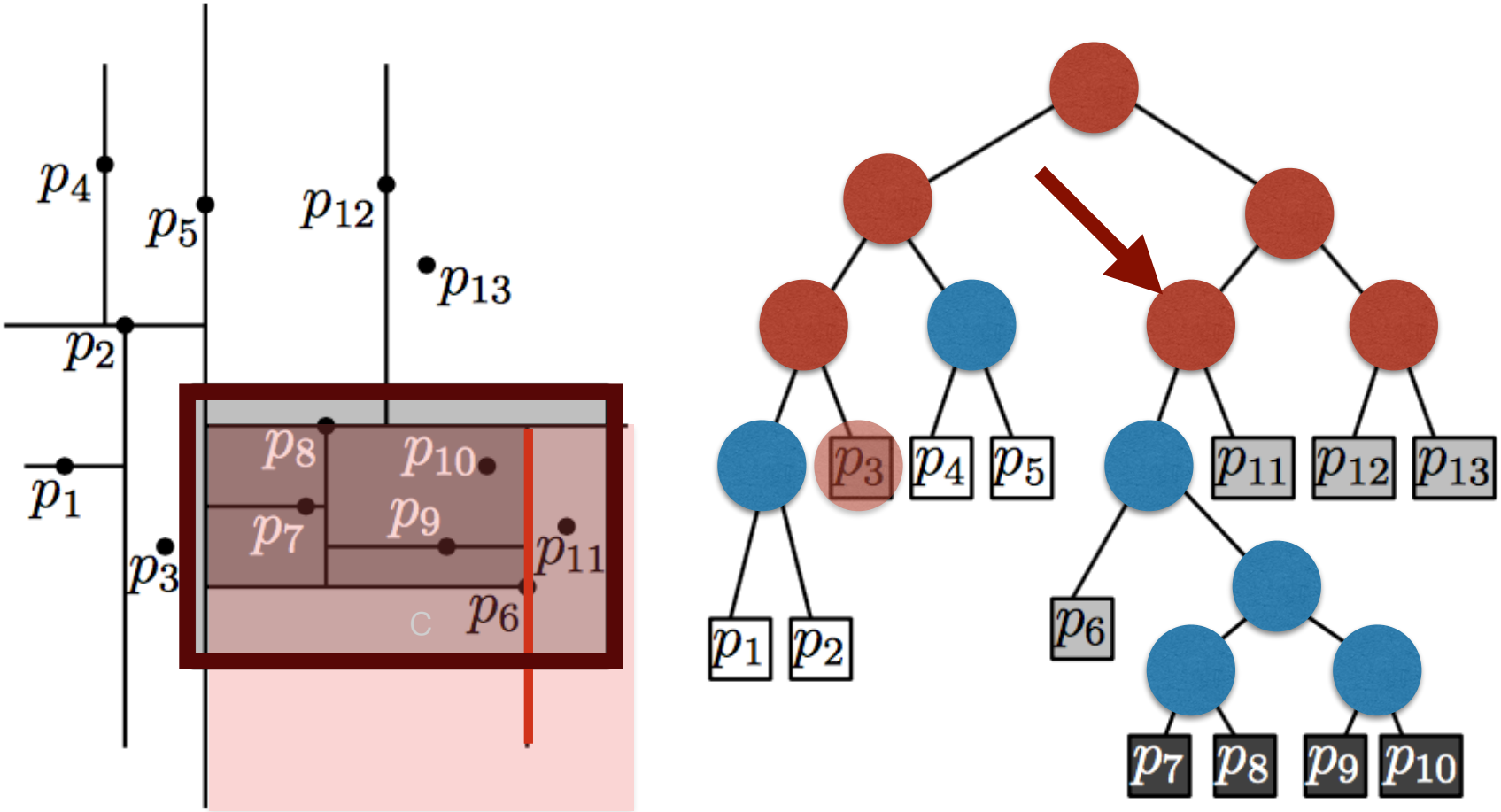

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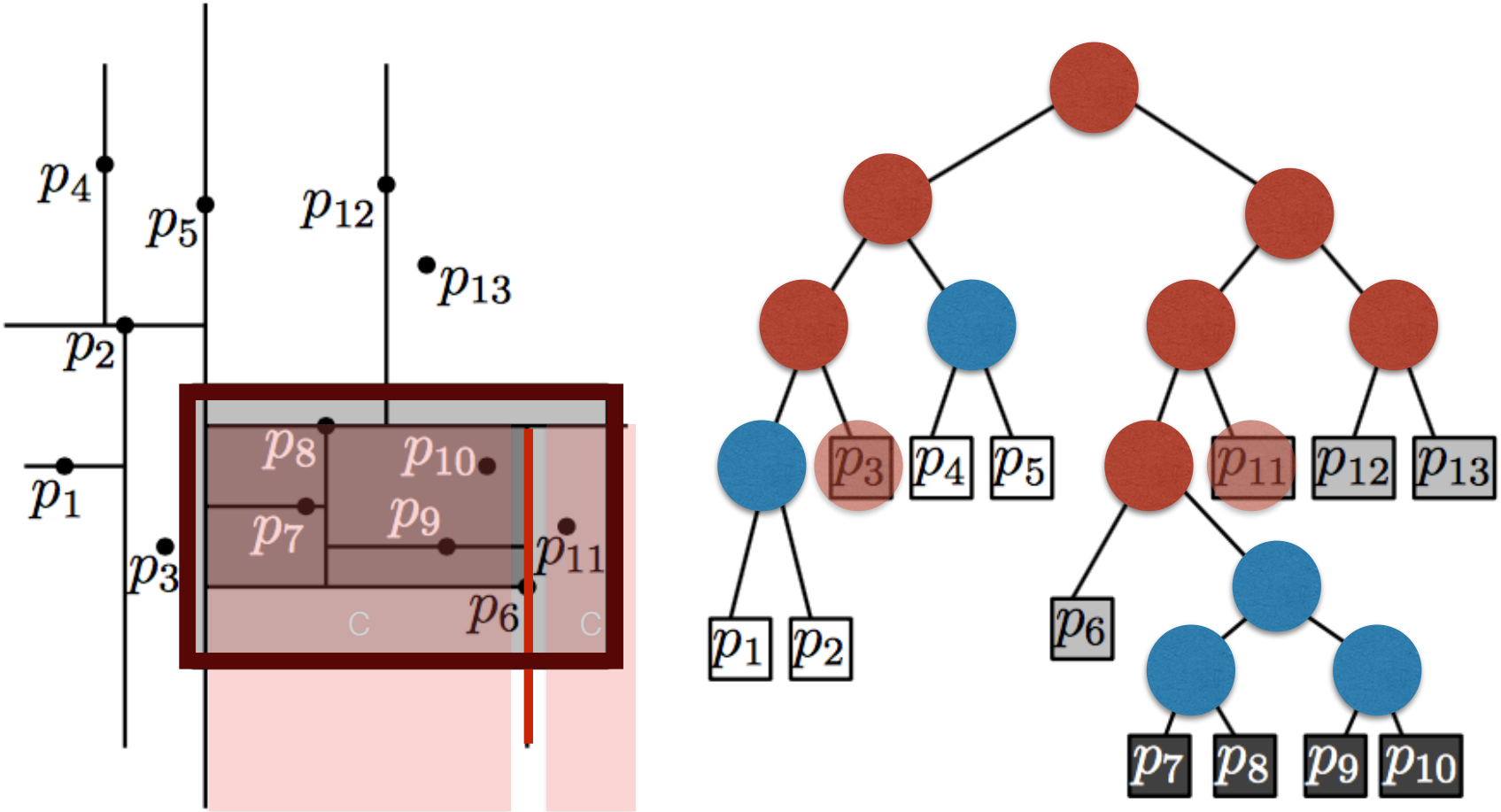

