• Ex 35.5 (Billingsley)

Suppose that $X_1, X_2, \cdots$ is a martingale with $E[X_n]=0$, $E[X_n^2]<\infty$.

Show that $E[(X_{n+r}-X_n)^2]=\sum_{k=1}^{r}E[(X_{n+k}-X_{n+k-1})^2]$

(the variance of the sum is the sum of the variance)

Assume that $\sum_n E[(X_n-X_{n-1})^2]<\infty$,

prove that X_n converges a.s.

Hint: by Thm 35.5 and then (see Thm 22.6) by Thm 35.3.

Pf: • Thm 35.5: The X_n is a submartingale with $k=\sup_n E[|X_n|]<\infty$,

then $X_n \to_{a.s.} X$, X is integrable.

• Thm 35.3: If the X_n is a submartingale, then for $\alpha>0$,

$$P\left[\max_{i\le n} X_i \ge \alpha\right] \le \frac{1}{\alpha}E[|X_n|].$$

(a). $E[(X_{n+r}-X_n)^2] = E\left[\left[\sum_{k=1}^{r}(X_{n+k}-X_{n+k-1})\right]^2\right]$

$= \sum_{k=1}^{r}E[(X_{n+k}-X_{n+k-1})^2] + 2\sum_{i=1}^{r}\sum_{\substack{j=1 \\ i<j}}^{r}E[(X_{n+i}-X_{n+i-1})(X_{n+j}-X_{n+j-1})]$

• For $i<j$, since $X_n\in L^2$,

$(X_{n+i}-X_{n+i-1})\in L^2$, $(X_{n+j}-X_{n+j-1})\in L^2$,

$E[|(X_{n+i}-X_{n+i-1})(X_{n+j}-X_{n+j-1})|]$

$\le E^{\frac{1}{2}}[(X_{n+i}-X_{n+i-1})^2]\,E^{\frac{1}{2}}[(X_{n+j}-X_{n+j-1})^2]<\infty$ by the C-S inequality.

Thus $(X_{n+i}-X_{n+i-1})(X_{n+j}-X_{n+j-1})\in L^1$, hence

• $E[(X_{n+i}-X_{n+i-1})(X_{n+j}-X_{n+j-1})] = E[E[(X_{n+i}-X_{n+i-1})(X_{n+j}-X_{n+j-1})\,|\,\mathcal{F}_{n+j-1}]]$

$= E[(X_{n+i}-X_{n+i-1})\,E[(X_{n+j}-X_{n+j-1})\,|\,\mathcal{F}_{n+j-1}]]$ by $X_{n+i}, X_{n+i-1}\in\mathcal{F}_{n+j-1}$

$= E[(X_{n+i}-X_{n+i-1})\cdot 0]=0$ by $E[X_{n+j}\,|\,\mathcal{F}_{n+j-1}]=X_{n+j-1}$.

$\Rightarrow E[(X_{n+r}-X_n)^2]=\sum_{k=1}^{r}E[(X_{n+k}-X_{n+k-1})^2]$.

(b) From (a), $E[(X_n-X_1)^2]=\sum_{k=1}^{n-1}E[(X_{k+1}-X_k)^2]$; and

$E[(X_n-X_1)^2]=E[X_n^2]+E[X_1^2]-2E[X_nX_1]$

$=E[X_n^2]+E[X_1^2]-2E[E[X_nX_1\,|\,\mathcal{F}_1]]$ by $X_nX_1\in L^1$.

$=E[X_n^2]-E[X_1^2]$ by $E[X_n\,|\,\mathcal{F}_1]=X_1$.

$\Rightarrow E[X_n^2]=E[X_1^2]+\sum_{k=1}^{n-1}E[(X_{k+1}-X_k)^2]$.

$\sup_n E[X_n^2] = E[X_1^2] + \sup_n \sum_{k=1}^{n-1} E[(X_{k+1}-X_k)^2]$

$= E[X_1^2] + \sum_n E[(X_n - X_{n-1})^2] < \infty$. by the hypothesis.

or (**) →

- Since the X_n is a martingale and $g(x) = x^2$ is convex, the X_n^2 is a submartingale.
  Hence by Thm 35.5, the martingale convergence thm,
  $X_n^2 \to_{a.s.} X^2$, X^2 is integrable.
- Since the X_n^2 is a submartingale, $E[X_n^2] \nearrow$;
  $\lim_n E[X_n^2] = \sup_n E[X_n^2] < \infty$, the $E[X_n^2]$ converges, then
  $E[X_{m+n}^2] - E[X_m^2] = E[(X_{m+n} - X_m)^2] \to 0$ as $m, n \to \infty$.
- Since the X_n is a martingale, so is the $Y_n = X_{m+n} - X_m$:
  $E[Y_{n+1} \mid \mathcal{F}_n] = E[X_{m+n+1} - X_m \mid \mathcal{F}_n]$ Set $\mathcal{F}_n = \sigma(X_m, \dots, X_{m+n})$
  $= E[X_{m+n+1} \mid \mathcal{F}_n] - X_m = X_{m+n} - X_m = Y_n$ by $X_m \in \mathcal{F}_n$
  Hence the $Y_m^2 = (X_{m+n} - X_m)^2$ is a submartingale.
- $P[\max_{1\le k \le r} |X_{m+n} - X_m| \ge \frac{1}{k}] = P[\max_{1\le k\le r} (X_{m+n} - X_m)^2 \ge \frac{1}{k^2}]$
  $\le k^2 E[(X_{m+r} - X_m)^2]$ by Thm 35.3.
- Since the set on the left is nondecreasing in r, letting $r \to \infty$,
  $P[\sup_{k\ge1} |X_{m+k} - X_m| \ge \frac{1}{k}] \le k^2 \lim_r E[(X_{m+r} - X_m)^2]$
  $\Rightarrow \lim_m P[\sup_{n\ge1} |X_{m+n} - X_m| \ge \frac{1}{k}] = 0$ (*)
  by $E[(X_{m+n} - X_m)^2] \to 0$ as $m, n \to \infty$.
- Let $E(m,k)$ be the set where $\sup_{j,k\ge m} |X_j - X_k| \ge \frac{2}{k}$.
  and put $E(k) = \cap_m E(m,k)$.
  Then $E(m,k) \downarrow E(k)$ and (*) implies that $P[E(k)] = 0$.
  Now $P[\bigcup_k E(k)] \le \sum_k P[E(k)] = 0$,
  where $\bigcup_k E(k)$ contains the set where the X_n is not fundamental,
  hence X_n converges a.s..

