Project 1

Public Finance in Macroeconomics

Handed in by the **Heterogeneous Geeks**

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Public Finance in Macroeconomics: Heterogenous Agent Models

at the Graduate School of Economics, Finance and Management

Problem 1: Simple Preliminaries

- The first two derivatives of the utility function are given by

$$u'(c) = (1-\sigma) \cdot \frac{1}{1-\sigma} c^{1-\sigma-1} = c^{-\sigma}$$

and

$$u''(c) = -\sigma c^{-\sigma-1}.$$

Therefore, the coefficient of relative risk aversion is

$$-\frac{u''(c)}{u'(c)} \cdot c = \frac{\sigma c^{-\sigma-1}}{c^{-\sigma}} \cdot c = \sigma.$$

The coefficient of relative risk aversion measures the risk aversion of the individual. In case of a CRRA utility function, it is constant and does not depend on the consumption level. Generally, it can be interpreted as the percentage change in marginal utility for consumption changes by one percent.

- The addition of a constant term is a monotonic transformation, consequently, the ordinal ranking is **not** changed. As a result, the utility function represents the same underlying preferences.

- Proof: $\log c$ is a special case of this CRRA utility function:
It holds that

$$\lim_{\sigma \to 0} \frac{c^{1-\sigma} - 1}{1-\sigma} = \frac{0}{0} \quad \forall t, s^t.$$

Thus, we can apply l'Hopital's rule. That is

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma} = \lim_{\sigma \to 1} \frac{\partial [c^{1-\sigma} - 1]}{\partial \sigma} \frac{\partial \sigma}{\partial 1 - \sigma} = \lim_{\sigma \to 1} \frac{-c^{1-\sigma} \log c}{-1} = \log c.$$

- The consumption growth pattern is such that

$$\bar{c}_0 = (1+g) \cdot c_0 \qquad \bar{c}_1 = (1+g) \cdot c_1$$

Consequently,

$$u(\bar{c}) = \frac{\bar{c}_0^{1-\sigma}}{1-\sigma} + \beta \frac{\bar{c}_1^{1-\sigma}}{1-\sigma} = \frac{[(1+g) \cdot c_0]^{1-\sigma}}{1-\sigma} + \beta \frac{[(1+g) \cdot c_1]^{1-\sigma}}{1-\sigma}$$

holds. Collecting terms yields

$$u(\bar{c}) = \frac{(1+g)^{1-\sigma} \cdot c_0^{1-\sigma}}{1-\sigma} + \beta \frac{(1+g)^{1-\sigma} \cdot c_1^{1-\sigma}}{1-\sigma} = (1+g)^{1-\sigma} \left[\frac{c_0^{1-\sigma}}{1-\sigma} + \beta \frac{c_1^{1-\sigma}}{1-\sigma} \right] = (1+g)^{1-\sigma} u(c).$$

This type of preference is called homothetic. In this case the degree of homogeneity is $1-\sigma$.

- The presented monotonic transformation is not homothetic.

  Again assume the following consumption growth pattern:

$$\bar{c}_0 = (1+g)\cdot c_0 \qquad \bar{c}_1 = (1+g)\cdot c_1$$

If the monotone transformation was homogenous, it should hold that:

$$\Rightarrow \quad \frac{[(1+g)\cdot c_0]^{1-\sigma}-1}{1-\sigma} + \beta\frac{[(1+g)\cdot c_1]^{1-\sigma}-1}{1-\sigma} = (1+g)^{1-\sigma}\cdot\left(\frac{c_0^{1-\sigma}-1}{1-\sigma} + \beta\frac{c_1^{1-\sigma}-1}{1-\sigma}\right)$$

$$\Rightarrow \quad [(1+g)\cdot c_0]^{1-\sigma} - 1 + \beta[(1+g)\cdot c_1]^{1-\sigma} - 1 = (1+g)^{1-\sigma}\cdot\left(c_0^{1-\sigma} - 1 + \beta c_1^{1-\sigma} - 1\right)$$

$$\Rightarrow \quad [(1+g)\cdot c_0]^{1-\sigma} + \beta[(1+g)\cdot c_1]^{1-\sigma} - 2 = [(1+g)\cdot c_0]^{1-\sigma} + \beta[(1+g)\cdot c_1]^{1-\sigma} - 2\cdot(1+g)^{1-\sigma}.$$

That is, $2 = 2\cdot(1+g)^{1-\sigma} \quad \Rightarrow \quad (1+g)^{1-\sigma} = 0 \quad \Rightarrow\Leftarrow \quad \forall g \neq 0 \qquad QED.$

As a consequence, it can be concluded that the monotone transformation is not homothetic.

