- Ex 6.5 (Billingsley)

Prove Poisson's thm: If $A_1, A_2, \dots$ are independent events,

$\bar{p}_n = \frac{1}{n}\sum_{i=1}^{n} P(A_i)$, $N_n = \sum_{i=1}^{n} I_{A_i}$, then

$n^{-1}N_n - \bar{p}_n \to_p 0$.

Pf: $N_n = \sum_{i=1}^{n} I_{A_i}$, where $E[I_{A_i}] = P(A_i)$, $Var[I_{A_i}] = P(A_i)(1-P(A_i)) < \infty$,

$$P[|n^{-1}N_n - \bar{p}_n| \geq \epsilon] \leq \frac{1}{\epsilon^2} E[(n^{-1}N_n - \bar{p}_n)^2]$$

$$= \frac{1}{\epsilon^2} Var[n^{-1}N_n] = \frac{1}{n^2\epsilon^2}\sum_{i=1}^{n} Var[I_{A_i}]$$

$$= \frac{1}{n^2\epsilon^2}\sum_{i=1}^{n}[P(A_i)(1-P(A_i))]$$

$$\leq \frac{1}{n^{\cancel{2}}\epsilon^2}\cancel{n}\cdot\frac{1}{4} \to 0 \quad \text{by } P(A_i)(1-P(A_i)) \leq \frac{1}{4}.$$

Hence $n^{-1}N_n - \bar{p}_n \to_p 0$.

#

• Ex 6.7 (Billingsley)

(a) Let $x_1, x_2, \cdots$ be a sequence of real numbers, and put

$S_n = x_1 + \cdots + x_n$.

Suppose that $n^{-2} S_{n^2} \to 0$ and that the $x_n$ are bounded,

and show that $n^{-1} S_n \to 0$

(b) Suppose that $n^{-2} S_{n^2} \to 0$ a.s. and that the $X_n$ are uniformly bounded.

($\sup_n |X_n(\omega)| < \infty$). Show that $n^{-1} S_n \to 0$ a.s.

Here the $X_n$ need not be identically distributed or even independent.

Pf: (a) Suppose that $n^{-2} S_{n^2} \to 0$.

If $M$ bounds the $|x_n|$, then for each $n \geq 1$,

if $k^2 \leq n < (k+1)^2$,

$$\left|\frac{1}{n} S_n - \frac{1}{k^2} S_{k^2}\right| = \left|\left(\frac{1}{n} - \frac{1}{k^2}\right) S_n + \frac{1}{k^2}(S_n - S_{k^2})\right|$$

$$\leq \left|\frac{1}{n} - \frac{1}{k^2}\right| |S_n| + \frac{1}{k^2} |S_n - S_{k^2}|$$

$$\leq \left|\frac{1}{n} - \frac{1}{k^2}\right| nM + \frac{1}{k^2}(n-k^2) M$$

$$= \left(\frac{1}{k^2} - \frac{1}{n}\right) nM + \frac{1}{k^2}(n-k^2) M$$

$$= \left(\frac{n}{k^2} - 1 + \frac{n}{k^2} - 1\right) M$$

$$= 2\left(\frac{n-k^2}{k^2}\right) M \to 0 \text{ as } n \to \infty.$$

Thus $|n^{-1} S_n - n^{-2} S_{n^2}| \to 0$ as $n \to \infty$.

$$|n^{-1} S_n| \leq |n^{-1} S_n - n^{-2} S_{n^2}| + |n^{-2} S_{n^2}| \to 0$$

Hence $n^{-1} S_n \to 0$.

(b) Suppose $n^{-2} S_{n^2} \to 0$ a.s., then there is a null set $N$ with $P[N] = 0$

s.t. $n^{-2} S_{n^2}(\omega) \to 0$ for $\omega \in N^c$.

Suppose also that $\sup_n |X_n(\omega)| < \infty$, uniformly bounded,

then from (a) we have that

$$n^{-1} S_n(\omega) \to 0 \text{ for } \omega \in N^c,$$

Hence $n^{-1} S_n \to 0$ a.s.

#

- Ex 6.8 (Billingsley)

Suppose that $X_1, X_2, \dots$ are independent and uniformly bounded, $E[X_n]=0$. Using only Ex 6.7, the first Borel-Cantelli lemma, and Chebyshev's inequality, prove that $n^{-1}S_n \to 0$ a.s.

Pf: • Suppose $E[X_n^2] \le M$ for all $n$, then

$$Var[S_n] = \sum_{i=1}^{n} E[X_n^2] \le Mn.$$

• It follows by Chebychev's inequality that, for $\epsilon > 0$,

$$P[|S_n| > n\epsilon] \le \frac{Mn}{n^2\epsilon^2} = \frac{M}{n\epsilon^2}$$

But $\sum_n P[|S_n| > n\epsilon] \le \sum \frac{M}{n\epsilon^2} = \infty$.

• However, if we confine ourselves to the subsequence $\{n^2\}$, then

$$\sum_n P[|S_{n^2}| > n^2\epsilon] \le \sum \frac{M}{n^2\epsilon^2} < \infty,$$

hence by the first Borel-Cantelli lemma we have

$$P[|S_{n^2}| > n^2\epsilon \text{ i.o.}] = 0, \text{ and consequently,}$$

$$\frac{S_{n^2}}{n^2} \to 0 \text{ a.s.}$$

• From Ex 6.7, since $\frac{S_{n^2}}{n^2} \to 0$ a.s. and the $X_n$ are uniformly bounded, we conclude that $\frac{S_n}{n} \to 0$ a.s.

#

- Ex 20.20 (Billingsley)

(a) Suppose that $f: R^2 \mapsto R^1$ is continuous. Show that $X_n \to_p X$ and $Y_n \to_p Y$ imply $f(X_n, Y_n) \to_p f(X, Y)$.

(b) Show that addition and multiplication preserve convergence in probability.

Pf: (a) • Given $\epsilon$ choose $M$ s.t. $P[|X|>M] < \epsilon$ and $P[|Y|>M] < \epsilon$,

• and then choose $\delta$ s.t. $|x|, |y| < M$, $|x-x'| < \delta$, and $|y-y'| < \delta$ imply that $|f(x', y') - f(x, y)| < \epsilon$. Then,

$$
\begin{aligned}
&P[|f(X_n, Y_n) - f(X, Y)| > \epsilon] \\
&\leq P([|X|>M] \cup [|Y|>M] \cup [|X_n - X| > \delta] \cup [|Y_n - Y| > \delta]) \\
&\leq P[|X|>M] + P[|Y|>M] + P[|X_n - X| > \delta] + P[|Y_n - Y| > \delta] \\
&\leq 2\epsilon + P[|X_n - X| > \delta] + P[|Y_n - Y| > \delta] \\
&\to 2\epsilon \quad \text{as } n \to \infty \quad \because X_n \to_p X \text{ and } Y_n \to_p Y.
\end{aligned}
$$

Since $\epsilon$ is arbitrary, the result follows.

