- <u>Definition</u> (Characteristic Functions)

A random variable X with distribution μ has characteristic function

$$\varphi(t) = E[e^{itX}]$$
$$= \int_{-\infty}^{\infty} e^{itx} \mu(dx)$$
$$= \int_{-\infty}^{\infty} \cos(tx)\,\mu(dx) + i\int_{-\infty}^{\infty} \sin(tx)\,\mu(dx).$$

Remark:

(i) If μ_1 and μ_2 have respectively characteristic function $\varphi_1(t)$ and $\varphi_2(t)$, then $\mu_1 * \mu_2$ has characteristic function $\varphi_1(t)\varphi_2(t)$.
Although convolution is essential to the study of sums of independent r.v.'s, it is a complicated operation, and it is often easier to study the products of the corresponding characteristic functions.

(ii) The characteristic function uniquely determine the distribution: "The Uniqueness Thm",
thus, no information is lost.

(iii) The pointwise convergence of characteristic function iff the weak convergence of the corresponding distributions.
"The Continuity Thm",
Thus, investigate the asymptotic distributions of sums of independent r.v.'s by means of their characteristic functions.

• <u>Proposition</u> (Properties of Chf's)

(a) $\varphi(0)=1$ and $|\varphi(t)|\le 1$ for all t, hence chf always exists.

(b) $\varphi(t)$ is uniformly continuous.

(c) If X has chf $\varphi(t)$, then $aX+b$ has chf

$$E[e^{it(aX+b)}]=e^{itb}\varphi(at)$$

(d) $-X$ has chf $\varphi(-t)$, which is the complex conjugate of $\varphi(t)$.

(e) If $X_1,\cdots,X_n$ are independent, then

$$E[e^{it\sum_{k=1}^n X_k}]=\prod_{k=1}^n E[e^{itX_k}]=\prod_{k=1}^n \varphi_k(t).$$

Pf: (a) $\varphi(0)=E[e^{i0X}]=E[e^0]=1$;

$$|\varphi(t)|=|E[e^{itX}]|\le E|e^{itX}|=E|\cos(tX)+i\sin(tX)|=E(\cos^2(tX)+\sin^2(tX))^{\frac{1}{2}}=1.$$

(b) For any t in R,

$$\begin{aligned}|\varphi(t+h)-\varphi(t)| &= |E[e^{i(t+h)X}]-E[e^{itX}]|\\ &= |E[e^{itX}(e^{ihX}-1)]|\\ &\le E[|e^{itX}||e^{ihX}-1|]=E[|e^{ihX}-1|] \quad \because |e^{itX}|=1\\ &\to 0 \quad \because \text{As } h\downarrow 0,\ e^{ihX}-1\downarrow 0 \text{ and } |e^{ihX}-1|\le|e^{ihX}|=1,\end{aligned}$$

the result follows by the bounded convergence thm.

Note that the upper bound is independent of t which accounts for the uniform continuity.

(c) $E[e^{it(aX+b)}]=e^{itb}E[e^{itaX}]=e^{itb}\varphi(at)$

(d) Take $a=-1$, $b=0$, the result follows:

$$\begin{aligned}E[e^{it(-X)}] &= \varphi(-t) \quad \text{from (c)}\\ &= E[\cos(-tX)]+i\,E[\sin(-tX)]\\ &= E[\cos(tX)]-i\,E[\sin(tX)]=\bar{\varphi}(t)\end{aligned}$$

(e) If X_i has chf $\varphi_i(t)$, $i=1,2$ and X_1 and X_2 are independent, then the chf of X_1+X_2 is

$$E[e^{it(X_1+X_2)}]=E[e^{itX_1}e^{itX_2}]=E[e^{itX_1}]E[e^{itX_2}]=\varphi_1(t)\varphi_2(t).$$

This extends to sums of three or more:

$$E[e^{it\sum_{k=1}^n X_k}]=\prod_{k=1}^n E[e^{itX_k}]=\prod_{k=1}^n\varphi_k(t).$$

#

- **Theorem** (Expansions of Chf's) PS: ㄋ不規ㄉ加絕對值

(a) $\left|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}\right| \le \min\left\{\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\right\}$

the first term on the right gives a sharp estimate for $|x|$ small,

the second term a sharp estimate for $|x|$ large

(b) If X has a moment of order n, replace x by tX,

$\left|\varphi(t) - \sum_{k=0}^{n} \frac{(it)^k}{k!} E[X^k]\right| \le E\left[\min\left\{\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\right\}\right]$ → only approximation

(c) If the mgf $M(t) = E[e^{tX}] < \infty$ for all t in R,

then all moments exist and $\lim_n \frac{|t|^n E[|X|^n]}{n!} = 0$,

hence $\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k]$. → true expansion.

Pf: (a) Integration by parts shows that, for $x > 0$,

- $\int_0^x (x-s)^n e^{is} ds = \frac{x^{n+1}}{n+1} + \frac{i}{n+1} \int_0^x (x-s)^{n+1} e^{is} ds$

$\because$ let $u = e^{is}$, $dv = (x-s)^n ds$; $du = ie^{is} ds$, $v = -\frac{(x-s)^{n+1}}{n+1}$.

$\therefore \int_0^x (x-s)^n e^{is} ds = \left[-e^{is} \frac{(x-s)^{n+1}}{n+1}\right]_0^x + \frac{i}{n+1} \int_0^x (x-s)^{n+1} e^{is} ds$.

$= \frac{x^{n+1}}{n+1} + \frac{i}{n+1} \int_0^x (x-s)^{n+1} e^{is} ds$.

it follows by induction that

- $e^{ix} = \sum_{k=0}^{n} \frac{(ix)^k}{k!} + \frac{i^{n+1}}{k!} \int_0^x (x-s)^n e^{is} ds$

$\because n=0$, $\int_0^x e^{is} ds = \frac{e^{ix}-1}{i} = x + i\int_0^x (x-s) e^{is} ds$,

$\therefore e^{ix} = 1 + ix + i^2 \int_0^x (x-s) e^{is} ds$

$= 1 + ix + i^2\left[\frac{x^2}{2} + \frac{i}{2}\int_0^x (x-s)^2 e^{is} ds\right]$

$= 1 + ix + \frac{(ix)^2}{2} + \frac{i^3}{2}\left[\frac{x^3}{3} + \frac{i}{3}\int_0^x (x-s)^3 e^{is} ds\right]$

$= \sum_{k=0}^{n} \frac{(ix)^k}{k!} + \frac{i^{n+1}}{n!} \int_0^x (x-s)^n e^{is} ds$. (a.1)

$\Rightarrow \left|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}\right| \le \frac{1}{n!}\int_0^x (x-s)^n ds = \frac{1}{n!}\int_0^x -(x-s)^n d(x-s)$

$= \frac{-1}{n!}\left[\frac{(x-s)^{n+1}}{n+1}\right]_0^x = \frac{x^{n+1}}{(n+1)!}$

- Replace n by $n-1$ in (a.1),

$e^{ix} = \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} + \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1} e^{is} ds$

$= \sum_{k=0}^{n} \frac{(ix)^k}{k!} - \frac{(ix)^n}{n!} + \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1} e^{is} ds$.

(next pg. cont.)

$$\Rightarrow \left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \le \frac{x^n}{n!} + \frac{1}{(n-1)!}\int_0^x (x-s)^n ds = \frac{2x^n}{n!}$$

Hence

$$\left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \le \min\left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}$$

(b) If X has moments of order n,

replace x by tX in (a) and take expectations.

(c) If $M(t) = E[e^{tX}]$ exists for all t in R, then

- $E[e^{|tX|}] = E[e^{|t|X} 1_{[X>0]}] + E[e^{-|t|X} 1_{[X<0]}]$

  $\le M(|t|) + M(-|t|) < \infty$, that is,

- $E[e^{|tX|}] = \sum_{k=0}^{\infty} \frac{|t|^k}{k!} E[|X|^k] < \infty$, all moments exist,

  and $\lim_n \frac{|t|^n}{n!} E[|X|^n] = 0$. for all t in R,

  $\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k]$.

\#.

• <u>Proposition</u> (Expansions of Chf and Error Term).

If $E[|X^n|]<\infty$, then

$$\varphi(t)=\sum_{k=0}^{n}\frac{(it)^k}{k!}E[X^k]+o(t^n),\quad t\to 0$$

Pf: $\because \left|\varphi(t)-\sum_{k=0}^{n}\frac{(it)^k}{k!}E[X^k]\right|\le E\left[\min\left\{\frac{|tX|^{n+1}}{(n+1)!},\frac{2|tX|^n}{n!}\right\}\right]$

$\therefore$ the error R is at most $|t|^n E\left[\min\left\{\frac{|t||X|^{n+1}}{(n+1)!},\frac{2|X|^n}{n!}\right\}\right]$.

