Fourier-Mukai transform on abelian variety

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Contents

Preface

Motivations and plans. Here are main references:

(1) [Mum70];
(2) [BL04].

Assumption. In this lecture note we always work over $\mathbb{C}$ for convenience.

1. Geometry of complex tori

1.1. **Complex tori.**

Definition 1.1.1. Let V be a complex vector space of dimension g and Λ be a lattice in V. The quotient $X = V/\Lambda$ is called a *complex tori* of dimension g.

Proposition 1.1.1. Let $h\colon X = V/\Lambda \to X' = V'/\Lambda$ be a holomorphic map between complex tori.

(1) There is a unique homomorphism $f\colon X \to X'$ such that $h(x) = f(x) + h(0)$ for all $x \in X$.
(2) There is a unique $\mathbb{C}$-linear map $F\colon V \to V'$ with $F(\Lambda) \subset \Lambda'$ inducing the homomorphism f.

In particular, there is an injective homomorphism of abelian groups

$$\begin{aligned} \rho_a\colon \operatorname{Hom}(X,X') &\to \operatorname{Hom}(V,V') \\ f &\mapsto F, \end{aligned}$$

and F is called *analytic representation* of f.

Proof. Let $f = t_{-h(0)} \circ h$. Then the composed map $f \circ \pi\colon V \to X \to X'$ can be lifted to a holomorphic map $F\colon V \to V'$ such that $F(0) = 0$.

$$\begin{tikzcd} V \arrow[rr, dashed, "F"] \arrow[dr, "f\circ\pi"'] & & V' \arrow[dl, "\pi'"] \\ & X' & \end{tikzcd}$$

This shows for all $\lambda \in \Lambda$ and $v \in V$ we have $F(v+\lambda) - F(v) \in \Lambda'$, and thus the continuity of $v \mapsto F(v+\lambda) - F(v)$ implies $F(v+\lambda) - F(v) = F(\lambda)$ holds for all $\lambda \in \Lambda$ and $v \in V$, and thus f is a homomorphism. Moreover, the partial derivatives of F are periodic and thus by Liouville's theorem it follows that F is $\mathbb{C}$-linear. The uniqueness of F and f is obvious. □

Definition 1.1.2.

(1) An *isogeny* of a complex tori X to a complex tori X' is by definition a surjective homomorphism $X \to X'$ with finite kernel.
(2) The *exponent* $e = e(f)$ of an isogeny f is defined to be the exponent of the finite group $\ker f$.

Definition 1.1.3. For a homomorphism $f\colon X \to X'$ of complex tori, the *degree* of f is defined to be order of $\ker f$, if it is finite, and 0 otherwise.

Definition 1.1.4. For any integer n, the homomorphism $n_X\colon X \to X$ is defined by $x \mapsto nx$, and $X_n := \ker n_X$ is called the *group of n-division points of X*

Proposition 1.1.2. Let X be a complex tori of dimension g. If $n \in \mathbb{Z}$ and $n \neq 0$, $X_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$. In particular, the degree of n_X is n^{2g}.

Proof. It is clear that

$$\ker n_X = \frac{1}{n}\Lambda/\Lambda = \Lambda/n\Lambda = (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

□

Proposition 1.1.3. For any isogeny $f: X \to X'$ of exponent e, there exists an isogeny $g: X \to X'$, unique to isomorphisms, with $g \circ f = e_X$ and $f \circ g = e_{X'}$.

Proof. Since $\ker f \subseteq \ker e_X$, there is a unique map $g: X' \to X$ such that $g \circ f = e_X$. Indeed, we define $g := e_X \circ f^{-1}: X' \to X$, which is well-defined as for any point $x' \in X$, the preimages of x' differs some element in $\ker e_X$. This provides a map $g: X' \to X$ such that $g \circ f = e_X$. It is unique, otherwise suppose there exists g_1, g_2 such that $g_i \circ f = e_X$. Then $(g_1 - g_2) \circ f = 0$, which implies $g_1 - g_2 = 0$, since f is surjective. Moreover, g is an isogeny, since both e_X and f are isogenies.

On the other hand, the kernel of g is contained in the kernel of $e_{X'}$. Indeed, for every $x' \in \ker g$, we may choose x such that $f(x) = x'$, and $x \in \ker e_X$ since $ex = g \circ f(x) = g(x') = 0$. Then

$$ex' = ef(x) = f(ex) = 0.$$

By the same argument $e_{X'} = f' \circ g$ for some isogeny $f': X \to X'$. Since

$$f' \circ e_X = f' \circ g \circ f = e_{X'} \circ f = f \circ e_X,$$

we have $f' = f$ since e_X is surjective. □

1.2. **Hodge structures.** Let X be a compact complex manifold of Kähler type[2]. Then there is the following Hodge decomposition

$$H^k(X,\mathbb{Z}) \cong \bigoplus_{p+q=k} H^{p,q}(X),$$

such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

For the complex tori case, there is additional description on its de Rham cohomology $H^k(X,\mathbb{Z})$. Suppose $X = \mathbb{C}^g/\Lambda$. Then we have the following commutative diagram

$$\begin{tikzcd} \mathbb{C}^g \arrow[r, two heads, "\pi"] \arrow[d, dash, "\cong"'] & \mathbb{C}^g/\Lambda \arrow[d, dash, "\cong"] \\ T_0X = V \arrow[r, "\exp"] & X. \end{tikzcd}$$

This implies that $\pi_1(X) = \Lambda$ and thus $H^1(X,\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z}) = \Lambda^*$.

If we forget the complex structure, topologically we have $X \cong (S^1)^{2g}$. Then

$$\begin{tikzcd} H^k(X,\mathbb{Z}) & \bigwedge^k H^1(X,\mathbb{Z}) \arrow[l, "\cong"'] \\ H^k((S^1)^{2g},\mathbb{Z}) \arrow[u, "\cong"] & \bigwedge^k H^1((S^1)^{2g},\mathbb{Z}). \arrow[l, "\cong"] \arrow[u, "\cong"'] \end{tikzcd}$$

[2]A compact complex manifold is of Kähler, if there exists a Kähler metric ω on X.

In other words, the k-th cohomology is determined by the 1-st cohomology group $H^1(X,\mathbb{Z})$.

In order to compute the Dolbeault cohomology, we equip $X=\mathbb{C}^g/\Lambda$ with a Kähler metric ω. Then by the theory of harmonic forms, there is an isomorphism

$$\mathcal{H}^{p,q}(X)=\{\Delta_{\mathrm{d}}(\alpha)=0\mid \alpha\in\mathcal{A}^{p,q}(X)\}\cong H^{p,q}(X).$$

Since $X=\mathbb{C}^g/\Lambda$ is a Lie group, its tangent bundle is trivial. Thus

$$\mathcal{A}^{p,q}(X)=\mathrm{span}_{C^\infty(X)}\{\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_p}\wedge\mathrm{d}\bar{z}^{j_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{j_q}\},$$

where $\{\mathrm{d}z^1,\dots,\mathrm{d}z^g\}$ is a basis of $H^0(X,\Omega^1_X)$.

Note that the above isomorphism is independent of the choice of Kähler metric, we choose the standard flat metric, that is, the metric induced by the Euclidean metric on $\mathbb{C}^g$. Suppose $\alpha=\sum_{|I|=p,|J|=q}f_{IJ}\mathrm{d}z_I\wedge\mathrm{d}\bar{z}_J$. Then

$$\Delta_{\mathrm{d}}(\alpha)=0\Longleftrightarrow\Delta f_{IJ}=0\Longleftrightarrow f_{IJ}\in\mathbb{C}.$$

This shows the Hodge number of complex tori $X=\mathbb{C}^g/\Lambda$ is

$$h^{p,q}(X)=\binom{g}{p}\times\binom{g}{q}.$$

1.3. **Line bundles on a complex tori.** In this section, we will show how to describe (holomorphic) line bundles on abelian varieties explicitly.

1.3.1. *Appell-Humbert theorem.* Let X be a complex tori defined by V/Λ, where $V=\mathbb{C}^g$ and $L\subseteq V$ is a lattice. Let $\mathcal{E}$ be a vector bundle on X, as there is a natural projection $\pi\colon V\to X$, the pullback bundle $\pi^*\mathcal{E}$ is a vector bundle on V. By Oka-Grauert principle[3], the pullback bundle $\pi^*\mathcal{E}$ is trivial, since V is contractible and Stein.

For line bundle cases, this fact can be proved algebraically by using the exponential sequence

$$0\to\mathbb{Z}\to\mathcal{O}_V\to\mathcal{O}_V^*\to 0.$$

Indeed, since $H^p(V,\mathbb{Z})=0$ for $p>0$ as V is contractible, and $H^p(V,\mathcal{O}_V)=0$ for $p>0$ as V is Stein, then by the long exact sequence induced by the exponential sequence, we have $H^1(V,\mathcal{O}_V^*)=0$, which shows every line bundle on V is trivial.

In this section, we want to introduce the classification of line bundles on X. Let $\mathcal{L}$ be a line bundle on X and fix an isomorphism $\pi^*\mathcal{L}\cong\mathcal{O}_V$. There is a natural Λ-action on $\pi^*\mathcal{L}$ such that the quotient of $\pi^*\mathcal{L}$ by Λ is $\mathcal{L}$. Since the only holomorphic automorphisms of a line bundle fixing the base are given by multiplication by non-vanishing holomorphic functions, then the action of Λ on $\mathbb{C}\times V$ can be written as

$$(\alpha,z)\mapsto(\phi_\lambda(\alpha),z+\lambda)$$

[3]In complex geometry, the Oka-Grauert principle states that over Stein complex manifolds, the non-abelian cohomology-classification of holomorphic vector bundles coincides with that of topological vector bundles.

for all $\lambda \in \Lambda$. where $\phi_\lambda \in H^0(V, \mathcal{O}_V^*)$. Moreover, it satisfies

$$\phi_{\lambda_1+\lambda_2} = \lambda_2^* \phi_{\lambda_1} \cdot \phi_{\lambda_2},$$

that is, $\{\phi_\lambda\}_{\lambda\in\Lambda}$ satisfies the cocycle condition, and thus $\{\phi_\lambda\}_{\lambda\in\Lambda}$ gives an element in $Z^1(\Lambda, H^0(V, \mathcal{O}_V^*))$.

