• Ex 25.5 (Billingsley)

Show that $\lim_n P[|X_n - X| > \epsilon] = 0$ implies that $P([X \leq x] \triangle [X_n \leq x]) \to 0$ if $P[X = x] = 0$.

Pf: Since $\lim_n P[|X_n - X| > \epsilon] = 0$ implies that

$$P[X_n \leq x] \to P[X \leq x] \text{ if } P[X = x] = 0, \text{ then}$$

$$P([X_n \leq x] \triangle [X \leq x]) \to P([X \leq x] \triangle [X \leq x]) = P[\phi] = 0.$$

#

• Ex 25.6 (Billingsley)

For arbitrary r.v.'s X_n there exists positive constants a_n s.t. $a_n X_n \Rightarrow 0$.

Pf: Set $a_n = 1/n\,O(E[|X_n|])$,

$$P[|a_n X_n| > \epsilon]$$

$$\leq \frac{1}{\epsilon} E[|a_n X_n|] = \frac{a_n}{\epsilon} E[|X_n|]$$

$$\sim \frac{1}{n\epsilon} \to 0 \text{ as } n \to \infty$$

Thus $a_n X_n \to_p 0$, since 0 is a constant, hence $a_n X_n \Rightarrow 0$.

#

• Ex 26.1 (Billingsley)

A r.v. has a lattice distribution if for some a and b, $b>0$, the lattice $[a+nb : n=0,\pm 1,\dots]$ supports the distribution of X.

Let X have chf φ.

(a) Show that a necessary condition to have a lattice distribution is that $|\varphi(t)|=1$ for some $t\neq 0$, i.e. $|\varphi(t_0)|=1$.

(b) Show that the condition is sufficient as well.

(c) Suppose that $|\varphi(t)|=|\varphi(t')|=1$ for incommensurable t and t' ($t\neq 0$, $t'\neq 0$, t/t' irrational).
Show that $P[X=c]=1$ for some constant c.

Pf: (b) If $|\varphi(t)|=1$ for some $t\neq 0$, then

$\varphi(t)=e^{ita}$ for some a, and

$$e^{ita}=\int e^{itX}\mu(dx),$$

$$1=\int e^{it(X-a)}\mu(dx),$$

$$0=\int\left(1-e^{it(X-a)}\right)\mu(dx)=\int\left(1-\cos t(X-a)+i\sin t(X-a)\right)\mu(dx)$$

$$=\int\left(1-\cos t(X-a)\right)\mu(dx)$$

Since the integral vanishes and $1-\cos t(X-a)\geq 0$ by $\cos t(X-a)\leq 1$

$1-\cos t(X-a)=0$ a.s., $\cos t(X-a)=1$ a.s.,

$t(X-a)=2n\pi$ a.s., $X=a+n\frac{2\pi}{t}$ a.s..

Hence X has a lattice distribution

(a) If X has a lattice distribution,

$$|\varphi(t)|=|E[e^{itX}]|=|E[e^{it(a+nb)}]|$$

$$=|E[e^{ita}]||E[e^{itnb}]|=1$$

(c) From (a), the mass of μ concentrates at points of the form $a+2\pi n/t$ and also at points of the form $a'+2\pi n/t'$.

If μ is positive at two distinct points, it follows that t/t' is rational, but we have a contradiction.

Hence μ is degenerate. #

• <u>Ex 26.2</u> (Billingsley)

If $\mu(-\infty, x] = \mu[-x, \infty)$ for all x

(which implies that $\mu(A) = \mu(-A)$ for all $A \in \mathcal{R}'$),

then μ is symmetric.

Show that this holds if and only if the chf is real.

Pf: "$\Rightarrow$" If μ is symmetric.

$$\varphi_X(t) = E[e^{itX}]$$

$$= \int_{-\infty}^{\infty} \cos tx\, \mu(dx) + i\int_{-\infty}^{\infty} \sin tx\, \mu(dx)$$

$$= \int_{-\infty}^{\infty} \cos tx\, \mu(dx) + i\left(\int_{-\infty}^{0} \sin tx\, \mu(dx) + \int_{0}^{\infty} \sin tx\, \mu(dx)\right)$$

$$= \int_{-\infty}^{\infty} \cos tx\, \mu(dx) + i\left(\int_{-\infty}^{0} \sin tx\, \mu(dx) + \int_{-\infty}^{0} \sin t(-y)\, \mu(d(-y))\right)$$

$$= \int_{-\infty}^{\infty} \cos tx\, \mu(dx) + i\left(\cancel{\int_{-\infty}^{0} \sin tx\, \mu(dx)} - \cancel{\int_{-\infty}^{0} \sin ty\, \mu(dy)}\right), \quad \mu(d(-y)) = \mu(dy)$$

$$= \int_{-\infty}^{\infty} \cos tx\, \mu(dx) \in \mathbb{R}$$

"$\Leftarrow$" If the chf is real.

$$\varphi_{-X}(t) = E[e^{it(-X)}]$$

$$= \int \cos t(-x)\, \mu(dx) + i\int \sin t(-x)\, \mu(dx)$$

$$= \int \cos tx\, \mu(dx) - i\int \sin tx\, \mu(dx) \quad \text{by} \begin{cases} \cos t(-x) = \cos tx \\ \sin t(-x) = -\sin tx. \end{cases}$$

$$= \overline{\varphi_X(t)} \equiv \varphi_X(t) \quad \text{by the hypothesis.}$$

Thus X and $-X$ have the same chf and

by the uniqueness thm of chf,

X and $-X$ have the distribution and hence X is symmetric r.v.

#

- <u>Ex 26.5</u> (Billingsley)

Show by Thm 26.1 and integration by parts that if μ has a density f with integrable derivative f', then $\varphi(t) = o(t^{-1})$ as $|t| \to \infty$.

Extend to higher derivatives.

Pf: · Thm 26.1: Riemann-Lebesgue Thm

If μ has a density, then $\varphi(t) \to 0$ as $|t| \to \infty$.

· If μ has a density f with integrable derivative f'.

