Moduli problems of vector bundles

Bowen Liu[1]

[1] liubw22@mails.tsinghua.edu.cn

Contents

1. MODULI PROBLEMS

1.1. **Functors of points.**

Definition 1.1.1. The *functor of points* of a scheme X is a contravariant functor $h_X := \mathrm{Hom}(-,X)\colon \mathrm{Sch} \to \mathrm{Set}$, and a morphism of schemes $f\colon X \to Y$ induces a natural transformation of funtors $h_f\colon h_X \to h_Y$, given by

$$\begin{aligned} h_f(Z)\colon h_X(Z) &\to h_Y(Z) \\ g &\mapsto f\circ g, \end{aligned}$$

where Z is a scheme.

Definition 1.1.2. The contravariant functors from schemes to sets are called *presheaves* on Sch and form a category, which is denoted by $\mathrm{Psh}(\mathrm{Sch}) = \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set})$.

Example 1.1.1. *For a scheme X, $h_X(\mathrm{Spec}\, k) = \mathrm{Hom}(\mathrm{Spec}\, k, X)$ is the set of k-points of X.*

Lemma 1.1.1 (Yoneda lemma)**.** Let $\mathscr{C}$ be any category. Then for any $C \in \mathscr{C}$ and any presheaf $\mathcal{F} \in \mathrm{Psh}(\mathscr{C})$, there is a bijection

$$\{\text{natural transformsations } \eta\colon h_C \to \mathcal{F}\} \longleftrightarrow \mathcal{F}(C),$$

which is given by $\eta \mapsto \eta_C(\mathrm{id}_{\mathscr{C}})$.

Proof. To see the surjectivity: For an object $s \in \mathcal{F}(C)$, we define $\eta\colon h_C \to \mathcal{F}$ defined as follows: For $C' \in \mathscr{C}$, consider

$$\begin{aligned} \eta_{C'}\colon h_C(C') &\to \mathcal{F}(C') \\ f &\mapsto \mathcal{F}(f)(s). \end{aligned}$$

(1) It's well-defined: Since $\mathcal{F}$ is a contravariant functor, then for $f\colon C' \to C$, we have $\mathcal{F}(f)\colon \mathcal{F}(C) \to \mathcal{F}(C')$, and thus $\mathcal{F}(f)(s) \in \mathcal{F}(C')$.
(2) It's a natural transformation: Since if we take $g\colon C'' \to C'$, and consider the following diagram

$$\begin{tikzcd} h_C(C') \arrow[r, "\eta_{C'}"] \arrow[d, "h_C(g)"] & \mathcal{F}(C') \arrow[d, "\mathcal{F}(g)"] \\ h_C(C'') \arrow[r, "\eta_{C''}"] & \mathcal{F}(C''). \end{tikzcd}$$

For arbitrary $f\colon C' \to C \in h_C(C')$, note that

$$\begin{aligned} \eta_{C''} \circ h_C(g) &= \eta_{C''}(f\circ g) \\ &= \mathcal{F}(f\circ g)(s) \\ &= \mathcal{F}(g)\circ \mathcal{F}(f)(s) \\ &= \mathcal{F}(g)\circ \eta_{C'}(f). \end{aligned}$$

Thus above diagram commutes, that is, η is a natural transformation.

By construction, we have

$$\eta_C(\mathrm{id}_C) = \mathcal{F}(\mathrm{id}_C)(s) = s.$$

This proves the surjectivity.

To see the injectivity: Suppose we have two natural transformation $\eta, \eta' : h_C \to \mathcal{F}$ such that $\eta_C(\mathrm{id}_C) = \eta'_C(\mathrm{id}_C)$. Then if we want to show $\eta = \eta'$, it suffices to show for arbitrary $C' \in \mathscr{C}$, we have $\eta_{C'} = \eta'_{C'}$. Let $g : C' \to C$. Then we have the following commutative diagram

$$\begin{array}{ccc} h_C(C) & \xrightarrow{\eta_C} & \mathcal{F}(C) \\ \downarrow{\scriptstyle h_C(g)} & & \downarrow{\scriptstyle \mathcal{F}(g)} \\ h_C(C') & \xrightarrow{\eta_{C'}} & \mathcal{F}(C'). \end{array}$$

It follows that

$$\mathcal{F}(g) \circ \eta_C(\mathrm{id}_C) = \eta_{C'} \circ h_C(g)(\mathrm{id}_C) = \eta_{C'}(g),$$

and by the same argument one has $\mathcal{F}(g) \circ \eta'_C(\mathrm{id}_C) = \eta'_{C'}(g)$. Hence

$$\eta_{C'}(g) = \mathcal{F}(g) \circ \eta_C(\mathrm{id}_C) = \mathcal{F}(g) \circ \eta'_C(\mathrm{id}_C) = \eta'_{C'}(g).$$

This completes the proof. □

Corollary 1.1.1. The functor $h : \mathscr{C} \to \mathrm{Psh}(\mathscr{C})$ is fully faithful.

Proof. A functor is called fully faithful if for every $C, C' \in \mathscr{C}$, there is the following bijection

$$\mathrm{Hom}_{\mathscr{C}}\left(C, C'\right) \leftrightarrow \mathrm{Hom}_{\mathrm{Psh}(\mathscr{C})}(h_C, h_{C'}).$$

Then take $\mathcal{F} = h_{C'}$ in Yoneda lemma to conclude. □

Definition 1.1.3. A presheaf $\mathcal{F} \in \mathrm{Psh}(\mathscr{C})$ is called *reprensentable* if there exists an object $C \in \mathscr{C}$ and a natural isomorphism $F \cong h_C$.

So it's natural to ask if every presheaf F is reprensentable by a scheme X? The answer is negative, as we will see. However, we are quite interested in answering this question for special functors, known as moduli functor.

1.2. **Moduli problems.** A moduli problem is a classification problem: we have a collection of objects and we want to classify them up to some equivalence. In fact, we want more than this: we want a moduli space encodes how these objects vary continously in families.

Definition 1.2.1. A *naive moduli problem* (in algebraic geometry) is a collection $\mathcal{A}$ of objects (in algebraic geometry) and an equivalence relation $\sim$ on $\mathcal{A}$.

Example 1.2.1.

1. Let $\mathcal{A}$ be the set of k-dimensional linear subspaces of an n-dimensional vector space and $\sim$ be equality.
2. Let $\mathcal{A}$ be the collection of vector bundles on a fixed scheme X and $\sim$ be the relation given by isomorphism of vector bundles.

Our aim is to find a scheme M whose k-points are in bijection with equivalence classes $\mathcal{A}/_\sim$. Furthermore, we want M to also encode how these objects vary continously in "families".

