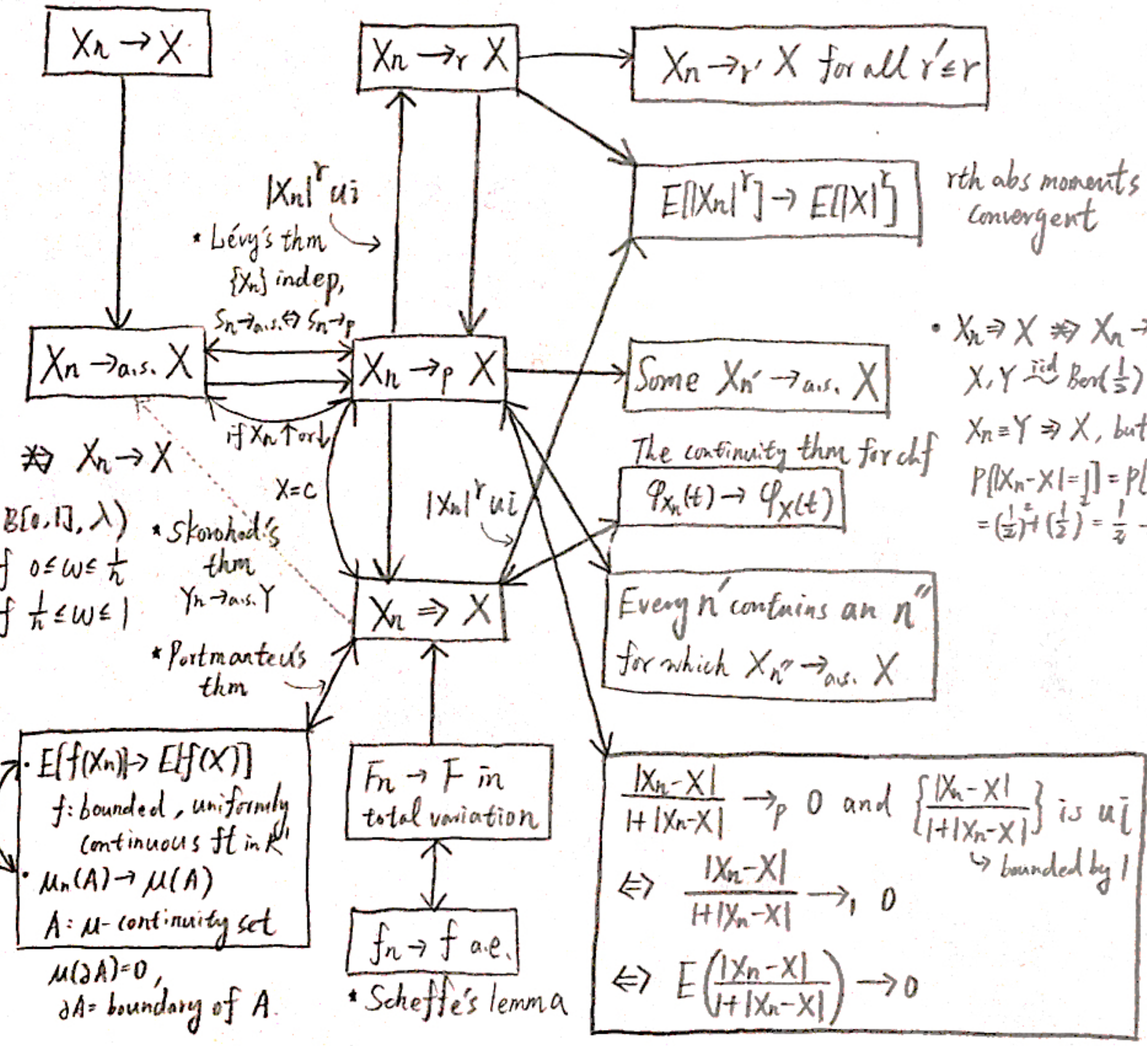
- $X_n \to_r X$, $X_n \to_p X$ $\not\Rightarrow$ $X_n \to_{a.s.} X$
  On $([0,1], \mathcal{B}[0,1], \lambda)$
  $X_n = 1$ if $\frac{j}{2^m} \le \omega \le \frac{j+1}{2^m}$, $n = 2^m + j$, $0 \le j \le 2^m - 1$; $X = 0$

- $X_n \to_{a.s.} X$, $X_n \to_p X$ $\not\Rightarrow$ $X_n \to_r X$
  On $([0,1], \mathcal{B}[0,1], \lambda)$
  $X_n = \begin{cases} 2^n & \text{if } 0 \le \omega \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \le \omega \le 1 \end{cases}$; $X = 0$



- $X_n \to_{a.s.} X$, $X_n \to_p X$ $\not\Rightarrow$ $X_n \to X$
  on $([0,1], \mathcal{B}[0,1], \lambda)$
  $X_n = \begin{cases} n, & \text{if } 0 \le \omega \le \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} \le \omega \le 1 \end{cases}$
  $X = 0$.

- $X_n \Rightarrow X$ $\not\Rightarrow$ $X_n \to_p X$
  $X, Y \overset{iid}{\sim} \mathrm{Ber}(\frac{1}{2})$.
  $X_n = Y \Rightarrow X$, but
  $P[|X_n - X| = 1] = P[|Y - X| = 1] = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2} \not\to 0$.

- $\{X_n\}$ is uniformly integrable (ui):
  $\begin{cases} X_n \in L_1 \ \forall n; \\ \sup_n \int_{[|X_n| > \alpha]} |X_n| \, dP \to 0 \text{ as } \alpha \to \infty \end{cases}$

  $\Leftrightarrow \begin{cases} \text{① uniformly continuous} \\ \quad \sup_n \int_A |X_n| \, dP < \epsilon \text{ if } P(A) < \delta \quad \forall A \in \Omega \\ \text{② uniformly bounded (mean)} \\ \quad \sup_n E[|X_n|] < \infty \end{cases}$

  and if $\sup_n E[|X_n|^{1+\delta}] < \infty$ for $\delta > 0$.

- Skorohod's thm:
  If $X_n \Rightarrow X$, then
  $\exists \{Y_n, n \ge 1, Y\}$ on the same $((0,1), \mathcal{B}(0,1), \lambda)$,
  with $Y_n \overset{d}{=} X_n$, $n \ge 1$, $Y \overset{d}{=} X$, s.t.
  $Y_n \to_{a.s.} Y$.

- <u>Definition</u> (Almost Sure Convergence)

  A statement about r.v.'s holds almost surely (a.s.) if there exists an event $N \in \mathcal{B}$ with $P(N)=0$ s.t. the statement holds if $\omega \in N^c$.