$\because \min\left\{\frac{|t||X|^{n+1}}{(n+1)!},\frac{2|X|^n}{n!}\right\}\le\frac{|t||X|^{n+1}}{(n+1)!}\to 0$ as $t\to 0$, and

$\min\left\{\frac{|t||X|^{n+1}}{(n+1)!},\frac{2|X|^n}{n!}\right\}\le\frac{2|X|^n}{n!}\in L_1$,

$\therefore R/t^n=E\left[\min\left\{\frac{|t||X|^{n+1}}{(n+1)!},\frac{2|X|^n}{n!}\right\}\right]\to 0$ as $t\to 0$ by DCT.

Hence $\varphi(t)=\sum_{k=1}^{n}\frac{(it)^k}{k!}E[X^k]+o(t^n)$.

#

- **Example** (Chf of the $N(0,1)$ r.v.)

If X has the standard normal distribution,
it chf is $\varphi(t) = e^{-\frac{t^2}{2}}$.

Sol 1: we show that the mgf $M(t)$ exists for all t in R, and hence

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k]$$

$$M(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2tx}{2}} dx$$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx = e^{\frac{t^2}{2}} < \infty \text{ for all } t \text{ in } R$$

and then

$$M(t) = E[e^{tX}] = \sum_{k=0}^{\infty} \frac{t^k}{k!} E[X^k] \quad ①$$

$$= e^{\frac{t^2}{2}} = \sum_{k=0}^{\infty} \frac{(\frac{t^2}{2})^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{t^{2k}}{k!}, \quad ②$$

coefficients of $t^{2k} = \frac{1}{(2k)!} E[X^{2k}] = \frac{1}{2^k} \frac{1}{k!}$ by ①, ②

So we conclude that

$$E[X^{2k}] = \frac{(2k)!}{k!} \left(\frac{1}{2}\right)^k = \frac{2^k k!}{k!} \left(\frac{1}{2}\right)^k (2k-1)\times(2k-3)\times\cdots\times 1 = 1\times 3\times\cdots\times(2k-1).$$

$$E[X^{2k+1}] = 0$$

So
$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^{2k}}{(2k)!} E[X^{2k}]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \frac{(2k)!}{k!} \left(\frac{1}{2}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{\left[-\frac{t^2}{2}\right]^k}{k!} = e^{-\frac{t^2}{2}}.$$

Sol 2: By direct computation.

#

• <u>Proposition</u> (Moments and Derivatives of Chf's)

If $E[|X^k|]<\infty$, then $\varphi^{(k)}(0)=i^k E[X^k]$.

Pf: ∵ $\varphi'(t)=\lim_{h\downarrow 0}\frac{\varphi(t+h)-\varphi(h)}{h}$

∴ We want to show that, as $h\downarrow 0$,

$\frac{\varphi(t+h)-\varphi(t)}{h}$ and $E[iXe^{itX}]$ are closed.

$$\Rightarrow \frac{\varphi(t+h)-\varphi(t)}{h}-E[iXe^{itX}]$$
$$=E\left[\frac{e^{i(t+h)X}-e^{itX}-ihXe^{itX}}{h}\right]$$
$$=E\left[e^{itX}\frac{e^{ihX}-1-ihX}{h}\right]$$

∵ $e^{ihX}-\sum_{k=0}^{1}\frac{(ih)^k}{k!}X^k\le\min\left\{\frac{(hX)^2}{2!},2h|X|\right\}$

∴ $\left|\frac{e^{ihX}-1-ihX}{h}\right|\le\min\left\{\frac{h\cdot X^2}{2!},2|X|\right\}\le 2|X|\in L_1$, and

$\left|\frac{e^{ihX}-1-ihX}{h}\right|\le\frac{hX^2}{2!}\to 0$ as $h\downarrow 0$, so $\frac{e^{ihX}-1-ihX}{h}\to 0$ as $h\downarrow 0$.

By dominated convergence thm,

$$\lim_{h\downarrow 0}\frac{\varphi(t+h)-\varphi(t)}{h}-E[iXe^{itX}]$$
$$=\lim_{h\downarrow 0}E\left[e^{itX}\frac{e^{ihX}-1-ihX}{h}\right]$$
$$=E\left[\lim_{h\downarrow 0}\left(e^{itX}\frac{e^{ihX}-1-hX}{h}\right)\right]=0,$$

That is, $\varphi'(t)=E[iXe^{itX}]$.

• Repeating this argument inductively gives

$\varphi^{(k)}(t)=E[(iX)^k e^{itX}]$,

• hence $\varphi^{(k)}(0)=i^k E[X^k]$.

Pf2: ∵ $\varphi(t)$ is uniformly continuous,

∴ the differentiation and integration signs are interchangeable,

hence $\varphi^{(k)}(t)=\frac{\partial^k}{\partial t^k}E[e^{itX}]=E[\frac{\partial^k}{\partial t^k}e^{itX}]=E[(iX)^k e^{itX}]$,

and the result follows.

#

· Theorem (The Riemann-Lebesgue Thm)

If μ has a density, then $\varphi(t) \to 0$ as $|t| \to \infty$.

Remark:

lightness of the tails of μ is reflected by smoothness of φ.

Pf: • The problem is to prove for integrable f that

$$\int f(x) e^{itx} dx \to 0 \text{ as } |t| \to \infty.$$

•∵ $\int |f| dx < \infty$,

∴ for any $\epsilon > 0$, there exists a step function $g = \sum_{k=1}^{n} \alpha_k I_{A_k}$ where $A_k = (a_k, b_k]$ for which $\int |f - g| dx < \epsilon$

•⇒ $\left| \int f(x) e^{itx} dx \right|$

$$\le \int |e^{itx}| \, |f(x) - g(x)| dx + \left| \int g(x) e^{itx} dx \right|$$

$$\le \epsilon + \sum_{k=1}^{n} |\alpha_k| \left| \int_{a_k}^{b_k} e^{itx} dx \right| \quad \text{by } |e^{itx}| = 1$$

$$= \epsilon + \sum_{k=1}^{n} |\alpha_k| \left| \frac{e^{itb_k} - e^{ita_k}}{it} \right| \to \epsilon \text{ as } |t| \to \infty.$$

$$\because \left| \frac{e^{itb_k} - e^{ita_k}}{it} \right| = \frac{|e^{ita_k}| \, |e^{it(b_k - a_k)} - 1|}{|i| \, |t|}$$

$$\le \frac{\min\{|t(b_k - a_k)|, 2\}}{|t|}$$

$$\le \frac{2}{|t|} \to 0 \text{ as } |t| \to \infty$$

Since this holds for arbitrary ϵ, the result follows.

#

• <u>Proposition</u> (An Important Analysis Result)

The improper Riemann integral exists and

$$\lim_{t\to\infty}\int_0^t \frac{\sin x}{x}\,dx = \frac{\pi}{2};$$

but $\frac{\sin x}{x}$ is not Lebesgue integrable over $(0,\infty)$,

thus $\lim_{t\to\infty}\int_0^t \frac{\sin x}{x}\,dx$ cannot be written $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Pf: (1) We first show $\frac{\sin x}{x}$ is not Lebesgue integrable over $(0,\infty)$.

$\because$ for $x\in(2n\pi+\frac{\pi}{4},\, 2n\pi+\frac{3\pi}{4}) = I_n$,

$$\left(\frac{\sin x}{x}\right)^+ \geq \frac{1}{\sqrt{2}}\frac{1}{(2n+1)\pi} = C_n,$$

$\therefore$ for $x\geq 0$:

$$\left(\frac{\sin x}{x}\right)^+ \geq \sum_{n=1}^{\infty} C_n 1_{I_n}(x)$$

$\Rightarrow$ its integral is

$$\sum_{n=1}^{\infty} C_n\, m(I_n) = \sum_n \frac{1}{\sqrt{2}(2n+1)\pi}\,\frac{\pi}{2} = \sum_n \frac{1}{\sqrt{2}(2n+1)2} = +\infty.$$

it follows that $E(f^+) = +\infty$. Similarly $E(f^-) = -\infty$;

therefore the Lebesgue integral, $E(f)$, does not exist.

(2) We now show $\lim_{t\to\infty}\int_0^t \frac{\sin x}{x}\,dx$ exists.