Definition 1.2.2. Let $(\mathcal{A},\sim)$ be a naive moduli problem. Then a *moduli problem* is given by

1. sets $\mathcal{A}_S$ of families over S and an equivalence relation $\sim_S$ on $\mathcal{A}_S$ for all schemes S.
2. pullback maps $f^*\colon \mathcal{A}_S \to \mathcal{A}_T$, for every morphism of schemes $f\colon T\to S$, such that
   (1) $(\mathcal{A}_{\operatorname{Spec} k}, \sim_{\operatorname{Spec} k}) = (\mathcal{A},\sim)$;
   (2) For the identity $\mathrm{id}\colon S\to S$ and any family $\mathcal{F}$ over S, we have $\mathrm{id}^*\mathcal{F}=\mathcal{F}$;
   (3) For a morphism $f\colon T\to S$ and equivalent families $\mathcal{F}\sim_S\mathcal{G}$, we have $f^*\mathcal{F}\sim_T f^*\mathcal{G}$.
   (4) For morphisms $f\colon T\to S$, $g: S\to R$, and a family $\mathcal{F}$ over R, we have an equivalence
$$(g\circ f)^*\mathcal{F}\sim_T f^*g^*\mathcal{F}$$

Notation 1.2.1. For a family $\mathcal{F}$ over S and a point $s\colon \operatorname{Spec} k\to S$, $\mathcal{F}_s := s^*\mathcal{F}$ denotes the corresponding family over $\operatorname{Spec} k$.

Corollary 1.2.1. A moduli problem defines a functor $\mathcal{M}\in \mathrm{Psh}(\mathrm{Sch})$, given by
$$\mathcal{M}(S) := \{\text{families over } S\}/_{\sim_S}$$
$$\mathcal{M}(f\colon T\to S) := f^*\colon \mathcal{M}(S)\to\mathcal{M}(T)$$

Example 1.2.2. *Consider the naive moduli problem given by vector bundles on a fixed scheme X up to isomorphism. Then this can be extended in two different ways. The natural notion for a family over S is a locally free sheaf $\mathcal{F}$ over $X\times S$ which is flat over S, but there are two possible ways to define relations:*
$$\mathcal{F}\sim'_S\mathcal{G}\iff\mathcal{F}\cong\mathcal{G}$$
$$\mathcal{F}\sim_S\mathcal{G}\iff\mathcal{F}\cong\mathcal{G}\otimes\pi_S^*\mathcal{L}$$
where $\mathcal{L}$ is a line bundle $\mathcal{L}\to S$ and $\pi_S\colon X\times S\to S$.

1.3. **Fine moduli spaces.** The ideal is when there is a scheme that reprensents our given moduli functor.

Definition 1.3.1. Let $\mathcal{M}\colon \mathrm{Sch}\to\mathrm{Set}$ be a moduli functor. Then a scheme M is a *fine moduli space* for $\mathcal{M}$ if it reprensents $\mathcal{M}$.

Remark 1.3.1. To be explicit, the scheme M is a fine moduli space for the moduli functor $\mathcal{M}$ if there is a natural isomorphism $\eta\colon \mathcal{M}\to h_M$. Thus for every scheme S, we have a bijection
$$\eta_S\colon \mathcal{M}(S) = \{\text{families over } S\}/_{\sim_S}\longleftrightarrow h_M(S) = \{\text{morphisms } S\to M\}$$
In particular, if $S=\operatorname{Spec} k$, then the k-points of M are in bijection with the set $\mathcal{A}/_\sim$. Moreover, if $T\to S$ is a morphism between schemes, then the following diagram commutes

$$\begin{array}{ccc} \mathcal{M}(S) & \xrightarrow{\eta_S} & h_M(S) \\ \downarrow & & \downarrow \\ \mathcal{M}(T) & \xrightarrow[\eta_T]{} & h_M(T). \end{array}$$

Definition 1.3.2. Let M be a fine moduli space for $\mathcal{M}$. Then the family $\mathcal{U} \in \mathcal{M}(M)$, determined by $\mathcal{U} := \eta_M^{-1}(\mathrm{id}_M)$, is called the *universal family*.

Remark 1.3.2. For any $\mathcal{F} \in \mathcal{M}(S)$, that is, a family over a scheme S, it corresponds to a morphism $f: S \to M$. On the other hand, the family $f^*\mathcal{U}$ corresponds to the morphism $\mathrm{id}_M \circ f$.

$$\begin{array}{ccc} f^*\mathcal{U} \in \mathcal{M}(S) & \xrightarrow{\eta_S} & h_M(S) \ni \mathrm{id}_M \circ f \\ \downarrow & & \downarrow \\ \mathcal{U} \in \mathcal{M}(M) & \xrightarrow[\eta_T]{} & h_M(M) \ni \mathrm{id}_M\,. \end{array}$$

This shows families $f^*\mathcal{U}$ and $\mathcal{F}$ correspond to the same morphism, and thus

$$f^*\mathcal{U} \sim_S \mathcal{F}.$$

This shows any family is equivalent to a family obtained by pulling back the universal family, and that's why $\mathcal{U}$ is called the universal family.

Proposition 1.3.1. If a fine moduli space for moduli functor $\mathcal{M}$ exists, then it is unique up to a unique isomorphism.

Proof. Suppose $(M,\eta),(M',\eta')$ are two fine moduli space for $\mathcal{M}$. Then they are related by unique isomorphisms

$$\eta'_M \circ (\eta_M)^{-1}(\mathrm{id}_M)\colon M \to M',$$
$$\eta_{M'} \circ (\eta'_{M'})^{-1}(\mathrm{id}_{M'})\colon M' \to M.$$

$\square$

Example 1.3.1. *In this example let's show that the projective space $\mathbb{P}^n$ can be interpreted as a fine moduli space for the moduli problem of lines through the origin $V := \mathbb{A}^{n+1}$. Firstly, we need to clearify the definition of the moduli functor in this setting. A family of lines through the origin in V over a scheme S is a line bundle $\mathcal{L}$ over S which is a subbundle of the trivial vector bundle $V \times S$ over S, and two families are equivalent if and only if they are equal.*

There is a tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \subseteq V \times \mathbb{P}^n$, and the dual line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ is generated by global sections $x_0,\dots,x_n$. Given any morphism $f: S \to \mathbb{P}^n$, the line bundle $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by the global sections $f^*(x_0),\dots,f^*(x_n)$. Hence there is a surjection $\mathcal{O}_S^{\oplus n+1} \twoheadrightarrow f^*\mathcal{O}_{\mathbb{P}^n}(1)$. Dualize the above surjection we obtain an inclusion $\mathcal{L} := f^*\mathcal{O}_{\mathbb{P}^n}(1) \hookrightarrow \mathcal{O}_S^{\oplus n+1} = V \times S$. This provides a family of lines through the origin in V over S.

Conversely, let $\mathcal{L} \subseteq V \times S$ be a family of lines through the origin in V over S. Then dualize this inclusion this provides a surjection $q: V^* \times S \twoheadrightarrow \mathcal{L}^*$. The vector *bundle $V^* \times S$ is globally generated by the global sections $\sigma_0, \ldots, \sigma_n$ corresponding to the dual basis of the standard basis on V. In particular, it provides a unique morphism*

$$\begin{aligned} f: S &\to \mathbb{P}^n \\ s &\mapsto [q \circ \sigma_0(s), \ldots, q \circ \sigma_n(s)]. \end{aligned}$$

Moreover, we hace $f^* \mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{L} \subseteq V \times S$. Indeed,

Hence, there is a bijective correspondence between morphisms $S \to \mathbb{P}^n$ and families of lines through the origin in V over S, and this bijection has functoriality. In particular, the projective space $\mathbb{P}^n$ is a fine moduli space for this moduli problem and tautological line bundle is the universal family.