(**): $\sup_n E[X_n^2] < \infty \Rightarrow$ the X_n are uniformly integrable $\Rightarrow \sup_n E[|X_n|] < \infty$,
by the martingale convergence thm, $X_n \to_{a.s.} X$, X is integrable.

#

- Ex 35.6 (Billingsley)

"Doob's decomposition":

Show that a submartingale X_n can be represented as

$X_n = Y_n + Z_n$ → decomposition of a submartingale,

where Y_n is a martingale, $0 \le Z_1 \le Z_2 \le \cdots$.

Remark:

a submartingale X_n minus a "compensator" Z_n is a martingale Y_n:

$Y_n = X_n - Z_n$.

Hint: Take $X_0 = 0$ and $\Delta_n = X_n - X_{n-1}$, and define $Z_n = \sum_{k=1}^{n} E[\Delta_k \mid \mathcal{F}_{k-1}]$ ($\mathcal{F}_0 = \{0, \Omega\}$)

Pf: • Define $X_0 = 0$, $\mathcal{F}_0 = \{0, \Omega\}$ and $\Delta_n = X_n - X_{n-1}$, and

- $Z_n = \sum_{k=1}^{n} E[\Delta_k \mid \mathcal{F}_{k-1}] = \sum_{k=1}^{n} \left(E[X_k \mid \mathcal{F}_{k-1}] - X_{k-1} \right)$

(i) $Z_1 = E[\Delta_1 \mid \mathcal{F}_0] = E[X_1 \mid \{0, \Omega\}]$

$= E[X_1] \ge E[X_0] = 0$ by the X_n is a submartingale

(ii) $Z_n - Z_{n-1} = E[\Delta_n \mid \mathcal{F}_{n-1}] \ge 0$ by the X_n is a submartingale.

Hence $0 \le Z_n \uparrow$ and Z_n is measurable $\mathcal{F}_{n-1}$.

- It suffices to show that

$Y_n = X_n - Z_n = X_n - \sum_{k=1}^{n} E[\Delta_k \mid \mathcal{F}_{k-1}]$ is a martingale.

(i) $\mathcal{F}_n \subset \mathcal{F}_{n-1}$ by the hypothesis.

(ii) Y_n is measurable $\mathcal{F}_n$ since X_n is measurable $\mathcal{F}_n$.

(iii) $E[|Y_n|] < \infty$ since

$E[|Y_n|] = E\left[\left| X_n - \sum_{k=1}^{n} E[\Delta_k \mid \mathcal{F}_{k-1}] \right| \right]$

$\le E[|X_n|] + \sum_{k=1}^{n} \left| E[\Delta_k \mid \mathcal{F}_{k-1}] \right| < \infty \because E[|X_n|] < \infty$.

(iv) $E[Y_{n+1} \mid \mathcal{F}_n] = Y_n$ a.s. since

$E[Y_{n+1} \mid \mathcal{F}_n] = E[X_{n+1} - Z_{n+1} \mid \mathcal{F}_n]$

$= E[X_{n+1} \mid \mathcal{F}_n] - Z_{n+1}$ by Z_{n+1} is measurable $\mathcal{F}_n$

$= \cancel{E[X_{n+1} \mid \mathcal{F}_n]} - (Z_n + \cancel{E[X_{n+1} \mid \mathcal{F}_n]} - X_n)$

$= X_n - Z_n = Y_n$.

#

- Ex 35.7 (Billingsley)

If $X_1, X_2, \cdots$ is a martingale and bounded either above or below, then

$\sup_n E[|X_n|] < \infty$.

Pf: (i) If $X_1, X_2, \cdots$ is bounded above,

$\sup_n X_n \le k < \infty$ for some k. Then $\sup_n E[X_n^+] < \infty$.

$$
\begin{aligned}
E[|X_n|] &= E[X_n^+] + E[X_n^-] \\
&= 2E[X_n^+] - (E[X_n^+] - E[X_n^-]) \\
&= 2E[X_n^+] - E[X_n] \\
&= 2E[X_n^+] - E[X_1] \quad \text{by the } X_n \text{ is a martingale.}
\end{aligned}
$$

$\sup_n E[|X_n|] = 2\sup_n E[X_n^+] - E[X_1] < \infty$

by $E[|X_1|] < \infty$, so $E[X_1] < \infty$.

(ii) If $X_1, X_2, \cdots$ is bounded below,

$\inf_n X_n \ge k > -\infty$ for some k. Then $\sup E[X_n^-] < \infty$.

$$
\begin{aligned}
E[|X_n|] &= E[X_n^+] + E[X_n^-] \\
&= E[X_n^+] - E[X_n^-] + 2E[X_n^-] \\
&= E[X_n] + 2E[X_n^-] \\
&= E[X_1] + 2E[X_n^-]. \quad \text{by the } X_n \text{ is a martingale.}
\end{aligned}
$$

$\sup_n E[|X_n|] = E[X_1] + 2\sup_n E[X_n^-] < \infty$

by $E[|X_1|] < \infty$, so $E[X_1] < \infty$.

#

- Ex 35.8 (Billingsley)

Let $X_n = \Delta_1 + \cdots + \Delta_n$ where the Δ_n are iid with $P[\Delta_n = \pm 1] = \frac{1}{2}$.

Let τ be the smallest n s.t. $X_n = 1$ and define X_n^* by

$$X_n^* = \begin{cases} X_n & \text{if } n \le \tau \\ X_\tau & \text{if } n \ge \tau \end{cases}$$

Show that the hypothesis of Thm 35.5 are satisfied by $\{X_n^*\}$ but that it is impossible to integrate to the limit.

Hint: Use Ex 35.7.

Pf: • Thm 35.5: $\{X_n\}$ is a submartingale with $\sup_n E[|X_n|] < \infty$, then

$X_n \to_{a.s.} X$, X is integrable

• Ex 35.7: If $\{X_n\}$ is a martingale and bounded above or below, then $\sup_n E[|X_n|] < \infty$.

• Since the X_n is a sum of iid r.v.'s Δ_n with $E[\Delta_n] = 0$, the X_n is a martingale.

• If τ is the smallest n s.t. $X_n = 1$, then τ is a stopping time.

so $X_n^* = \begin{cases} X_n & \text{if } n \le \tau \\ X_\tau & \text{if } n \ge \tau \end{cases}$ is also a martingale.

