Exercise 8

Bowen Hua

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1 Laplacian Smoothing

1.1 (A)

The laplacian $L = W - A$ can be decomposed into a diagonal matrix W and a off-diagonal matrix $-A$. Therefore, we only have to show that the diagonal elements of $D^T D$ form W and the off-diagonal elements of $D^T D$ form $-A$.

For the diagonal elements, $L_{ii} = D_i \cdot D_i$ where D_i is the i-th column of the D matrix and $\cdot$ denotes inner product. From the definition of the oriented edge matrix D we know that $D_{ki} = -1$ if there is an edge k whose ending node is i and $D_{ki} = 1$ if edge k's beginning node is i. Therefore, $L_{ii} = D_i \cdot D_i$ is just the degree of node i. That is, $L_{ii} = W_i$.

For the off-diagonal elements, $L_{ij} = D_i \cdot D_j$. We only need to examine the indices k's for which $D_{ki} \neq 0$ and $D_{kj} \neq 0$. This only happens when edge k connects node i and node j, in which case $D_{ki} D_{kj} = -1$. This implies that $L_{ij} = -Aij$ for $i \neq j$.

1.2 (B)

Given the Laplacian smoothing problem:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\|y - x\|_2^2 + \frac{\lambda}{2} x^T L x \,, \tag{1}$$

we first show that it is convex.

The first term is a norm, which is always convex. The second term is convex iff L is positive semidefinite. This is true because L is diagonally dominant and symmetric with nonnegative diagonal elements.

With convexity, we can safely invoke the first-order optimality condition, which leads to:

$$(\lambda D^T D + I)\hat{x} = y. \tag{2}$$