Example 1.3.2.

1.4. **Coarse moduli spaces.**

Definition 1.4.1. A *coarse moduli space* for a moduli functor $\mathcal{M}$ is a scheme M and a natural transformation of funtors $\eta\colon \mathcal{M} \to h_M$ such that

1. $\eta_{\operatorname{Spec} k}\colon \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is bijective;
2. For any scheme N and natural transformation $\nu\colon M \to h_N$, there exists a unique morphism of schemes $f: M \to N$ such that $\nu = h_f \circ \eta$, where $h_f: h_M \to h_N$ is the corresponding natural transformation of presheaves.

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\eta"] \arrow[dr, "\nu"'] & & h_M \arrow[dl, "h_f"] \\ & h_N & \end{tikzcd}$$

Remark 1.4.1. A coarse moduli space for $\mathcal{M}$ is unique up to unique isomorphism.

Proposition 1.4.1. Let (M, η) be a corase moduli space for a moduli problem $\mathcal{M}$. Then (M, η) is a fine moduli space if and only if

1. there exists a family $\mathcal{U}$ over M such that $\eta_M(\mathcal{U}) = \mathrm{id}_M$;
2. for families $\mathcal{F}$ and $\mathcal{G}$ over a scheme S, we have $\mathcal{F} \sim_S \mathcal{G}$ if and only if $\eta_S(\mathcal{F}) = \eta_S(\mathcal{G})$.

2. ALGEBRAIC GROUP

2.1. **Definition and examples.**

Definition 2.1.1 (algebraic group)**.** An algebraic group is a scheme G over k, with

1. multiplication $m\colon G\times G\to G$;
2. identity $e\colon \operatorname{Spec}k\to G$;
3. inversion $i\colon G\to G$.

such that the following diagrams commute

$$\begin{tikzcd} G\times G\times G \arrow[r] \arrow[d] & G\times G \arrow[d] \\ G\times G \arrow[r] & G \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}k\times G \arrow[r, "e\times\mathrm{id}"] \arrow[dr] & G\times G \arrow[d, "m"] & G\times\operatorname{Spec}k \arrow[l, "\mathrm{id}\times e"'] \arrow[dl] \\ & G & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "i\times\mathrm{id}"] \arrow[d] & G\times G \arrow[d, "m"] & G \arrow[l, "\mathrm{id}\times i"'] \arrow[d] \\ \operatorname{Spec}k \arrow[r, "e"] & G & \operatorname{Spec}k \arrow[l, "e"] \end{tikzcd}$$

Definition 2.1.2 (affine algebraic group)**.** G is an affine algebraic group, if underlying scheme is affine.

Definition 2.1.3 (group variety)**.** G is a group variety, if the underlying scheme G is a variety.

Definition 2.1.4 (homomorphism of algebraic groups)**.** A homomorphism of algebraic groups G and H is a morphism of schemes $f: G\to H$ such that the following diagram commute

$$\begin{tikzcd} G\times G \arrow[r, "m_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H\times H \arrow[r, "m_H"] & H \end{tikzcd}$$

Definition 2.1.5 (algebraic subgroup)**.** An algebraic subgroup of G is a closed subscheme H such that $H\hookrightarrow G$ is a homomorphism of algebraic groups.

Theorem 2.1.1 (Hopf)**.** There exists an equivalence of categories

$$\{\text{affine algebraic groups}\}\longleftrightarrow\{\text{finitely generated Hopf algebra over } k\}$$
$$G\mapsto \mathcal{O}(G)$$

where $\mathcal{O}(G)$ is the k-algebra of regular functions on G.

Example 2.1.1 (additive group)**.** *The additive group* $\mathbb{G}_a=\operatorname{Spec}k[t]$ *over* k *is an affine algebraic group, with underlying variety* $\mathbb{A}^1$*, and whose group structure is* *given by*

$$m^*(t) = t \otimes 1 + 1 \otimes t$$
$$e^*(t) = 0$$
$$i^*(t) = -t$$

Example 2.1.2 (multiplicative group)**.** *The multiplicative group* $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ *over* k *is the algebraic group whose underlying variety is* $\mathbb{A}^1 \setminus \{0\}$*, and whose group structure is given by*

$$m^*(t) = t \otimes t$$
$$e^*(t) = 1$$
$$i^*(t) = t^{-1}$$

Example 2.1.3 (general linear group)**.** *The general linear group* $\mathrm{GL}(n,k)$ *is an open subvariety of* $\mathbb{A}^{n^2}$ *cut out by the condition that determinant is non-zero. It's an affine variety with coordinate ring* $k[x_{ij} : 1 \le i, j \le n]_{\det(x_{ij})}$*. The co-group operations are defined by*

$$m^*(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}$$
$$e^*(x_{ij}) = 1$$
$$i^*(x_{ij}) = (x_{ij})^{-1}_{ij}$$

where $(x_{ij})^{-1}_{ij}$ *is the regular function on* $\mathrm{GL}(n,k)$ *by taking the* (i,j)*-entry of the inverse of a matrix.*

Definition 2.1.6 (linear algebraic group)**.** A linear algebraic group is an algebraic subgroup of $\mathrm{GL}(n,k)$.

Example 2.1.4. $\mathrm{SL}(n,k)$ *and* $\mathrm{O}(n)$ *are linear algebraic groups.*

Definition 2.1.7 (linear representation)**.** A linear representation of algebraic group G on a finite dimensional k-vector space V is a homomorphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$.

2.2. **Representation theory of tori.**

Definition 2.2.1 (tori)**.** Let G be an affine algebraic group over k.

1. G is a torus if $G \cong G_m^n$ for some $n > 0$.
2. A torus of G is a subgroup scheme of G which is a torus.
3. A maximal torus of G is a torus $T \subset G$ which is not contained in any other torus.

Definition 2.2.2 (character group)**.** For a torus T. $X^*(T) := \mathrm{Hom}(T, G_m)$ is called character group and $X_*(T) := \mathrm{Hom}(G_m, T)$ is called cocharacter group.

Theorem 2.2.1. The map

$$\theta : \mathbb{Z}^n \to X^*(G_m^n)$$
$$(a_1, \dots, a_n) \mapsto \{(t_1, \dots, t_n) \mapsto (t_1^{a_1}, \dots, t_n^{a_n})\}$$

Proof. It's clear well-defined and injective. To see surjectivity, for $\phi \in X^*(G_m^n)$, where

$$\phi^*: k[t,t^{-1}] \to k[t_1,\dots,t_n,t_1^{-1},\dots,t_n^{-1}]$$
$$t \mapsto \sum c_a t_1^{a_1}\dots t_n^{a_n}$$

and

$$m^*\phi^*(t) = \phi^*(t)\otimes\phi^*(t)$$

where □

Proposition 2.2.1. Let $T = G_m^n$, for a finite dimensional representation $\rho: T \to \mathrm{GL}(V)$, there exists a weight space decomposition $V = \bigoplus_{\chi\in X^*(T)} V_\chi$, where

$$V_\chi = \{v \in V \mid tv = \chi(t)v, \forall t\}$$

Remark 2.2.1. linear representation of T is one to one correspondence to $X^*(T)$-graded V spaces.