There is an equivalent relation $\sim$ on $Z^1(\Lambda, H^0(V, \mathcal{O}_V^*))$ defined by $\{\phi_\lambda\} \sim \{\phi'_\lambda\}$ if and only if there exists $f \in H^0(V, \mathcal{O}_V^*)$ such that for all $\lambda \in \Lambda$, we have

$$\phi'_\lambda \cdot \phi_\lambda^{-1} = \lambda^*(f) \cdot f^{-1},$$

and the quotient group of $Z^1(\Lambda, H^0(V, \mathcal{O}_V^*))/_\sim$ is denoted by $H^1(\Lambda, H^0(V, \mathcal{O}_V^*))$. After passing to this cohomology group, actually there is the following isomorphism

$$\begin{aligned} H^1(\Lambda, H^0(V, \mathcal{O}_V^*)) &\xrightarrow{\cong} H^1(X, \mathcal{O}_X^*) \\ [\{\phi_\lambda\}_{\lambda\in\Lambda}] &\to [\mathcal{L}]. \end{aligned}$$

Thus, in order to classify all line bundles, it suffices to have an effective way to produce elements in $H^1(\Lambda, H^0(V, \mathcal{O}_V^*))$. Recall that for a Hermitian form h on V, the real part $\mathrm{Re}h$ is symmetric and the imaginary part $E := \mathrm{Im}h$ is alternating. Also, E preserves the complex structure of V, that is, $E(\sqrt{-1}x, \sqrt{-1}y) = E(x, y)$ for all $x, y \in V$.

Definition 1.3.1. Let $V = \mathbb{C}^g$ and $\Lambda \subseteq V$ be a lattice. A Hermitian form h on V satisfies the *integrality condition*, if

$$E : \Lambda \times \Lambda \to \mathbb{Z}.$$

Lemma 1.3.1. Let h be a Hermitian form on V satisfying the integrality condition and $E = \mathrm{Im}h$.

(1) There exists $\alpha : \Lambda \to \mathrm{U}(1)$ such that for any $\lambda_1, \lambda_2 \in \Lambda$, we have

$$\frac{\alpha(\lambda_1 + \lambda_2)}{\alpha(\lambda_1) \cdot \alpha(\lambda_2)} = e^{\sqrt{-1}\pi E(\lambda_1, \lambda_2)} \in \{\pm 1\}.$$

(2) For $\lambda \in \Lambda$, if we define

$$\phi_\lambda(z) = \alpha(\lambda) \cdot e^{\pi h(z,\lambda) + \frac{1}{2}\pi h(\lambda,\lambda)} \in H^0(V, \mathcal{O}_V^*),$$

then $\{\phi_\lambda\} \in Z^1(\Lambda, H^0(V, \mathcal{O}_V^*))$.

(3) There is a commutative diagram

$$\begin{tikzcd} {[\mathcal{L}] \in H^1(X, \mathcal{O}_X^*)} \arrow[r, "\delta"] \arrow[d, two heads, "\pi^*"'] & H^2(X, \mathbb{Z}) \arrow[d, two heads, "\pi^*"] \\ {[\{\phi_\lambda\}_{\lambda\in\Lambda}] \in H^1(\Lambda, H^0(V, \mathcal{O}_V^*))} \arrow[r, "\delta"] & H^2(\Lambda, \mathbb{Z}), \end{tikzcd}$$

such that $c_1(\mathcal{L}) = E$ under the identification $H^2(X, \mathbb{Z}) \cong \bigwedge^2 \Lambda^*$, where $\mathcal{L}$ is the line bundle corresponding to $\{\phi_\lambda\}_{\lambda\in\Lambda}$.

Proof. For (1). Suppose that the rank of Λ is two and take a basis $\{e, f\}$ of Λ. Then define a map

$$\begin{aligned} \delta\colon \Lambda &\to \mathbb{R} \\ ne + mf &\mapsto \frac{1}{2} nmE(e,f). \end{aligned}$$

For $\lambda_1, \lambda_2 \in \Lambda$, we may write it as

$$\begin{aligned} \lambda_1 &= ae + bf \\ \lambda_2 &= ce + df, \end{aligned}$$

and thus by definition of δ it gives

$$\begin{aligned} \delta(\lambda_1 + \lambda_2) &= \frac{1}{2}(a+c)(b+d)E(e,f) \\ \delta(\lambda_1) &= \frac{1}{2} abE(e,f) \\ \delta(\lambda_2) &= \frac{1}{2} cdE(e,f). \end{aligned}$$

A direct computation shows that

$$\begin{aligned} \delta(\lambda_1 + \lambda_2) - \delta(\lambda_1) - \delta(\lambda_1) &= \frac{1}{2}(ad + bc)E(e,f) \\ &\equiv \frac{1}{2}(ad - bc)E(e,f) \pmod 1 \\ &\equiv \frac{1}{2} E(\lambda_1, \lambda_2) \pmod 1. \end{aligned} \tag{1.1}$$

This shows that $\alpha = e^{2\pi\sqrt{-1}\delta}\colon \Lambda \to \mathrm{U}(1)$ satisfies

$$\frac{\alpha(\lambda_1 + \lambda_2)}{\alpha(\lambda_1)\cdot\alpha(\lambda_2)} = e^{\sqrt{-1}\pi E(\lambda_1,\lambda_2)} \in \{\pm 1\}.$$

In the general case, we choose a symplectic basis $\{e_1, f_1, e_2, f_2, \ldots, e_g, f_g\}$ of Λ and write $\Lambda = \bigoplus_{i=1}^g \Lambda_i$ as an orthogonal decomposition with respect to E, where $\Lambda_i = \mathrm{span}_{\mathbb{Z}}\{e_i, f_i\}$. Then a similar computation yields that $\delta\colon \Lambda \to \mathbb{R}$ defined by

$$\delta\left(\sum_{i=1}^g (n_i e_i + m_i f_i)\right) = \frac{1}{2}\sum_{i=1}^g n_i m_i E(e_i, f_i)$$

satisfy (1.1), and we can also define $\alpha = e^{2\pi\sqrt{-1}\delta}\colon \Lambda \to \mathrm{U}(1)$, which satisfies the desired property.

For (2). By definition, we have

$$\begin{aligned} \phi_{\lambda_1+\lambda_2}(z) &= \alpha(\lambda_1 + \lambda_2) e^{\pi h(z, \lambda_1+\lambda_2) + \frac{1}{2}\pi h(\lambda_1+\lambda_2, \lambda_1+\lambda_2)} \\ \phi_{\lambda_1}(z + \lambda_2) &= \alpha(\lambda_1) e^{\pi h(z+\lambda_2, \lambda_1) + \frac{1}{2}\pi h(\lambda_1, \lambda_1)} \\ \phi_{\lambda_2}(z) &= \alpha(\lambda_2) e^{\pi h(z, \lambda_2) + \frac{1}{2}\pi h(\lambda_2, \lambda_2)}. \end{aligned}$$

Thus

$$\begin{aligned}\phi_{\lambda_1}(z+\lambda_2)\phi_{\lambda_2}(z) &= \alpha(\lambda_1)\alpha(\lambda_2)e^{\pi\left(h(z+\lambda_2,\lambda_1)+h(z,\lambda_2)+\frac{1}{2}h(\lambda_1,\lambda_1)+\frac{1}{2}h(\lambda_2,\lambda_2)\right)}\\ &= \alpha(\lambda_1+\lambda_2)e^{-\sqrt{-1}\pi E(\lambda_1,\lambda_2)+\pi\left(h(z+\lambda_2,\lambda_1)+h(z,\lambda_2)+\frac{1}{2}h(\lambda_1,\lambda_1)+\frac{1}{2}h(\lambda_2,\lambda_2)\right)}\\ &= \alpha(\lambda_1+\lambda_2)e^{\pi\left(h(z,\lambda_1+\lambda_2))+\frac{1}{2}h(\lambda_1+\lambda_2,\lambda_1+\lambda_2)\right)}e^{-\sqrt{-1}\pi E(\lambda_1,\lambda_2)+\frac{1}{2}h(\lambda_2,\lambda_1)-\frac{1}{2}h(\lambda_1,\lambda_2)}.\end{aligned}$$

Note that

$$\begin{aligned}-\sqrt{-1}\pi E(\lambda_1,\lambda_2)+\frac{1}{2}h(\lambda_2,\lambda_1)-\frac{1}{2}h(\lambda_1,\lambda_2) &= -\sqrt{-1}\pi E(\lambda_1,\lambda_2)+\sqrt{-1}E(\lambda_2,\lambda_1)\\ &= -2\sqrt{-1}\pi E(\lambda_1,\lambda_2)\in 2\pi\sqrt{-1}\,\mathbb{Z}.\end{aligned}$$

This shows

$$\phi_{\lambda_1+\lambda_2}(z)=\phi_{\lambda_1}(z+\lambda_2)\phi_{\lambda_2}(z).$$

For (3). By the exponential sequence

$$0\to\mathbb{Z}\to\mathcal{O}_V\to\mathcal{O}_V^*\to 0,$$

there is the following short exact sequence

$$0\to\mathbb{Z}\to H\to H^0(V,\mathcal{O}_V^*)\to 0, \tag{1.2}$$

since $H^1(V,\mathbb{Z})=0$. Moreover, since V is contractible and Stein, we have $H^i(V,\mathcal{O}_V^*)=0$ for $i\geq 1$. Thus by Appendix to §2 of [Mum70], we get natural isomorphisms as vertical maps

$$\begin{tikzcd} H^1(X,\mathcal{O}_X^*) \arrow[r, "\delta"] \arrow[d, dash, "\cong"'] & H^2(X,\mathbb{Z}) \arrow[d, dash, "\cong"] \\ H^1(\Lambda,H^0(V,\mathcal{O}_V^*)) \arrow[r, "\delta"] & H^2(\Lambda,\mathbb{Z}), \end{tikzcd}$$

and the commutativity can be checked by using a small open covering of X.