(b) Since $f(X, Y) = X + Y$ and $f(X, Y) = XY$ are continuous, the result follows.

#

· Ex 20.21 ( Billingsley)

Suppose that the sequence $\{X_n\}$ is fundamental in probability in the sense that for $\epsilon$ positive there exists $N_\epsilon$ s.t.

$$P[|X_m - X_n| > \epsilon] < \epsilon \text{ for } m, n > N_\epsilon$$

(a) Prove there is a subsequence $\{X_{n_k}\}$ and a r.v. $X$ s.t. $\lim_k X_{n_k} = X$ with probability 1.

(b) Show that $X_n \to_p X$

pf: (a) · Choose increasing $n_k$ s.t.

$$P[|X_m - X_n| > 2^{-k}] < 2^{-k} \text{ for } m, n > n_k$$

Consequently, we have

$$P[|X_{n_{k+1}} - X_{n_k}| > 2^{-k}] < 2^{-k}, \text{ thus}$$

$$\cdot \sum_{k=1}^{\infty} P[|X_{n_{k+1}} - X_{n_k}| > 2^{-k}] < \sum_{k=1}^{\infty} 2^{-k} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 < \infty,$$

The Borel-Cantelli lemma implies that

$$P\left(\limsup_{k\to\infty} [|X_{n_{k+1}} - X_{n_k}| > 2^{-k}]\right) = 0.$$

For $\omega \in N^c$, $|X_{n_{k+1}} - X_{n_k}| \le 2^{-k}$ for all large $k$ and thus

· for any $k > l$ large, we get

$$|X_{n_k}(\omega) - X_{n_l}(\omega)| \le \sum_{j=l}^{k-1} |X_{n_{j+1}} - X_{n_j}| \le \sum_{j=l} 2^{-j} = 2 \cdot 2^{-l},$$

hence $\{X_{n_k}(\omega)\}$ is a Cauchy sequence of real numbers.

Completeness of real line implies $\lim_j X_{n_j}(\omega)$ exists,

that is, $\omega \in N^c$ implies $\lim_j X_{n_j}(\omega)$ exists.

This means that $\{X_{n_j}\}$ converges a.s. and we call the limit $X$.

(b) · $\because$ $[|X_n - X_{n_j}| < \frac{\epsilon}{2}]$ and $[|X_{n_j} - X| < \frac{\epsilon}{2}]$ implies $[|X_n - X| < \epsilon]$,

$$\therefore P[|X_n - X| > \epsilon] \le P[|X_n - X_{n_j}| > \tfrac{\epsilon}{2}] + P[|X_{n_j} - X| > \tfrac{\epsilon}{2}]$$

· For $\epsilon$ positive, pick $n, n_j$ so large that the fundamental in probability implies that $P[|X_n - X_{n_j}| > \frac{\epsilon}{2}] < \frac{\epsilon}{2}$.

· Since $X_{n_j} \to X$ with probability 1 implies $X_{n_j} \to_p X$,

$$P[|X_{n_j} - X| > \tfrac{\epsilon}{2}] < \tfrac{\epsilon}{2} \text{ for large } n_j.$$

The result follows. #

- Ex 20.22 (Billingsley)

Suppose that $\{X_n\}$ is monotone and that $X_n \to_p X$.
Show $X_n \to X$ with probability 1.

Pf: WLOG, assume $X_1 \le X_2 \le \cdots$.

- $X_n \to_p X$ implies that there is a subsequence $\{X_{n_k}\}$ s.t. $X_{n_k} \to X$ with probability 1.

  For $\omega \in N^c$ we have $X(\omega) - X_{n_k}(\omega) < \epsilon$ for all $k \ge k_0(\omega)$.

- The monotonicity implies that $X(\omega) - X_n(\omega) < \epsilon$ for all $n \ge k_0(\omega)$.

$$
\begin{aligned}
&P([|X_n - X| > \epsilon] \text{ i.o.}) \\
&= P([X - X_n > \epsilon] \text{ i.o.}) \\
&= 1 - P(\liminf_n [X - X_n < \epsilon]) \\
&= 1 - 1 = 0.
\end{aligned}
$$

Hence the result follows. #

- Ex 20.23 (Billingsley)

Let $\{X_n\}$ be a sequence of r.v.'s and let $\bar{X}_n = n^{-1}\sum_{k=1}^{n} X_K$. Show that

(i) If $\lim_n X_n = 0$ a.s., then $\lim_n \bar{X}_n = 0$ a.s.

(ii) If $\sup_n E|X_n|^r < \infty$ and $X_n \to 0$ in $L_r$, then $\bar{X}_n \to 0$ in $L_r$, where $r \geq 1$. the result in part (ii) may not be true for $r \in (0,1)$.

(iii) $X_n \to_p 0$ may not imply $\bar{X}_n \to_p 0$.

Pf : (i) • It suffices that show that

if $\{x_n\}$ is a sequence of real numbers satisfying $\lim_n x_n = 0$, then $\lim_n \bar{x}_n = 0$.

• Assume that $\lim_n x_n = 0$. Then $M = \sup_n |x_n| < \infty$, and for any $\epsilon > 0$, there is an $N$ s.t. $|x_n| \leq \epsilon$ for all $n > N$. Then for $n > \max\{N, NM/\epsilon\}$,

$$\begin{aligned} \left|\frac{1}{n}\sum_{i=1}^{n} x_i\right| &\leq \frac{1}{n}\left(\sum_{i=1}^{N}|x_i| + \sum_{i=N+1}^{n}|x_i|\right) \\ &\leq \frac{1}{n}\left(\sum_{i=1}^{N} M + \sum_{i=N+1}^{n}\epsilon\right) \\ &= \frac{NM}{n} + \frac{\epsilon(n-N)}{n} \\ &\leq \epsilon + \epsilon = 2\epsilon \end{aligned}$$

(ii) • It suffices to show that $E|\bar{X}_n|^r \to 0$ as $n \to \infty$, $r \geq 1$.