• For $n \le \tau$, $X_n^* = X_n < 1$;

For $n \ge \tau$, $X_n^* = X_\tau = 1$;

Hence X_n^* is bounded above,

from Ex 35.7 we have that $\sup_n E[|X_n^*|] < \infty$.

• It follows from the martingale convergence thm, Thm 35.5,

$X_n^* \to_{a.s.} X^*$, X^* is integrable.

• Suppose it is capable to integrable to the limit

$0 = E[X_1] = E[X_n^*]$ by the X_n^* is a martingale

$\to E[\lim_n X_n^*]$

$= E[X_\tau] = 1$ by $X_n^* = X_\tau$ for $n \ge \tau$. a contradiction!

Hence it is impossible to integrate to the limit. #

• Ex 35.9 (Billingsley) "Partial of the Martingale Stopping Thm"

Let $X_1, X_2, \ldots$ be a martingale, and assume that $|X_1(\omega)|$ and $|X_n(\omega) - X_{n-1}(\omega)|$ are bounded by a constant indep. of ω and n.

Let τ be a stopping time with "finite mean".

Show that X_τ is integrable and that $E[X_\tau] = E[X_1]$.

Pf: (a) Suppose that $|X_1| \le K$, $|X_n - X_{n-1}| \le K$. Set $X_0 = 0$, $\mathcal{F}_0 = \{0, \Omega\}$.

$$|X_\tau| = \left|\sum_{n=1}^{\tau}(X_n - X_{n-1})\right|$$
$$\le |X_1| + \sum_{n=2}^{\tau}|X_n - X_{n-1}|$$
$$\le K + (\tau - 1)K = \tau K$$

$E|X_\tau| \le E[\tau K] = K E[\tau] < \infty$ by the hypothesis: $E[\tau] < \infty$

Hence X_τ is integrable.

(b) Since $X_\tau I_{[\tau \le k]} \to_{a.s.} X_\tau$ as $k \to \infty$; and

$E\left[|X_\tau I_{[\tau \le k]}|\right] \le E[|X_\tau|] < \infty$, then

by the dominated convergence thm,

$$\lim_{k\to\infty} E[X_\tau I_{[\tau \le k]}] = E\left[\lim_{k\to\infty} X_\tau I_{[\tau \le k]}\right] = E[X_\tau]$$

• $E[X_\tau I_{[\tau \le k]}] = \int_{[\tau \le k]} X_\tau \, dP$

$$= \sum_{i=1}^{k} \int_{[\tau = i]} X_i \, dP$$
$$= \sum_{i=1}^{k} \left(\int_{[\tau \ge i]} X_i \, dP - \int_{[\tau \ge i+1]} X_i \, dP\right)$$
$$= \int_{[\tau \ge 1]} X_1 \, dP - \int_{[\tau \ge k+1]} X_k \, dP \quad \text{by expansion only}$$
$$= E[X_1] - \int_{[\tau \ge k+1]} X_k \, dP \quad \text{by } [\tau \ge 1] = \Omega$$
$$= E[X_1] - \int_{[\tau > k]} X_{k+1} \, dP \quad \text{by } [\tau > k] = [\tau \le k]^c \in \mathcal{F}_k \subset \mathcal{F}_{k+1}, \{X_n\} \text{ martingale}$$

• By $\left|\int_{[\tau > k]} X_{k+1} \, dP\right| \le \int_{[\tau > k]} |X_{k+1}| \, dP$

$$\le K(k+1) P[\tau > k] \quad \text{by } |X_{k+1}| \le K(k+1)$$
$$\le K(k+1) \frac{1}{k} \int_{[\tau > k]} \tau \, dP \to K \cdot 1 \cdot 0 = 0 \quad \text{as } k \to \infty.$$

Hence $E[X_1] = E[X_\tau]$. #

- Ex 35.10 (Billingsley)

Use Ex 35.8 and Ex 35.9 to show that the τ is Ex 35.8 has infinite mean.

Thus the waiting time until a symmetric random walk moves one step from the starting point has infinite expected value.

Pf: If τ has finite mean, i.e. $E[\tau]<\infty$,

Then from Ex 35.9 it follows that

X_τ is integrable and $E[X_\tau]=E[X_1]$.

But from Ex 35.8 we have $E[X_\tau]\neq E[X_1]$, a contradiction.

Hence τ has infinite mean, $E[\tau]=\infty$.

#.

- Ex 35.15 (Billingsley)

Suppose that $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $A \in \mathcal{F}_\infty$, and prove that

$P[A \| \mathcal{F}_n] \to I_A$ a.s.

Pf: • Thm 35.6: If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and Z is integrable, then

$$E[Z \| \mathcal{F}_n] \to E[Z \| \mathcal{F}_\infty] \text{ a.s.}$$

• Since $A \in \mathcal{F}_\infty$, I_A is measurable $\mathcal{F}_\infty$, and since I_A is bounded, it is integrable;

It follows by Thm 35.6 that

$$E[I_A \| \mathcal{F}_n] \to E[I_A \| \mathcal{F}_\infty] \text{ a.s.},$$

hence that

$$P[A \| \mathcal{F}_n] \to I_A \text{ a.s.}$$

#

· Ex 35.17 (Billingsley)

Suppose that θ has an arbitrary distribution,
and suppose that, conditionally on θ,
the r.v.'s $Y_1, Y_2, \dots$ are iid with normally distributed
with mean θ and variance σ^2.
Construct such a sequence $\{\theta, Y_1, Y_2, \dots\}$.
Prove that

$E[\theta \mid Y_1, \dots, Y_n] \to \theta$ with probability 1

pf: · Let $Z_1, Z_2, \dots$ are iid $n(0, \sigma^2)$ r.v.'s and indep of θ with $E[|\theta|] < \infty$.

Let $Y_i = Z_i + \theta$, hence
the $Y_i \mid \theta$ are iid $n(\theta, \sigma^2)$.

· Let $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

Since $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and θ is integrable, by Thm 35.6 we have

$E[\theta \mid Y_1, \dots, Y_n] \to E[\theta \mid \mathcal{F}_\infty]$ a.s.

· Since $E[Y_n] = \theta$ exists, by the SLLN we have

$\frac{Y_1 + \cdots + Y_n}{n} \to_{a.s.} \theta$,

and thus θ is measurable $\mathcal{F}_\infty$.

Hence $E[\theta \mid Y_1, \dots, Y_n] \to E[\theta \mid \mathcal{F}_\infty] = \theta$ a.s.