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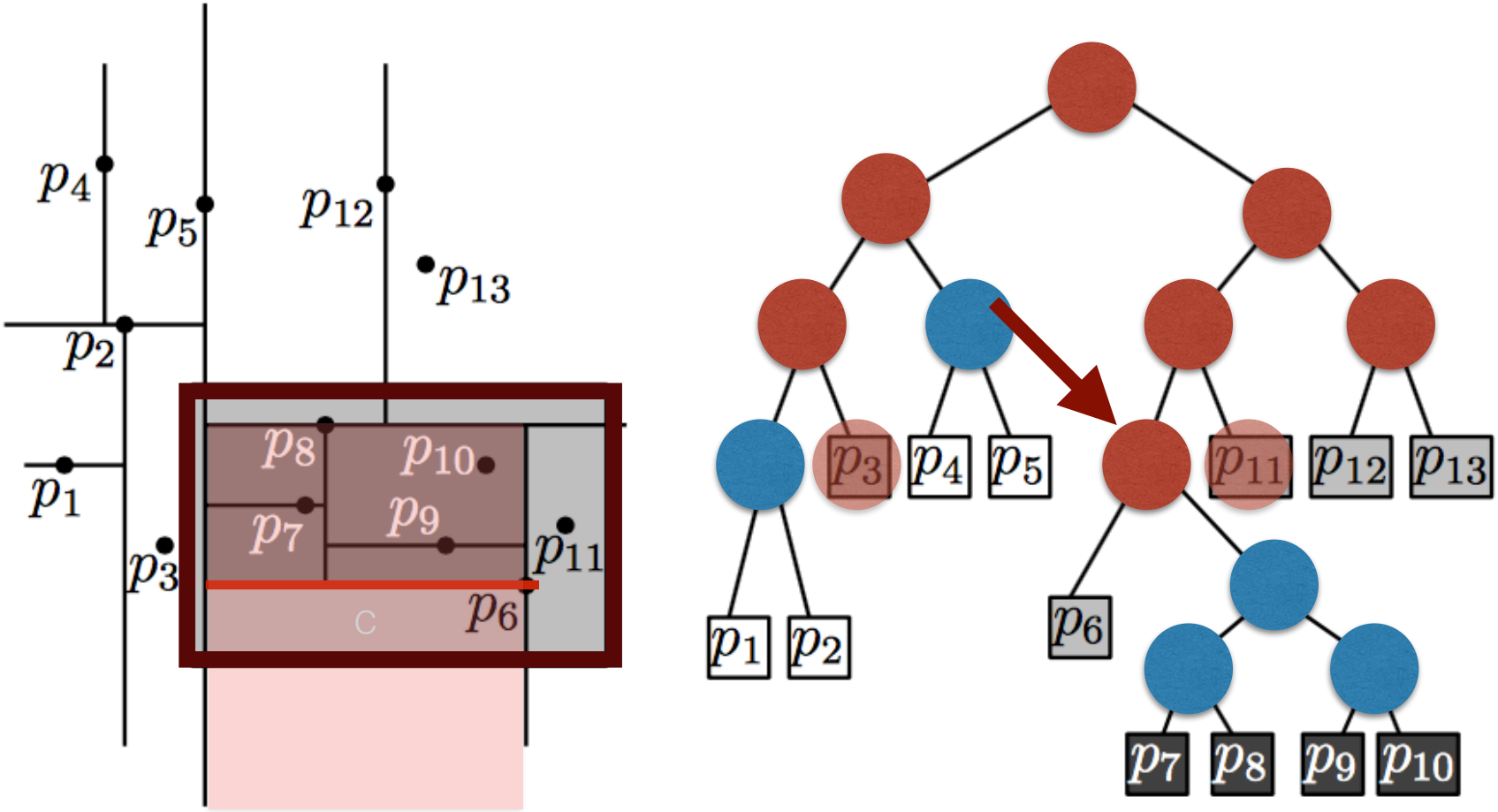

