

ON CHERN INEQUALITIES

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ABSTRACT. It is a lecture note for presentations given in the course "topics in algebraic geometry" (Spring, 2024), which is organized by Mao Sheng.

This lecture note includes classical theory about the slope stability and Bogomolov-Gieseker inequality. Also, we give a brief introduction to the generic semipositive positive of Miyaoka and some results in [Miy87].

CONTENTS

0.	Preface	2
0.1.	Conventions	2
0.2.	Notations	2
1.	Preliminaries	3
1.1.	Torsion-freeness and relexivity	3
1.2.	Chow ring	3
1.3.	Chern classes	5
1.4.	Cones of divisors and curves	6
1.5.	Asymptotic Riemann-Roch	8
2.	Techniques	9
2.1.	Semistable sheaves	9
2.2.	A numerical criterion for semistability on curves	12
2.3.	Mumford-Mehta-Ramanathan's thmsub	16
2.4.	The Bogomolov-Gieseker inequality for semistable sheaves	16
2.5.	Semistability in positive and mixed characteristic	20
2.6.	Generic semipositive theorem for cotangent bundle	21
3.	Results	23
3.1.	Semipositivity of $3c_2 - c_1^2$	23
3.2.	Non-negativity of the Kodaira dimension of minimal threefolds	23
	References	24

0. PREFACE

0.1. **Conventions.**

- (1) An (algebraic) variety over a field k is an integral separated scheme of finite type over k .
- (2) A subvariety of a variety is a closed subscheme which is a variety.
- (3) A curve, surface or a threefold means a variety of dimension 1, 2 or 3.
- (4) A point on a scheme will always be a closed point.

0.2. **Notations.**

1. PRELIMINARIES

In this section, unless otherwise specified, X always denotes a variety of dimension n over an algebraically closed field k .

1.1. Torsion-freeness and reflexivity.

1.1.1. Torsion-freeness.

Definition 1.1.1. An \mathcal{O}_X -module \mathcal{F} is said to be *locally free sheaf* if there is an open covering $\{U_i\}$ of X such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$ holds for every U_i .

Definition 1.1.2. An \mathcal{O}_X -module \mathcal{F} is said to be *coherent sheaf* if

- (1) \mathcal{F} is of finite type.
- (2) For every open subset $U \subseteq X$ and every morphism $\alpha: \mathcal{O}_U^r \rightarrow \mathcal{F}|_U$, the kernel of α is of finite type.

Definition 1.1.3. A coherent sheaf \mathcal{F} on X is *torsion-free* if a stalk \mathcal{F}_x is a torsion-free $\mathcal{O}_{X,x}$ -module for every $x \in X$.

Definition 1.1.4. A coherent subsheaf \mathcal{F} of a torsion-free sheaf \mathcal{E} is said to be *saturated* if the quotient \mathcal{E}/\mathcal{F} is again torsion-free.

Proposition 1.1.1 ([GD71, Proposition 8.4.5]). Let X, Y be two varieties and $f: X \rightarrow Y$ be a dominant morphism. Then for any torsion-free \mathcal{O}_X -module \mathcal{F} , the direct image $f_*\mathcal{F}$ is a torsion-free \mathcal{O}_Y -module.

Proposition 1.1.2 ([Ish14, Proposition 5.1.7]). Let X be a normal variety. Then every torsion-free sheaf is locally free outside a set of codimension two.

Corollary 1.1.1. Every torsion-free sheaf on a smooth curve is locally free.

1.1.2. Reflexivity.

Definition 1.1.5. A coherent \mathcal{O}_X -module \mathcal{F} is said to be *reflexive* if the canonical homomorphism $\mathcal{F} \rightarrow \mathcal{F}^{**}$ is an isomorphism.

Proposition 1.1.3. Every locally free sheaf is reflexive, and every reflexive sheaf is torsion-free.

Proof. It follows from the definitions. □

Proposition 1.1.4 ([Kob87, Proposition 5.5.18]). The dual sheaf of any coherent sheaf is reflexive.

1.2. Chow ring.

1.2.1. Cycles.

Definition 1.2.1. A k -cycle on X is a \mathbb{Z} -linear combination of irreducible subvarieties of dimension k .

Notation 1.2.1. The group of all k -cycles on X is denoted by $Z_k(X)$.

Definition 1.2.2. A *Weil divisor* on X is an $(n - 1)$ -cycle.

Definition 1.2.3. A *Cartier divisor* on X is a global section of quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$.

Definition 1.2.4. A k -cycle α on X is defined to be *rationaly equivalent to zero* if there are finitely many $(k + 1)$ -dimensional irreducible subvarieties $W_i \subseteq X$ and non-zero rational functions. $f_i \in \mathbb{C}(W_i)$ such that

$$\alpha = \sum_i [\text{div}_{W_i}(f_i)],$$

where $\text{div}_{W_i}(f_i)$ is the divisor of the rational functions f_i on W_i .

Definition 1.2.5. The group of k -cycles modulo rational equivalences is defined to be $A_k(X)$, which is said to be the k -th *Chow group*.

Example 1.2.1. $A_{n-1}(X)$ is the group of *Weil divisors modulo linear equivalence*.

Notation 1.2.2. The group of Cartier divisors modulo linear equivalence is denoted by $\text{Pic}(X)$.

Remark 1.2.1. There is a group homomorphism from $\text{Pic}(X)$ to $A_{n-1}(X)$. In general it's neither injective nor surjective, but it's injective when X is normal and an isomorphism when X is smooth.

Definition 1.2.6. The group of *cycles of codimension k modulo rational equivalence* is defined to be $A^k(X) := A_{n-k}(X)$.

1.2.2. The intersection pairing.

Theorem 1.2.1 ([Har77, Appendix A.1]). Let X be a smooth variety. There is a unique intesection product $A^r(X) \times A^s(X) \rightarrow A^{r+s}(X)$ for each r, s satisfying the axioms listed below

- (1) The intersection pairing makes makes $A^*(X)$ into a commutative associated graded ring with identity. It's called the *Chow ring* of X .
- (2) For any morphism $f: X \rightarrow Y$, $f^*: A^*(Y) \rightarrow A^*(X)$ is a ring homomorphism. If $g: Y \rightarrow Z$ is another morphism, then $f^* \circ g^* = (g \circ f)^*$.
- (3) If $f: X \rightarrow Y$ is a proper morphism, $f_*: A^*(X) \rightarrow A^*(Y)$ is a homomorphism of graded groups. If $g: Y \rightarrow Z$ is another proper morphism, then $g_* \circ f_* = (g \circ f)_*$.

- (4) If $f: X \rightarrow Y$ is a proper morphism, $x \in A^*(X)$ and $y \in A^*(Y)$, then

$$f_*(x \cdot f^*y) = f_*(x) \cdot y.$$

This is said to be the *projection formula*.

- (5) If Y, Z are cycles on X , and if $\Delta: X \rightarrow X \times X$ is the diagonal morphism, then

$$Y \cdot Z = \Delta^*(Y \times Z).$$

- (6) If Y and Z are subvarieties of X which intersect properly (meaning that every irreducible component of $Y \cap Z$ has codimension equal to $\text{codim } Y + \text{codim } Z$), then

$$Y \cdot Z = \sum i(Y, Z; W_j) W_j,$$

where the sum runs over the irreducible components W_j of $Y \cap Z$, and where the integer $i(Y, Z; W_j)$ depends only on a neighborhood of the generic point of W_j on X , which is said to be the *local intersection multiplicity* of Y and Z along W_j .

- (7) If Y is a subvariety of X , and Z is an effective Cartier divisor meeting Y properly, then $Y \cdot Z$ is just the cycle associated to the Cartier divisor $Y \cap Z$ on Y , which is defined by restricting the local equation of Z to Y .