$t\varphi(t) = t\int_{-\infty}^{\infty} e^{itx} f(x)\,dx$ let $u = f(x)$, $dv = e^{itx}dx$; $du = f'(x)\,dx$, $v = \frac{1}{it}e^{itx}$

$= \frac{t}{it}\left(e^{itx}f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{itx} f'(x)\,dx\right)$

$= i\int_{-\infty}^{\infty} e^{itx} f'(x)\,dx$, by $e^{itx}f(x) = 0$ as $x = \pm\infty$.

The above is true since $|e^{itx}| \le 1$ and $f(\pm\infty) = 0$ by $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$t\varphi(t) = i\int_{-\infty}^{\infty} e^{itx} f'(x)\,dx \to 0$ as $|t| \to \infty$ by Thm 26.1 since $f'(x)$ is integrable.

Hence $\varphi(t) = o(t^{-1})$.

· If μ has a density f with integrable derivative $f^{(n)}$.

$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$

$= \frac{i}{t}\int_{-\infty}^{\infty} e^{itx} f'(x)\,dx$ by the similar argument above.

$= \left(\frac{i}{t}\right)^2 \int_{-\infty}^{\infty} e^{itx} f''(x)\,dx$

$\vdots$

$= \left(\frac{i}{t}\right)^n \int_{-\infty}^{\infty} e^{itx} f^{(n)}(x)\,dx$

$t^n\varphi(t) = i^n\int_{-\infty}^{\infty} e^{itx} f^{(n)}(x)\,dx \to 0$ as $|t| \to \infty$ by Thm 26.1 since $f^{(n)}(x)$ is integrable.

Hence $\varphi(t) = o(t^{-n})$.

#

• <u>Ex 26.6</u> (Billingsley)

Show for independent r.v.'s uniformly distributed over $(-1,1)$ that $X_1+\cdots+X_n$ has density $\pi^{-1}\int_0^\infty (\sin t/t)^n \cos tx\,dt$ for $n\ge 2$.

Pf: • (26.20): The inversion formula: suppose $\int_{-\infty}^{\infty}|\varphi(t)|\,dt<\infty$, i.e. $\varphi(t)$ integrable.

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(t)\,dt$$

• Suppose that the X_n are iid $U(-1,1)$ r.v.'s.

$$\varphi_{X_1}(t)=\int_{-1}^{1} e^{itx}\tfrac{1}{2}\,dx=\frac{1}{2it}\int_{-1}^{1} e^{itx}\,d(itx)$$

$$=\frac{1}{2it}\left[e^{itx}\right]_{-1}^{1}=\frac{1}{2it}\left(e^{it}-e^{-it}\right)$$

$$=\frac{1}{2it}\left((\cos t+i\sin t)-(\cos t-i\sin t)\right)$$

$$=\frac{1}{\cancel{2i}t}\cancel{2i}\sin t=\frac{\sin t}{t}$$

Since $X_1,\ldots,X_n$ are independent, let $S_n=X_1+\cdots+X_n$,

$$\varphi_{S_n}(t)=\prod_{i=1}^{n}\varphi_{X_i}(t)=\left(\frac{\sin t}{t}\right)^n.$$

• Since $|\varphi_{S_n}(t)|=\left|\left(\frac{\sin t}{t}\right)^n\right|\le\frac{1}{|t|^n}$, $t\neq 0$,

so $\varphi_{S_n}(t)$ is integrable for $n\ge 2$.

• $f_{S_n}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_{S_n}(t)\,dt$ by the inversion formula for integrable $\varphi(t)$.

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\left(\frac{\sin t}{t}\right)^n dt$$

$$=\frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\cos tx\left(\frac{\sin t}{t}\right)^n dt-i\int_{-\infty}^{\infty}\sin tx\left(\frac{\sin t}{t}\right)^n dt\right)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^n\cos tx\,dt-\frac{i}{2\pi}\int_{-\infty}^{\infty}\sin tx\,\varphi_{S_n}(t)\,dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^n\cos tx\,dt$$

• $\int_{-\infty}^{\infty}\sin tx\,\varphi_{S_n}(t)\,dt=0$ since

$\varphi_{S_n}(t)=\varphi_{S_n}(-t)$ and $\sin tx$ is an odd function.

#

• Ex 26.9 (Billingsley)

Use chf for a simple proof that the family of Cauchy distribution defined by $f_u(x) = \frac{1}{\pi}\frac{u}{u^2+x^2}$, $-\infty < x < \infty$, is closed under convolution. Do the same for the normal, gamma, Poisson distribution.

Pf: (a) Suppose that $X \sim \text{Cauchy}(a_1, b_1)$, $Y \sim \text{Cauchy}(a_2, b_2)$, $X \perp\!\!\!\perp Y$

For $Z \sim \text{Cauchy}(0,1)$, $\varphi_Z(t) = e^{-|t|}$.

$$\varphi_X(t) = E[e^{itX}] = E[e^{it(a_1+b_1Z)}] \quad \text{by } Z = \frac{X-a_1}{b_1},\ X = a_1+b_1Z$$

$$= e^{ita_1} E[e^{itb_1Z}]$$

$$= e^{ita_1 - |b_1t|}.$$

$$\varphi_Y(t) = e^{ita_2 - |b_2t|}.$$

$$\Rightarrow \varphi_{X+Y}(t) = e^{ita_1 - |b_1t|} \cdot e^{ita_2 - |b_2t|} \quad \text{by } X \perp\!\!\!\perp Y$$

$$= e^{it(a_1+a_2) - |(b_1+b_2)t|}$$

The Cauchy (a, b) family is closed under convolution.

(b) Suppose that $X \sim n(\mu_1, \sigma_1^2)$, $Y \sim n(\mu_2, \sigma_2^2)$, $X \perp\!\!\!\perp Y$,

$$\varphi_X(t) = e^{i\mu_1 t - \frac{1}{2}\sigma_1^2 t^2}$$

$$\varphi_Y(t) = e^{i\mu_2 t - \frac{1}{2}\sigma_2^2 t^2}$$

$$\Rightarrow \varphi_{X+Y}(t) = e^{i\mu_1 t - \frac{1}{2}\sigma_1^2 t^2} \cdot e^{i\mu_2 t - \frac{1}{2}\sigma_2^2 t^2} \quad \text{by } X \perp\!\!\!\perp Y$$

$$= e^{i(\mu_1+\mu_2)t - \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$$

The $n(\mu, \sigma^2)$ family is closed under convolution.