- <u>Definition</u> ($X_n \xrightarrow{a.s.} X$)

  $\{X_n\}$ converges a.s. to X if for any $\varepsilon > 0$,

  $$P([|X_n - X| > \varepsilon] \text{ i.o.}) = P\left(\limsup_{n\to\infty} [|X_n - X| > \varepsilon]\right) = 0$$

  $$\Leftrightarrow P\left(\liminf_{n\to\infty} [|X_n - X| \le \varepsilon]\right) = P([|X_n - X| \le \varepsilon] \text{ eventually}) = 1.$$

  Since ε is arbitrary,

  $$P(|X_n - X| \to 0) = P(X_n - X \to 0) = P(X_n \to X) = 1.$$

- <u>Example</u> (Converges a.s. but NOT Converge Everywhere)

  Let $(\Omega, \mathcal{B}, P)$ be the Lebesgue unit interval $([0,1], \mathcal{B}([0,1]), \lambda)$ where λ is the Lebesgue measure. Define

  $$X_n(\omega) = \begin{cases} n & \text{if } 0 \le \omega \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < \omega \le 1. \end{cases}$$

  - $X_n \xrightarrow{a.s.} 0$ since

    if $N = \{0\}$, then $\omega \in N^c$ implies $X_n(\omega) \to 0$.

  - $X_n \not\to 0$ since

    $X_n(0) = n \to \infty$.

    #

- <u>Definition</u> ($X_n \xrightarrow{P} X$).

$\{X_n\}$ converges in probability (i.p.) to X if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P[|X_n - X| > \varepsilon] = 0.$$

$\Leftrightarrow$ for any $\varepsilon > 0$, $\delta > 0$, $\exists\, n_0 = n_0(\varepsilon, \delta)$ s.t.

$$P[|X_n - X| > \varepsilon] < \delta \text{ for } n \geq n_0$$

Note: convergence i.p. is weaker.

- <u>Example</u> (Converge i.p. but NOT Converge a.s.)

$$X_n(\omega) = \begin{cases} 1, & \text{if } \frac{j}{2^m} \leq \omega < \frac{j+1}{2^m} \\ 0, & \text{o.w.} \end{cases}$$

where $n = 2^m + j$, $0 \leq j \leq 2^m - 1$.

That is,

$X_1 = 1_{[0,1]}$,

$X_2 = 1_{[0,\frac{1}{2}]}$, $X_3 = 1_{[\frac{1}{2},1]}$,

$X_4 = 1_{[0,\frac{1}{3}]}$, $X_5 = 1_{[\frac{1}{3},\frac{2}{3}]}$, $X_6 = 1_{[\frac{2}{3},1]}$.

$\vdots \quad \vdots \quad \vdots$

- $X_n \xrightarrow{P} 0$
- $X_n \overset{a.s.}{\not\to} 0$ since

for any $\omega \in [0,1]$, $X_n(\omega) \not\to 0$

since $X_n(\omega) = 1$ for infinitely many values of n. #

- <u>Theorem</u> (Converge a.s. implies Converge i.p.)

Let $\{X_n, n \geq 1, X\}$ be r.v.'s on $(\Omega, \mathcal{B}, P)$, then

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X.$$

Pf: By Fatou's Lemma, for any $\varepsilon > 0$,

$$0 = P\left(\liminf_{n\to\infty} [|X_n - X| > \varepsilon]\right) \leq \liminf_{n\to\infty} P[|X_n - X| > \varepsilon]$$

$$\leq \limsup_{n\to\infty} P[|X_n - X| > \varepsilon] < P\left(\limsup_{n\to\infty} [|X_n - X| > \varepsilon]\right) = 0$$

Since $X_n \xrightarrow{a.s.} X$, $P\left(\limsup_{n\to\infty} [|X_n - X| > \varepsilon]\right) = 0 \Rightarrow P\left(\liminf_{n\to\infty} [|X_n - X| > \varepsilon]\right) = 0$. #

- Theorem (Convergence i.p. iff Convergence a.s. along a Subsequence)

Suppose that $\{X_n, n \geq 1, X\}$ are real-valued r.v.'s.

$X_n \xrightarrow{P} X$ iff each subsequence $\{X_{n_k}\}$ contain

a further subsequence $\{X_{n_{k(i)}}\}$ s.t. $X_{n_{k(i)}} \xrightarrow{a.s.} X$ as $i \to \infty$.

Pf: If $X_n \xrightarrow{P} X$, then given $\{n_k\}$, choose a subsequence $\{n_{k(i)}\}$

s.t. $k \geq k(i)$ implies that $P[|X_{n_k} - X| \geq 2^{-i}] < 2^{-i}$.

By the Borel-Cantelli Lemma,

$$\sum_i P[|X_{n_{k(i)}} - X| > 2^{-i}] < \sum_i 2^{-i} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 < \infty,$$

$$\Rightarrow P([|X_{n_{k(i)}} - X| > 2^{-i}] \text{ i.o.}) = 0.$$

That is, $|X_{n_{k(i)}} - X| \leq 2^{-i}$ for large i.

Since i is arbitrary, $X_{n_{k(i)}} \xrightarrow{a.s.} X$.

If $X_n \not\xrightarrow{P} X$, there is some positive ε s.t.

$P[|X_{n_k} - X| > \varepsilon] > \varepsilon$ holds along some sequence $\{n_k\}$.

No subsequence of $\{X_{n_k}\}$ can converge to X i.p.,

and hence none can converge to X a.s.

#

- <u>Definition</u> (Norm on L_p, L_p Metric, $X_n \xrightarrow{L_p} X$)

  A norm on the space L_p is

  $$\|X\|_p := (E|X|^p)^{1/p}, \quad 1 \le p < \infty.$$

  L_p metric (induced by norm) is

  $$d_p(X,Y) = (E|X-Y|^p)^{1/p}, \quad 1 \le p < \infty.$$

  A sequence $\{X_n\}$ of r.v.'s converges in L_p to X, $X_n \xrightarrow{L_p} X$, if

  $$E(|X_n - X|^p) \to 0 \text{ as } n \to \infty, \quad 0 < p < \infty.$$

  That is, $d_p(X_n, X) = (E|X_n - X|^p)^{1/p} \to 0, \ 1 \le p < \infty.$

- <u>Example</u> ($\bar{X}_n \xrightarrow{L_p} \mu$, $p=2$)

  Suppose $\{X_n\}$ is an iid sequence of r.v.'s with $E(X_n)=\mu$, $Var(X_n)=\sigma^2<\infty$.

  Then $\bar{X}_n = \sum_{i=1}^{n} X_i/n \xrightarrow{L_2} \mu$.

  Pf: $E\left(\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right)^2 = \frac{1}{n^2} E(S_n - n\mu)^2 = \frac{1}{n^2} Var(S_n) = \frac{n\sigma^2}{n^2} \to 0$ as $n \to \infty$.

  #

- <u>Example</u> (L_p Convergence but NOT a.s. Convergence)

  $\Leftrightarrow$ <u>Example</u> (Converges i.p. but NOT a.s. Convergence)

  $$X_n(\omega) = \begin{cases} 1 & \text{if } \frac{j}{2^m} \le \omega < \frac{j+1}{2^m} \\ 0 & \text{o.w.} \end{cases}$$

  where $n = 2^m + j$, $0 \le j \le 2^m - 1$

  That is,

  $X_1 = 1_{[0,1]}$

  $X_2 = 1_{[0,\frac{1}{2}]}$, $X_3 = 1_{[\frac{1}{2},1]}$,

  $\vdots \qquad \vdots$

  - $X_n \xrightarrow{L_p} 0$ for any p.