• For $r \geq 1$, $\varphi(x) = |x|^r$ is a convex function. Then

$$|\bar{X}_n|^r = \varphi(\bar{X}_n) = \varphi\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \leq \frac{1}{n}\sum_{i=1}^{n}\varphi(X_i) = \frac{1}{n}\sum_{i=1}^{n}|X_i|^r.$$

$$E|\bar{X}_n|^r \leq n^{-1}\sum_{i=1}^{n} E|X_i|^r$$

• When $\lim_n E|X_n|^r = 0$, $\lim_n n^{-1}\sum_{i=1}^{n} E|X_i|^r = 0$ from (i). Hence $\lim_n E|\bar{X}_n|^r = 0$, that is, $\bar{X}_n \to 0$ in $L_r$.

(iii) • Consider the sequence of independent r.v.'s defined s.t.

$$P[X_n = 2^n] = \frac{1}{n} \text{ and } P[X_n = 0] = 1 - \frac{1}{n}.$$

• So $P[X_n = 0] = 1 - \frac{1}{n} \to 1$ as $n \to \infty$, $X_n \to_p 0$.

• To show $\bar{X}_n \not\to_p 0$, we will show that $P[|\bar{X}_n| \leq 1] \not\to 1$. Take $N$ s.t $2^{\frac{N}{2}} > N$ (any $N \geq 4$ will suffice). For $n \geq N$, we have

$$\begin{aligned} P[|\bar{X}_n| \leq 1] &\leq P\left[X_k = 0 \text{ whenever } \tfrac{n+1}{2} < k \leq n\right] \\ &\leq \left(1 - \frac{1}{n}\right)^{\frac{n}{2}} \to e^{-\frac{1}{2}}. \end{aligned}$$

#

• Ex 21.2 (Billingsley)

Show that, if $X$ has the standard normal distribution, then

$$E[|X|^{2n+1}] = 2^n n! \sqrt{2/\pi}.$$

Pf: $E[|X|^{2n+1}]$

$$= \int_{-\infty}^{\infty} |x|^{2n+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \sqrt{\tfrac{2}{\pi}} \int_0^{\infty} x^{2n+1} e^{-\frac{x^2}{2}} dx \quad \text{let } u = x^{2n},\ dv = -e^{-\frac{x^2}{2}} d(-\tfrac{x^2}{2});\ du = 2n x^{2n-1},\ v = -e^{-\frac{x^2}{2}}$$

$$= \sqrt{\tfrac{2}{\pi}} \left[-\overset{0}{\cancel{x^{2n} e^{-\frac{x^2}{2}}}}\right]_0^{\infty} + \sqrt{\tfrac{2}{\pi}}\, 2n \int_0^{\infty} x^{2n-1} e^{-\frac{x^2}{2}} dx$$

$$\text{let } u = x^{2n-2},\ dv = -e^{-\frac{x^2}{2}} d(-\tfrac{x^2}{2});\ du = (2n-2) x^{2n-3},\ v = -e^{-\frac{x^2}{2}}$$

$$= \sqrt{\tfrac{2}{\pi}}\, 2n \left\{ \left[-\overset{0}{\cancel{x^{2n-2} e^{-\frac{x^2}{2}}}}\right]_0^{\infty} + \int_0^{\infty} 2(n-1) x^{2n-3} e^{-\frac{x^2}{2}} dx \right.$$

$$= \sqrt{\tfrac{2}{\pi}}\, 2^2 n(n-1) \int_0^{\infty} x^{2n-3} e^{-\frac{x^2}{2}} dx$$

$\vdots$

$$= \sqrt{\tfrac{2}{\pi}}\, 2^n n(n-1) \times \cdots \times 1 \int_0^{\infty} x e^{-\frac{x^2}{2}} dx$$

$$= \sqrt{\tfrac{2}{\pi}}\, 2^n \cdot n! \int_0^{\infty} -e^{-\frac{x^2}{2}} d(-\tfrac{x^2}{2})$$

$$= \sqrt{\tfrac{2}{\pi}}\, 2^n \cdot n! \left[-e^{-\frac{x^2}{2}}\right]_0^{\infty} = \sqrt{\tfrac{2}{\pi}}\, 2^n \cdot n!.$$

#

• <u>Ex 21.21</u> (Billingsley)

Let $X_1, X_2, \dots$ be "identically distributed" r.v.'s with finite second moment.
Show that $n P[|X_1| \geq \epsilon\sqrt{n}] \to 0$ and $n^{-\frac{1}{2}} \max_{k \leq n} |X_k| \to_p 0$.

Pf: (a) • $n P[|X_1| \geq \epsilon\sqrt{n}]$

$$= n \int_{|X_1| \geq \epsilon\sqrt{n}} dP = n \int_{\frac{|X_1|}{\epsilon\sqrt{n}} \geq 1} 1 \, dP$$

$$\leq n \int_{\frac{|X_1|}{\epsilon\sqrt{n}} \geq 1} \frac{X_1^2}{\epsilon n} dP \quad \text{by } X_1^2 \in L_1 \text{, finite second moment.}$$

$$= E\left[X_1^2 1_{[|X_1| \geq \epsilon\sqrt{n}]}\right]$$

• Since $X_1^2 1_{[|X_1| \geq \epsilon\sqrt{n}]} \leq X_1^2 \in L_1$, $X_1^2 1_{[|X_1| \geq \epsilon\sqrt{n}]} \to 0$ as $n \to \infty$,
the dominated convergence thm implies that
$n P[|X_1| \geq \epsilon\sqrt{n}] \leq E\left[X_1^2 1_{[|X_1| \geq \epsilon\sqrt{n}]}\right] \to E[0] = 0$ as $n \to \infty$.

Hence $n P[|X_1| \geq \epsilon\sqrt{n}] \to 0$ as $n \to \infty$.

(b) $P\left[n^{-\frac{1}{2}} \max_{k \leq n} |X_k| \geq \epsilon\right]$

$$= P\left[\max_{k \leq n} |X_k| \geq \sqrt{n}\epsilon\right]$$

$$\leq P\left(\bigcup_{k \leq n} [|X_k| \geq \sqrt{n}\epsilon]\right)$$

$$\leq \sum_{k=1}^{n} P[|X_k| \geq \sqrt{n}\epsilon]$$

$$= n P[|X_1| \geq \sqrt{n}\epsilon] \to 0 \quad \because \{X_n\} \text{ identically distributed and from (a).}$$

Hence $n^{-\frac{1}{2}} \max_{k \leq n} |X_k| \to_p 0$.

#

- ~~Ex 22.2~~ (Billingsley)

(a) Assume $\{X_n\}$ independent, and define $X_n^{(c)} = X_n 1_{[|X_n| \le c]}$.