Remark 1.2.2. If X is not smooth, the intersection pairing also makes sense in some settings, see [Ful98] for more details. For example, on any variety (or scheme), there is always an intersection pairing

$$\text{Pic}(X) \times A^k(X) \rightarrow A^{k+1}(X).$$

1.3. Chern classes.

1.3.1. Chern classes of locally free sheaf.

Definition 1.3.1. A locally free sheaf \mathcal{E} of rank r on X has *Chern classes* $c_i(\mathcal{E}) \in A^i(X)$ for all $0 \leq i \leq r$, which is defined by

$$\sum_{i=0}^r (-1)^i \pi^* c_i(\mathcal{E}) \xi^{r-i} = 0$$

in $A^r(\mathbb{P}(\mathcal{E}))$, where $\xi \in A^1(\mathbb{P}(\mathcal{E}))$ be the class of the divisor corresponding to $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ and $\pi: \mathbb{P}(\mathcal{E}) \rightarrow X$ be the projection.

Definition 1.3.2. Let \mathcal{E} be a locally free sheaf of rank r on X . The *total Chern class* is

$$c(\mathcal{E}) = c_0(\mathcal{E}) + \cdots + c_r(\mathcal{E}) \in A^*(X).$$

Proposition 1.3.1 ([Har77, Appendix A.3]).

- (1) $c_0(\mathcal{E}) = 1$ for any \mathcal{E} and $c_1(\mathcal{O}_X) = 1$ for any X .

- (2) If $f: X \rightarrow Y$ is a morphism and \mathcal{E} is locally free on Y , then $c_i(f^*\mathcal{E}) = f^*(c_i(\mathcal{E}))$.
- (3) If $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ is an exact sequence, then $c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G})$.
- (4) $c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$, where \mathcal{E}^\vee is the dual of \mathcal{E} .
- (5) $c_1(\wedge^r \mathcal{E}) = c_1(\mathcal{E})$ when \mathcal{E} has rank r .
- (6) If D is a Cartier divisor on X , then

$$c_1(\mathcal{O}_X(D)) = D.$$

1.3.2. Chern classes of coherent sheaf. Let $F(X)$ be the free abelian group generated by the set of coherent sheaves (up to isomorphisms, otherwise it's not a set) on X , that is, an element of $F(X)$ is a formal linear combination $\sum_i n_i \mathcal{F}_i$, where $n_i \in \mathbb{Z}$ and \mathcal{F}_i is coherent. Let

$$(E) \quad 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

be an exact sequence of sheaves, and we associate the element $Q(E) = \mathcal{F} - \mathcal{F}' - \mathcal{F}''$ of $F(X)$ to this exact sequence.

Definition 1.3.3. The *group of classes of sheaves* $K(X)$ on X is defined to be the quotient of $F(X)$ by the subgroup generated by the $Q(E)$, where E runs over all short exact sequences.

Definition 1.3.4. Let $F_1(X)$ be the free group generated by the set of locally free sheaves (up to isomorphisms), and $K_1(X)$ be the quotient of $F_1(X)$ by the subgroup generated by the $Q(E)$, where E runs over all short exact sequences of locally free sheaves.

Theorem 1.3.1 ([BS58]). Let X be a smooth quasi-projective variety. Then the homomorphism $\epsilon: K_1(X) \rightarrow K(X)$ is a bijection.

Corollary 1.3.1. The definition of Chern classes can be extended to arbitrary coherent sheaves.

1.4. Cones of divisors and curves.

1.4.1. The cones of divisors.

Definition 1.4.1. For two Cartier divisors D_1, D_2 on X , D_1 is *numerically equivalent* to D_2 if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curves C .

Definition 1.4.2. The *Néron-Severi group* $N^1(X)_{\mathbb{Z}}$ is the quotient group of $\text{Pic}(X)$ by numerical equivalence, and

$$N^1(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \quad N^1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Theorem 1.4.1. The Néron-Severi group $N^1(X)_{\mathbb{Z}}$ is a free abelian group of finite rank, and its rank is said to be the *Picard number*.

Definition 1.4.3. For two 1-cycles C, C' on X , C is *numerically equivalent* to C' if they have the same intersection number with every Cartier divisor.

Notation 1.4.1. The quotient group of $Z_1(X)$ by numerical equivalence is denoted by $N_1(X)_{\mathbb{Z}}$, and

$$N_1(X)_{\mathbb{Q}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \quad N_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Remark 1.4.1. The intersection pairing

$$N^1(X)_{\mathbb{Z}} \times N_1(X)_{\mathbb{Z}} \rightarrow \mathbb{Z}$$

is by definition non-degenerate.

Definition 1.4.4. The *cone of effective curves* $\text{NE}(X)_{\mathbb{R}} \subseteq N_1(X)_{\mathbb{R}}$ is the cone spanned by non-negative linear combinations of curves, and $\overline{\text{NE}}(X)_{\mathbb{R}}$ is the *cone of pseudo-effective curves*, where $N_1(X)_{\mathbb{R}}$ is endowed with its usual topology as a \mathbb{R} -vector space.

1.4.2. *Nef cones and ample cones.*

Definition 1.4.5. A Cartier divisor on X is *nef (numerically effective)* if it has non-negative intersection with every irreducible curve on X .

Definition 1.4.6. The ample classes in $N^1(X)_{\mathbb{R}}$ forms an open cone $\text{NA}(X)_{\mathbb{R}}$, which is said to be *ample cone*.

Definition 1.4.7. The nef classes in $N^1(X)_{\mathbb{R}}$ forms a closed cone $\text{Nef}(X)_{\mathbb{R}}$, which is said to be *nef cone*.

Theorem 1.4.2 ([Laz04, Theorem 1.4.23]). Let X be a projective variety.

- (1) The closure of the ample cone is the nef cone;
- (2) The interior of nef cone is the ample cone.

Theorem 1.4.3 ([Laz04, Theorem 1.4.28 and Theorem 1.4.29]). Let X be a projective variety.

- (1) The pseudo-effective cone is the closed cone dual to the nef cone, that is,

$$\overline{\text{NE}}(X)_{\mathbb{R}} = \{\gamma \in N_1(X)_{\mathbb{R}} \mid D \cdot \gamma \geq 0, \quad \forall D \in \overline{\text{NA}}(X)_{\mathbb{R}}\}.$$

- (2)

$$\text{NA}(X)_{\mathbb{R}} = \{\gamma \in N^1(X)_{\mathbb{R}} \mid D \cdot \gamma > 0, \quad \forall D \in \overline{\text{NE}}(X)_{\mathbb{R}} - \{0\}\}.$$

1.5. Asymptotic Riemann-Roch.

Theorem 1.5.1 ([Laz04, Theorem 1.1.24]). Let X be a projective variety of dimension n and D be a Cartier divisor on X . Then

$$\chi(X, \mathcal{O}(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1}).$$

More generally, for any coherent sheaf \mathcal{F} on X ,

$$\chi(X, \mathcal{F} \otimes \mathcal{O}_X(mD)) = \mathrm{rk} \mathcal{F} \cdot \frac{D^n}{n!} m^n + O(m^{n-1}).$$

2. TECHNIQUES

2.1. Semistable sheaves. Let X be a normal projective variety of dimension n over an algebraically closed field k of arbitrary characteristic.

Definition 2.1.1. The *average first Chern class* of a torsion-free sheaf \mathcal{E} is

$$\delta(\mathcal{E}) = \frac{c_1(\mathcal{E})}{\mathrm{rk} \mathcal{E}} \in A^1(X)_{\mathbb{Q}}.$$

Definition 2.1.2. For a given $(n-1)$ -tuple $\mathfrak{A} = (H_1, \dots, H_{n-1}) \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$, the *average degree* with respect to \mathfrak{A} is the rational number $\delta_{\mathfrak{A}}(\mathcal{E}) = \delta(\mathcal{E})H_1 \dots H_{n-1}$.

