- <u>Ex 27.12</u> (Billingsley)

There can be asymptotic normality even if there are no moments at all. Construct a simple example.

Pf: • Let $X_K = (1-\frac{1}{k^2}) Y_K + \frac{1}{k^2} Z_K$

where the Y_K is iid $N(0,1)$ and the Z_K is iid Cauchy$(0,1)$, and let $S_n = \sum_{k=1}^{n} X_K$.

- $E[|X_K|^r] = \infty$, for $r \geq 1$
- $$\frac{S_n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (1-\frac{1}{k^2}) Y_K + \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{k^2} Z_K$$
$$= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_K - \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{k^2} Y_K + \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{k^2} Z_K .$$
- (i) Since the Y_K are iid $N(0,1)$ with mean 0 and var. 1, the Lindeberg-Levy thm implies that
$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_K \Rightarrow N$$

(ii) $$\varphi_{\sum_{k=1}^{n}\frac{1}{k^2}Y_K}(t) = E\left[e^{it\sum_{k=1}^{n}\frac{1}{k^2}Y_K}\right] = \prod_{k=1}^{n} E\left[e^{it\frac{1}{k^2}Y_K}\right]$$
$$= \prod_{k=1}^{n} e^{-\frac{1}{2}(\frac{t}{k^2})^2} = e^{-\frac{t^2}{2}\sum_{k=1}^{n}\frac{1}{k^4}}$$
$$\to e^{-\frac{t^2}{2}\sum_{k=1}^{\infty}\frac{1}{k^4}} \text{ as } n\to\infty \quad \because \sum_{k=1}^{\infty}\frac{1}{k^4} < \infty .$$

The continuity thm of chf implies that
$$\sum_{k=1}^{n} \frac{1}{k^2} Y_K \Rightarrow N(0, \sum_{k=1}^{\infty}\frac{1}{k^4})$$

Since $\frac{1}{\sqrt{n}} \to 0$ as $n\to\infty$,

$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{k^2} Y_K \Rightarrow 0$, that is, $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{k^2} Y_K \to_p 0$ by Slutsky's thm.

(iii) Since the Z_K are iid Cauchy$(0,1)$,

$\varphi_{Z_K}(t) = e^{-|t|}$

$$\varphi_{\sum_{k=1}^{n}\frac{1}{k^2}Z_K}(t) = E\left[e^{it\sum_{k=1}^{n}\frac{1}{k^2}Z_K}\right] = \prod_{k=1}^{n} E\left[e^{it\frac{1}{k^2}Z_K}\right]$$
$$= \prod_{k=1}^{n} e^{-\frac{|t|}{k^2}} = e^{-|t|\sum_{k=1}^{n}\frac{1}{k^2}}$$
$$\to e^{-\frac{t^2}{2}\frac{\pi^2}{6}} \text{ as } n\to\infty \quad \because \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

The continuity thm of chf implies that
$$\sum_{k=1}^{n} \frac{1}{k^2} Z_K \Rightarrow \text{Cauchy}(0, \frac{\pi^2}{6}).$$

With the similar argument implies $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{k^2} Z_K \to_p 0$.

- By (i), (ii) and (iii), $\frac{S_n}{\sqrt{n}} \Rightarrow N$. #

• Ex 27.14 (Billingsley)

"The CLT for a random number of summands."

Let $X_1, X_2, \cdots$ be iid r.v.'s with mean 0 and variance σ^2, and.

let $S_n = X_1 + \cdots + X_n$.

For each positive t, let ν_t be a r.v. assuming positive integers as values;

it need not be independent of the X_n.

Suppose that there exist positive constants a_t and θ s.t.

$$a_t \to \infty, \quad \frac{\nu_t}{a_t} \Rightarrow \theta \quad \text{as } t \to \infty.$$

Show by the following steps that

$$\frac{S_{\nu_t}}{\sigma\sqrt{\nu_t}} \Rightarrow N, \quad \frac{S_{\nu_t}}{\sigma\sqrt{\theta a_t}} \Rightarrow N.$$

(a) Show that it may be assumed that $\theta = 1$ and the a_t are integers.

(b) Show that it suffices to prove the second relation above.

(c) Show that it suffices to prove $(S_{\nu_t} - S_{a_t})/\sqrt{a_t} \Rightarrow 0$.

(d) Show that

$$P\left[|S_{\nu_t} - S_{a_t}| \ge \epsilon\sqrt{a_t}\right] \le P\left[|\nu_t - a_t| \ge \epsilon^3 a_t\right] + P\left[\max_{|k-a_t| \le \epsilon^3 a_t} |S_k - S_{a_t}| \ge \epsilon\sqrt{a_t}\right]$$

and conclude from Kolmogorov's inequality that

the last probability is at most $2\epsilon\sigma^2$.

Pf: (a). WLOG, we may assume that $\theta = 1$;

Since $a_t \le \lceil a_t \rceil \le a_t + 1$, then $\frac{a_t}{\lceil a_t \rceil} \to 1$ as $t \to \infty$.

$$\frac{\nu_t}{\lceil a_t \rceil} = \frac{\nu_t}{a_t}\frac{a_t}{\lceil a_t \rceil} \Rightarrow 1 \text{ by Slutsky's thm.}$$

(b) Suppose $\theta = 1$ and the a_t are integers.

If $\frac{S_{\nu_t}}{\sigma\sqrt{a_t}} \Rightarrow N$, then

$$\frac{S_{\nu_t}}{\sigma\sqrt{\nu_t}} = \frac{S_{\nu_t}}{\sigma\sqrt{a_t}}\sqrt{\frac{a_t}{\nu_t}} \Rightarrow N \quad \because \frac{\nu_t}{a_t} \Rightarrow 1, \sqrt{\frac{a_t}{\nu_t}} \Rightarrow 1. \text{ by Slutsky's thm.}$$

(c) $$\frac{S_{\nu_t}}{\sigma\sqrt{a_t}} = \frac{S_{a_t}}{\sigma\sqrt{a_t}} + \frac{S_{\nu_t} - S_{a_t}}{\sigma\sqrt{a_t}}$$

Since $S_{a_t} = X_1 + \cdots + X_{a_t}$ with $E[X_k] = 0$ and $Var[X_k] = \sigma^2$,

the Lindeberg-Levy thm implies that

$$\frac{S_{a_t}}{\sigma^2\sqrt{a_t}} \Rightarrow N.$$

Thus it suffices to prove $\frac{S_{\nu_t} - S_{a_t}}{\sigma\sqrt{a_t}} \Rightarrow 0$, i.e. $\frac{S_{\nu_t} - S_{a_t}}{\sqrt{a_t}} \Rightarrow 0$.

(d). $P\left[\left|\frac{S_{\nu_t}-S_{a_t}}{\sqrt{a_t}}\right|\geq\epsilon\right] = P\left[|S_{\nu_t}-S_{a_t}|\geq\epsilon\sqrt{a_t}\right]$

$$= \sum_{k=1}^{\infty} P\left[\nu_t=k,\ |S_k-S_{a_t}|\geq\epsilon\sqrt{a_t}\right]$$

$$= \sum_{|k-a_t|>\epsilon^3 a_t} P\left[\nu_t=k,\ |S_k-S_{a_t}|\geq\epsilon\sqrt{a_t}\right]$$

$$+ \sum_{|k-a_t|\leq\epsilon^3 a_t} P\left[\nu_t=k,\ |S_k-S_{a_t}|\geq\epsilon\sqrt{a_t}\right]$$

$$\leq \sum_{|k-a_t|>\epsilon^3 a_t} P\left[\nu_t=k\right]$$

$$+ \sum_{|k-a_t|\leq\epsilon^3 a_t} P\left[\nu_t=k,\ \max_{|k-a_t|\leq\epsilon^3 a_t}|S_k-S_{a_t}|\geq\epsilon\sqrt{a_t}\right]$$

$$\leq P\left[|\nu_t-a_t|>\epsilon^3 a_t\right]$$

$$+ P\left[\max_{|k-a_t|\leq\epsilon^3 a_t}|S_k-S_{a_t}|\geq\epsilon\sqrt{a_t}\right],$$