By the commutativity of the diagram, in order to compute the first Chern class of $\mathcal{L}$ corresponding to $\{\phi_\lambda\}_{\lambda\in\Lambda}\in Z^1(\Lambda,H^0(V,\mathcal{O}_V^*))$, it suffices to compute $\delta\left(\{\phi_\lambda\}_{\lambda\in\Lambda}\right)$. By the short exact sequence (1.2), we have $Z^1(\Lambda,H)\twoheadrightarrow Z^1(\Lambda,H^0(V,\mathcal{O}_V^*))$, that is, there exists $\{f_\lambda\}_{\lambda\in\Lambda}\in Z^1(\Lambda,H)$ such that $\exp(2\pi\sqrt{-1}f_\lambda)=\phi_\lambda$. For $\{f_\lambda\}_{\lambda\in\Lambda}$, we have

$$\delta(f_\lambda)(\lambda_1,\lambda_2)(z)=f_{\lambda_2}(z+\lambda_1)-f_{\lambda_1+\lambda_2}(z)+f_{\lambda_1}(z)\in\mathbb{Z}.$$

Then use the following fact

$$\begin{tikzcd} Z^2(\Lambda,\mathbb{Z}) \arrow[r, "A"] \arrow[d, two heads] & \mathrm{Hom}(\bigwedge^2 L,\mathbb{Z}) \arrow[r, "\cong"] & \bigwedge^2 L^* \arrow[r, "\cong"] & H^2(X,\mathbb{Z}) \\ H^2(\Lambda,\mathbb{Z}), \arrow[urr, "\cong"] & & & \end{tikzcd}$$

where for $F\in Z^2(\Lambda,\mathbb{Z})$, we have $A(F)(\lambda_1,\lambda_2):=F(\lambda_1,\lambda_2)-F(\lambda_2,\lambda_1)$.

Thus we get

$$\begin{aligned}
\delta(\{\phi_\lambda\}_{\lambda\in\Lambda})(\lambda_1,\lambda_2) =& f_{\lambda_2}(z+\lambda_1)-f_{\lambda_1+\lambda_2}(z)+f_{\lambda_1}(z)-f_{\lambda_1}(z+\lambda_2)+f_{\lambda_1+\lambda_2}(z)-f_{\lambda_2}(z)\\
=& f_{\lambda_2}(z+\lambda_1)-f_{\lambda_1}(z+\lambda_2)+f_{\lambda_1}(z)-f_{\lambda_2}(z)\\
=& \frac{1}{2\pi\sqrt{-1}}\log\alpha(\lambda_2)+\frac{1}{2\pi\sqrt{-1}}\left(\pi h(z+\lambda_1,\lambda_2)+\frac{1}{2}\pi h(\lambda_2,\lambda_1)\right)\\
&-\frac{1}{2\pi\sqrt{-1}}\log\alpha(\lambda_1)-\frac{1}{2\pi\sqrt{-1}}\left(\pi h(z+\lambda_2,\lambda_1)+\frac{1}{2}\pi h(\lambda_1,\lambda_2)\right)\\
&+\frac{1}{2\pi\sqrt{-1}}\log\alpha(\lambda_1)+\frac{1}{2\pi\sqrt{-1}}\left(\pi h(z,\lambda_1)+\frac{\pi}{h}(\lambda_1,\lambda_2)\right)\\
&-\frac{1}{2\pi\sqrt{-1}}\log\alpha(\lambda_2)-\frac{1}{2\pi\sqrt{-1}}\left(\pi h(z,\lambda_2)+\frac{1}{2}\pi h(\lambda_2,\lambda_2)\right)\\
=& \frac{1}{2\sqrt{-1}}(h(\lambda_1,\lambda_2)-h(\lambda_2,\lambda_1))\\
=& E(\lambda_1,\lambda_2).
\end{aligned}$$

□

Notation 1.3.1. Since the construction of $\{\phi_\lambda\}_{\lambda\in\Lambda}$ depends on Hermitian metric h and α, we write $\mathcal{L}(h,\alpha)$ to denote the line bundle determined by h and α.

Lemma 1.3.2.

$$\mathcal{L}(h_1,\alpha_1)\otimes\mathcal{L}(h_1,\alpha_1)=\mathcal{L}(h_1+h_2,\alpha_1\cdot\alpha_2).$$

Theorem 1.3.1 (Appell-Humbert)**.** Any line bundle on X is isomorphic to a unique $\mathcal{L}(h,\alpha)$.

Remark 1.3.1. In other words, if we set

$\mathrm{Herm}^{\mathrm{int}}(V)=\{h\colon V\times V\to\mathbb{C}\mid h$ is a Hermitian metric satisfying the integrable condition$\}$

and

$\widetilde{\mathrm{Herm}}^{\mathrm{int}}(V)=\{(h,\alpha)\mid h\in\mathrm{Herm}^{\mathrm{int}}(V),\ \alpha\colon\Lambda\to\mathrm{U}(1)$ such that $\alpha(\lambda_1+\lambda_2)=e^{\pi\sqrt{-1}\mathrm{Im}h(\lambda_1+\lambda_2)}\alpha(\lambda_1)\cdot\alpha(\lambda_2)\}$,

then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Lambda,\mathrm{U}(1)) \arrow[r] \arrow[d, "\cong"] & \widetilde{\mathrm{Herm}}^{\mathrm{int}}(V) \arrow[r] \arrow[d, "\cong"] & \mathrm{Herm}^{\mathrm{int}}(V) \arrow[r] \arrow[d, "\cong"] & 0\\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & H^{1,1}(X,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

1.3.2. *Type of a line bundle.*

Lemma 1.3.3. Let $X=\mathbb{C}^g/\Lambda$ be a complex tori and h be a Hermitian form which satisfies the integrality condition. Then for the symplectic form $E=\mathrm{Im}h$, there exists a basis $\{e_1,f_1,\dots,e_g,f_g\}$ of Λ such that E is of blocked diagonal matrix

$$\mathrm{diag}\{E_1,\dots,E_{g'},0,\dots,0\},$$

where

$$E_i = \begin{pmatrix} 0 & n_i \\ -n_i & 0 \end{pmatrix}$$

and $n_i \in \mathbb{Z}$ and $0 < n_1 \mid n_2 \mid \cdots \mid n_{g'}$ are called elementary divisors.

Proof. If $E \equiv 0$, then there is nothing to prove, so we may assume $E \not\equiv 0$. Consider the set

$$\{E(\ell,\ell') \mid \ell,\ell' \in \Lambda\} \subset \mathbb{Z}.$$

Since $\mathbb{Z}$ is ordered, there exists a pair $\{e_1, f_1\} \subset L$ such that $E(e_1, f_1) > 0$ is the minimal among

$$\{E(\ell,\ell') \mid \ell,\ell' \in \Lambda\} \cap \mathbb{Z}_+ \neq \varnothing$$

Let $\Lambda_1 = \operatorname{span}_{\mathbb{Z}}\{e_1, f_1\}$ and $\Lambda_1^\perp = \{\ell \in \Lambda \mid E(\ell, L_1) = 0\}$. It's clear that $\Lambda_1 \cap \Lambda_1^\perp = \{0\}$. For any $\ell \in \Lambda$, consider $a, b \in \mathbb{Q}$ such that

$$\widetilde{\ell} := \ell - a e_1 - b f_1$$

such that $E(\widetilde{\ell}, e_1) = E(\widetilde{\ell}, f_1) = 0$. Clearly we have

$$a = \frac{E(\widetilde{\ell}, f_1)}{E(e_1, f_1)}, \quad b = \frac{E(e_1, \widetilde{\ell})}{E(e_1, f_1)}.$$

Now we claim that $a, b \in \mathbb{Z}$. Indeed, suppose on contrary and write $E(e_1, f_1) = n$ and $E(e_1, \widetilde{f}_1) = m$ such that $n \nmid m$. Then there exist $c, d \in \mathbb{Z}$ such that

$$0 < cn + dm = (n, m) < n.$$

Therefore $E(e_1, cf_1 + d\widetilde{\ell}) = cE(e, f_1) + dE(e, \widetilde{\ell}) < n$, which contradicts to the choice of e_1, f_1. This shows $b \in \mathbb{Z}$, and by the same argument we can show $a \in \mathbb{Z}$. The claims implies

$$\Lambda = \Lambda_1 \oplus \Lambda_1^\perp.$$

By induction one shows the existence of a basis $\{e_1, f_1, \ldots, e_{g'}, f_{g'}, \ldots, e_g, f_g\}$ of Λ such that

$$\Lambda = \bigoplus_{i=1}^{g} \Lambda_i,$$

where $\Lambda_i = \operatorname{span}_{\mathbb{Z}}\{e_i, f_i\}$. If we define $n_i = E(e_i, f_i) \in \mathbb{Z}$, then $0 < n_1 \mid n_2 \mid \cdots \mid n_{g'}$ and $n_k = 0$ for $k > g'$.

Definition 1.3.2. Let $X = V/\Lambda$ be a complex tori and $\mathcal{L} = \mathcal{L}(h, \alpha)$ for $(h, \alpha) \in \widetilde{\operatorname{Herm}}^{\operatorname{int}}(V)$. The collection of elementary divisors $\{n_1, \ldots, n_g\}$ of $E = \operatorname{Im} h$ is called the *type* of $\mathcal{L}$.

1.4. **The dual complex tori.** Let $X = V/\Lambda$ be a complex tori of dimension g. Consider the $\mathbb{C}$-vector space $V^\vee := \operatorname{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ of $\mathbb{C}$-anti-linear forms $\ell : V \to \mathbb{C}$. The underlying $\mathbb{R}$-vector space of $V^\vee$ is isomorphic to $\operatorname{Hom}_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{R})$ by $\ell \mapsto \operatorname{Im}\ell$, and the inverse map is given by $k \mapsto \ell(z) := -k(\sqrt{-1}z) + \sqrt{-1}k(z)$. Hence the canonical $\mathbb{R}$-bilinear form

$$\begin{aligned} \langle \text{-}, \text{-} \rangle : V^\vee \times V &\to \mathbb{R} \\ (\ell, v) &\mapsto \operatorname{Im}\ell(v), \end{aligned}$$

is non-degenerate, and this implies $\Lambda^\vee := \{\ell \in V^\vee \mid \langle \ell, \Lambda \rangle \subseteq \mathbb{Z}\}$ is a lattice.

Definition 1.4.1. The *dual complex tori* is defined as

$$\widehat{X} := V^\vee/\Lambda^\vee.$$

Proposition 1.4.1. $\widehat{X} \cong \mathrm{Pic}^0(X)$.