Definition 2.1.3. A torsion-free sheaf \mathcal{E} is said to be *semistable* if

$$\delta_{\mathfrak{A}}(\mathcal{F}) \leq \delta_{\mathfrak{A}}(\mathcal{E})$$

for every non-zero subsheaf \mathcal{F} of \mathcal{E} .

Notation 2.1. If $\mathfrak{A} = ([H], \dots, [H])$, we use the terminology H -semistable instead of \mathfrak{A} -semistable.

Theorem 2.1.1 ([HN75]). Let \mathcal{E} be a torsion-free sheaf on X and $\mathfrak{A} \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$. Then there exists a unique filtration $\Sigma_{\mathfrak{A}}$,

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_s = \mathcal{E},$$

which is called the *Harder-Narasimhan filtration*, such that

- (1) $\mathrm{Gr}_i(\Sigma_{\mathfrak{A}}) = \mathcal{E}_i / \mathcal{E}_{i+1}$ is a torsion-free \mathfrak{A} -semistable sheaf;
- (2) $\delta_{\mathfrak{A}}(\mathrm{Gr}_i(\Sigma_{\mathfrak{A}}))$ is a strictly decreasing function in i .

Sketch. Here we only give a sketch of proof of the existence. Put $\delta_{\mathfrak{A}}^{\max}(\mathcal{E}) := \sup\{\delta_{\mathfrak{A}}(\mathcal{F}) \mid 0 \neq \mathcal{F} \subseteq \mathcal{E} \text{ a coherent subsheaf}\}$. Then we need to prove that

- (1) $\delta_{\mathfrak{A}}^{\max}(\mathcal{E}) < \infty$;
- (2) There exists a saturated subsheaf $\mathcal{F}_1 \subseteq \mathcal{E}$ with maximal slope.

After that, suppose both \mathcal{F}_1 and \mathcal{F}_2 coherent subsheaves of rank r_1 and r_2 with maximal slope. By the following exact sequence

$$0 \rightarrow \mathcal{F}_1 \cap \mathcal{F}_2 \rightarrow \mathcal{F}_1 \oplus \mathcal{F}_2 \rightarrow \mathcal{F}_1 + \mathcal{F}_2 \rightarrow 0,$$

one has

$$\begin{aligned} c_1(\mathcal{F}_1 + \mathcal{F}_2) &= c_1(\mathcal{F}_1) + c_1(\mathcal{F}_2) - c_1(\mathcal{F}_1 \cap \mathcal{F}_2) \\ \mathrm{rk}(\mathcal{F}_1 + \mathcal{F}_2) &= \mathrm{rk}(\mathcal{F}_1) + \mathrm{rk}(\mathcal{F}_2) - \mathrm{rk}(\mathcal{F}_1 \cap \mathcal{F}_2). \end{aligned}$$

Then

$$\begin{aligned} \mathrm{rk}(\mathcal{F}_1 + \mathcal{F}_2)\delta_{\mathfrak{A}}(\mathcal{F}_1 + \mathcal{F}_2) &= r_1\delta_{\mathfrak{A}}(\mathcal{F}_1) + r_2\delta_{\mathfrak{A}}(\mathcal{F}_2) - \mathrm{rk}(\mathcal{F}_1 \cap \mathcal{F}_2)\delta_{\mathfrak{A}}(\mathcal{F}_1 \cap \mathcal{F}_2) \\ &\geq (r_1 + r_2)\delta_{\mathfrak{A}}^{\max}(\mathcal{E}) - \mathrm{rk}(\mathcal{F}_1 \cap \mathcal{F}_2)\delta_{\mathfrak{A}}^{\max}(\mathcal{E}) \\ &= \mathrm{rk}(\mathcal{F}_1 + \mathcal{F}_2)\delta_{\mathfrak{A}}^{\max}(\mathcal{E}). \end{aligned}$$

This shows $\mathcal{F}_1 + \mathcal{F}_2$ also has the maximal slope. By adding all these subsheaves together, this gives the *maximal \mathfrak{A} -destabilizing subsheaf* \mathcal{E}_1 . We repeat above process to obtain the maximal \mathfrak{A} -destabilizing subsheaf of $\mathcal{E}/\mathcal{E}_1$, and consider its preimage to obtain \mathcal{E}_2 , that is, $\mathcal{E}_2/\mathcal{E}_1 = (\mathcal{E}/\mathcal{E}_1)_1$. It remains to show $\delta_{\mathfrak{A}}(\mathcal{E}_1) > \delta_{\mathfrak{A}}(\mathcal{E}_2/\mathcal{E}_1)$. Indeed, otherwise we would have $\delta_{\mathfrak{A}}(\mathcal{E}_1) \leq \delta_{\mathfrak{A}}(\mathcal{E}_2)$, a contradiction. \square

Remark 2.1.1. The maximal \mathfrak{A} -destabilizing subsheaf of \mathcal{E} is characterized by the following properties:

- (1) $\delta_{\mathfrak{A}}(\mathcal{E}_1) \geq \delta_{\mathfrak{A}}(\mathcal{F})$ for every coherent subsheaf \mathcal{F} of \mathcal{E} ;
- (2) If $\delta_{\mathfrak{A}}(\mathcal{E}_1) = \delta_{\mathfrak{A}}(\mathcal{F})$ for $\mathcal{F} \subset \mathcal{E}$, then $\mathcal{F} \subset \mathcal{E}_1$.

Remark 2.1.2. The \mathfrak{A} -semistable filtration of the dual sheaf \mathcal{E}^* is essentially the same as that of \mathcal{E} , with each entry substituted by the duals of the quotient $\mathcal{E}/\mathcal{E}_{s-i}$.

Theorem 2.1.2. Let $\mathcal{E}_1^{\mathfrak{A}} \subset \mathcal{E}$ denote the maximal \mathfrak{A} -destabilizing subsheaf for $\mathfrak{A} \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$.

- (1) Let L be a closed affine segment joining $\mathfrak{A} \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$ and $\mathfrak{C} \in \overline{\mathrm{NA}}(X)_{\mathbb{R}}^{n-1}$ and $\mathfrak{B} = (1-t)\mathfrak{A} + t\mathfrak{C}$ be a \mathbb{Q} -rational point on L . Then $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) = \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ whenever $0 < t < \epsilon$, where ϵ is a positive constant depends continuously on \mathfrak{C} provided \mathcal{E} and \mathfrak{A} is fixed.
- (2) Let $K \subset \overline{\mathrm{NA}}(X)_{\mathbb{R}}^{n-1}$ be a compact subset and $\mathfrak{A} \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$ is away from K . Let $\mathfrak{A} \sharp K$ stands the union of the segments joining \mathfrak{A} and K . Then there exists an open neighborhood $U \subset N^1(X)_{\mathbb{R}}$ of \mathfrak{A} such that $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) = \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ for every $\mathfrak{B} \in U \cap (\mathfrak{A} \sharp K) \cap \overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$.
- (3) If $\mathfrak{A} \in \mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$, then there exists an open neighborhood $U \subset \mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$ of \mathfrak{A} such that $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) = \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ for every $\mathfrak{B} \in U$.

Proof. For simplicity, we show the case $n = 2$ only, as the proof is quite similar for higher dimensions, and we denote $\mathfrak{C} = H \in \overline{\mathrm{NA}}(X)_{\mathbb{Q}}$.

- (1). At first we prove that $\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}) \leq c$ for some constant c depending on \mathcal{E} , and on \mathfrak{C} continuously.

- (a) If $\mathcal{E}^*(H)$ is globally generated, that is, there exists a surjective morphism $\mathcal{O}_X^{\oplus N} \rightarrow \mathcal{E}^*(H)$ for some integer N . By taking dual we

have an injective morphism $\mathcal{E} \rightarrow \mathcal{O}_X^{\oplus N}(H)$, and thus

$$\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}) \leq c,$$

where c is a constant depending on \mathcal{E} , and on \mathfrak{C} continuously.