• Since $|k-a_t|\leq\epsilon^3 a_t$ implies that $a_t(1-\epsilon^3)\leq k\leq a_t(1+\epsilon^3)$,

and, $\max_{a_t\leq k\leq a_t(1+\epsilon^3)}|S_k-S_{a_t}|<\epsilon\sqrt{a_t}$ and $\max_{a_t(1-\epsilon^3)\leq k\leq a_t}|S_k-S_{a_t}|<\epsilon\sqrt{a_t}$

implies that

$$\max_{|k-a_t|\leq\epsilon^3 a_t}|S_k-S_{a_t}|<\epsilon\sqrt{a_t},$$

by Kolmogorov's inequality, it follows that $S_{a_t(1+\epsilon^3)}-S_{a_t}$

$$P\left[\max_{a_t\leq k\leq a_t(1+\epsilon^3)}|S_k-S_{a_t}|\geq\epsilon\sqrt{a_t}\right]\leq\frac{1}{\epsilon^2 a_t}\mathrm{Var}\left[S_{a_t\epsilon^3}\right]=\frac{a_t\epsilon^3\sigma^2}{\epsilon^2 a_t}=\epsilon\sigma^2;$$

The similar argument implies that

$$P\left[\max_{a_t(1-\epsilon^3)\leq k\leq a_t}|S_k-S_{a_t}|\geq\epsilon\sqrt{a_t}\right]\leq\epsilon\sigma^2;$$

• Since $\frac{\nu_t}{a_t}\Rightarrow 1$, then $P\left[\left|\frac{\nu_t}{a_t}-1\right|>\epsilon^3\right]=P\left[|\nu_t-a_t|>\epsilon^3 a_t\right]\to 0$ as $t\to\infty$.

Thus $P\left[\left|\frac{S_{\nu_t}-S_{a_t}}{\sqrt{a_t}}\right|\geq\epsilon\right]\to 2\epsilon\sigma^2$ as $t\to\infty$,

Since ϵ is arbitrary, the result follows. #

• Ex 27.15 (Billingsley)

"A central limit thm in renewal theory"

Let $X_1, X_2, \dots$ be iid positive r.v.'s with mean m and variance σ^2, and as in Ex 23.10 let $N_t = \max\{n : S_n \le t\}$.

Prove by the following steps that

$$\frac{N_t - tm^{-1}}{\sigma t^{1/2} m^{-3/2}} \Rightarrow N.$$

(a) Show by the results in Ex 21.21 and 23.10 that

$$\frac{S_{N_t} - t}{\sqrt{t}} \Rightarrow 0.$$

(b) Show that it suffices to prove that

$$\frac{N_t - S_{N_t} m^{-1}}{\sigma t^{1/2} m^{-3/2}} = \frac{-(S_{N_t} - mN_t)}{\sigma t^{1/2} m^{-1/2}} \Rightarrow N.$$

(c) Show (Ex 23.10) that $N_t/t \Rightarrow m^{-1}$, and apply the thm in Ex 27.14.

Pf: • Ex 23.10: Suppose the X_n are positive and $S_n/n \to m$ with probability 1. Then $\lim_t N_t/t = 1/m$ with probability 1.

• Ex 21.21: Let the X_n be identically distributed r.v.'s with finite second moments.
Then $n P[|X_1| \ge \epsilon\sqrt{n}] \to 0$ and $\frac{1}{\sqrt{n}} \max_{k \le n} |X_k| \to_p 0$.

• Ex 27.14: Let the X_n be iid with mean 0 and variance σ^2, and let $S_n = X_1 + \cdots + X_n$. Suppose $\frac{N_t}{t} \Rightarrow \frac{1}{m}$ as $t \to \infty$, then $\frac{S_{N_t}}{\sigma\sqrt{t/m}} \Rightarrow N$ as $t \to \infty$.

(a) Since $S_{N_t} \le t < S_{N_t+1} = S_{N_t} + X_{N_t+1}$

$$\frac{S_{N_t} - t}{\sqrt{t}} \le 0 < \frac{S_{N_t} - t}{\sqrt{t}} + \frac{X_{N_t+1}}{\sqrt{t}}$$

From Ex 21.21 we have $\sum_n P[|X_n|/\sqrt{n} \ge \epsilon] = 0$,
by the Borel-Cantelli lemma we have $P[\frac{|X_n|}{\sqrt{n}} \ge \epsilon \text{ i.o.}] = 0$,
that is, $\frac{|X_n|}{\sqrt{n}} \to 0$ with probability 1.

$$\frac{X_{N_t+1}}{\sqrt{t}} = \frac{X_{N_t+1}}{\sqrt{N_t+1}} \sqrt{\frac{N_t+1}{t}} \to 0 \times \sqrt{m^{-1}} = 0 \text{ with probability } 1,$$

Thus $\frac{S_{N_t} - t}{\sqrt{t}} \to 0$ with probability 1, and hence $\frac{S_{N_t} - t}{\sqrt{t}} \Rightarrow 0$.

(b) Since

$$\frac{N_t - tm^{-1}}{\sigma t^{1/2} m^{-3/2}} = \frac{N_t - S_{N_t} m^{-1}}{\sigma t^{1/2} m^{-3/2}} + \frac{m^{-1}(S_{N_t} - t)}{\sigma t^{1/2} m^{-3/2}}$$

$$= \frac{-(S_{N_t} - mN_t)}{\sigma t^{1/2} m^{-1/2}} + \frac{1}{\sigma m^{-1/2}} \frac{S_{N_t} - t}{\sqrt{t}}$$

From (a) we have that $\frac{S_{N_t} - t}{\sqrt{t}} \Rightarrow 0$,

So it suffices to show that

$$\frac{S_{N_t} - mN_t}{\sigma t^{1/2} m^{-1/2}} \Rightarrow N, \text{ then } \frac{-(S_{N_t} - mN_t)}{\sigma t^{1/2} m^{-1/2}} \Rightarrow N, \text{ too.}$$

(c) From Ex 23.10 we have that $N_t/t \to m^{-1}$ with probability 1, thus $N_t/t \Rightarrow m^{-1}$, and then apply the thm in Ex 27.14 that

$$\frac{S_{N_t} - mN_t}{\sigma\sqrt{m^{-1}t}} = \frac{S_{N_t} - mN_t}{\sigma t^{1/2} m^{-1/2}} \Rightarrow N.$$

Hence the result follows.

#

• Ex 27.16 (Billingsley)

Show that

$$\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}du \sim \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}} \quad \text{as } x\to\infty$$

Pf: $\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}du \to 0$ as $x\to\infty$,

$\frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}} \to 0$ as $x\to\infty$.

$$\lim_{x\to\infty}\frac{\int_x^\infty e^{-\frac{u^2}{2}}du}{\frac{1}{x}e^{-\frac{x^2}{2}}}$$

$$= \lim_{x\to\infty}\frac{-e^{-\frac{x^2}{2}}}{-\frac{1}{x^2}e^{-\frac{x^2}{2}}+\frac{1}{x}(-x)e^{-\frac{x^2}{2}}}$$

$$= \lim_{x\to\infty}\frac{1}{\frac{1}{x^2}+1} = 1$$

Hence the result follows. #

• <u>Ex 27.17</u> (Billingsley)

Suppose that the X_n are iid with mean 0 and variance 1, and suppose that $a_n \to \infty$.

Formally combine the CLT and (27.28) to obtain

$$P[S_n \ge a_n\sqrt{n}] \sim \frac{1}{\sqrt{2\pi}}\frac{1}{a_n}e^{-\frac{a_n^2}{2}} = e^{-a_n^2(1+\xi_n)/2},$$

where $\xi_n \to 0$ if $a_n \to \infty$.

For a case in which this does hold, see Thm 9.4

Pf: • The Lindeberg-Lévy CLT:

Suppose that $\{X_n\}$ is iid with mean c and finite variance σ^2.

If $S_n = X_1 + \cdots + X_n$, then $\frac{S_n - nc}{\sigma\sqrt{n}} \Rightarrow N$.