#

• Ex 1.12 (Shao)

Let X and Y be independent r.v.'s satisfying
$E|X+Y|^a < \infty$ for some $a > 0$.
Show that $E|X|^a < \infty$.

Pf: • By 2^p-inequality,

$$E|X|^a = E|X+c-c|^a$$
$$\le 2^p\left(E|X+c|^a + E|c|^a\right)$$

• It suffices to show $E|X+c|^a < \infty$. Let c s.t. $P(Y>c)>0$, $P(Y\le c)>0$.

• $$\infty > E|X+Y|^a \ge E\left(|X+Y|^a I_{[Y>c,\, X+c>0]}\right) + E\left(|X+Y|^a I_{[Y\le c,\, X+c\le 0]}\right)$$
$$\ge E\left(|X+c|^a I_{[Y>c,\, X+c>0]}\right) + E\left(|X+c|^a I_{[Y\le c,\, X+c\le 0]}\right)$$
$$= P(Y>c)\, E\left(|X+c|^a I_{[X+c>0]}\right) + P(Y\le c)\, E\left(|X+c|^a I_{[X+c\le 0]}\right)$$

where the last equality follows from the independence of X and Y.

• Thus $E\left(|X+c|^a I_{[X+c>0]}\right)$, $E\left(|X+c|^a I_{[X+c\le 0]}\right) < \infty$,
hence $E\left(|X+c|^a\right) = E\left(|X+c|^a I_{[X+c>0]}\right) + E\left(|X+c|^a I_{[X+c\le 0]}\right) < \infty$.

#

· <u>Ex 1.20</u> (Shao)

Show that a r.v. X is independent of itself iff X is a constant a.s.
Can X and $f(X)$ be independent, where f is a Borel function.

Pf: (a) "$\Leftarrow$" If $X=c$ a.s. for a constant $c \in R$.

For any $A \in \mathcal{B}$ and $B \in \mathcal{B}$,

$$P(X\in A, X\in B) = I_A(c)\, I_B(c) = P(X\in A)P(X\in B)$$

Hence X and X are independent.

"$\Rightarrow$" If X is independent of itself.

Then for any $t \in R$,

$$P(X\le t) = P(X\le t, X\le t)$$
$$= [P(X\le t)]^2$$

Thus $P(X\le t)=0$ or 1

Since $P(X\le t)$ is an increasing function of t,

there must be a $c \in R$ s.t.

$$P(X\le c)=1 \text{ and } P(X<c)=0.$$

Hence $X=c$ a.s.

(b) If X and $f(X)$ are independent,

then so are $f(X)$ and $f(X)$.

From (a) we have that

this occurs iff $f(X)$ is a constant a.s.

#.

- Ex 1.30 (Shao)

Find an example of two r.v.'s X and Y s.t. X and Y are not independent but their chf's φ_X and φ_Y satisfy

$$\varphi_X(t)\varphi_Y(t)=\varphi_{X+Y}(t)$$

Pf: Let $X=Y$ be a r.v. having Cauchy$(0,1)$ with $\varphi_X(t)=\varphi_Y(t)=e^{-|t|}$.

Since $X=Y$ are not degenerate, then X, Y are not independent.

Then the chf of $X+Y=2X$ is

$$\varphi_{X+Y}(t)=E[e^{it(2X)}]$$
$$=\varphi_X(2t)=e^{-|2t|}=e^{-|t|}e^{-|t|}$$
$$=\varphi_X(t)\varphi_Y(t).$$

#

- Ex 1.33 (Shao)

Let X and Y be independent r.v.'s.

Show that if X and $X-Y$ are independent, then X must be degenerate.

Pf: • Since X and Y are independent, so are $-X$ and Y. Hence

$$\varphi_{Y-X}(t)=\varphi_Y(t)\varphi_{-X}(t)=\varphi_Y(t)\varphi_X(-t).$$

• If X and $X-Y$ are independent, so are X and $Y-X$. Then

$$\varphi_Y(t)=\varphi_{X+(Y-X)}(t)=\varphi_X(t)\varphi_{Y-X}(t)$$
$$=\varphi_X(t)\varphi_X(-t)\varphi_Y(t).$$

Since $\varphi_Y(0)=1$ and φ_Y is continuous, $\varphi_Y(t)\neq 0$ for a neighborhood of 0

$$\varphi_X(t)\varphi_X(-t)=\varphi_X(t)\overline{\varphi_X(t)}$$
$$=|\varphi_X(t)|^2=1 \text{ on this neighborhood of } 0$$

Thus $|\varphi_X(t)|=1$ for all $t\neq 0$, and hence $X=c$ a.s. for some c.

#

• Ex 1.48 (Shao)

Let X_n be a r.v. and m_n be a median of X_n,
Show that if $X_n \Rightarrow X$ for a r.v. X, then
any limit point of m_n is a median of X.

Pf: • Let μ_n and μ be the distributions of X_n and X, respectively.

• WLOG, assume that $\lim_n m_n = m$.

• For $\epsilon > 0$ s.t. $\mu\{m-\epsilon\} = \mu\{m+\epsilon\} = 0$, that is,
they are continuity points of the distribution of X.
For sufficiently large n,
$|m_n - m| < \epsilon$, and thus $m-\epsilon < m_n < m+\epsilon$.
Since m_n is a median of X_n,
$\frac{1}{2} \le P(X_n \le m_n) \le P(X_n \le m+\epsilon)$,
$\frac{1}{2} \le P(X_n \ge m_n) \le P(X_n \ge m-\epsilon)$.

• Let $n \to \infty$, we have that
$\frac{1}{2} \le P(X \le m+\epsilon)$,
$\frac{1}{2} \le P(X \ge m-\epsilon)$,

• Let $\epsilon \to 0$, we have that
$\frac{1}{2} \le P(X \le m)$,
$\frac{1}{2} \le P(X \ge m)$.

Hence m is a median of X. #

- Ex 1.60 (Shao)

Let $U_1, U_2, \dots$ be iid uniform$(0,1)$ and

$$Y_n = \left(\prod_{i=1}^{n} U_i\right)^{-\frac{1}{n}}.$$

Show that $\sqrt{n}(Y_n - e) \Rightarrow N(0, e^2)$

Pf: • Suppose $Y_n = \left(\prod_{i=1}^{n} U_i\right)^{-\frac{1}{n}}$,

$$\log Y_n = -\frac{1}{n}\sum_{i=1}^{n}\log U_i = \frac{1}{n}\sum_{i=1}^{n}(-\log U_i).$$

• Let $X_n = -\log U_n$. $U_n = e^{-X_n}$, $J = e^{-X_n} > 0$,

$f_{X_n}(x) = e^{-x}$, $x > 0$,

thus $X_1, X_2, \dots$ be iid exponential(1) with $E[X_n] = 1$, $Var[X_n] = 1$.