Problem 2: The Negishi Method

- The social planner problem can be written as

$$\begin{aligned}
&\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t[\alpha \ln c_t^1 + (1-\alpha)\ln c_t^2] \\
&\text{s.t.} \\
&c_t^i \geq 0 \qquad \forall i, t \\
&c_t^1 + c_t^2 = 2 \quad \forall t.
\end{aligned}$$

Step 1: The Lagrangian (with scaled multiplier $\frac{\mu_t}{2}$) is

$$\mathbb{L} = \sum_{t=0}^{\infty} \beta^t[\alpha \ln c_t^1 + (1-\alpha)\ln c_t^2] + \frac{\mu_t}{2}[2 - c_t^1 - c_t^2].$$

The FOCs are given by

$$\frac{\alpha\beta^t}{c_t^1} - \frac{\mu_t}{2} = 0 \qquad\qquad \Rightarrow \qquad \frac{\mu_t}{2} = \frac{\alpha\beta^t}{c_t^1}$$

and

$$\frac{(1-\alpha)\beta^t}{c_t^2} - \frac{\mu_t}{2} = 0 \qquad \Rightarrow \qquad \frac{\mu_t}{2} = \frac{(1-\alpha)\beta^t}{c_t^2}.$$

It follows that

$$\frac{(1-\alpha)\beta^t}{c_t^2} = \frac{\alpha\beta^t}{c_t^1} \qquad \Rightarrow \qquad \frac{c_t^1}{c_t^2} = \frac{\alpha}{1-\alpha}.$$

Substituting the budget constraint yields

$$c_1^t + c_t^2 = 2 \quad \Rightarrow \quad c_1^t + \frac{1-\alpha}{\alpha}c_t^1 = 2 \quad \Rightarrow \quad \left(1 + \frac{1-\alpha}{\alpha}\right) \cdot c_t^1 = 2 \quad \Rightarrow \quad c_t^1(\alpha) = 2\alpha.$$

Thus, $c_t^2(\alpha) = 2(1-\alpha)$.

Step 2: The transfer function, that is the amount of resources that agent i must receive (or give) in order to be able to afford the planner's allocation given α and her resources y_t^i in period t is

$$t^i(\alpha) = \sum_t \beta^t \left(c_t^i(\alpha) - y_t^i\right).$$

That is

$$\begin{aligned} t^1(\alpha) &= \sum_t 2\beta^t\alpha - \sum_t \beta^t y_t^1 \\ t^2(\alpha) &= \sum_t 2\beta^t(1-\alpha) - \sum_t \beta^t y_t^2. \end{aligned}$$

Notice that

$$\begin{aligned} \sum_t \beta^t y_t^1 &= \cdot(0 + 2\beta^2 + 0 + 2\beta^4 + \ldots) = 2 \cdot \sum \beta^{2t} \\ \sum_t \beta^t y_t^2 &= \cdot(2\beta + 0 + 2\beta^3 + \ldots) = 2 \cdot \sum \beta^{2t-1} = 2\beta \cdot \sum \beta^2 t, \end{aligned}$$

Therefore,

$$t^1(\alpha) = 2\alpha \cdot \sum_t \beta^t - 2 \cdot \sum_t \beta^{2t} = \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2}$$
$$t^2(\alpha) = 2(1-\alpha) \cdot \sum_t \beta^t - 2\beta \cdot \sum_t \beta^{2t} = \frac{2(1-\alpha)}{1-\beta} - \frac{2\beta}{1-\beta^2}.$$