• (27.28):

$$\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}du \sim \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}} \quad \text{as } x \to \infty$$

• Thm 9.4: "Variant of the Iterated Logarithm."

Let $S_n = X_1 + \cdots + X_n$, where the X_n are iid simple r.v.'s with mean 0 and variance 1.

If a_n are constants satisfying

$$a_n \to \infty, \quad \frac{a_n}{\sqrt{n}} \to 0,$$

then

$$P[S_n \ge a_n\sqrt{n}] = e^{-a_n^2(1+\xi_n)/2}$$

for a sequence $\xi_n \to 0$.

• Since the X_n are iid with mean 0 and variance 1, it follows by the Lindeberg-Lévy CLT that

$\frac{S_n}{\sqrt{n}} \Rightarrow N$; that is,

$$P[\frac{S_n}{\sqrt{n}} \ge a_n] = P[S_n \ge a_n\sqrt{n}]$$

$$\to \frac{1}{\sqrt{2\pi}}\int_{a_n}^\infty e^{-\frac{u^2}{2}}du \sim \frac{1}{\sqrt{2\pi}}\frac{1}{a_n}e^{-\frac{a_n^2}{2}} \quad \text{as } a_n \to \infty.$$

$$= e^{-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log a_n^2 - \frac{a_n^2}{2}}$$

$$= e^{-\frac{a_n^2}{2}\left(1 + \frac{\log(2\pi) + \log a_n^2}{a_n^2}\right)} \equiv e^{-\frac{a_n^2}{2}(1+\xi_n)}$$

where $\xi_n = \frac{\log(2\pi) + \log a_n^2}{a_n^2} \to 0$ as $a_n \to \infty$. #

• Ex 27.18 (Billingsley)

"Stirling's formula."

Let $S_n = X_1 + \cdots + X_n$, where the X_n are iid and has the Poisson distribution with parameter 1; Poisson(1).

Prove successively:

(a) $E\left[\left(\frac{S_n - n}{\sqrt{n}}\right)^-\right] = e^{-n}\sum_{k=0}^{n}\left(\frac{n-k}{\sqrt{n}}\right)\frac{n^k}{k!} = \frac{n^{n+(1/2)}e^{-n}}{n!}$

(b) $\left(\frac{S_n - n}{\sqrt{n}}\right)^- \Rightarrow N^-$

(c) $E\left[\left(\frac{S_n - n}{\sqrt{n}}\right)^-\right] \to E[N^-] = \frac{1}{\sqrt{2\pi}}$

(d) $n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$.

pf: • Ex 21.2: If $X \sim N(0,1)$, $E[|X|^{2n+1}] = \sqrt{\frac{2}{\pi}}\, 2^n n!$

(a) Since the $X_n \overset{iid}{\sim}$ Poisson(1), then $S_n \sim$ Poisson(n)

$$
\begin{aligned}
E\left[\left(\frac{S_n - n}{\sqrt{n}}\right)^-\right] &= \sum_{k=0}^{n}\left(-\frac{k-n}{\sqrt{n}}\right) e^{-n}\frac{n^k}{k!} \\
&= e^{-n}\sum_{k=0}^{n}\left(\frac{n-k}{\sqrt{n}}\right)\frac{n^k}{k!} \\
&= \frac{e^{-n}}{\sqrt{n}}\left(\sum_{k=0}^{n}\frac{n^{k+1}}{k!} - \sum_{k=0}^{n}\frac{kn^k}{k!}\right) \\
&= \frac{e^{-n}}{\sqrt{n}}\left[\left(n + n^2 + \frac{n^3}{2!} + \cdots + \frac{n^{n+1}}{n!}\right) - \left(n + n^2 + \frac{n^3}{2!} + \cdots + \frac{n^n}{(n-1)!}\right)\right] \\
&= \frac{n^{n+\frac{1}{2}}e^{-n}}{n!}
\end{aligned}
$$

(b) • Since the $X_n \overset{iid}{\sim}$ Poisson(1) with mean 1 and variance 1, by the Lindeberg-Lévy thm,

$$\frac{S_n - n}{\sqrt{n}} \Rightarrow N.$$

• Let $h(\omega) = \begin{cases} -h(\omega) & \text{if } h(\omega) \le 0 \\ 0 & \text{if } h(\omega) \ge 0 \end{cases}$, then h is a continuous ft, by the mapping thm it follow that

$$\left(\frac{S_n - n}{\sqrt{n}}\right)^- \Rightarrow N^-.$$

(next pg. cont.)

(c) It suffices to show that

(i) the $\left(\frac{S_n-n}{\sqrt{n}}\right)^-$ are ui; and.

(ii) $E[N^-]=\frac{1}{\sqrt{2\pi}}$.

(i) • Since $E\left[\left(\frac{S_n-n}{\sqrt{n}}\right)^2\right]=\frac{1}{n}E[(S_n-n)^2]=\frac{1}{n}Var[S_n]=1$, so

$$\sup_n E\left[\left(\frac{S_n-n}{\sqrt{n}}\right)^2\right]=1<\infty,$$

• then the $\frac{S_n-n}{\sqrt{n}}$ are ui, and the $\left|\frac{S_n-n}{\sqrt{n}}\right|$ are ui;

• Since $\left(\frac{S_n-n}{\sqrt{n}}\right)^-\le\left|\frac{S_n-n}{\sqrt{n}}\right|$ for all n,

it follows that the $\left(\frac{S_n-n}{\sqrt{n}}\right)^-$ are ui,

hence the result follows.

(ii)
$$E[N^-]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}-x\,e^{-\frac{x^2}{2}}dx$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{-\frac{x^2}{2}}d\left(-\frac{x^2}{2}\right)$$
$$=\frac{1}{\sqrt{2\pi}}\left[e^{-\frac{x^2}{2}}\right]_{-\infty}^{0}=\frac{1}{\sqrt{2\pi}}$$

(d) From (a) and (c) we have that

$$E\left[\left(\frac{S_n-n}{\sqrt{n}}\right)^-\right]=\frac{n^{n+\frac{1}{2}}e^{-n}}{n!}\to E[N^-]=\frac{1}{\sqrt{2\pi}},$$

then $\lim_{n\to\infty}\frac{\sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n}}{n!}=1$, that is,

$$n!\sim\sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n}.$$

#

• <u>Ex 34.3</u> (Billingsley)

Show that the independence of X and Y implies $E[Y|X]=E[Y]$, which in turn implies that $E[XY]=E[X]E[Y]$, uncorrelated.
Show by examples in an Ω of three points that the reverse implications are both false.

Pf: (a) $E[Y|X]\equiv E[Y|\sigma(X)]$,

for $A\in\sigma(X)$

$\int_A E[Y|X]\,dP$

$=\int_{[X\in A]} Y\,dP=\int Y I_{[X\in A]}\,dP$ $\quad\because A\equiv[X\in A]$

$=\int Y\,dP\times\int_{[X\in A]}dP$ $\quad\because$ X and Y are independent.

$=E[Y]\times\int_{[X\in A]}dP$

$=\int_A E[Y]\,dP$

Hence $E[Y|X]=E[Y]$ with probability 1.

(b) $E[XY]=E[E[XY|\sigma(X)]]\equiv E[E[XY|X]]$

$=E[XE[Y|X]]=E[X\cdot E[Y]]$ $\quad\because E[Y|X]=E[Y]$ with probability 1

$=E[X]\cdot E[Y]$

(c) ① If $(X,Y)=(0,0),(1,-1),(1,1)$ w.p. $\frac{1}{3}$ each $\Rightarrow$ dependent.

But we have that

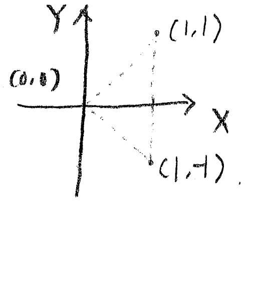
$E[Y|X=0]=0\times P[Y=0|X=0]=0$;

$E[Y|X=1]=1\times P[Y=1|X=1]+(-1)\times P[Y=-1|X=1]$

$=1\times\frac{1}{2}+(-1)\times\frac{1}{2}=0$.