By the Lindeberg-Levy thm we have that

$$\frac{\bar{X}_n - 1}{1/\sqrt{n}} = \sqrt{n}(\bar{X}_n - 1) \Rightarrow N(0,1).$$

• Note that

$$Y_n = e^{\bar{X}_n} \approx e^1 + e^1(\bar{X}_n - 1) \quad \text{by Taylor's series expansion}$$

$$\sqrt{n}\, e^{-1}(Y_n - e) \approx \sqrt{n}(\bar{X}_n - 1) \Rightarrow N(0,1)$$

$$\sqrt{n}(Y_n - e) \Rightarrow N(0, e^2) \quad \text{by Slutsky's thm.}$$

#.

• Ex1.61 (Shao)

Suppose that X_n is a r.v. having binomial(n, θ).

Define $Y_n = \begin{cases} \log(\frac{X_n}{n}) & \text{when } X_n \geq 1 \\ 1 & \text{when } X_n = 0. \end{cases}$

Show that

(a) $\lim_n Y_n = \log\theta$ a.s.

(b) $\sqrt{n}(Y_n - \log\theta) \Rightarrow N(0, \frac{1-\theta}{\theta})$

pf: • Let $Z_1, Z_2, \cdots$ be iid Bernoulli(θ), then

$X_n = \sum_{i=1}^{n} Z_i$ where $E[Z_i] = \theta$, $Var[Z_i] = \theta(1-\theta) < \infty$,

(a) For any $\epsilon > 0$,

$$P\left[\left|\frac{X_n}{n} - \theta\right| \geq \epsilon\right] \leq \frac{1}{\epsilon^4} E\left|\frac{X_n}{n} - \theta\right|^4$$

$$= \frac{\theta^4(1-\theta) + (1-\theta)^4\theta}{\epsilon^4 n^3} + \frac{\theta^2(1-\theta)^2(n-1)}{\epsilon^4 n^3}$$

Thus

$$\sum_{n=1}^{\infty} P\left[\left|\frac{X_n}{n} - \theta\right| \geq \epsilon\right] < \infty.$$

By the Borel-Cantelli lemma it follows that

$$P\left[\left|\frac{X_n}{n} - \theta\right| \geq \epsilon \text{ i.o.}\right] = 0$$

Hence $\frac{X_n}{n} \to \theta$ a.s.

• Define $W_n = I_{[X_n \neq 0]} \frac{X_n}{n}$, then

$$Y_n = \log(W_n + e I_{[X_n = 0]})$$

• We want to show that $\lim_n I_{[X_n=0]} = 0$ a.s, and hence $\lim_n I_{[X_n \neq 0]} = 1$ a.s.

$$\sum_n P[I_{[X_n=0]} > \epsilon] = \sum_n P[X_n = 0]$$

$$= \sum_n (1-\theta)^n = \frac{1-\theta}{1-(1-\theta)} = \frac{1-\theta}{\theta} < \infty,$$

by the Borel-Cantelli lemma it follows that $\lim_n I_{[X_n=0]} = 0$ a.s.

thus $\lim_n I_{[X_n \neq 0]} = 1$ a.s, and hence

$$\lim_n Y_n = \log\left(\lim_n \left(I_{[X_n \neq 0]} \frac{X_n}{n} + e I_{[X_n = 0]}\right)\right) \text{ a.s.}$$

$$= \log\theta \text{ a.s. by } \frac{X_n}{n} \to_{a.s.} \theta.$$

(b) • Since $X_n = \sum_{i=1}^{n} Z_i$ where Z_i is iid with $E[Z_i] = \theta$, $Var[Z_i] = \theta(1-\theta)$

by the Lindeberg-Levy thm it follows that

$$\frac{X_n - n\theta}{\sqrt{n\,\theta(1-\theta)}} = \sqrt{n}\,\frac{X_n/n - \theta}{\sqrt{\theta(1-\theta)}} \Rightarrow N(0,1),$$

$$\sqrt{n}\left(\frac{X_n}{n} - \theta\right) \Rightarrow N(0, \theta(1-\theta))$$

• $Y_n = \log W_n + I_{[X_n=0]}$,

$$\sqrt{n}(Y_n - \log\theta) = \sqrt{n}(\log W_n - \log\theta) + \sqrt{n}\, I_{[X_n=0]}$$

• $\sqrt{n}(W_n - \theta) = \sqrt{n}\left(\frac{X_n}{n} - \theta\right) - \sqrt{n}\, I_{[X_n=0]}\,\frac{X_n}{n}$

$\Rightarrow N(0, \theta(1-\theta)) - 0\cdot\theta = N(0, \theta(1-\theta))$ if $\sqrt{n}\, I_{[X_n=0]} \to_p 0$.

$\log W_n \approx \log\theta + \frac{1}{\theta}(W_n - \theta)$ by Taylor's series expansion

$\sqrt{n}\,\theta(\log W_n - \log\theta) \approx \sqrt{n}(W_n - \theta) \Rightarrow N(0, \theta(1-\theta))$, by Slutsky's thm.

hence, $\sqrt{n}(\log W_n - \log\theta) \Rightarrow N\left(0, \frac{1-\theta}{\theta}\right)$

• It suffices to show that

$$\sqrt{n}\, I_{[X_n=0]} \to_p 0,$$

and by Slutsky's thm the result follows.

$\sum_n P(\sqrt{n}\, I_{[X_n=0]} > \epsilon) = \sum_{n=1}^{\infty} P(X_n = 0) < \infty$ by the same argument above.

thus by the Borel-Cantelli lemma it follows that

$P(\sqrt{n}\, I_{[X_n=0]} > \epsilon \;\; i.o.) = 0$,

and hence $\sqrt{n}\, I_{[X_n=0]} = 0$ a.s., then $\sqrt{n}\, I_{[X_n=0]} \to_p 0$.