(c) Suppose that $X \sim \text{gamma}(\alpha_1, \beta)$, $Y \sim \text{gamma}(\alpha_2, \beta)$, $X \perp\!\!\!\perp Y$.

$$\varphi_X(t) = \int_0^\infty e^{itx} \frac{1}{\Gamma(\alpha_1)\beta^{\alpha_1}} x^{\alpha_1-1} e^{-\frac{x}{\beta}}\, dx$$

$$= \frac{(\frac{1}{\beta}-it)^{-\alpha_1}}{\beta^{\alpha_1}} \int_0^\infty \frac{(\frac{1}{\beta}-it)^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} e^{-(\frac{1}{\beta}-it)x}\, dx$$

$$= (1-i\beta t)^{-\alpha_1}$$

$$\varphi_Y(t) = (1-i\beta t)^{-\alpha_2}$$

$$\Rightarrow \varphi_{X+Y}(t) = (1-i\beta t)^{-(\alpha_1+\alpha_2)} \quad \text{by } X \perp\!\!\!\perp Y$$

The gamma(α, β) family, with β fixed, is closed under convolution.

(d) Suppose $X \sim Poisson(\lambda_1)$, $Y \sim Poisson(\lambda_2)$, $X \perp\!\!\!\perp Y$

$$\varphi_X(t) = \sum_{x=0}^{\infty} e^{itx} e^{-\lambda_1} \frac{\lambda_1^x}{x!}$$

$$= e^{-\lambda_1} \sum_{x=0}^{\infty} \frac{(e^{it}\lambda_1)^x}{x!} = e^{\lambda_1(e^{it}-1)}$$

$$\varphi_Y(t) = e^{\lambda_2(e^{it}-1)}$$

$$\Rightarrow \varphi_{X+Y}(t) = e^{(\lambda_1+\lambda_2)(e^{it}-1)} \quad \text{by } X \perp\!\!\!\perp Y$$

The Poisson(λ) family is closed under convolution.

#

- **Ex 26.10** (Billingsley)

Suppose that $F_n \Rightarrow F$ and that the chf's are dominated by an integrable function.
Show that F has a density that is the limit of the densities of the F_n.

Pf: • Thm 26.3: The continuity thm of chf:
Let μ_n, μ be probability measures with chfs φ_n, φ.
A necessary and sufficient condition for $\mu_n \Rightarrow \mu$ is that $\varphi_n(t) \to \varphi(t)$ for each t.

• (26.20): The inversion formula: suppose $\int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$, then we have density f s.t.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt.$$

• Suppose that $F_n \Rightarrow F$, then $\mu_n \Rightarrow \mu$,
by the continuity thm we have that $\varphi_n(t) \to \varphi(t)$ for each t

• Suppose $|\varphi_n(t)| \le Z \in L_1$, by the dominated convergence thm we have

$$\begin{aligned} f_n(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi_n(t) dt \\ &\to \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \lim_n \varphi_n(t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt \\ &= f(x). \end{aligned}$$

#

• Ex 26.12 (Billingsley)

Show that

$$\mu\{a\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,dt \qquad (26.30)$$

Hint: by the kind of argument leading to (26.16)

Pf: • Let $I_T = \frac{1}{2T}\int_{-T}^{T} e^{-ita}\left[\int_{-\infty}^{\infty} e^{itx}\,\mu(dx)\right]dt$

$$= \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} e^{it(x-a)}\,\mu(dx)\,dt$$

$$= \frac{1}{2T}\int_{-\infty}^{\infty}\int_{-T}^{T} e^{it(x-a)}\,dt\,\mu(dx) \quad \text{by } e^{it(x-a)} \geq 0, \text{ Fubini's Thm.}$$

• If $x \neq a$,

$$\int_{-T}^{T} e^{it(x-a)}\,dt = \frac{1}{i(x-a)}\int_{-T}^{T} e^{it(x-a)}\,d(it(x-a))$$

$$= \frac{1}{i(x-a)}\left[e^{it(x-a)}\right]_{-T}^{T} = \frac{1}{i(x-a)}\left(e^{iT(x-a)} - e^{-iT(x-a)}\right)$$

$$= \frac{2\sin T(x-a)}{x-a};$$

• If $x = a$,

$$\int_{-T}^{T} e^{it(x-a)}\,dt = \int_{-T}^{T} dt = 2T$$

$$\Rightarrow I_T = \frac{1}{\cancel{2}T}\int_{-\infty}^{\infty} \frac{\cancel{2}\sin T(x-a)}{x-a} I_{[x\neq a]}\,\mu(dx) + \frac{1}{\cancel{2T}}\int_{-\infty}^{\infty} \cancel{2T}\, I_{[x=a]}\,\mu(dx)$$

$$= \int_{-\infty}^{\infty} \frac{\sin T(x-a)}{T(x-a)} I_{[x\neq a]}\,\mu(dx) + \mu\{a\}.$$

$\because \left|\frac{\sin T(x-a)}{T(x-a)} I_{[x\neq a]}\right| \leq 1$ and $\frac{\sin T(x-a)}{T(x-a)} I_{[x\neq a]} \to 0$ as $T\to\infty$.

$\therefore$ It follows by the bounded convergence thm that

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} \frac{\sin T(x-a)}{T(x-a)} I_{[x\neq a]}\,\mu(dx)$$

$$= \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{\sin T(x-a)}{T(x-a)} I_{[x\neq a]}\,\mu(dx) = 0.$$

Hence $\mu\{a\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,dt.$

#

• Ex 26.14 (Billingsley)

Show that μ has no point mass if $\varphi^2(t)$ is integrable.

Pf: • (26.31): $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 dt = \sum_k (\mu\{x_k\})^2$.