    $E|X_1|^p = 1$, $E|X_2|^p = \frac{1}{2}$, $\cdots$, $E|X_4|^p = \frac{1}{4}$, $\cdots$, so $E|X_n|^p \to 0$

  - $X_n \overset{a.s.}{\nrightarrow} 0$

    for any $\omega \in [0,1]$, $X_n(\omega) \nrightarrow 0$.

    Since $X_n(\omega) = 1$ for infinitely many value of n. #

- **Theorem** (L_p Convergence implies Convergence i.p.)

For $p>0$, if $X_n \xrightarrow{L_p} X$ then $X_n \xrightarrow{P} X$.

Pf: By Chebychev's inequality, for $\varepsilon>0$,

$$P[|X_n - X| > \varepsilon] \le \frac{E(|X_n-X|^p)}{\varepsilon^p} \to 0 \text{ as } n\to\infty$$

since $X_n \xrightarrow{L_p} X$, i.e. $E(|X_n - X|^p) \to 0$.

#

- **Example** (Convergence $\begin{cases} a.s. \\ i.p. \end{cases}$ but NOT L_p Convergence)

cf Example (Converges a.s but NOT Converges Everywhere)

Let $(\Omega, \mathcal{B}, P) = ([0,1], \mathcal{B}[0,1], \lambda)$, λ is the Lebesgue measure.

$$X_n(\omega) = \begin{cases} 2^n & \text{if } 0 \le \omega < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \le \omega \le 1 \end{cases} \quad \text{i.e. } X_n(\omega) = 2^n 1_{[0,\frac{1}{n})}(\omega),\ 0 \le \omega \le 1$$

- $X_n \xrightarrow{P} 0$ since

  $P[|X_n| > \epsilon] = P([0, \frac{1}{n})) = \frac{1}{n} \to 0$ as $n \to \infty$.

- $X_n \not\xrightarrow{L_p} 0$ since

  $E(|X_n|^p) = 2^{np}\frac{1}{n} \to \infty$ as $n\to\infty$.

- $X_n \xrightarrow{a.s.} 0$ since

  if $N = \{0\}$, then $\omega \in N^c$ implies $X_n(\omega) \to 0$ as $n \to \infty$.

#

• **Definition** (Uniform Integrability)

A family $\{X_n, n \in \mathcal{N}\}$ of L_1 r.v.'s indexed by $\mathcal{N}$, an index set, is uniformly integrable (ui) if

$$\sup_{n\in\mathcal{N}} \int_{[|X_n|\geq \alpha]} |X_n|\,dP = \sup_{n\in\mathcal{N}} E\left[|X_n| 1_{[|X_n|\geq\alpha]}\right] \to 0 \text{ as } \alpha\to\infty$$

That is,

$$\lim_{\alpha\to\infty} \sup_{n\in\mathcal{N}} \int_{[|X_n|\geq\alpha]} |X_n|\,dP = 0.$$

That is,

$$\int_{[|X_n|\geq\alpha]} |X_n|\,dP \to 0 \text{ as } \alpha\to\infty \text{ uniformly in } n\in\mathcal{N}$$

• **Example** (Criteria for Various Families to Be ui.)

(1) If the family consists of one element, i.e. $T=\{1\}$, then $\{X_1\}$ is ui.

$$\int_{[|X_1|\geq a]} |X_1|\,dP \to 0 \text{ as } a\to\infty.$$

pf: $\because$ $|X_1| 1_{[|X_1|\geq a]} \downarrow 0$ as $a\to\infty$ & $|X_1| 1_{[|X_1|\geq a]} \leq |X_1| \in L_1$

$\therefore$ By DCT, $E(|X_1| 1_{[|X_1|\geq a]}) = \int_{[|X_1|\geq a]} |X_1|\,dP \to E(0) = 0$ as $a\to\infty$. #

(2) Dominated family

If there exists a dominating r.v. Y s.t. $|X_t| \leq Y \in L_1$, for all $t\in T$, then $\{X_t, t\in T\}$ is ui.

pf: $\sup_{t\in T} \int_{[|X_t|\geq a]} |X_t|\,dP \leq \int_{[|Y|\geq a]} |Y|\,dP \to 0$ as $a\to\infty$ from (1). #

(3) Finite family.

Suppose $X_t \in L_1$ for $t=1,\cdots,n$. Then $\{X_t, t\in\{1,\cdots,n\}\}$ is ui.

pf: $\because |X_t| \leq \sum_{i=1}^{n} |X_i| \in L_1$, then appling (2). #

(next pg cont.)

(4) More domination.

Suppose for each $t \in T$ that $X_t \in L_1$, $Y_t \in L_1$ and $|X_t| \le |Y_t|$.

Then if $\{Y_t\}$ is ui, so is $\{X_t\}$ ui.

Pf: $\sup_{t\in T} \int_{[|X_t| \ge a]} |X_t| \, dP \le \sup_{t\in T} \int_{[|Y_t| \ge a]} |Y_t| \, dP \to 0$ as $a \to \infty$. #

(5) Crystall Ball Condition.

For $p>0$, the family $\{|X_n|^p\}$ is ui if

$\sup_n E(|X_n|^{p+\delta}) < \infty$ for some $\delta > 0$.

Pf: $\sup_n \int_{[|X_n|^p \ge a]} |X_n|^p \, dP$

$$= \sup_n \int_{[|\frac{X_n}{a^{1/p}}| \ge 1]} |X_n|^p \cdot 1 \, dP$$

$$= \sup_n \int_{[\frac{|X_n|^\delta}{a^{\delta/p}} \ge 1]} |X_n|^p \cdot 1 \, dP$$

$$\le \sup_n \int_{[\frac{|X_n|^\delta}{a^{\delta/p}} > 1]} \frac{|X_n|^\delta}{a^{\delta/p}} |X_n|^p \, dP$$

$$\le a^{-\delta/p} \sup_n E(|X_n|^{p+\delta})$$

$\to 0$ as $a \to \infty$ $\because \sup_n E(|X_n|^{p+\delta}) < \infty$. #

Remark:

- The $|X_n|^p$, $p>1$, is ui implies the X_n is ui.
- By (5), the X_n is ui if $\sup_n E[|X_n|^{1+\delta}] < \infty$ for some $\delta > 0$.

(6) $\{X_n\}$ is ui if there is an integrable r.v. Z, $Z \in L_1$, s.t

$P[|X_n| \ge t] \le P[|Z| \ge t]$ for $t > 0$

Pf: $\cdot$ (2.1.10): $\int_{[|X|>\alpha]} |X| \, dP = \alpha P[|X|>\alpha] + \int_\alpha^\infty P[|X|>t] \, dt$ by $E(X) = \int P(X>t) \, dt$

$|X| \, 1_{[|X|>\alpha]}$

- $\int_{[|X_n|>\alpha]} |X_n| \, dP = \alpha P[|X_n|>\alpha] + \int_\alpha^\infty P[|X_n|>t] \, dt$

$$\le \alpha P[|Z|>\alpha] + \int_\alpha^\infty P[|Z|>t] \, dt$$

$$= \int_{[|Z|>\alpha]} |Z| \, dP$$

- $\sup_n \int_{[|X_n|>\alpha]} |X_n| \, dP \le \int_{[|Z|>\alpha]} |Z| \, dP \to 0$ for $\alpha \to \infty$.