#

- <u>Example</u> (Permutation, Cycles)
  - Every permutation can be written as a product of cycles.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 7 & 4 & 6 & 2 & 3 \end{pmatrix} = (1\,5\,6\,2)(3\,7)(4) \to \text{cyclic representation}$$

$$\Rightarrow 1 \to 5 \to 6 \to 2 \to \overset{①}{1}\,;\ 3 \to 7 \to \overset{②}{3}\,;\ 4 \to \overset{③}{4}$$

- Let Ω_n consists of the $n!$ permutations of $1, 2, \dots, n$, all equally probable;
  $\mathcal{F}_n$ contains all subsets of Ω_n; and
  $P(A)$ is the fraction of points in A.
- Let $X_k(\omega) = 1$ or 0 according as the kth element in the cyclic representation of the permutation ω completes a cycle or not.
  $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$. is the number of cycles in ω.

$$\Rightarrow X_1 = X_2 = X_3 = X_5 = 0;\ X_4 = X_6 = X_7 = 1;\ S_7 = 3.$$

- $X_1, \dots, X_n$ are independent and $P[X_k = 1] = 1/(n-k+1)$

Pf: $\because X_1(\omega) = 1$ iff the random permutation ω sends 1 to itself,
$P[X_1(\omega) = 1] = 1/n$.

- If $X_1(\omega) = 1$, then the image of 2 is one of $2, \dots, n$, and $X_2(\omega) = 1$ iff this image is in fact 2,
  $P[X_2(\omega) = 1 \mid X_1(\omega) = 1] = 1/(n-1)$;

  If $X_1(\omega) = 0$, then ω sends 1 to some $i \neq 1$, so that the image of i is one of $1, \dots, i-1, i+1, \dots, n$, and
  $X_2(\omega) = 1$ iff this image is in fact 1;
  $P[X_2(\omega) = 1 \mid X_1(\omega) = 0] = 1/(n-1)$;

  thus $P[X_2(\omega) = 1 \mid X_1(\omega) = i] = P[X_2(\omega) = 1] = 1/(n-1)$, $i = 1, 2$.

  Hence by induction the result follows.

#

- <u>Example</u> (Weak Law for Cycles of Permutations)
  - Let Ω_n consist of the $n!$ permutations of $1, 2, \dots, n$, all equally probable.
  - $X_{nk}(\omega)$ be 1 or 0 accordings the kth element in the cyclic representation of $\omega \in \Omega_n$ completes a cycle or not.
  - $X_{n1}, \dots, X_{nn}$ are independent and

    $$P[X_{nk}=1] = \frac{1}{n-k+1};$$

    $S_n = \sum_{k=1}^{n} X_{nk}$ is the number of cycles.
  - Let $m_{nk} = E[X_{nk}] = \frac{1}{n-k+1}$,

    $$\sigma_{nk}^2 = m_{nk}(1-m_{nk})$$

    If $L_n = \sum_{k=1}^{n} \frac{1}{k}$,

    $$E[S_n] = \sum_{k=1}^{n} m_{nk} = \sum_{k=1}^{n} \frac{1}{n-k+1} = \sum_{k=1}^{n} \frac{1}{k} = L_n,$$

    $$Var[S_n] = \sum_{k=1}^{n} m_{nk}(1-m_{nk}) < \sum_{k=1}^{n} m_{nk} = L_n,$$
  - So $P\left[\left|\frac{S_n - L_n}{L_n}\right| \geq \epsilon\right]$

    $$\leq \frac{E[(S_n-L_n)^2]}{\epsilon^2 L_n^2} = \frac{Var[S_n]}{\epsilon^2 L_n^2} \quad \text{by Chebychev's inequality.}$$

    $$< \frac{L_n}{\epsilon^2 L_n^{2}} \to 0 \text{ as } n \to \infty \quad \because n \to \infty \text{ implies } L_n = \sum_{k=1}^{n} \frac{1}{k} \to \infty.$$

    Thus $\frac{S_n}{L_n} \to_p 1$, 1 can be regarded as a r.v. on any probability space.

    Since $L_n = \log n + O(1)$, we have

    $$\frac{S_n}{\log n} = \frac{S_n}{L_n}\frac{L_n}{\log n} = \frac{S_n}{L_n}\frac{\log n + O(1)}{\log n} \to_p 1 \times 1 = 1. \text{ by Slutsky's thm.}$$

    thus $\frac{S_n}{\log n} \Rightarrow 1$.
  - Since Ω_n changes with n, there cannot be a strong law. #.

- **Example** (Goncharov's Thm: CLT for Cycles of Permutation)
  - Let Ω_n consist of the $n!$ permutations of $1, 2, \dots, n$, all equally probable.
  - $X_{nk}(\omega) = 1$ or 0 accordings the kth element in the cyclic presentation of $\omega \in \Omega_n$ completes a cycle or not.
  - $X_{n1}, \dots, X_{nn}$ are independent and
    $$P[X_{nk} = 1] = \frac{1}{n-k+1};$$
  - $S_n = \sum_{k=1}^{n} X_{nk}$ is the number of cycles.

    Let $m_{nk} = E[X_{nk}] = \frac{1}{n-k+1}$,
    $$\sigma_{nk}^2 = m_{nk}(1-m_{nk}),$$
    If $L_n = \sum_{k=1}^{n} \frac{1}{k}$,
    $$E[S_n] = \sum_{k=1}^{n} m_{nk} = \sum_{k=1}^{n} \frac{1}{n-k+1} = \sum_{k=1}^{n} \frac{1}{k} = L_n,$$
    $$s_n^2 = \sum_{k=1}^{n} m_{nk}(1-m_{nk}) = L_n - \sum_{k=1}^{n} \left(\frac{1}{n-k+1}\right)^2 = L_n + O(1).$$
  - Let $X_{nk}' = X_{nk} - \frac{1}{n-k+1}$, then $X_{n1}', \dots, X_{nn}'$ are independent,
    $$E[X_{nk}'] = 0, \quad \sigma_{nk}^2 = m_{nk}(1-m_{nk}), \quad s_n^2 = L_n + O(1).$$
  - Since $|X_{nk}'| \le 1$, uniformly bounded; and $s_n^2 \to \infty$,
    $$\sum_{k=1}^{n} \frac{1}{s_n^2} \int_{|X_{nk}'| \ge \epsilon s_n} X_{nk}'^2 \, dP$$
    $$\le \sum_{k=1}^{n} \frac{1}{s_n^2} \int |X_{nk}'|^2 \frac{|X_{nk}'|}{\epsilon s_n} \, dP$$
    $$\le \sum_{k=1}^{n} \frac{1}{s_n^3} \cdot 1 \cdot \int |X_{nk}'|^2 \, dP$$
    $$= \frac{s_n^2}{s_n^3} = \frac{1}{s_n} \to 0 \text{ as } n \to \infty,$$
    which is Lindeberg's condition, then
    $$\frac{\sum_{k=1}^{n}\left(X_{nk} - \frac{1}{n-k+1}\right)}{s_n} = \frac{S_n - L_n}{s_n} \Rightarrow N;$$
    Since $L_n = \log n + O(1)$, $s_n = \sqrt{L_n} + O(1) = \sqrt{\log n} + O(1)$,
    $$\frac{S_n - \log n}{\sqrt{\log n}} = \left(\frac{S_n - L_n}{s_n} + \frac{L_n - \log n}{s_n}\right) \frac{s_n}{\sqrt{\log n}}$$
    $$\Rightarrow (N + 0) \times 1 = N \quad \text{by Slutsky's thm.}$$