Proof. Note that each character $\chi: T \to G_m \subset \mathbb{A}^1$, then $\chi \in \mathcal{O}(T) = k[t_1,\dots,t_n,t_1^{-1},\dots,t_n^{-1}]$, and $\chi^*(T) = \mathbb{Z}^n$ forms a basis of $\mathcal{O}(T)$.

Given a representation $\phi: T \to \mathrm{GL}(V)$, choose a basis $\{e_{ij}\}$ of $\mathrm{End}\,V$, define $\phi_{ij}: T \to \mathrm{GL}(V) \subset \mathrm{End}(V) \xrightarrow{pr_{ij}} k$, that is

$$\phi = \sum \phi_{ij} e_{ij}$$

If we denote $\phi_{ij} = c_\chi^{ij}\chi$, then

$$\phi = \sum \phi_{ij} e_{ij} = \sum\sum c_\chi^{ij}\chi e_{ij} = \sum A_\chi \chi$$

where

$$A_\chi = c_\chi^{ij} e_{ij} \in \mathrm{End}\,V$$

Need to check

1. $\mathrm{im}\,A_\chi = V_\chi$.
2. $\sum A_\chi = \mathrm{id}$.
3. $\mathbb{A}_\psi A_\chi = \delta_{\chi\psi} A_\psi$.

Define $\oplus A_\chi \to V$ is an isomorphism □

2.3. **Group actions.**

Definition 2.3.1 (group action)**.** An action of an algebraic group G on a scheme X is a morphism of scheme $\sigma: G \times X \to X$ such that the following diagrams commute

$$\begin{tikzcd} \mathrm{Spec}\,k \times X \arrow[r, "e\times \mathrm{id}_X"] \arrow[dr] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} G\times G\times X \arrow[r, "\mathrm{id}_G\times\sigma"] \arrow[d, "m_G\times \mathrm{id}_X"'] & G\times X \arrow[d, "\sigma"] \\ G\times X \arrow[r, "\sigma"] & X \end{tikzcd}$$

Remark 2.3.1. If X is an affine scheme over k, and $\mathcal{O}(X)$ is its algebra of regular functions, then an action of G on X give rise to a coaction homomorphism of k-algebras, that is

$$\sigma^* : \mathcal{O}(X) \to \mathcal{O}(G \times X)$$
$$f \mapsto \sum h_i \otimes f_i$$

This gives a homomorphism $G \to \mathrm{Aut}(\mathcal{O}(X))$ where the k-algebra automorphism of $\mathcal{O}(X)$ corresponding to $g \in G$ is given by

$$f \mapsto h_i(g) \otimes f_i$$

Definition 2.3.2 (G-equivariant morphism)**.** Let $\sigma_X : G \times X \to X$ and $\sigma_Y : G \times Y \to Y$ be group actions on schemes X and Y, a morphism $f : X \to Y$ is G-equivariant if the following diagram commute

$$\begin{array}{ccc} G \times X & \xrightarrow{\mathrm{id}_G \times f} & G \times Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Definition 2.3.3 (rational group action)**.** An action of G on a k-algebra A is rational if every element of A is contained in a finite dimensional G-invariant linear subspace of A.

Lemma 2.3.1. Let G be an affine algebraic group acting on an affine scheme X, then every $f \in \mathcal{O}(X)$ is contained in a finite dimensional G-invariant subspace of $\mathcal{O}(X)$. Furthermore, for any finite dimensional vector subspace W of $\mathcal{O}(G)$, there is a finite dimensional G-invariant vector subspace V of $\mathcal{O}(X)$ containing W.

Theorem 2.3.1. Any affine algebraic group over k is a linear algebraic group.

2.4. **Orbits and stabilisers.**

Definition 2.4.1. Let G be a linear algebraic group acting on a scheme X by $\sigma : G \times X \to X$.

1. The orbit Gx of $x \in X(k)$ is the set theoretic image of $\sigma_x : G \times \operatorname{Spec} k \overset{\mathrm{id} \times x}{\hookrightarrow} G \times X \to \overset{\sigma}{\to} X$
2.

Proposition 2.4.1. Let G be a linear algebraic group variety, acting on a variety X.

1. The orbits of closed points are locally closed subsets of X. Hence, it can be identified with the reduced locally closed subschemes.
2. The boundary $\overline{GX(k)} \backslash GX(k)$ is a union of orbits of strictly smaller dimension. In particular, orbits of minimal dimensional are closed and each orbit closure contains a closed orbit.

Definition 2.4.2. An action of an affine algebraic group G on a scheme X is closed if all G-orbits in X are closed.

Theorem 2.4.1. Let G be an affine algebraic group acting on a scheme X. For $x \in X(k)$, we have

$$\dim G = \dim G_x + \dim Gx$$

Proposition 2.4.2. Let G be an affine algebraic group acting on a scheme X by a morphism $\sigma : G \times X \to X$. Then the dimension of the stabiliser subgroup (resp. orbit) viewed as a function $X \to \mathbb{N}$ is upper semi-continous (resp. lower semi-continuous), that is for every n,

$$\{x \in X \mid \dim G_x \geq n\} \text{ and } \{x \in X \mid \dim Gx \leq n\}$$

are closed in X.

3. Quotients

3.1. **Categorical quotient.**

Definition 3.1.1 (invariant morphism)**.** $\varphi\colon X \to Z$ is G-invariant, if φ is G-equivariant with Z equipped with trivial G-action.

Definition 3.1.2 (categorical quotient)**.** A categorical quotient for the action of G on a scheme X is a G-invariant morphism $\varphi\colon X \to Y$ such that for any G-invariant morphism $f : X \to Z$, there is the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & Z \\ & Y \arrow[ur, dashed] & \end{tikzcd}$$

3.2. **Good quotient.**

Definition 3.2.1 (good quotient)**.** A morphism $\varphi\colon X \to Y$ is a good quotient for the action of G on X if

1. φ is G-invariant.
2. φ is surjective.
3. If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))^G \subset \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism.
4. If W_1 and W_2 are disjoint G-invariant closed subsets, then closures of $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.
5. φ is affine.

Definition 3.2.2 (geometric quotient)**.** A good quotient is called a geometric quotient, if the preimage of each point is a single orbit.

Proposition 3.2.1. A good quotient is a categorical quotient.

Corollary 3.2.1. Let G be an affine algebraic group acting on a scheme X and $\varphi\colon X \to Y$ a good quotient, then

1. $\overline{G\cdot x_1} \cap \overline{G\cdot x_2} \neq \varnothing$ if and only if $\varphi(x_1) = \varphi(x_2)$.
2. For each $y \in Y$, the preimage $\varphi^{-1}(y)$ contains a unique closed orbit. In particular, if the action is closed, then φ is a geometric quotient.

Proof. For (1). As φ is constant on orbit closures, then $\varphi(x_1) = \varphi(x_2)$ if $\overline{G\cdot x_1} \cap \overline{G\cdot x_2} \neq \varnothing$. Conversely, if $\varphi(x_1) = \varphi(x_2)$, $\square$

Remark 3.2.1.