$\because$ $\int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x}\,dx$ alternates in sign and

$\left|\frac{\sin x}{x}\right|$ decreases to 0,

$$\therefore \lim_{t\to\infty}\int_0^t \frac{\sin x}{x}\,dx = \lim_{k\to\infty}\sum_{n=1}^{k}\left[\int_{(n-1)\pi}^{n\pi}\frac{\sin x}{x}\,dx\right] \text{ exists.}$$

(3) We finally show $\lim_{t\to\infty}\int_0^t \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

$\because$ $\int_0^t \frac{\sin x}{x}\,dx = \int_0^t \sin x\left[\int_0^\infty e^{-ux}\,du\right]dx$ and

$\int_0^t\int_0^\infty |e^{-ux}\sin x|\,du\,dx = \int_0^t |\sin x|\frac{1}{x}\,dx \leq t < \infty$ by $\sin x \leq x$.

$\therefore$ Fubini's thm applies to the integration of $e^{-ux}\sin x$ over $(0,t)\times(0,\infty)$:

$$\Rightarrow \int_0^t \frac{\sin x}{x}\,dx = \int_0^\infty\left[\int_0^t e^{-ux}\sin x\,dx\right]du$$

(next pg. cont.)

Let the inner integral is $I_t(u)$

$I_t(u) = \int_0^t e^{-ux}\sin x\,dx$

let $a = e^{-ux}$, $db = \sin x\,dx$; $da = -ue^{-ux}dx$, $b = -\cos x$

$= \left[-e^{-ux}\cos x\right]_0^t - \int_0^t ue^{-ux}\cos x\,dx$

let $a = ue^{-ux}$, $db = \cos x\,dx$; $da = -u^2e^{-ux}$, $b = \sin x$

$= \left[1 - e^{-ut}\cos t\right] - \left\{\left[ue^{-ux}\sin x\right]_0^t + \int_0^t u^2e^{-ux}\sin x\,dx\right\}$

$= 1 - e^{-ut}(u\sin t + \cos t) - u^2 I_t(u)$

$\Rightarrow I_t(u) = \frac{1}{1+u^2}\left[1 - e^{-ut}(u\sin t + \cos t)\right]$

$\Rightarrow \int_0^t \frac{\sin x}{x}dx = \int_0^U \frac{du}{1+u^2} - \int_0^\infty \frac{e^{-ut}}{1+u^2}(u\sin t + \cos t)du$

$\because \int_0^\infty \frac{du}{1+u^2}$ let $u = \tan\theta$, $du = \sec^2\theta\,d\theta$,

$= \int_0^\pi \frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)} = \frac{\pi}{2} \to \frac{\pi}{2}$ as $t \to \infty$, and.

$\int_0^\infty \frac{e^{-ut}}{1+u^2}(u\sin t + \cos t)\,du$ let $s = ut$, $u = \frac{s}{t}$, $du = \frac{1}{t}ds$

$= \int_0^\infty \frac{e^{-s}}{t^2+s^2}(s\sin t + t\cos t)ds \to 0$ as $t \to \infty$.

$\therefore$ the improper Riemann integral is

$\lim_{t\to\infty} \int_0^t \frac{\sin x}{x}dx = \frac{\pi}{2}$.

#

- <u>Proposition</u> (An Important Analytic Fact, Cont)

If $\operatorname{sgn}\theta = +1, 0,$ or -1 as θ is positive, 0, or negative, then

$$\int_0^T \frac{\sin t\theta}{t}\,dt = \operatorname{sgn}\theta \cdot S(T\cdot|\theta|) \to \operatorname{sgn}\theta\cdot\frac{\pi}{2} \text{ as } T\to\infty$$

where $S(T) = \int_0^T \frac{\sin t}{t}\,dt$

Pf: ① If $\theta > 0$,

$$\int_0^T \frac{\sin t\theta}{t}\,dt \quad \text{let } \beta = t\theta,\ t = \frac{1}{\theta}\beta,\ dt = \frac{1}{\theta}\,d\beta$$

$$= \int_0^{T\theta} \frac{\sin\beta}{\beta}\,\cancel{\theta}\cdot\frac{1}{\cancel{\theta}}\,d\beta$$

$$= \operatorname{sgn}\theta\cdot S(T|\theta|)$$

② If $\theta < 0$,

$$\int_0^T \frac{\sin t\theta}{t}\,dt \quad \text{let } \beta = -t\theta,\ t = -\frac{1}{\theta}\beta,\ dt = -\frac{1}{\theta}\,d\beta$$

$$= \int_0^{-T\theta} \frac{-\sin\beta}{\beta}(+\theta)(+\frac{1}{\theta})\,d\beta$$

$$= -\int_0^{T|\theta|} \frac{\sin\beta}{\beta}\,d\beta = \operatorname{sgn}\theta\cdot S(T|\theta|)$$

③ If $\theta = 0$,

$$\int_0^T \frac{\sin t\theta}{t}\,dt = 0 = \operatorname{sgn}\theta\cdot S(T|\theta|).$$

$\Rightarrow$ For any θ, improper Riemann integral exists:

$$\int_0^T \frac{\sin t\theta}{t}\,dt = \operatorname{sgn}\theta\cdot S(T|\theta|) \to \operatorname{sgn}\theta\cdot\frac{\pi}{2}.$$

#

- Remark:

$$\int_0^T \frac{\sin t\theta}{t}\,dt \to \begin{cases} \frac{\pi}{2} & \text{if } \theta > 0, \\ 0 & \text{if } \theta = 0 \\ -\frac{\pi}{2} & \text{if } \theta < 0. \end{cases} \quad \text{as } T\to\infty.$$

- Theorem (Inversion Formula and Uniqueness Thm)

If the probability measure μ has the characteristic function φ, then

$$\mu((a,b)) + \frac{1}{2}\mu\{a\} + \frac{1}{2}\mu\{b\}$$
$$= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\varphi(t)\,dt = \mu(a,b] \text{ if } \mu\{a\}=\mu\{b\}=0.$$

Distinct measures cannot have the same chf $\Rightarrow$ The Uniqueness Thm

Pf: • Let $I_T = \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\varphi(t)\,dt$

$$= \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\left[\int_{-\infty}^{\infty} e^{itx}\mu(dx)\right]dt$$

$$= \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty} \frac{e^{it(x-a)}-e^{it(x-b)}}{it}\mu(dx)\,dt,$$

$$\because \left|\frac{e^{it(x-a)}-e^{it(x-b)}}{it}\right| = \frac{|e^{it(x-b)}||e^{it(a-b)}-1|}{|t|}$$
$$\le \frac{\min\{|t(a-b)|,2\}}{|t|}$$
$$\le \frac{|t(a-b)|}{|t|} = |a-b|.$$

$$\therefore \int_{-T}^{T}\int_{-\infty}^{\infty}\left|\frac{e^{it(x-a)}-e^{it(x-b)}}{it}\right|\mu(dx)\,dt$$
$$\le |a-b|\int_{-T}^{T}\int_{-\infty}^{\infty}\mu(dx)\,dt = 2T|a-b| < \infty.$$

• Hence by Fubini's thm,

$$I_T = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-T}^{T}\frac{e^{it(x-a)}-e^{it(x-b)}}{it}\,dt\right]\mu(dx)$$

• Since sin s and cos s are odd and even,

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-T}^{T}\left[\frac{\cancel{i}\sin t(x-a)}{\cancel{i}t} - \frac{\cancel{i}\sin t(x-b)}{\cancel{i}t}\right]dt\right\}\mu(dx)$$

$$= \int_{-\infty}^{\infty}\left\{\int_{0}^{T}\left[\frac{1}{\pi}\frac{\sin t(x-a)}{t}dt - \frac{1}{\pi}\frac{\sin t(x-b)}{t}\right]dt\right\}\mu(dx)$$

$$= \int_{-\infty}^{\infty}\left[\frac{sgn(x-a)}{\pi}S(T\cdot|x-a|) - \frac{sgn(x-b)}{\pi}S(T\cdot|x-b|)\right]\mu(dx)$$

$$\because S(T\cdot|x-a|) = \int_0^{T\cdot|x-a|}\frac{\sin t}{t}dt \to \lim_{T\to\infty}\int_0^{T\cdot|x-a|}\frac{\sin t}{t}dt = \frac{\pi}{2}$$

$\therefore S(T\cdot|x-a|)$ is bounded,

(next pg. cont.)