#

- <u>Example</u> (Records, Ranks, Renyi Thm).

Let $\{X_n\}$ be iid r.v.'s with continuous distribution function $F(x)$.

(i) record times

$L(1)=1$,

$L(n)=\min\{k: X_k > X_{L(n-1)}\}$, $n \geq 2$.

(ii) record values

$X_{L(n)}$, $n \geq 1$

(iii) associated counting process $\{\mu(n), n\geq 1\}$:

$\mu(n) = \#$ records among $X_1, \cdots, X_n = \max\{k: L(k)\leq n\}$

$= \sum_{k=1}^{n} I_k$, where $I_k = 1$ if X_k is record; $I_k = 0$ o.w.

(iv) ranks (of X_k among $X_1, \cdots, X_k$)

$R_k = \sum_{j=1}^{k} I_{[X_j \geq X_k]}$.

Results:

(a) The continuity of F implies $P[X_i = X_j]=0$, then $P(\text{Ties})=0$.

$P(\text{Ties}) = P\left(\bigcup_{i\neq j}[X_i = X_j]\right) = \lim_{n\to\infty} P\left(\bigcup_{i\neq j}^{n}[X_i = X_j]\right)$

$\leq \lim_{n\to\infty} \sum_{i\neq j}^{n} P[X_i = X_j] = 0$

(b) <u>Renyi theorem</u>

(i) The ranks $R_1, \cdots, R_n$ are independent and

$P[R_k = j] = \frac{1}{k}$. for $j=1,\cdots,k$, $k=1,\cdots,n$.

(ii) The indicators $I_1, \cdots, I_n$ are independent and. $P[I_k = 1] = \frac{1}{k}$.

pf: (i) $P[R_1 = r_1, \cdots, R_n = r_n] = \frac{1}{n!}$,

$P[R_n = r_n] = \sum_{r_1,\cdots,r_{n-1}} P[R_1 = r_1, \cdots, R_n = r_n] = \frac{(n-1)!}{n!} = \frac{1}{n}$.

so $P[R_1 = r_1, \cdots, R_n = r_n] = \prod_{k=1}^{n} P[R_k = r_k]$.

(ii) $P[I_k = 1] = P[R_k = 1] = \frac{1}{k}$.

(next pg. cont.).

(c) The probability of infinitely many records is 1, i.e.

$P([I_n = 1] \text{ i.o.}) = 1$

pf: $\because \{I_k\}$ are independent r.v.'s and

$$\sum_{n=1}^{\infty} P[I_n=1] = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

$\therefore$ by the Borel zero-one law,

$P([I_n = 1] \text{ i.o.}) = 1$.

(d) The probability of infinitely many double records is 0, i.e.

Let $D_n = 1$ if X_{n-1} and X_n both are records; $D_n = 0$ o.w.

$P([D_n = 1] \text{ i.o.}) = 0$.

pf: • Note first $\{D_n, n \geq 2\}$ are not independent.

• $P(D_n=1) = P(I_n=1, I_{n-1}=1) = P(I_n=1)P(I_{n-1}=1) = \frac{1}{n(n-1)}$.

$$\sum_{n=2}^{\infty} P(D_n=1) = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lim_{m\to\infty} \sum_{n=2}^{m}\left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= \lim_{m\to\infty}\left(1 - \frac{1}{m}\right) = 1 < \infty.$$

$\therefore$ by the Borel-Cantelli lemma,

$P([D_n = 1] \text{ i.o.}) = 0$.

PS: (i) The expected no. of double records

$$E\left(\sum_{n=2}^{\infty} D_n\right) = \sum_{n=2}^{\infty} E(D_n) = \sum_{n=2}^{\infty} P(D_n=1) = 1$$

(ii) Since double records seem to be rare events,

the total no. of double records, $\sum_{n=2}^{\infty} D_n \sim \text{Poisson}(1)$.

#

- **Theorem** (Strong Laws for Record Times and Counting Process)

Let the record times be

$L(1)=1$

$L(n) = \min\{k : X_k > X_{L(n-1)}\}$

and the record counts be

$\mu(n) = \sum_{k=1}^{n} I_k$, where $I_k = 1$ if X_k is record.

Then we have

(1) $\frac{\mu(n)}{\log n} \to_{a.s.} 1$,

(2) $\frac{\log L(n)}{n} \to_{a.s.} 1$.

Pf: (1) Recall that I_k are independent $Ber(\frac{1}{k})$ r.v.'s.

$\because \sum_{k=1}^{n} Var\left(\frac{I_k}{\log k}\right) = \sum_{k=1}^{n} \frac{\frac{1}{k}(1-\frac{1}{k})}{(\log k)^2} \le \sum_{k=1}^{n} \frac{1}{k(\log k)^2} < \infty$,

$\therefore$ by the Kolmogorov convergence criterion, it follows that

$\sum_{k=1}^{n}\left(\frac{I_k - \frac{1}{k}}{\log k}\right)$ converges a.s.

Then, by the Kronecker's lemma,

$\frac{1}{\log n}\sum_{k=1}^{n}(I_k - \frac{1}{k}) \to_{a.s.} 0$ as $n\to\infty$.

Since $\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k} \to 1$ as $n\to\infty$, hence

$\frac{1}{\log n}\sum_{k=1}^{n} I_k = \frac{\mu(n)}{\log n} \to_{a.s.} 1$. as $n\to\infty$.

(2) • Note $\mu(L(n)) = n$.