• By (26.31),

$$\sum_k (\mu\{x_k\})^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 dt$$

$$\le \limsup_{T\to\infty} \frac{1}{2T} \left(\limsup_{T\to\infty} \int_{-T}^{T} |\varphi(t)|^2 dt \right)$$

$$= \limsup_{T\to\infty} \frac{1}{2T} \int_{\mathbb{R}} |\varphi(t)|^2 dt = 0 \quad \text{by } \int_{\mathbb{R}} |\varphi(t)|^2 dt < \infty.$$

Since $\sum_k (\mu\{x_k\})^2$ is nonnegative,

thus $\sum_k (\mu\{x_k\})^2 = 0$ and hence $\mu(\{x_k\}) = 0$ for all k.

#

• Ex 26.13 (Billingsley)

Let $x_1, x_2, \cdots$ be the points of positive μ-measure.

By the following steps prove that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2 dt = \sum_k (\mu\{x_k\})^2. \quad (26.31)$$

Let X and Y be independent and have chf φ.

(a) Show by (26.30) that the left side of (26.31) is $P[X-Y=0]$.

(b) Show (Thm 20.3) that $P[X-Y=0] = \int_{-\infty}^{\infty} P[X=y]\,\mu(dy) = \sum_k (\mu\{x_k\})^2$.

Pf: • (26.30): $\mu\{a\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,dt$

• Thm 20.3: If X and Y are independent random vectors with distributions μ and ν in R^j and R^k, then

$$P[(X,Y)\in B] = \int_{R^j} P[(x,Y)\in B]\,\mu(dx), \quad B\in \mathcal{R}^{j+k}, \text{ and}$$

$$P[X\in A, (X,Y)\in B] = \int_A P[(x,Y)\in B]\,\mu(dx), \quad A\in \mathcal{R}^j,\ B\in \mathcal{R}^{j+k}.$$

(a)
$$\begin{aligned}
P[X-Y=0] &= \mu_{X-Y}\{0\} \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-it0}\varphi_X(t)\varphi_{-Y}(t)\,dt \quad \text{by } \varphi_{X-Y}(t) = \varphi_X(t)\varphi_{-Y}(t) \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi(t)\overline{\varphi(t)}\,dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2 dt
\end{aligned}$$

(b)
$$\begin{aligned}
P[X-Y=0] &= E[I_{[X-Y=0]}] \\
&= E[E[I_{[X-Y=0]} \mid Y]] \\
&= \int_{-\infty}^{\infty} P[X=y]\,\mu(dy) \\
&= \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \mu\{x_k\} I_{[x_k=y]}\,\mu(dy). \\
&= \sum_{k=1}^{\infty} \mu\{x_k\}\int_{-\infty}^{\infty} I_{[x_k=y]}\,\mu(dy) \quad \text{by Fubini's Thm: } \mu\{x_k\} I_{[x_k=y]} \ge 0 \\
&= \sum_{k=1}^{\infty} \mu\{x_k\}\cdot\mu\{x_k\} \\
&= \sum_{k=1}^{\infty} (\mu\{x_k\})^2.
\end{aligned}$$

#

• Ex 26.20 (Billingsley)

Use the continuity thm to prove the result in Ex 25.2 concerning the convergence of the binomial distribution to the Poisson.

Pf: • The continuity thm: Let μ_n, μ be probability measures with chf's φ_n, φ. A necessary and sufficient condition for $\mu_n \Rightarrow \mu$ is that $\varphi_n(t) \to \varphi(t)$ for each t.

• Let μ_n be binomimal $(n, \frac{\lambda}{n})$,

$$\varphi_n(t) = \sum_{k=0}^{n} e^{itk} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\lambda}{n} e^{it}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$
$$= \left(1-\frac{\lambda}{n}+\frac{\lambda}{n}e^{it}\right)^n$$

• Let μ be Poisson (λ),

$$\varphi(t) = \sum_{k=0}^{\infty} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!}$$
$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!}$$
$$= e^{\lambda(e^{it}-1)}$$

• $$\lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}+\frac{\lambda}{n}e^{it}\right)^n$$
$$= \lim_{n\to\infty} \left(1+\frac{\lambda(e^{it}-1)}{n}\right)^n$$
$$= e^{\lambda(e^{it}-1)} = \varphi(t)$$

• By the continuity thm, it follows that

$\mu_n \Rightarrow \mu$,

This is the Poisson approximation to the binomial.

#

• <u>Ex 26.21</u> (Billingsley)

According to Ex 25.8, if $X_n \Rightarrow X$, $a_n \to a$, $b_n \to b$,

then $a_n X_n + b_n \Rightarrow aX+b$.

Prove this by means of chf's.

Pf: • $E\left[e^{it(a_n X_n + b_n)}\right] = e^{itb_n} E\left[e^{ita_n X_n}\right] = e^{itb_n} \varphi_n(ta_n)$

• Since e^{itx} is a continuous function of x,

so $e^{itb_n} \to e^{itb}$.

• $\varphi_n(ta_n) \to \varphi(ta)$ since

$|\varphi_n(ta_n) - \varphi(ta)| \le |\varphi_n(ta_n) - \varphi_n(ta)| + |\varphi_n(ta) - \varphi(ta)| \to 0$

by (i) $ta_n \to ta$, so $|\varphi_n(ta_n) - \varphi_n(ta)| \to 0$, $\because$ φ_n is uniformly continuous.

(ii) $X_n \Rightarrow X$, thus $\mu_n \Rightarrow \mu$, and hence $|\varphi_n(ta) - \varphi(ta)| \to 0$.

• $E\left[e^{it(a_n X_n + b_n)}\right] = e^{itb_n} \varphi_n(ta_n)$

$\to e^{itb} \varphi(ta) = E\left[e^{it(aX+b)}\right]$,

by the continuity thm it follows that

$a_n X_n + b_n \Rightarrow aX+b$.

#.

• Ex 27.1 (Billingsley)

Prove Thm 23.2 by means of chf.

Hint: Use (27.5) to compare the chf of $\sum_{k=1}^{r_n} Z_k$ with $\exp[Z_k p_{nk}(e^{it}-1)]$.