(1) says that the good quotient space Y is the set of orbits modulo $\sim$, where $G\cdot x_1 \sim G\cdot x_2$ if and only if $\overline{G\cdot x_1} \cap \overline{G\cdot x_2} \neq \varnothing$.
(2) says that there exists a unique representative, that is a closed orbit in each equivalent class.

Example 3.2.1. *Consider $\mathbb{G}_m$ acting on $\mathbb{A}^2$ via $t(x,y) = (tx, t^{-1}y)$, we claim*

$$\begin{aligned}\varphi\colon \mathbb{A}^2 &\to A\\ (x,y) &\mapsto xy\end{aligned}$$

is a good quotient.

For (4). If W is G-invariant closed, then it's union of orbits:

1. *finitely many orbits, then it's union of orbits closure*
2. *infinitely many orbits, then $I(W) \subset (xy-\alpha_i)$ for infinitely many α_i, then $b \in I(W)$ divides by infinitely many $(xy-\alpha_i)$, that is $I(W) = 0$.*

For (5).

Remark 3.2.2. Note that φ is not a geometric quotient, since $\varphi^{-1}(0)$ contains three orbits, so it gives an example for good quotient which is not geometric quotient. There is also an example for categorical quotient which is not good quotient, see [**?**].

Proposition 3.2.2. The geometric or good quotient is local in the target.

4. Reductivity

4.1. **Reductive groups.** Let G be an algebraic group acting on a finitely generated k-algebra A, then Hilbert's 14 problem when does A^G is finitely generated. It fails for general algebraic group, and that's a motivation for reductive group.

In differential geometry, there is the following categorical equivalent

$$\{\text{real compact Lie group}\} \longleftrightarrow \{\text{complex reductive group}\}$$

From a compact Lie group K, one can construct a real algebraic group $K_{\mathbb{R}}$, then consider its $\mathbb{C}$-points to obtain a complex reductive group $K_{\mathbb{C}}$

Example 4.1.1. *The complexification of* $\mathrm{U}(n)$ *is* $\mathrm{GL}(n,\mathbb{C})$ *and the complexification of* $\mathrm{SU}(n)$ *is* $\mathrm{SL}(n,\mathbb{C})$.

Example 4.1.2. *For* $\mathrm{U}(1) = \{z \in \mathbb{C} \mid z\overline{z} = 1\}$, *one has*

$$K_{\mathbb{R}} = \mathrm{Spec}\,\frac{\mathbb{R}[x,y]}{(x^2+y^2-1)}$$
$$K_{\mathbb{C}} = \{(x,y) \in \mathbb{C}^2 \mid x^2+y^2 = 1\} = \mathbb{C}^*$$

Definition 4.1.1. Let G be an affine algebraic group, an element $g \in G$ is semisimple(unipotent), if there exists a faithful representation $\rho\colon G \to \mathrm{GL}(n)$ such that $\rho(g)$ is semisimple(unipotent).

Definition 4.1.2. An affine algebraic group is unipotent if every non-trivial reprensentation $\rho\colon G \to \mathrm{GL}(V)$ has a non-zero G-invariant vector.

Definition 4.1.3. Let $\mathcal{U}_n$ denote the upper triangular matrices with diagonamal elements 1.

Proposition 4.1.1. The following are equivalent

1. G is unipotent;
2. For all representation $\rho\colon G \to \mathrm{GL}(V)$, there exists a basis of V such that

$$\rho(G) \subset \mathcal{U}_n \subset \mathrm{GL}_n \cong \mathrm{GL}(V)$$

3. G is isomorphic to a subgroup of $\mathcal{U}_n$.

Proof. For (2) to (1): If $\{e_1,\dots,e_n\}$ is a basis of V such that $\rho(G) \subseteq \mathcal{U}_n$, then e_1 is fixed.

For (1) to (2):

For (2) to (3): It's trivial, since we already know affine implies linear algebraic group.

For (3) to (2): □

Remark 4.1.1. If G is a unipotent affine algebraic group, then every $g \in G(k)$ is unipotent. If G is smooth, then the converse is true.

Proposition 4.1.2. An algebraic group G is seperated.

Thus group variety is a reduced algebraic group, and a reduced algebraic group is the same as smooth algebraic group.

Theorem 4.1.1 (Catier)**.** If $\operatorname{char} k = 0$, then every affine algebraic group over k is smooth.

Definition 4.1.4 (reductive)**.** An affine algebraic group G is reductive if it's smooth and every smooth connected unipotent normal subgroup of G is trivial.

Example 4.1.3. GL_n *is reductive.*

Example 4.1.4. $\mathbb{G}_a$ *is not reductive, since* $\mathbb{G}_a$ *is isomorphic to a subgroup of* $\mathcal{U}_2$*, that is*

$$c \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

Example 4.1.5. *Torus* $\mathbb{G}_m^n$ *is linearly reductive, since by previous result, one has*

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi$$

where $V_\chi = \{v \in V \mid tv = \chi(t)v\}$*. So any* $\langle w \rangle \subset V_\chi$ *is a representation, then*

$$V_\chi = \oplus_{r_i} V_\chi^{r_i}$$

This completes the proof.

Proposition 4.1.3. The following are equivalent

1. G is linearly reductive;
2. For all $\rho \colon G \to \mathrm{GL}(V)$, any G-invariant subspace $V' \subseteq V$ admits a G-invariant complement;
3. For any surjection of G-representations $\phi \colon V \to W$, the induced map $\phi^G : V^G \to W^G$ is surjective.
4.

Example 4.1.6. $\mathbb{G}_a$ *is not linearly reductive. Use (2), consider*

$$\begin{aligned} \rho \colon \mathbb{G}_a &\to \mathrm{GL}_2 \\ c &\mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Consider $V_1 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \subseteq k^2$*, then there is no* G*-stable complement.*

4.2. **Reynolds operator.**

Definition 4.2.1 (Reynolds operator)**.** Let G be a group acting on a k-algebra A, a G-equivariant linear map $R_A : A \to A^G$ is called Reynolds operator if it's projection from A to A^G.

Example 4.2.1. *Let G be a finite group acting on a k-vector space V with* $\operatorname{char} k \nmid |G|$*, then Reynolds operator is given by*

$$R_V : V \to V^G$$
$$v \mapsto \frac{1}{|G|}\sum_{g\in G} gv$$

Lemma 4.2.1. Let G be a linearly reductive group acting rationally on a finitely generated k-algebra, then there exists a Reynolds operator $R_A : A \to A^G$.

Corollary 4.2.1. Let A,B be k-algebras admitting rational action of a linearly reductive group G, with Reynolds operators R_A and R_B, then any G-equivariant homomorphism $h : A \to B$, the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow{\scriptstyle R_A} & & \downarrow{\scriptstyle R_B} \\ A^G & \xrightarrow{h} & B^G \end{array}$$

Corollary 4.2.2. Let A be a k-algebras admitting a rational action of a linearly reductive group G with Reynolds operator R_A, then for $a \in A^G, b \in A$, one has $R_A(ab) = aR_A(b)$.