- and hence the integrand of I_T, $I(T,x,a,b)$, where $I(T,x,a,b) = \frac{\operatorname{sgn}(x-a)}{\pi} S(T\cdot|x-a|) - \frac{\operatorname{sgn}(x-b)}{\pi} S(T\cdot|x-b|)$ is bounded and converges as $T\to\infty$ to the function

- $$\lim_{T\to\infty} I(T,x,a,b) = \begin{cases} -\frac{1}{2} - (-\frac{1}{2}) = 0 & \text{for } x<a \\ 0 - (-\frac{1}{2}) = \frac{1}{2} & \text{for } x=a \\ \frac{1}{2} - (-\frac{1}{2}) = 1 & \text{for } a<x<b \\ \frac{1}{2} - 0 = \frac{1}{2} & \text{for } x=b \\ \frac{1}{2} - \frac{1}{2} = 0 & \text{for } x>b \end{cases}$$

- Thus, by dominated convergence thm,

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\,dt \\ &= \lim_{T\to\infty} \int_{-\infty}^{\infty} I(T,x,a,b)\,\mu(dx) \\ &= \int_{-\infty}^{\infty} \lim_{T\to\infty} I(T,x,a,b)\,\mu(dx) \\ &= \mu((a,b)) + \tfrac{1}{2}\mu\{a\} + \tfrac{1}{2}\mu\{b\} \Rightarrow \text{the inversion formula.} \end{aligned}$$

- The inversion formula will imply uniqueness: if μ and ν have the same chf, then

$\mu(a,b] = \nu(a,b]$ if $\mu\{a\} = \nu\{a\} = \mu\{b\} = \nu\{b\} = 0$,

but such intervals $(a,b]$ form a π-system generating $\mathcal{R}^1$, hence $\mu = \nu$.

#

- **Theorem** (Inversion Formula for Continuous Density $f = F'$)

If $\int_{-\infty}^{\infty} |\varphi(t)|\,dt < \infty$, then X has an absolutely continuous distribution with a continuous density $f = F'$, where

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(t)\,dt \qquad (\text{provided } \varphi(t) \text{ is integrable w.r.t. } t).$$

Pf:
- By hypothesis on φ, the integral in inversion formula can be extended over R^1:

$$\int_{-\infty}^{\infty} \left|\frac{e^{-ita} - e^{-itb}}{it}\right| |\varphi(t)|\,dt$$
$$= \int_{-\infty}^{\infty} \frac{|e^{it(b-a)} - 1|}{|t|} |e^{-itb}|\,|\varphi(t)|\,dt$$
$$\le (b-a)\int_{-\infty}^{\infty}|\varphi(t)|\,dt < \infty \qquad \because\ |e^{it(b-a)} - 1| \le \min\{|t|(b-a), 2\} \le |t|(b-a).$$

- There can be no point mass:

$$\mu((a,b)) + \tfrac{1}{2}\mu\{a\} + \tfrac{1}{2}\mu\{b\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it}\varphi(t)\,dt \quad \text{by the integral can be extended over } R^1.$$
$$\le \frac{(b-a)}{2\pi}\int_{-\infty}^{\infty}|\varphi(t)|\,dt \quad \text{from above.}$$
$$\Rightarrow \tfrac{1}{2}\mu\{a\} + \tfrac{1}{2}\mu\{a\} = \mu\{a\} \to 0 \text{ as } b \to a \text{ for all } a.$$

- The corresponding distribution function satisfies

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx} - e^{-it(x+h)}}{ith}\varphi(t)\,dt \quad \text{by } F(x+h) - F(x) = \mu(\overset{a}{\underset{\uparrow}{x}}, \overset{b}{\underset{\uparrow}{x+h}}].$$

$$\because \left|\frac{e^{-itx} - e^{-it(x+h)}}{ith}\varphi(t)\right|$$
$$= \frac{|e^{ith} - 1|}{|th|}|e^{-it(x+h)}|\,|\varphi(t)| \le |\varphi(t)| \le 1,\quad |e^{ith} - 1| \le \min\{|th|, 2\} \le |th|.$$

and $\dfrac{e^{-itx} - e^{-it(x+h)}}{ith}\varphi(t) \to e^{-itx}\varphi(t)$ as $h \to 0$.

$\therefore$ by the bounded convergence thm,

$$f(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(t)\,dt.$$

\#

- <u>Theorem</u> (The Continuity Thm).
  Let μ_n, μ be probability measures with chf's φ_n, φ.
  A necessary and sufficient condition for
  $\mu_n \Rightarrow \mu$ is that $\varphi_n(t) \to \varphi(t)$ for each t.

Remark:
chf's can be used to study limit distributions.

Pf: "$\Rightarrow$". For each t, e^{itx} is a bounded and continuous ft in x,
the Portmanteau theorem implies that
$$\mu_n \Rightarrow \mu \Rightarrow \int e^{itx}\mu_n(dx) = \varphi_n(t) \to \int e^{itx}\mu(dx) = \varphi(t).$$

"$\Leftarrow$" $\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\,dt \quad u>0.$ by $\varphi_n(0)=1$, φ_n is continuous at 0.
$$= \frac{1}{u}\int_{-u}^{u}\left[\int_{-\infty}^{\infty}(1-e^{itx})\,\mu_n(dx)\right]dt$$

- $\because |e^{itx}-1| \le \min\{|tx|, 2\} \le 2,$
$$\int_{-u}^{u}\left[\int_{-\infty}^{\infty}|1-e^{itx}|\,\mu_n(dx)\right]dt \le 2\cdot 2u = 4u < \infty$$
$\therefore$ Fubini's thm is applicable;

- $$= \frac{1}{u}\int_{-\infty}^{\infty}\left[\int_{-u}^{u}(1-e^{itx})\,dt\right]\mu_n(dx)$$
$$= \frac{1}{u}\int_{-\infty}^{\infty}\left[t-\frac{e^{itx}}{ix}\right]_{-u}^{u}\mu_n(dx)$$
$$= \frac{1}{u}\int_{-\infty}^{\infty}\left[2u-\frac{e^{iux}-e^{-iux}}{ix}\right]\mu_n(dx)$$
$$= \frac{1}{u}\int_{-\infty}^{\infty}\left\{2u-\frac{1}{ix}\left[\cancel{\cos(ux)}+i\sin(ux)-\cancel{\cos(-ux)}-i\sin(-ux)\right]\right\}\mu_n(dx)$$
$$= 2\int_{-\infty}^{\infty}\left(1-\frac{\sin ux}{ux}\right)\mu_n(dx)$$

- $\ge 2\int_{|x|\ge 2/u}\left(1-\frac{1}{|ux|}\right)\mu_n(dx) \quad \because \frac{\sin ux}{ux} \le \left|\frac{\sin ux}{ux}\right| \le \frac{1}{|ux|} \le \frac{1}{2}$

- $\ge \int_{|x|\ge 2/u}\mu_n(dx) = \mu_n\left[x: |x| \ge \frac{2}{u}\right] \quad \because 1-\frac{1}{|ux|} \ge 1-\frac{1}{2} = \frac{1}{2}$

Note that the first integral is real.

(next pg. cont.)

• Since φ is continuous at 0 and $\varphi(0)=1$,

there is for positive ϵ a u for which $u^{-1}\int_{-u}^{u}(1-\varphi(t))dt < \epsilon$.

Since φ_n converges to φ, $|\varphi_n(t)| \le 1$,

the bounded convergence thm implies that

there exists an n_0 s.t. $u^{-1}\int_{-u}^{u}(1-\varphi_n(t))dt < 2\epsilon$ for $n \ge n_0$.

If $a = \frac{2}{u}$, then $\mu_n[x: |x| \ge a] < 2\epsilon$ for $n \ge n_0$,

choose $b > a$ s.t

$\mu_n[x: |x| \ge b] < 2\epsilon$ for $n=1, \dots, n_0-1$, and

$\mu_n[x: |x| \ge b] < 2\epsilon$ for $n \ge n_0$,

thus $\mu_n(-b,b) > 1-2\epsilon$ for all n,

hence $\{\mu_n\}$ is tight.

• $\mu_n \Rightarrow \mu$ will follow if it is shown that

each subsequence $\{\mu_{n_k}\}$ that converges weakly at all converges weakly to μ.

• But if $\mu_{n_k} \Rightarrow \nu$ as $k \to \infty$, then by the necessity of the thm,

ν has chf $\lim_k \varphi_{n_k}(t) = \varphi(t)$ since, by hypothesis, $\varphi_n(t) \to \varphi(t)$

• By the inversion formula, ν and μ must coincide.

#.