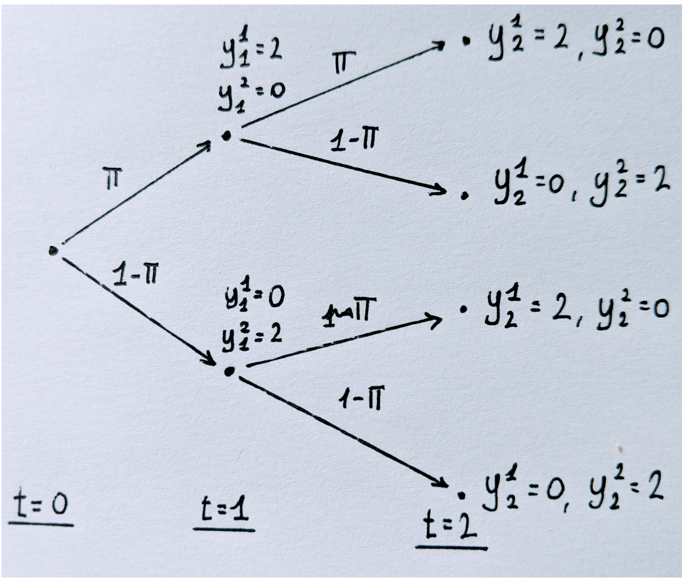
Step 3: In a competitive equilibrium, there are no transfers, thus

$$t^1(\alpha) = 0 \quad \Rightarrow \quad \frac{2\alpha}{1-\beta} = \frac{2}{1-\beta^2} \quad \Rightarrow \quad \alpha = \frac{1-\beta}{1-\beta^2} = \frac{1-\beta}{(1-\beta)\cdot(1+\beta)} = \frac{1}{1-\beta}$$
$$t^2(\alpha) = 0 \quad \Rightarrow \quad \frac{2(1-\alpha)}{1-\beta} = \frac{2\beta}{1-\beta^2} \quad \Rightarrow \quad 1-\alpha = \frac{\beta(1-\beta)}{1-\beta^2} = \frac{\beta}{1-\beta}.$$

The competitive equilibrium is given by

$$c_t^{1*} = \frac{2}{1+\beta}$$
$$c_t^{2*} = \frac{2\beta}{1+\beta}.$$

- The tree for both agents is given by



Note that, hereby, we assume that both agents share a common income process. π is not determined independently for both agents but determines both agent's income at the same time.

- The new maximization problem is

$$
\max_{\{c_t^1, c_t^2\}_{t=1}^2} \sum_{t=0}^{2} \sum_{s^t} \beta^t \pi(s^t) [\alpha \ln c_t^1(s^t) + (1-\alpha) \ln c_t^2(s^t)]
$$

s.t.

$$
c_t^i(s^t) \geq 0 \qquad \forall i, t
$$

$$
c_t^1(s^t) + c_t^2(s^t) = y_t^1(s^t) + y_t^2(s^t) \quad \forall t.
$$

Notice that the resource constraint is such that

$$
c_t^1(s^t) + c_t^2(s^t) = y_t^1(s^t) + y_t^2(s^t) = [\pi \cdot 2 + (1-\pi) \cdot 0] + [(1-\pi) \cdot 2 + \pi \cdot 0] = 2 \qquad \forall s^t, \forall t \in (1,2),
$$

That is, the amount of resources available to the planner in each period is equal to 2 with probability 1. Therefore, the planner's problem - and thus the optimal consumption level - is neither affected by past (or future) history nor by the probability of each state. The problem is identical to the prior one. Repeating the steps yields

$$
c_t^1(\alpha) = 2\alpha
$$

$$
c_t^2(\alpha) = 2(1-\alpha).
$$

- The Lagrangian of the previous exercise is

$$
\mathbb{L} = \sum_{t=0}^{2} \sum_{s^t} \beta^t \pi(s^t) \left[\alpha \ln c_t^1(s^t) + (1-\alpha) \ln c_t^2(s^t) + \frac{\mu_t}{2} \left(2 - c_t^1(s^t) - c_t^2(s^t) \right) \right].
$$