Proof. For $a \in A^G$, the homomorphism l_a given by

$$l_a : A \to A$$
$$b \mapsto ab$$

is G-equivariant, then above Corollary completes the proof. □

Lemma 4.2.2. Let A be a k-algebra admitting a rational action of a linearly reductive group G with Reynolds operator R_A. Then for any ideal $I \subset A^G$, we have $IA \cap A^G = I$. More generally, if $\{I_j\}_{j\in J}$ are a set of ideals in A^G, then we have

$$(\sum_{j\in J} I_j A) \cap A^G = \sum_{j\in J} I_j$$

In particular, if A is noetherian, then so is A^G.

Theorem 4.2.1 (Hilbert, Mumford)**.** Let G be a linearly reductive group acting rationally on a finitely generated k-algebra A, then A^G is finitely generated.

Theorem 4.2.2 (Popov)**.** An affine algebraic group G over k is reductive if and only if for every rational G-action on a finitely generated k-algebra A, A^G is finitely generated.

5. WHAT'S GIT?

5.1. **Introduction.** It is the study of quotients in the context of algebraic geometry. Many objects we want to take a quotient always have some sort of geometric structures, and GIT allow us to construct quotients that preserve geometric structure.

We let a group G act on a geometric object X. The action of G gives a partition of X into G-orbits, so we can take quotient X/G. However, it is not always the case that the set of G-orbits has a geometric structure. We take the situation in smooth manifold for an example:

Suppose G is a Lie group and X is a smooth manifold, the quotient X/G will not always have the structure of a smooth manifold(For example, the presence of non-closed orbits, usually gives a non-Hausdorff quotient). However, if G acts properly and freely, then X/G has a smooth manifold structure, such that natural projection $\pi : X \to X/G$ is a smooth submanifold.

GIT consider the same thing under the context of algebraic geometry. As we can see in smooth manifold, only certain types of group (compared with Lie group) and group actions (compared with properly and freely) are allowed in the construction of GIT.

Now we consider a quite simple example, to catch the ideal of GIT.

Example 5.1.1. Let M_n be the $n \times n$ matrices over $\mathbb{C}$, then we can give M_n a geometric structure by regarding it as an affine variety. Consider the conjugate action of GL_n on M_n. Can we regard $\mathrm{M}_n/\mathrm{GL}_n$ as a variety? The answer is yes and we will show it later.

However, good thing does not happen always, consider

Example 5.1.2. Let $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by $\lambda.(x, y) := (\lambda x, \lambda y)$. The $\mathbb{C}^\times$-orbits are $\{(\lambda x, \lambda y) : \lambda \in \mathbb{C}^\times, (x, y) \neq (0,0)\}$ as well as the origin $\{(0,0)\}$. However, this set of orbits can not have a structure of variety.

So we need to be more careful when we constructing quotients in the category of varieties.

Definition 5.1.1. A morphism $f : X \to Y$ is called G-invariant morphism, if it is constant on orbits.

Definition 5.1.2. In any category, we call a G-invariant morphism $\pi : X \to Y$ is categorical quotient of X by G, when for any G-invariant morphism $f : X \to Z$, we have that f factors uniquely through π. That is, there exists a unique $\overline{f}$ such that $\overline{f} \circ \pi = f$, for any G-invariant morphism f.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \\ & Y \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Since categorical quotient is defined by its universal property, so it is unique when it exists. However, for a quotient in the category of varieties, simple being a categorical quotient may not have a good geometric properties, so we need to define good categorical quotient.

If G acts on a variety X, then we can get an action on the regular functions on X as follows. For $f \in \mathcal{O}(U), U \subset X$, we define

$$g.f(x) = f(g^{-1}.x)$$

For the types of group action we are interested in, we require

$$g.f \in \mathcal{O}(U), \quad \forall f \in \mathcal{O}(U)$$

Definition 5.1.3. A surjective G-invariant map of varieties $p : X \to Y$ is called a good categorical quotient of X by G, if the following three properties holds

1. For all open $U \subset Y$, $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))^G$ is an isomorphism.
2. If $W \subseteq X$ is closed and G-invariant, then $p(W) \subset Y$ is closed.
3. If $V_1, V_2 \subseteq X$ are closed, G-invariants, and $V_1 \cap V_2 = \varnothing$, then $p(V_1) \cap p(V_2) = \varnothing$.

Remark 5.1.1. Note that the first requirement implies a good categorical quotient must be a categorical one: If $f : X \to Z$ is a G-invariant morphism, then $f^* : \mathcal{O}(Z) \to \mathcal{O}(X)$ must embed in $\mathcal{O}(X)^G$. If p is a good categorical quotient, then p^* is an isomorphism to $\mathcal{O}(X)^G$, so

$$\begin{tikzcd}
\mathcal{O}(Z) \arrow[rr, "f^*"] \arrow[dr, dotted, "\overline{f^*}"'] & & \mathcal{O}(X)^G \arrow[r, hook] & \mathcal{O}(X) \\
& \mathcal{O}(Y) \arrow[ur, hook, two heads, "p^*"'] \arrow[u, phantom, "\circlearrowright"] & &
\end{tikzcd}$$

So f^* can factor through $\mathcal{O}(Y)$, and this factoring is unique since p^* is an isomorphism. By the anti-equivalence of category, the dual $f = \overline{f} \circ p$ is a unique factoring of f through p.

Remark 5.1.2. As we can see in the above Remark7.1.6, the first requirement already implies categorical quotient, the more restrictions intend to avoid bad situation in geometry, such as Example7.1.2.

We denote by $X//G$ the good categorical quotient, or GIT quotient, of a variety X by a group G.

In the following, we will first construct GIT quotient in affine case, and this serves as a guide for projective case: we want to glue affine quotients to get projective one, since every projective variety admits an affine covering. Unfortunately, we can not cover the whole of a projective variety, which leads to the concept of semistability.

It's natural to define $X//G = \operatorname{Spec} \mathcal{O}(X)^G$ in affine cases, since $X = \operatorname{Spec} \mathcal{O}(X)$, so G-invariant regular functions may representate the quotient we desire, but for this we require that $\mathcal{O}(X)^G$ is finitely generated.

Historically, whether the ring of invariants is finitely generated or not is knowns as Hilbert's 14-th problem. For general linear group over $\mathbb{C}$, Hilbert showed that the invariant rings are always finitely generated. However, Nagata gave an counterexample that $\mathscr{O}(X)^G$ is not finitely generated, and proved that for any reductive group, $\mathscr{O}^G$ is finitely generated.

That's why we come to reductive groups now!

5.2. **Reductive groups.** Now we focus on the reductive group which we can use to construct GIT quotient. We will define when a linear algebraic group is reductive and give some properties of it.

Definition 5.2.1. A (linear) algebraic group is a subgroup of $\mathrm{GL}(n,k)$ which is an irreducible algebraic set.

Example 5.2.1. The set of unitary matrices with determinant 1

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc - 1 = 0 \right\}$$

is an algebraic group[2].

Example 5.2.2. $k^\times$ is also an algebraic group, by the embedding $\lambda \to \lambda I$

Example 5.2.3. General linear group $\mathrm{GL}(n,k)$ is an algebraic group[3].