• <u>Corollary</u> (Cor's of the Continuity Thm)

We observe that, to show that $\{\mu_n\}$ is tight,

only $\varphi(0)=1$ and the continuity of φ at 0 were used; so:

(1) Suppose that $\lim_n \varphi_n(t) = g(t)$ for each t ($\therefore g(0)=1$),

where the limit g is continuous at 0

Then there exists a μ s.t. $\mu_n \Rightarrow \mu$, and μ has chf g

(2) Suppose that $\lim_n \varphi_n(t) = g(t)$ for each t and $\{\mu_n\}$ is tight,

then there exists a μ s.t. $\mu_n \Rightarrow \mu$, and μ has chf g.

- **Lemma** (Logarithm for Complex Numbers Can Be Avoided).

Let $z_1, \dots, z_m$ and $w_1, \dots, w_m$ be complex numbers of "modulus at most 1", that is, $|z_k| \le 1$, $|w_k| \le 1$ for all $k = 1, \dots, m$; then

$$|z_1 \cdots z_m - w_1 \cdots w_m| \le \sum_{k=1}^{m} |z_k - w_k|.$$

Pf: For $m=1$, this is obvious;

Suppose this holds for $m-1$;

$\because z_1 \cdots z_m - w_1 \cdots w_m$

$= (z_1 - w_1) z_2 \cdots z_m - w_1 (z_2 \cdots z_m - w_2 \cdots w_m)$

$\therefore |z_1 \cdots z_m - w_1 \cdots w_m|$

$\le |z_1 - w_1| |z_2| \cdots |z_m| + |w_1| |z_2 \cdots z_m - w_2 \cdots w_m|$

$\le |z_1 - w_1| + \sum_{k=2}^{m} |z_k - w_k|$ $\quad \because |z_k| \le 1, |w_k| \le 1$ for all k

$= \sum_{k=1}^{m} |z_k - w_k|.$

#

• Theorem (The Lindeberg-Lévy Thm: The CLT for iid r.v.'s)

Suppose that $\{X_n\}$ is an "iid" sequence of r.v.'s with "mean" μ and "finite positive variance" σ^2, i.e. "the first two moments $< \infty$".

If $S_n = X_1 + \cdots + X_n$, then $\frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow N$.

Remark:

• By Slutsky's thm,

$\frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow N$ and $\frac{\sigma}{\sqrt{n}} \to 0$ implies $n^{-1}S_n \Rightarrow \mu$, i.e. $n^{-1}S_n \to_p \mu$.

Thus the CLT and the SLLN refine the WLLN in different directions:

$$\begin{array}{ccccc} \text{SLLN} & \leftarrow & \text{WLLN} & \rightarrow & \text{CLT} \\ n^{-1}S_n \to_{a.s.} \mu & & n^{-1}S_n \to_p \mu & & \frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow N \\ & & \text{iff} & & \\ & & n^{-1}S_n \Rightarrow \mu & & \end{array}$$

Pf: • WLOG, let $E[X_k] = 0$, $E[X_k^2] = 1$.

$\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n}{\sqrt{n}}$,

• $\frac{S_n}{\sqrt{n}}$ has the chf $E[e^{it\frac{S_n}{\sqrt{n}}}] = E[e^{i\frac{t}{\sqrt{n}}\sum_{k=1}^{n} X_k}] = \varphi^n(\frac{t}{\sqrt{n}})$.

• By the continuity thm, the problem is to show

$\varphi^n(\frac{t}{\sqrt{n}}) \to E[e^{itN}] = e^{-\frac{t^2}{2}}$, the chf of N

• ∵ the first two moments exists,

$\therefore \varphi(\frac{t}{\sqrt{n}}) = 1 + \frac{it\,E[X_k]}{\sqrt{n}} + \frac{i^2t^2 E[X_k^2]}{2n} + o(\frac{t^2}{n})$

$= 1 - \frac{t^2}{2n} + o(\frac{t^2}{n})$, t: fixed, n: vary $\Rightarrow n\,o(\frac{t^2}{n}) \to 0$ as $n \to \infty$

• ∵ $\left|\varphi^n(\frac{t}{\sqrt{n}}) - (1 - \frac{t^2/2}{n})^n\right| \le \sum_{k=1}^{n} \left|\varphi(\frac{t}{\sqrt{n}}) - (1 - \frac{t^2/2}{n})\right|$ ∵ $|\varphi(\frac{t}{\sqrt{n}})|, |1 - \frac{t^2/2}{n}| \le 1$

$= n\left|\varphi(\frac{t}{\sqrt{n}}) - (1 - \frac{t^2/2}{n})\right|$

$= n \cdot o(\frac{t^2}{n}) \to 0$ as $n \to \infty$

• and $(1 - \frac{t^2/2}{n})^n \to e^{-\frac{t^2}{2}}$, the chf of N, the result follows.

#

- <u>Theorem</u> (The Delta Method)

Suppose $\{X_n\}$ is an "iid" sequence of r.v.'s with "mean μ and var σ^2,"
and set $S_n = \sum_{k=1}^{n} X_k$, $\bar{X}_n = n^{-1} S_n$.

(a) If $g'(\mu) \neq 0$, then
$$\sqrt{n}\left(g(\bar{X}_n) - g(\mu)\right) \Rightarrow N(0, \sigma^2 [g'(\mu)]^2)$$

(b) If $g'(\mu) = 0$ and $g''(\mu) \neq 0$, then
$$n\left(g(\bar{X}_n) - g(\mu)\right) \Rightarrow \frac{1}{2} \sigma^2 g''(\mu) \chi^2(1).$$

Pf: • Since the first two moments exist, by the Lindeberg-Lévy CLT,
$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow N, \text{ i.e. } \sqrt{n}(\bar{X}_n - \mu) \Rightarrow N(0, \sigma^2)$$

• By Skorohod's thm, there exist on a single probability space r.v.'s $\bar{Y}_n$ and Y with $\bar{Y}_n \stackrel{d}{=} \bar{X}_n$ and $Y \stackrel{d}{=} N(0, \sigma^2)$ s.t.
$$\sqrt{n}(\bar{Y}_n - \mu) \to_{a.s.} Y$$

(a) If $g'(\mu) \neq 0$, then by Taylor's series expansion,
$$g(\bar{Y}_n) \approx g(\mu) + g'(\mu)(\bar{Y}_n - \mu)$$
$$\sqrt{n}[g'(\mu)]^{-1}\left(g(\bar{Y}_n) - g(\mu)\right) = \sqrt{n}(\bar{Y}_n - \mu) \to_{a.s.} Y$$
$$\sqrt{n}\left(g(\bar{Y}_n) - g(\mu)\right) \to_{a.s.} [g'(\mu)] \cdot Y \stackrel{d}{=} N(0, \sigma^2 [g'(\mu)]^2)$$

Since $\bar{Y}_n$ has the distribution of $\bar{X}_n$, it follows that
$$\sqrt{n}\left(g(\bar{X}_n) - g(\mu)\right) \Rightarrow N(0, \sigma^2 [g'(\mu)]^2).$$

(b) If $g'(\mu) = 0$ and $g''(\mu) \neq 0$, then by Taylor's series expansion,
$$g(\bar{Y}_n) \approx g(\mu) + \overset{0}{g'(\mu)}(\bar{Y}_n - \mu) + \frac{1}{2} g''(\mu)(\bar{Y}_n - \mu)^2 = g(\mu) + \frac{1}{2} g''(\mu)(\bar{Y}_n - \mu)^2$$
$$n \, 2 [g''(\mu)]^{-1}\left(g(\bar{Y}_n) - g(\mu)\right) = [\sqrt{n}(\bar{Y}_n - \mu)]^2 \to_{a.s.} Y^2$$
$$n\left(g(\bar{Y}_n) - g(\mu)\right) \to_{a.s.} \frac{1}{2}[g''(\mu)] Y^2 \stackrel{d}{=} \frac{1}{2}\sigma^2 [g''(\mu)] \chi^2(1)$$

Since $\bar{Y}_n$ has the distribution of $\bar{X}_n$, it follows that
$$n\left(g(\bar{X}_n) - g(\mu)\right) \Rightarrow \frac{1}{2}\sigma^2 g''(\mu) \chi^2(1).$$

#

- **Example** (Application of the Delta Method)

Suppose that one want to estimate the parameter α of an exponential distribution, with density $f(x) = \alpha e^{-\alpha x}$, $x>0$, on the basis of an independent sample $X_1, \cdots, X_n$.

As $n \to \infty$, the sample mean $\bar{X}_n \to_p \frac{1}{\alpha}$, the mean of the distribution, by WLLN, hence it is natural to use $\frac{1}{\bar{X}_n}$ to estimate α itself.

