RIEMANNIAN GEOMETRY

BOWEN LIU

Contents

0. Preface 5
0.1. About this lecture 5
0.2. To readers 5
0.3. Some notations and conventions 7

Part 1. Preliminaries 8
1. Basic notions 8
1.1. Vector bundle 8
1.2. Riemannian manifold 9
2. Connection 11
2.1. Motivation of connection 11
2.2. Examples 12
2.3. Levi-Civita connection 14
3. Tensor computations 15
3.1. Induced connection on tensor 15
3.2. Type change of tensor 18
3.3. Induced metric on tensor 19
3.4. Trace of tensor 19
4. Geodesic and normal coordinate 21
4.1. Geodesic 21
4.2. Normal coordinate 21
4.3. Hopf-Rinow's Theorem 24

Part 2. Curvature 26
5. Riemannian Curvature 26
5.1. Curvature form and curvature tensor 26
5.2. Bianchi identities 31
6. Other curvatures 33
6.1. Sectional curvature 33
6.2. Ricci curvature and scalar curvature 35
7. Basic models 37
7.1. Einstein manifold 37
7.2. Sphere 39
7.3. Lie group with invariant metric 40

Part 3. Bochner's technique 43
8. Hodge theory on Riemannian manifold 43
8.1. Inner product on differential forms 43
8.2. Hodge star operator 45
8.3. Computations of adjoint operator 48
8.4. Divergence 51
8.5. Conformal Laplacian 54
8.6. Hodge theorem and corollaries 56
9. Bochner's technique 57
9.1. Bochner formula 57
9.2. Obstruction to the existence of Killing fields 58
9.3. Obstruction to the existence of harmonic 1-forms 60

Part 4. Minimal length curve problem 62
10. Pullback connection 62
10.1. Pullback and pushforward 62
10.2. Pullback connection 62
10.3. Pullback curvature 64
10.4. Parallel transport 65
10.5. Second fundamental form 66
11. Variation formulas of curves 69
11.1. First variation formula 69
11.2. Second variation formula 72
12. Jacobi field I: as the null space 75
12.1. First properties 75
12.2. Conjugate points 77
12.3. Locally minimal geodesic 77
13. Cut locus and injective radius 81
13.1. Cut locus 81
13.2. Injective radius 83
14. Jacobi field II: a useful tool 86
14.1. Taylor expansion of Jacobi field 86
14.2. Gauss lemma 88
14.3. Jacobi fields on constant sectional curvature manifold 90
14.4. Polar decomposition of metric with constant sectional curvature 91
14.5. A criterion for constant sectional curvature space 93

Part 5. Harmonic maps 96
15. Harmonic map 96
15.1. Harmonic map and totally geodesic 96
15.2. First variation of smooth map 97
15.3. Second variation formula of harmonic map 98
15.4. Bochner formula for harmonic map 100

Part 6. Topology of Riemannian manifold 101

16. Isometry group 101
16.1. Isometry and local isometry 101
16.2. Properties of Killing field 102
16.3. Cartan-Ambrose-Hicks theorem 105
17. Riemannian covering 107
17.1. Riemannian covering 107
18. Topology of non-positive sectional curvature manifold 109
18.1. Cartan-Hadamard manifold 109
18.2. Cartan's torsion-free theorem 112
18.3. Preissmann's Theorem 113
18.4. Other facts 118
19. Topology of positive curvature manifold 119
19.1. Myers' theorem 119
19.2. Synge's theorem 119
19.3. Other facts 120
20. Topology of constant sectional curvature manifold 122
20.1. Hopf's theorem 122

Part 7. Comparison theorems 124
21. Comparison theorems based on sectional curvature 124
21.1. Rauch comparison 124
21.2. Hessian comparison 129
22. Comparison theorems based on Ricci curvature 132
22.1. Local Laplacian comparison 132
22.2. Maximal principle 135
22.3. Global Laplacian comparison 137
22.4. Volume comparison 139
23. Splitting theorem 148
23.1. Geodesic rays 148
23.2. Buseman function 148
23.3. Splitting theorem and its corollaries 154

Part 8. Appendix 156
Appendix A. Fundamental group 156
A.1. Homotopy 156
A.2. Fundamental group 157
Appendix B. Covering space 161
B.1. Proper map 161
B.2. The lifting theorems 162
B.3. The classification of the covering spaces 165
B.4. The structure of the deck transformation group 166
B.5. Covering of manifold 167
B.6. Orientable double covering 168
Appendix C. Group action 169
C.1. G-set 169

C.2. Continuous action 169
C.3. Properly discontinuous action 171
Appendix D. Review of smooth manifolds 174
D.1. Submersions, Immersions, and Embeddings 174
D.2. Submanifold 175
D.3. Lie group 176
D.4. Killing form 178
D.5. Lie group action 180
D.6. Distributions and Foliations 182
References 183

0. Preface

0.1. About this lecture.

0.2. To readers.

This note is divided into several parts:

(1) In the **First** part, we firstly introduce connections on a vector bundle E in different viewpoints. Given a connection on E, one can construct connection on its dual bundle E^*, tensor product $E \otimes E^*$ and so on. When E is chosen to be tangent bundle equipped with a Riemannian metric, there is a unique connection which is compatible with metric and torsion-free, which is called Levi-Civita connection.

A section of tensor products of tangent bundle with its dual bundle is called a tensor, and tensor computation is a powerful tool of Riemannian geometry, so we collect some basic properties and operations about tensor in section 2.

However, tensor computation may be quite complicated in general. To give a neat local computation for tensor, we introduce geodesic in section 3 in order to introduce normal coordinate. By the way we also introduce Hopf-Rinow's theorem about completeness.

(2) The **Second** part is about curvature. We introduce curvature using two different views: curvature form and curvature tensor and prove Bianchi identities in these two views. We also introduce Ricci identity for tensor, which is a crucial step in Bochner's technique. In the end we introduce some other important curvatures such as sectional curvature, Ricci curvature and scalar curvature.

(3) The **Third** part is about Bochner's technique, which is one of the most important techniques in modern Riemannian geometry. Holding this technical, we can see how does bounded Ricci curvature appear as an obstruction to the existence of Killing fields and harmonic 1-forms. Aside these, we also introduce Hodge theory, which allows us to use harmonic 1-forms to represent elements in the first homology group. Then Bochner's technique gives a kind of vanishing theorem.

(4) The goal of **Fourth part** is to solve the following question: Given two points p, q, what' the length-minimizing curve connecting p, q in a Riemannian manifold?". To answer this, we consider the arc-length functional, and

   (a) First variation formula implies geodesics are critical points of arc-length functional.

   (b) Second variation formula implies if a geodesic contains no interior conjugate points, then it's locally minimum of arc-length functional.

   Along the way we develop the tools of index form and Jacobi fields, which are also quite important in the following parts.

(5) The **Fifth part** generalizes geodesic and Hessian of smooth function to some extent. In this part we define what is second fundamental form,

and when a smooth map between Riemannian manifold is harmonic map. Finally, we consider its variation and Bochner's formula.

(6) The **Sixth part** introduces how does curvature condition control the topology of the whole manifold. We mainly consider the following three cases:
  (a) A Riemannian manifold M with non-positive sectional curvature is $K(\pi_1(M),1)$, that is M is covered by $\mathbb{R}^n$. A fact in topology says if a finite dimensional CW complex is a $K(G,1)$ space for some group G, we must have G is torsion-free. Here Cartan's torsion-free theorem gives a neat proof of this fact by using deck transformations and some basic facts about Lie group action. Furthermore, $\pi_1(M)$ deserves many other interesting properties:
    I Preissmann's theorem says if M is compact with negative sectional curvature, then any non-trivial Abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$ and $\pi_1(M)$ itself is not abelian.
    II Byers' theorem says more: if M is compact with negative sectional curvature, then any non-trivial solvable subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.
  (b) A Riemannian manifold with curvature lower bounded is also quite interesting.
    I Myers' theorem says a Riemannian manifold with positive Ricci curvature is compact, and with finite fundamental group. However, it's meaningless to consider what will happen if Ricci curvature is upper bounded, since every Riemannian n-manifold admits a complete metric with $\mathrm{Ric} < 0$ if $n \geq 3$.
    II Synge's theorem says a little about fundamental group of Riemannian manifold M with positive sectional curvature and even dimension: If it's orientable, then it's simply-connected, otherwise $\pi_1(M) = \mathbb{Z}_2$.
  (c) Finally, a celebrated theorem of Hopf implies every Riemannian manifold with constant sectional curvature is covered by three basic models, which are called space forms.