Hence $E[Y|X]=0$.

$E[Y]=0\times\frac{1}{3}+(-1)\times\frac{1}{3}+1\times\frac{1}{3}=0=E[Y|X]$.



dependent;
but $E[Y|X]=E[Y]=0$,
and hence $E[XY]=E[X]E[Y]=0$, uncorrelated.

(c) ② If $(X,Y)=(-1,1),(0,-2),(1,1)$ w.p. $\frac{1}{3}$ each $\Rightarrow$ dependent.

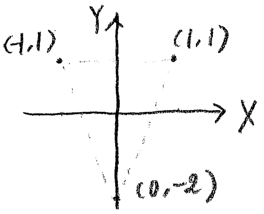
then $E[X]=(-1)\times\frac{1}{3}+0\times\frac{1}{3}+1\times\frac{1}{3}=0$;

$E[Y]=1\times\frac{1}{3}+(-2)\times\frac{1}{3}+1\times\frac{1}{3}=0$;

$E[XY]=(-1)(1)\frac{1}{3}+(0)(-2)\frac{1}{3}+(1)(1)\frac{1}{3}=0$,

hence $E[XY]=E[X]E[Y]=0$. $\Rightarrow$ uncorrelated. But,

$E[Y|X]=Y\neq 0=E[Y]$.



dependent;

$E[Y|X]\neq E[Y]$,

but $E[XY]=E[X]E[Y]$, uncorrelated.

#.

• Ex 34.4 (Billingsley)

(a) Let B be an event with $P(B)>0$,
and define a probability measure P_0 by $P_0(A)=P(A|B)$.
show that $P_0[A\|\mathcal{G}]=P[A\cap B\|\mathcal{G}]/P[B\|\mathcal{G}]$
on a set of P_0-measure 1.

(b) Suppose that $\mathcal{H}$ is generated by a partition $B_1, B_2, \dots$,
and let $\mathcal{G}\vee\mathcal{H}=\sigma(\mathcal{G}\cup\mathcal{H})$.
show that with probability 1,

$$P[A\|\mathcal{G}\vee\mathcal{H}]=\sum_i I_{B_i}\frac{P[A\cap B_i\|\mathcal{G}]}{P[B_i\|\mathcal{G}]}$$

pf: ① First show that

$$\int f\,dP_0=\frac{1}{P(B)}\int_B f\,dP$$

(i) If $f=I_G$, indicator function,

$$\int I_G\,dP_0=1\times P_0(G)+0=P[G|B]$$
$$=\frac{P(G\cap B)}{P(B)}=\frac{1}{P(B)}\int_B I_G\,dP.$$

(ii) If $f=\sum_{i=1}^n a_i I_{G_i}$, simple function,

$$\int\sum_{i=1}^n a_i I_{G_i}\,dP_0=\sum_{i=1}^n a_i\int I_{G_i}\,dP_0$$
$$=\frac{1}{P(B)}\sum_{i=1}^n a_i\int_B I_{G_i}\,dP$$
$$=\frac{1}{P(B)}\int_B\sum_{i=1}^n a_i I_{G_i}\,dP.$$

(iii) If $f\geq 0$, nonnegative function,
there exist simple functions f_n s.t. $0\leq f_n\uparrow f$,

$$\int f\,dP_0=\lim_n\int f_n\,dP_0=\lim_n\frac{1}{P(B)}\int_B f_n\,dP$$
$$=\lim_n\frac{1}{P(B)}\int f_n I_B\,dP=\frac{1}{P(B)}\int f I_B\,dP \text{ by the MCT}$$
$$=\frac{1}{P(B)}\int_B f\,dP.$$

(iv) If f is integrable function, $f=f^+-f^-$, $f^+,f^-\geq 0$,

$$\int f\,dP_0=\int f^+\,dP_0-\int f^-\,dP_0=\frac{1}{P(B)}\int_B f^+\,dP-\frac{1}{P(B)}\int_B f^-\,dP$$
$$=\frac{1}{P(B)}\int_B(f^+-f^-)\,dP=\frac{1}{P(B)}\int_B f\,dP.$$

② $P[B|\mathcal{G}]>0$ on a set of P_0-measure 1.

If $P[B|G]=0$ for some $G\in\mathcal{G}$ where $P_0(G)=P[G|B]>0$.

$$P[B|G]=\frac{P(B\cap G)}{P(G)}=0,\ \text{so } P(B\cap G)=0.$$

But then

$$P_0(G)=P[G|B]=\frac{P(G\cap B)}{P(B)}=0,\ \text{a contradiction}$$

(a) It suffices to show that

$$\int_G P_0[A|\mathcal{G}]\,P[B|\mathcal{G}]\,dP_0=\int_G P[A\cap B|\mathcal{G}]\,dP_0,\ G\in\mathcal{G}.$$

$$\int_G P_0[A|\mathcal{G}]\,P[B|\mathcal{G}]\,dP_0$$

$$=\int_G I_A\,P[B|\mathcal{G}]\,dP_0=\frac{1}{P(B)}\int_B I_A I_G\,P[B|\mathcal{G}]\,dP$$

$$=\frac{1}{P(B)}\int_B I_A I_G I_B\,dP=\frac{1}{P(B)}\int_G I_A I_B I_B\,dP$$

$$=\frac{1}{P(B)}\int_G P[A\cap B|\mathcal{G}]\,I_B\,dP=\frac{1}{P(B)}\int_B P[A\cap B|\mathcal{G}]\,I_G\,dP$$

$$=\int_G P[A\cap B|\mathcal{G}]\,dP_0.$$

(b) • If $P_i(A)=P(A|B_i)$, then for $C=G\cap B_i$, $G\in\mathcal{G}, B_i\in\mathcal{H}$,

$$\int_{G\cap B_i} P_i[A|\mathcal{G}]\,dP$$

$$=P(B_i)\int_G P_i[A|\mathcal{G}]\,dP_i=P(B_i)\,P_i(A\cap G)$$

$$=P(B_i)\int I_A I_G\,dP_i=\int_{B_i} I_A I_G\,dP$$

$$=\int_{G\cap B_i} I_A\,dP=\int_{G\cap B_i} P[A|\mathcal{G}\vee\mathcal{H}]\,dP$$

• Therefore, $\int_C I_{B_i} P_i[A|\mathcal{G}]\,dP=\int_C I_{B_i} P[A|\mathcal{G}\vee\mathcal{H}]\,dP$ if $C=G\cap B_i$, and of course this holds for $C=G\cap B_j$ if $j\neq i$.

• But C's of this form constitute a π-system generating $\mathcal{G}\vee\mathcal{H}$, and hence $I_{B_i} P_i[A|\mathcal{G}]=I_{B_i} P[A|\mathcal{G}\vee\mathcal{H}]$ on a set of P-measure 1.

$$P[A|\mathcal{G}\vee\mathcal{H}]=\sum_i I_{B_i} P[A|\mathcal{G}\vee\mathcal{H}]=\sum_i I_{B_i} P_i[A|\mathcal{G}]$$

$$=\sum_i I_{B_i}\frac{P[A\cap B_i|\mathcal{G}]}{P[B_i|\mathcal{G}]}\quad \text{from (a)}$$

with probability 1.

#

• Ex 34.5 (Billingsley)

The equation (34.5) was proved by showing that the left side is a version of the ride side.

Prove it by showing that the right side is a version of the left side.