- (b) If H is ample, then there exists some integer m such that mH is globally generated, and thus in this case $\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}) \leq c$ for some constant c depending on \mathcal{E} , and on \mathfrak{C} continuously.
- (c) If $H \in \overline{\text{NA}}(X)_{\mathbb{Q}}$, the same result still holds, as a nef divisor is a limit of ample divisors.

Furthermore, we put $c' = \delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{A}})$. By the definition of the maximal destabilizing sheaves, we get

$$\delta_{\mathfrak{B}}(\mathcal{E}_1^{\mathfrak{A}}) \leq \delta_{\mathfrak{B}}(\mathcal{E}_1^{\mathfrak{B}}).$$

As $\delta_{\mathfrak{B}}$ is a linear function in $\mathfrak{B} = (1-t)\mathfrak{A} + t\mathfrak{C}$, this inequality is rewritten as

$$(1-t)\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}}) + t\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{A}}) \leq (1-t)\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) + t\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}).$$

Hence

$$\begin{aligned} \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) &\leq \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}}) \leq \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) + \frac{t}{1-t}(\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}) - \delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{A}})) \\ &\leq \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) + \frac{t}{1-t}(c - c'). \end{aligned}$$

Note that $\delta(\mathcal{E}_1^{\mathfrak{A}}), \delta(\mathcal{E}_1^{\mathfrak{B}}) \in (1/r!)A^1(X)_{\mathbb{Z}}$ and $\mathfrak{A} \in (1/m)N^1(X)_{\mathbb{Z}}$ for some positive integer m . Therefore, if

$$\frac{t}{1-t}(c - c') < \frac{1}{r!m},$$

then $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}}) = \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}})$.

(2). Let U be the open ball centered at \mathfrak{A} with radius $r = \inf_{\mathfrak{C} \in K} \epsilon(\mathcal{E}, \mathfrak{A}, \mathfrak{C})d(\mathfrak{A}, \mathfrak{C})$, where d stands for Euclidean metric.

(3). Let $K \subset \text{NA}(X)_{\mathbb{R}}^{n-1}$ be a sphere centered at \mathfrak{A} and apply (2). \square

Corollary 2.1.1. Given a compact subset $K \subset \overline{\text{NA}}(X)_{\mathbb{R}}^{n-1}$ and $\mathfrak{A} \in \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-1}$ is away from K , the \mathfrak{B} -semistable filtration is a refinement of \mathfrak{A} -semistable filtration for all $\mathfrak{B} \in (\mathfrak{A} \sharp K)_{\mathbb{Q}}$ sufficiently near \mathfrak{A} .

Proof. By (2) of above theorem, we have $\mathcal{E}_1^{\mathfrak{B}} \subseteq \mathcal{E}_1^{\mathfrak{A}}$ for all $\mathfrak{B} \in (\mathfrak{A} \sharp K)_{\mathbb{Q}}$ sufficiently near \mathfrak{A} . If \mathcal{E} is semistable, it's clear that the \mathfrak{B} -semistable filtration of \mathcal{E} is a refinement of \mathfrak{A} -semistable filtration of \mathcal{E} , and the general case is obtained by repeating above process for each semistable grade $\mathcal{E}_i/\mathcal{E}_{i+1}$. \square

Corollary 2.1.2. Let \mathcal{E} be a torsion-free sheaf on X . Then the function $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ is a continuous, piecewise multilinear function on $\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$, and continuous on any \mathbb{Q} -rational segment of $\overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$.

Proof. Note that if both \mathfrak{A} and \mathfrak{B} in $\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$, then there exists some open neighborhood of \mathfrak{A} containing \mathfrak{B} , and there also exists some open neighborhood of \mathfrak{B} containing \mathfrak{A} . By the symmetry we have $\mathcal{E}_1^{\mathfrak{B}} = \mathcal{E}_1^{\mathfrak{A}}$, and thus $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ is continuous on $\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$. The same argument shows $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ is also continuous in any rational segment of $\overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$. \square

2.2. A numerical criterion for semistability on curves. Throughout this section, the ground field k is always an algebraically closed field with characteristic 0 except Lemma 2.2.1, and C is a smooth complete curve.

2.2.1. Projective bundle on curves. Let \mathcal{E} be a locally free sheaf of rank r on C and $\pi: \mathbb{P}(\mathcal{E}) \rightarrow C$ the associated projective bundle with tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Definition 2.2.1. The normalized hyperplane class $\lambda_{\mathcal{E}}$ is the numerical class of $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) - \pi^* \delta(\mathcal{E}) \in N^1(\mathbb{P}(\mathcal{E}))_{\mathbb{Q}}$.

Remark 2.2.1. The normalized hyperplane class $\lambda_{\mathcal{E}}$ is uniquely determined by two properties:

- (1) $\lambda_{\mathcal{E}}^r = 0$.
- (2) $\lambda_{\mathcal{E}}$ on each fiber is numerically equivalent to the hyperplane.

Proposition 2.2.1. The class of relative anti-canonical divisor $-K_{\mathbb{P}(\mathcal{E})} + \pi^* K_C$ equals $r\lambda_{\mathcal{E}}$.

Proof. It follows from the relative Euler sequence, that is,

$$0 \rightarrow \Omega_{\mathbb{P}(\mathcal{E})/C}^1 \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \otimes \pi^* \mathcal{E}^* \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})} \rightarrow 0.$$

\square

Proposition 2.2.2 ([Har77, Exercise III 8.4]).

- (1) $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)) = \mathcal{S}^{\ell} \mathcal{E}$ for $\ell \geq 0$ and $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)) = 0$ for $\ell < 0$.
- (2) $R^i \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)) = 0$ for $0 < i < n$.
- (3) $R^n \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)) = 0$ for $\ell > -n - 1$.

Proposition 2.2.3 ([Har77, Proposition V 2.3]). The Néron-Severi group of $\mathbb{P}(\mathcal{E})$ is

$$N^1(\mathbb{P}(\mathcal{E})) = \mathbb{R} \lambda_{\mathcal{E}} \oplus \pi^* N^1(X),$$

and the group of numerically equivalent 1-cycles is

$$N_1(\mathbb{P}(\mathcal{E})) = \lambda_{\mathcal{E}}^{r-2} N^1(\mathbb{P}(\mathcal{E})).$$

2.2.2. *Criterion.*

Lemma 2.2.1. Let f be a seperable surjective k -morphism of a smooth complete curve C' onto C . Then a locally free sheaf \mathcal{E} is semistable if and only if $f^*\mathcal{E}$ is semistable.

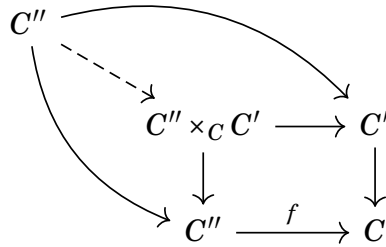
Proof. Firstly let's prove if part. Let $\mathcal{G} \subseteq \mathcal{E}$ be a non-zero subsheaf. Then $\delta(f^*\mathcal{G}) \leq \delta(f^*\mathcal{E})$ as $f^*\mathcal{E}$ is semistable, and thus $\delta(\mathcal{G}) \leq \delta(\mathcal{E})$.