(7) The **Seventh part** is also about curvature, but it shows how to use comparison in curvatures to obtain comparison in other objects, such as length, metrics, volume and Hessian or Laplacian operators. A philosophy is that the larger" curvature is, the smaller other thing is. It also gives us some rigidity theorem, an interesting result is that Cheng's theorem.
  (a) If (M,g) be a Riemannian n-manifold with $\mathrm{Ric}(g) \geq (n-1)kg$ for some constant $k > 0$, then Myers' theorem implies $\mathrm{diam}(M) \leq \pi/\sqrt{k}$. If $\mathrm{diam}(M) = \pi/\sqrt{k}$, then Cheng's theorem says (M,g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$ with standard metric.

0.3. **Some notations and conventions.**

0.3.1. *Conventions.*

1. Einstein summation is always used.
2. Unless otherwise specified, we always work with the category of smooth (real) manifolds which are connected.

0.3.2. *Notations about smooth manifolds.*

1. M is used to denote a smooth manifold, and $x \in M$ denotes its point.
2. TM and Ω^k_M are used to denote tangent bundle and bundle of k-forms over M respectively.
3. $\Omega^k_M(E)$ is used to denote bundle of k-forms over M valued E.
4. v is used to denote vector in tangent space.
5. X is used to denote a vector field on M, and X_x denotes the value of X at point $x \in M$.
6. α is used to denote a k-form on M, and α_x denotes the value of α at point $x \in M$.
7. For a vector bundle E over M, $C^\infty(E,M)$ is used to denote its sections.
8. $\mathfrak{X}(M)$ is used to denote the set of all vector fields on M.
9. $\frac{\partial f}{\partial x^i}$ or $\partial_i f$ is used to denote the partial derivative of a smooth function $f: \mathbb{R}^n \to \mathbb{R}$ with respect to x^i, where $\{x^i\}_{i=1}^n$ is a coordinate of $\mathbb{R}^n$.

0.3.3. *Notations about Riemannian manifolds.*

1. (M,g) is used to denote a Riemannian manifold, that is a smooth manifold M together with a Riemannian metric g.
2. $\langle -,- \rangle_g$ is used to denote a Riemannian metric g, or directly $\langle -,- \rangle$ if there is no ambiguity.

Part 1. Preliminaries

1. Basic notions

1.1. Vector bundle.

Definition 1.1.1 (vector bundle)**.** Let M be a smooth manifold. A (real) vector bundle E of rank r on M consists of the following data:

(1) E is a smooth manifold with surjective map $\pi\colon E\to M$, such that
  (1) For all $x\in M$, fibre E_x is a $\mathbb{R}$-vector space of dimension r.
  (2) For all $x\in M$, there exists $x\in U\subseteq M$ and there is a diffeomorphism $\varphi\colon \pi^{-1}(U)\to U\times\mathbb{R}^r$ such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & U \\ & U\times\mathbb{R}^r \arrow[ur, "p_1"'] \arrow[r, "p_2"] & \mathbb{R}^r \end{tikzcd}$$

  and for all $y\in U$, $E_y\xrightarrow{p_2\circ\varphi}\mathbb{R}^r$ is a $\mathbb{R}$-vector space isomorphism. The pair (U,φ) is called a trivialization of E over U.

Remark 1.1.1 (transition functions). Let $(U_\alpha,\varphi_\alpha),(U_\beta,\varphi_\beta)$ be local trivializations. Then $\varphi_\alpha\circ\varphi_\beta^{-1}\colon (U_\alpha\cap U_\beta)\times\mathbb{R}^r\to(U_\alpha\cap U_\beta)\times\mathbb{R}^r$ induces a smooth map

$$g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to \mathrm{GL}(r,\mathbb{R}),$$

where $g_{\alpha\beta}$ is called transition function. Furthermore, it satisfies

$$\begin{aligned} g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}&=\mathrm{id}\quad \text{on } U_\alpha\cap U_\beta\cap U_\gamma\\ g_{\alpha\alpha}&=\mathrm{id}\quad \text{on } U_\alpha. \end{aligned}$$

In fact, transition functions contain all information about vector bundle since a vector bundle is locally trivial, so how are these trivial pieces glued together really matters.

Definition 1.1.2 (vector bundle)**.** Let M be a smooth manifold. A (real) vector bundle E of rank r on M consists of the following data:

(1) open covering $\{U_\alpha\}$ of M.
(2) smooth functions $\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to\mathrm{GL}(r,\mathbb{R})\}$ satisfies

$$\begin{aligned} g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}&=\mathrm{id}\quad \text{on } U_\alpha\cap U_\beta\cap U_\gamma\\ g_{\alpha\alpha}&=\mathrm{id}\quad \text{on } U_\alpha. \end{aligned}$$

Remark 1.1.2. The two definitions above are equivalent. The first definition implies the second clearly. The converse is a standard constructive method: If we already have an open covering and a set of transition functions, the vector bundle E is defined to be the quotient of the disjoint union $\coprod_\alpha(U_\alpha\times\mathbb{R}^r)$ by the equivalence relation that puts $(p',v')\in U_\beta\times\mathbb{R}^r$ equivalent to $(p,v)\in U_\alpha\times\mathbb{R}^r$ if and only if $p=p'$ and $v'=g_{\alpha\beta}(p)v$.

Exercise 1.1.1. *Show that the total space of a smooth vector bundle E is a smooth manifold.*

Example 1.1.1 (trivial bundle)**.** *Let M be a smooth manifold. Then $M\times\mathbb{R}^r$ is called trivial vector bundle of rank r on M.*

Definition 1.1.3 (subbundle)**.** Let M be a smooth manifold and $\pi\colon E\to M$ be a vector bundle. A subset $F\subseteq E$ is called a subbundle of rank s, if

(1) For all $x\in M$, $F\cap E_x$ is a subspace of E_x with dimension s.
(2) $\pi|_F\colon F\to M$ is a vector bundle.

Definition 1.1.4 (section)**.** Let M be a smooth manifold and $\pi\colon E\to M$ be a vector bundle. A section of E is a smooth map $s\colon M\to E$ such that $\pi\circ s=\mathrm{id}_M$.

Definition 1.1.5 (tensor)**.** A section of $\bigotimes^s TM\otimes\bigotimes^r T^*M$ is called an (s,r)-tensor.

Example 1.1.2. *A smooth function f is a $(0,0)$-tensor.*

Example 1.1.3. *A vector field X is an $(1,0)$-tensor.*

Example 1.1.4. *A 1-form ω is a $(0,1)$-tensor.*

1.2. **Riemannian manifold.**

Definition 1.2.1. A Riemannian metric is a $(0,2)$-tensor, and a Riemannian manifold (M,g) is a smooth manifold M together with a Riemannian metric g.

1.2.1. *Basic examples.*

Example 1.2.1 (sphere)**.** *Let $\mathbb{S}^n(R)$ denote n-dimensional sphere with radius R. There is a natural inclusion $f\colon\mathbb{S}^n(R)\hookrightarrow(\mathbb{R}^{n+1},g_{can})$, and we can use f to pullback g_{can} to obtain a metric on $\mathbb{S}^n(R)$, denoted by g. Given a local chart (U,φ,x^i), we can write*

$$f(x^1,\dots,x^n)=\left(x^1,\dots,x^n,\sqrt{R^2-\sum_{i=1}^n(x^i)^2}\right).$$

For any $\frac{\partial}{\partial x^i}$, we have

$$\mathrm{d}f\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial f^j}{\partial x^i}\frac{\partial}{\partial x^j}=\frac{\partial}{\partial x^i}-\frac{x^i}{\sqrt{R^2-\sum_{i=1}^n(x^i)^2}}\frac{\partial}{\partial x^{n+1}}.$$

Thus for any two $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial x^j}$ we have

$$\begin{aligned}
g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)&=g_{can}\left(\mathrm{d}f\frac{\partial}{\partial x^i},\mathrm{d}f\frac{\partial}{\partial x^j}\right)\\
&=g_{can}\left(\frac{\partial}{\partial x^i}-\frac{x^i}{\sqrt{R^2-\sum_{i=1}^n(x^i)^2}}\frac{\partial}{\partial x^{n+1}},\frac{\partial}{\partial x^j}-\frac{x^j}{\sqrt{R^2-\sum_{i=1}^n(x^i)^2}}\frac{\partial}{\partial x^{n+1}}\right)\\
&=\delta_{ij}+\frac{x^ix^j}{R^2-\sum_{i=1}^n(x^i)^2},
\end{aligned}$$

which implies

$$g_{ij} = \delta_{ij} + \frac{x^i x^j}{T^2},$$

where $T^2 = R^2 - \sum_{i=1}^n (x^i)^2$.