Pf: $\because$ (34.5): If X is integrable and the σ-fields $\mathcal{G}_1$ and $\mathcal{G}_2$ s.t. $\mathcal{G}_1 \subset \mathcal{G}_2$, then

$$E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[X|\mathcal{G}_1] \text{ with probability } 1.$$

• "$\Rightarrow$" For $G \in \mathcal{G}_1$,

$$\int_G E[E[X|\mathcal{G}_2]|\mathcal{G}_1]\,dP$$

$$= \int_G E[X|\mathcal{G}_2]\,dP = \int_G X\,dP \quad \because G \in \mathcal{G}_1 \subset \mathcal{G}_2$$

$$= \int_G E[X|\mathcal{G}_1]\,dP$$

• "$\Leftarrow$" For $G \in \mathcal{G}_1$, then $G \in \mathcal{G}_2$,

$$\int_G E[X|\mathcal{G}_1]\,dP$$

$$= \int_G X\,dP = \int_G E[X|\mathcal{G}_2]\,dP$$

$$= \int_G E[E[X|\mathcal{G}_2]|\mathcal{G}_1]\,dP.$$

#

- <u>Ex 34.6</u> (Billingsley)

Prove for bounded X and Y that $E[Y\,E[X|\mathcal{G}]] = E[X\,E[Y|\mathcal{G}]]$.

Pf: • Since X and Y are bounded,

then X and Y are integrable, and hence

there exist $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$ s.t.

(i) $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$ are measurable $\mathcal{G}$ and integrable.

(ii) $\int_G E[X|\mathcal{G}]\,dP = \int_G X\,dP$ for $G \in \mathcal{G}$,

$\int_G E[Y|\mathcal{G}]\,dP = \int_G Y\,dP$ for $G \in \mathcal{G}$.

• Then $Y\,E[X|\mathcal{G}]$ and $X\,E[Y|\mathcal{G}]$ are also integrable,

and hence $E[Y\,E[X|\mathcal{G}]]$, $E[X\,E[Y|\mathcal{G}]]$ exist.

• $E[Y\,E[X|\mathcal{G}]]$

$= \int Y\,E[X|\mathcal{G}]\,dP = \int E[Y\,E[X|\mathcal{G}]|\mathcal{G}]\,dP$

$= \int E[X|\mathcal{G}]\,E[Y|\mathcal{G}]\,dP \quad \because E[X|\mathcal{G}]$ is measurable $\mathcal{G}$.

$= \int E[X\,E[Y|\mathcal{G}]|\mathcal{G}]\,dP \quad \because E[Y|\mathcal{G}]$ is measurable $\mathcal{G}$.

$= E[X\,E[Y|\mathcal{G}]]$.

#

- $\underline{Ex\ 34.8}$ (Billingsley)

Assume that X is nonnegative but not necessarily integrable.
Show that it is still possible to define a nonnegative r.v. $E[X|\mathcal{G}]$, measurable $\mathcal{G}$, s.t. (34.1) holds:

$$\int_G E[X|\mathcal{G}]\,dP = \int_G X\,dP, \quad G \in \mathcal{G}$$

Prove versions of the monotone convergence thm and Fatou's lemma.

Pf: (a) Suppose $X \geq 0$, define a measure ν s.t

$$\nu(G) = \int_G X\,dP, \quad G \in \mathcal{G}$$

If ν and P are σ-finite measures s.t. $\nu \ll P$, then there exists a nonnegative r.v. $E[X|\mathcal{G}]$, s.t

$$\nu(G) = \int_G E[X|\mathcal{G}]\,dP, \quad G \in \mathcal{G}$$

Hence $E[X|\mathcal{G}]$ is measurable $\mathcal{G}$ and

$$\int_G E[X|\mathcal{G}]\,dP = \int_G X\,dP, \quad G \in \mathcal{G}.$$

(b) Version of the monotone convergence thm (MCT):

$0 \leq X_n \uparrow X$ implies

$E[X_n|\mathcal{G}] \uparrow E[X|\mathcal{G}]$ with probability 1.

- If $X \leq Y$ with probability 1, then

  $E[X|\mathcal{G}] \leq E[Y|\mathcal{G}]$ with probability 1.

- Thus we have

  $E[X_n|\mathcal{G}] \leq E[X_{n+1}|\mathcal{G}] \leq E[X|\mathcal{G}]$ a.s. for all n,

Hence $\lim_n E[X_n|\mathcal{G}]$ exists a.s., and

$$\int_G \lim_n E[X_n|\mathcal{G}]\,dP = \lim_n \int_G E[X_n|\mathcal{G}]\,dP \quad \text{by the MCT.}$$

$$= \lim_n \int_G X_n\,dP = \int_G X\,dP \quad \text{by the MCT}$$

$$= \int_G E[X|\mathcal{G}]\,dP$$

It follows that $\lim_n E[X_n|\mathcal{G}] = E[X|\mathcal{G}]$ a.s.

(next pg. cont.)

(c) Version of Fatou's lemma

$0 \le X_n$ with probability 1, then

$E[\liminf_n X_n | \mathcal{G}] \le \liminf_n E[X_n | \mathcal{G}]$ with probability 1

$E[\liminf_n X_n | \mathcal{G}] = E[\lim_n \inf_{k \ge n} X_n | \mathcal{G}]$

$= \lim_n E[\inf_{k \ge n} X_k | \mathcal{G}]$ $\quad \inf_{k \ge n} X_n \uparrow$ in n and by part (b)

$\le \lim_n \inf_{k \ge n} E[X_k | \mathcal{G}]$ $\quad X_k \ge \inf_{k \ge n} X_k$, so $E[X_k | \mathcal{G}] \ge E[\inf_{k \ge n} X_k | \mathcal{G}]$, $\forall k$

$= \liminf_n E[X_n | \mathcal{G}]$.

#

- Ex 34.9 (Billingsley)

(a) show for nonnegative X that
$$E[X|\mathcal{G}] = \int_0^\infty P[X>t|\mathcal{G}]\,dt \quad \text{a.s.}$$

(b) Generalize Markov's inequality:
$$P[|X|\ge \alpha|\mathcal{G}] \le \alpha^{-k} E[|X|^k|\mathcal{G}] \quad \text{a.s.}$$

(c) Generalize Chebychev's inequality:
$$P[|X-E[X|\mathcal{G}]| \ge \alpha|\mathcal{G}] \le \frac{1}{\alpha^2} E[(X-E[X|\mathcal{G}])^2|\mathcal{G}] \quad \text{a.s.}$$

(d) Generalize Hölder's inequality:
If $p^{-1}+q^{-1}=1$, $1<p,q<\infty$, then
$$E[|XY||\mathcal{G}] \le E^{1/p}[|X|^p|\mathcal{G}]\cdot E^{1/q}[|Y|^q|\mathcal{G}] \quad \text{a.s.}$$

Pf: • Thm 34.5: Let $\mu(\cdot,\omega)$ be a conditional distribution w.r.t. $\mathcal{G}$ of a r.v. X. If $\phi: R^1\to R^1$ is a Borel function for which $\phi(X)$ is integrable, then $\int_{R^1}\phi(x)\,\mu(dx,\omega)$ is a version of $E[\phi(X)|\mathcal{G}]_\omega$.

(a) By Thm 34.5 we have that $\mu(G,\omega)=P[G|\mathcal{G}]_\omega$ a.s..
$$\int X\,d\mu(\cdot,\omega) = \int_0^\infty x\,\mu(dx,\omega)$$
$$= \int_0^\infty\int_0^x dt\,\mu(dx,\omega)$$
$$= \int_0^\infty\int_t^\infty \mu(dx,\omega)\,dt \quad \text{by Fubini's thm.}$$
$$= \int_0^\infty \mu(X>t,\omega)\,dt$$
$$= \int_0^\infty P[X>t|\mathcal{G}]_\omega\,dt \quad \text{a.s.}$$
Hence the result follows if $E[X|\mathcal{G}]_\omega = \int X\,d\mu(\cdot,\omega)$ a.s.
This is certainly true if $X=I_G$ for $G\in\mathcal{F}$.
By the linearity of the conditional expectation,
this also holds when X is a nonnegative simple function.
For general $X\ge 0$, there exists simple functions $\{X_n\}$ s.t. $0\le X_n\uparrow X$ a.s.
$$E[X|\mathcal{G}]_\omega = \lim_n E[X_n|\mathcal{G}]_\omega = \lim_n \int X_n\,d\mu(\cdot,\omega) = \int X\,d\mu(\cdot,\omega) \quad \text{a.s.}$$
For integrable $X=X^+-X^-$, the result follows similarly.