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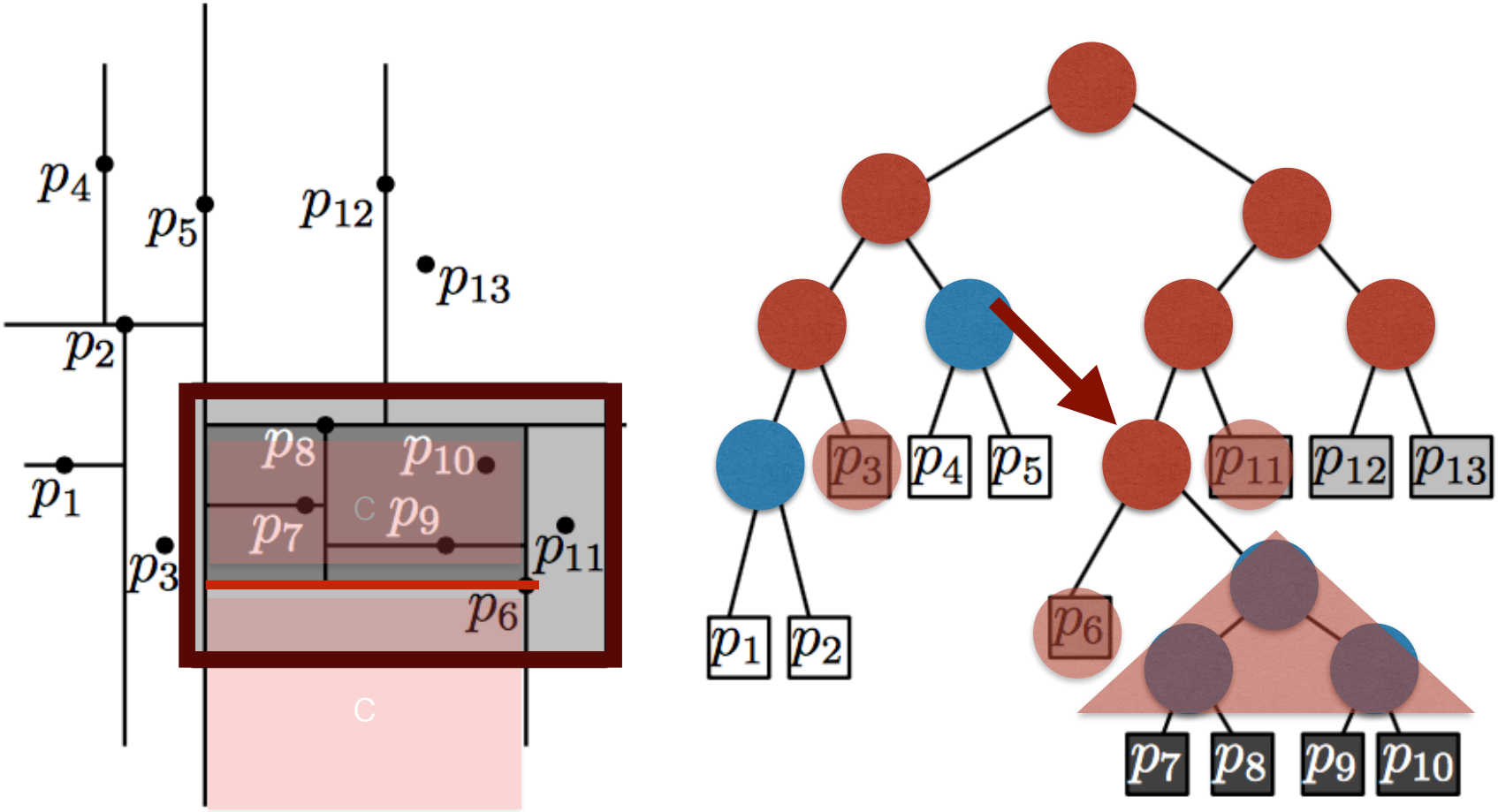