Example 1.2.2 (hyperbolic upper plane)**.** *Let* $\mathbb{H}^n(R) = \{(x^1, \dots, x^{n-1}, y) \in \mathbb{R}^n \mid y > 0\}$ *with metric*

$$g = R^2 \frac{\delta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}y \otimes \mathrm{d}y}{y^2}.$$

Example 1.2.3 (Poincaré disk)**.** *Let* $\mathbb{B}^n(R) = \{x \in \mathbb{R}^n \mid |x| < R\}$ *with metric*

$$g = 4R^4 \frac{\delta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j}{(R^2 - |x|^2)^2}.$$

1.2.2. *Lie group with invariant metric.*

Definition 1.2.2 (left-invariant metric)**.** A Riemannian metric $\langle \cdot, \cdot \rangle$ on a Lie group G is called left-invariant if

$$\langle (\mathrm{d}L_g)X, (\mathrm{d}L_g)Y \rangle = \langle X, Y \rangle$$

holds for arbitrary $g \in G$ and vector fields X, Y.

Remark 1.2.1. Similarly we can define a right-invariant metric, and a Riemannian metric which is both left-invariant and right-invariant is called bi-invariant metric.

Proposition 1.2.1. There is a bijective correspondence between left-invariant metrics on a Lie group G, and inner products on the Lie algebra $\mathfrak{g}$ of G.

Proof. Given an inner product $\langle \cdot, \cdot \rangle_e$ on Lie algebra $\mathfrak{g}$, then we have an inner product on G defined as follows

$$\langle X_g, Y_g \rangle := \langle (\mathrm{d}L_{g^{-1}})X_g, (\mathrm{d}L_{g^{-1}})Y_g \rangle_e,$$

where X, Y are two vector fields on G. It's left-invariant, since

$$\begin{aligned} \langle (\mathrm{d}L_h)X_g, (\mathrm{d}L_h)Y_g \rangle &= \langle (\mathrm{d}L_{(hg)^{-1}}) \circ (\mathrm{d}L_h)X_g, (\mathrm{d}L_{(hg)^{-1}}) \circ (\mathrm{d}L_h)Y_g \rangle_e \\ &= \langle (\mathrm{d}L_{g^{-1}})X_g, (\mathrm{d}L_{g^{-1}})Y_g \rangle_e. \end{aligned}$$

Conversely, if we have a left-invariant inner product $\langle \cdot, \cdot \rangle$ on G, then it's clear we have an inner product on $\mathfrak{g}$, by just considering its value at identity. Furthermore, these two constructions are inverse to each other, this completes the proof. □

Proposition 1.2.2. There is a bijective correspondence between bi-invariant metrics on a Lie group G, and Ad-invariant inner products on the Lie algebra $\mathfrak{g}$ of G.

Proof. Given an Ad-invariant inner product $\langle\cdot,\cdot\rangle_e$ on the Lie algebra $\mathfrak{g}$, by Proposition 1.2.1, there is a left-invariant metric $\langle\cdot,\cdot\rangle$ on G, it suffices to check it's also right-invariant:

$$\begin{aligned}\langle(\mathrm{d}R_h)X_g,(\mathrm{d}R_h)Y_g\rangle &= \langle(\mathrm{d}L_{(gh)^{-1}})\circ(\mathrm{d}R_h)X_g,(\mathrm{d}L_{(gh)^{-1}})\circ(\mathrm{d}R_h)Y_g\rangle_e\\ &= \langle\mathrm{Ad}(h^{-1})(\mathrm{d}L_{g^{-1}})X_g,\mathrm{Ad}(h^{-1})(\mathrm{d}L_{g^{-1}})Y_g\rangle_e\\ &= \langle(\mathrm{d}L_{g^{-1}})X_g,(\mathrm{d}L_{g^{-1}})Y_g\rangle_e.\end{aligned}$$

Conversely, if we start with a bi-invariant metric, then its restriction to the Lie algebra is an Ad-invariant metric, since $\mathrm{Ad}(g)$ is exactly the differential of $L_g\circ R_{g^{-1}}$. □

2. Connection

Connection is a very basic concept in the study of the geometry of vector bundle. In this section, firstly we introduce the motivation of connection on vector bundle. If we choose vector bundle to be the tangent bundle of a Riemannian manifold, then we can discuss the compatibility of the connection with the Riemannian metric and torsion-freeness, which will lead to the Levi-Civita connection.

2.1. **Motivation of connection.**

Given a vector bundle E on a smooth manifold M, the motivation of connection on E arise to take the derivative" of a section s of E in a given direction. It's quite natural to ask such a question, since when we learn calculus, we already know how to take the derivative of a smooth function $f: M\to\mathbb{R}^m$, to obtain a 1-form.

If we change our language in a fancy way, any smooth function $f: M\to\mathbb{R}^m$ is a section of the trivial vector bundle $M\times\mathbb{R}^m$ as follows:

$$x\mapsto(x,f(x)),$$

and its derivative $\mathrm{d}f$ is a section of $T^*M\otimes(M\times\mathbb{R}^m)$. Then taking derivatives can be seen as the following operator:

$$\nabla: C^\infty(M,M\times\mathbb{R}^m)\to C^\infty(M,T^*M\otimes(M\times\mathbb{R}^m)).$$

This motivates us why we define a connection on a vector bundle as follows:

Definition 2.1.1 (connection)**.** Let E be a vector bundle on a smooth manifold M. A connection ∇ on a vector bundle E is a $\mathbb{R}$-linear operator

$$\nabla: C^\infty(M,E)\to C^\infty(M,T^*M\otimes E)$$

satisfying the Leibniz rule $\nabla(fs)=\mathrm{d}f\otimes s+f\nabla s$, where $f\in C^\infty(M)$ and $s\in C^\infty(M,E)$.

Remark 2.1.1 (local form). Let $\{e_\alpha\}$ be a local frame of E. For any local section $s=s^\alpha e_\alpha$ of E, the Leibniz rule implies

$$\nabla(s^\alpha e_\alpha)=\mathrm{d}s^\alpha e_\alpha+s^\alpha\nabla e_\alpha.$$

Thus, the connection ∇ is determined by $\nabla e_\alpha=\omega_\alpha^\beta\otimes e_\beta$, where ω_α^β are 1-forms. For simplicity, we always write $\omega_\alpha^\beta\otimes e_\beta$ as $\omega_\alpha^\beta e_\beta$ if there is no ambiguity.

The collection of 1-forms $\{\omega_\alpha^\beta\}$ forms a matrix ω, with the lower index is the row index and the upper index is the column index.

Remark 2.1.2 (Christoffel symbol). Let $\{e_\alpha\}$ be a local frame of vector bundle E and ω be the matrix of connection 1-forms. If we write $\omega_\alpha^\beta = \Gamma_{i\alpha}^\beta \mathrm{d}x^i$, then $\Gamma_{i\alpha}^\beta$ is called the Christoffel symbol.

Let X be a vector field which is locally written as $X = X^i \frac{\partial}{\partial x^i}$ s be a section of E which is locally written as $s^\alpha e_\alpha$. Then

$$\begin{aligned}
\nabla_X s &= \nabla_{X^i \frac{\partial}{\partial x^i}} s^\alpha e_\alpha \\
&= X^i \nabla_{\frac{\partial}{\partial x^i}} s^\alpha e_\alpha \\
&= X^i s^\alpha \nabla_{\frac{\partial}{\partial x^i}} e_\alpha + X^i \frac{\partial s^\alpha}{\partial x^i} e_\alpha \\
&= \left(X^i s^\alpha \Gamma_{i\alpha}^\beta + X(s^\beta) \right) e_\beta.
\end{aligned}$$

Remark 2.1.3 (connection and covariant derivative). Some authors may also use the terminology covariant derivative". Here we make a clarification: We give two definitions of connection ∇ on a vector bundle E. Given a section s of E and a vector field X, we call $\nabla_X s$ the covariant derivative of s with respect to X. In fact, one can see the connection and covariant derivative as the same thing, just with different terminology.

2.2. **Examples.** Given a vector bundle E over a smooth manifold M, one can construct many vector bundles by algebraic method, such as considering its dual bundle E^*, tensor product $E \otimes E$ and so on. Now let's see if we already have a connection on E, how to induce some connections on these vector bundles.

2.2.1. *Induced connection on dual bundle.* Let E be a vector bundle equipped with connection ∇^E, induced connection on dual bundle E^* is defined as follows

$$\mathrm{d}(s,t) = (\nabla^{E^*} s, t) + (s, \nabla^E t),$$

where s, t are sections of E^* and E respectively, and (-,-) denotes the dual pairing. Suppose $\{e_\alpha\}$ is a local frame of E with dual frame $\{e^\alpha\}$, then

$$0 = ((\omega^*)_\gamma^\alpha e^\gamma, e_\beta) + (e^\alpha, \omega_\beta^\gamma e_\gamma),$$

that is,

$$(\omega^*)_\beta^\alpha + \omega_\beta^\alpha = 0,$$

where ω and ω^* are connection 1-forms of ∇^E and ∇^{E^*} respectively.