(b) By Thm 34.5, too,

$$P[|X| \geq \alpha \mid \mathcal{G}]_\omega = \mu(|x| \geq \alpha, \omega)$$

$$= \int_{|x|^k \geq \alpha^k} d\mu(\cdot, \omega) \leq \alpha^{-k} \int |x|^k d\mu(\cdot, \omega) \qquad \frac{|x|^k}{\alpha^k} \geq 1$$

$$= \alpha^{-k} \int |x|^k \mu(dx, \omega) = \alpha^{-k} E[|X|^k \mid \mathcal{G}]_\omega \text{ a.s.}$$

(c) Replace X by $(X - E[X|\mathcal{G}])$ in part (b) with $k=2$,

$$P[|X - E[X|\mathcal{G}]| \geq \alpha \mid \mathcal{G}]_\omega \leq \alpha^{-2} E[(X - E[X|\mathcal{G}])^2 \mid \mathcal{G}] \text{ a.s.}$$

(d) Suppose $p^{-1} + q^{-1} = 1$, $1 < p, q < \infty$, then for positive a, b,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Replace a by $|X| / E^{1/p}[|X|^p|\mathcal{G}]$, b by $|Y| / E^{1/q}[|Y|^q|\mathcal{G}]$,

$$\frac{|XY|}{E^{1/p}[|X|^p|\mathcal{G}] E^{1/q}[|Y|^q|\mathcal{G}]} \leq p^{-1} \frac{|X|^p}{E[|X|^p|\mathcal{G}]} + q^{-1} \frac{|Y|^q}{E[|Y|^q|\mathcal{G}]},$$

$$|XY| \leq p^{-1}|X|^p E^{1/p - 1}[|X|^p|\mathcal{G}] E^{1/q}[|Y|^q|\mathcal{G}] + q^{-1}|Y|^q E^{1/p}[|X|^p|\mathcal{G}] E^{1/q - 1}[|Y|^q|\mathcal{G}]$$

$$E[|XY| \mid \mathcal{G}]$$

$$\leq p^{-1} E^{1/p - 1}[|X|^p|\mathcal{G}] E^{1/q}[|Y|^q|\mathcal{G}] \cdot E[|X|^p|\mathcal{G}]$$

$$+ q^{-1} E^{1/p}[|X|^p|\mathcal{G}] E^{1/q - 1}[|Y|^q|\mathcal{G}] \cdot E[|Y|^q|\mathcal{G}]$$

$$= \left(\frac{1}{p} + \frac{1}{q}\right) E^{1/p}[|X|^p|\mathcal{G}] E^{1/q}[|Y|^q|\mathcal{G}]$$

$$= E^{1/p}[|X|^p|\mathcal{G}] E^{1/q}[|Y|^q|\mathcal{G}].$$

#.

• Ex 34.10 (Billingsley)

(a) Show that, if $\mathcal{G}_1 \subset \mathcal{G}_2$ and $E[X^2] < \infty$, then

$$E[(X - E[X|\mathcal{G}_2])^2] \le E[(X - E[X|\mathcal{G}_1])^2].$$

The dispersion of X about its conditional mean becomes smaller as the σ-field grows.

(b) Define $Var[X|\mathcal{G}] = E[(X - E[X|\mathcal{G}])^2|\mathcal{G}]$

Prove that $Var[X] = E[Var[X|\mathcal{G}]] + Var[E[X|\mathcal{G}]]$.

Pf: • Thm 34.4 : If X is integrable, $\mathcal{G}_1 \subset \mathcal{G}_2$, then

$$E[E[X|\mathcal{G}_1]|\mathcal{G}_2] = E[X|\mathcal{G}_1] = E[E[X|\mathcal{G}_2]|\mathcal{G}_1]$$

(a) • If $Y = X - E[X|\mathcal{G}_1]$

$$X - E[X|\mathcal{G}_2] = X - E[X|\mathcal{G}_1] - (E[X|\mathcal{G}_2] - E[X|\mathcal{G}_1])$$

$$= Y - (E[X|\mathcal{G}_2] - E[E[X|\mathcal{G}_1]|\mathcal{G}_2])$$

$$= Y - E[(X - E[X|\mathcal{G}_1])|\mathcal{G}_2]$$

$$= Y - E[Y|\mathcal{G}_2].$$

• $E[(X - E[X|\mathcal{G}_2])^2|\mathcal{G}_2] = E[(Y - E[Y|\mathcal{G}_2])^2|\mathcal{G}_2]$

$$= E[Y^2|\mathcal{G}_2] - E^2[Y|\mathcal{G}_2]$$

$$\le E[Y^2|\mathcal{G}_2] = E[(X - E[X|\mathcal{G}_1])^2|\mathcal{G}_2]$$

• $E[(X - E[X|\mathcal{G}_2])^2] = E[E[(X - E[X|\mathcal{G}_2])^2|\mathcal{G}_2]]$

$$\le E[E[(X - E[X|\mathcal{G}_1])^2|\mathcal{G}_2]]$$

$$= E[(X - E[X|\mathcal{G}_1])^2]$$

(b) $Var[X] = E[(X - E[X])^2]$

$$= E[((X - E[X|\mathcal{G}]) - (E[X] - E[X|\mathcal{G}]))^2]$$

$$= E[(X - E[X|\mathcal{G}])^2] + E[(E[X] - E[X|\mathcal{G}])^2] - 2E[(X - E[X|\mathcal{G}])(E[X] - E[X|\mathcal{G}])]$$

① $E[(X - E[X|\mathcal{G}])(E[X] - E[X|\mathcal{G}])]$

$$= E[E[(X - E[X|\mathcal{G}])(E[X] - E[X|\mathcal{G}])|\mathcal{G}]]$$

$$= E[(E[X] - E[X|\mathcal{G}])\, E[(X - E[X|\mathcal{G}])|\mathcal{G}]]$$

$$= E[(E[X] - E[X|\mathcal{G}])(E[X|\mathcal{G}] - E[X|\mathcal{G}])] = 0.$$

② $E[(X-E[X|\mathcal{G}])^2]$

$= E[E[(X-E[X|\mathcal{G}])^2|\mathcal{G}]]$

$= E[Var[X|\mathcal{G}]]$.

③ $E[(E[X]-E[X|\mathcal{G}])^2]$

$= E[(E[X|\mathcal{G}]-E[E[X|\mathcal{G}]])^2]$

$= Var[E[X|\mathcal{G}]]$.

$\Rightarrow Var[X] = E[Var[X|\mathcal{G}]] + Var[E[X|\mathcal{G}]]$.

#

• Ex 34.11 (Billingsley)

Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be σ-fields in $\mathcal{F}$, let $\mathcal{G}_{ij} = \sigma(\mathcal{G}_i \cup \mathcal{G}_j)$, and let $A_i \in \mathcal{G}_i$, $i=1,2,3$. The three conditions are equivalent:

(i) $P[A_3 \| \mathcal{G}_{12}] = P[A_3 \| \mathcal{G}_2]$ for all A_3

(ii) $P[A_1 \cap A_3 \| \mathcal{G}_2] = P[A_1 \| \mathcal{G}_2] P[A_3 \| \mathcal{G}_2]$ for all A_1, A_3

(iii) $P[A_1 \| \mathcal{G}_{23}] = P[A_1 \| \mathcal{G}_2]$ for all A_1.

If $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are interpreted as descriptions of the past, present, future, respectively,

(i) is a general version of the Markov property: the conditional probability of a future event A_3 given the past and present $\mathcal{G}_{12}$ is the same as the conditional probability given the present $\mathcal{G}_2$ alone.

(iii) is the same with time reversed.

(ii) says that past and future events A_1 and A_3 are conditionally independent given the present $\mathcal{G}_2$.

Pf: • $P[A_1 \cap A_3 \| \mathcal{G}_2] = E[I_{A_1} I_{A_3} \| \mathcal{G}_2]$

$= E[E[I_{A_1} I_{A_3} \| \mathcal{G}_{12}] \| \mathcal{G}_2]$

$= E[I_{A_1} P[A_3 \| \mathcal{G}_{12}] \| \mathcal{G}_2]$ $\quad A_1 \in \mathcal{G}_{12} = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$.