Hence the X_n is ui. #

• <u>Theorem</u> (Characterization of ui)

Let $\{X_t, t\in T\}$ be L_1 r.v.'s. This family is ui iff "both"

(a) "Uniformly continuous":

For all $\epsilon>0$, there exists $\delta=\delta(\epsilon)$, indep of A, s.t.

$$\sup_{t\in T}\int_A |X_t|\,dP<\epsilon \text{ if } P(A)<\delta,\ \forall A\in\Omega.$$

and

(b) "Uniformly bounded mean":

$$\sup_{t\in T} E(|X_t|)<\infty$$

Pf: "$\Rightarrow$" Suppose $\{X_t\}$ is ui. For any $X\in L_1$ and $a>0$,

$$\int_A |X|\,dP=\int_{A[|X|\le a]}|X|\,dP+\int_{A[|X|\ge a]}|X|\,dP$$

$$\le a\,P(A)+\int_{[|X|\ge a]}|X|\,dP$$

So

$$\sup_{t\in T}\int_A |X_t|\,dP\le a\,P(A)+\sup_{t\in T}\int_{[|X_t|\ge a]}|X_t|\,dP$$

Insert $A=\Omega$, pick "a" so large s.t. $\sup_{t\in T}\int_{[|X_t|\ge a]}|X_t|\,dP<1$.

$$\sup_{t\in T}\int_\Omega |X_t|\,dP=\sup_{t\in T}E(|X_t|)$$

$$\le a\,P(A)+1<\infty$$

Hence we get (b).

To get (a) pick "a" so large that

$$\sup_{t\in T}\int_{[|X_t|>a]}|X_t|\,dP\le\frac{\epsilon}{2}.$$

If $P(A)\le\dfrac{\epsilon/2}{a}\equiv\delta$,

then $\sup_{t\in T}\int_A |X_t|\,dP\le\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$

(next pg. cont.)

"$\Leftarrow$" Suppose (a), (b) hold.

Chebychev's inequality implies that

$$\sup_{t\in T} P[|X_t|>a] \le \sup_{t\in T} \frac{E(|X_t|)}{a} = const/a \text{ from (b).}$$

Now we apply (a): Given $\epsilon>0$, there exists δ s.t. when $P(A)<\delta$, we have $\sup_{t\in T}\int_A |X_t|\,dP < \epsilon$

Pick "a" so large that $\sup_{t\in T} P[|X_t|>a] < \delta$, that is, $P[|X_t|>a]<\delta$ for all $t\in T$.

Then we get

$$\sup_{t\in T}\int_{[|X_t|\ge a]} |X_t|\,dP < \epsilon$$

which is the ui property.

#

- <u>Example</u> ($\sup_n E(|X_n|) < \infty$ but NOT ui)

Let $\{X_n\}$ be a seq of r.v.'s s.t.

$$X_n = \begin{cases} n & \text{w.p. } p \\ 0 & \text{w.p. } q = 1-p \end{cases}$$

Find a value of $p = p_n$ s.t.

$1 = E(X_n) = np + 0$, $p = \frac{1}{n}$, $q = 1 - \frac{1}{n}$.

- $\sup_n E(|X_n|) < \infty$.

Since $X_n \geq 0$, $E(|X_n|) = E(X_n) = 1 \Rightarrow \sup_n E(|X_n|) = 1 < \infty$.

- The family is not ui.

$$\int_{[|X_n| \geq a]} |X_n| \, dP = \begin{cases} 1 & \text{if } |X_n| = X_n = n \geq a \\ 0 & \text{if o.w.} \end{cases}$$

Thus,

$$\sup_n \int_{[|X_n| \geq a]} |X_n| \, dP = 1 \not\to 0.$$

- We check (a) does not hold: NOT uniformly continuous.

$P[|X_n| > a] = P[X_n > a] = \frac{1}{n} \to 0$ as $a \leq n \to \infty$

Pick "a" so large s.t. $P[|X_n| > a] < \delta$ for all n,

but we get, $\exists \epsilon > 0$, s.t. $\sup_n \int_{[|X_n| > a]} |X_n| \, dP = 1 > \epsilon$, not $< \epsilon$. #

- **Theorem** (Addition, Product, and ui)

(a) If $\{X_n, n\geq 1\}$ and $\{Y_n, n\geq 1\}$ are ui, then so is $\{X_n+Y_n, n\geq 1\}$.

(b) Let p and q are conjugate: $1<p,q<\infty$, $p^{-1}+q^{-1}=1$.

If $\{|X_n|^p, n\geq 1\}$ and $\{|Y_n|^q, n\geq 1\}$ are ui, then so is $\{X_nY_n, n\geq 1\}$.

Pf: (a) We want to show that

$$\lim_{\alpha\to\infty}\sup_n E|X_n+Y_n|1_{[|X_n+Y_n|>2\alpha]}\to 0.$$

$\because$ $|X_n+Y_n|1_{[|X_n+Y_n|>2\alpha]}$, $\alpha>0$.

$\leq 2(|X_n|\vee|Y_n|)I_{[2(|X_n|\vee|Y_n|)>2\alpha]}$ $\because |X_n+Y_n|\leq 2|X_n|\vee|Y_n|$

$\leq 2|X_n|1_{[|X_n|>\alpha]}+2|Y_n|1_{[|Y_n|>\alpha]}$

Since $2(|X_n|\vee|Y_n|)I_{[|X_n|\vee|Y_n|>\alpha]}=2|X_n|I_{[|X_n|>\alpha]}$ if $|X_n|\vee|Y_n|=|X_n|$

$=2|Y_n|I_{[|Y_n|>\alpha]}$ if $|X_n|\vee|Y_n|=|Y_n|$

$\therefore$ $\sup_n E(|X_n+Y_n|1_{[|X_n+Y_n|>2\alpha]})$

$\leq 2\sup_n E(|X_n|1_{[|X_n|>\alpha]})+2\sup_n E(|Y_n|1_{[|Y_n|>\alpha]})\to 0$ as $\alpha\to\infty$.

Since $\{X_n\}$ and $\{Y_n\}$ are ui

(b) We want to show that

$$\lim_{\alpha\to\infty}\sup_n E|X_nY_n|1_{[|X_nY_n|>\alpha]}\to 0.$$

$\because$ $|X_nY_n|1_{[|X_nY_n|>\alpha]}\leq |X_n||Y_n|1_{[|X_n|>\sqrt{\alpha}]}+|X_n||Y_n|1_{[|Y_n|>\sqrt{\alpha}]}$

Since $\alpha<|X_nY_n|<|X_n|^2$ if $|Y_n|<|X_n|$

$<|Y_n|^2$ if $|X_n|<|Y_n|$.