Remark 2.2.1 (another viewpoint of torsion-freeness). Let ∇ be a connection on TM, given by the Christoffel symbol Γ_{ij}^k. Then induced connection on T^*M is given by

$$\nabla \mathrm{d}x^k = -\Gamma_{ij}^k \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

Given a section s of T^*M, there is a natural 2-form obtained from taking the exterior derivative $\mathrm{d}s$, and note that $\bigwedge^2 T^*M$ is just the skew-symmetrization of $T^*M\otimes T^*M$, so it's natural to require the skew-symmetrization of ∇s is $\mathrm{d}s$. Locally, it suffices to require the skew-symmetrization of $\nabla \mathrm{d}x^k=0$, that is,

$$-\Gamma_{ij}^k \mathrm{d}x^i\wedge \mathrm{d}x^j=0.$$

This shows ∇ is a torsion-free connection if and only if the skew-symmetrization of ∇s is $\mathrm{d}s$.

2.2.2. *Induced connection on tensor product.* Let E,F be two vector bundles equipped with connection ∇^E,∇^F respectively. Then the induced connection on $E\oplus F$ is given by

$$\nabla^{E\otimes F}(s\otimes t):=\nabla^E s\otimes t+s\otimes\nabla^F t,$$

where s,t are sections of E,F respectively.

2.2.3. *Induced connection on wedge product.* Let E be a vector bundle equipped with connection ∇, there is an induced connection on $\bigwedge^2 E$, since it's a subbundle of $\bigotimes^2 E$. To be explicit

$$\begin{aligned}\nabla^{\wedge^2 E}(s\wedge t):=&\nabla^{\otimes^2 E}(s\otimes t-t\otimes s)\\ =&\nabla s\otimes t+s\otimes\nabla t-\nabla t\otimes s-t\otimes\nabla s\\ =&\nabla s\wedge t+s\wedge\nabla t,\end{aligned}$$

where s,t are sections of E.

Remark 2.2.2. In general case, there is an induced connection on $\otimes^k E$, given by

$$\nabla^{\otimes^k E}(s_1\otimes\cdots\otimes s_k)=\sum_{i=1}^k s_1\otimes\cdots\otimes\nabla s_i\otimes\cdots\otimes s_k,$$

where $s_1,\dots,s_k$ are sections of E. Its restriction on $\bigwedge^k E$ gives a connection on $\bigwedge^k E$, that is,

$$\nabla^{\otimes^k E}(s_1\wedge\cdots\wedge s_k)=\sum_{i=1}^k s_1\wedge\cdots\wedge\nabla s_i\wedge\cdots\wedge s_k.$$

2.2.4. *Induced connection on endomorphism bundle.* Let E be a vector bundle equipped with connection ∇^E, there is an induced connection ∇ on $\mathrm{End}E$, since we have $\mathrm{End}E\cong E\otimes E^*$. Suppose $\{e_\alpha\}$ is a local frame of E with dual frame $\{e^\alpha\}$, for section s of $E\otimes E^*$ locally written as $s=s_\beta^\alpha e_\alpha\otimes e^\beta$, A direct computation shows

$$\begin{aligned}\nabla^{E\otimes E^*}(s_\beta^\alpha e_\alpha\otimes e^\beta)&=\mathrm{d}s_\beta^\alpha e_\alpha\otimes e^\beta+s_\beta^\alpha(\nabla^E e_\alpha\otimes e^\beta+e_\alpha\otimes\nabla^{E^*}e^\beta)\\ &=\mathrm{d}s_\beta^\alpha e_\alpha\otimes e^\beta+s_\beta^\alpha\omega_\alpha^\gamma e_\gamma\otimes e^\beta-s_\beta^\alpha\omega_\gamma^\beta e_\alpha\otimes e^\gamma\\ &=(\mathrm{d}s_\beta^\alpha+s_\beta^\alpha\omega_\alpha^\gamma-\omega_\gamma^\beta s_\beta^\alpha)e_\alpha\otimes e^\beta.\end{aligned}$$

Thus, in matrix notation we have

$$\nabla s=\mathrm{d}s+s\omega-\omega s.$$

There is another way to construct a connection on $E\otimes E^*$: For any section s of $E\otimes E^*$, we have a function $s(e^\alpha, e_\beta)$. Then the induced connection on $E\otimes E^*$ can be defined as

$$\mathrm{d}s(e^\alpha, e_\beta) = \nabla^{E\otimes E^*} s(e^\alpha, e_\beta) + s(\nabla^{E^*} e^\alpha, e_\beta) + s(e^\alpha, \nabla^E e_\beta).$$

If we write $s = s^\alpha_\beta e_\alpha \otimes e^\beta$, then

$$\begin{aligned}\mathrm{d}(s^\alpha_\beta) &= (\nabla s)^\alpha_\beta + s(-\omega^\alpha_\gamma e^\gamma, e_\beta) + s(e^\alpha, \omega^\gamma_\beta e_\gamma)\\ &= (\nabla s)^\alpha_\beta - s^\gamma_\beta \omega^\alpha_\gamma + \omega^\gamma_\beta s^\alpha_\gamma,\end{aligned}$$

which implies induced connection on endomorphism bundle obtained from these two ways are the same.

2.3. **Levi-Civita connection.**

2.3.1. *Compatibility with metric.* Now consider a vector bundle E with a metric g, which can be locally written as $g_{\alpha\beta} e^\alpha \otimes e^\beta$. If there is a connection ∇ on E, it's natural to ask it to be compatible with metric.

Definition 2.3.1 (compatibility with metric)**.** A connection ∇ on a vector bundle E is compatible with the metric g if for any two sections s, t of E, we have

$$\mathrm{d}g(s,t) = g(\nabla s, t) + g(s, \nabla t).$$

Remark 2.3.1 (local form). Let $\{e_\alpha\}$ be a local frame of E. A direct computation shows

$$\begin{aligned}\mathrm{d}g_{\alpha\beta} &= \mathrm{d}g(e_\alpha, e_\beta)\\ &= g(\nabla e_\alpha, e_\beta) + g(e_\alpha, \nabla e_\beta)\\ &= \omega^\gamma_\alpha g_{\gamma\beta} + g_{\alpha\gamma}\omega^\gamma_\beta\end{aligned}$$

for all α, β, which is equivalent to

$$\frac{\partial}{\partial x^i} g_{\alpha\beta} = \Gamma^\gamma_{i\alpha} g_{\gamma\beta} + \Gamma^\gamma_{i\beta} g_{\alpha\gamma}$$

for all i, α, β. In matrix notation we have

$$\mathrm{d}g = \omega g + g\omega^t.$$

2.3.2. *Torsion-freeness.* Now let's choose our vector bundle E to be the tangent bundle of a Riemannian manifold (M, g).

Definition 2.3.2 (torsion-freeness)**.** A connection ∇ of TM is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

where X, Y are vector fields.

Remark 2.3.2 (local form). If we choose $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}$, then we have

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} &= (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial}{\partial x^k}\\ &= 0,\end{aligned}$$

which is equivalent to saying Γ^k_{ij} is symmetric in i and j.

2.3.3. *Levi-Civita connection.* There are infinitely many connection on the tangent bundle of a Riemannian manifold, but an interesting thing is that there is only one of them which is both compatible with the Riemannian metric and torsion-freeness.

Note that compatibility with Riemannian metric g implies

$$\begin{aligned} Xg(Y,Z) &= g(\nabla_X Y,Z)+g(Y,\nabla_X Z) \\ Yg(Z,X) &= g(\nabla_Y Z,X)+g(Z,\nabla_Y X) \\ Zg(X,Y) &= g(\nabla_Z X,Y)+g(X,\nabla_Z Y), \end{aligned}$$

where X,Y and Z are vector fields. Adding the first two equations, subtracting the third, and using the torsion-free condition, we will see

$$Xg(Y,Z)+Yg(Z,X)-Zg(X,Y) = g([X,Z],Y)+g([Y,Z],X)+g([X,Y],Z)+2g(Z,\nabla_Y X),$$

thus

$$g(Z,\nabla_Y X) = \frac{1}{2}(Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g([X,Z],Y)-g([Y,Z],X)-g([X,Y],Z)),$$

which implies $\nabla_X Y$ is uniquely determined. The above formula is also called the Koszul formula.