Definition 5.2.2. A linear algebraic group G over k is reductive if every representation $\rho : G \to \mathrm{GL}(n,k)$ has a decomposition as a direct sum of irreducible representations.

In fact, many classical groups such as $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C})$ are reductive, now we give a proof of $\mathbb{C}^\times$ is a reductive group, the proof given here follows [5].

Proposition 5.2.1. The multiplicative group $\mathbb{C}^\times$ is reductive.

Proof. Let $\rho : \mathbb{C}^\times \to \mathrm{GL}(n,\mathbb{C})$ be a representation of $\mathbb{C}^\times$, we will show ρ has a decomposition as a direct sum of irreducible representations. Assume ρ is not irreducible.

Let $\langle , \rangle$ denote the standard inner product on $V = \mathbb{C}^n$, then define

$$\langle x; y \rangle := \int_0^{2\pi} \langle \rho(e^{i\theta})x, \rho(e^{i\theta})y \rangle d\theta$$

[2] In general, special linear group $\mathrm{SL}(n)$ is always an algebraic group by considering the irreducible polynomial $\det -1$.

[3] We can check this by introducing a new variable T and consider irreducible polynomial $T \cdot \det -1$ with $n^2 + 1$ variables.

This form has the following property; $\langle\rho(g)x;\rho(g)y\rangle = \langle x;y\rangle$, where $x,y \in V, g = e^{i\psi} \in S = \{z \in \mathbb{C}^\times : |z| = 1\}$. Indeed,

$$\begin{aligned}\langle\rho(e^{i\psi})x;\rho(e^{i\psi})y\rangle &= \int_0^{2\pi} \langle\rho(e^{i\theta}\rho(e^{i\psi}))x, \rho(e^{i\theta})\rho(e^{i\psi})y\rangle d\theta \\ &= \int_0^{2\pi} \langle\rho(e^{i(\theta+\psi)})x, \rho(e^{i(\theta+\psi)})y\rangle d\theta \\ &= \int_0^{2\pi} \langle\rho(e^{i\phi})x, \rho(e^{i\phi})y\rangle d\phi, \text{ where } \phi = \theta+\psi \\ &= \langle x;y\rangle\end{aligned}$$

And also note that $\langle;\rangle$ is an inner product. If ρ is not irreducible, then there exists some $\mathbb{C}^\times$-invariant subspace U of V, let $W = U^\perp$ be the orthogonal complement of U with respect to $\langle;\rangle$. Then we can see W is S-invariant as follows

$$\begin{aligned}\langle u;\rho(g)w\rangle &= \left\langle \rho\left(g^{-1}\right)u; \rho\left(g^{-1}\right)\rho(g)w\right\rangle \\ &= \left\langle \rho\left(g^{-1}\right)u; w\right\rangle \\ &= 0\end{aligned}$$

where $w \in W, u \in U, g \in S$. The last equality holds since U is S-invariant. What we need to do is to show W is $\mathbb{C}^\times$-invaraint.

Let N be the subset of $\mathbb{C}^\times$ which leaves W invariant, it contains S obviously. We will show that this set is closed in the Zariski topology. If we can do this, since all Zariski closed subset in $\mathbb{C}^\times$ are finite sets and whole space, so we can conclude $N = \mathbb{C}^\times$, as desired.

Let $W = \text{span}\{e_1,\ldots,e_r\}$, and extends this basis to a basis $\{e_1,\ldots,e_n\}$ of V. Then we can regard W as solutions of equations

$$\langle v, e_i\rangle = 0, \quad i = r+1,\ldots,n$$

these define polynomials which take the coordinate of v as variables, which we call it f_i, so we can see W as a zero set of $\{f_{r+1},\ldots,f_n\}$.

For each $i \in \{1,\ldots,r\}, j \in \{r+1,\ldots,n\}$, consider the set $\{T \in \mathrm{GL}(V) \mid f_j(Te_i) = 0\}$. If we fix i,j, this set is the zero set of a polynomial in the coordinates of T. So it's a closed set in $\mathrm{GL}(V)$, with respect to Zariski topology. Then we have $\{T \in \mathrm{GL}(V) \mid Te_i \in W\} = \bigcap_{j=r+1}^n \{T \in \mathrm{GL}(V) \mid f_j(Te_i) = 0\}$ is closed, so

$$\{T \in \mathrm{GL}(V) \mid Te_i \in W, \forall i \in \{1,\ldots,r\}\} = \bigcap_{i=1}^r \{T \in \mathrm{GL}(V) \mid Te_i \in W\}$$

is closed, thus we have

$$\begin{aligned}\{T \in \mathrm{GL}(V) \mid Tw \in W, \forall w \in W\} &= \{T \in GL(V) : T(\lambda_1 e_1 + \ldots + \lambda_r e_r) \in U \text{ for all } \lambda_i \in \mathbb{C}\} \\ &= \{T \in GL(V) : \lambda_1(Te_1) + \ldots + \lambda_r(Te_r) \in U \text{ for all } \lambda_i \in \mathbb{C}\} \\ &= \{T \in GL(V) : Te_i \in W \text{ for each } i \in \{1,2,\ldots,r\}\}\end{aligned}$$

is closed with respect to Zariski topology, so $N = \rho^{-1}(\{T \in \mathrm{GL}(V) \mid Tw \in W, \forall w \in W\})$ is closed, as we desired. □

Thanks to Maschke's theorem, we also have the following result.

Proposition 5.2.2. Let G be a finite group, then G is reductive.

Now we many examples of reductive alegbraic group, so we can define how to act on a variety X properly.

Definition 5.2.3. For a reductive alegbraic group, we say that G acts rationally on a variety X if it acts by a morphism of varieties $G \times X \to X$

But why we need reductive groups? and why this action? There are two key properties which might answer these questions.

Lemma 5.2.1. Let G be a reductive group acting rationally on an affine variety X, then $\mathscr{O}(X)^G$ is finitely generated.

Proof. See [6] for a proof. □

The following lemma is used in the construction of GIT quotient. It allows us to find a G-invariant function which separates disjoint G-invariant sets.

Lemma 5.2.2. Let G be a reductive group acting rationally on an affine variety $X \subset \mathbb{A}^n$. Let Z_1, Z_2 be two closed G-invariant subsets of X with $Z_1 \cap Z_2 = \varnothing$. Then there exists a G-invariant function $F \in \mathscr{O}(X)^G$ such that $F(Z_1) = 1, F(Z_2) = 0$.

Proof. See [5] for a proof. □

5.3. **The affine quotient.** We now have enough tools to construct the quotient of an affine variety by a reductive group.

For an affine variety X, the quotient of X by a reductive group G is just $\mathrm{Spec}\, \mathscr{O}(X)^G$. We will prove that this construction satisfies the required conditions being a good categorical quotient.

Theorem 5.3.1. Let X be an affine variety and G be a reductive group acting rationally on X. Let $p^* : \mathscr{O}(X)^G \to \mathscr{O}(X)$ be defined by the inclusion $\mathscr{O}(X)^G \subseteq \mathscr{O}(X)$. Then the dual of this map, $p : X \to Y := \mathrm{Spec}\, \mathscr{O}(X)^G$ is a good categorical quotient.