Pf: • Thm 23.2: The Poisson Approximation:

Suppose that for each n, $Z_{n1}, \ldots, Z_{nr_n}$ are independent r.v.'s with $P[Z_{nk}=1]=p_{nk}$, $P[Z_{nk}=0]=1-p_{nk}$.

If $\sum_{k=1}^{r_n} p_{nk} \to \lambda \geq 0$, $\max_{1\leq k\leq r_n} p_{nk} \to 0$, then

$$P\left[\sum_{k=1}^{r_n} Z_{nk}=i\right] \to e^{-\lambda}\frac{\lambda^i}{i!}, \quad i=0,1,2,\ldots.$$

•(27.5): Let $z_1,\ldots,z_m$ and $w_1,\ldots,w_m$ be complex numbers of modulus 1; then

$$|z_1\cdots z_m - w_1\cdots w_m| \leq \sum_{k=1}^{m} |z_k - w_k|$$

•The chf of a Poisson(λ) is

$$\varphi(t)=\sum_{k=0}^{\infty} e^{itk} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^{it})^k}{k!} = e^{-\lambda+\lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

Since $\sum_{k=1}^{r_n} p_{nk}\to\lambda$, it suffices to compare the chf of $\sum_{k=1}^{r_n} Z_{nk}$ with $e^{\sum_{k=1}^{r_n} p_{nk}(e^{it}-1)}$.

Since $Z_{n1},\ldots,Z_{nr_n}$ are independent,

$$E\left[e^{it\sum_{k=1}^{r_n} Z_{nk}}\right] = \prod_{k=1}^{r_n} E\left[e^{itZ_{nk}}\right] = \prod_{k=1}^{r_n}\left[(1-p_{nk})+p_{nk}e^{it}\right]$$

Since $E[e^{itZ_{nk}}]=1-p_{nk}+p_{nk}e^{it}$ is a chf, $|1-p_{nk}+p_{nk}e^{it}|\leq 1$; and suppose X_{nk} is a Poisson(p_{nk}) r.v., then it's chf is

$$E\left[e^{itX_{nk}}\right]=e^{p_{nk}(e^{it}-1)}, \text{ hence } |e^{p_{nk}(e^{it}-1)}|\leq 1,$$

So by (27.5)

$$\left|\prod_{k=1}^{r_n}\left[(1-p_{nk})+p_{nk}e^{it}\right] - e^{\sum_{k=1}^{r_n} p_{nk}(e^{it}-1)}\right|$$

$$\leq \sum_{k=1}^{r_n}\left|(1-p_{nk})+p_{nk}e^{it} - e^{p_{nk}(e^{it}-1)}\right| = \sum_{k=1}^{r_n}\left|e^{p_{nk}(e^{it}-1)} - 1 - p_{nk}(e^{it}-1)\right|$$

$$\leq \sum_{k=1}^{r_n}\min\left\{\frac{1}{2!}p_{nk}^2(e^{it}-1)^2,\ 2|p_{nk}(e^{it}-1)|\right\} \leq \sum_{k=1}^{r_n}\frac{1}{2}p_{nk}^2(e^{it}-1)^2$$

$$\leq \frac{1}{2}\left(\max_{1\leq k\leq r_n} p_{nk}\right)2^2\sum_{k=1}^{r_n} p_{nk} \quad \because |e^{it}-1|\leq|e^{it}|+1=2$$

$$\to 0\cdot 2\cdot\lambda = 0 \text{ as } n\to\infty.$$

So $\left|\prod_{k=1}^{r_n}\left[(1-p_{nk})+p_{nk}e^{it}\right] - e^{\lambda(e^{it}-1)}\right|$

$$\leq \left|\prod_{k=1}^{r_n}\left[(1-p_{nk})+p_{nk}e^{it}\right] - e^{\sum_{k=1}^{r_n} p_{nk}(e^{it}-1)}\right| + \left|e^{\sum_{k=1}^{r_n} p_{nk}(e^{it}-1)} - e^{\lambda(e^{it}-1)}\right| \to 0.$$

(next pg. cont.)

- By the inversion formula for $\mu\{a\}$:

$$\mu\{a\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,dt.$$

Since we have $\varphi_n(t) = \prod_{k=1}^{r_n}\left[(1-p_{nk}) + p_{nk}(e^{it}-1)\right] \to \varphi(t) = e^{\lambda(e^{it}-1)}$.

$$\lim_{n\to\infty} P\left[\sum_{k=1}^{r_n} Z_{nk} = a\right]$$

$$= \lim_{n\to\infty}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi_n(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\lim_{n\to\infty}\varphi_n(t)\,dt \quad \because |e^{-ita}\varphi_n(t)| \le 1 \text{ and bounded convergence thm.}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,dt = e^{-\lambda}\frac{\lambda^a}{a!}.$$

#

• Ex 27.2 (Billingsley)

If $\{X_n\}$ is independent and the X_n all have the same distribution with finite first moment, then $n^{-1}S_n \to_{a.s.} E[X_1]$ (Thm 22.1).

so that $n^{-1}S_n \Rightarrow E[X_1]$. Prove the latter by chf's.

Hint: Use (27.5)

Pf: • (27.5): Let $z_1, \dots, z_m$ and $w_1, \dots, w_m$ be complex numbers of modulus 1; then

$$|z_1 \cdots z_m - w_1 \cdots w_m| \le \sum_{k=1}^{m} |z_k - w_k|$$

• Let $m = E[X_1]$. $E[e^{itm}] = e^{itm}$.

It suffices to compare $E[e^{itS_n/n}]$ with e^{itm}.