Remark 2.3.3 (local form). Firstly, compatibility implies

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki}^{\ell} g_{\ell j} + \Gamma_{kj}^{\ell} g_{i\ell}.$$

By permuting i,j,k we obtain the following two equations:

$$\begin{aligned} \frac{\partial g_{jk}}{\partial x^i} &= \Gamma_{ij}^{\ell} g_{\ell k} + \Gamma_{ik}^{\ell} g_{j\ell} \\ \frac{\partial g_{ki}}{\partial x^j} &= \Gamma_{jk}^{\ell} g_{\ell i} + \Gamma_{ji}^{\ell} g_{k\ell}. \end{aligned}$$

By the symmetry of Γ_{ij}^{ℓ} in i,j and the symmetry of g_{ij}, we have

$$2\Gamma_{ij}^{\ell} g_{\ell k} = \frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}.$$

If we use (g^{ij}) to denote the inverse matrix of (g_{ij}), then we have

$$\Gamma_{ij}^{\ell} = \frac{1}{2} g^{k\ell}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right),$$

which implies the Christoffel symbol is completely determined by the Riemannian metric and its first order partial derivatives.

3. Tensor computations

3.1. **Induced connection on tensor.** Let M be a smooth manifold.

Definition 3.1.1. For an (s,r)-tensor T, ∇T is an $(s,r+1)$-tensor, which is defined by

$$\begin{aligned}\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}}) :=& \frac{\partial}{\partial x^i}T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}}) \\ &-\sum_{l=1}^{s}T(\mathrm{d}x^{j_1},\dots,\nabla_{\frac{\partial}{\partial x^i}}\mathrm{d}x^{j_\ell},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}}) \\ &-\sum_{m=1}^{r}T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^{i_m}},\dots,\frac{\partial}{\partial x^{i_r}})\end{aligned}$$

Definition 3.1.2 (parallel tensor)**.** A tensor T is called parallel if $\nabla T=0$.

Definition 3.1.3 (covariant derivative of tensor)**.** For an (s,r)-tensor T, the covariant derivative of T with respect to vector field X, which is an (s,r)-tensor, is defined as

$$\nabla_X T := \nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},X,\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}}).$$

Remark 3.1.1 (local form). If we write an (s,r)-tensor T locally as

$$T^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r},$$

and $(s,r+1)$-tensor ∇T locally as

$$\nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}\otimes\mathrm{d}x^{i}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r},$$

Then by definition we have

$$\nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}=\frac{\partial T^{j_1\dots j_s}_{i_1\dots i_r}}{\partial x^i}+\sum_{l=1}^{s}\Gamma^{j_\ell}_{iq}T^{j_1\dots j_{l-1}qj_{l+1}\dots j_s}_{i_1\dots i_r}-\sum_{m=1}^{r}\Gamma^{q}_{ii_m}T^{j_1\dots j_s}_{i_1\dots i_{m-1}qi_{m+1}\dots i_r}.$$

Example 3.1.1. *Consider* $(0,0)$*-tensor* f*, that is, a smooth function. Then* ∇f *is a* $(0,1)$*-tensor, given by*

$$\nabla f=\nabla_i f\,\mathrm{d}x^i.$$

By definition $\nabla_i f=\frac{\partial f}{\partial x^i}$*, it coincides with our usual notations.*

Inductively, we can define $\nabla^2 T$ to be $\nabla(\nabla T)$, which is an $(s,r+2)$-tensor, and locally write it as

$$\nabla^2 T=\nabla^2_{k,i}T^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}\otimes\mathrm{d}x^{k}\otimes\mathrm{d}x^{i}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r}.$$

Now there is a natural question: Since $\nabla^2_{k,i}T$ is an (s,r)-tensor, and $\nabla_k\nabla_i T$ is also an (s,r)-tensor, are they equal? Unfortunately, it's false in general.

Example 3.1.2. *For* $(0,0)$*-tensor* f*, by definition we have* $\nabla^2 f$ *is* $\nabla(\nabla_i f \mathrm{d}x^i)$*, which is called the Hessian of* f*, denoted by* $\operatorname{Hess} f$*. More explicitly*

$$\begin{aligned}\operatorname{Hess} f &= \nabla(\nabla_i f \mathrm{d}x^i)\\ &= \frac{\partial \nabla_i f}{\partial x^k}\mathrm{d}x^k \otimes \mathrm{d}x^i - \nabla_i f \Gamma^i_{kj}\mathrm{d}x^k \otimes \mathrm{d}x^j\\ &= (\frac{\partial^2 f}{\partial x^k \partial x^i} - \Gamma^j_{ki}\frac{\partial f}{\partial x^j})\mathrm{d}x^k \otimes \mathrm{d}x^i,\end{aligned}$$

that is, $\nabla^2_{k,i} f = \frac{\partial^2 f}{\partial x^k \partial x^i} - \Gamma^j_{ki}\frac{\partial f}{\partial x^j}$*. On the other hand, we have* $\nabla_k \nabla_i f = \frac{\partial^2 f}{\partial x^k \partial x^i}$*. This shows in general* $\nabla^2_{k,i} \neq \nabla_k \nabla_i f$*.*

Proposition 3.1.1.

$$\nabla^2_{k,i} T^{j_1\dots j_s}_{i_1\dots i_r} = \nabla_k \nabla_i T^{j_1\dots j_s}_{i_1\dots i_r} - \Gamma^j_{ki}\nabla_j T^{j_1\dots j_s}_{i_1\dots i_r}.$$

Proof. Direct computation shows

$$\begin{aligned}\nabla^2_{k,i} T^{j_1\dots j_s}_{i_1\dots i_r} =& \nabla^2 T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})\\ =& \nabla_{\frac{\partial}{\partial x^k}} \nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})\\ =& \underbrace{\frac{\partial}{\partial x^k}\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})}_{\text{part I}}\\ & \underbrace{-\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})}_{\text{part II}}\\ & \underbrace{-\sum_{l=1}^{s}\nabla T(\mathrm{d}x^{j_1},\dots,\nabla_{\frac{\partial}{\partial x^k}}\mathrm{d}x^{j_\ell},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})}_{\text{part III}}\\ & \underbrace{-\sum_{m=1}^{r}\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^{i_m}},\dots,\frac{\partial}{\partial x^{i_r}})}_{\text{part IV}}.\end{aligned}$$

Note that

(1) Part I + part III + part IV is $\nabla_k \nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}$.

(2) Part II is $\Gamma^j_{ki}\nabla_j T^{j_1\dots j_s}_{i_1\dots i_r}$.

□

Remark 3.1.2 (another viewpoint of compatibility). Note that we can regard our Riemannian metric g as a $(0,2)$-tensor. Recall our definition for compatibility is for any two vector fields X,Y we have

$$\mathrm{d}g(X,Y) = g(\nabla X, Y) + g(X, \nabla Y).$$

Or more explicitly for vector field Z, we have

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

However, by definition of ∇g we have

$$\nabla_Z g(X,Y) = Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y),$$

which shows that compatibility is equivalent to $\nabla g = 0$.

3.2. **Type change of tensor.** In general, for an (s,r)-tensor, we can change its type into any type of $(s-k, r+k)$ for all k such that $s-k \geq 0, r+k \geq 0$, since TM is canonically isomorphic to T^*M, which is called music isomorphism. More explicitly, for any vector field X, it gives a 1-form by

$$X^\flat : Y \mapsto g(X,Y),$$

where Y is a vector field. Locally we have

$$\begin{aligned} g(\frac{\partial}{\partial x^i}, Y) &= g(\frac{\partial}{\partial x^i}, \mathrm{d}x^j(Y)\frac{\partial}{\partial x^j}) \\ &= \mathrm{d}x^j(Y) g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\ &= g_{ij}\mathrm{d}x^j(Y), \end{aligned}$$

that is, $(\frac{\partial}{\partial x^i})^\flat = g_{ij}\mathrm{d}x^j$ of T^*M. Similarly, for any 1-form ω, it can be regarded as a section of $TM = T^{**}M$ as follows

$$\omega^\sharp : \beta \mapsto g(\omega, \beta),$$

and locally we have

$$\begin{aligned} g(\mathrm{d}x^j, \beta) &= g(\mathrm{d}x^j, \beta(\frac{\partial}{\partial x^i})\mathrm{d}x^i) \\ &= \beta(\frac{\partial}{\partial x^i}) g^{ij}, \end{aligned}$$

that is, $(\mathrm{d}x^j)^\sharp = g^{ij}\frac{\partial}{\partial x^i}$. In a summary, we have the so-called music isomorphism locally looks like

$$\begin{aligned} \flat : TM &\to T^*M & \sharp : T^*M &\to TM \\ \frac{\partial}{\partial x^i} &\mapsto g_{ij}\mathrm{d}x^j & \mathrm{d}x^j &\mapsto g^{ij}\frac{\partial}{\partial x^i} \end{aligned}$$

Example 3.2.1. *For a smooth function f, ∇f is a $(0,1)$-tensor, locally written as*

$$\nabla f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i.$$

Then we can change its type into $(1,0)$, that is,

$$\nabla f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}.$$

More generally, for a 1-form ω, locally looks like $\omega_i \mathrm{d}x^i$. Then we can change it into an $(1,0)$-tensor, and it locally looks like

$$\omega^\sharp = g^{ij}\omega_i \frac{\partial}{\partial x^j}.$$

Example 3.2.2 (induced metric on T^*M)**.** *Recall that a Riemannian metric g is a $(0,2)$-tensor, locally written as*

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

Then we can change its type into $(2,0)$, that is,

$$g_{ij}g^{ik}g^{j\ell}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^\ell} = \delta^k_j g^{j\ell}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^\ell} = g^{k\ell}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^\ell},$$

that is, a metric on T^*M.

