Advanced Network Analysis

A Local Structure Graph Model

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Readings

- Emily Casleton, Daniel Nordman, and Mark S. Kaiser. A local structure model for network analysis. *Statistics and Its Interface*, 2(10), 2017.
- Olga V. Chyzh and Mark S. Kaiser. Network analysis using a local structure graph model. *Political Analysis*, 27(4):397{414, 2019.

Motivation

- Do edges among red nodes affect the formation of edges among turquoise nodes?
- If nodes 6 and 7 formed and edge, would that make it more likely that nodes 3 and 4 form an edge?

The Estimator

- Suppose i is an potential edge in a network of potential edges (realized and unrealized);

- Then $s_i = (u_i, v_i)$ is i's location in Cartesian space.

- Denote the binary random variable, $y(s_i) = y_i$, so that:

$$y(s_i) = \begin{cases} 1 & \text{if edge } s_i \text{ is present} \\ 0 & \text{if edge } s_i \text{ is absent.} \end{cases}$$

- Define i's neighbors as $N_i = \{s_j : s_j \text{ is a neighbor of } s_i\}$.
- Make a Markov assumption of conditional spatial independence:

$$f(y(s_i)|\mathbf{y}(s_j) : s_j \neq s_i) = f(y(s_i)|\mathbf{y}(N_i))$$

- If connectivities between edges are continuous, then the Markov assumption is redundant.

The DV

```
##       edge_id Y  
##  [1,] "12"    "1"
##  [2,] "13"    "1"
##  [3,] "14"    "1"
##  [4,] "15"    "0"
##  [5,] "16"    "0"
##  [6,] "17"    "0"
##  [7,] "21"    "0"
##  [8,] "23"    "1"
##  [9,] "24"    "0"
## [10,] "25"    "0"
```

Set Up the Connectivity Matrix, W

- Start with an adjacency matrix among all potential edge-pairs.
- Code two edges as connected if they connect opposite-colored pairs of same-colored nodes.

	54	56	57	61	62
12	0	1	1	0	0
13	0	1	1	0	0
14	0	1	1	0	0
15	0	0	0	0	0
16	0	0	0	0	0

Set Up the Connectivity Matrix, W

```r
W[1:10, 25:35]
```

```
##    51 52 53 54 56 57 61 62 63 64 65
## 12  0  0  0  0  1  1  0  0  0  0  1
## 13  0  0  0  0  1  1  0  0  0  0  1
## 14  0  0  0  0  1  1  0  0  0  0  1
## 15  0  0  0  0  0  0  0  0  0  0  0
## 16  0  0  0  0  0  0  0  0  0  0  0
## 17  0  0  0  0  0  0  0  0  0  0  0
## 21  0  0  0  0  1  1  0  0  0  0  1
## 23  0  0  0  0  1  1  0  0  0  0  1
## 24  0  0  0  0  1  1  0  0  0  0  1
## 25  0  0  0  0  0  0  0  0  0  0  0
```

The Binary Conditional Distribution

$$\mathrm{P}(Y_i = y_i | \boldsymbol{y}(N_i)) = \exp[A_i(\boldsymbol{y}(N_i))y_i - B_i(\boldsymbol{y}(N_i))], \tag{1}$$

where A_i is a natural parameter function and $B_i = \log[1 + \exp(A_i(y(N_i)))]$, and $\mathbf{y}\ (N_i)$ is a vector of values of the binary random variables (edges) of i's neighbors.

The Natural Parameter Function

$$A_i\left(\boldsymbol{y}\left(N_i\right)\right)=\log\left(\frac{\kappa_i}{1-\kappa_i}\right)+\eta\sum_{j\in N_i}w_{ij}\left(y_j-\kappa_j\right), \tag{2}$$

where $\log\left(\frac{\kappa_i}{1-\kappa_i}\right)=\boldsymbol{X}_i^T\boldsymbol{\beta}$, $\boldsymbol{X}_i$ is a column vector of k exogenous covariates, $\boldsymbol{\beta}$ is a k by 1 vector of estimation parameters, w_{ij} is the ij^{th} element of a matrix of connectivities among edges \textbf{W}, and η is its parameter.

- When $y_j > \kappa_j$, then the dependence term makes a positive contribution to $A_i(\mathbf{y}(N_i))$---complementary processes;
- When $y_j < \kappa_j$, then the dependence term makes a negative contribution to $A_i(\mathbf{y}(N_i))$---substitution-type processes;
- Key condition: $w_{ij} = w_{ji}$.
- Model does not require (prohibits) row-standardization of $\mathbf{w}$.