Proof. □

Now we give a concrete example to show how powerful the GIT construction is, and gives the answer to the Example7.1.1 we mentioned at first.

Example 5.3.1. Consider the set X of 2×2 matrices over $\mathbb{C}$, embedded in $\mathbb{C}^4$ by

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \mapsto (w, x, y, z)$$

It is an affine variety obviously, and consider the general linear group acts on it by conjugate action, then as the theorem above implies

$$X//G = \mathrm{Spec}\, k[w, z, y, z]^G$$

We know that there are two important invariants under conjugate action, that is, determinant and trace. In this case they are $\det = wz - xy$ and $\operatorname{tr} = w + z$, so we have an obvious inclusion

$$k[wz-xy, w+z] \subset k[w,x,y,z]^G$$

We will show that we in fact have equality.

Let $\lambda \in \mathbb{C}^\times$ be arbitrary and consider the matrix $A = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$. For all matrices $M = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, we can calculate as follows

$$\begin{aligned} A^{-1}MA &= \begin{pmatrix} 0 & -\frac{1}{\lambda} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \\ &= \begin{pmatrix} z & \frac{y}{\lambda} \\ \lambda x & w \end{pmatrix} \end{aligned}$$

Let $f \in k[w,x,y,z]^G$, i.e. we require f satisfy that $f(w,x,y,z) = A.f(M) = f(A.M) = f(A^{-1}MA) = f(z, \frac{y}{\lambda}, \lambda x, w)$. That is

$$f(w,x,y,z) = f\left(z, \frac{y}{\lambda}, \lambda x, w\right)$$

From this equality, we can make the following observations

1. x must appear in the form xy to cancel λ in $A.f$.
2. z and w must appear in an symmetric way, i.e. must in the forms of $z+w$ or zw.

So we conclude $f \in k[xy, wz, z+w]$.

Similarly consider matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. And after the same calculation we can get

$$f(w,x,y,z) = f(w-x, w+x-y-z, y, y+z)$$

As we already have $f \in k[xy, wz, w+z]$, we can reformulate this requirement into

$$f(xy, wz, w+z) = f\left(wy + xy - y^2 - z, wy + wz - y^2 - yz, w+z\right)$$

We can see that this formular holds only when extra terms in $B.f$ must cancel with each other, which implies $f \in k[wz-xy, w+z]$, as desired. So we have the construction

$$\begin{aligned} X/\!/G &= \operatorname{Spec} k[w,x,y,z]^G \\ &= \operatorname{Spec} k[wz-xy, w+z] \\ &= \operatorname{Spec} k[u,v] \\ &= \mathbb{C}^2 \end{aligned}$$

Remark 5.3.1. There is a high-dimensional analogous: if $\mathrm{GL}(n,\mathbb{C})$ acts on $M(n,\mathbb{C})$ by conjugate action, then

$$M(n,\mathbb{C})/\!/\mathrm{GL}(n,\mathbb{C}) = \mathbb{C}^n$$

See [7] for more details.

5.4. **The projective quotient.** Now we construct projective quotient by gluing together affine quotients.

Let X be a projective variety, then X can be covered by some affine varieties X_{f_i}. In order to construct GIT quotient of X by G, it's natural for us to take quotient for every affine variety of G of the form $X_{f_i}//G = \mathrm{Spec}(\mathscr{O}(X_{f_i})^G)$, and cover the projective quotient by them. To do this, we need an action of G on the coordinates of X.

Our approach is to embed X in $\mathbb{P}^m$ for some m such that the action of G can be extended to a linear action on $\mathbb{A}^{m+1}$. This is called a linearisation of the action of G.

Definition 5.4.1. Let the group G act rationally on a projective variety X. Let $\varphi : X \hookrightarrow \mathbb{P}^m$ be an embedding of X that extends the group action, i.e. we have a rationally group action on $\mathbb{P}^m$ such that $\varphi(g.x) = g.\varphi(x)$. Let $\pi : \mathbb{A}^{m+1} \to \mathbb{P}^m$ be the natural projection. A linearisation of the action of G with respect to φ is a linear action of G on $\mathbb{A}^{m+1}$ that is compatible with the action of G on X in the following sense

1. For any $y \in \mathbb{A}^{m+1}, g \in G$
$$\pi(g.y) = g.(\pi(y))$$
2. For all $g \in G$, the map
$$\mathbb{A}^{m+1} \to \mathbb{A}^{m+1}, \quad y \mapsto g.y$$
is linear.

We write φ_G for a linearisation of the action of G with respect to φ.

Remark 5.4.1. Note that such action induces an action of G on $\mathscr{O}(X)$. we have $\mathscr{O}(X) \cong k[x_0,\dots,x_m]/I$ for some homogeneous ideal I, since X is isomorphic to the image $\varphi(X) \subseteq \mathbb{P}^m$. Using the fact that G acts on $k[x_0,\dots,k_m]$ by $g.f(x_0,\dots,x_m) := f(g^{-1}.(x_0,\dots,x_m))$, we can know that G also acts on $\mathscr{O}(X)$, and it's well-defined, since $g.f' \in I$ for $f' \in I$.

Example 5.4.1. Let $\mathbb{C}^\times$ act on $\mathbb{P}^1$ by $\lambda.(x_0, x_1) = (x_0 : \lambda x_1)$. A linearisation can be given by the obvious action on $\mathbb{A}^2$ with $\lambda.(x_0, x_1) = (x_0, \lambda x_1)$.

The above example illustrates a quite important issue when we are constructing projective quotient: good categorical quotient may not exist. The only possible G-invariant morphism sends all orbits to a point, since $(1,0),(0,1)$ are both in the closure of $(1,t)$. But this fails to seperate closed orbits, so is not a good categorical quotient.

The solution to such problem is to take an open G-invariant subset which has a good categorical quotient. We desire this subset to be covered by G-invariant open affine subsets so that we can cover the quotient by gluing together affine quotients. This leads us to the notion of semistability,

Definition 5.4.2. Let G be a reductive group acting on a projective variety X which has an embedding $\varphi : X \to \mathbb{P}^m$. A point $x \in X$ is called semistable (with respect to the linearisation φ_G) if there exists some G-invariant homogeneous polynomial f of degree greater than 0 in $\mathscr{O}(X)$, such that $f(x) \neq 0$ and X_f is affine.

Remark 5.4.2. Write $X^{\text{as}}(\varphi_G)$ for the set of semistable points of X with respect to φ_G, or just X^{as} when it's not ambiguous.

For Example7.4.3, the set of semistable points of X with respect to φ_G is $X^{\text{as}} = X_{x_0} = \mathbb{P}^1 \backslash \{(0:1)\}$. On this subset, the map to a point $p : X^{\text{as}} \to \mathbb{P}^0$ is indeed a good categorical quotient.

Theorem 5.4.1. Let G be a reductive group acting rationally on a projective variety X embedded in $\mathbb{P}^m$ with a linearisation φ_G. Let R be the coordinate ring of X, then there is a good categorical quotient

$$p : X^{\text{as}}(\varphi_G) \to X^{\text{as}(\varphi_G)} /\!/ G \cong \operatorname{Proj} R^G$$