Conversely, suppose \mathcal{E} is semistable. Without lose of generality we may assume f is a Galois morphism with Galois group G , which acts on $f^*\mathcal{E}$. If $f^*\mathcal{E}$ is not semistable and \mathcal{F}_1 be the maximal destabilizing subbundle of $f^*\mathcal{E}$. For any $g \in G$, we have $g^*\mathcal{F}_1 = \mathcal{F}_1$ as the maximal destabilizing subsheaf is unique. Hence there exists a subbundle \mathcal{E}_1 of \mathcal{E} such that $f^*\mathcal{E}_1 = \mathcal{F}_1$, and by if part \mathcal{E}_1 is semistable. On the other hand, by semistability we have $\mathcal{E}_1 = \mathcal{E}$, and thus $\mathcal{F}_1 = f^*\mathcal{E}$. This completes the proof. \square

Theorem 2.2.1. The following conditions are equivalent:

- (1) \mathcal{E} is semistable;
- (2) $\lambda_{\mathcal{E}}$ is nef;
- (3) $\overline{\text{NA}}(\mathbb{P}(\mathcal{E})) = \mathbb{R}_+ \lambda_{\mathcal{E}} \oplus \mathbb{R}_+ \pi^* d$, where d is a positive generator of $N^1(C)_{\mathbb{Z}} \cong \mathbb{Z}$;
- (4) $\overline{\text{NE}}(\mathbb{P}(\mathcal{E})) = \mathbb{R}_+ \lambda_{\mathcal{E}}^{r-1} \oplus \mathbb{R}_+ \lambda_{\mathcal{E}}^{r-2} \pi^* d$;
- (5) Every effective divisor on $\mathbb{P}(\mathcal{E})$ is nef.

Proof. (1) to (2). If $\lambda_{\mathcal{E}}$ is not nef, then there exists an irreducible curve $C' \subset \mathbb{P}(\mathcal{E})$ with $C' \lambda_{\mathcal{E}} < 0$. It's clear¹ that C' is mapped surjectively onto C . Let C'' be the normalization of C' and $f: C'' \rightarrow C$ be the composition of $C'' \rightarrow C' \rightarrow C$. Then by the base change $f: C'' \rightarrow C$, the multi-section C' becomes a union of cross sections C''_i on the projective bundle $\mathbb{P}(f^*\mathcal{E})$ over C'' , and $C''_i \lambda_{f^*\mathcal{E}}$ is evidently negative since $C' \lambda_{\mathcal{E}} < 0$. For a section $s: C'' \rightarrow C''_i \subset \mathbb{P}(f^*\mathcal{E})$, it gives a line bundle $\mathcal{L} = s^* \mathcal{O}_{\mathbb{P}(f^*\mathcal{E})}(1)$ on C'' , which has degree $C''_i c_1(\mathcal{O}_{\mathbb{P}(f^*\mathcal{E})}(1)) = C''_i \lambda_{f^*\mathcal{E}} + \delta(f^*\mathcal{E}) < \delta(f^*\mathcal{E})$. On the other hand, there is a surjective morphism $f^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ and thus $f^*\mathcal{E}$ is unstable. Then by Lemma 2.2.1 one has \mathcal{E} is unstable.



¹Otherwise we have $C' \lambda_{\mathcal{E}} > 0$.

(2) to (4). If $\lambda_{\mathcal{E}}^{r-2}(a\lambda_{\mathcal{E}} + b\pi^*d)$ is pseudo-effective and $\lambda_{\mathcal{E}}$ is nef, then

$$b = \lambda_{\mathcal{E}}^{r-1}(a\lambda_{\mathcal{E}} + b\pi^*d) \geq 0.$$

On the other hand, since π^*d is semiample, we have π^*d is also nef. Thus

$$a = \pi^*d(a\lambda_{\mathcal{E}}^{r-1} + b\lambda_{\mathcal{E}}^{r-2}\pi^*d) \geq 0.$$

The equivalent between (3) and (4) is straightforward since the nef cone is the closed cone dual to the pseudo-effective cone (Theorem 1.4.3).

(3) and (4) to (5). By (3), we have $\lambda_{\mathcal{E}} + \epsilon\pi^*d$ is ample for any positive real number ϵ , since it lies in the interior of the nef cone. Assume $a\lambda_{\mathcal{E}} + b\pi^*d$ is an effective divisor. Then the 1-cycles $(a\lambda_{\mathcal{E}} + b\pi^*d)(\lambda_{\mathcal{E}} + \epsilon\pi^*d)^{r-2}$ is also effective, and thus their limit $(a\lambda_{\mathcal{E}} + b\pi^*d)\lambda_{\mathcal{E}}^{r-2}$ is pseudo-effective. Then by (4) one has $a, b \geq 0$, and thus $a\lambda_{\mathcal{E}} + b\pi^*d$ is nef by (3).

(5) to (1). Suppose that \mathcal{E} is unstable and let \mathcal{E}_1 be the maximal destabilizing subbundle. Let α be a rational number with $\delta(\mathcal{E}_1) > \alpha > \delta(\mathcal{E})$. Then by the Riemann-Roch theorem,

$$\begin{aligned} H^0(C, \mathcal{S}^N \mathcal{E}_1(-N\alpha d)) &\subseteq H^0(C, \mathcal{S}^N \mathcal{E}(-N\alpha d)) \\ &\cong H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(N) \otimes \pi^* \mathcal{O}_C(-N\alpha d)) \end{aligned}$$

is non-trivial for sufficiently large N . Then $N\{\lambda_{\mathcal{E}} + (\delta(\mathcal{E}) - \alpha)\pi^*d\}$ is effective but clearly not nef. \square

2.2.3. Semipositive and semistability.

Definition 2.2.2. Let D be a \mathbb{Q} -Cartier divisor on C . A \mathbb{Q} -torsion-free sheaf $\mathcal{F} = \mathcal{E}(D)$ is said to be *ample* or *semipositive* if $\xi + \pi^*D$ is ample or nef, where $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

Definition 2.2.3. A \mathbb{Q} -torsion-free sheaf \mathcal{F} is said to be *negative* or *seminegative* if \mathcal{F}^* is ample or semipositive.

Proposition 2.2.4. The direct sums, tensor products, symmetric products and exterior products of ample (or semipositive) \mathbb{Q} -torsion-free sheaves are all ample (or semipositive).

Theorem 2.2.2. Let \mathcal{E} be a vector bundle on C . Then \mathcal{E} is semistable if and only if $\mathcal{E}(-\delta(\mathcal{E}))$ is semipositive.

Proof. It follows from Theorem 2.2.1. \square

Corollary 2.2.1. Let \mathcal{E} be a vector bundle on C . Then \mathcal{E} is semistable if and only if $\mathcal{E}(-\delta(\mathcal{E}))$ is seminegative.

Proof. It suffices to note that \mathcal{E} is semistable if and only if \mathcal{E}^* is semistable. \square

Corollary 2.2.2.

- (1) The \mathbb{Q} -vector bundle $\mathcal{E}(-D)$ is seminegative if and only if $\deg D \geq \deg \delta(\mathcal{E}_1)$, where \mathcal{E}_1 is the maximal destabilizing subsheaf of \mathcal{E} .
- (2) The \mathbb{Q} -vector bundle $\mathcal{E}(-D)$ is negative if and only if $\deg D > \deg \delta(\mathcal{E}_1)$, where \mathcal{E}_1 is the maximal destabilizing subsheaf of \mathcal{E} .
- (3) The \mathbb{Q} -vector bundle $\mathcal{E}(D)$ is semipositive if and only if $\deg D \geq \deg \delta((\mathcal{E}^*)_1)$.
- (4) The \mathbb{Q} -vector bundle $\mathcal{E}(D)$ is positive if and only if $\deg D > \deg \delta((\mathcal{E}^*)_1)$.

Proof. For simplicity we only prove the first statement, and the proof is quite similar for others.

Let $\mathcal{E}_1 \subset \dots \subset \mathcal{E}_s = \mathcal{E}$ be the semistable filtration of \mathcal{E} . Since $\mathcal{G}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable and $\deg \delta(\mathcal{G}_i)$ is decreasing in i , one has $\mathcal{G}_i(-\delta(\mathcal{E}_1))$ is seminegative for all i , and thus $\mathcal{E}(-\delta(\mathcal{E}_1))$ is seminegative. If $\deg D \geq \deg \delta(\mathcal{E}_1)$, then $\mathcal{E}(-D)$ is also seminegative.