Prove that for $\sum |X_n|$ to converge a.s.

it is necessary that $\sum P[|X_n| > c]$ and $\sum E[|X_n^{(c)}|]$ converge for all positive $c$,

sufficient that they converge for some positive $c$.

(b) If the three series $\sum P[|X_n| > c]$ and $\sum E[X_n^{(c)}]$ and $\sum Var[X_n^{(c)}]$ converge but $\sum E[|X_n^{(c)}|] = \infty$, then there is probability 1 that $\sum X_n$ converges conditionally but not absolutely.

Pf: (a) $|X_n^{(c)}| = | X_n 1_{[|X_n| \le c]} | = |X_n| 1_{[|X_n| \le c]} = |X_n|^{(c)}$

$$
\begin{aligned}
\sum Var[|X_n^{(c)}|] &= \sum Var[|X_n|^{(c)}] \\
&= \sum \{ E[|X_n|^{(c)2}] - (E[|X_n|^{(c)}])^2 \} \\
&\le \sum E[|X_n|^{(c)2}] \qquad |X_n|^{(c)2} = |X_n|^2 1_{[|X_n| \le c]} \\
&\le c \sum E[|X_n|^{(c)}] \\
&< \infty \text{ by hypothesis } \sum E[|X_n^{(c)}|] < \infty
\end{aligned}
$$

Hence by the three-series thm, the result follows.

(b) If the three series converge, then $\sum X_n$ converges a.s.

But if $\sum E[|X_n^{(c)}|] = \infty$, then $\sum |X_n|$ diverges a.s. by $\{X_n\}$ independent.

Hence there is probability 1,

$\sum X_n$ converges conditionally but not absolutely.

#

Remark:

- $\sum E[|X_n^{(c)}|] < \infty \Rightarrow \sum Var[|X_n^{(c)}|] < \infty$.
- For $\{X_n\}$ independent with $X_n \ge 0$ for all $n$,

$\sum X_n$ converges a.s. iff $\begin{cases} \sum P[|X_n| > c] < \infty \\ \sum E[|X_n^{(c)}|] = \sum E[X_n^{(c)}] < \infty . \end{cases}$

- Ex 22.3 (Billingsley)

(a) Generalize the Borel-Cantelli lemma:

Suppose $X_n$ are nonnegative.

If $\Sigma E[X_n] < \infty$, then $\Sigma X_n$ converges a.s.

If the $X_n$ are independent and uniformly bounded,

and if $\Sigma E[X_n] = \infty$, then $\Sigma X_n$ diverges with probability 1.

(b) Construct independent, nonnegative $X_n$ s.t.

$\Sigma X_n$ converges a.s. but $\Sigma E[X_n] = \infty$. i.e. not uniformly bounded.

For an extreme example, arrange that

$P[X_n > 0 \text{ i.o.}] = 0$ but $E[X_n] \neq 0$.

Pf: (a) ① Suppose $\Sigma E[X_n] < \infty$.

$$\because E\left[\sum_{i=1}^{\infty} X_i\right] = E\left[\lim_n \sum_{i=1}^{n} X_i\right]$$

$$= \lim_n E\left[\sum_{i=1}^{n} X_i\right] \quad \because X_i \geq 0,\ \sum_{i=1}^{n} X_i \uparrow \sum_{i=1}^{\infty} X_i,$$

then by the monotone convergence thm

$$= \lim_n \sum_{i=1}^{n} E[X_i]$$

$$= \sum_{i=1}^{\infty} E[X_i] < \infty$$

and $\sum_{i=1}^{\infty} X_i \geq 0$.

$\therefore \Sigma X_n$ converges a.s. i.e. $\Sigma X_n < \infty$ a.s.

Since, let $A = [\omega : \sum_n X_n = \infty]$

if $P(A) > 0$, then

$$E\left[\sum_{i=1}^{\infty} X_i\right] = E\left[\sum_{i=1}^{\infty} X_i 1_A\right] + E\left[\sum_{i=1}^{\infty} X_i 1_{A^c}\right]$$

$$\geq E\left[\sum_{i=1}^{\infty} X_i 1_A\right] = \infty.$$

Thus we get a contradiction, so $P[\Sigma X_n = \infty] = 0$.

② $\because$ The $X_n$ are independent r.v.'s $\therefore [\Sigma X_n \text{ converges}]$ is a tail event.

- Suppose $P[\Sigma X_n \text{ converges}] = 1$.

Then by the three-series thm, for all $c > 0$

$\Sigma P[|X_n| \geq c]$, $\Sigma E[X_n^{(c)}]$, $\Sigma Var[X_n^{(c)}]$ converge.

- Since $X_n$ is uniformly bounded, there is a $C_0 > 0$ s.t.

$P[|X_n| \leq C_0] = 1$ for all $n$, that is, $P[X_n = X_n^{(C_0)}] = 1$ for all $n$.

Thus $\sum_n E[X_n^{(C_0)}] = \sum_n E[X_n] = \infty$, we get a contradiction.

Hence $P[\Sigma X_n \text{ diverges}] = 1$.

(b) Suppose $\{X_n\}$ with

$$P[X_n = 2^n] = \frac{1}{2^n} \text{ and } P[X_n = 0] = 1 - \frac{1}{2^n}.$$

Since $\sum_n P[X_n > 0] = \sum_n \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 < \infty$,

thus $P[X_n > 0 \text{ i.o.}] = 0$, $P\left[\liminf_n [X_n = 0]\right] = 1$ by the Borel-Cantelli lemma.

That is, $\exists N$ s.t. $X_n = 0$ for $n \geq N$ a.s.

Hence $\sum_n X_n$ converges a.s.

But $\sum_n E[X_n] = \sum_n 2^n \frac{1}{2^n} = \sum_n 1 = \infty$. #

• <u>Ex 22.4</u> (Billingsley)

Show that under the hypothesis of Thm 22.6 that $\Sigma X_n$ has finite variance and extend Thm 22.4 to infinite sequences.

Pf: • Thm 22.6 : $\{X_n\}$ indep, $E[X_n]=0$, $\Sigma Var[X_n]<\infty$

$\Rightarrow \Sigma X_n$ converges a.s., $\rightarrow$ Kolmogorov's convergence criterion.