• (i) ⇒ (ii): Suppose $P[A_3 \| \mathcal{G}_{12}] = P[A_3 \| \mathcal{G}_2]$, then

$P[A_1 \cap A_3 \| \mathcal{G}_2] = E[I_{A_1} P[A_3 \| \mathcal{G}_{12}] \| \mathcal{G}_2]$

$= E[I_{A_1} P[A_3 \| \mathcal{G}_2] \| \mathcal{G}_2] = P[A_1 \| \mathcal{G}_2] P[A_3 \| \mathcal{G}_2]$.

• (ii) ⇒ (i): Suppose $P[A_1 \cap A_3 \| \mathcal{G}_2] = P[A_1 \| \mathcal{G}_2] P[A_3 \| \mathcal{G}_2]$, then

$E[I_{A_1} P[A_3 \| \mathcal{G}_{12}] \| \mathcal{G}_2] = E[I_{A_1} P[A_3 \| \mathcal{G}_2] \| \mathcal{G}_2]$, so

$\int_{A_1 \cap A_2} P[A_3 \| \mathcal{G}_2] \, dP = \int_{A_1 \cap A_2} P[A_3 \| \mathcal{G}_{12}] \, dP$

The set $A_1 \cap A_2$ form a π-system generating $\mathcal{G}_{12}$, hence $P[A_3 \| \mathcal{G}_2] = P[A_3 \| \mathcal{G}_{12}]$ a.s.

• (i) ⇔ (iii) is trivial.

#

· <u>Ex 34.12</u> (Billingsley)

Use Example 33.10 to calculate $P[N_s=k \mid N_u, u\ge t]$ $(s\le t)$ for the Poisson process

Pf: · Ex 33.10: The Poisson process has the Markov property:

$$P[N_u\in H \mid N_{t_1},\dots,N_{t_k}] = P[N_u\in H \mid N_{t_k}]$$

if $0\le t_1\le\cdots\le t_k\le u$. Then

$$P[N_u\in H \mid N_s, s\le t] = P[N_u\in H \mid N_t], \quad t\le u.$$

· Ex 34.11: $\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3$ in $\mathcal{F}$, $\mathcal{G}_{ij}=\sigma(\mathcal{G}_i\cup\mathcal{G}_j)$, $A_i\in\mathcal{G}_i$.

The three conditions are equivalent:

(i) $P[A_3\mid\mathcal{G}_{12}] = P[A_3\mid\mathcal{G}_2]$

(ii) $P[A_1\cap A_3\mid\mathcal{G}_2] = P[A_1\mid\mathcal{G}_2]\,P[A_3\mid\mathcal{G}_2]$

(iii) $P[A_1\mid\mathcal{G}_{23}] = P[A_1\mid\mathcal{G}_2]$.

· Suppose $\{N_t: t\ge 0\}$ is a Poisson process. Then it has the Markov property:

$$P[N_u=h \mid N_s, s\le t] = P[N_u=h\mid N_t], \quad t\le u.$$

Thus by Ex 34.11 we have that

$$P[N_s=k \mid N_u, u\ge t] = P[N_s=k\mid N_t].\quad s\le t.$$

<u>Ex 33.7</u> $\to$ $P[N_s=k\mid N_t=n] = \dfrac{P[N_s=k, N_t=n]}{P[N_t=n]}$ $\quad k\le n$.

$$= \frac{P[N_s=k, N_t-N_s=n-k]}{P[N_t=n]} = \frac{P[N_s=k]\,P[N_t-N_s=n-k]}{P[N_t=n]} \quad \text{indep. increments}$$

$$= \frac{e^{-\lambda s}(\lambda s)^k/k! \times P[N_{t-s}=n-k]}{e^{-\lambda t}(\lambda t)^n/n!} \quad N_t\sim\text{Poisson}(\lambda t)\ \&\ \text{stationary increments.}$$

$$= \frac{\cancel{e}^{-\lambda s}(\lambda s)^k/k! \times \cancel{e}^{-\lambda(t-s)}(\lambda(t-s))^{n-k}/(n-k)!}{\cancel{e}^{-\lambda t}(\lambda t)^n/n!}$$

$$= \frac{n!}{k!(n-k)!}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}.$$

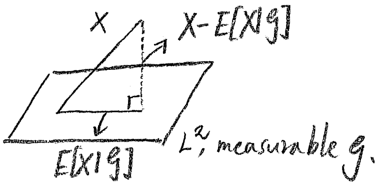
Hence $N_s=k\mid N_t \sim \text{binomial}(N_t, \frac{s}{t})$. #

• Ex 34.13 (Billingsley)

Let L^2 be the Hilbert space of square-integrable r.v.'s on $(\Omega, \mathcal{F}, P)$.
For $\mathcal{G}$ a σ-field in $\mathcal{F}$,
let $M_{\mathcal{G}}$ be the subspace of elements of L^2 that are measurable $\mathcal{G}$.
show that the operator $P_{\mathcal{G}}$ defined for $X \in L^2$ by

$$P_{\mathcal{G}} X = E[X|\mathcal{G}]$$

is the perpendicular projection on $M_{\mathcal{G}}$.



Pf: It suffices to show that

(i) $P_{\mathcal{G}} X = E[X|\mathcal{G}]$ is measurable $\mathcal{G}$

(ii) $P_{\mathcal{G}} X = E[X|\mathcal{G}] \in L^2$

(iii) $\langle X - E[X|\mathcal{G}], E[X|\mathcal{G}] \rangle = 0$

(i) $E[X|\mathcal{G}]$ is measurable $\mathcal{G}$ by the def of conditional expectation

(ii) $E[|E[X|\mathcal{G}]|^2] = E[E^2[X|\mathcal{G}]]$

$\leq E[E[X^2|\mathcal{G}]]$ by the version of Jensen's inequality: $g(x) = x^2$ convex

$= E[X^2] < \infty$ by $X \in L^2$.

(iii) $\langle X - E[X|\mathcal{G}], E[X|\mathcal{G}] \rangle$

$= E[(X - E[X|\mathcal{G}])\, E[X|\mathcal{G}]]$

$= E[E[(X - E[X|\mathcal{G}])\, E[X|\mathcal{G}] \,|\, \mathcal{G}]]$

$= E[E[X|\mathcal{G}]\, \overbrace{(E[X|\mathcal{G}] - E[X|\mathcal{G}])}^{0}]$

$= 0.$

#

• Ex 35.1 (Billingsley)

Suppose that $\Delta_1, \Delta_2, \dots$ are independent r.v.'s with mean 0.
Let $X_1 = \Delta_1$ and $X_{n+1} = X_n + \Delta_{n+1} f_n(X_1, \dots, X_n)$,
and suppose that the X_n are integrable.
Show that $\{X_n\}$ is a martingale.
The martingales of gambling have this form.

Pf: • $\{X_n\}$ is a martingale:

(i) $\mathcal{F}_n \subset \mathcal{F}_{n+1}$

(ii) X_n is measurable $\mathcal{F}_n$

(iii) $E[|X_n|] < \infty$

(iv) $E[X_{n+1} | \mathcal{F}_n] = X_n$ a.s.

• Since $X_1 = \Delta_1$, $X_{n+1} = X_n + \Delta_{n+1} f_n(X_1, \dots, X_n)$,

$X_1 = \Delta_1$,

$X_2 = X_1 + \Delta_2 f_1(X_1) = \Delta_1 + \Delta_2 f_1(\Delta_1)$

$\vdots$

So $\mathcal{F}_n = \sigma(\Delta_1, \dots, \Delta_n) = \sigma(X_1, \dots, X_n)$, then

(i) $\mathcal{F}_n \subset \mathcal{F}_{n+1}$;

(ii) X_n is measurable $\mathcal{F}_n$;

(iii) $E[|X_n|] < \infty$ by hypothesis;

(iv) $E[X_{n+1} | \mathcal{F}_n] = E[X_n + \Delta_{n+1} f_n(X_1, \dots, X_n) | \mathcal{F}_n]$

$= X_n + f_n(X_1, \dots, X_n) E[\Delta_{n+1} | \mathcal{F}_n]$

$= X_n + f_n(X_1, \dots, X_n) E[\Delta_{n+1}]$ by the Δ_n are independent.