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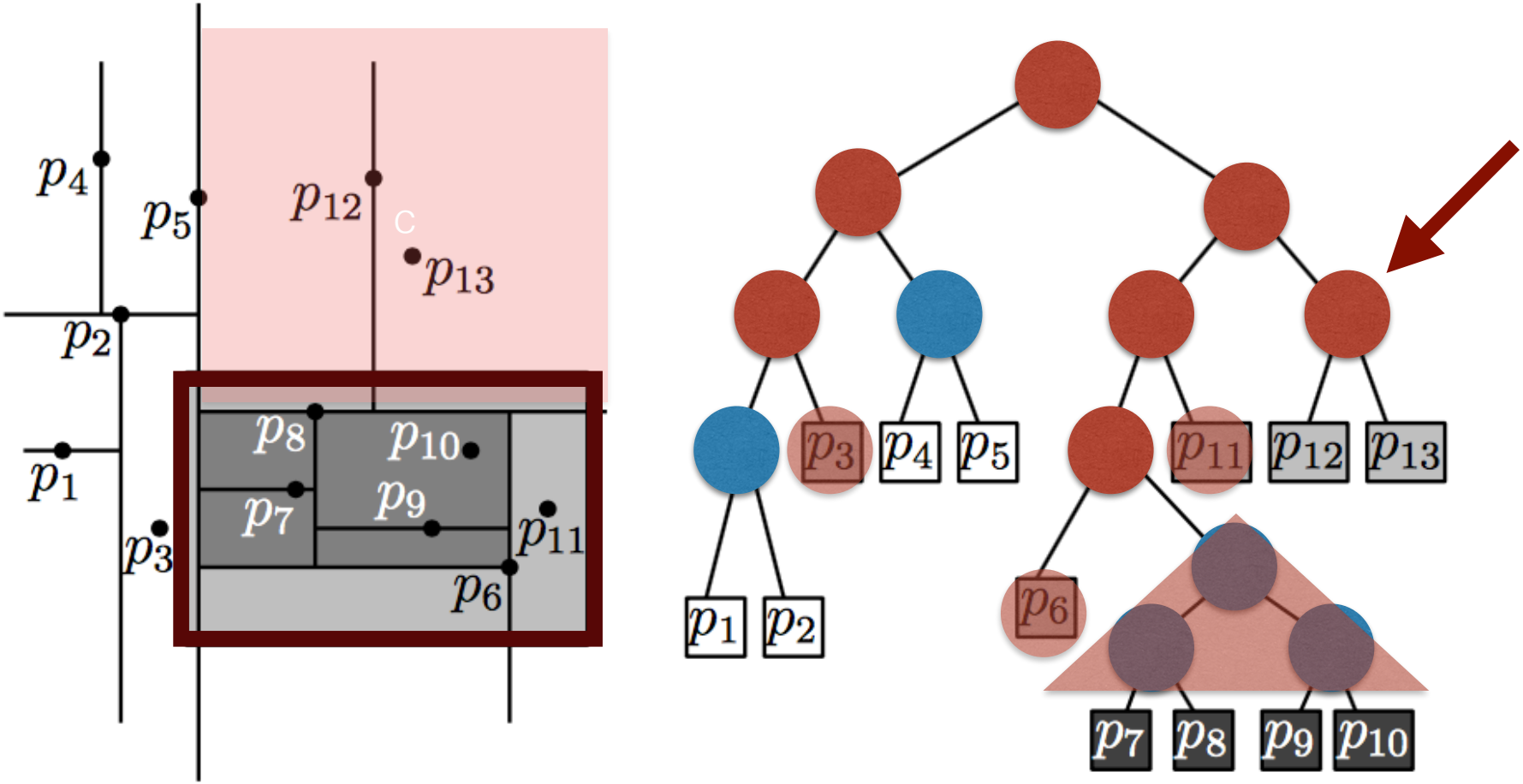

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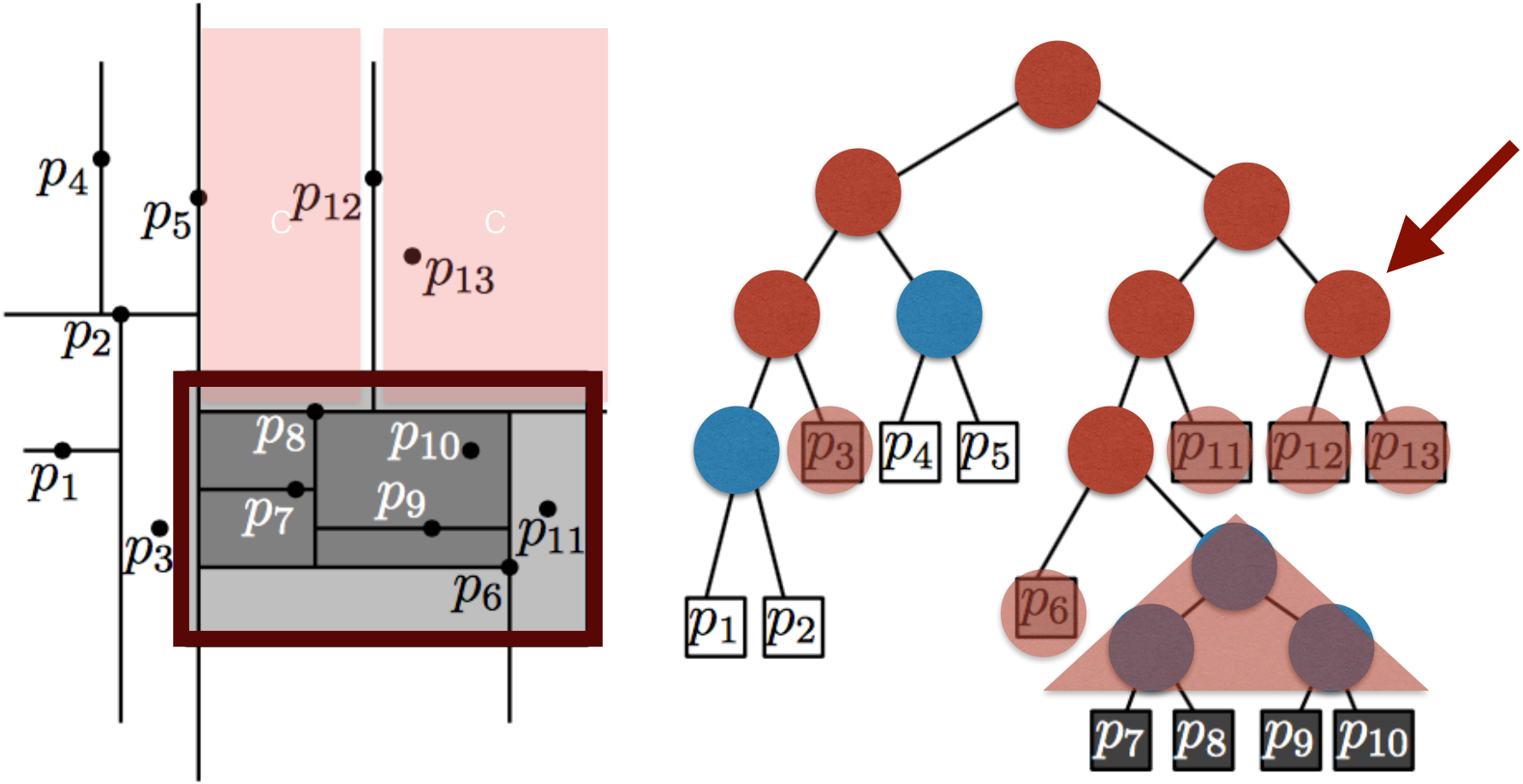