• Thm 22.4 : $\{X_1, \cdots, X_n\}$ indep, $E[X_n]=0$, $Var[X_n]<\infty$,

for $\alpha>0$,

$$P\left[\max_{1\le k\le n} |S_k| \ge \alpha\right] \le \frac{1}{\alpha^2} Var[S_n] \rightarrow \text{Kolmogorov's inequality.}$$

(a)• To show that

$$\begin{cases} \{X_n\} \text{ indep} \\ E[X_n]=0 \\ \Sigma Var[X_n]<\infty \end{cases} \Rightarrow Var[\Sigma X_n]<\infty,$$

• By Thm 22.6, $\Sigma X_n$ converges a.s..

$$Var\left[\sum_{i=1}^{\infty} X_i\right] = E\left[\left(\sum_{i=1}^{\infty} X_i\right)^2\right] - E^2\left[\sum_{i=1}^{\infty} X_i\right]$$

$$\le E\left[\left(\sum_{i=1}^{\infty} X_i\right)^2\right] = E\left[\lim_n \left(\sum_{i=1}^{n} X_i\right)^2\right]$$

$$\because \sum_{i=1}^{n} X_i \rightarrow_{a.s.} \sum_{i=1}^{\infty} X_i, \text{ so } \left(\sum_{i=1}^{n} X_i\right)^2 \rightarrow_{a.s.} \left(\sum_{i=1}^{\infty} X_i\right)^2$$

$$= E\left[\liminf_{n\to\infty} \left(\sum_{i=1}^{n} X_i\right)^2\right]$$

$$\le \liminf_{n\to\infty} E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] \quad \text{by Fatou's lemma} : \left(\sum_{i=1}^{n} X_i\right)^2 \ge 0.$$

$$= \liminf_{n\to\infty} \left(\sum_{i=1}^{n} E[X_i^2]\right) \quad \because \{X_n\} \text{ indep } \& \ E[X_n]=0.$$

$$= \sum_n Var[X_n] < \infty \quad \text{by hypothesis.}$$

(b)• To show that

Suppose $X_1, X_2, \ldots$ are independent with $E[X_n]=0$, $Var[X_n]<\infty$.

For $\alpha>0$,

$$P\left[\sup_{k\ge 1} |S_k| \ge \alpha\right] \le \frac{1}{\alpha^2} Var\left[\sum_n X_n\right].$$

• Since $\left[\max_{1\le k\le n} |S_k| \ge \alpha\right] \uparrow \left[\sup_{k\ge 1} |S_k| \ge \alpha\right]$, hence

$P\left[\max_{1\le k\le n} |S_k| \ge \alpha\right] \uparrow P\left[\sup_{k\ge 1} |S_k| \ge \alpha\right]$, and

$Var\left[\sum_{i=1}^{n} X_i\right] \uparrow Var\left[\sum_n X_n\right] < \infty$ from (a),

the result follows.

#

• Ex 22.5 (Billingsley)

Suppose that $X_1, X_2, \dots$ are independent, each with the Cauchy distribution with density $f_u(x) = \frac{1}{\pi}\frac{u}{u^2+x^2}$, $-\infty < x < \infty$ for $u>0$.

(a) Show that $n^{-1}\sum_{k=1}^{n} X_k$ does not converge a.s..

Contrast with Thm 22.1.

(b) Show that $P[n^{-1}\max_{k\le n} X_k \le x] \to e^{-u/\pi x}$ for $x>0$.

Relate to Thm 14.3.

Pf: (a) • Thm 22.1: $\{X_n\}$ iid with $E[|X_n|]<\infty \Rightarrow \frac{S_n}{n} \to_{a.s.} E[X_1]$ → Kolmogorov's SLLN.

• Suppose $Y_n = \frac{S_n}{n} \to_{a.s.} Y$, $Y$ is a r.v..

Since $Y = \lim_n \frac{S_n}{n} = \lim_n \frac{\sum_{k=m}^{n} X_k}{n}$ for all $m$,

$Y$ is measurable $\sigma(X_m, X_{m+1}, \cdots)$ for all $m$.

$Y$ is measurable $\mathcal{T} = \bigcap_m \sigma(X_m, X_{m+1}, \dots)$, so $Y$ is a tail r.v.

Ex 22.1 implies that there is a constant $a$ s.t. $P[Y=a]=1$, so $Y_n \to_{a.s.} a$;

Ex 26.9 implies that $Y_n = \frac{S_n}{n} \Rightarrow$ Cauchy$(0,u)$,

thus we get a contradiction,

then $Y_n = \frac{S_n}{n}$ does not converge a.s. i.e. $P[\frac{S_n}{n}$ converges$] \ne 1, = 0$.

(b) • Thm 14.3: The class of extreme distribution functions consists exactly of the distribution functions of the types

(14.22) $F_1(x) = e^{-e^{-x}}$

(14.24) $F_{2,\alpha}(x) = \begin{cases} 0 & \text{if } x<0 \\ e^{-x^{-\alpha}} & \text{if } x\ge 0 \end{cases}$

(14.25) $F_{3,\alpha}(x) = \begin{cases} e^{-(-x)^{\alpha}} & \text{if } x\le 0 \\ 1 & \text{if } x\ge 0 \end{cases}$

• For $x>0$.

$$P[n^{-1}\max_{k\le n} X_k \le x] = P[\max_{k\le n} X_k \le nx] = \prod_{k=1}^{n} P[X_k \le nx]$$

$$= F^n(nx) = e^{n\log F(nx)} \to e^{\lim_n \frac{\log F(nx)}{1/n}}$$

$$= e^{\lim_n \frac{\frac{1}{F(nx)}xf(nx)}{-1/n^2}} = e^{\lim_n \frac{-n^2x}{F(nx)}\frac{1}{\pi}\frac{u}{u^2+n^2x^2}}$$

$$= e^{\lim_n -\frac{x}{F(nx)}\frac{1}{\pi}\frac{u}{u^2/n^2+x^2}} = e^{-\frac{u}{\pi x}},$$

$$\Rightarrow P[n^{-1}\max_{k\le n} X_k \le \tfrac{ux}{\pi}] \to e^{-\frac{u}{\pi}\frac{\pi}{ux}} = e^{-x^{-1}}, \quad x\ge 0,$$

that is $F^n(\frac{nu}{\pi}x) \Rightarrow F_{2,1}(x) = \begin{cases} 0 & \text{if } x<0, \\ e^{-x^{-1}} & \text{if } x\ge 0. \end{cases}$ (14.24)

#

- <u>Ex 22.6</u> (Billingsley)

If $X_1, X_2, \ldots$ are i.i.d. r.v.'s, and if $P[X_1 \geq 0] = 1$ and $P[X_1 > 0] > 0$, then $\sum_n X_n = \infty$ a.s..

Deduce this from Thm 22.1 and its Corollary and also directly: find a positive $\epsilon$ s.t. $X_n > \epsilon$ i.o. with probability 1.