$= X_n + f_n(X_1, \dots, X_n) \cdot 0 = X_n$.

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- Ex 35.2 (Billingsley)

Let $Y_1, Y_2, \cdots$ be independent r.v.'s with mean 0 and variance σ^2.

Let $X_n = (\sum_{k=1}^{n} Y_k)^2 - n\sigma^2$, show that $\{X_n\}$ is a martingale.

Pf: Let $\mathcal{F}_n = \sigma(Y_1, \cdots, Y_n)$. Then

(i) $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

Since $X_n = (\sum_{k=1}^{n} Y_k)^2 - n\sigma^2$, then

(ii) X_n is measurable $\mathcal{F}_n$.

(iii) $E[|X_n|] < \infty$, since

$$E[|X_n|] = E\left[\left|(\sum_{k=1}^{n} Y_k)^2 - n\sigma^2\right|\right]$$
$$\leq E\left[(\sum_{k=1}^{n} Y_k)^2\right] + n\sigma^2$$
$$= Var(\sum_{k=1}^{n} Y_k) + n\sigma^2 = 2n\sigma^2 < \infty.$$

(iv) $E[X_{n+1} \mid \mathcal{F}_n] = E\left[\sum_{k=1}^{n+1} Y_k^2 - (n+1)\sigma^2 \mid \mathcal{F}_n\right]$ by $E[(\sum_{k=1}^{n+1} Y_k)^2] = \sum_{k=1}^{n+1} E[Y_k^2]$

$$= E\left[\sum_{k=1}^{n+1} Y_k^2 \mid \mathcal{F}_n\right] - (n+1)\sigma^2$$
$$= \sum_{k=1}^{n} Y_k^2 + E[Y_{n+1}^2 \mid \mathcal{F}_n] - (n+1)\sigma^2$$
$$= \sum_{k=1}^{n} Y_k^2 + \sigma^2 - (n+1)\sigma^2 \quad \text{by the } Y_n \text{ are indep.}$$
$$= \sum_{k=1}^{n} Y_k^2 - n\sigma^2 = X_n.$$

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- $\underline{Ex\ 35.3}$ (Billingsley)

Suppose that $\{Y_n\}$ is a finite-state Markov chain with transition matrix $[p_{ij}]$.

* Suppose that $\sum_j p_{ij}\, x(j) = \lambda\, x(i)$ for all i

(the $x(i)$ are the component of a right eigenvector of $[p_{ij}]$.)

The $X_n = \lambda^{-n} x(Y_n)$ is a martingale.

Pf: Let $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$, then

(i) $\mathcal{F}_n \subset \mathcal{F}_{n+1}$

(ii) The $X_n = \lambda^{-n} x(Y_n)$ is measurable $\mathcal{F}_n$

(iii) $E[|X_n|] < \infty$, since

$$E[|X_n|] = \lambda^{-n} E[|x(Y_n)|]$$
$$= \lambda^{-n} \sum_j |x(j)|\, P[Y_n = j] < \infty \quad \text{by } \{Y_n\} \text{ is finite state}$$

(iv)
$$E[X_{n+1} \mid \mathcal{F}_n] = E[\lambda^{-(n+1)} x(Y_{n+1}) \mid \mathcal{F}_n]$$
$$= \lambda^{-(n+1)} E[x(Y_{n+1}) \mid Y_1, \dots, Y_n]$$
$$= \lambda^{-(n+1)} E[x(Y_{n+1}) \mid Y_n] \quad \text{a.s. by } \{Y_n\} \text{ is a Markov chain.}$$
$$= \lambda^{-(n+1)} \sum_j P_{Y_n j}\, x(j)$$
$$= \lambda^{-(n+1)} \lambda\, x(Y_n) \quad \text{by the } x(i) \text{ are right eigenvectors of } [p_{ij}].$$
$$= \lambda^{-n} x(Y_n) = X_n$$

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- **Ex 35.4 (Billingsley)**

Suppose that $Y_1, \cdots, Y_n$ are independent, positive r.v.'s, and $E[Y_n]=1$.
Put $X_n = Y_1 \cdots Y_n$

(a) show that $\{X_n\}$ is a martingale,
and converges with probability 1 to an integrable X.

(b) Suppose specifically that Y_n assumes values $\frac{1}{2}$ and $\frac{3}{2}$ w.p. $\frac{1}{2}$ each.
Show that $X=0$ a.s.
This gives an example where
$E[\prod_{n=1}^{\infty} Y_n] \neq \prod_{n=1}^{\infty} E[Y_n]$
for independent, integrable, positive r.v.'s.
Show, however, that $E[\prod_{n=1}^{\infty} Y_n] \leq \prod_{n=1}^{\infty} E[Y_n]$ always holds

Pf: (a) · Let $\mathcal{F}_n = \sigma(Y_1, \cdots, Y_n)$. The $X_n = Y_1 \cdots Y_n$ is a martingale:

(i) $\mathcal{F}_n \subset \mathcal{F}_{n+1}$

(ii) $X_n = Y_1 \cdots Y_n$ is measurable $\mathcal{F}_n$.

(iii) $E[|X_n|] = E[|Y_1 \cdots Y_n|]$
$= \prod_{i=1}^{n} E[Y_n] = 1 < \infty$ by the Y_n are indep., positive, $E[Y_n]=1$

(iv) $E[X_{n+1} | \mathcal{F}_n] = E[Y_1 \cdots Y_{n+1} | \mathcal{F}_n]$
$= Y_1 \cdots Y_n \, E[Y_{n+1} | \mathcal{F}_n]$
$= Y_1 \cdots Y_n \, E[Y_{n+1}]$ by the Y_n are indep.
$= Y_1 \cdots Y_n = X_n$

- Since the X_n is a martingale, the X_n is also a submartingale;
$K = \sup_n E[|X_n|] = \sup_n 1 = 1 < \infty$,
hence by the martingale convergence thm,
$\lim_n X_n = X$ a.s. and X is integrable; $E[|X|] < \infty$.

(next pg. cont.)

(b) • Suppose $Y_n = \begin{cases} 1/2 & \text{w.p. } \frac{1}{2} \\ 3/2 & \text{w.p. } \frac{1}{2} \end{cases}$

• $X_n = Y_1 \cdots Y_n = \dfrac{3^{S_n}}{2^n}$, $S_n = \sum_{i=1}^{n} I_{[Y_n = \frac{3}{2}]}$.

$\log X_n = S_n \log 3 - n \log 2$,

$\log X_n / n = \frac{S_n}{n} \log 3 - \log 2$.

By the SLLN, $\frac{S_n}{n} \rightarrow_{a.s.} P[Y_n = \frac{3}{2}] = \frac{1}{2}$.

$\log X_n / n \rightarrow_{a.s.} \frac{1}{2} \log 3 - \log 2 < 0$

$X_n = e^{\log X_n} = e^{n (\log X_n / n)} \rightarrow_{a.s.} 0$ by $n \to \infty$, $\log X_n / n \to k < 0$.

• $E[\prod_{n=1}^{\infty} Y_n] = E[\lim_n \prod_{i=1}^{n} Y_i]$

$= E[\lim_n X_n] = E[0] = 0$; but

$\prod_{n=1}^{\infty} E[Y_n] = \prod_{n=1}^{\infty} 1 = 1 \neq 0 = E[\prod_{n=1}^{\infty} Y_n]$.

• $E[\prod_{n=1}^{\infty} Y_n] = E[\lim_n X_n]$

$= E[\liminf_n X_n]$

$\leq \liminf_n E[X_n]$ by Fatou's thm.

$= \liminf_n \prod_{i=1}^{n} E[Y_i]$ by the Y_n are indep.

$= \prod_{n=1}^{\infty} E[Y_n]$ by $E[Y_n] = 1$ for all n.

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