Proof. By Appell-Humbert Theorem (Theorem 1.3.1) the map

$$\begin{aligned} \mathrm{Hom}(\Lambda, \mathrm{U}(1)) &\to \mathrm{Pic}^0(X) \\ \alpha &\mapsto \mathcal{L}(0,\alpha) \end{aligned}$$

is an isomorphism. On the other hand, the non-degeneracy of the form $\langle \text{-}, \text{-} \rangle$ implies

$$\begin{aligned} V^\vee &\to \mathrm{Hom}(\Lambda, \mathrm{U}(1)) \\ \ell &\mapsto e^{2\pi\sqrt{-1}\langle \ell, \text{-} \rangle} \end{aligned}$$

is surjective, and the kernel of this homomorphism is exactly $\Lambda^\vee$. As a consequence, it induces an isomorphism $\widehat{X} \to \mathrm{Pic}^0(X)$. □

Lemma 1.4.1. Let $\mathcal{L} = \mathcal{L}(h,\alpha)$ be a line bundle on X and $x_0 \in X$ with $z_0 \in V$ as a lifting of x_0. Then

$$T_{x_0}^* \mathcal{L}(h,\alpha) = \mathcal{L}(h, \alpha e^{2\pi\sqrt{-1}\mathrm{Im}h(z_0,\text{-})}),$$

where $T_{x_0}: X \to X$ is the translation defined by $y \mapsto y + x_0$.

Proof. Since z_0 is a lifting of x_0, then the translation T_{z_0} on V induces the translation T_{x_0} on X, and the induced map of T_{x_0} on the fundamental group Λ of X is identity. Hence if $\{\phi_\lambda\}_{\lambda\in\Lambda}$ is the cocycle class of $\mathcal{L}$, then

$$(\mathrm{id}_\Lambda \times T_{z_0})^* \phi_\lambda = \alpha(\lambda) e^{\pi h(z_0,\lambda)} e^{\pi(h(z,\lambda)+\frac{1}{2}h(\lambda,\lambda))}$$

is the cocycle class of $T_x^*\mathcal{L}$. But $\alpha(\lambda)e^{\pi h(z_0,\text{-})}$ may not be a map from $\Lambda \to \mathrm{U}(1)$, so we need to choose another representative in the cocycle class. Recall that $\phi'_\lambda \sim \phi_\lambda$ if and only if there exists $g \in \Gamma(V, \mathcal{O}_V^*)$ such that $\phi'_\lambda(z) = \phi_\lambda(z) g(z+\lambda) g(z)^{-1}$. If we choose $g(z) = e^{-\pi h(z,z_0)}$, then

$$(\mathrm{id}_\Lambda \times T_{z_0})^* \phi_\lambda g(z+\lambda) g(z)^{-1} = \alpha(\lambda) e^{2\pi\sqrt{-1}\mathrm{Im}h(z_0,\lambda)} e^{\pi h(z,\lambda)+\frac{\pi}{2}h(\lambda,\lambda)},$$

where $\alpha(\lambda)e^{2\pi\sqrt{-1}\mathrm{Im}h(z_0,\lambda)}: \Lambda \to \mathrm{U}(1)$. This shows

$$T_{x_0}^* \mathcal{L}(h,\alpha) = \mathcal{L}(h, \alpha e^{2\pi\sqrt{-1}\mathrm{Im}h(z_0,\text{-})}).$$

□

Corollary 1.4.1. The map $\phi_{\mathcal{L}}$ only depends on the first Chern class of $\mathcal{L}$.

Proof. It follows from Lemma 1.4.1 immediately. □

Corollary 1.4.2. Let $\mathcal{L}$ be a line bundle on X. Then

$$\begin{aligned} \phi_{\mathcal{L}}: X &\to \mathrm{Pic}^0(X) \\ x &\mapsto T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}, \end{aligned}$$

is a group homomorphism.

Proof. By Lemma 1.4.1 we know that $\mathcal{L}$ and $T_x^*\mathcal{L}$ have the same first Chern class for any $x \in X$. As a consequence, $T_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(X)$. □

Corollary 1.4.3. Let $\mathcal{L} = \mathcal{L}(h,\alpha)$ be a line bundle on X. Then map

$$\begin{aligned} \phi_h : V &\to V^\vee \\ z &\mapsto h(z,\text{-}) \end{aligned}$$

is the analytic representation of $\phi_{\mathcal{L}}$.

Proof. By Lemma 1.4.1 we get

$$\begin{aligned} t_x^*\mathcal{L} \otimes \mathcal{L}^{-1} &= \mathcal{L}(0, e^{2\pi\sqrt{-1}\mathrm{Im}h(z,\text{-})}) \\ &= \mathcal{L}(0, e^{2\pi\sqrt{-1}\langle \phi_h(v),\text{-}\rangle}). \end{aligned}$$

Comparing this with the isomorphism $\widehat{X} \to \mathrm{Pic}^0(X) = \mathrm{Hom}(\Lambda, \mathrm{U}(1))$ in Proposition 1.4.1 gives the assertion. □

Definition 1.4.2. A line bundle $\mathcal{L}$ on X is called *non-degenerate*, if $K(\mathcal{L})$ is finite.

Proposition 1.4.2. Let $\mathcal{L} = \mathcal{L}(h,\alpha)$ for $(h,\alpha) \in \widetilde{\mathrm{Herm}}^{\mathrm{int}}(V)$ be a line bundle. Then the following statements are equivalent:

(1) $\mathcal{L}$ is non-degenerate;
(2) $\phi_{\mathcal{L}} : X \to \widehat{X}$ is an isogeny;
(3) $\deg \phi_{\mathcal{L}} = \det(\mathrm{Im}h) \neq 0$.

Proof. If $\mathcal{L}$ is non-degenerate, then by definition $K(\mathcal{L})$ is finite and $\phi_{\mathcal{L}} : X \to \widehat{X}$ is surjective, as X is of the same dimension as $\widehat{X}$. Conversely, if $\phi_{\mathcal{L}}$ is an isogeny, then $K(\mathcal{L}) = \ker \phi_{\mathcal{L}}$ is finite by definition of isogeny. This shows (1) is equivalent to (2).

For (1) and (3): Note that by definition of degree, $\deg \phi_{\mathcal{L}} \neq 0$ if and only if $\ker \phi_{\mathcal{L}}$ is finite. Now it remains to show $\deg \phi_{\mathcal{L}} = \det(\mathrm{Im}h)$. Let

$$\Lambda(\mathcal{L}) = \{v \in V \mid \mathrm{Im}h(v,\Lambda) \subseteq \mathbb{Z}\}.$$

Then by Lemma 1.4.1 it is easy to see $K(\mathcal{L}) = \Lambda(\mathcal{L})/\Lambda$, and

$$\deg \phi_{\mathcal{L}} = \left|\frac{\Lambda(\mathcal{L})}{\Lambda}\right| = \det(\mathrm{Im}h)$$

follows from elementary linear algebra. □

Theorem 1.4.1 (seesaw theorem)**.** Let X, Y be varieties with X is complete and $\mathcal{L}$ be a line bundle on $X \times Y$. Then

(1) $Y_1 = \{y \in Y \mid \mathcal{L}|_{X\times\{y\}} \cong \mathcal{O}_{X\times\{y\}}\}$ is a Zariski closed subset of Y.
(2) There exists a line bundle $\mathcal{M}$ on Y_1 such that $\mathcal{L}|_{X\times Y_1} \cong p_Y^*\mathcal{M}$.

Corollary 1.4.4. Let X, Y be varieties with X is complete and $\mathcal{L}$ be a line bundle on $X \times Y$. If $\mathcal{L}|_{X\times\{y\}}$ is trivial for all y out of an open dense subset of Y and $\mathcal{L}|_{\{x_0\}\times Y}$ is trivial for some $x_0 \in X$, then $\mathcal{L}$ is trivial.

Corollary 1.4.5. $K(\mathcal{L})$ is a Zariski closed subset.

Proof. Recall that $x \in K(\mathcal{L})$ if and only if $T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ is trivial. If we denote $\widetilde{\mathcal{L}} = m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1}$, where $m\colon X \times X \to X$ is the addition on X, then $\widetilde{\mathcal{L}}|_{X \times \{x\}} = T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$, and thus $K(\mathcal{L})$ is a Zariski closed subset of X by Theorem 1.4.1. □

Corollary 1.4.6. Let $\mathcal{L} \in \mathrm{Pic}^0(X)$ be a line bundle which is not trivial. Then $H^k(X, \mathcal{L}) = 0$ for all $k \in \mathbb{Z}_{\geq 0}$.

Proof. Firstly, $H^0(X, \mathcal{L}) = 0$, otherwise $\mathcal{L} \cong \mathcal{O}_X$. Let k be the smallest integer such that $H^k(X, \mathcal{L}) \neq 0$. Then

$$H^k(X \times X, m^*\mathcal{L}) \neq 0$$

On the other hand,

$$H^k(X \times X, m^*\mathcal{L}) \cong H^k(X \times X, p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}) \cong \bigoplus_{i+j=k} H^i(X, \mathcal{L}) \otimes H^j(X, \mathcal{L}) = 0,$$

a contradiction. □

1.5. **Cohomology of line bundles on complex tori.**

1.5.1. *Global section.*

Proposition 1.5.1. Let $X = \mathbb{C}^g/\Lambda$ be a complex tori and $\mathcal{L} \cong \mathcal{L}(h, \alpha)$ for $(h, \alpha) \in \widetilde{\mathrm{Herm}}^{\mathrm{int}}(V)$.

(1) If there exists $0 \neq \omega \in V$ such that $h(\omega, \omega) < 0$, then $H^0(X, \mathcal{L}) = 0$.
(2) If $h > 0$, then $h^0(X, \mathcal{L}) = \sqrt{\det E}$, where $E = \mathrm{Im} h$.
(3) If $h \geq 0$ and the null space N of h is not trivial, then the natural map $\eta\colon V \to V' := V/N$ induces an epimorphism $\eta\colon X \to X' := V'/(\Lambda/N \cap \Lambda)$ of complex tori. Moreover, there exists a line bundle $\mathcal{L}' := \mathcal{L}(h', \alpha')$ over X' with $h' > 0$, such that $\mathcal{L} = \eta^*\mathcal{L}'$. In particular, $\mathcal{L}$ cannot be ample.