Conversely, if $\deg D$ is smaller than $\deg \delta(\mathcal{E}_1)$ for a \mathbb{Q} -divisor D , then $\mathcal{E}(-D)$, containing an ample \mathbb{Q} -vector bundle $\mathcal{E}_1(-D)$, is never seminegative. \square

Corollary 2.2.3. A semistable vector bundle \mathcal{E} on C is ample (resp. semipositive, seminegative, negative) if and only if its degree is positive (resp. semipositive, seminegative, negative).

Proof. Take $D = 0$ in Corollary 2.2.2. \square

Corollary 2.2.4. Let \mathcal{E} and \mathcal{F} be semistable bundles on C . Then $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ are also semistable.

Proof. It follows from the semipositive bundle tensor with semipositive bundle is still semipositive. \square

Corollary 2.2.5. Let \mathcal{E} and \mathcal{F} be two vector bundles. Then $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is negative if and only if $\deg \delta(\mathcal{F}_1) + \deg \delta((\mathcal{E}^*)_1) < 0$. As a consequence, $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}/\mathcal{E}_1)$ is negative.

Proof. For the first part, note that $\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \mathcal{E}^* \otimes \mathcal{F}$ and take $D = 0$ in Corollary 2.2.2. For the half part, it suffices to note $(\mathcal{E}/\mathcal{E}_1)_1 = \mathcal{E}_2/\mathcal{E}_1$. \square

Corollary 2.2.6. A vector bundle is semistable if and only if $\mathcal{S}^n \mathcal{E}$ is semistable, where $n \geq 2$.

Proposition 2.2.5. Let \mathcal{E} be a vector bundle on C . The following conditions are equivalent:

- (1) \mathcal{E} is semistable;
- (2) $\mathcal{E}(-D)$ is negative with D is a \mathbb{Q} -divisor of degree $\delta(\mathcal{E}) + (1/2r!)$.

Proof. The implication (1) to (2) follows from Corollary 2.2.1.

Conversely, assume (2) and let \mathcal{E}_1 be the maximal destabilizing subsheaf. Then by Corollary 2.2.2 we have $\mathcal{E}(-D)$ is negative if and only if $\deg D > \deg \delta(\mathcal{E}_1)$ so that

$$\delta(\mathcal{E}) \leq \delta(\mathcal{E}_1) < \delta(\mathcal{E}) + \frac{1}{2r!}.$$

On the other hand, both $\deg \delta(\mathcal{E}_1)$ and $\deg \delta(\mathcal{E})$ sit in $(1/r!)\mathbb{Z}$. Hence we have $\deg \delta(\mathcal{E}_1) = \deg \delta(\mathcal{E})$, and thus $\mathcal{E}_1 \cong \mathcal{E}$. \square

Corollary 2.2.7. Let $\mathcal{C} \rightarrow T$ be a proper smooth family of irreducible curves, where \mathcal{C} and T are k -varieties. Let \mathcal{E} be a vector bundle on \mathcal{C} . Then the set

$$S(T) = \{t \in T \mid \mathcal{E} \text{ is semistable on } C_t\}$$

is a Zariski open subset of T .

2.3. Mumford-Mehta-Ramanathan's thmsub.

Theorem 2.3.1 ([MR82]). Let X be a complex normal projective variety of dimension n and \mathcal{E} be a torsion-free sheaf. Let H_1, \dots, H_{n-1} be ample Cartier divisors. Then for sufficiently large integers m_1, \dots, m_{n-1} , the maximal destabilizing subsheaf \mathcal{F} of $\mathcal{E}|_C$ extends to a saturated subsheaf² of \mathcal{E} on X if C is a general complete intersection curve of $|m_i H_i|$'s.

2.4. The Bogomolov-Gieseker inequality for semistable sheaves. In this section, the ground field k is always algebraically closed of characteristic zero.

Lemma 2.4.1. Let X be a normal projective variety of dimension n and $\mathfrak{A} \in \text{NA}(X)_{\mathbb{R}}^{n-1}$. Let \mathcal{E} be an \mathfrak{A} -semistable torsion-free sheaf on X , with its first Chern class being a \mathbb{Q} -Cartier divisor. Let D be a non-zero effective Cartier divisor on X . Then

$$H^0(X, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}) - D)) = 0$$

for every positive integer t such that $tc_1(\mathcal{E})$ is an integral Cartier divisor.

Proof. For a generic curve C in X , by Theorem 2.3.1 one has $\mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}))|_C$ is semistable since $\mathcal{S}^{rt} \mathcal{E}$ is semistable. If

$$H^0(X, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}) - D)) \neq 0,$$

²Such an extension of \mathcal{F} is necessarily the maximal (H_1, \dots, H_{n-1}) -destabilizing subsheaf of \mathcal{E} and hence unique.

then there is an inclusion $\mathcal{O}_C(D) \rightarrow \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E}))|_C$. But $\deg \delta(\mathcal{O}_C(D)) > 0$ since D is effective and $\mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E})) = \mathcal{S}^{rt}\{\mathcal{E}(-\delta(\mathcal{E}))\}$ has degree zero on every curve. This contradicts to $\mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E}))|_C$ is semistable. \square

Corollary 2.4.1. Let things be as Lemma 2.4.1 and L be a fixed Cartier divisor. Then $h^0(X, \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E}) + L))$ is bounded by a polynomial of degree $r - 1$ in t .

Proof. For simplicity of the notation, put $\mathcal{F}^t = \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E}))$. The proof is by induction on the dimension n of X . If $n = 1$, let D be a divisor of degree d such that $D - L$ is effective. Then there is a natural exact sequence

$$H^0(X, \mathcal{F}^t(L - D)) \rightarrow H^0(X, \mathcal{F}^t(L)) \rightarrow H^0(D, \mathcal{F}^t(L))$$

of which the first term vanishes by Lemma 2.4.1, where the last term is a k -vector space of dimension $d \binom{rt+r-1}{rt} = d \binom{rt+r-1}{r-1}$. This completes the proof of $n = 1$.

For $n \geq 2$, let $\mathfrak{A} \equiv (H_1, \dots, H_{n-1}) \pmod{(\mathbb{Q}_+^\times)^{n-1}}$, where H_i is integral and ample. Let Y be a general hyperplane section in $|mH_{n-1}|$ for sufficiently large m such that $\mathcal{E}|_Y$ is (H_1, \dots, H_{n-2}) -semistable on Y and $Y - L$ is ample. Note that such a number m , though possible very large, is independent of t . Consider the exact sequence

$$H^0(X, \mathcal{F}^t(L - Y)) \rightarrow H^0(X, \mathcal{F}^t(L)) \rightarrow H^0(Y, \mathcal{F}^t(L)).$$

The first term vanishes by Lemma 2.4.1 and the dimension of the last term is bounded by a polynomial of degree $r - 1$ by the induction hypothesis. This completes the proof. \square

Theorem 2.4.1 (The Bogomolov-Gieseker inequality). Let S be a smooth projective surface over k . If \mathcal{E} is an H -semistable torsion-free sheaf of rank r on S , where H is an ample divisor, then

$$(r - 1)c_1^2(\mathcal{E}) \leq 2rc_2(\mathcal{E}).$$

Proof. From Corollary 2.4.1, it follows that neither $h^0(S, \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E})))$ nor $h^2(S, \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E}))) = h^0(S, \mathcal{S}^{rt}\mathcal{E}^*(-tc_1(\mathcal{E}^*)) + K_S)$ grows like t^{r+1} . Hence we obtain the inequality

$$\chi(S, \mathcal{S}^{rt}\mathcal{E}(-tc_1(\mathcal{E}))) \leq \text{polynomial of degree } r \text{ in } t.$$