The FOCs for a generic $t \in (1,2)$, that is, for $c_t^1(s^t)$ and $c_t^1(s^t)$ are

$$
\frac{\alpha \beta^t \pi(s^t)}{c_t^1(s^t)} = \frac{\mu_t}{2}
$$

$$
\frac{(1-\alpha) \beta^t \pi(s^t)}{c_t^1(s^t)} = \frac{\mu_t}{2}.
$$

Substituting the optimal consumption yields

$$
\frac{\mu_t}{2} = \frac{\alpha \beta^t \pi(s^t)}{2\alpha} \quad \Rightarrow \quad \mu_t(s^t) = \beta^t \pi(s^t) QED.
$$

The values for each t and s^t are

$$
\begin{aligned}
&\mu_0(\pi) = 1 \\
&\mu_1(\pi) = \beta\pi \\
&\mu_1(1-\pi) = \beta(1-\pi) \\
&\mu_2(\pi,\pi) = (\beta\pi)^2 \\
&\mu_2(\pi, 1-\pi) = \beta^2\pi(1-\pi) \\
&\mu_2(1-\pi,\pi) = \beta^2\pi(1-\pi) \\
&\mu_2(1-\pi,1-\pi) = \beta^2(1-\pi)^2.
\end{aligned}
$$

- As before, the transfer function is

$$
t^i(\alpha) = \sum_t \sum_{s^t} \mu_t(s^t)\left(c_t^i(\alpha) - y_t^i\right),
$$

that is

$$
t^1(\alpha) = (2\alpha-1)+2\beta\pi(\alpha-1)+2\beta(1-\pi)\alpha+2(\beta\pi)^2(\alpha-1)+2\beta^2\pi(1-\pi)(2\alpha-1)+2\beta^2(1-\pi)^2\alpha.
$$

Setting $t^1(\alpha) = 0$ yields

$$
(2\alpha-1)+2\beta\pi(\alpha-1)+2\beta(1-\pi)\alpha+2(\beta\pi)^2(\alpha-1)+2\beta^2\pi(1-\pi)(2\alpha-1)+2\beta^2(1-\pi)^2\alpha = 0.
$$

That is:

$$
\begin{aligned}
2\alpha &= \frac{1+2\beta\pi+2(\beta\pi)^2+2\beta^2\pi(1-\pi)}{1+\beta\pi+\beta(1-\pi)+(\beta\pi)^2+2\beta^2\pi(1-\pi)+\beta^2(1-\pi)^2} \\
&= \frac{1+2\beta\pi+2\beta^2\pi}{1+\beta+\beta^2\pi^2+2\beta^2\pi+\beta^2(1-2\pi-\pi^2)} \\
&= \frac{1+2\pi(\beta+\beta^2)}{1+\beta+\beta^2}
\end{aligned}
$$

and

$$
\begin{aligned}
2(1-\alpha) = 2 - 2\alpha = 2 - \frac{1+2\pi(\beta+\beta^2)}{1+\beta+\beta^2} \\
&= \frac{2+2\beta+2\beta^2-1-2\pi(\beta+\beta^2)}{1+\beta+\beta^2} \\
&= \frac{1+2(1-\pi)(\beta+\beta^2)}{1+\beta+\beta^2} QED.
\end{aligned}
$$

- Since in a competitive equilibrium agents can consume only what they can afford, the planner's α that sustains such an equilibrium must represent the differences in the discounted expected stream of incomes between the two agents. If $\pi = 0.5$, then the agents have the same expected income for each period. Therefore, their expected income stream is identical, independent of the discount factor.

- If $\pi > 0.5$, then agent 1 is endowed, in expectation, with a larger income compared with agent 2. Therefore, he will be able to consume more than the other agent $\Leftrightarrow$ Pareto equilibrium with weight $\alpha > 0.5$.

- Solving for sequential market equilibrium price:

  The intertemporal budget constraint is

$$
c_t^i(s^t) + \sum_{s_{t+1}} a_{t+1}^i(s_{t+1}, s^t) q_t(s_{t+1}|s^t) \leq y_t^i(s^t) + a_t^i(s^t).
$$

  No-Ponzi condition is

$$
-a_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1}).
$$

  The Lagrangian for the consumer problem will be given by

$$
\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \Bigg\{ \beta^t u(c_t^i(s^t))\pi_t(s^t) + \eta_t^i(s^t) \Bigg[y_t^i(s^t) + a_t^i(s^t) - c_t^i(s^t) - \sum_{s^{t+1}} a_t^i(s_{t+1}, s^t) q_t(s_{t+1}|s^t) \Bigg] \\
+ \sum_{s^{t+1}} \nu_t^i(s_{t+1}, s^t) \Bigg[A_{t+1}^i(s^{t+1}) + a_{t+1}^i(s^{t+1}) \Bigg] \Bigg\}
\end{aligned}
$$

  for a given initial wealth $a_0^i(s_0)$.