Proof. For a line bundle $\mathcal{L}(h, \alpha)$, the global sections of $\mathcal{L}$ are identified with holomorphic functions on $V = \mathbb{C}^g$ with automorphic factor. To be explicit,

$$H^0(X, \mathcal{L}) = \{\theta(z) \in H^0(V, \mathcal{O}_V) \mid \theta(z + \ell) = \theta(z) \cdot \phi_\ell(z)\},$$

where

$$\phi_\ell(z) = e^{\pi h(z,\ell) + \frac{\pi}{2} h(\ell,\ell)}.$$

For (1). Suppose there exists $0 \neq w \in V$ such that $h(w, w) < 0$. Let W be a complex subspace of V of positive dimension containing w and K be a compact subset of V such that $V = K + L$. Let $z_0 \in V$ and $w \in W$, and write $w = k + \ell$, where $k \in K$ and $\ell \in \Lambda$. We have

$$\begin{aligned}|\theta(z_0 + w)| &= |\theta(z_0 + k + \ell)| \\ &= |\theta(z_0 + k)| e^{\pi \mathrm{Re} h(z_0 + k, \ell) + \frac{\pi}{h}(\ell,\ell)}.\end{aligned}$$

Note that

$$\begin{aligned}\mathrm{Re} h(z_0 + k, \ell) + \frac{1}{2} h(\ell, \ell) &= \mathrm{Re} h(z_0 + k, w) - \mathrm{Re} h(z_0 + k, k) + \frac{1}{2} h(w, w) + \frac{1}{2} h(k, k) - \mathrm{Re} h(w, k) \\ &= \frac{1}{2} h(w, w) + \mathrm{Re} h(z_0, w) + c(k, z_0).\end{aligned}$$

Of the terms on the right, for fixed z_0, the first term is a real negative definite quadratic form in w, the second term is linear in w and the third term is bounded since K is compact. This shows $|\theta(z_0+w)|$ tends to $-\infty$ as $w \to \infty$ in W. By applying the maximum principle to $|\theta(z_0+w)|$ as a function of w, we conclude[4] $\theta(z_0+w)=0$, hence $\theta \equiv 0$. This shows $\mathcal{L}(h,\alpha)$ has no non-zero global section.

For (2). By Lemma 1.3.1, we have $c_1(\mathcal{L}) = \mathrm{Im} h$. Thus $h>0$ is equivalent to $\mathcal{L}$ is an ample line bundle. By Kodaira vanishing theorem, we have $H^i(X,\mathcal{L}) = 0$ for all $i>0$, and thus $\chi(X,\mathcal{L}) = h^0(X,\mathcal{L})$. By Riemann-Roch theorem, we have

$$\begin{aligned} h^0(X,\mathcal{L}) &= \deg(\mathrm{ch}(\mathcal{L})\,\mathrm{td}(X)) \\ &= \frac{1}{g!}c_1(\mathcal{L})^g, \end{aligned}$$

where $\mathrm{td}(X) = \mathrm{td}(\mathcal{T}_X) = 1$ as $\mathcal{T}_X$ is trivial. We take $\{\mathrm{d}x^i, \mathrm{d}y^i\}$ to be dual basis to a symplectic basis $\{e_i, f_i\}$ for $L = H_1(X,\mathbb{Z})$. Then by (3) of Lemma 1.3.1, we have

$$c_1(\mathcal{L}) = E = \sum_{i=1}^{g} n_i \mathrm{d}x^i \wedge \mathrm{d}y^i.$$

In particular, we have

$$\begin{aligned} c_1(\mathcal{L})^g &= \bigwedge^g \left(\sum_{i=1}^{g} n_i \mathrm{d}x^i \wedge \mathrm{d}y^i \right) \\ &= g! \prod_{i=1}^{g} n_i \mathrm{d}x^1 \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}x^g \wedge \mathrm{d}y^g. \end{aligned}$$

Therefore, $h^0(X,\mathcal{L}(h,\alpha)) = \prod_{i=1}^{g} n_i = \sqrt{\det E}$.

For (3). Since the symplectic form $E = \mathrm{Im} h$ is not non-degenerates, by Lemma 1.3.3 we decompose the lattice Λ as $\Lambda = \Lambda' \oplus \Lambda_{\mathrm{null}}$, and decompose V as $V = V' \oplus V_{\mathrm{null}}$, where $V' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{C}$ and $V_{\mathrm{null}} = \Lambda_{\mathrm{null}} \otimes_{\mathbb{Z}} \mathbb{C}$. Therefore, we can set

$$h' = h|_{V' \times V'}, \quad \alpha' = \alpha|_{V'}, \quad \alpha_{\mathrm{null}} = \alpha|_{V_{\mathrm{null}}}.$$

If we set $X = V'/\Lambda'$ and $X_{\mathrm{null}} = V_{\mathrm{null}}/\Lambda_{\mathrm{null}}$, then $\mathcal{L}' = \mathcal{L}'(h',\alpha')$ is a line bundle on X' and $\mathcal{L}_{\mathrm{null}} \in \mathrm{Pic}^0(X_{\mathrm{null}})$ is a degree zero line bundle given by α_{null}.

Let $p\colon X \to X'$ and $q\colon X \to X_{\mathrm{null}}$ be the natural projections. Then

$$\mathcal{L} \cong p^*\mathcal{L}' \otimes q^*\mathcal{L}_{\mathrm{null}}$$

As p is connected and proper, then

$$H^0(X,\mathcal{L}) = H^0(X', p_*(p^*\mathcal{L}' \otimes q^*\mathcal{L}_{\mathrm{null}})) = H^0(X',\mathcal{L}') \oplus H^0(X_{\mathrm{null}},\mathcal{L}''),$$

since $p_*(p^*\mathcal{L}' \otimes q^*\mathcal{L}_{\mathrm{null}}) = \mathcal{L}' \oplus p_*q^*\mathcal{L}_{\mathrm{null}} = \mathcal{L}' \oplus \mathcal{O}_{X'}^{\oplus H^0(X_{\mathrm{null}},\mathcal{L}_{\mathrm{null}})}$. □

[4] If there exists w_0 such that $|\theta(z_0+w_0)| \neq 0$, let's say $|\theta(z_0+w_0)| = M > 0$, then for a compact set K containing w_0, by maximum principle we know that the maximum of $|\theta(z_0+w)|$ attains at the boundary ∂K, but we can always choose K such that $|\theta(z_0+w)| < M/2$ for $w \in \partial K$, as $|\theta(z_0+w)|$ tends to 0 as w tends to ∞, a contradiction.

1.5.2. *Riemann-Roch theorem.*

Theorem 1.5.1. For all line bundles $\mathcal{L}$ on X, if $\mathcal{L} \cong \mathcal{O}_X(D)$, we have

$$\chi(\mathcal{L}) = \frac{D^g}{g!}$$
$$\chi(\mathcal{L})^2 = \deg \phi_{\mathcal{L}}.$$

1.5.3. *Mumford-Kempf vanishing.* In this section we will show the following results:

Theorem 1.5.2. Let $\mathcal{L}$ be a non-degenerate line bundle on an abelian variety X. Then

(1) there exists a unique integer $0 \le i = i(\mathcal{L}) \le g$ such that $H^p(X,\mathcal{L}) = 0$ for all $p \neq i$ and $H^i(X,\mathcal{L}) \neq 0$;
(2) let H be an ample line bundle and $p(n) = \chi(H^{\otimes n} \otimes \mathcal{L})$ be the Hilbert polynomial. Then all roots of $p(n)$ are real numbers and $i(\mathcal{L})$ equals to the number of positive roots of $p(n)$ counted with multiplicity.

Corollary 1.5.1. Let $X = V/\Lambda$ be an abelian variety and $\mathcal{L} = \mathcal{L}(h,\alpha)$ be a non-degenerate line bundle on X. The index $i(\mathcal{L})$ equals the number of negative eigenvalues of h.

Example 1.5.1. *If $\mathcal{L}$ is ample, and we simply take $H = \mathcal{L}$, then*

$$p(n) = \chi(\mathcal{L}^{\otimes(n+1)}) = \frac{\mathcal{L}^g}{g!}(n+1)^g.$$

In particular, $p(n)$ has no positive root and thus $i(\mathcal{L}) = 0$. This coincides with previous result, as we know for any ample line bundle $\mathcal{L}$ we have $h^i(X,\mathcal{L}) = 0$ for all $i > 0$ and if $\mathcal{L} = \mathcal{L}(h,\alpha)$, then $h^0(X,\mathcal{L}) = \sqrt{\det E}$, where $E = \mathrm{Im} h$.

1.6. **Poincaré bundle.** Let X be an abelian variety over $\mathbb{C}$. The dual abelian variety $\widehat{X}$ and Poincaré bundle $\mathcal{P}$ on $X \times \widehat{X}$ is introduced as the solution to the problem of representing the *Picard functor*. To be precisely, it is a functor which to any variety T associates the group of equivalence classes of line bundles on $X \times T$, where two line bundles are identified when they are isomorphic up to tensoring by the pullback of a line bundle on T.

The fact that $\widehat{X}$ represents the Picard functor means that there is a universal line bundle $\mathcal{P}$ on $X \times \widehat{X}$, called the *Poincaré bundle*. Universality means that given a variety T and a line bundle $\mathcal{L}$ on $X \times T$, whose restrictions to the fibers of $p_T : X \times T \to T$ have vanishing first Chern class, there exists a unique morphism $f : T \to \widehat{X}$ such that $\mathcal{L} \cong (\mathrm{id}_X \times f)^*\mathcal{P} \otimes p_T^*\mathcal{N}$, where $\mathcal{N}$ is a line bundle on T.

Theorem 1.6.1. There exists a unique Poincaré bundle $\mathcal{P}_X$ on $X \times \widehat{X}$, uniquely determined up to isomorphisms by the following properties:

(1) If a point $\xi \in \widehat{X}$ corresponds to a line bundle $\mathcal{L}$ on X, then

$$\mathcal{P}_\xi := \mathcal{P}_X|_{X| \times \{\xi\}} \cong \mathcal{L}.$$

(2) $\mathcal{P}_X|_{\{0\}\times\widehat{X}}$ is trivial.

Proof. Consider a Hermitian form form $h\colon (V\times V^\vee)\times(V\times V^\vee)\to\mathbb{C}$ defined by

$$h((v_1,\ell_1),(v_2,\ell_2)) = \overline{\ell_2(v_1)} + \ell_1(v_2).$$

and $\alpha\colon \Lambda\times\Lambda^\vee\to\mathbb{C}$ defined by

$$\alpha(\lambda,\ell_0) = e^{\pi\sqrt{-1}\mathrm{Im}\ell_0(\lambda)}.$$

By definition of $\Lambda^\vee$, we have h satisfies the integrality condition and α is a semicharacter with respect to h. Then by Appell-Humbert theorem the pair (h,α) defines a line bundle on $X\times\widehat{X}$.