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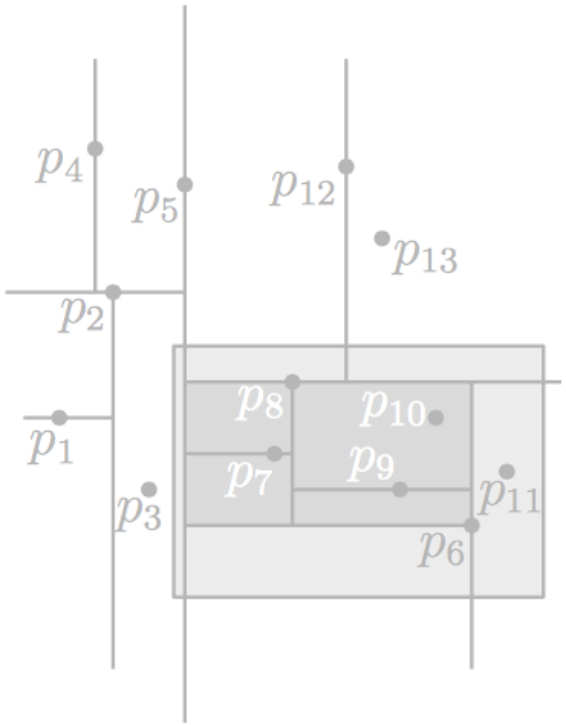
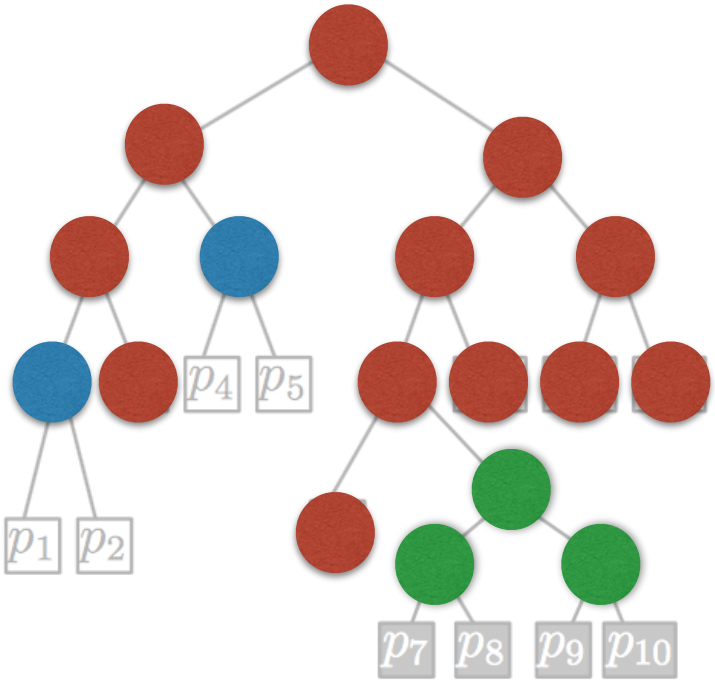
never visited by the query

visited may or may not have points to report

visited and whole subtree is output





points guaranteed to be in the range

The time to answer a range search = O(green nodes + red nodes)

Furthermore, nb.green nodes = O(k), where k = size of output

never visited by the query

visited may or may not have points to report

visited and whole subtree is output

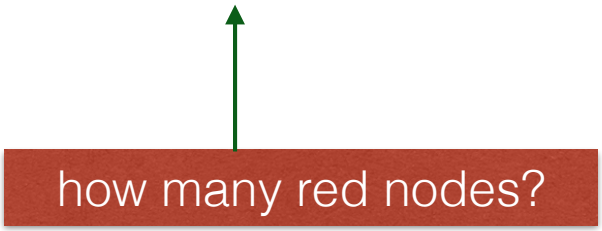
Claim:

region(v) does not intersect the range

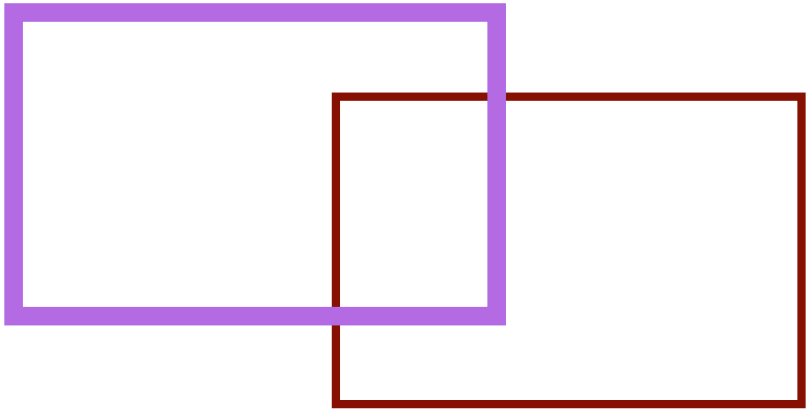
region(v) intersects the range but is not included in the range