Estimation

$$\log PL = \sum_i \{y_i \log(p_i) + (1 - y_i) \log(1 - p_i))\}, \tag{3}$$

where:

$$p_i = \frac{\exp[A_i(y(N_i))]}{1 + \exp[A_i(y(N_i))]} \tag{4}$$

Let's Program This

```
#Likelihood
loglik<-function(par,W,Y){
b0<-par[1]
eta<-par[2]
xbeta<-b0
kappa<-exp(xbeta)/(1+exp(xbeta)) #logit of Xb
A_i=log(kappa/(1-kappa))+eta*W%*%(Y-kappa) #Eqn 2
p_i<- exp(A_i)/(1+exp(A_i)) #Eqn 1, also Eqn 4
PL<-Y*log(p_i)+(1-Y)*log(1-p_i) #Eqn 3
ell <- -sum(PL)
#cat("ell",ell, fill=TRUE)
return(ell)
}
```

Let's Estimate

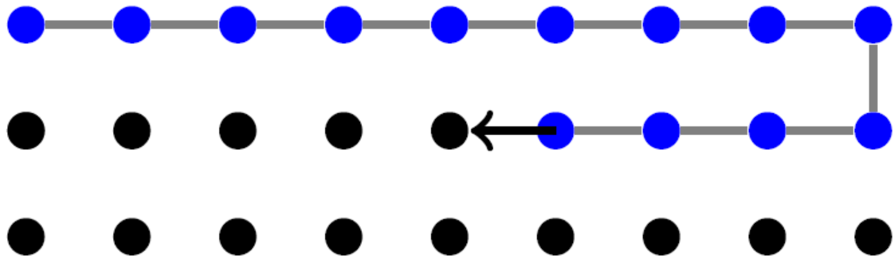
```r
m1<-optim(par=c(0,0),loglik,W=Wmat,Y=Y)
m1
```

```
## $par
## [1] -3.1572208  0.8955375
##
## $value
## [1] 13.72339
##
## $counts
## function gradient
##       95       NA
##
## $convergence
## [1] 0
##
## $message
## NULL
```

Estimating Standard Errors

Need to simulate Y based on our parameter estimates:

- Start from a vector of initial $y_0 = \{y_{01}, y_{02}, \ldots, y_{0n}\}$ drawn from a binomial distribution;
- Moving row-wise, for i = 1, ..., n, individually simulate/update each observation as a function of previously simulated observations:



- n individual updates provide 1 full Gibbs iteration
- Use the result of step 2 as the new initial values and repeat step 2 to obtain M simulated networks y (can burn-in, thin, etc.).

Simulate Networks Based on m1

1. Function `spatbin.genone` simulates/updates a value for y for a single observation--step 2
2. Function `spatbin.onegibbs` applies `spatbin.genone` to update every observation of y--step 3
3. Function `spatbin.genfield` applies `spatbin.onegibbs` to generate M networks --step 4

```
spatbin.genone<-function(coeffs,w,curys){
b0<-coeffs[1]
eta<-coeffs[2]
xbeta<- b0
kappa<-exp(xbeta)/(1+exp(xbeta))
A_i=log(kappa/(1-kappa))+eta*w%*%(curys-kappa)
p_i<- exp(A_i)/(1+exp(A_i))
y<- rbinom(n=length(curys), size=1, prob=p_i)
return(y)
}
```

```r
spatbin.onegibbs<-function(coeffs,w,curys){
cnt<-0
n<-length(curys)
newys<-NULL
repeat{
    cnt<-cnt+1
    ny<-spatbin.genone(coeffs=coeffs,w=w,curys=curys)
    curys[cnt]<-ny[cnt]
    if(cnt==n) break
    }
newys<-curys
return(newys)
}
```

```r
spatbin.genfield<-function(coeffs,w,y0s,M){
curys<-y0s
cnt<-0
res<-as.data.frame(y0s)
repeat{
    cnt<-cnt+1
    newys<-spatbin.onegibbs(coeffs=coeffs,w=w,curys=curys)
    curys<-newys
    res<-cbind(res,curys)
    if(cnt==M) break
    }

return(res)
}
```

```r
n<-length(Y)
y0s=rbinom(n=n, size=1, prob=.5)
sims<-spatbin.genfield(coeffs=m1$par,w=W,y0s=y0s,M=1000)
#Take every 10th simulated network, i.e. burnin=10, thinning=10
sims<-sims[,seq(from=10, to=ncol(sims),by=10)]
```

Obtaining Standard Errors

1. Estimate the model on simulated networks (after burnin and thinning);
2. The standard errors are the standard deviations of the estimated coefficients.

Obtaining Standard Errors

```
sim_est<-function(Y){
 res<-optim(par=m1$par,loglik,W=W,Y=as.matrix(Y))
 return(c(res$par,res$convergence))
}

library(parallel)
sim_est<-do.call("rbind",mclapply(sims, sim_est))
#Drop results if didn't converge (models that converged have converge
sim_est<-sim_est[sim_est[,3]==0,]

#Get sds of the estimates:
boot_se<-apply(sim_est,2,sd)
mytable<-cbind("coeff"=m1$par,"se"=boot_se[-3],"z-value"=(m1$par/boot
mytable
```

```
##              coeff        se     z-value
## [1,] -3.1572208  13.80388 -0.22871980
## [2,]  0.8955375 111.93631  0.00800042
```