How good is the estimate?

Since the first two moments exist with mean $\frac{1}{\alpha}$ and var $\frac{1}{\alpha^2}$, by the Lindeberg-Lévy CLT,

$$\frac{S_n - n\frac{1}{\alpha}}{\frac{1}{\alpha}\sqrt{n}} \Rightarrow N, \text{ i.e. } \alpha\sqrt{n}\left(\bar{X}_n - \frac{1}{\alpha}\right) \Rightarrow N.$$

Thus, $\bar{X}_n$ is approximately normally distributed with mean $\frac{1}{\alpha}$ and stdev $\frac{1}{\alpha\sqrt{n}}$.

By Skorohod's thm, there exist on a single probability space r.v.s. $\bar{Y}_n$ and Y with $\bar{Y}_n \stackrel{d}{=} \bar{X}_n$ and $Y \stackrel{d}{=} N$ s.t.

$$\alpha\sqrt{n}\left(\bar{Y}_n - \frac{1}{\alpha}\right) \to_{a.s.} Y$$

Let $g(\bar{Y}_n) = \frac{1}{\bar{Y}_n}$, since $g'(\mu) = -\frac{1}{\mu^2} \neq 0$, by Taylor series expansion,

$$\frac{1}{\bar{Y}_n} = \frac{1}{\mu} - \frac{1}{\mu^2}(\bar{Y}_n - \mu) = \alpha - \alpha^2\left(\bar{Y}_n - \frac{1}{\alpha}\right)$$

$$\alpha\sqrt{n}\,\alpha^{-2}\left(\frac{1}{\bar{Y}_n} - \alpha\right) = \alpha^{-1}\sqrt{n}\left(\frac{1}{\bar{Y}_n} - \alpha\right)$$

$$= -\alpha\sqrt{n}\left(\bar{Y}_n - \frac{1}{\alpha}\right) \to_{a.s.} -Y \stackrel{d}{=} N$$

Since $\bar{Y}_n$ has the distribution of $\bar{X}_n$, it follows that

$$\frac{\sqrt{n}}{\alpha}\left(\frac{1}{\bar{X}_n} - \alpha\right) \Rightarrow N.$$

Thus, $\frac{1}{\bar{X}_n}$ is approximated normally distributed with mean α and stdev $= \frac{\alpha}{\sqrt{n}}$

#

- Definition (A Triangular Array)

  Suppose that for each n,

  $X_{n1}, \cdots, X_{nr_n}$ are independent;

  The probability space for the sequence may change with n, i.e. $(\Omega_n, \mathcal{F}_n, P_n)$.

  Such a collection is called a triangular array of r.v.'s.

- Definition (The Lindeberg Condition)

  Suppose that for triangular arrays,

  $E[X_{nk}]=0, \quad \sigma_{nk}^2 = E[X_{nk}^2], \quad S_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2$;

  The Lindeberg condition: for $\epsilon > 0$,

  $$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{S_n^2} \int_{|X_{nk}| \geq \epsilon S_n} X_{nk}^2 \, dP = 0.$$

  Remark:

  - To establish the asymptotic normality of $S_n = \sum_{k=1}^{r_n} X_{nk}$:
    - expanding the chf of each X_{nk} to second order terms;
    - and estimating the remainder.

    A successful remainder estimate is possible under the assumption of the Lindeberg condition.
  - The assumption that X_{nk} has mean 0 entails no loss of generality:

    Set $X'_{nk} = X_{nk} - E[X_{nk}]$, then $E[X'_{nk}]=0$.

• <u>Theorem</u> (The Lindeberg - Feller CLT)

Suppose that for each n the sequence $X_{n1}, \dots, X_{nr_n}$ is "independent" and satisfy $E[X_{nk}]=0$, $\sigma_{nk}^2 = E[X_{nk}^2]$, $s_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2$.

If the Lindeberg condition holds: for all $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{|X_{nk}| \geq \epsilon s_n} X_{nk}^2 \, dP = 0,$$

then $S_n / s_n \Rightarrow N$

Pf: • Replacing X_{nk} by X_{nk}/s_n show that there is WLOG in assuming $s_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2 = 1$, so $s_n = 1$

• To establish $S_n \Rightarrow N$, the objective is to show

$$\prod_{k=1}^{r_n} \varphi_{nk}(t) = e^{-\frac{t^2}{2}} + o(1) = \prod_{k=1}^{r_n} e^{-\frac{1}{2}t^2\sigma_{nk}^2} + o(1)$$

and then by the continuity thm the result follows.

• Since the first two moments exist, the chf of X_{nk} satisfies

$$\left| \varphi_{nk}(t) - \left(1 + it E[X_{nk}] - \tfrac{1}{2} t^2 E[X_{nk}^2]\right) \right|$$

$$= \left| \varphi_{nk}(t) - \left(1 - \tfrac{1}{2} t^2 \sigma_{nk}^2\right) \right|$$

$$\leq E\left[\min\{|tX_{nk}|^3, |tX_{nk}|^2\}\right]$$

$$\leq \int_{|X_{nk}|<\epsilon} |tX_{nk}|^3 \, dP + \int_{|X_{nk}| \geq \epsilon} |tX_{nk}|^2 \, dP$$

$$\leq \epsilon |t|^3 \sigma_{nk}^2 + t^2 \int_{|X_{nk}| \geq \epsilon} X_{nk}^2 \, dP$$

• We then check $|1 - \frac{1}{2}t^2\sigma_{nk}^2| \leq 1$ for all k

$\because \sigma_{nk}^2 = E[X_{nk}^2]$

$$= \int_{|X_{nk}|<\epsilon} X_{nk}^2 \, dP + \int_{|X_{nk}| \geq \epsilon} X_{nk}^2 \, dP$$

$$\leq \epsilon^2 + \int_{|X_{nk}| \geq \epsilon} X_{nk}^2 \, dP.$$

$$\therefore \max_{1 \leq k \leq r_n} \sigma_{nk}^2 \leq r_n \epsilon^2 + \sum_{k=1}^{r_n} \int_{|X_{nk}| \geq \epsilon} X_{nk}^2 \, dP \to 0$$

$\because \epsilon$ is arbitrary and by the Lindeberg condition.

Hence for large enough n, $1 - \frac{1}{2}t^2\sigma_{nk}^2$ are all between 0 and 1.

(next pg. cont.)

- The chf of S_n satisfies

$$\left|\prod_{k=1}^{r_n} \varphi_{nk}(t) - \prod_{k=1}^{r_n}\left(1-\frac{1}{2}t^2\sigma_{nk}^2\right)\right|$$

$$\leq \sum_{k=1}^{r_n} \left|\varphi_{nk}(t) - \left(1-\frac{1}{2}t^2\sigma_{nk}^2\right)\right| \quad \because |\varphi_{nk}(t)| \leq 1,\ \left|1-\frac{1}{2}t^2\sigma_{nk}^2\right| \leq 1 \text{ for large } n.$$

$$\leq \epsilon |t|^3 \sum_{k=1}^{r_n} \sigma_{nk}^2 + \frac{1}{t^2}\sum_{k=1}^{r_n}\int_{|X_{nk}|\geq\epsilon} X_{nk}^2\,dP$$

$$\to \epsilon |t|^3 \quad \because \sum_{k=1}^{r_n}\sigma_{nk}^2 = s_n = 1 \text{ and the Lindeberg condition.}$$

Since ϵ is arbitrary,

$$\left|\prod_{k=1}^{r_n} \varphi_{nk}(t) - \prod_{k=1}^{r_n}\left(1-\frac{1}{2}t^2\sigma_{nk}^2\right)\right| \to 0$$

- The objective now is to show

$$\left|\prod_{k=1}^{r_n} e^{-\frac{1}{2}t^2\sigma_{nk}^2} - \prod_{k=1}^{r_n}\left(1-\frac{1}{2}t^2\sigma_{nk}^2\right)\right| \to 0:$$

$$\left|\prod_{k=1}^{r_n} e^{-\frac{1}{2}t^2\sigma_{nk}^2} - \prod_{k=1}^{r_n}\left(1-\frac{1}{2}t^2\sigma_{nk}^2\right)\right|$$

$$\leq \sum_{k=1}^{r_n}\left|e^{-\frac{1}{2}t^2\sigma_{nk}^2} - \left(1-\frac{1}{2}t^2\sigma_{nk}^2\right)\right| \quad \because \left|e^{-\frac{1}{2}t^2\sigma_{nk}^2}\right| \leq 1,\ \left|1-\frac{1}{2}t^2\sigma_{nk}^2\right| \leq 1 \text{ for large } n$$

$$\leq \sum_{k=1}^{r_n}\left(\frac{1}{4}t^4\sigma_{nk}^4\right) \quad \because |e^z-1-z| \leq \min\left\{\frac{2|z|^2}{2!}, \frac{|z|^3}{3!}\right\} \leq |z|^2$$

$$= \frac{1}{4}t^4\sum_{k=1}^{r_n}\sigma_{nk}^4$$

$$\leq \frac{1}{4}t^4\left(\max_{1\leq k\leq r_n}\sigma_{nk}^2\right)\left(\sum_{k=1}^{r_n}\sigma_{nk}^2\right) \to 0 \quad \because s_n^2 = 1 \text{ and } \max_{1\leq k\leq r_n}\sigma_{nk}^2 \to 0.$$

- Thus the result follows.