Now it remains to check (1) and (2). Let $\{\phi_{(\lambda,\ell_0)}\colon (\Lambda\times\Lambda^\vee)\times(V\times V^\vee)\to\mathbb{C}^*\}$ be the cocycle corresponding to (h,α), which is given by

$$\phi_{(\lambda,\ell_0)}((v,\ell)) = \alpha(\lambda,\ell_0)e^{\pi h((v,\ell),(\lambda,\ell_0))+\frac{\pi}{2}h((\lambda,\ell_0),(\lambda,\ell_0))}.$$

For (1): If $\mathcal{L}\in\widehat{X}=\mathrm{Pic}^0(X)$, then there is an $\ell\in V^\vee$ such that $\mathcal{L}$ is given by the pair $(0,e^{2\pi\sqrt{-1}\mathrm{Im}\ell})$. On the other hand, the restriction $\mathcal{P}_X|_{X\times\{\mathcal{L}\}}$ is given by the restriction $\phi|_{\Lambda\times\{0\}\times V\times\{\ell\}}$, that is,

$$\phi_{(\lambda,0)}(v,\ell) = e^{\pi\ell(\lambda)},$$

where $\lambda\in\Lambda$ and $v\in V$. Since the complex structure on $V^\vee$ is the dual complex structure on V, we have $e^{\pi\overline{\ell(v)}}$ is a nowhere vanishing holomorphic function on V. Then $\phi_{(\lambda,0)}(v,\ell)$ is equivalent to the cocycle of $\mathcal{L}$, since

$$\phi_{(\lambda,0)}(v,\ell)e^{-\pi\overline{\ell(v+\lambda)}}e^{\pi\overline{\ell(v)}} = e^{2\pi\sqrt{-1}\mathrm{Im}\ell(\lambda)}$$

holds for all $\lambda\in\Lambda$.

For (2): The restriction of $\mathcal{P}_X|_{\{0\}\times\widehat{X}}$ is given by

$$\phi_{(0,\ell_0)}(0,\ell) = 1$$

for all $\ell_0\in\Lambda^\vee$ and all $\ell\in V^\vee$, which is the trivial line bundle on $\{0\}\times\widehat{X}$.

This shows the existence of the Poincaré bundle, and the uniqueness follows from Seesaw principle (Corollary 1.4.4). □

Lemma 1.6.1. Identifying $X=\widehat{\widehat{X}}$, the homomorphism

$$\begin{aligned}\phi_{\mathcal{P}_X}\colon X\times\widehat{X} &\to \widehat{X}\times X\\ z &\mapsto t_z^*\mathcal{P}_X\otimes\mathcal{P}_X^{-1}\end{aligned}$$

coincides with the isomorphism $s\colon X\times\widehat{X}\to\widehat{X}\times X$ defined by $(x,\widehat{x})\mapsto(\widehat{x},x)$. In particular, $\phi_{\mathcal{P}_X}^*\mathcal{P}_{\widehat{X}} = \mathcal{P}_X$.

Proof. In the proof of Theorem 1.6.1, we show that the Hermitian form $h = c_1(\mathcal{P}_X)$ is

$$\begin{aligned}h\colon (V\times V^\vee)\times(V\times V^\vee) &\to \mathbb{C}\\ (v_1,\ell_1),(v_2,\ell_2) &\mapsto \overline{\ell_2(v_1)}+\ell_1(v_2).\end{aligned}$$

For all $(v,\ell)\in V\times V^\vee$, we have $h((v,\ell),\text{-})\in\mathrm{Hom}_{\overline{\mathbb{C}}}(V\times V^\vee,\mathbb{C})$. Under the identification $\mathrm{Hom}_{\overline{\mathbb{C}}}(V\times V^\vee,\mathbb{C})\cong V^\vee\times V$, we have $h((v,\ell),\text{-})=(\ell,v)$. By Corollary

1.4.3 we have $h((v,\ell),\text{-})$ is the analytic representation of $\phi_{\mathcal{P}_X}$. This implies $\phi_{\mathcal{P}_X} = s$, and $s^*\mathcal{P}_{\widehat{X}} = \mathcal{P}_X$ follows from the universal property of Poincaré bundle. $\square$

Lemma 1.6.2. $\left((-1)_X \times 1_{\widehat{X}}\right)^* \mathcal{P}_X = \left(1_X \times (-1)_{\widehat{X}}\right)^* \mathcal{P}_X \cong \mathcal{P}_X^{-1}$.

Proof. Note that

$$\left(1_X \times (-1)_{\widehat{X}}\right)^* \mathcal{P}_X|_{X\times\{\widehat{x}\}} = \mathcal{P}_X|_{X\times\{-\widehat{x}\}} = \mathcal{P}_{-\widehat{x}} = \mathcal{P}_{\widehat{x}}^{-1}.$$

and $\left(1_X \times (-1)_{\widehat{X}}\right)^* \mathcal{P}_X|_{\{0\}\times\widehat{X}}$ is trivial. Then by Corollary 1.4.4 we have

$$\left(1_X \times (-1)_{\widehat{X}}\right)^* \mathcal{P}_X = \mathcal{P}_X^{-1}.$$

The same argument yields $\left((-1)_X \times 1_{\widehat{X}}\right)^* \mathcal{P}_X = \mathcal{P}_X^{-1}$. $\square$

Lemma 1.6.3. $t^*_{(x,\widehat{x})}\mathcal{P}_X \cong \mathcal{P}_X \otimes p_1^*\mathcal{P}_{\widehat{x}} \otimes p_2^*\mathcal{P}_x$ for all $(x,\widehat{x}) \in X \times \widehat{X}$, where p_1, p_2 denote the projections of $X \times \widehat{X}$ onto its factors.

Proof. By Seesaw principle one can show

$$t^*_{(0,\widehat{x})}\mathcal{P}_X = \mathcal{P}_X \otimes p_1^*\mathcal{P}_{\widehat{x}}$$

holds for all $\widehat{x} \in \widehat{X}$. On the other hand, for any $x \in X$ we have

$$t^*_{(x,0)}\mathcal{P}_X = t^*_{(x,0)}s^*\mathcal{P}_{\widehat{X}} = s^*\mathcal{P}_{\widehat{X}} \otimes s^*p_1^*\mathcal{P}_x = \mathcal{P}_x \otimes p_2^*\mathcal{P}_x.$$

Combine both statements gives the assertion. $\square$

Proposition 1.6.1. The Poincaré bundle $\mathcal{P}_X$ is a symmetric non-degenerate line bundle on $X \times \widehat{X}$ of type $(1,\ldots,1)$ and index $i(\mathcal{P}_X) = g$.

Proof. By Lemma 1.6.2 we have $\mathcal{P}_X$ is symmetric. By Lemma 1.6.3 we have

$$t^*_{(x,\widehat{x})}\mathcal{P}_X = \mathcal{P}_X$$

if and only if $x = \widehat{x} = 0$. This shows $K(\mathcal{P}_X) = 0$ and thus $\mathcal{P}_X$ is non-degenerate of type $(1,\ldots,1)$.

For the index, by Corollary 1.5.1 the index $i(\mathcal{P}_X)$ is the number of the negative eigenvalues of the Hermitian form $c_1(\mathcal{P}_X)$ on $V \times V^\vee$. By Lemma 1.6.2 we have

$$((-1)_V \times 1_{V^\vee})^* c_1(\mathcal{P}_X) = c_1(\mathcal{P}_X) = -c_1(\mathcal{P}_X).$$

Since it is non-degenerate, it must have $g = \frac{1}{2}\dim(V \times V^\vee)$ negative eigenvalues. This completes the proof. $\square$

Corollary 1.6.1.

$$h^i(\mathcal{P}_X) = \begin{cases} 1 & i = g, \\ 0 & i \neq g. \end{cases}$$

Proof. It follows directly from the definition of index. $\square$

Corollary 1.6.2.

$$\mathbf{R}^j p_{i*}\mathcal{P}_X = \begin{cases} \mathbb{C}_0, & j = g \text{ for } i = 1,2; \\ 0, & j \neq g \text{ for } i = 1,2. \end{cases}$$

Here $\mathbb{C}_0$ is the skyscraper sheaf on X respectively $\widehat{X}$ with support 0 and fiber $\mathbb{C}$.

Corollary 1.6.3. Let $e_1,\dots,e_{2g}$ be a basis of $H^1(X,\mathbb{Z})$ and $e_1^*,\dots,e_{2g}^* \in H^1(X,\mathbb{Z})^*$ be the dual basis. Denote $f_i := c_1(\mathcal{P}_X)(e_i^*)$. Then

$$c_1(\mathcal{P}_X) = \sum_{i=1}^{2g} e_i \otimes f_i \in H^1(X,\mathbb{Z}) \otimes H^1(\widehat{X},\mathbb{Z}).$$

Proof. By Lemma 1.6.2 we have

$$((-1)_V \times 1_{V^\vee})^* c_1(\mathcal{P}_X) = c_1(\mathcal{P}_X^{-1}) = -c_1(\mathcal{P}_X).$$

But $(-1)_X \times 1_{\widehat{X}}$ induces the identity on $H^2(X,\mathbb{Z}) \otimes H^0(\widehat{X},\mathbb{Z})$ as well as on $H^0(X,\mathbb{Z}) \otimes H^2(\widehat{X},\mathbb{Z})$. This shows $c_1(\mathcal{P}_X) \in H^1(X,\mathbb{Z}) \otimes H^1(\widehat{X},\mathbb{Z})$.

Since $\mathcal{P}_X$ is non-degenerate, the first Chern class $c_1(\mathcal{P}_X)$ induces an isomorphism $H^1(X,\mathbb{Z})^* \to H^1(\widehat{X},\mathbb{Z})$ and thus $\{f_i\}$ gives a basis of $H^1(\widehat{X},\mathbb{Z})$. Since $c_1(\mathcal{P}_X) \in H^1(X,\mathbb{Z}) \otimes H^1(\widehat{X},\mathbb{Z})$, we write it as

$$c_1(\mathcal{P}_X) = \sum_{i,j=1}^{2g} c_{ij} e_i \otimes f_j,$$

where $c_{ij} \in \mathbb{Z}$. Then

$$f_k = c_1(\mathcal{P}_X)(e_k^*) = \sum_{i,j} c_{ij}(e_i \otimes f_j)(e_k^*) = c_{ij} e_k^*(e_i) f_j = \sum c_{kj} f_j.$$

This shows $c_{kj} = \delta_{jk}$. □

At the last of this section, we show some applications of Poincaré bundle. The first application is the following equivalence for line bundles are the same.