  The FOCs w.r.t. $c_t^i(s^t)$ and $a_t^i(s_{t+1}, s^t)$ are

$$
\beta^t u'(c_t^i(s^t))\pi_t(s^t) - \eta_t^i(s^t) = 0
$$

$$
-\eta_t^i(s^t) q_t(s_{t+1}|s^t) + \nu_t^i(s_{t+1}, s^t) + \eta_t^i(s_{t+1}, s^t) = 0.
$$

  Due to the the INADA conditions, the debt limit constraint will not be binding at the optimum.

$$\implies \nu_t^i(s_{t+1}, s^t) = 0, \forall t, s^t, s_{t+1} \implies -\eta_t^i(s^t) q_t(s_{t+1}|s^t) + \eta_t^i(s_{t+1}, s^t) = 0$$

We have

$$\beta^t u'(c_t^i(s^t))\pi_t(s^t) - \eta_t^i(s^t) = 0$$

$$-\eta_t^i(s^t) q_t(s_{t+1}|s^t) + \eta_t^i(s_{t+1}, s^t) = 0.$$

Substituting the first equation into the second yields

$$q_t(s_{t+1}|s^t) = \beta \frac{u'(c_{t+1}^i(s^{t+1}))}{u'(c_t^i(s^t))} \pi_t(s_{t+1}|s^t) \quad \forall t, s^t, s_{t+1}. \tag{1}$$

Notice that from the FOCs of the Arrow-Debreu market (given in lecture), for agent i, we have that

$$\beta^t u'(c_t^i(s^t))\pi_t(s^t) = \mu_i p_t(s^t)$$

$$\beta^{t+1} u'(c_{t+1}^i(s^{t+1}))\pi_t(s^{t+1}) = \mu_i p_{t+1}(s^{t+1}).$$

Dividing the second equation by the first gives

$$\frac{p_{t+1}(s^{t+1})}{p_t(s^t)} = \beta \frac{u'(c_{t+1}^i(s^{t+1}))}{u'(c_t^i(s^t))} \pi_t(s_{t+1}|s^t) \quad \forall t, s^t, s_{t+1}. \tag{2}$$

From (1) and (2):

$$q_t(s_{t+1}|s^t) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} \quad QED.$$

- Using the result

$$\mu_t(s^t) = p_t(s^t) = \beta^t \pi(s^t),$$

it follows that

$$q_1(s^1) = \mu_1(s^1)$$
$$q_1(s^1, s^2) = \frac{\mu_1(s^2)}{\mu_1(s^1)}.$$

Therefore,

$$
\begin{aligned}
q_1(\pi) &= \beta\pi \\
q_1(1-\pi) &= \beta(1-\pi) \\
q_2(\pi,\pi) &= \frac{(\beta\pi)^2}{\beta\pi} = \beta\pi \\
q_2(1-\pi,\pi) &= \frac{\beta^2\pi(1-\pi)}{\beta(1-\pi)} = \beta\pi \\
q_2(\pi,1-\pi) &= \frac{\beta^2\pi(1-\pi)}{\beta\pi} = \beta(1-\pi) \\
q_2(1-\pi,1-\pi) &= \frac{\beta^2(1-\pi)^2}{\beta(1-\pi)} = \beta(1-\pi).
\end{aligned}
$$

The intuition for these prices is that they ensure the no arbitrage condition. For example, in period 0 you can buy one unit of consumption delivered in $t+1$ at price p_{t+1}. Then you can sell this unit in period t to earn q_t units of consumption delivered in period t. Hence, in period 0, you can sell q_t units of consumption delivered in t at price p_t, earn $p_t q_t$ in period 0. By the no arbitrage condition(and noting that the income process has no memory):

$$-p_{t+1} + p_t q_t = 0 \quad QED.$$