#

- <u>Summary</u>

Lindeberg's Condition iff $\begin{cases} S_n/s_n \Rightarrow N, \text{ where } N \sim N(0,1) \\ \max\limits_{1\leq k\leq r_n} \frac{\sigma_{nk}^2}{s_n^2} \to 0 \end{cases}$

- <u>Corollary</u> (The Lindeberg-Feller CLT holds for iid r.v.'s)

Suppose that $X_{nk} = X_k$ and $r_n = n$,

where the entire sequence $\{X_k\}$ is iid with mean 0 and variance σ^2.

Then the Lindeberg condition reduces to

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n\sigma^2} \int_{|X_k| \geq \epsilon\sigma\sqrt{n}} X_k^2 \, dP = 0,$$

which holds because

$$\sum_{k=1}^{n} \frac{1}{n\sigma^2} \int_{|X_k| \geq \epsilon\sigma\sqrt{n}} X_k^2 \, dP$$

$$= \frac{1}{\sigma^2} \int_{|X_k| \geq \epsilon\sigma\sqrt{n}} X_k^2 \, dP \qquad \because \{X_k\} \text{ is iid}$$

$$= \frac{1}{\sigma^2} \int X_k^2 1_{[|X_k| \geq \epsilon\sigma\sqrt{n}]} \, dP$$

Since $X_k^2 1_{[|X_k| \geq \epsilon\sigma\sqrt{n}]} \downarrow 0$ and $X_k^2 1_{|X_k| \geq \epsilon\sigma\sqrt{n}} \leq X_k^2 \in L_1$,

then by dominated convergence thm it follows that

the Lindeberg condition holds and hence

by the Lindeberg-Feller CLT, $S_n/s_n = S_n/\sigma\sqrt{n} \Rightarrow N$.

#

- <u>Theorem</u> (The Lyapounov Condition and CLT)

Suppose that for each n the sequence $X_{n1}, \cdots X_{nr_n}$ is independent and satisfy $E[X_{nk}]=0$, $\sigma_{nk}^2 = E[X_{nk}^2]$, $s_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2$.

If the $|X_{nk}|^{2+\delta}$ are integrable for some $\delta > 0$ and that the Lyapounov condition holds:

$$\lim_n \sum_{k=1}^{r_n} \frac{1}{s_n^{2+\delta}} E\left[|X_{nk}|^{2+\delta}\right] = 0,$$

then the Lindeberg condition holds, and hence $S_n/s_n \Rightarrow N$.

Pf: Check the Lindeberg condition:

$$\sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{|X_{nk}| \geq \epsilon s_n} X_{nk}^2 \, dP$$

$$= \sum_{k=1}^{r_n} \int_{\left|\frac{X_{nk}}{\epsilon s_n}\right| \geq 1} \left|\frac{X_{nk}}{s_n}\right|^2 \cdot 1 \, dP$$

$$\leq \frac{1}{\epsilon^\delta} \sum_{k=1}^{r_n} \int \left|\frac{X_{nk}}{s_n}\right|^2 \left|\frac{X_{nk}}{s_n}\right|^\delta dP$$

$$= \frac{1}{\epsilon^\delta} \sum_{k=1}^{r_n} \frac{1}{s_n^{2+\delta}} E\left[|X_{nk}|^{2+\delta}\right]$$

$\to 0$ as $n \to \infty$ by the Lyapounov condition.

Thus the Lindeberg condition holds, and hence $S_n/s_n \Rightarrow N$.

#

- <u>Summary</u>

$$\begin{array}{ccccc}
\text{Lyapounov's condition} & \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} & \text{Lindeberg condition} & \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} & S_n/s_n \Rightarrow N, \text{ where } N \sim N(0,1). \\
 & & \Downarrow \not\Uparrow & & \\
 & & \max\limits_{1 \leq k \leq r_n} \frac{\sigma_{nk}^2}{s_n^2} \to 0 & & \\
 & & \Downarrow \not\Uparrow & & \\
 & & s_n^2 \to \infty. & &
\end{array}$$

• <u>Proposition</u> (CLT for Uniformly Bounded r.v.'s with $s_n^2 \to \infty$)

Suppose that $X_1, X_2, \dots$ are "independent" and "uniformly bounded", i.e. $|X_n| \le K$ for all n, and have mean 0.
If the variance s_n^2 of $S_n = \sum_{k=1}^{n} X_k$ goes to infinity, i.e. "$s_n^2 \to \infty$",
then $S_n / s_n \Rightarrow N$

Pf: If K bounds the X_n, i.e. $|X_n| \le K$ for all n, then.

- $\sum_{k=1}^{n} \frac{1}{s_n^3} E[|X_k|^3] \le \sum_{k=1}^{n} \frac{1}{s_n^3} K E[|X_k|^2] = \frac{K}{s_n} \to 0 \quad \because s_n \to \infty$

which is the Lyapounov condition for $\delta = 1$,
thus the Lindeberg condition holds,
and hence $S_n / s_n \Rightarrow N$.

#.

- <u>Example</u> (Lindeberg's Condition holds, but NOT Lyapounov's Condition)

If X_n are iid r.v.'s with density

* $$f(x) = \frac{c}{|x|^3(\log|x|)^2}, \text{ if } |x| > 2$$

- Note $E[X_n] = 0$ since X_n is symmetric and $E[|X_n|] < \infty$; and

$$Var[X_n] \equiv \sigma^2 = 2\int_2^\infty \frac{cx^2}{x^3(\log x)^2}\,dx$$

$$= 2\int_2^\infty \frac{c}{(\log x)^2}\,d(\log x)$$

$$= 2\left[-\frac{c}{\log x}\right]_2^\infty = \frac{2c}{\log 2} < \infty.$$

$\Rightarrow$ Lindeberg's condition holds, and hence $S_n/s_n \Rightarrow N$.

- There is no $\delta > 0$ s.t. the Lyapounov condition holds.

$$E[|X_k|^{2+\delta}]$$

$$= 2\int_2^\infty \frac{cx^{2+\delta}}{x^3(\log x)^2}\,dx$$

$$= 2c\int_2^\infty \frac{x^\delta}{x(\log x)^2}\,dx$$

let $y = \log x$, $x = e^y$, $dx = e^y dy$

$$= 2c\int_{\log 2}^\infty \frac{e^{\delta y}}{y^2}\,dy = \infty \text{ for all } \delta > 0.$$

#

- **Proposition** (Uniform Asymptotic Negligibility, UAN)

The Lindeberg condition $\Rightarrow$ uniform asymptotic negligibility.

$$P\left[\max_{1\le k\le r_n} \frac{|X_{nk}|}{s_n} > \epsilon\right] \to 0 \text{ as } n\to\infty.$$

Remark:

No individual summand may dominate another one;
the summands are called uniformly asymptotically negligible.

Pf: $P\left[\max_{1\le k\le r_n} |X_{nk}|/s_n > \epsilon\right]$

$$\le P\left(\bigcup_{1\le k\le r_n}\left[|X_{nk}|/s_n > \epsilon\right]\right)$$

$$\le \sum_{k=1}^{r_n} P\left[|X_{nk}|/s_n > \epsilon\right]$$

$$= \sum_{k=1}^{r_n} E\left[1_{[|X_{nk}|/\epsilon s_n > 1]}\right]$$

$$\le \frac{1}{\epsilon^2}\sum_{k=1}^{r_n}\frac{1}{s_n^2}\int_{|X_{nk}|>\epsilon s_n} X_{nk}^2\,dP$$

$\to 0$ as $n\to\infty$ by the Lindeberg condition.