region(v) is completely included in the range

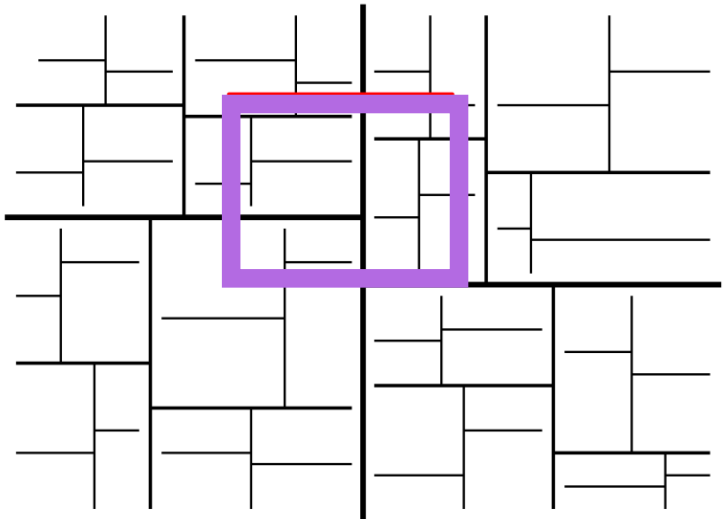
The time to answer a range search = O(red nodes) + O(k)

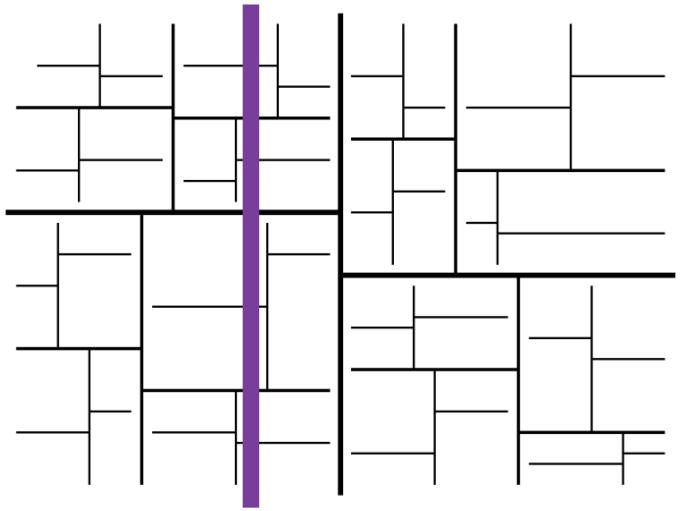


How many red nodes?



nb. red nodes = nb. nodes such that the boundary of their region intersects the boundary of the range



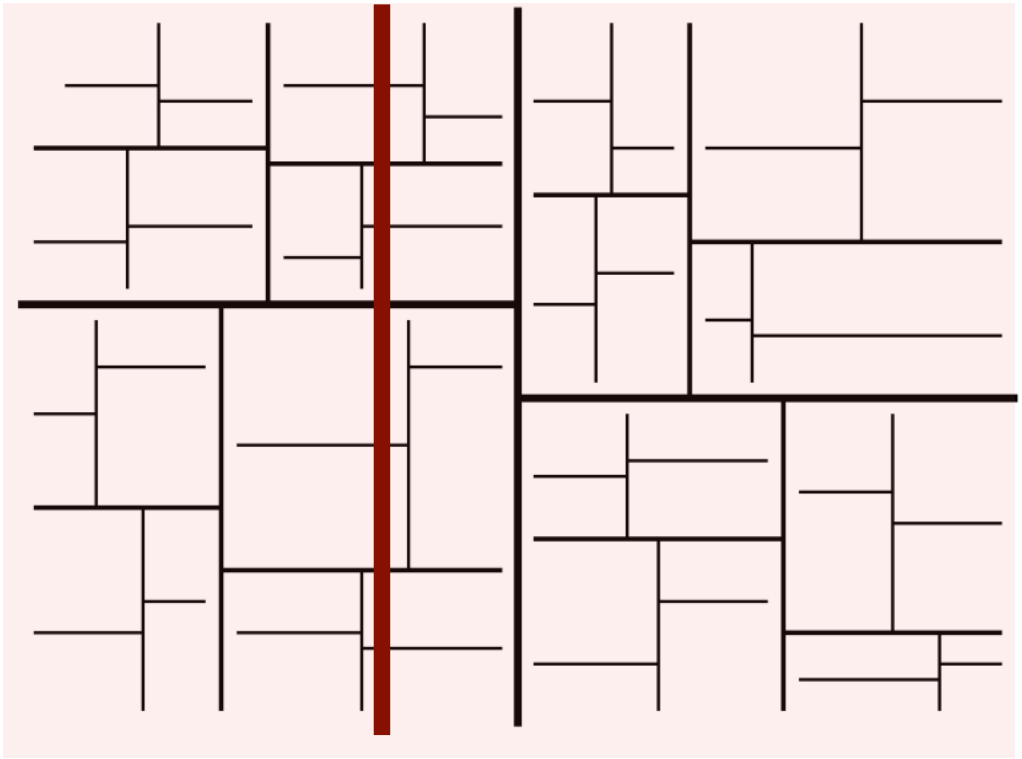


Simplified problem: We'll count the number of nodes whose region intersects a vertical line l.