Since $E[e^{itS_n/n}] = E[e^{itn^{-1}\sum_{k=1}^{n} X_k}] = \prod_{k=1}^{n} E[e^{i\frac{t}{n}X_1}] = \varphi^n(\frac{t}{n})$, $\because$ X_n are iid,

• $|\varphi^n(\frac{t}{n}) - e^{itm}|$

$$\le \sum_{k=1}^{n} |\varphi(\tfrac{t}{n}) - e^{i\frac{t}{n}m}| \quad \because |\varphi(\tfrac{t}{n})| \le 1, |e^{i\frac{t}{n}m}| \le 1$$

$$= n|\varphi(\tfrac{t}{n}) - e^{i\frac{t}{n}m}|$$

$$\le n|\varphi(\tfrac{t}{n}) - 1 - i\tfrac{t}{n}m| + n|e^{i\frac{t}{n}m} - 1 - i\tfrac{t}{n}m|$$

• Since $n|\varphi(\frac{t}{n}) - 1 - i\frac{t}{n}m|$

$$\le n\, E\left[|e^{i\frac{t}{n}X_1} - 1 - i\tfrac{t}{n}X_1|\right]$$

$$\le E\left[\min\left\{n \cdot \frac{1}{2}\frac{t^2X_1^2}{n^2},\ n \cdot 2 \cdot \frac{|tX_1|}{n}\right\}\right],$$

$\because$ $n|e^{i\frac{t}{n}X_1} - 1 - i\frac{t}{n}X_1| \le 2|t||X_1| \in L_1$ $\because$ $E[|X_1|] < \infty$; and

$n|e^{i\frac{t}{n}X_1} - 1 - i\frac{t}{n}X_1| \le \frac{1}{2}\frac{t^2X^2}{n} \to 0$ as $n \to \infty$,

$\therefore$ the dominated convergence thm implies that

$n|\varphi(\frac{t}{n}) - 1 - i\frac{t}{n}m| \to 0$ as $n \to \infty$; and

• Since $n|e^{i\frac{t}{n}m} - 1 - i\frac{t}{n}m| \le \min\{\frac{t^2m^2}{2n}, 2|tm|\} \le \frac{t^2m^2}{2n} \to 0$ as $n \to \infty$,

hence $|\varphi^n(\frac{t}{n}) - e^{itm}| \to 0$ as $n \to \infty$,

the continuity thm implies that $n^{-1}S_n \Rightarrow m = E[X_1]$.

#

- Ex 27.3 (Billingsley)

For a Poisson r.v. Y_λ with mean λ, show that
$(Y_\lambda-\lambda)/\sqrt{\lambda} \Rightarrow N$ as $\lambda\to\infty$.

Pf: • To deal with values of λ are not integers,
let N_1, N_2, N_3 be independent Poisson with means
$\lfloor\lambda\rfloor$, $\lambda-\lfloor\lambda\rfloor$, $\lfloor\lambda\rfloor+1-\lambda$,
If we set $S_{\lfloor\lambda\rfloor}=N_1$, $Y_\lambda=N_1+N_2$, $S_{\lfloor\lambda\rfloor+1}=N_1+N_2+N_3$

• We first show that, under $S_n\sim$ Poisson (n),
$(S_n-n)/\sqrt{n}\Rightarrow N$ as $n\to\infty$
where $S_n=\sum_{k=1}^{n}X_k$, $X_k \overset{iid}{\sim}$ Poisson(1) with $E[X_k]=1$, $Var[X_k]=1$.
By the CLT for iid r.v.'s the result follows.

• Since $S_{\lfloor\lambda\rfloor}\le Y_\lambda\le S_{\lfloor\lambda\rfloor+1}$, it follows that
$(Y_\lambda-\lambda)/\sqrt{\lambda}\Rightarrow N$ as $\lambda\to\infty$.

#.

- <u>Ex 27.4</u> (Billingsley)

Suppose that $|X_{nk}| \le M_n$ with probability 1 and $M_n / s_n \to 0$.
Verify Lyapounov's condition and then Lindeberg's condition.

Pf: • For each n the sequence $X_{n1}, \cdots, X_{nr_n}$ is independent and

$$E[X_{nk}] = 0, \ \sigma_{nk}^2 = E[X_{nk}^2], \ s_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2 .$$

• Lyapounov's condition:

$|X_{nk}|^{2+\delta}$ are integrable for some $\delta > 0$, and

$$\lim_n \sum_{k=1}^{r_n} \frac{1}{s_n^{2+\delta}} E\left[|X_{nk}|^{2+\delta}\right] = 0.$$

• Lindeberg's condition:

$$\lim_n \sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{|X_{nk}| \ge \epsilon s_n} X_{nk}^2 \, dP = 0.$$

(a)
$$\lim_n \sum_{k=1}^{r_n} \frac{1}{s_n^3} E\left[|X_{nk}|^3\right]$$
$$\le \lim_n \sum_{k=1}^{r_n} \frac{1}{s_n^3} M_n \cdot E\left[|X_{nk}|^2\right]$$
$$= \lim_n \frac{M_n}{s_n} \to 0 \text{ by hypothesis.}$$

which is Lyapounov's condition for $\delta = 1$.

(b) Then Lindeberg's condition follows:

$$\lim_n \sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{|X_{nk}| \ge \epsilon s_n} X_{nk}^2 \, dP$$
$$\le \lim_n \sum_{k=1}^{r_n} \frac{1}{s_n^2} \int |X_{nk}|^2 \frac{|X_{nk}|}{\epsilon s_n} \, dP$$
$$\le \lim_n \sum_{k=1}^{r_n} \frac{1}{s_n^2} \frac{M_n}{\epsilon s_n} E[X_{nk}^2]$$
$$= \lim_n \frac{M_n}{\epsilon s_n} \to 0 \text{ by hypothesis.}$$

#

- Ex 27.5 (Billingsley)

Suppose that the r.v.'s in any single row of the triangular array are identically distributed.

To what do Lindeberg's and Lyapounov's conditions reduce?