By Hölder's inequality,

$E|X_nY_n|1_{[|X_nY_n|>\alpha]}\leq \|X_n1_{|X_n|>\sqrt{\alpha}}\|_p\|Y_n\|_q+\|X_n\|_p\|Y_n1_{|Y_n|>\sqrt{\alpha}}\|_q$

$\leq \|X_n1_{|X_n|>\sqrt{\alpha}}\|_p\sup_n\|Y_n\|_q+\sup_n\|X_n\|_p\|Y_n1_{|Y_n|>\sqrt{\alpha}}\|_q$

$\to 0$ as $\alpha\to\infty$ uniformly in n.

Since $\{|X_n|^p\}$ and $\{|Y_n|^q\}$ are ui, that is, for $\{X_n\}$,

$\sup_n E|X_n|^p<\infty \Rightarrow \sup_n(E|X_n|^p)^{1/p}=\sup_n\|X_n\|_p<\infty$, and

$\sup_n\|X_n1_{|X_n|>\sqrt{\alpha}}\|_p=\sup_n E(|X_n|^p1_{|X_n|>\sqrt{\alpha}})^{1/p}\to 0$ as $\alpha\to\infty$. #

- **Theorem** (Convergence Mode, ui, and Moment Convergence)

(1) If $X_n \xrightarrow{a.s.} X$ or $X_n \xrightarrow{P} X$. Let $r>0$ s.t. $E|X_n|^r<\infty$ for all n.

The following are equivalent:

(a) $\{|X_n|^r, n\geq 1\}$ is ui;

(b) $X_n \xrightarrow{L_r} X$;

(c) $E|X_n|^r \to E|X|^r$.

(2) If $X_n \Rightarrow X$. Let $r>0$ s.t. $E|X_n|^r<\infty$ for all n.

The following are equivalent:

(d) $\{|X_n|^r, n\geq 1\}$ is ui;

(e) $E|X_n|^r \to E|X|^r$

Remark: • $X_n \xrightarrow{L_r} X$ implies that $E[|X_n|^r] \to E[|X|^r]$ by (b) $\Rightarrow$ (c).

• $X_n \Rightarrow X$ and $\{X_n\}$ is ui $\not\Rightarrow$ $X_n \xrightarrow{L_r} X$

Pf: (a) $\Rightarrow$ (b). By Fatou's lemma and uniform bounded $E(|X_n|)$ for ui

$$E|X|^r = E(\liminf_{n\to\infty} |X_n|^r) \leq \liminf_{n\to\infty} E|X_n|^r \leq \sup_n E|X_n|^r < \infty.$$

Thus $X \in L_r$, too.

$\because \{|X_n|^r, n\geq 1\}$ is ui, $\{|X|^r\}$ is ui $\therefore \{|X_n-X|^r, n\geq 1\}$ is ui, too. (Result)

$$E|X_n-X|^r = E|X_n-X|^r 1_{[|X_n-X|^r\leq\varepsilon]} + E|X_n-X|^r 1_{[|X_n-X|^r>\varepsilon]}$$

$$\leq \varepsilon + \int_{[|X_n-X|^r>\varepsilon]} |X_n-X|^r dP$$

$\because X_n \xrightarrow{a.s.} X$ or $X_n \xrightarrow{P} X$, $\therefore P[|X_n-X|^r>\varepsilon] \to 0$ as $n\to\infty$

and since uniform continuity for ui

$\therefore \limsup_n E|X_n-X|^r \leq \varepsilon + 0 = \varepsilon$

This being true for arbitrary ε, the result follows: $X_n \xrightarrow{L_r} X$.

(next pg. cont).

(b) $\Rightarrow$ (c).

- For $0<r\le 1$. By the Cr-inequality,

$$E(|X_n|^r) \le E|X_n-X|^r + E|X|^r$$

Change the roles of X_n and X, we get

$$E(|X|^r) \le E|X_n-X|^r + E|X_n|^r$$

$$\Rightarrow \left|E|X_n|^r - E|X|^r\right| \le E|X_n-X|^r \to 0 \quad \because X_n \xrightarrow{L_r} X.$$

- For $r>1$, by Minkowski's inequality the result follows.

(c) $\Rightarrow$ (a)

Let $A>0$ and consider a bounded, uniformly continuous ft f_A:

$$f_A(x) \begin{cases} = |x|^r & \text{for } |x|^r \le A \\ \le |x|^r & \text{for } A<|x|^r\le A+1 \\ = 0 & \text{for } |x|^r > A+1 \end{cases}$$

Hence we have

$$\liminf_{n\to\infty} \int_{|X_n|^r\le A+1} |X_n|^r \, dP$$

$$\ge \lim_{n\to\infty} E[f_A(X_n)] = E[f_A(X)] \quad \because X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \Rightarrow X.$$

Note. we just need $X_n \Rightarrow X$.

$$\ge \int_{|X|^r\le A} |X|^r \, dP$$

Substracting from (c), we get

$$\limsup_{n\to\infty} \int_{|X_n|^r>A+1} |X_n|^r \, dP \le \int_{|X|^r>A} |X|^r \, dP \to 0 \text{ as } A\to\infty$$

This means: for any $\epsilon>0$, there exists $A_0=A_0(\epsilon)$ and $n_0=n_0(A_0(\epsilon))$ s.t.

$$\sup_{n>n_0} \int_{|X_n|^r>A+1} |X_n|^r \, dP < \epsilon, \text{ provided } A>A_0.$$

Since each $|X_n|^r$ is integrable, there exists $A_1=A_1(\epsilon)$ s.t.

the supremum above may be taken over all $n\ge 1$ provided that $A>A_0 \vee A_1$

This establish (a), and complete the proof of the theorem.

(2) $\because X_n \Rightarrow X$, $\therefore X_n^r \Rightarrow X^r$.

By Skorohod's thm: $\exists\, Y_n^r \overset{d}{=} X_n^r$, $Y^r \overset{d}{=} X^r$ s.t. $Y_n^r \xrightarrow{a.s.} Y^r$.

Since $\{X_n^r\}$ is ui, so is $\{Y_n^r\}$.

From (1): (a) $\Rightarrow$ (c), we get (d) $\Rightarrow$ (e): $E|X_n|^r = E|Y_n|^r \to E|Y|^r = E|X|^r$.

From the proof of (1): (c) $\Rightarrow$ (a), we get (e) $\Rightarrow$ (d).

#

• <u>Theorem</u> (A Necessary and Sufficient Condition for Convergence i.p.)