On the other hand, we have

$$\begin{aligned}
\chi(S, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}))) &\stackrel{(1)}{=} \chi(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(rt) \otimes \pi^* \mathcal{O}_S(-tc_1(\mathcal{E}))) \\
&\stackrel{(2)}{=} \frac{t^{r+1}}{(r+1)!} \{rc_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) - \pi^* c_1(\mathcal{E})\}^{r+1} + O(t^r) \\
&\stackrel{(3)}{=} \frac{(rt)^{r+1}}{(r+1)!} \left\{ -c_2(\mathcal{E}) + \frac{r-1}{2r} c_1^2(\mathcal{E}) \right\} + O(t^r),
\end{aligned}$$

where

- (1) holds from the projection formula;
- (2) holds from by the asymptotic Riemann-Roch theorem (Theorem 1.5.1);
- (3) holds from the following standard computation

$$\begin{aligned}
\left\{ \xi - \frac{\pi^* c_1(\mathcal{E})}{r} \right\}^{r+1} &= \left\{ \xi^r - \pi^* c_1(\mathcal{E}) \xi^{r-1} + \frac{r-1}{2r} \pi^* c_1^2(\mathcal{E}) \xi^{r-2} \right\} \left\{ \xi - \frac{\pi^* c_1(\mathcal{E})}{r} \right\} \\
&= \left\{ \pi^* c_2(\mathcal{E}) \xi^{r-2} + \frac{r-1}{2r} \pi^* c_1^2(\mathcal{E}) \xi^{r-2} \right\} \left\{ \xi - \frac{\pi^* c_1(\mathcal{E})}{r} \right\} \\
&= \left\{ -c_2(\mathcal{E}) + \frac{r-1}{2r} c_1^2(\mathcal{E}) \right\},
\end{aligned}$$

where $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

This completes the proof. \square

Corollary 2.4.2. Let \mathcal{E} be a locally free sheaf of rank r on a smooth projective surface S . Let L be an ample integral divisor on S such that $\mathcal{E}(-\delta(\mathcal{E}) + L)$ is ample and $\mathcal{E}(-\delta(\mathcal{E}) - L)$ is negative (as \mathbb{Q} -vector bundles). Assume the inequality $2rc_2(\mathcal{E}) < (r-1)c_1^2(\mathcal{E})$ and put

$$\alpha = \frac{(r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E})}{6r^2(r+1)L^2} \in \mathbb{Q}.$$

Then

- (1) Either $\mathcal{S}^t \mathcal{E}(-t\delta(\mathcal{E}))$ or $\mathcal{S}^t \mathcal{E}^*(-t\delta(\mathcal{E}^*))$ contains the ample line bundle $\mathcal{O}_S(t\alpha L)$, where t is any very large integer such that $t\delta(\mathcal{E})$ and $t\alpha$ are integral.
- (2) For any nef divisor D , the maximal D -destabilizing subsheaf \mathcal{E}_1^D has normalized degree not less than

$$\delta_D(\mathcal{E}) + \frac{\alpha LD}{r}$$

with respect to D .

Proof. (1). For simplicity, we put $\mathcal{F} = \mathcal{E}(-\delta(\mathcal{E}))$. By the standard computation we have

$$\chi(S, \mathcal{S}^t \mathcal{F}) = \frac{t^{r+1}}{(r+1)!} \left\{ -c_2(\mathcal{E}) + \frac{r-1}{2r} c_1^2(\mathcal{E}) \right\} + O(t^r).$$

Hence, by the Serre duality, we infer that $h^0(S, \mathcal{S}^t \mathcal{F})$ or $h^0(S, \mathcal{S}^t \mathcal{F}^*)$ is

$$\geq \frac{t^{r+1}}{4r(r+1)!} \{ (r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E}) \} + O(t^r).$$

Assume the first case and consider the following natural exact sequences

$$\begin{aligned} 0 \rightarrow H^0(S, \mathcal{S}^t \mathcal{F}(-t\alpha L)) &\rightarrow H^0(S, \mathcal{S}^t \mathcal{F}) \rightarrow H^0(C, \mathcal{S}^t \mathcal{F}), \\ 0 \rightarrow H^0(C, \mathcal{S}^t \mathcal{F}(-tL)) &\rightarrow H^0(C, \mathcal{S}^t \mathcal{F}) \rightarrow H^0(D, \mathcal{S}^t \mathcal{F}), \end{aligned}$$

where C is a general curve linearly equivalent to $t\alpha L$ and D is a 0-cycle of degree $t^2\alpha L^2$. The first term of the second sequence vanishes as $\mathcal{S}^t \mathcal{F}(-tL) = \mathcal{S}^t \{\mathcal{F}(-L)\}$ is negative. Hence $h^0(C, \mathcal{S}^t \mathcal{F})$ is bounded by

$$\begin{aligned} t^2\alpha(\text{rk } \mathcal{S}^t \mathcal{F})L^2 &\equiv \frac{\alpha t^{r+1}}{(r-1)!} L^2 \\ &\equiv \frac{t^{r+1}}{6(r+1)!r} \{ (r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E}) \} \pmod{O(t^r)}. \end{aligned}$$

This shows $H^0(S, \mathcal{S}^t \mathcal{F}(-t\alpha L))$ is non-zero whenever t is very large in view of the first exact sequence, and thus such a non-zero global section gives the inclusion $\mathcal{O}_S(t\alpha L) \hookrightarrow \mathcal{S}^t \mathcal{F}$. Similarly, the second case will yield $H^0(S, \mathcal{S}^t \mathcal{F}^*(-t\alpha L)) \neq 0$.

(2). It suffices to consider the following cases:

(a) If $\mathcal{S}^t \mathcal{F}$ contains $\mathcal{O}_S(t\alpha L)$, then

$$\delta_D(\mathcal{E}_1^D) - \delta_D(\mathcal{E}) \geq \alpha LD.$$

(b) If $\mathcal{S}^t \mathcal{F}^*$ contains $\mathcal{O}_S(t\alpha L)$, then

$$\delta_D(\mathcal{E}_1^D) - \delta_D(\mathcal{E}) \geq \frac{1}{r} \left\{ \delta_D((\mathcal{E}^*)_1^D) - \delta_D(\mathcal{E}^*) \right\} \geq \frac{\alpha LD}{r}.$$

This completes the proof. \square

Corollary 2.4.3. Let \mathcal{E} be a torsion-free sheaf of rank r on a normal projective variety X of dimension n and H_1, \dots, H_{n-2} be ample Cartier divisors. Let D be a nef Cartier divisor on X . Assume that $H_1 \dots H_{n-2} D$ is not numerically trivial. If \mathcal{E} is (H_1, \dots, H_{n-2}, D) -semistable, then

$$(r-1)c_1^2(\mathcal{E})H_1 \dots H_{n-2} \leq 2rc_2(\mathcal{E})H_1 \dots H_{n-2}.$$

Proof. Suppose the contrary. We may assume that \mathcal{E} is a vector bundle in codimension 2 by taking double dual. Fix an ample divisor H_0 such that $\mathcal{E}(-\delta(\mathcal{E}) + H_0)$ and $\mathcal{E}^*(\delta(\mathcal{E}) + H_0)$ are both ample. Let H be an arbitrary ample divisor. Then by Theorem 2.3.1 there exist positive integers m_1, \dots, m_{n-2} depending on H such that $H|_S$ -semistable filtration of $\mathcal{E}|_S$ coincides with the restriction of (H_1, \dots, H_{n-2}, H) -semistable filtration of \mathcal{E} to a generic complete intersection surface $S = (m_1 H_1) \dots (m_{n-2} H_{n-2})$.