Pf: • Thm 22.1: If $X_1, X_2, \ldots$ are iid and have finite mean $E[X_1]$, then $S_n/n \to E[X_1]$ a.s. $\to$ Kolmogorov's SLLN.

• Corollary: Suppose that $X_1, X_2, \ldots$ are iid and $E[X_1^-] < \infty$, $E[X_1^+] = \infty$ (so that $E[X_1] = \infty$). Then $n^{-1}\sum_{k=1}^{n} X_k \to \infty$ with probability 1.

(a) • If $E[X_1] < \infty$, then Thm 22.1 implies that $n^{-1}\sum_{k=1}^{n} X_k \to E[X_1]$ a.s., since $n \nearrow \infty$, $\sum_{k=1}^{n} X_k \to \sum_n X_n = \infty$ a.s.;

• If $E[X_1] = \infty$, then corollary implies that $n^{-1}\sum_{k=1}^{n} X_k \to \infty$ a.s, since $n \nearrow \infty$, $\sum_{k=1}^{n} X_k \to \sum_n X_n = \infty$ a.s..

(b) • Since $P[X_1 \geq 0] = 1$ and $P[X_1 > 0] > 0$, there is a $\epsilon > 0$ s.t. $P[X_1 > \epsilon] = \frac{1}{2}$.

• So $\sum_n P[X_n > \epsilon] = \sum_n \frac{1}{2} = \infty$;

Since $\{X_n\}$ is iid, the second Borel-Cantelli lemma implies

$P[X_n > \epsilon \text{ i.o.}] = 1$,

since $\epsilon$ is arbitrary, we have

$P[X_n > 0 \text{ i.o.}] = 1$, that is,

with probability 1, there is a $N$ s.t. $X_n > 0$ for $n > N$;

Hence $\sum_n X_n = \infty$ a.s.

#

- Ex 22.7 (Billingsley)

Suppose that $X_1, X_2, \ldots$ are i.i.d. and $E[|X_1|] = \infty$.
Use (21.9) to show that $\sum_n P[|X_n| \geq an] = \infty$ for each $a$,
and conclude that $\sup_n n^{-1}|X_n| = \infty$ with probability 1.
Now show that $\sup_n n^{-1}|S_n| = \infty$ with probability 1.
Compare this with the corollary to Thm 22.1.

Pf: · (21.9): $E[X] = \int_0^\infty P[X > t]\,dt$ for $X$ is nonnegative.

· Corollary: $\{X_n\}$ iid, $E[X_1^-] < \infty$, $E[X_1^+] = \infty$ (so that $E[X_1] = \infty$).

$\Rightarrow n^{-1}\sum_{k=1}^{n} X_k \to \infty$ with probability 1

(a)
$$\begin{aligned}
\infty &= E[|X_1|] \\
&= \int_0^\infty P[|X_1| > t]\,dt \qquad \because |X_1| > 0. \\
&= \sum_{n=1}^{\infty} \int_{a(n-1)}^{an} P[|X_1| > t]\,dt \\
&\leq \sum_{n=1}^{\infty} P[|X_n| > a(n-1)] \cdot \int_{a(n-1)}^{an} dt \qquad \because P[|X_1| > t] \downarrow \text{ as } t \uparrow \text{ \& } \{X_n\} \text{ iid.} \\
&= \left\{1 + \sum_{n=1}^{\infty} P[|X_n| > an]\right\} \cdot a \\
&= a + a\sum_{n=1}^{\infty} P[|X_n| > an]
\end{aligned}$$

Hence $\sum_{n=1}^{\infty} P[|X_n| > an] = \infty$. for each $a$

(b) Since $\{X_n\}$ is iid, the second Borel-Cantelli lemma implies that

$$P[|X_n| > an \text{ i.o.}] = 1 = P\left[\limsup_{n\to\infty} \frac{|X_n|}{n} > a\right].$$

This means for each $a$ there is a null set $N(a)$ s.t. if $\omega \in N(a)^c$, then $\limsup_{n\to\infty} \frac{|X_n|}{n} > a$.

Let $N = \bigcup_{a=1}^{\infty} N(a)$; then $N$ is a null set, if $\omega \in N^c$,
$\limsup_{n\to\infty} \frac{|X_n|}{n} > a$ for every $a$; hence $\sup_n \frac{|X_n|}{n} = \infty$. a.s.

(c) Since $|S_n - S_{n-1}| = |X_n| \leq |S_n| + |S_{n-1}|$, and
$|S_n| \leq \frac{1}{2}an$ and $|S_{n-1}| \leq \frac{1}{2}an$ imply $|X_n| \leq an$,
So $P\left[\frac{|X_n|}{n} > a \text{ i.o.}\right] = 1$ implies $P\left[\frac{|S_n|}{n} > \frac{1}{2}a \text{ i.o.}\right] = 1$.
By the argument above we have $\sup_n n^{-1}|S_n| = \infty$ a.s. for $E[|X_n|] = \infty$
Whereas the corollary implies that $n^{-1}S_n \to \infty$ a.s. for $E[X_n] = \infty$.

• Ex 22.9 (Billingsley)

Let $Z_n$ be 1 or 0 according as at time $n$ there is or is not a record. Let $R_n = Z_1 + \cdots + Z_n$ be the number of records up to time $n$.
Show that $R_n / \log n \to_p 1$.

Pf: The $Z_K$ are independent Bernoulli $(\frac{1}{K})$, where $Z_K = 1$ if the $k$th time is a record.

$$E[R_n] = \sum_{K=1}^{n} \frac{1}{K} \sim \log n$$

$$Var[R_n] = \sum_{K=1}^{n} \frac{1}{K}(1 - \frac{1}{K}) = \sum_{K=1}^{n} \frac{1}{K} - \sum_{K=1}^{n} \frac{1}{K^2} \sim \log n$$

$$P\left[\left|\frac{R_n - E[R_n]}{\log n}\right| \geq \epsilon\right]$$

$$\leq \frac{1}{\epsilon^2} \frac{1}{(\log n)^2} E\left[(R_n - E[R_n])^2\right] \quad \text{by Chebychev's inequality}$$

$$= \frac{1}{\epsilon^2} \frac{1}{(\log n)^2} Var[R_n] \sim \frac{1}{\epsilon^2} \frac{1}{\log n} \to 0 \text{ as } n \to \infty$$

Thus $\frac{R_n - E[R_n]}{\log n} \to_p 0$;

Since $$\frac{R_n - \log n}{\log n} = \frac{R_n - E[R_n]}{\log n} + \frac{E[R_n] - \log n}{\log n}$$

$$\to_p 0 + 0 = 0 \text{ as } n \to \infty.$$

Hence $\frac{R_n}{\log n} \to_p 1$ as $n \to \infty$.