Number of nodes v such that region(v) intersects a vertical line l?

We'll think recursively, starting at the root:

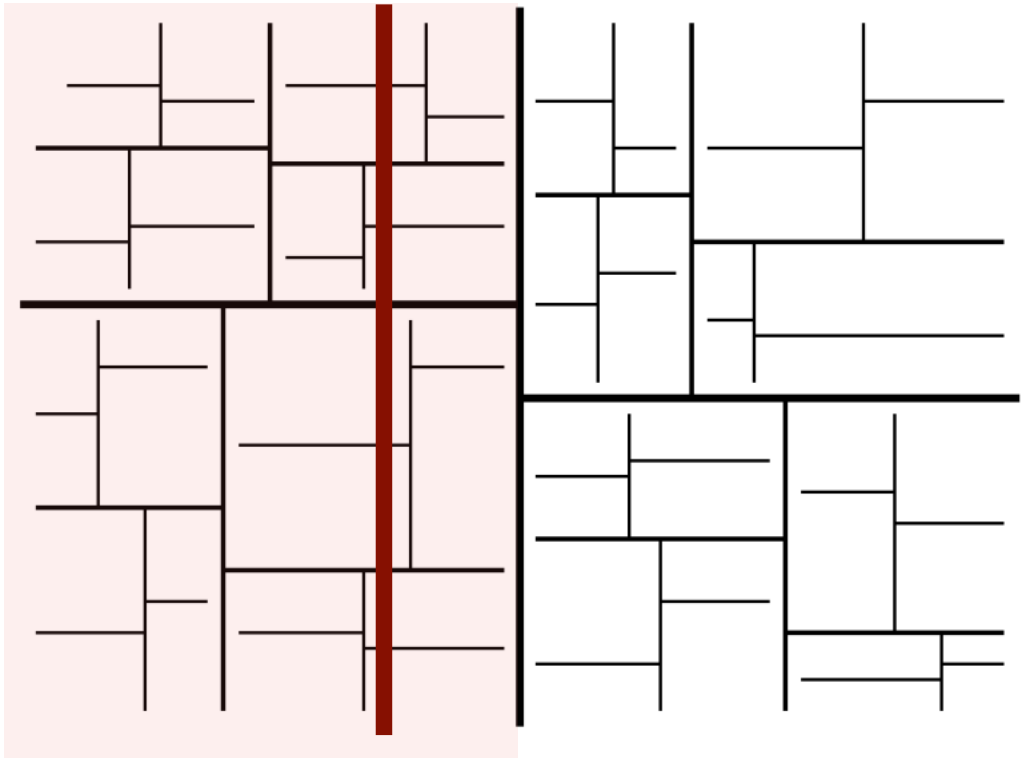
- depth=0: region(root) intersects l $+1$



Number of nodes v such that region(v) intersects a vertical line l?

We'll think recursively, starting at the root:

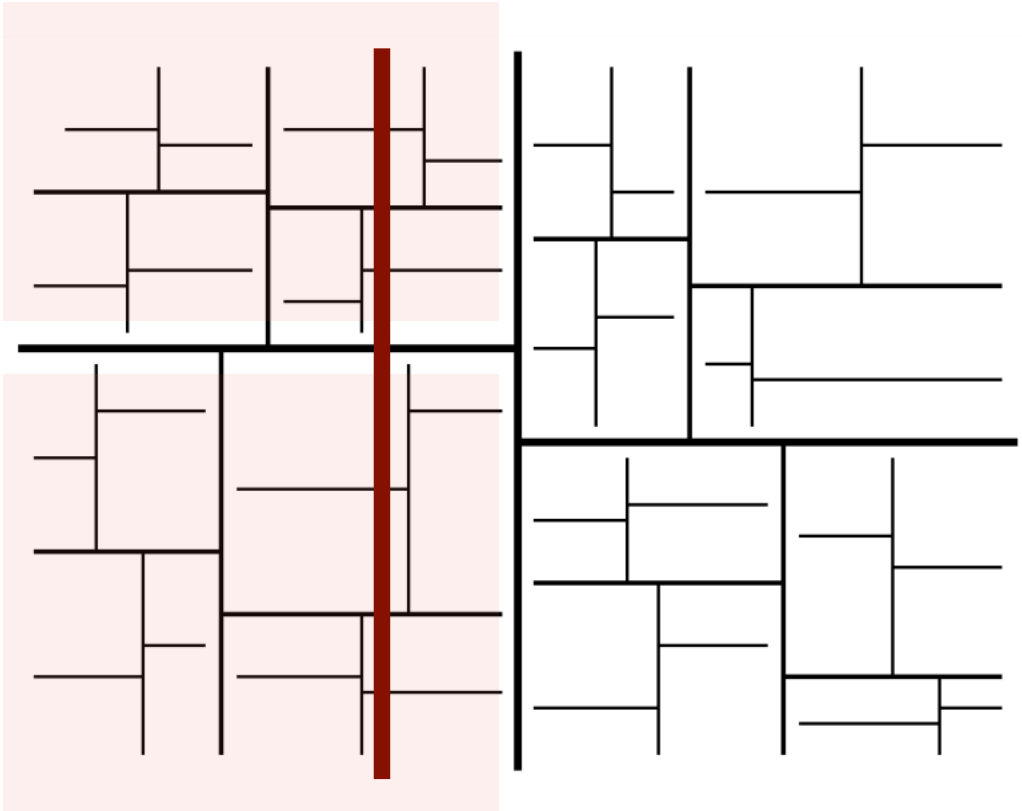
- depth=0: region(root) intersects l $+1$
- depth=1: only one of {left, right} child intersects l $+1$



Number of nodes v such that region(v) intersects a vertical line l?