Pf: • For each n the sequence $X_{n1}, \dots, X_{nr_n}$ is i.i.d. and

$$E[X_{nk}]=0,\ \sigma_{nk}^2 = E[X_{nk}^2] = \sigma_n^2,\ S_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2 = r_n \sigma_n^2$$

· Lindeberg's condition:

$$\lim_n \sum_{k=1}^{r_n} \frac{1}{S_n^2} \int_{|X_{nk}| \geq \epsilon S_n} X_{nk}^2 \, dP$$

$$= \lim_n \sum_{k=1}^{r_n} \frac{1}{r_n \sigma_n^2} \int_{|X_{n1}| \geq \epsilon \sigma_n \sqrt{r_n}} X_{n1}^2 \, dP \quad \because \text{ for each } n,\ X_{nk} \text{ iid.}$$

$$= \lim_n \frac{1}{\sigma_n^2} \int_{|X_{n1}| \geq \epsilon \sigma_n \sqrt{r_n}} X_{n1}^2 \, dP \to 0 \text{ as } n \to \infty.$$

· Lyapounov's condition:

$$\lim_n \sum_{k=1}^{r_n} \frac{1}{S_n^{2+\delta}} E[|X_{nk}|^{2+\delta}]$$

$$= \lim_n \sum_{k=1}^{r_n} \frac{1}{(r_n \sigma_n^2)^{1+\frac{\delta}{2}}} E[|X_{n1}|^{2+\delta}]$$

$$= \lim_n \frac{r_n}{r_n^{1+\delta} \sigma_n^{2+\delta}} E[|X_{n1}|^{2+\delta}]$$

$$= \lim_n \frac{1}{r_n^{\delta} \sigma_n^{2+\delta}} E[|X_{n1}|^{2+\delta}] \to 0 \text{ as } n \to \infty.$$

#

- Ex 27.6 (Billingsley)

Suppose that $Z_1, Z_2, \ldots$ are iid with mean 0 and var 1, and suppose $X_{nk} = \sigma_{nk} Z_k$

Write down Lindeberg's condition and show that it holds if $\max_{k \le r_n} \sigma_{nk}^2 = o\left(\sum_{k=1}^{r_n} \sigma_{nk}^2\right)$

Pf: • For each n the sequence $X_{n1}, \ldots, X_{nr_n}$ is independent and.

$E[X_{nk}] = \sigma_{nk} E[Z_k] = 0$;

$E[X_{nk}^2] = \sigma_{nk}^2 E[Z_k^2] = \sigma_{nk}^2$;

$s_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2$.

• $\max_{k \le r_n} \sigma_{nk}^2 = o\left(\sum_{k=1}^{r_n} \sigma_{nk}^2\right) = o(s_n^2) \Rightarrow \max_{k \le r_n} \frac{\sigma_{nk}^2}{s_n^2} \to 0$.

• Lindeberg's condition is

$$\sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{|X_{nk}| \ge \epsilon s_n} X_{nk}^2 \, dP$$

$$= \sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{\sigma_{nk}^2 |Z_k|^2 \ge \epsilon^2 s_n^2} \sigma_{nk}^2 Z_k^2 \, dP$$

$$= \sum_{k=1}^{r_n} \frac{\sigma_{nk}^2}{s_n^2} \int_{|Z_1|^2 \ge \epsilon \frac{s_n^2}{\sigma_{nk}^2}} Z_1^2 \, dP \qquad \text{by } \{Z_i\} \text{ is iid}$$

$$\le \left(\int_{|Z_1|^2 \ge \epsilon \frac{s_n^2}{\max_{k \le r_n} \sigma_{nk}^2}} Z_1^2 \, dP\right)\left(\underbrace{\sum_{k=1}^{r_n} \frac{\sigma_{nk}^2}{s_n^2}}_{1}\right)$$

$\because \frac{\max_{k \le r_n} \sigma_{nk}^2}{s_n^2} \to 0$, $\frac{s_n^2}{\max_{k \le r_n} \sigma_{nk}^2} \to \infty$,

$\therefore Z_1^2 \mathbb{1}_{|Z_1|^2 \ge \epsilon \frac{s_n^2}{\max_{k \le r_n} \sigma_{nk}^2}} \to Z_1^2 \cdot 0 = 0$, and

$Z_1^2 \mathbb{1}_{|Z_1|^2 \ge \epsilon \frac{s_n^2}{\max_{k \le r_n} \sigma_{nk}^2}} \le Z_1^2 \in L_1$,

So the dominated convergence thm implies that

$$\int_{|Z_1|^2 \ge \epsilon \frac{s_n^2}{\max_{k \le r_n} \sigma_{nk}^2}} Z_1^2 \, dP \to 0 \text{ as } n \to \infty,$$

hence the Lindeberg condition holds. #

• <u>Ex 27.9</u> (Billingsley)

Let S_n be the number of inversions in a random permutation on n letters. Prove a central limit thm for S_n.

Pf: • $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 7 & 4 & 6 & 2 & 3 \end{pmatrix}$

Let X_{nk} be the number of smaller letters lying to the right of letter k.

$\Rightarrow X_{71}=0,\ X_{72}=0,\ X_{73}=0,\ X_{74}=2,\ X_{75}=4,\ X_{76}=2,\ X_{77}=4.$

• $\because X_{n1}(\omega)=0$;

$X_{n2}(\omega)=0$ or 1;

$\vdots$

$X_{nk}(\omega)=0,1,\dots,k-1$

$\vdots$

$X_{nn}(\omega)=0,1,\dots,n-1$

$\therefore (X_{n1}(\omega),\dots,X_{nn}(\omega))$ have $1\times 2\times\cdots\times n = n!$ possibilities.

Then $P[X_{n1}=x_1,\dots,X_{nn}=x_n]=\frac{1}{n!}$.

• $X_{n1},\dots,X_{nk}$ are independent and

$P[X_{nk}=x_k]=\frac{1}{k}$, for $x_k=0,1,\dots,k-1$; $k=1,\dots,n$:

Since $P[X_{nn}=x_n]=\sum_{x_1,\dots,x_{n-1}} P[X_{n1}=x_1,\dots,X_{nn}=x_n]=\frac{(n-1)!}{n!}=\frac{1}{n}$,

so $P[X_{n1}=x_1,\dots,X_{nn}=x_n]=\prod_{k=1}^{n} P[X_{nk}=x_k]$.

• Let $S_n=\sum_{k=1}^{n} X_{nk}$; and $X'_{nk}=X_{nk}-E[X_{nk}]$ are independent r.v.'s with

$E[X'_{nk}]=0$, $s_n^2=\sum_{k=1}^{n}\sigma_{nk}^2$, where $\sigma_{nk}^2=E[X_{nk}^2]-(E[X_{nk}])^2$.