#

• Ex 22.8 (Billingsley)

"Wald's equation". Let $X_1, X_2, \cdots$ be i.i.d. with finite mean, and put $S_n = X_1 + \cdots + X_n$.

Suppose that $\tau$ is a stopping time:

$\tau$ has positive integers as values and $[\tau = n] \in \sigma(X_1, \cdots, X_n)$.

Suppose also that $E[\tau] < \infty$.

(a) Prove that

$$E[S_\tau] = E[X_1]\, E[\tau].$$

(b) Suppose that $X_n$ is $\pm 1$ with probabilities $p$ and $q$, $p \neq q$, let $\tau$ be the first $n$ for which $S_n$ is $-a$ or $b$ $(a, b > 0)$, and calculate $E[\tau]$. This gives the expected duration of the game in the "gambler's ruin problem" for unequal $p$ and $q$.

Pf: (a) • Let $S_\tau = S_\tau^+ - S_\tau^-$, and

$$S_\tau^+ = \sum_{k=1}^{\tau} X_k^+ = \sum_{k=1}^{\infty} I_{[k \le \tau]} X_k^+ \; ; \; S_\tau^- = \sum_{k=1}^{\infty} I_{[k \le \tau]} X_k^-.$$

• Since $[\tau \ge k] = [\tau < k]^c = [\tau \le k-1]^c \in \sigma(X_1, \cdots, X_{k-1})$, and $X_k^+ \in \sigma(X_k)$, so $I_{[\tau \ge k]} \perp\!\!\!\perp X_k^+$, $\because X_n$ are i.i.d.

$$\begin{aligned}
• \; E[S_\tau^+] &= E\left[\sum_{k=1}^{\infty} I_{[k \le \tau]} X_k^+\right] \\
&= \sum_{k=1}^{\infty} E\left[I_{[k \le \tau]} X_k^+\right] \quad \because I_{[k \le \tau]} X_k^+ \ge 0 \text{ and by MCT.} \\
&= \sum_{k=1}^{\infty} E\left[I_{[k \le \tau]}\right] E[X_k^+] \quad \because I_{[\tau \ge k]} \perp\!\!\!\perp X_k^+. \\
&= E[X_1^+] \sum_{k=1}^{\infty} P[\tau \ge k] \quad \because X_n \text{ are iid} \\
&= E[X_1^+]\, E[\tau]
\end{aligned}$$

• By the argument above we have

$E[S_\tau^-] = E[X_1^-]\, E[\tau]$. So that

$$\begin{aligned}
E[S_\tau] &= E[S_\tau^+] - E[S_\tau^-] \\
&= (E[X_1^+] - E[X_1^-])\, E[\tau] \\
&= E[X_1]\, E[\tau].
\end{aligned}$$

(next pg. cont.)

(b) • Let $\tau$ be the first $n$ for which $S_n = -a$ or $b$ $(a, b > 0)$

$\tau = \min\{n : \sum_{k=1}^{n} X_k = -a \text{ or } \sum_{k=1}^{n} X_k = b\}$,

and let $S_0 = a$, Total $= a+b \equiv N$,

• and let $P_{ij}$ be the transition probability from state $i$ to state $j$.

$P_{00} = 1$, $P_{NN} = 1$, $P_{i,i+1} = p = 1 - P_{i,i-1}$, $i = 1, \dots, N-1$.

• Let $f_i = P[\text{eventually reach } N \mid S_0 = i]$

Since $f_i = p f_{i+1} + q f_{i-1}$, $q = 1-p$, by conditioning on the first step.

$$f_i - p f_i = p(f_{i+1} - f_i) + q f_{i-1},$$

$$\Rightarrow f_{i+1} - f_i = \frac{q}{p}(f_i - f_{i-1}) = \left(\frac{q}{p}\right)^2 (f_{i-1} - f_{i-2})$$

$$= \cdots = \left(\frac{q}{p}\right)^i (f_1 - f_0) = \left(\frac{q}{p}\right)^i f_1 \quad \because f_0 = 0.$$

$$f_i - f_{i-1} = \left(\frac{q}{p}\right)^{i-1} f_1$$

$$\vdots$$

$$f_1 - f_0 = \left(\frac{q}{p}\right)^0 f_1 = f_1$$

$$\Rightarrow f_i = \left[1 + \left(\frac{q}{p}\right)^1 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right] f_1, \quad \text{let } \rho = q/p$$

$$= \begin{cases} \frac{1-\rho^i}{1-\rho} f_1 & \text{if } p \neq q \\ i f_1 & \text{if } p = q \end{cases}$$

$$= \begin{cases} \frac{1-\rho^i}{1-\rho^N} & \text{if } p \neq q \quad \because 1 = f_N = \frac{1-\rho^N}{1-\rho} f_1 \Rightarrow f_1 = \frac{1-\rho}{1-\rho^N} \\ \frac{i}{N} & \text{if } p = q \quad \because 1 = f_N = N f_1 \Rightarrow f_1 = \frac{1}{N}. \end{cases}$$

Hence $f_a = \frac{1-\rho^a}{1-\rho^{a+b}} = P[\text{eventually reach } a+b \mid S_0 = a]$ for $p \neq q$.

• By Wald's equation: $E[S_\tau] = E[X_1]\,E[\tau]$, where

$E[X_1] = 1 \cdot p + (-1) \cdot q = p - q$;

$E[S_\tau] = E[\sum_{k=1}^{\tau} X_k] = b \cdot f_a + (-a)(1 - f_a) = \frac{(a+b)(1-\rho^a)}{(1-\rho^{a+b})} - a$;

$E[\tau] = \frac{a+b}{p-q} \frac{1-\rho^a}{1-\rho^{a+b}} - \frac{a}{p-q}$, $\rho = q/p \neq 1$.

#

• Ex 22.12 (Billingsley)

Prove (what is essentially "Kolmogorov's zero-one law") that if $A$ is independent of a $\pi$-system $\mathcal{P}$ and $A \in \sigma(\mathcal{P})$, then $P(A)$ is either 0 or 1.

Pf: • $\pi$-system: ($\pi$) $A, B \in \mathcal{P}$ implies $A \cap B \in \mathcal{P}$.

• $\lambda$-system: ($\lambda_1$) $\Omega \in \mathcal{L}$;

($\lambda_2$) $A \in \mathcal{L}$ implies $A^c \in \mathcal{L}$

($\lambda_3$) $A_1, A_2, \cdots \in \mathcal{L}$ and $A_n \cap A_m = \emptyset$ for $m \neq n$ imply $\cup_n A_n \in \mathcal{L}$.