Since $\{L(n)\ge k\} = \{\mu(k)\le n\}$ $(k>n)$,

for any finite $k>0$,

$P[L(n)<k] = P[\mu(k)>n] \to 0$ as $n\to\infty$,

thus $L(n)\to_p \infty$, since $L(n)$ is monotone increasing, $L(n)\to_{a.s.}\infty$,

• Hence by random index strong law, from (1) we know that

$\frac{\mu(L(n))}{\log L(n)} = \frac{n}{\log L(n)} \to_{a.s.} 1$ as $n\to\infty$; hence

$\frac{\log L(n)}{n} \to_{a.s.} 1$ as $n\to\infty$. #

- <u>Theorem</u> (CLT for Record Times and Counting Process)

Let the record times be

$L(1)=1$

$L(n)=\min\{k: X_k > X_{L(n-1)}\}$

and the record counts be

$\mu(n)=\sum_{k=1}^{n} I_k$, where $I_k=1$ if X_k is record

Then we have

(1) $\dfrac{\mu(n)-\log n}{\sqrt{\log n}} \Rightarrow N$.

(2) $\dfrac{\log L(n)-n}{\sqrt{n}} \Rightarrow N$

Pf: (1) Recall that I_k are independent $Ber(\frac{1}{k})$ r.v.'s with

$E[I_k]=\frac{1}{k}$ and $Var[I_k]=\frac{1}{k}(1-\frac{1}{k})$.

Then $I_k-\frac{1}{k}$ are independent and have mean 0, and.

$I_k' \equiv I_k-\frac{1}{k}$ is uniformly bounded by 1, and.

$s_n^2=\sum_{k=1}^{n} Var(I_k')=\sum_{k=1}^{n}\frac{1}{k}(1-\frac{1}{k})=\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k^2}\sim \log n \to \infty$.

Hence the Lyapounov condition holds for $\delta=1$:

$\sum_{k=1}^{n}\frac{1}{s_n^3}E[|I_k'|^3] \le \sum_{k=1}^{n}\frac{1}{s_n^3}1\cdot E[|I_k'|^2]=\frac{1}{s_n}\to 0$ as $n\to\infty$.

thus the Lindeberg condition holds and hence

$\dfrac{\mu(n)-E[\mu(n)]}{\sqrt{Var(\mu(n))}} \Rightarrow N$.

$\because$ $E[\mu(n)]=\sum_{k=1}^{n}\frac{1}{k}\sim \log n$, $Var(\mu(n))=\sum_{k=1}^{n}\frac{1}{k}(1-\frac{1}{k})\sim \log n$.

$\therefore$ $\dfrac{\mu(n)-\log n}{\sqrt{\log n}} \Rightarrow N$.

(2) Note $\mu(L(n))=n$.

By the generalized Anscombe thm and (1) we know that

$\dfrac{\mu(L(n))-\log L(n)}{\sqrt{\log L(n)}} \Rightarrow N$, so $\dfrac{n-\log L(n)}{\sqrt{\log L(n)}} \Rightarrow N$.

Since $\frac{\log L(n)}{n} \to_{a.s.} 1$, then by Slutsky's thm,

$\dfrac{n-\log L(n)}{\cancel{\sqrt{\log L(n)}}}\dfrac{\cancel{\sqrt{\log L(n)}}}{\sqrt{n}} \Rightarrow N$, hence $\dfrac{\log L(n)-n}{\sqrt{n}} \Rightarrow N$. #

- Ex 4.24 (Karr)

$E[|X|^r] < \infty$ iff $\sum_{k=1}^{\infty} k^{r-1} P\{|X| \ge k\} < \infty$, $r > 0$.

Pf: • $E(|X|^r) = \int_0^\infty P(|X|^r > t)\,dt$

$= \int_0^\infty P(|X| > t^{\frac{1}{r}})\,dt$

$= \int_0^\infty P(|X| \ge s)\,ds^r$

$= r\int_0^\infty s^{r-1} P(|X| \ge s)\,ds.$

• "$\Leftarrow$": If $\sum_{k=1}^{\infty} k^{r-1} P\{|X| \ge k\} < \infty$,

$E(|X|^r) = r\int_0^\infty s^{r-1} P(|X| \ge s)\,ds$

$= r\sum_{k=1}^{\infty} \int_{k-1}^{k} s^{r-1} P(|X| \ge s)\,ds.$

$\ge \sum_{k=1}^{\infty} P\{|X| \ge k\} \int_{k-1}^{k} r s^{r-1}\,ds.$

$= \sum_{k=1}^{\infty} [k^r - (k-1)^r]\, P\{|X| \ge k\}$

$\approx r\sum_{k=1}^{\infty} k^{r-1} P\{|X| \ge k\} < \infty$ by $k^r - (k-1)^r = k^r(1-(1-\frac{1}{k})^r)$

$\approx k^r \cdot r\frac{1}{k} = rk^{r-1}$

• "$\Rightarrow$": If $E(|X|^r) < \infty$,

$E(|X|^r) = r\sum_{k=0}^{\infty} \int_{k}^{k+1} x^{r-1} P\{|X| > x\}\,dx$

$\le \sum_{k=0}^{\infty} P\{|X| > k\} \int_{k}^{k+1} r x^{r-1}\,dx$

$= \sum_{k=0}^{\infty} [(k+1)^r - k^r]\, P\{|X| > k\}$

$\approx r\sum_{k=0}^{\infty} (k+1)^{r-1} P\{|X| > k\}.$

① If $1 \le r < \infty$

$E(|X|^r) \approx r\sum_{k=0}^{\infty} k^{r-1} P\{|X| \ge k\} + r\sum_{k=0}^{\infty} c_2 k^{r-2} P\{|X| \ge k\} + \cdots + r\sum_{k=0}^{\infty} c_r P\{|X| \ge k\}$

$< \infty$ by $\sum_{k=0}^{\infty} P\{|X| \ge k\} < \sum_{k=0}^{\infty} k P\{|X| \ge k\} < \cdots < \sum_{k=0}^{\infty} k^{r-1} P\{|X| \ge k\} < \infty$

② If $0 < r < 1$

$E(|X|^r) \le r\sum_{k=0}^{\infty} (k+1)^{r-1} P\{|X| > k\} \le r\sum_{k=0}^{\infty} k^{r-1} P\{|X| > k\}$, by $(k+1)^{r-1} \le k^{r}$ for $0 < r < 1$.

#