3.3. **Induced metric on tensor.** If g is a Riemannian metric, then its $(2,0)$-type is a metric on T^*M. Now we can induce a metric on $T^*M\otimes T^*M$ as follows: Take two $(0,2)$-tensors T,S and write them locally as $T = T_{ij}\mathrm{d}x^i\otimes \mathrm{d}x^j, S = S_{k\ell}\mathrm{d}x^k\otimes\mathrm{d}x^\ell$. Then

$$\begin{aligned} g(T,S) &= T_{ij}S_{k\ell}g(\mathrm{d}x^i\otimes\mathrm{d}x^j, \mathrm{d}x^k\otimes\mathrm{d}x^\ell) \\ &:= T_{ij}S_{k\ell}g^{ik}g^{j\ell}. \end{aligned}$$

Remark 3.3.1. In general, we also have induced metric on $\bigotimes^k T^*M$, and on Ω^k_M, which will be used later in Hodge theory.

Proposition 3.3.1. If connection ∇ on vector bundle T^*M is compatible with metric g on it, then induced connection on $T^*M\otimes T^*M$ is compatible with induced metric g on it.

Proof. It suffices to check

$$\frac{\partial}{\partial x^m}g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^\ell) = g(\nabla_{\frac{\partial}{\partial x^m}}\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^\ell)+g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\nabla_{\frac{\partial}{\partial x^m}}\mathrm{d}x^k\otimes\mathrm{d}x^\ell).$$

By compatibility of ∇ and g, we have

$$\frac{\partial g^{ij}}{\partial x^k} = -\Gamma^i_{k\ell}g^{\ell j} - \Gamma^j_{k\ell}g^{i\ell}.$$

Thus a direct computation shows

$$\begin{aligned} \frac{\partial}{\partial x^m}g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^\ell) &= -(\Gamma^i_{mn}g^{nk}+\Gamma^k_{mn}g^{in})g^{j\ell} - g^{ik}(\Gamma^j_{mn}g^{n\ell}+\Gamma^\ell_{mn}g^{jn}) \\ g(\nabla_{\frac{\partial}{\partial x^m}}\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^\ell) &= -\Gamma^i_{mn}g^{nk}g^{j\ell} - \Gamma^j_{mn}g^{ik}g^{n\ell} \\ g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\nabla_{\frac{\partial}{\partial x^m}}\mathrm{d}x^k\otimes\mathrm{d}x^\ell) &= -\Gamma^k_{mn}g^{in}g^{j\ell} - \Gamma^\ell_{mn}g^{ik}g^{jn}. \end{aligned}$$

This yields the desired result. □

3.4. **Trace of tensor.** Let's see a simple example: For an $(1,1)$-tensor T, we can define its trace", since there is a natural isomorphism between $TM\otimes T^*M$ and $\mathrm{End}(TM)$. Thus, we can take its trace in the sense of matrix. To be explicit, if we locally write T as $T = T^i_j\frac{\partial}{\partial x^i}\otimes\mathrm{d}x^j$, then trace of T, denoted by $\mathrm{tr}_g T$, is defined as T^i_i.

If T is not in $(1,1)$-type, then we change it into $(1,1)$-type and then take trace:

(1) If $T = T_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$, then $T = g^{ik}T_{ij}\frac{\partial}{\partial x^k} \otimes \mathrm{d}x^j$. Thus, $\operatorname{tr}_g T = g^{ij}T_{ij}$.
(2) If $T = T^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$, then $T = g_{kj}T^{ij}\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^k$, that is, $\operatorname{tr}_g T = g_{ij}T^{ij}$.

In general, if a tensor of type (r,s) with $r+s=2n$, we can change its type into (n,n) and take trace n times to obtain a number. Later we will see we obtain Ricci curvature by taking trace of curvature, and we obtain scalar curvature by taking trace of Ricci curvature.

Remark 3.4.1 (scalar laplacian). For a smooth function $f\colon M \to \mathbb{R}$, $\nabla^2 f$ is a $(0,2)$-form, locally looks like

$$\nabla^2_{i,j} f \,\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

Then its trace looks like

$$\operatorname{tr}_g \nabla^2 f = g^{ij}\nabla^2_{i,j} f.$$

That's called scalar Laplacian of f, denoted by Δf.

Remark 3.4.2. If g is induced metric on $(0,2)$-tensor, then for any $(0,2)$-tensor T, we have

$$\begin{aligned} g(g,T) &= g(g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, T_{k\ell}\mathrm{d}x^k \otimes \mathrm{d}x^\ell) \\ &= g_{ij}T_{k\ell}g^{ik}g^{j\ell} \\ &= \delta^k_j g^{j\ell}T_{k\ell} \\ &= g^{k\ell}T_{k\ell} \\ &= \operatorname{tr}_g T. \end{aligned}$$

Proposition 3.4.1 (magic formula)**.** For a $(0,2)$-tensor T, we have

$$X(\operatorname{tr}_g T) = g(g, \nabla_X T).$$

Proof. From above remark we can see $\operatorname{tr}_g T = g(g,T)$, then ∇ is compatible with metric completes the proof. □

Remark 3.4.3 (local form). Locally we have

$$\nabla_i(g^{jk}T_{jk}) = g^{jk}(\nabla_i T_{jk}),$$

that is, g^{jk} can "pass through" covariant derivative, which is called magic formula".

4. Geodesic and normal coordinate

In this section we always assume (M,g) is a Riemannian manifold equipped with Levi-Civita connection ∇.

4.1. **Geodesic.**

Definition 4.1.1 (Geodesic)**.** A smooth curve $\gamma\colon(-\varepsilon,\varepsilon)\to M$ is called a geodesic, if for each local coordinate $\{x^i\}$, it satisfies

$$\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t}+\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\Gamma^k_{ij}\circ\gamma=0,$$

where $\gamma^i=x^i\circ\gamma$.

Remark 4.1.1. In Section 10, we will give a definition of geodesic by using pullback connection.

Theorem 4.1.1. For any $p\in M, v\in T_pM$, there exists $\varepsilon>0$ and a geodesic $\gamma\colon(-\varepsilon,\varepsilon)\to M$ such that

$$\begin{aligned}\gamma(0)&=p\\ \gamma'(0)&=v.\end{aligned}$$

Moreover, any two such geodesics agree on their common domain.

Proof. Follows from standard result in ODEs' theory. □

Remark 4.1.2. Note that standard result in ODEs' theory only guarantees the short time existence of geodesic. If we use I to denote the maximal interval such that γ can be defined on it, then in general $I\neq\mathbb{R}$.

Notation 4.1.1. For $v\in T_pM$, γ_v denotes the unique geodesic such that $\gamma(0)=p$ and $\gamma'(0)=v$.

Lemma 4.1.1. For each $p\in M, v\in T_pM$ and $c,t\in\mathbb{R}$, one has

$$\gamma_{cv}(t)=\gamma_v(ct),$$

whenever either side is defined.

Proof. It's clear by uniqueness. □

Definition 4.1.2. For any $p\in M$, V_p is a subspace of T_pM defined by

$$V_p:=\{v\in T_pM\mid\gamma_v(1)\text{ is defined}\}.$$

Remark 4.1.3. From Lemma 4.1.1, $v\in V_p$ if $|v|<\varepsilon$ for sufficiently small $\varepsilon>0$.

4.2. **Normal coordinate.**

4.2.1. *Exponential map and geodesical ball.*

Definition 4.2.1 (exponential map)**.** For $p\in M$, the exponential map at point p is the map

$$\begin{aligned}\exp_p\colon V_p&\to M\\ v&\mapsto \gamma_v(1).\end{aligned}$$

Theorem 4.2.1. The exponential map $\exp_p$ maps a neighborhood $0\in T_pM$ diffeomorphically onto a neighborhood of $p\in M$.