By Corollary 2.4.3, we have

$$\begin{aligned} \delta(\mathcal{E}_1^{(\mathfrak{B}, H)})SH - \delta(\mathcal{E}^{(\mathfrak{B}, H)})SH &= \delta_H((\mathcal{E}|_S)_1^H) - \delta_H(\mathcal{E}|_S) \\ &\geq c \{(r-1)c_1^2(\mathcal{E}|_S) - 2rc_2(\mathcal{E}|_S)\}(H, H_0)_S / (H_0^2)_S \\ &= c \{[(r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E})]S\} HH_0 S / H_0^2 S, \end{aligned}$$

where $\mathfrak{B} = (H_1, \dots, H_{n-2})$ and c is a positive constant. Therefore, by dividing out both sides by $m_1 \dots m_{n-2}$, we obtain the inequality

$$\delta_{(\mathfrak{B}, H)}(\mathcal{E}_1^{(\mathfrak{B}, H)}) \geq \delta_{(\mathfrak{B}, H)}(\mathcal{E}) + cHH_0H_1 \dots H_{n-2}.$$

By the continuity of the function $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ on a segment joining (\mathfrak{B}, D) and (\mathfrak{B}, H) , we have

$$\delta_{(\mathfrak{B}, D)}(\mathcal{E}_1^{(\mathfrak{B}, D)}) \geq \delta_{(\mathfrak{B}, H)}(\mathcal{E}) + cDH_0H_1 \dots H_{n-2} > \delta_{(\mathfrak{B}, D)}(\mathcal{E}),$$

a contradiction. \square

Corollary 2.4.4. Let \mathcal{E} be a torsion-free sheaf of rank r on a normal projective variety X of dimension n and H_1, \dots, H_{n-2} be ample Cartier divisors. If

$$\{(r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E})\}H_1 \dots H_{n-2} > 0,$$

then \mathcal{E} is (H_1, \dots, H_{n-2}, D) -unstable for any non-zero nef divisor D .

2.5. Semistability in positive and mixed characteristic.

2.5.1. Semistability in positive characteristic. Let C be a smooth complete curve over an algebraically closed field k of characteristic $p > 0$ and $F: C \rightarrow C$ be the absolute Frobenius morphism.

Definition 2.5.1. A vector bundle \mathcal{E} on C is said to be *strongly semistable* if, for every positive integer s , $(F^s)^*\mathcal{E}$ is semistable.

Remark 2.5.1. If C is an elliptic curve, it's known that every semistable bundle is strongly semistable, but that is not the case when $g(C) \geq 2$.

Proposition 2.5.1. If \mathcal{E} is strongly semistable on C , then $f^*\mathcal{E}$ is semistable for any surjective k -morphism $f: C' \rightarrow C$.

Proof. Let C'' be a smooth model of the separable closure of C . The natural projection $C' \rightarrow C''$ is pure inseparable and hence $C' = F^{-s}C''$ for some non-negative integer s (Proposition IV 2.5 of [Har77]). Thus we get the commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{F^s} & C'' \\ g \downarrow & & \downarrow h \\ F^{-s}C & \xrightarrow{F^s} & C \end{array}$$

Since \mathcal{E} is strongly semistable, we have $(F^s)^*$ is semistable on $F^{-s}C$, and thus $f^*\mathcal{E} = g^*(F^s)^*\mathcal{E}$ is also semistable by Lemma 2.2.1 as g is separable. \square

Remark 2.5.2. The Theorem 2.2.1 and its corollaries still hold in positive characteristic if the “semistability” is substituted by “strong semistability”.

2.5.2. Semistability in mixed characteristic. Let X be a smooth projective variety over a noetherian integral domain R of characteristic zero and \mathcal{E} be a torsion-free sheaf on X . Fix $\mathfrak{A} \in \overline{\text{NA}}(X/R)_{\mathbb{Q}}^{n-1}$, where n is the relative dimension of X . Then the set of geometric points $t \in \text{Spec} R$ such that \mathcal{E}_t/X_t is \mathfrak{A} -semistable forms an open subset.

On the contrary, we know very little about the strong semistability of the reductions of a semistable sheaves.

Question 2.5.1. Let C be an irreducible smooth projective curve over a noetherian integral domain R of characteristic zero. Assume that a locally free sheaf \mathcal{E} on C is \mathfrak{A} -semistable on the generic fibre C_* . Let S be the set of primes of positive characteristic on $\text{Spec} R$ such that \mathcal{E} is strongly semistable. Is S a dense subset of $\text{Spec} R$?

2.6. Generic semipositive theorem for cotangent bundle. From now on, all varieties are defined over an algebraically closed field k of characteristic 0. Let X be a normal projective variety of dimension n .

Definition 2.6.1. Let $\mathfrak{B} \in \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-2}$.

- (1) A torsion-free sheaf \mathcal{E} on X is said to be *generically \mathfrak{B} -seminegative* if, for every numerically effective \mathbb{Q} -Cartier divisor D on X , its maximal (\mathfrak{B}, D) -destabilizing subsheaf \mathcal{E}_1 satisfies $\delta_{(\mathfrak{B}, D)}(\mathcal{E}_1) < 0$.
- (2) A torsion-free sheaf \mathcal{E} on X is said to be *generically \mathfrak{B} -semipositive* if \mathcal{E}^* is generically \mathfrak{B} -seminegative.

Lemma 2.6.1. Let \mathcal{E} be a torsion-free sheaf on X and

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \mathcal{E}$$

be the (\mathfrak{B}, D) -semistable filtration of \mathcal{E} and put $\alpha_i = \delta_{(\mathfrak{B}, D)}(\mathcal{E}_i/\mathcal{E}_{i-1})$. Then $\alpha_1 > \cdots > \alpha_s \geq 0$ for every $D \in \overline{\text{NA}}(X)_{\mathbb{Q}}$ if \mathcal{E} is generically \mathfrak{B} -semipositive.

Proof. It follows from the definition. \square

Theorem 2.6.1. Let $\mathfrak{B} = (H_1, \dots, H_{n-2}) \in \text{NA}(X)_{\mathbb{Q}}^{n-2}$ and \mathcal{E} be a generically \mathfrak{B} -semipositive torsion-free sheaf on X . Then

$$c_2(\mathcal{E})H_1 \dots H_{n-2} \geq 0$$

holds.

Theorem 2.6.2. Suppose $\mathfrak{B} = (H_1, \dots, H_{n-2}) \in \text{NA}(X)_{\mathbb{Q}}^{n-2}$. Then the torsion-free sheaf $\rho_* \Omega_{X'}^1$ is generically \mathfrak{B} -semipositive unless X is uniruled, where $\rho: X' \rightarrow X$ denotes an arbitrary resolution.

3. RESULTS

3.1. Semipositivity of $3c_2 - c_1^2$.

Proposition 3.1.1. Let X be a non-uniruled, normal projective variety of dimension n with \mathbb{Q} -Cartier canonical divisor K_X which is nef. Let $\mathfrak{B} \in \text{NA}(X)_{\mathbb{Q}}^{n-2}$ such that $K_X^2|\mathfrak{B}|$ is positive. Then

$$\{3c_2(\mathcal{E}) - c_1(\mathcal{E})^2\}|\mathfrak{B}| \geq 0,$$

where $\mathcal{E} = \rho_*\Omega_{X'}^1$, and $\rho: X' \rightarrow X$ is an arbitrary resolution.

3.2. Non-negativity of the Kodaira dimension of minimal threefolds.

3.2.1. The Gorenstein case.

Theorem 3.2.1. Let X be a normal projective Gorenstein threefold with only canonical singularities (X is Gorenstein if and only if K_X is a Cartier divisor). Assume K_X is nef. Then the Euler characteristic $\chi(X, \mathcal{O}_X)$ is non-negative. In particular, either $h^0(X, \mathcal{O}_X(K_X))$ or $h^1(X, \mathcal{O}_X)$ is non-zero, and thus $\kappa(X) \geq 0$.

3.2.2. The K_X^2 is numerically non-trivial case.

Theorem 3.2.2. Let X be a normal projective Gorenstein threefold with only isolated singularities. Assume the \mathbb{Q} -Cartier divisor K_X is nef and K_X^2 is numerically non-trivial. Then $\kappa(X) \geq 0$.

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