Application: International Alliances

```
#Open the data:
data("ally_data")
ally_data$tot_trade<-log(ally_data$tot_trade+1)
ally_data<-ally_data[ally_data$year==2007,]
ally_data[1:5,]
```

```
##     ccode1 ccode2 edge defense mil_ratio tot_trade joint_dem year
## 62       2     20    1       1 0.7231990  2.653716         1 2007
## 75       2     31    2       1 0.9936821  2.206597         1 2007
## 227      2     42    5       1 0.9553047  2.331096         1 2007
## 268      2     51    6       1 0.9852507  2.215345         1 2007
## 305      2     52    7       1 0.9901861  2.335300         1 2007
```

```
#Prepare W:
W2007 <- W
W2007[1:5,1:5]
```

```
##        edge_diff1   edge_diff2   edge_diff5   edge_diff6   edge_diff7
## [1,] 0.000000e+00 1.777676e-06 1.858189e-06 2.092406e-06 2.086917e-06
## [2,] 1.777676e-06 0.000000e+00 8.051235e-08 3.147301e-07 3.092406e-07
## [3,] 1.858189e-06 8.051235e-08 0.000000e+00 2.342178e-07 2.287283e-07
## [4,] 2.092406e-06 3.147301e-07 2.342178e-07 0.000000e+00 5.489462e-09
```

Likelihood (1 X)

```
#Likelihood
loglik<-function(par,X,W,Y){
b0<-par[1]
b1<-par[2]
eta<-par[3]
xbeta<-b0+b1*X
kappa<-exp(xbeta)/(1+exp(xbeta)) #logit of Xb
A_i=log(kappa/(1-kappa))+eta*W%*%(Y-kappa) #Eqn 2
p_i<- exp(A_i)/(1+exp(A_i)) #Eqn 1, also Eqn 4
PL<-Y*log(p_i)+(1-Y)*log(1-p_i) #Eqn 3
ell <- -sum(PL)
#cat("ell",ell, fill=TRUE)
return(ell)
}
```

Let's Estimate

```
X=ally_data$tot_trade
Y=ally_data$defense
m1<-optim(par=c(0,0,0),loglik,X=X,W=W,Y=Y)
m1
```

```
## $par
## [1] -1.686244323 -0.007318607  0.936193299
##
## $value
## [1] 470.8606
##
## $counts
## function gradient
##       88       NA
##
## $convergence
## [1] 0
##
## $message
## NULL
```

Standard Errors

```
spatbin.genone<-function(coeffs,X,w,curys){
b0<-coeffs[1]
b1<-coeffs[2]
eta<-coeffs[3]
xbeta<- b0+b1*X
kappa<-exp(xbeta)/(1+exp(xbeta))
A_i=log(kappa/(1-kappa))+eta*w%*%(curys-kappa)
p_i<- exp(A_i)/(1+exp(A_i))
y<- rbinom(n=length(curys), size=1, prob=p_i)
return(y)
}
```

```
spatbin.onegibbs<-function(coeffs,X,w,curys){
cnt<-0
n<-length(curys)
newys<-NULL
repeat{
    cnt<-cnt+1
    ny<-spatbin.genone(coeffs=coeffs,X=X,w=w,curys=curys)
    curys[cnt]<-ny[cnt]
    if(cnt==n) break
    }
newys<-curys
return(newys)
}
```

```r
spatbin.genfield<-function(coeffs,X,w,y0s,M){
curys<-y0s
cnt<-0
res<-as.data.frame(y0s)
repeat{
    cnt<-cnt+1
    newys<-spatbin.onegibbs(coeffs=coeffs,X=X,w=w,curys=curys)
    curys<-newys
    res<-cbind(res,curys)
    if(cnt==M) break
    }

return(res)
}
```

Simulate 1000 Random Networks

```
n<-length(Y)
y0s=rbinom(n=n, size=1, prob=.5)
sims<-spatbin.genfield(coeffs=m1$par,X=X,w=W,y0s=y0s,M=1000)
#Take every 10th simulated network, i.e. burnin=10, thinning=10
sims<-sims[,seq(from=10, to=ncol(sims),by=10)]
saveRDS(sims, "sims.rds")
```

Estimate an LSGM on Each of the Simulated Networks

```
sims<-readRDS("data/sims.rds")
sim_est<-function(Y){
 res<-optim(par=m1$par,loglik,X=X,W=W,Y=as.matrix(Y))
 return(c(res$par,res$convergence))
}

library(parallel)
sim_est<-do.call("rbind",mclapply(sims, sim_est))
#Drop results if didn't converge (models that converged have converge
sim_est<-sim_est[sim_est[,4]==0,]
saveRDS(sim_est,"sim_est.rds")
```

Calculate SEs and Make a Table

```
#Get sds of the estimates:
sim_est<-readRDS("data/sim_est.rds")
boot_se<-apply(sim_est,2,sd)
mytable<-cbind("coeff"=m1$par,"se"=boot_se[-4],"z-value"=(m1$par/boot
mytable
```

```
##             coeff           se      z-value
## [1,] -1.686244323 2.318147e-01 -7.2741041749
## [2,] -0.007318607 3.253597e-02 -0.2249389529
## [3,]  0.936193299 2.832822e+03  0.0003304808
```