Let $\{X_n, X, n\ge 1\}$ be r.v.'s. Then

$$X_n \xrightarrow{P} X \text{ as } n\to\infty \text{ iff } E\left(\frac{|X_n-X|}{1+|X_n-X|}\right)\to 0 \text{ as } n\to\infty.$$

Pf: Since $\frac{|x|}{1+|x|}$ is bounded by 1, i.e. $0<\frac{|x|}{1+|x|}<1$,

then $\left\{\frac{|X_n-X|}{1+|X_n-X|}\right\}$ is ui. Since

$$\frac{|X_n-X|}{1+|X_n-X|}\xrightarrow{P} 0 \text{ and } \left\{\frac{|X_n-X|}{1+|X_n-X|}\right\} \text{ is ui}$$

$$\Leftrightarrow \frac{|X_n-X|}{1+|X_n-X|}\xrightarrow{L_1} 0 \quad\Leftrightarrow\quad E\left(\frac{|X_n-X|}{1+|X_n-X|}\right)\to 0.$$

It suffices to show that

$$\frac{|X_n-X|}{1+|X_n-X|}\xrightarrow{P} 0 \Leftrightarrow X_n-X\xrightarrow{P} 0.$$

Since

$$P\left[\frac{|X_n-X|}{1+|X_n-X|}>\epsilon\right]\le P\left[|X_n-X|>\epsilon\right]\le P\left[\frac{|X_n-X|}{1+|X_n-X|}>\frac{\epsilon}{1+\epsilon}\right]$$

$$\because |X_n-X|>\epsilon \Rightarrow |X_n-X|+\epsilon|X_n-X|>\epsilon+\epsilon|X_n-X|$$

$$\Rightarrow |X_n-X|(1+\epsilon)>\epsilon(1+|X_n-X|)$$

$$\Rightarrow \frac{|X_n-X|}{1+|X_n-X|}>\frac{\epsilon}{1+\epsilon},$$

the result follows.

#

• Theorem (Addition preserves $\begin{cases} a.s. \\ i.p. \\ L_2, L_1 \end{cases}$ Convergence)

If $\{X_n, X, Y_n, Y, n\geq 1\}$ are r.v.'s.

(a) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then $X_n+Y_n \xrightarrow{a.s.} X+Y$.

(b) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_n+Y_n \xrightarrow{P} X+Y$

(c) If $X_n \xrightarrow{L_2} X$ and $Y_n \xrightarrow{L_2} Y$, then $X_n+Y_n \xrightarrow{L_2} X+Y$

(d) If $X_n \xrightarrow{L_1} X$ and $Y_n \xrightarrow{L_1} Y$, then $X_n+Y_n \xrightarrow{L_1} X+Y$

} general p is ok.

Note. If $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$, $X_n+Y_n \not\Rightarrow X+Y$.

We only get Slutsky's thm:

If $X_n \Rightarrow X$ and $Y_n \xrightarrow{P} c$, then $X_n+Y_n \xrightarrow{d} X+c$.

Pf: (a) $\exists N_X \in \mathcal{B}, N_Y \in \mathcal{B}$ with $P(N_X)=0$, $P(N_Y)=0$ s.t.

for $\omega \in N_X^c$, $X_n(\omega) \to X(\omega)$; for $\omega \in N_Y^c$, $Y_n(\omega) \to Y(\omega)$

We first check $P((N_X^c \cap N_Y^c)^c) = 0$

$P((N_X^c \cap N_Y^c)^c) = P(N_X \cup N_Y) \leq P(N_X) + P(N_Y) = 0$

Set $N = (N_X^c \cap N_Y^c)^c$, then $P(N)=0$ and for $\omega \in N^c = N_X^c \cap N_Y^c$,

$X_n(\omega) + Y_n(\omega) \to X(\omega) + Y(\omega)$.

Hence $X_n+Y_n \xrightarrow{a.s.} X+Y$.

(b) $\because$ $[|X_n-X| \leq \frac{\epsilon}{2}] \cap [|Y_n-Y| \leq \frac{\epsilon}{2}]$

$\Rightarrow [|X_n-X| + |Y_n-Y| \leq \epsilon] \Rightarrow [|(X_n-X)+(Y_n-Y)| \leq \epsilon]$

$\therefore$ $[|(X_n-X)+(Y_n-Y)| > \epsilon] \Rightarrow [|X_n-X| > \frac{\epsilon}{2}] \cup [|Y_n-Y| > \frac{\epsilon}{2}]$

$\Rightarrow P[|(X_n-X)+(Y_n-Y)| > \epsilon] \leq P[|X_n-X| > \frac{\epsilon}{2}] + P[|Y_n-Y| > \frac{\epsilon}{2}] \to 0$.

(c) By Minkowski's thm for general p

$\|(X_n-X)+(Y_n-Y)\|_p \leq \|X_n-X\|_p + \|Y_n-Y\|_p \to 0$. #

- **Theorem** (Product preserves $\begin{cases} a.s. \\ i.p. \\ L_2, \text{only} \end{cases}$ Convergence)

(a) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then $X_nY_n \xrightarrow{a.s.} XY$.

(b) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_nY_n \xrightarrow{P} XY$

(c) If $X_n \xrightarrow{L_2} X$ and $Y_n \xrightarrow{L_2} Y$, then $X_nY_n \xrightarrow{L_1} XY$

Note. $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$ but $X_nY_n \not\Rightarrow XY$

Only Slutsky's thm: $X_n \Rightarrow X$ and $Y_n \xrightarrow{P} c$, then $X_nY_n \Rightarrow cX$.

Pf: (a) $\exists N_X \in \mathcal{B}, N_Y \in \mathcal{B}$ with $P(N_X)=0$, $P(N_Y)=0$ s.t.

for $\omega \in N_X^c$, $X_n(\omega) \to X(\omega)$; for $\omega \in N_Y^c$, $Y_n(\omega) \to Y(\omega)$

Set $N = (N_X^c \cap N_Y^c)^c$, we know that $P(N)=0$ and

for $\omega \in N^c$, $X_n(\omega)Y_n(\omega) \to X(\omega)Y(\omega)$

Hence $X_nY_n \xrightarrow{a.s.} XY$.

(b) $\because X_n \xrightarrow{P} X$. there exists a subsequence $\{n_k'\} \subset \{n_k\}$ s.t. $X_{n_k'} \xrightarrow{a.s.} X$

$\because Y_n \xrightarrow{P} Y$, there exists a subsequence $\{n'_{k(i)}\} \subset \{n_k'\}$ s.t.

$X_{n'_{k(i)}} \xrightarrow{a.s.} X$ and $Y_{n'_{k(i)}} \xrightarrow{a.s.} Y$.

Thus $X_{n'_{k(i)}} Y_{n'_{k(i)}} \xrightarrow{a.s.} XY$ from (a).

Hence $X_nY_n \xrightarrow{P} XY$.

(c). $E|X_nY_n - XY| = E|X_nY_n - X_nY + X_nY - XY|$

$\leq E[|X_n||Y_n-Y|] + E[|X_n-X||Y|]$ by triangle inequality

$\leq \|X_n\|_2 \|Y_n - Y\|_2 + \|X_n - X\|_2 \|Y\|_2$ by Schwartz's inequality

$\because X_n \xrightarrow{L_2} X$, $\|X_n\|_2 = (E|X_n|^2)^{\frac{1}{2}} \to (E|X|^2)^{\frac{1}{2}} < \infty$ and

$Y_n \xrightarrow{L_2} Y$, $\|Y_n - Y\|_2 = (E|Y_n - Y|^2)^{\frac{1}{2}} \to 0$,

$\therefore$ The first term converges to 0.