Proposition 1.6.2. For line bundles $\mathcal{L}_1, \mathcal{L}_2$ on X, the following statements are equivalent:

(1) $\mathcal{L}_1$ and $\mathcal{L}_2$ are algebraically equivalent;
(2) $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \in \mathrm{Pic}^0(X)$;
(3) $\phi_{\mathcal{L}_1} = \phi_{\mathcal{L}_2}$;
(4) $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$.

Proof. Firstly, it is clear that (2) is equivalent to (4) by Appell-Humbert Theorem (Theorem 1.3.1), and (3) is equivalent to (4) by Corollary 1.4.1.

For (1) → (4): Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are algebraically equivalent. Then by definition there exists a smooth irreducible variety T and a line bundle $\mathcal{M}$

on $X \times T$ such that $\mathcal{L}_1 = \mathcal{M}|_{X\times\{t_1\}}$ and $\mathcal{L}_2 = \mathcal{M}|_{X\times\{t_2\}}$, where t_1, t_2 are closed points of T. Then the map

$$\begin{aligned} T &\to H^2(X,\mathbb{Z}) \\ t &\mapsto c_1(\mathcal{M}|_{X\times\{t\}}) \end{aligned}$$

is a constant map, as T is connected and $H^2(X,\mathbb{Z})$ is discrete. Thus $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$.

For (2) → (1): Suppose $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \in \mathrm{Pic}^0(X)$. Define $\mathcal{M} = \mathcal{P} \otimes p_1^*\mathcal{L}_1$ on $X \times \widehat{X}$, where $p_1 : X \times \widehat{X}$ is the projection onto the first factor. Since $\mathcal{M}|_{X\times\{\mathcal{L}_1\otimes\mathcal{L}_2^{-1}\}} = \mathcal{L}_1$ and $\mathcal{M}|_{X\times\{0\}} = \mathcal{L}_2$, this shows □

The second application is the following criterion for a homomorphism $f : X \to \widehat{X}$ to be of the form $\phi_{\mathcal{L}}$ for some line bundle $\mathcal{L}$ on X.

Theorem 1.6.2. Let $f : X \to \widehat{X}$ be a homomorphism with analytic representation $F : V \to \overline{\Omega}$. The following statement are equivalent:

(1) $f = \phi_{\mathcal{L}}$ for some line bundle $\mathcal{L} \in \mathrm{Pic}(X)$;
(2) The form $F : V \times V \to \mathbb{C}$ defined by $(v,w) \mapsto F(v)(w)$ is Hermitian.

Before that, we need the following lemma:

Lemma 1.6.4 ([BL04, Lemma 2.5.6])**.** For a line bundle $\mathcal{M}$ on X and an integer n, the following statements are equivalent:

(1) $\mathcal{M} = \mathcal{L}^n$ for some line bundle $\mathcal{L}$ on X;
(2) $X_n := \ker n_X \subseteq K(\mathcal{M})$.

Proof of Theorem 1.6.2. It is clear that (1) implies (2) as the analytic representation of $\phi_{\mathcal{L}}$ is given by $c_1(\mathcal{L})$ by Corollary 1.4.3.

Conversely, suppose $F : V \times V \to \mathbb{C}$ is Hermitian. Let $\mathcal{M}$ be the pullback of the Poincaré bundle $\mathcal{P}_X$ under the homomorphism $(\mathrm{id}_X, f) : X \to X \times \widehat{X}$. The claim is $2f = \phi_{\mathcal{M}}$. Indeed, let h be the Hermitian form of Poincaré bundle. Then

$$\begin{aligned} (v,w) \mapsto (\mathrm{id}_V, F)^* h(v,w) &= h((v,F(v)),(w,F(w))) \\ &= \overline{F(w)(v)} + F(v)(w) \\ &= 2F(v)(w) \end{aligned}$$

is the Hermitian form of $\mathcal{M}$. As $(\mathrm{id}_V, F)^*\phi_h$ is the analytic representation of $\phi_{\mathcal{M}}$ and $2F$ is the analytic representation of $2f$, we have $2f = \phi_{\mathcal{M}}$, and thus $X_2 \subseteq K(\mathcal{M})$. By Lemma 1.6.4 there exists a line bundle $\mathcal{L}$ such that $\mathcal{M} = \mathcal{L}^2$. This shows

$$2f = \phi_{\mathcal{L}^2} = 2\phi_{\mathcal{L}}$$

and thus $f = \phi_{\mathcal{L}}$ since $\mathrm{Hom}(X, \widehat{X})$ is torsion-free. □

1.7. **Dual polarization.** Let $(X, \mathcal{L})$ be a polarized abelian variety of dimension g, that is, $\mathcal{L}$ is an ample line bundle. In this section we introduce there is a natural way to define a dual polarization on the dual abelian variety $\widehat{X}$.

Proposition 1.7.1 ([BL04, Proposition 14.4.1])**.** Suppose $\mathcal{L}$ is of type $(d_1, \dots, d_g)$. There is a unique polarization $\widehat{\mathcal{L}}$ on $\widehat{X}$ characterized by the following equivalent properties:

(1) $\phi_{\mathcal{L}}^* \widehat{\mathcal{L}} = \mathcal{L}^{d_1 d_g}$;

(2) $\phi_{\widehat{\mathcal{L}}} \circ \phi_{\mathcal{L}} = d_1 d_g \operatorname{id}_X$.

The line bundle $\widehat{\mathcal{L}}$ is called the *dual polarization* and $(\widehat{X}, \widehat{\mathcal{L}})$ is called the *dual polazied abelian variety*.

2. Fourier-Mukai transform

2.1. **Fourier-Mukai transform on abelian variety.** Let X be a smooth complex projective variety. The symbol $D^b(X)$ denotes the bounded derived category of the abelian category of coherent sheaves on X. If $f: X \to Y$ is a proper[5] morphism algebraic varieties, we denote by $\mathbf{R}f_*: D^b(X) \to D^b(Y)$ and $\mathbf{L}f^*: D^b(Y) \to D^b(X)$ the associated derived functors.

Remark 2.1.1. If f is flat, then the inverse image functor is exact and thus it does not need to be derived, so that we have $\mathbf{L}f^* = f^*$. Similarly, if f is an affine morphism, then the direct image functor does not need to be derived, and thus $\mathbf{R}f_* = f_*$.

Definition 2.1.1. Let X, Y be smooth proper algebraic varieties over k and projections of the Cartesian product $X \times Y$ onto the factors X, Y are denoted by π_X, π_Y respectively. Let $\mathcal{K}^\bullet$ be an object in the derived category $D^b(X \times Y)$. We define the functor

$$\Phi^{\mathcal{K}^\bullet}_{X\to Y}: D^b(X) \to D^b(Y)$$

by letting

$$\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{E}^\bullet) = \mathbf{R}\pi_{Y*}(\pi_X^*\mathcal{E}^\bullet \otimes \mathcal{K}^\bullet).$$

The complex $\mathcal{K}^\bullet$ is called the *kernel* of the functor, and $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is called the associated *integral functor*.

Definition 2.1.2. Let X, Y be smooth proper algebraic varieties over k and $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ be an integral functor.

(1) A complex $\mathcal{F}^\bullet \in D^b(X)$ satisfies the *WIT$_i$ condition* if there is a coherent sheaf $\mathcal{G}$ on Y such that $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{F}^\bullet) \cong \mathcal{G}[-i]$ in $D^b(Y)$.
(2) A complex $\mathcal{F}^\bullet \in D^b(X)$ satisfies the *IT$_i$ condition* if there is a locally free sheaf $\mathcal{G}$ on Y such that $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{F}^\bullet) \cong \mathcal{G}[-i]$ in $D^b(Y)$.

In the remaining part of this section, we assume X is an abelian variety of dimension g and $\widehat{X}$ is the dual abelian variety.

Definition 2.1.3. The *Fourier-Mukai transform* on X is defined as

$$\mathcal{S} := \Phi^{\mathcal{P}_X}_{X\to\widehat{X}}: D^b(X) \to D^b(\widehat{X}).$$

The *dual Fourier-Mukai transform* is the functor

$$\widehat{\mathcal{S}} := \Phi^{\mathcal{P}_{\widehat{X}}}_{X\to\widehat{X}}: D^b(\widehat{X}) \to D^b(X).$$

Theorem 2.1.1 ([BBHR09, Theorem 3.3])**.** For any $\mathcal{F}^\bullet \in D^b(X)$, there is an isomorphism

$$\widehat{\mathcal{S}} \circ \mathcal{S}(\mathcal{F}^\bullet) \cong \mathcal{F}^\bullet[-g].$$

[5]The properness is necessary, otherwise the direct image of a coherent sheaf may fail to be coherent.

Remark 2.1.2. If $\mathcal{F}$ is sheaf in degree zero, then there is a convergent spectral sequence

$$E_2^{p,q} = \widehat{\mathcal{S}}^p(\mathcal{S}^q(\mathcal{F})) \Rightarrow \begin{cases} \mathcal{F}, & p+q=0 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.1.1. For any $\mathcal{F}^\bullet \in D^b(X)$ and $\xi \in \widehat{X}$, there exists a canonical isomorphism

$$\mathbf{R}\Gamma(X, \mathcal{F}^\bullet \otimes \mathcal{P}_\xi) \cong \mathcal{S}(\mathcal{F}^\bullet)|_{\{\xi\}}$$

Similarly, for any $\mathcal{G}^\bullet \in D^b(\widehat{X})$, there is an isomorphism

$$\mathbf{R}\Gamma(X, \widehat{\mathcal{S}}(\mathcal{G}^\bullet) \otimes \mathcal{P}_\xi) \cong \mathcal{G}^\bullet[-g]|_{\{\xi\}}.$$

Proof. Consider the following Cartesian diagram

$$\begin{array}{ccc} X \times \{\xi\} & \longrightarrow & X \times \widehat{X} \\ \downarrow & & \downarrow \\ \{\xi\} & \longrightarrow & \widehat{X} \end{array}$$

By base change theorem it gives

$$\begin{aligned} \mathcal{S}(\mathcal{F}^\bullet)|_{\{\xi\}} &\cong \left(\mathbf{R}p_{2*}(p_1^*\mathcal{F}^\bullet \otimes \mathcal{P}_X)\right)|_{\{\xi\}} \\ &\cong \mathbf{R}\Gamma(X, \left(p_1^*\mathcal{F}^\bullet \otimes \mathcal{P}_X\right)|_{X\times\{\xi\}}) \\ &\cong \mathbf{R}\Gamma(X, \mathcal{F}^\bullet \otimes \mathcal{P}_\xi). \end{aligned}$$

This gives the proof of the first statement, and the second statement follows from the fact $\mathcal{S} \circ \widehat{\mathcal{S}} \cong [-g]$. ☐

Definition 2.1.4. If $\mathcal{F}$ be a WIT-sheaf of index i on X, then the coherent sheaf $\widehat{\mathcal{F}} := \mathcal{S}^i(\mathcal{F})$ is called *Fourier-Mukai transform* of $\mathcal{F}$.

