Derivative of Entropy Function for Feedback LCA

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1 Softmax

We first will start with the softmax function. The softmax z_i of an element u_i from a vector $\mathbf{u}$ is defined as:

$$z_i = \frac{e^{u_i}}{\sum_j e^{u_j}} \tag{1}$$

2 Derivative of Softmax

The derivative of z_i with respect to u_j is different for when $i = j$ versus when $i \neq j$ as we will see when we derive them.

Part 1: $i = j$

$$\begin{aligned}
\frac{\partial z_i}{\partial u_j} &= \frac{\partial \left(\frac{e^{u_i}}{\sum_j e^{u_j}} \right)}{\partial u_j} \\
&= \frac{\left(e^{u_i}\right) \sum_j e^{u_j} - \left(e^{u_i}\right)^2}{\left(\sum_j e^{u_j}\right)^2} \\
&= \frac{e^{u_i}}{\sum_j e^{u_j}} - \left(\frac{e^{u_i}}{\sum_j e^{u_j}} \right)^2 \\
&= z_i - \left(z_i\right)^2 \\
&= z_i \left(1 - z_i\right) \qquad (2)
\end{aligned}$$

Part 2: $i \neq j$

$$
\begin{aligned}
\frac{\partial z_i}{\partial u_j} &= \frac{\partial\left(\frac{e^{u_i}}{\sum_j e^{u_j}}\right)}{\partial u_j} \\
&= \frac{0\sum_j e^{u_j} - \left(e^{u_i}\right)e^{u_j}}{\left(\sum_j e^{u_j}\right)^2} \\
&= -\left(\frac{e^{u_i}}{\sum_j e^{u_j}}\right)\frac{e^{u_j}}{\sum_j e^{u_j}} \\
&= -z_i z_j
\end{aligned} \tag{3}
$$

Now that we know the derivative of the softmax function, we can use what we know to make deriving the derivative of the entropy function much easier.

3 Entropy Function

The entropy function E over distribution $\mathbf{z}$ with discrete elements z_j as described in equation 1 is defined as:

$$E = -\sum_j z_j \ln z_j \tag{4}$$

4 Derivative of Entropy Function

Let's combine what we know from the derivative of the softmax function to derive the derivative of the entropy function with respect to u_i's.

$$
\begin{aligned}
\frac{\partial E}{\partial u_i} = \frac{\partial \left(-\sum_j z_j \ln z_j\right)}{\partial u_i} &= -\sum_j \frac{\partial \left(z_j \ln z_j\right)}{\partial u_i} \\
&= -\left[\frac{\partial \left(z_i \ln z_i\right)}{\partial u_i} + \sum_{j \neq i} \frac{\partial \left(z_j \ln z_j\right)}{\partial u_i}\right] \\
&= -\left[\left\{z_i \left(1 - z_i\right)\left(\ln z_i\right) + \frac{z_i \left(z_i \left(1 - z_i\right)\right)}{z_i}\right\} + \left\{\sum_{j \neq i} -z_i z_j \left(\ln z_j\right) + z_j \frac{\left(-z_i z_j\right)}{z_j}\right\}\right] \\
&= -z_i \left(1 - z_i\right)\left(\ln z_i + 1\right) + \sum_{j \neq i} z_i z_j \left(\ln z_j + 1\right) \\
&= z_i \left[-\left(\ln z_i + 1\right) + z_i \left(\ln z_i + 1\right) + \sum_{j \neq i} z_j \left(\ln z_j + 1\right)\right] \\
&= z_i \left[-\left(\ln z_i + 1\right) + \sum_j z_j \left(\ln z_j + 1\right)\right]
\end{aligned}
$$