$\because X_n \xrightarrow{L_2} X$, $\|X_n - X\|_2 \to 0$ and

$Y_n \xrightarrow{L_2} Y$, $Y \in L_2$ by definition, thus $\|Y\|_2 < \infty$,

$\therefore$ The second term converges to 0.

#

• <u>Corollary</u> (Continuous ft preserves the $\left\{\begin{matrix} \text{a.s.} \\ \text{i.p.} \\ \text{in dist.}\end{matrix}\right.$ Convergence)

(a) If $X_n \xrightarrow{a.s.} X$, $g: \mathbb{R} \mapsto \mathbb{R}$ is continuous ft, then $g(X_n) \xrightarrow{a.s.} g(X)$

(b) If $X_n \xrightarrow{P} X$, $g: \mathbb{R} \mapsto \mathbb{R}$ is continuous ft, then $g(X_n) \xrightarrow{P} g(X)$.

(c) If $X_n \Rightarrow X$, $g: \mathbb{R} \mapsto \mathbb{R}$ is a continuous ft, then $g(X_n) \Rightarrow g(X)$

General case: if $g: \mathbb{R} \mapsto \mathbb{R}$ with $P[X \in D_g] = 0$. This is the mapping thm.

Pf: (a) If $X_n \xrightarrow{a.s.} X$, there exists $N \in \mathcal{B}$ with $P(N)=0$ s.t.

if $\omega \in N^c$: $X_n(\omega) \to X(\omega)$, in $\mathbb{R}$,

and hence by continuity,

if $\omega \in N^c$: $g(X_n(\omega)) \to g(X(\omega))$.

This is a.s. convergence of $\{g(X_n)\}$.

(b) Let $\{g(X_{n_k})\}$ be a subsequence of $\{g(X_n)\}$.

To show $g(X_n) \xrightarrow{P} g(X)$ it suffices to find an a.s. convergent further subsequence $\{g(X_{n_{k(i)}})\}$.

Since $X_n \xrightarrow{P} X$, we know $\{X_{n_k}\}$ has some a.s. convergent subsequence $\{X_{n_{k(i)}}\}$ s.t. $X_{n_{k(i)}} \xrightarrow{a.s.} X$.

From (a) we get $g(X_{n_{k(i)}}) \xrightarrow{a.s.} g(X)$,

and the result follows.

(c) We show the general case holds.

If $X_n \Rightarrow X$. by Skorohod's thm,

$\exists\, Y_n, n\geq 1, Y$ with $Y_n \stackrel{d}{=} X_n$, $Y \stackrel{d}{=} X$ s.t. $Y_n \xrightarrow{a.s.} Y$.

If $Y(\omega) \notin D_g$, $g(Y_n(\omega)) \xrightarrow{a.s.} g(Y(\omega))$, thus $g(Y_n(\omega)) \Rightarrow g(Y(\omega))$.

Hence $g(X_n) \stackrel{d}{=} g(Y_n) \Rightarrow g(Y) \stackrel{d}{=} g(X)$.

#

- <u>Theorem</u> (Modulus Inequality)

  If $X \in L_1$, $|E(X)| \le E(|X|)$

  Pf: $|E(X)| = |E(X^+) - E(X^-)| \le |E(X^+)| + |E(X^-)| = E(X^+) + E(X^-) = E(|X|)$

  Pf 2: $-|X| \le X \le |X| \Rightarrow -E|X| \le E(X) \le E|X| \Rightarrow |E(X)| \le E(|X|)$.

  #.

- <u>Theorem</u> (Markov Inequality)

  If $X \in L_1$. For any $\lambda > 0$,

  $$P[|X| \ge \lambda] \le \frac{E(|X|)}{\lambda}$$

  Pf: $1 \cdot 1_{[\frac{|X|}{\lambda} \ge 1]} \le \frac{|X|}{\lambda} 1_{[\frac{|X|}{\lambda} \ge 1]} \le \frac{|X|}{\lambda}$

  Taking expectations through the inequalities.

  #

- <u>Theorem</u> (Chebychev Inequality)

  If $X \in L_2$. For $\lambda > 0$,

  $$P[|X - E(X)| > \lambda] \le \frac{Var(X)}{\lambda^2}.$$

  Pf: $P[|X - E(X)| > \lambda] = P[(X - E(X))^2 > \lambda^2]$

  $\le \frac{1}{\lambda^2} E(X - E(X))^2$ by Markov inequality.

  $= \frac{1}{\lambda^2} Var(X)$.

  #

- <u>Theorem</u> (2^p-Inequality and C_p-Inequality)

(a) 2^p-inequality.

$$E(|X+Y|^p) \le 2^p(E|X|^p + E|Y|^p), \text{ for } 0<p<\infty.$$

(b) C_p-inequality.

$$E(|X+Y|^p) \le C_p(E|X|^p + E|Y|^p)$$

where $C_p = \begin{cases} 1 & \text{if } 0<p\le 1 \\ 2^{p-1} & \text{if } 1<p<\infty \end{cases}$

Pf: (a) For $p>0$,

$$|X+Y|^p \le (|X|+|Y|)^p \le (2\cdot|X|\vee|Y|)^p \le 2^p(|X|^p+|Y|^p)$$

Hence

$$E|X+Y|^p \le 2^p(E|X|^p + E|Y|^p).$$

(b) • For $0<p\le 1$,

Since $x^{1/p} \le x$ for any $0<x<1$. Hence

$$\left(\frac{|X|^p}{|X|^p+|Y|^p}\right)^{1/p} + \left(\frac{|Y|^p}{|X|^p+|Y|^p}\right)^{1/p} \le \frac{|X|^p}{|X|^p+|Y|^p} + \frac{|Y|^p}{|X|^p+|Y|^p} = 1$$

$$\Rightarrow |X|+|Y| \le (|X|^p+|Y|^p)^{1/p} \Rightarrow (|X|+|Y|)^p \le |X|^p+|Y|^p$$

So $E|X+Y|^p \le E(|X|+|Y|)^p \le (E|X|^p + E|Y|^p)$.

• For $1<p<\infty$,

Since $\varphi(x) = |x|^p$ is a convex function, i.e. $\varphi(rx+(1-r)y) \le r\varphi(x)+(1-r)\varphi(y)$

$$\left|\frac{|X|+|Y|}{2}\right|^p \le \frac{1}{2}|X|^p + \frac{1}{2}|Y|^p$$

$$(|X|+|Y|)^p \le 2^{p-1}|X|^p+|Y|^p$$

So $E|X+Y|^p \le E(|X|+|Y|)^p \le 2^{p-1}(E|X|^p+E|Y|^p)$.

#.