We'll think recursively, starting at the root:

- depth=0: region(root) intersects l +1
- depth=1: only one of {left, right} child intersects l +1
- depth=2: both {left, right} child intersect l recurse



Number of nodes v such that region(v) intersects a vertical line l?

We'll think recursively, starting at the root:

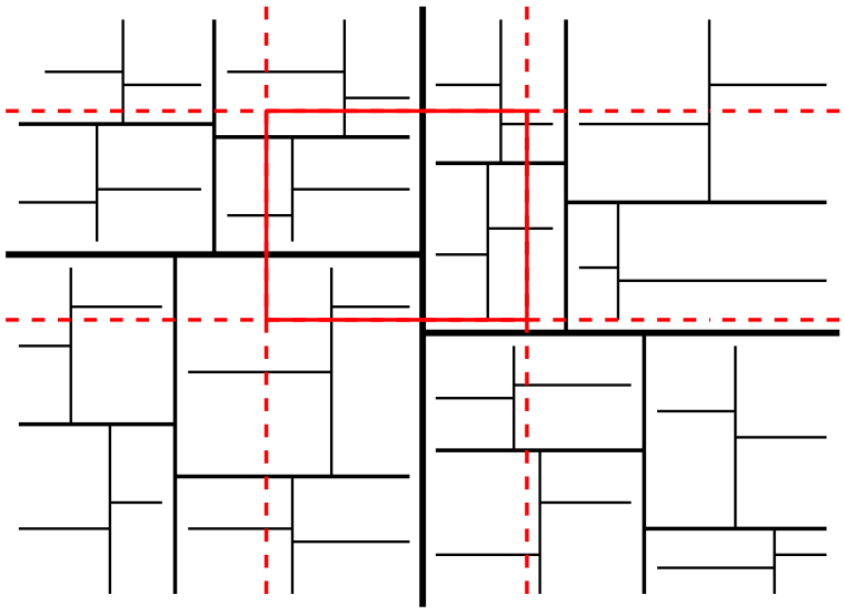
- depth=0: region(root) intersects l $+1$
- depth=1: only one of {left, right} child intersects l $+1$
- depth=2: both {left, right} child intersect l recurse

- Let G(n) = nb. of nodes in a kdtree of n points whose regions interest a vertical line l.
- Then $G(n) = 2 + 2G(n/4)$, and $G(1) = 1$
- This solves to $G(n) = O(\sqrt{n})$

Theorem: Any vertical or horizontal line stabs $O(\sqrt{n})$ regions in the kdtree.

What we know so far:

- The number of red nodes (regions intersected) if the query were a vertical line is $O(\sqrt{n})$
- The same is true if it were a horizontal line
- How about a query rectangle?



If the boundary of a region intersects the range => it must intersect at least one of the two vertical and two horizontal lines

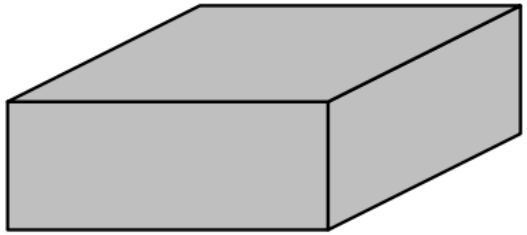
- Theorem: The number of nodes in the kdtree whose region intersects a query range is at most $4 \times O(\sqrt{n}) = O(\sqrt{n})$

kd-trees in 2D

Theorem: The kd-tree for a set of n points in 2D

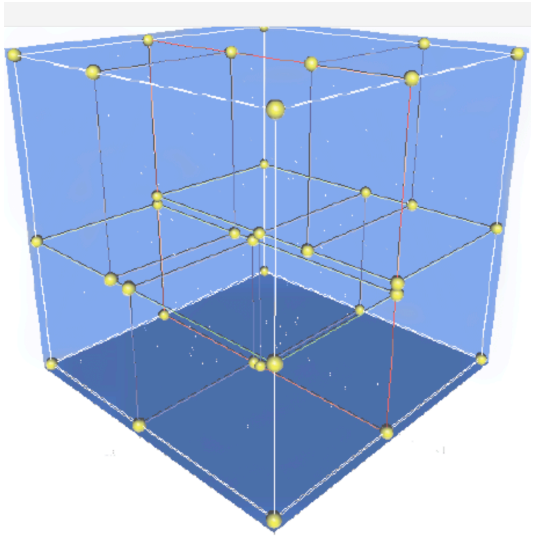
- uses $O(n)$ space
- can be built in $O(n \lg n)$ time
- 2-dimensional range query can be answered in $O(\sqrt{n} + k)$ time, where k is the nb. points reported.

kd-trees in 3D



- A 3D range query is a cube
- A 3D kd-tree alternates splits on x-, y- and z-dimensions

- Construction: Same as in 2D
- Answering range queries: Exactly the same as in 2D
- Analysis:
  - Let $G_3(n)$ = nb. of nodes in a kdtree of n points whose regions interest a vertical line l.
  - Then $G_3(n) = 4G_3(n/8) + O(1)$, and $G(1) = 1$
  - This solves to $G_3(n) = O(n^{2/3})$
  - 3D-range queries in $O(n^{2/3} + k)$



kd-trees in higher dimensions

Theorem: The kd-tree for a set of n points in d-space:

- uses $O(n)$ space
- can be built in $O(n \lg n)$ time
- d-dimensional range query can be answered in $O(n^{1-1/d} + k)$ time, where k is the nb. points reported.