Proof. Note that

$$(\mathrm{d}\exp_p)_0\colon T_0(T_pM)\to T_pM$$

and if we identify $T_0(T_pM)$ with T_pM by using $0+tv\mapsto v$, then $(\mathrm{d}\exp_p)_0$ then becomes a linear map from T_pM to itself. By inverse function theorem, it suffices to check $(\mathrm{d}\exp_p)_0$ is an identity map. For all $v\in T_pM$, A direct computation shows

$$\begin{aligned}(\mathrm{d}\exp_p)_0(v)&\overset{(1)}{=}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\exp_p(0+tv)\\ &=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma_{tv}(1)\\ &=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma_v(t)\\ &=\gamma_v'(0)\\ &=v,\end{aligned}$$

where (1) holds from our identification $T_0(T_pM)\cong T_pM$ and definition of differential. ☐

Definition 4.2.2 (geodesic ball)**.** For $p\in M$, if $B(0,\delta)$ is a ball in T_pM such that $\exp_p$ is a diffeomorphism on $B(0,\delta)$, then $B(p,\delta):=\exp_p(B(0,\delta))$ is called a geodesic ball centered at p.

Theorem 4.2.2 (normal coordinate)**.** For each $p\in M$, there exists a local coordinate centered at p such that $g_{ij}(0)=\delta_{ij}$, which is called normal coordinate.

Proof. Let $\{\frac{\partial}{\partial x^1}\big|_p,\dots,\frac{\partial}{\partial x^n}\big|_p\}$ be an orthonormal basis of T_pM with respect to Riemannian metric g, and fix the following linear isomorphism

$$\begin{aligned}\Phi\colon T_pM&\to\mathbb{R}^n\\ v^i\frac{\partial}{\partial x^i}\Big|_p&\mapsto(v^1,\dots,v^n).\end{aligned}$$

By Theorem 4.2.1 there exists a neighborhood U of p (for example, the geodesic ball) which is mapped by $\Phi\circ\exp_p^{-1}$ diffeomorphically onto a neighborhood of $0\in\mathbb{R}^n$, so $(\Phi\circ\exp_p^{-1},U,p)$ gives a local coordinate centered at p.

Let e_i denotes $(0,\dots,\underbrace{1}_{i-\text{th}},\dots,0)\in\mathbb{R}^n$. Then

$$\begin{aligned} g_{ij}(0) &= \langle \mathrm{d}(\exp_p\circ\Phi^{-1})_0 e_i, \mathrm{d}(\exp_p\circ\Phi^{-1})_0 e_j\rangle_p \\ &\overset{(1)}{=} \langle (\mathrm{d}\exp_p)_0 \frac{\partial}{\partial x^i}\Big|_p, (\mathrm{d}\exp_p)_0 \frac{\partial}{\partial x^j}\Big|_p\rangle_p \\ &\overset{(2)}{=} \langle \frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p\rangle_p \\ &\overset{(3)}{=} \delta_{ij}, \end{aligned}$$

where

(1) holds from Φ is a linear map, thus $\mathrm{d}\Phi^{-1}=\Phi^{-1}$.
(2) holds from Theorem 4.2.1.
(3) holds from our choice of $\{\frac{\partial}{\partial x^i}\Big|_p\}$.

□

Theorem 4.2.3. In normal coordinate (x^i,U,p), we have

$$\Gamma_{ij}^k(0)=0.$$

Proof. For arbitrary $v=(v^1,\dots,v^n)\in\mathbb{R}^n$, $\gamma(t)=\exp_p(t\Phi^{-1}(v))$ is a geodesic with $\gamma(0)=p$ and $\gamma'(0)=\Phi^{-1}(v)$. In normal coordinate (x^i,U,p) one has $\gamma(t)=(tv^1,\dots,tv^n)$, thus geodesic equation is given by

$$\Gamma_{ij}^k(tv)v^iv^j=0$$

By Evaluating this expression at $t=0$ one has $\Gamma_{ij}^k(0)v^iv^j=0$ for arbitrary index k, and take $v=\frac{1}{2}(e_i+e_j)$ to conclude $\Gamma_{ij}^k(0)=0$ for all i,j,k. □

Corollary 4.2.1. In normal coordinate (x^i,U,p), the Taylor expansion of g_{ij} around zero is

$$g_{ij}(x)=\delta_{ij}+O(|x|^2).$$

Proof. It follows from

$$\frac{\partial g_{ij}}{\partial x^k}=\Gamma_{ki}^\ell(0)g_{\ell j}(0)+\Gamma_{kj}^\ell(0)g_{i\ell}(0)=0.$$

□

4.2.2. *Application of normal coordinate.* Tensor computation is one of the hallmarks of Riemannian geometry, but sometimes there is a way to avoid some complicated computations if you don't want to do it. In this section we collect some useful tools which provide a neat to way to compute.

The idea is, if we want to prove an identity of tensors, it suffices to check it pointwisely, since zero tensor is independent of the choice of coordinates.

Thus, the normal coordinate is a good tool to use, since by Theorem 4.2.2, one has

$$\begin{aligned} x^i(p) &= 0, \\ g_{ij}(p) &= \delta_{ij}, \\ \Gamma_{ij}^k(p) &= 0. \end{aligned}$$

Example 4.2.1. *For an (s,r)-tensor T, from Proposition 3.1.1 one has*

$$\nabla^2_{k,i} T^{j_1\dots j_s}_{i_1\dots i_r} = \nabla_k \nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}$$

under normal coordinate. In particular, Hessian of a smooth function f can be written as $\nabla_k \nabla_i f \mathrm{d}x^k \otimes \mathrm{d}x^i$ locally, which is relatively easier to compute.

Proposition 4.2.1. Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇, and $\{\frac{\partial}{\partial x^i}\}$ is a local frame of TM with dual basis $\{\mathrm{d}x^i\}$. Then

$$\mathrm{d} = g^{ij} \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^j}}.$$

Proof. Firstly note that exterior derivative is independent of the choice of coordinates, and A direct computation also shows $g^{ij}\mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^j}}$ is also independent of the choice of coordinates. Now it suffices to compute in normal coordinate, that is, to show $\mathrm{d} = \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}}$. For arbitrary k-form ω, without lose of generality we may write it as $f \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k$. Then

$$\begin{aligned} \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}} \omega &= \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}} (f \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k) \\ &= \mathrm{d}x^i \wedge \frac{\partial f}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k \\ &= \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k \\ &= \mathrm{d}\omega. \end{aligned}$$

□

4.3. **Hopf-Rinow's Theorem.**

In this section we will figure out when the exponential map is defined on the whole space T_pM.

Definition 4.3.1 (geodesically complete)**.** A Riemannian manifold M is geodesically complete if for all $p \in M$, the exponential map $\exp_p$ is defined on the whole T_pM.

At this stage it's convenient to introduce a distance function on a Riemannian manifold M which is not necessarily geodesic complete as follows: For $p, q \in M$, consider all the piecewise smooth curves joining p and q. Since M is connected, such curves always exist (cover a continuous curve joining p and q by a finite number of coordinates neighborhood and replace each piece contained in a coordinate neighborhood by a smooth one).

Definition 4.3.2 (distance)**.** Let (M,g) be a Riemannian manifold, $p, q \in M$, the distance between p and q is defined by the infimum of the lengths of all piecewise smooth curves joining p and q, denoted by $\mathrm{dist}(p,q)$.

Proposition 4.3.1. The topology induced by distance function on M coincides with the original topology on M.

Proof. [Car92, Proposition 2.6]. □

Theorem 4.3.1 (Hopf-Rinow)**.** Let (M, g) be a Riemannian manifold and $p \in M$. The following statements are equivalent.

(1) M is geodesically complete.
(2) The closed and bounded sets of M are compact.
(3) M is complete as a topological space.

In addition, any of statements above implies that for any $p, q \in M$, there exists a geodesic joining p and q with length $\operatorname{dist}(p, q)$.

Proof. [Car92, Theorem 2.8]. □

Remark 4.3.1. Note that (2) is equivalent to (3) is a basic fact in general topology.

Definition 4.3.3 (complete)**.** A Riemannian manifold is complete, if it's geodesically complete, or it's complete as a topological space.

Corollary 4.3.1. If M is compact, then it's complete.