#

- **Proposition**

$$\max_{1\le k\le r_n}\frac{\sigma_{nk}^2}{s_n^2}\to 0 \Rightarrow s_n^2\to\infty$$

Pf: $\exists m$ s.t. $\sigma_{nm}^2 > 0$,

$$\frac{\sigma_{nm}^2}{s_n^2} \le \max_{1\le k\le r_n}\frac{\sigma_{nk}^2}{s_n^2} \to 0 \text{ as } n\to\infty$$

Hence $s_n^2\to\infty$ as $n\to\infty$.

#

- <u>Proposition</u> ($s_n^2 \nearrow \infty$ is Necessary for Non-trivial Result)
  - Suppose the X_n are independent r.v.'s with variance $\sigma_n^2 < \infty$.
    Let $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$, if $s_n^2 \nearrow s^2 < \infty$, then for $n > m$
    $$E[(S_n - S_m)^2] = \sum_{k=m+1}^{n} \sigma_k^2 \to 0 \text{ as } n, m \to \infty$$
    Thus $S_n \xrightarrow{L^2} S$ and hence $S_n \Rightarrow S$, in addition, without normalization.
  - The same argument applied to $S_n(j) = \sum_{\substack{k=1 \\ k \neq j}}^{n} X_k$ shows that
    $S_n(j) \Rightarrow S(j)$, which means that
    $S(j) + X_j \overset{d}{=} S$.
  - The Cramer theorem implies that
    if S is normal, then X_j must be normal,
    and since j is arbitrary, every summand must be normal!

- <u>Appendix</u> (Cramer's Theorem)
  If X and Y are independent, non-degenerate r.v.'s, and
  if $X+Y$ is normal, then X and Y are normal.

  \#

· <u>Example</u> $\left(S_n/s_n \Rightarrow N \text{ but NOT} \begin{cases} \text{the Lindeberg condition} \\ \max\limits_{1\le k\le r_n} \frac{\sigma_{nk}^2}{s_n^2} \to 0 \text{ as } n\to\infty \end{cases} \right)$

We know that the converse of the Lindeberg-Feller CLT is
if $S_n/s_n \Rightarrow N$ and $\max\limits_{k\le r_n} \frac{\sigma_{nk}^2}{s_n^2} \to 0$, then the Lindeberg condition holds.
But this converse fails without $\max\limits_{1\le k\le r_n} \frac{\sigma_{nk}^2}{s_n^2} \to 0$.

Remark:
If the variances, instead of decreasing rapidly,
increase very rapidly, we obtain another exception.

Pf: • Take $X_{nk} = X_k$ "normal" with mean 0 and variance $\sigma_{nk}^2 = \sigma_k^2$,
where $\sigma_1^2 = 1$ and $\sigma_n^2 = n s_{n-1}^2$

① We first check $S_n/s_n \Rightarrow N$:
Since S_n/s_n is a linear combination of normal r.v.'s X_k,
so S_n/s_n is normally distributed, with

$$E[S_n/s_n] = \frac{1}{s_n}\sum_{k=1}^{n} E[X_k] = 0,$$

$$Var[S_n/s_n] = \frac{1}{s_n^2}\sum_{k=1}^{n} \sigma_k^2 = 1,$$

hence $S_n/s_n \sim N(0,1)$, for all n, and hence $S_n/s_n \Rightarrow N$.

② We now show that $\max\limits_{k\le n} \frac{\sigma_k^2}{s_n^2} \not\to 0$

$\because \sigma_1^2 = 1 = s_1^2$,

$\sigma_2^2 = 2s_1^2 = 2 \Rightarrow s_2^2 = 1+2$,

$\sigma_3^2 = 3s_2^2 = 3(1+2)$,

$\vdots$

So $\sigma_n^2 \uparrow \infty$ as $n\to\infty$, and hence $s_n^2 \nearrow \infty$.

$$\therefore \max_{k\le n} \frac{\sigma_k^2}{s_n^2} = \frac{\sigma_n^2}{s_n^2} = \frac{\sigma_n^2}{s_{n-1}^2+\sigma_n^2} = \frac{1}{\sigma_n^2/n+\sigma_n^2} = \frac{n}{1+n} \to 1 \ne 0,$$

③ Since $\max\limits_{k\le n} \frac{\sigma_k^2}{s_n^2} \to 0$ does not hold,
neither can the Lindeberg condition.

#

- <u>Example</u> $\left(s_n^2 \nearrow s^2 < \infty,\ S_n/s_n \Rightarrow N \text{ but NOT} \begin{cases} \text{the Lindeberg Condition} \\ \max\limits_{1\le k\le n} \frac{\sigma_{nk}^2}{s_n^2} \to 0 \text{ as } n\to\infty \end{cases} \right)$

Suppose that X_n are independent "normal" r.v.'s with mean 0 and $s_n^2 \nearrow s < \infty$.

- Since S_n/s_n are linear combination of normal r.v.'s with

$$E\left[\frac{S_n}{s_n}\right] = \frac{1}{s_n}\sum_{k=1}^{n} E[X_k] = 0,$$

$$Var\left(\frac{S_n}{s_n}\right) = \frac{1}{s_n^2}\sum_{k=1}^{n} E[X_k^2] = 1,$$

Thus $S_n/s_n \stackrel{d}{=} N$ for all n, and hence $S_n/s_n \Rightarrow N$.

- The Lindeberg condition does NOT hold.

$$\sum_{k=1}^{n} \frac{1}{s_n^2} \int_{|X_k|>\epsilon s_n} X_k^2 \, dP$$

$$\ge \frac{1}{s_n^2} \int_{|X_1|>\epsilon s_n} X_1^2 \, dP$$

$$\ge \frac{1}{s^2} \int_{|X_1|>\epsilon s} X_1^2 \, dP > 0.$$

- $\max\limits_{1\le k\le n} \frac{\sigma_k^2}{s_n^2} \to 0$ does NOT hold.

If $\sigma_k^2 = \frac{1}{2^k}$ for all k, then $s_n^2 = \frac{\frac{1}{2}(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} = 1-(\frac{1}{2})^n \nearrow 1 = s^2 < \infty$, so

$$\max_{1\le k\le n} \frac{\sigma_k^2}{s_n^2} = \frac{\frac{1}{2}}{1-(\frac{1}{2})^n} \to \frac{1}{2} \neq 0 \text{ as } n\to\infty.$$

#

- **Example** (Asymptotic Normality holds and $\max_{1\le k\le n} \frac{\sigma_k^2}{s_n^2} \to 0$, but NOT the Lindeberg Condition)

- Note: $\frac{S_n}{s_n} \Rightarrow N(0,c)$, $c \neq 1$, the issue is not the $N(0,1)$.

- Suppose that Y_n are iid r.v.'s with mean 0 and variance 1, and Z_n are independent r.v.'s with

$$P[Z_k = k] = P[Z_k = -k] = \frac{1}{2k^2},\ P[Z_k = 0] = 1 - \frac{1}{k^2}.$$

Set $X_k = Y_k + Z_k$, $S_n = \sum_{k=1}^{n} X_k$.

- Since Y_k are iid r.v.'s with mean 0 and variance 1, the Lindeberg-Lévy thm implies that

$$\frac{\sum_{k=1}^{n} Y_k}{\sqrt{n}} \Rightarrow N.$$

- $\sum_{n=1}^{\infty} P[|Z_n| > \epsilon] = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, and then by the Borel-Cantelli lemma,

$P[Z_n \neq 0 \text{ i.o.}] = 0$, so $P[Z_n = 0 \text{ eventually}] = 1$, i.e. $Z_n = 0$ a.s.

$$\Rightarrow \frac{\sum_{k=1}^{n} Z_k}{\sqrt{n}} \to_{a.s.} 0.$$

- By Slutsky's thm,

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{n}} = \frac{\sum_{k=1}^{n} Y_k + \sum_{k=1}^{n} Z_k}{\sqrt{n}} \Rightarrow N.$$

- Since $E[Z_k] = 0$, $Var(Z_k) = 2k^2 \frac{1}{2k^2} = 1$,

so $E[X_k] = E[Y_k] + E[Z_k] = 0$, $Var(X_k) = Var(Y_k) + Var(Z_k) = 1 + 1 = 2$.

Then $\frac{S_n}{s_n} = \frac{\sum_{k=1}^{n} X_k}{\sqrt{2n}} \Rightarrow N(0, \frac{1}{2}) \neq N$ by Slutsky's thm.

- Check $\max_{1\le k\le n} \frac{\sigma_k^2}{s_n^2} = \frac{2}{2n} \to 0$ as $n \to \infty$.

#