$E[X_{nk}^2]=\sum_{x=0}^{k-1} x^2\cdot\frac{1}{k}=\frac{1}{k}\frac{(k-1)k(2(k-1)+1)}{6}=\frac{(k-1)(2k-1)}{6}$;

$E[X_{nk}]=\sum_{x=0}^{k-1} x\cdot\frac{1}{k}=\frac{1}{k}\frac{(k-1)k}{2}=\frac{k-1}{2}$.

$\sigma_{nk}^2=\frac{(k-1)(2k-1)}{6}-\left(\frac{k-1}{2}\right)^2=\frac{(k-1)(k+1)}{12}=\frac{k^2-1}{12}$

$s_n^2=\sum_{k=1}^{n}\sigma_{nk}^2=\frac{1}{12}\sum_{k=1}^{n}(k^2-1)=\frac{1}{12}\left(\frac{n(n+1)(2n+1)}{6}-\frac{(n+1)n}{2}\right)=\frac{n(n^2-1)}{36}$.

(next pg. cont.)

• Since $|X_{nk}| \le n$, uniformly bounded for each k,

$$\frac{n}{s_n} = \frac{n}{\left(\frac{n(n^2-1)}{36}\right)^{\frac{3}{2}}} = n\, O(n^{-\frac{3}{2}}) = O(n^{-\frac{1}{2}}) \to 0 \text{ as } n \to \infty.$$

Ex 27.4 implies that the Lindeberg condition holds, then

$$\frac{S_n - E[S_n]}{s_n} \Rightarrow N;$$

Since $E[S_n] = \sum_{k=1}^{n} \frac{(k-1)}{2} = \frac{1}{2}\frac{(n-1)n}{2} = \frac{n(n-1)}{4} = \frac{n^2}{4} + O(n)$,

$$s_n = \left(\frac{n(n^2-1)}{36}\right)^{\frac{1}{2}} = \sqrt{\frac{n^3}{36}} + O(\sqrt{n}).$$

$$\frac{S_n - \frac{n^2}{4}}{\sqrt{n^3/36}} = \left(\frac{S_n - E[S_n]}{s_n} + \frac{E[S_n] - \frac{n^2}{4}}{s_n}\right)\frac{s_n}{\sqrt{n^3/36}}$$

$$= \left(\frac{S_n - E[S_n]}{s_n} + \frac{O(n)}{\sqrt{n^3/64} + O(\sqrt{n})}\right)\frac{\sqrt{n^3/36} + O(\sqrt{n})}{\sqrt{n^3/36}}$$

$$\Rightarrow (N + 0) \times 1 = N \text{ as } n \to \infty.$$

#

- <u>Ex 27.11</u> (Billingsley)

Suppose independent X_n have density $|x|^{-3}$ outside $(-1,+1)$.
show that $(n\log n)^{-1/2} S_n \Rightarrow N$.

Pf: • $f(x) = |x|^{-3}$ if $|x|>1$.

$$E[|X_n|] = \int_{-\infty}^{\infty} |x|\cdot|x|^{-3}dx$$
$$= 2\int_1^{\infty} x^{-2}dx = 2\left[\frac{x^{-1}}{-1}\right]_1^{\infty} = 2<\infty,$$

$\Rightarrow E[X_n]=0$ since X_n is symmetric r.v.

But $Var[X_n] = E[X_n^2] = 2\int_1^{\infty} x^2\cdot x^{-3}dx$
$$= 2\int_1^{\infty} x^{-1}dx = 2[\log x]_1^{\infty} = \infty.$$

• Let $Y_{nk} = X_k 1_{[|X_k|\le\sqrt{n}]}$,

$S_n = X_1+\cdots+X_n$, $S_n' = Y_{n1}+\cdots+Y_{nn}$.

$\Rightarrow$ The Y_{nk} is independent r.v.s with

$E[Y_{nk}]=0$ $\because$ Y_{nk} is bounded and symmetric,

$\sigma_{nk}^2 = E[Y_{nk}^2]$
$$= 2\int_1^{\sqrt{n}} x^2\cdot x^{-3}dx = 2\int_1^{\sqrt{n}} x^{-1}dx$$
$$= 2[\log x]_1^{\sqrt{n}} = 2\log\sqrt{n} = \log n.$$

$s_n^2 = \sum_{k=1}^n \sigma_{nk}^2 = n\log n$.

Since $|Y_{nk}|\le\sqrt{n}$ for each k, uniformly bounded,

$$\sum_{k=1}^n \frac{1}{s_n^3}E[|Y_{nk}|^3] \le \sum_{k=1}^n \frac{\sqrt{n}}{s_n^3}E[Y_{nk}^2] = \frac{\sqrt{n}}{s_n} = \frac{\sqrt{n}}{\sqrt{n\log n}} \to 0 \text{ as } n\to\infty.$$

which is the Lyapounov condition for $\delta=1$, then

$$\frac{S_n'}{(n\log n)^{\frac{1}{2}}} \Rightarrow N.$$

• Note $\frac{S_n}{(n\log n)^{\frac12}} = \frac{S_n'}{(n\log n)^{\frac12}} + \frac{S_n - S_n'}{(n\log n)^{\frac12}}$, and

Since
$$P\left[\left|\frac{S_n-S_n'}{(n\log n)^{\frac12}}\right|>\epsilon\right] \le \frac{1}{\epsilon(n\log n)^{\frac12}} E\left[\sum_{k=1}^n X_k 1_{[|X_k|>\sqrt{n}]}\right]$$
$$= \frac{2n}{\epsilon(n\log n)^{\frac12}}\int_{\sqrt{n}}^{\infty} x\cdot x^{-3}dx$$
$$= \frac{2n}{\epsilon(n\log n)^{\frac12}}\times\frac{1}{\sqrt{n}} \to 0 \text{ as } n\to\infty.$$

Thus $\frac{S_n-S_n'}{(n\log n)^{\frac12}} \to_p 0$, and hence $\frac{S_n}{(n\log n)^{\frac12}} \Rightarrow N$. #