Lemma 2.1.2. Let $\mathcal{F}$ be a coherent sheaf on X such that

$$H^j(X, \mathcal{P}_\xi \otimes \mathcal{F}) = 0 \quad \text{for all } \xi \in \widehat{X} \text{ and } j \neq i.$$

Then $\mathcal{F}$ is an IT_i-sheaf.

Proof. Note that $\left(\mathcal{P}_X \otimes p_1^*\mathcal{F}\right)|_{X\times\{\xi\}} = \mathcal{P}_\xi \otimes \mathcal{F}$ implies

$$H^j(X \times \{\xi\}, \left(\mathcal{P}_X \otimes p_1^*\mathcal{F}\right)|_{X\times\{\xi\}}) = 0$$

for $j \neq i$. Then the assertion follows from cohomology and base change theorem ([Har77, Chapter III, Theorem 12.11]). ☐

Proposition 2.1.1. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of coherent sheaves on X with $\mathcal{F}$ and $\mathcal{H}$ are WIT_i-sheaves. Then $\mathcal{G}$ is also a WIT_i-sheaf and

$$0 \to \widehat{\mathcal{F}} \to \widehat{\mathcal{G}} \to \widehat{\mathcal{H}} \to 0$$

is exact.

Proof. Since the projection p_1 is flat, the sequence

$$0 \to \mathcal{P}_X \otimes p_1^*\mathcal{F} \to \mathcal{P}_X \otimes p_1^*\mathcal{G} \to \mathcal{P}_X \otimes p_1^*\mathcal{H} \to 0$$

is again exact. Then the result follows from the long exact sequence of cohomology for the functor p_{2*}. □

Example 2.1.1. *Let X be an abelian variety and $\mathbb{C}_x$ be the skyscraper sheaf on X with support $x \in X$. Then $\mathbb{C}_x$ is an IT_0-sheaf, since $H^j(X, \mathcal{P}_\xi \otimes \mathbb{C}_x) = 0$ for all $j > 0$ and $\xi \in \widehat{X}$. Its Fourier-Mukai transform is given by*

$$\widehat{\mathbb{C}}_x = p_{2*}(\mathcal{P}_X \otimes p_1^*\mathbb{C}_x) = \mathcal{P}_X|_{\{x\}\times\widehat{X}} = \mathcal{P}_x.$$

By Theorem 2.1.1 we know that for any $\mathcal{P}_x \in \mathrm{Pic}(\widehat{X})$, it is a WIT_g-sheaf, but it is not IT_g. Moreover, by Proposition 2.1.1 we know that every skyscraper sheaf of finite support is an IT_0-sheaf.

Definition 2.1.5. A vector bundle $\mathcal{U}$ on X is called *unipotent*, if it admits a filtration

$$0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_r = \mathcal{U}_0$$

such that $\mathcal{U}_i/\mathcal{U}_{i-1} \cong \mathcal{O}_X$ for all $i = 1, \ldots, r$.

Proposition 2.1.2. A vector bundle $\mathcal{U}$ on X is unipotent if and only if $\mathcal{U}$ is a WIT_g-sheaf with $\mathrm{supp}(\widehat{U}) = \{0\} \subset \widehat{X}$.

Proof. Suppose $\mathcal{U}$ is a unipotent bundle of rank r. If $r = 1$, then the assertion follows from Example 2.1.1. If $r > 1$ and the assertion holds for all unipotent bundle of rank $< r$. Then consider the short exact sequence

$$0 \to \mathcal{U}_{r-1} \to \mathcal{U} \to \mathcal{O}_X \to 0$$

and apply Proposition 2.1.1.

Suppose $\mathcal{U}$ is a WIT_g-sheaf and $\mathrm{supp}(\widehat{U}) = \{0\}$ is of length n. If $n = 1$, then the assertion follows from Example 2.1.1. If $n > 1$, and the assertion holds for all cases with length $< n$. Then consider the short exact sequence

$$0 \to \mathcal{V} \to \widehat{\mathcal{U}} \to \mathbb{C}_0 \to 0$$

and apply Proposition 2.1.1. □

2.2. **Atiyah's classification.** Let E be an elliptic curve and $e \in E$ be the base point, which defines a line bundle $\mathcal{O}_E(e)$ of degree one. The map $\phi_{\mathcal{O}_E(e)}: E \to \widehat{E}$ is an isomorphism, since by Theorem 1.5.1 its degree equals the square of $\chi(E, \mathcal{O}_E(e)) = H^0(E, \mathcal{O}_E(e)) = 1$. In this section, we always identify E with $\widehat{E}$ by this isomorphism, and under this identification, the Fourier-Mukai transform is an auto-equivalence $D^b(E) \to D^b(E)$.

In [Ati57], M. Atiyah provides a classification of all indecomposable vector bundles on E. In this section, we introduce this classification as an application of Fourier-Mukai transform.

Lemma 2.2.1. For any $\mathcal{F}^\bullet \in D^b(E)$, we have

$$\deg(\mathcal{S}(\mathcal{F}^\bullet)) = -\mathrm{rk}(\mathcal{F}^\bullet), \quad \mathrm{rk}(\mathcal{S}(\mathcal{F}^\bullet)) = \deg(\mathcal{F}^\bullet).$$

Proof. By Lemma 2.1.1 we have

$$\mathbf{R}\Gamma(E,\mathcal{S}(\mathcal{F}^\bullet)) = \mathcal{F}^\bullet[-1]|_{\{e\}}, \quad \mathbf{R}\Gamma(E,\mathcal{F}^\bullet) = \mathcal{S}(\mathcal{F}^\bullet)|_{\{e\}}.$$

By counting dimensions we have

$$\chi(E,\mathcal{S}(\mathcal{F}^\bullet)) = \sum_i \dim \mathbf{R}^i\Gamma(E,\mathcal{S}(\mathcal{F}^\bullet)) = -\operatorname{rk}(\mathcal{F}^\bullet)$$
$$\chi(E,\mathcal{F}^\bullet) = \sum_i \dim \mathbf{R}^i\Gamma(E,\mathcal{F}^\bullet) = \operatorname{rk}(\mathcal{S}(\mathcal{F}^\bullet)).$$

On the other hand, by the Riemann-Roch theorem we have $\chi(E,\mathcal{F}^\bullet) = \deg(\mathcal{F}^\bullet)$. This completes the proof. □

Lemma 2.2.2. Let $\mathcal{E}$ be an vector bundle on E. Then the Harder-Narasimhan filtration of $\mathcal{E}$ splits.

Proof. Suppose

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

is the Harder-Narasimhan filtration of $\mathcal{E}$. Then

$$\operatorname{Ext}^1(\mathcal{E}_1, \mathcal{E}/\mathcal{E}_1) = \operatorname{Hom}(\mathcal{E}_1, \mathcal{E}/\mathcal{E}_1) = 0$$

as $\mathcal{E}_1$ is semistable and $\mu_{\max}(\mathcal{E}/\mathcal{E}_1) = \mu(\mathcal{E}_2/\mathcal{E}_1) < \mu(\mathcal{E}_1)$. □

Corollary 2.2.1. Every indecomposable vector bundle on E is semistable.

Lemma 2.2.3. Let $\mathcal{F}$ be a semistable vector bundle on E with $\mu := \mu(\mathcal{F}) < 0$. Then $\mathcal{S}(\mathcal{F})[1]$ is a semistable vector bundle on E with $\mu(\mathcal{S}(\mathcal{E})) = -\mu^{-1}$.

Proof. Without lose of generality, we may assume $\mathcal{F}$ is indecomposable. As $\mathcal{F}$ has negative slope, we have

$$H^0(E, \mathcal{F}\otimes\mathcal{L}) = \operatorname{Hom}(\mathcal{L}^{-1}, \mathcal{F}) = 0$$

for all $\mathcal{L} \in \operatorname{Pic}^0(E)$, since indecomposable bundle on elliptic curve is semistable. Then by Lemma 2.1.2 we know that $\mathcal{F}$ is an IT_1-sheaf, and thus $\mathcal{S}(\mathcal{F})[1]$ is a vector bundle, and $\mathcal{S}(\mathcal{F})$ is indecomposable, since $\mathcal{F}$ is indecomposable. The slope of $\mathcal{S}(\mathcal{F})$ follows from Lemma 2.2.1. □

Theorem 2.2.1 ([Ati57])**.** For any $\mu \in \mathbb{Q}$, let $\operatorname{Vect}(E)_\mu$ be the category of semistable bundles on E with slope μ. Then there is an equivalence between $\operatorname{Vect}(E)_\mu$ and $\operatorname{Vect}(E)_0$.

Proof. If $\mu = 0$, then there is nothing to prove. If not, then by using Lemma 2.2.1 we are allowed to replace μ by $-\mu^{-1}$. Moreover, tensoring with $\mathcal{O}_E(e)$ is also an equivalent between $\operatorname{Vect}(E)_\mu$ and $\operatorname{Vect}(E)_{\mu+1}$. Now it suffices to show why this process eventually reaches $\mu = 0$. Recall that $\mathrm{SL}(2,\mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{P}^1_{\mathbb{Q}}$ transitively, by the following way: S acts on $\mathbb{P}^1_{\mathbb{Q}}$ via $\mu \mapsto \mu^{-1}$ and T acts on $\mathbb{P}^1_{\mathbb{Q}}$ via $\mu \mapsto \mu+1$. This coincides with the effect of Fourier-Mukai transform and tensoring with line bundle $\mathcal{O}_E(e)$, and thus this completes the proof. $\square$

Corollary 2.2.2. For any $\mu \in \mathbb{Q}$, there exists a semistable bundle with slope μ on E.

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