• <u>Theorem</u> (Hölder's Inequality)

For $1<p,q<\infty$, p and q are conjugate indices if $\frac{1}{p}+\frac{1}{q}=1$,

and if $E(|X|^p)<\infty$, $E(|Y|^q)<\infty$, i.e. $X\in L^p$, $Y\in L^q$

then XY is integrable and

$$|E(XY)|\le E(|XY|)\le (E|X|^p)^{1/p}(E|Y|^q)^{1/q}\equiv \|X\|_p\cdot\|Y\|_q$$

Remark: For $p=q=2 \Rightarrow$ Schwartz's Inequality

Pf: • If $E(|X|^p)=0$, then $X=0$ a.s. Thus $XY=0$ a.s and $E|XY|=0$

Similarly if $E(|Y|^q)=0$.

• So suppose the right side is positive.

If a and b are positive, there exist s and t s.t.

$$a=e^{\frac{s}{p}},\quad b=e^{\frac{t}{q}}$$

Since e^x is convex,

$$e^{\frac{1}{p}s+\frac{1}{q}t}\le \frac{1}{p}e^s+\frac{1}{q}e^t$$

$$ab\le \frac{a^p}{p}+\frac{b^q}{q}$$

Replace a by $|X|/\|X\|_p$, b by $|Y|/\|Y\|_q$

$$\frac{|XY|}{\|X\|_p\|Y\|_q}\le p^{-1}\left(\frac{|X|^p}{\|X\|_p}\right)^p+q^{-1}\left(\frac{|Y|^q}{\|Y\|_q}\right)^q$$

Taking expectations,

$$\frac{E|XY|}{\|X\|_p\|Y\|_q}\le p^{-1}+q^{-1}=1 \Rightarrow E|XY|\le \|X\|_p\cdot\|Y\|_q$$

#

- Theorem (Schwartz Inequality)

Suppose $X, Y \in L_2$. Then

$$|E(XY)| \le E(|XY|) \le \sqrt{E(X^2)E(Y^2)}$$

Pf: By Hölder's inequality for $p = q = 2$

Pf 2: For any $t \in \mathbb{R}$,

$$0 \le E(X-tY)^2 = E(X^2) - 2t\,E(XY) + t^2 E(Y^2) \equiv g(t).$$

$$g'(t) = -2E(XY) + 2t\,E(Y^2) = 0$$

$$\Rightarrow t = \frac{E(XY)}{E(Y^2)}.$$

Substitute this value of t,

$$0 \le E(X^2) - 2\left(\frac{E(XY)}{E(Y^2)}\right)E(XY) + \left(\frac{E(XY)}{E(Y^2)}\right)^2 E(Y^2)$$

$$= E(X^2) - 2(E(XY))^2/E(Y^2) + (E(XY))^2/E(Y^2)$$

$$= E(X^2) - (E(XY))^2/E(Y^2)$$

$$\Rightarrow E(XY) \le \sqrt{E(X^2)E(Y^2)}$$

Set $X \equiv |X|$, $Y \equiv |Y|$,

$$E(|XY|) \le \sqrt{E(X^2)E(Y^2)}.$$

#

• <u>Theorem</u> (Minkowski Inequality).

For $1 \le p < \infty$, suppose $X, Y \in L_p$. Then $X+Y \in L_p$ and

$$\|X+Y\|_p \le \|X\|_p + \|Y\|_p$$

Pf: • To show $X+Y \in L_p$, i.e. L_p is closed under addition,

$$E[|X+Y|^p] \le 2^p \left(E[|X|^p] + E[|Y|^p]\right) < \infty \quad \text{by } 2^p\text{-inequality}.$$

• If $p=1$, Minkowski inequality holds from the triangle inequality

So assume $1 < p < \infty$. Let q be conjugate to p s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

Thus $p-1 = p/q$.

$$\begin{aligned}
\|X+Y\|_p^p = E(|X+Y|^p) &= E\left(|X+Y||X+Y|^{p-1}\right) \\
&\le E\left(|X||X+Y|^{p-1}\right) + E\left(|Y||X+Y|^{p-1}\right) \quad \text{by triangle inequality} \\
&\le \|X\|_p \||X+Y|^{p/q}\|_q + \|Y\|_p \||X+Y|^{p/q}\|_q \quad \text{by Hölder inequality.} \\
&= \left(\|X\|_p + \|Y\|_p\right) \||X+Y|^{p/q}\|_q \\
&= \left(\|X\|_p + \|Y\|_p\right) \left(E(|X+Y|^p)\right)^{q^{-1}} \\
&= \left(\|X\|_p + \|Y\|_p\right) \left(\|X+Y\|_p\right)^{p/q} \\
&= \left(\|X\|_p + \|Y\|_p\right) \left(\|X+Y\|_p\right)^{p-1}
\end{aligned}$$

Thus

$$\|X+Y\|_p \le \|X\|_p + \|Y\|_p.$$

#

• Remark:

Minkowski's inequality is triangle inequality in L^p space.

• <u>Theorem</u> (Lyapounov's Inequality)

Let $0<\alpha<\beta$, then

$$(E|X|^{\alpha})^{1/\alpha} \le (E|X|^{\beta})^{1/\beta}, \text{ i.e. } \|X\|_{\alpha} \le \|X\|_{\beta}.$$

Note 1. From Lyapounov's inequality,

$$X \in L_{\beta} \Rightarrow X \in L_{\alpha} \text{ for } 0<\alpha<\beta$$

2. $X_n \xrightarrow{L_\beta} X \Rightarrow X_n \xrightarrow{L_\alpha} X$ for $0<\alpha<\beta$.

Since $E(|X_n-X|^{\alpha}) \le E(|X_n-X|^{\beta})^{\alpha/\beta} \to 0$ if $X_n \xrightarrow{L_\beta} X$.

Pf: By Hölder's inequality,

for $p,q>1$, p and q are conjugate, $\frac{1}{p}+\frac{1}{q}=1$. Then

$$E(|XY|) \le \|X\|_p \cdot \|Y\|_q$$

Set $Y\equiv 1$, $E(|X|) \le \|X\|_p = (E|X|^p)^{1/p}$

Set $X \equiv |X|^{\alpha}$, $E(|X|^{\alpha}) \le (E|X|^{\alpha p})^{1/p}$

Let $\alpha p=\beta$, $1<p=\frac{\beta}{\alpha} \Rightarrow \beta>\alpha$

$$E(|X|^{\alpha}) \le (E(|X|^{\beta}))^{\alpha/\beta}.$$

$\Rightarrow (E|X|^{\alpha})^{1/\alpha} \le (E|X|^{\beta})^{1/\beta}$, that is, $\|X\|_{\alpha}<\|X\|_{\beta}$, $0<\alpha<\beta$.

#

• <u>Theorem</u> (Jensen's Inequality)

If φ is convex on an interval containg the range of X, and if $E(|X|)<\infty$ and $E(|\varphi(X)|)<\infty$. Then

$$\varphi(E[X]) \le E[\varphi(X)]$$

Pf: Let $l(X)=t(X-E[X])+\varphi(E[X])$ be a supporting line through $(E[X], \varphi(E[X]))$, which is a line lying entirely under the graph of $\varphi(X)$, $l(X)\le\varphi(X)$

$$E[\varphi(X)] \ge E[l(X)] = l(E[X]) = \varphi(E[X]).$$

#