• Dynkin's $\pi$-$\lambda$ thm:

If $\mathcal{P}$ is a $\pi$-system and $\mathcal{L}$ is a $\lambda$-system, then $\mathcal{P} \subset \mathcal{L}$ implies $\sigma(\mathcal{P}) \subset \mathcal{L}$.

• Let $\mathcal{L} = \{B : A \perp\!\!\!\perp B,\ B \in \sigma(\mathcal{P})\}$, then $\mathcal{L} \subset \sigma(\mathcal{P})$.

Since for $B \in \mathcal{P} \subset \sigma(\mathcal{P})$ we have $A \cap B$, then $\mathcal{P} \subset \mathcal{L}$;

• Since ($\lambda_1$) $\Omega \in \mathcal{L}$ $\because$ $P(A)P(\Omega) = P(A \cap \Omega)$, $\Omega \in \sigma(\mathcal{P})$.

($\lambda_2$) $B \in \mathcal{L} \Rightarrow A \perp\!\!\!\perp B,\ B \in \sigma(\mathcal{P})$

$\Rightarrow A \perp\!\!\!\perp B^c,\ B^c \in \sigma(\mathcal{P})$

$\Rightarrow B^c \in \mathcal{L}$

($\lambda_3$) $B_1, B_2, \ldots \in \mathcal{L}$, $B_n$'s are disjoint

$$\Rightarrow P[A \cap (\cup_n B_n)] = P[\cup_n (A \cap B_n)] = \sum_n P[A \cap B_n]$$

$$= \sum_n P[A]P[B_n] = P[A] \sum_n P[B_n]$$

$$= P[A]\, P[\cup_n B_n]$$

$\Rightarrow \cup_n B_n \in \mathcal{L}$,

• Thus $\mathcal{L}$ is a $\lambda$-system; by Dynkin's $\pi$-$\lambda$ thm, $\sigma(\mathcal{P}) \subset \mathcal{L}$

Hence $\mathcal{L} = \sigma(\mathcal{P})$, and then $A \perp\!\!\!\perp \sigma(\mathcal{P})$.

• $A \in \sigma(\mathcal{P})$ implies that $A$ is independent of itself:

$P(A \cap A) = P(A)P(A) \Rightarrow P(A) = 0$ or $1$.

This is essentially Kolmogorov's zero-one law.

#

• Ex 23.9 (Billingsley)

If the waiting times $X_n$ are independent and exponentially distributed with parameter $\alpha$, then $S_n/n \to \alpha^{-1}$ with probability 1, by the SLLN.
From $\lim_{t\to\infty} N_t = \infty$ and $S_{N_t} \le t \le S_{N_t+1}$ deduce that $\lim_{t\to\infty} N_t/t = \alpha$ with probability 1.

Pf: (i) $f(x) = \alpha e^{-\alpha x}$, $0<x<\infty$.

$$E[X] = \int_0^\infty x\alpha e^{-\alpha x} dx \quad \text{let } u = x,\ dv = -e^{-\alpha x} d(-\alpha x);\ du = dx,\ v = -e^{-\alpha x}$$
$$= \left[-x e^{-\alpha x}\right]_0^\infty + \int_0^\infty e^{-\alpha x} dx$$
$$= -\alpha^{-1} \int_0^\infty e^{-\alpha x} d(-\alpha x)$$
$$= -\alpha^{-1} \left[e^{-\alpha x}\right]_0^\infty = \alpha^{-1}$$

Hence it follows by the SLLN that
$$\frac{S_n}{n} \to \alpha^{-1} \text{ with probability 1.}$$

(ii) $P\left[\lim_{t\to\infty} N(t) < \infty\right]$
$$= P\left[X_n = \infty \text{ for some } n\right]$$
$$= P\left(\bigcup_n [X_n = \infty]\right)$$
$$\le \sum_n P[X_n = \infty] = 0$$

Hence $P\left[\lim_{t\to\infty} N(t) = \infty\right] = 1$.

(iii) Since $S_{N_t} \le t \le S_{N_t+1}$,
$$\frac{S_{N_t}}{N_t} \le \frac{t}{N_t} \le \frac{S_{N_t+1}}{N_t+1}\,\frac{N_t+1}{N_t}$$

It follows by the SLLN that
$$\frac{S_{N_t}}{N_t} \to \alpha^{-1} \text{ as } t\to\infty \text{ with probability 1}$$

we have, by the same reasoning,
$$\frac{S_{N_t+1}}{N_t+1}\,\frac{N_t+1}{N_t} \to \alpha^{-1}\cdot 1 = \alpha^{-1} \text{ as } t\to\infty \text{ with probability 1.}$$

Thus $\frac{t}{N_t} \to \alpha^{-1}$ as $t\to\infty$ and hence $\frac{N_t}{t} \to \alpha$ a.s. as $t\to\infty$.

#

- Ex 23.10 (Billingsley) "The law of large numbers in renewal theory"

(a) Suppose that $X_1, X_2, \dots$ are positive, and assume directly that $S_n/n \to m$ with probability 1, as happens if the $X_n$ are iid with mean $m$. Show that $\lim_t N_t/t = 1/m$ with probability 1.

(b) Suppose now that $S_n/n \to \infty$ with probability 1, as happens if the $X_n$ are iid and have infinite mean. Show that $\lim_t N_t/t = 0$ with probability 1.

Pf: Since $S_{N_t} \le t \le S_{N_t+1}$, and $\lim_t N_t = \infty$,

$$\frac{S_{N_t}}{N_t} \le \frac{t}{N_t} \le \frac{S_{N_t+1}}{N_t+1}\,\frac{N_t+1}{N_t}$$

(a) Suppose $\lim_n S_n/n = m$ with probability 1, then

$\frac{S_{N_t}}{N_t} \to m$ with probability 1;

$\frac{S_{N_t+1}}{N_t+1}\,\frac{N_t+1}{N_t} \to m$ with probability;

Thus $\frac{t}{N_t} \to m$ with probability 1,

and hence $\frac{N_t}{t} \to m^{-1}$ with probability 1

(b) Suppose $\lim_n S_n/n = \infty$ with probability 1, then.

$\frac{S_{N_t}}{N_t} \to \infty$, $\frac{S_{N_t+1}}{N_t+1}\,\frac{N_t+1}{N_t} \to \infty$ with probability 1,

thus $\frac{t}{N_t} \to \infty$ with probability 1,

and hence $\frac{N_t}{t} \to 0$ with probability 1.

#