Part 2. Curvature

5. Riemannian Curvature

5.1. Curvature form and curvature tensor.

5.1.1. *Curvature form.* Let (M,g) be a Riemannian manifold with connection ∇ of a vector bundle E over M. Now we're going to extend the connection to something called the exterior derivative defined on sections of vector bundle valued k-forms as follows

$$\begin{aligned}\mathrm{d}^\nabla: C^\infty(M,\Omega_M^k\otimes E)&\to C^\infty(M,\Omega_M^{k+1}\otimes E)\\ \omega\otimes e&\mapsto \mathrm{d}\omega\otimes e+(-1)^k\omega\wedge\nabla e.\end{aligned}$$

Suppose $\{e_\alpha\}$ is a local frame of E. Then

$$\begin{aligned}(\mathrm{d}^\nabla)^2(s^\alpha e_\alpha)&=\mathrm{d}^\nabla(\mathrm{d}s^\alpha\otimes e_\alpha+s^\alpha\omega_\alpha^\beta\otimes e_\beta)\\ &=-\mathrm{d}s^\alpha\wedge\omega_\alpha^\beta\otimes e_\beta+\mathrm{d}(s^\alpha\omega_\alpha^\beta)\otimes e_\beta-s^\alpha\omega_\alpha^\beta\wedge\omega_\beta^\gamma\otimes e_\gamma\\ &=s^\alpha(\mathrm{d}\omega_\alpha^\beta-\omega_\alpha^\gamma\wedge\omega_\gamma^\beta)\otimes e_\beta.\\ (\mathrm{d}^\nabla)^2(e_\alpha)&=\mathrm{d}^\nabla(\omega_\alpha^\beta\otimes e_\beta)\\ &=\mathrm{d}\omega_\alpha^\beta\otimes e_\beta-\omega_\alpha^\beta\wedge\nabla e_\beta\\ &=\mathrm{d}\omega_\alpha^\beta\otimes e_\beta-\omega_\alpha^\beta\wedge\omega_\beta^\gamma\otimes e_\gamma\\ &=(\mathrm{d}\omega_\alpha^\beta-\omega_\alpha^\gamma\wedge\omega_\gamma^\beta)\otimes e_\beta.\end{aligned}$$

This shows $(\mathrm{d}^\nabla)^2$ is a global section of $\Omega_M^2\otimes\mathrm{End}E$, that is, it's compatible with change of basis. Indeed, for two local frames $e,\widetilde{e}$ such that $\widetilde{e}=ge$, we have

$$\begin{aligned}g(\mathrm{d}^\nabla)^2e&=(\mathrm{d}^\nabla)^2ge\\ &=(\mathrm{d}^\nabla)^2\widetilde{e}\\ &=(\mathrm{d}\widetilde{\omega}-\widetilde{\omega}\wedge\widetilde{\omega})\widetilde{e}\\ &=(\mathrm{d}\widetilde{\omega}-\widetilde{\omega}\wedge\widetilde{\omega})ge,\end{aligned}$$

which implies

$$g^{-1}(\mathrm{d}\widetilde{\omega}-\widetilde{\omega}\wedge\widetilde{\omega})g=\mathrm{d}\omega-\omega\wedge\omega.$$

In general case, for $s\in C^\infty(M,\Omega_M^k\otimes E)$, locally written as $s=s^\alpha e_\alpha$, where s^α is a k-form, a direct computation also shows

$$(\mathrm{d}^\nabla)^2(s^\alpha e_\alpha)=s^\alpha\wedge(\mathrm{d}^\nabla)^2(e_\alpha).$$

Definition 5.1.1 (curvature form)**.** Let E be a vector bundle over M equipped with connection ∇. Then there exists a section $\Omega\in C^\infty(M,\Omega_M^2\otimes\mathrm{End}E)$, called curvature form, such that

$$(\mathrm{d}^\nabla)^2s=\Omega\wedge s$$

for all $s\in C^\infty(X,\Omega_M^k\otimes E)$.

Remark 5.1.1 (local form). Suppose $\{e_\alpha\}$ is a local frame of E and write $\Omega = \Omega_\alpha^\beta e_\beta \otimes e^\alpha$, where

$$\Omega_\alpha^\beta = \Omega_{ij\alpha}^\beta \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Then $\Omega = \mathrm{d}\omega - \omega \wedge \omega$ shows

$$\begin{aligned}\Omega_{ij\alpha}^\beta \mathrm{d}x^i \wedge \mathrm{d}x^j &= \Omega_\alpha^\beta \\ &= \mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta \\ &= \mathrm{d}(\Gamma_{i\alpha}^\beta \mathrm{d}x^i) - (\Gamma_{i\alpha}^\gamma \mathrm{d}x^i) \wedge (\Gamma_{j\gamma}^\beta \mathrm{d}x^j) \\ &= (-\partial_j \Gamma_{i\alpha}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta)\mathrm{d}x^i \wedge \mathrm{d}x^j.\end{aligned}$$

In other words, we have $\Omega_{ij\alpha}^\beta = -(\partial_j \Gamma_{i\alpha}^\beta + \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta)$.

Remark 5.1.2. In physicists' language, a connection is a "field", the curvature is the "strength" of the field, and choosing a local frame is called "fixing the gauge". The reason for these names comes from H. Weyl's work on rewriting Maxwell's equations.

5.1.2. *Curvature tensor.* Another standard way to define curvature is to define it as follows:

$$\begin{aligned} R: TM \times TM \times E &\to E \\ (X, Y, s) &\mapsto R(X,Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.\end{aligned}$$

It's easy to prove that R we defined above is a tensor, that is, R is a section of $T^*M \otimes T^*M \otimes \mathrm{End}E$.

Remark 5.1.3 (local form). Let $\{e_\alpha\}$ be a local frame of E. Then locally

$$R = R_{ij\alpha}^\beta \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes e^\alpha \otimes e_\beta.$$

A direct computation shows that

$$\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} e_\alpha = \nabla_{\frac{\partial}{\partial x^i}} (\Gamma_{j\alpha}^\beta e_\beta) = \partial_i \Gamma_{j\alpha}^\beta e_\beta + \Gamma_{j\alpha}^\beta \Gamma_{i\beta}^\gamma e_\gamma = (\partial_i \Gamma_{j\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta) e_\beta.$$

Thus

$$R_\alpha^\beta = (\partial_i \Gamma_{j\alpha}^\beta - \partial_j \Gamma_{i\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta)\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

In particular, when the curvature form is regarded as a tensor, it coincides with curvature tensor since

$$\begin{aligned}\Omega_{ij\alpha}^\beta \mathrm{d}x^i \wedge \mathrm{d}x^j + \Omega_{ji\alpha}^\beta \mathrm{d}x^j \wedge \mathrm{d}x^i =& -(\partial_j \Gamma_{i\alpha}^\beta + \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta)(\mathrm{d}x^i \otimes \mathrm{d}x^j - \mathrm{d}x^j \otimes \mathrm{d}x^i) \\ &+ (\partial_i \Gamma_{j\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta)(\mathrm{d}x^j \otimes \mathrm{d}x^i - \mathrm{d}x^i \otimes \mathrm{d}x^j) \\ =& R_{ij\alpha}^\beta \mathrm{d}x^i \otimes \mathrm{d}x^j + R_{ji\alpha}^\beta \mathrm{d}x^j \otimes \mathrm{d}x^i\end{aligned}$$

5.1.3. *Riemannian curvature.* Let (M,g) be a Riemannian manifold and ∇ be the Levi-Civita connection on TM. Then R is a $(1,3)$-tensor, which is locally written as

$$R_{ijk}^{r}\mathrm{d}x^i\otimes\mathrm{d}x^j\otimes\mathrm{d}x^k\otimes\frac{\partial}{\partial x^r}.$$

Now let's give an explicit expression of $R_{ijk\ell}=g_{\ell r}R_{ijk}^{r}$. By definition we have

$$\begin{aligned}
R_{ijk\ell}&=R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}\right)\\
&=\left\langle\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}-\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}\right\rangle\\
&=\partial_i\left\langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}\right\rangle-\left\langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k},\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^\ell}\right\rangle\\
&\quad-\left(\partial_j\left\langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}\right\rangle-\left\langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^\ell}\right\rangle\right)\\
&=\underbrace{\partial_i\left\langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}\right\rangle-\partial_j\left\langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}\right\rangle}_{\text{part I}}\\
&\quad+\underbrace{\left\langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^\ell}\right\rangle-\left\langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k},\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^\ell}\right\rangle}_{\text{part II}}.
\end{aligned}$$

For part II, we have

$$g_{rs}(\Gamma_{ik}^{r}\Gamma_{j\ell}^{s}-\Gamma_{jk}^{r}\Gamma_{i\ell}^{s}).$$

For part I, note that

$$\begin{aligned}
\partial_i(\Gamma_{jk}^{r}g_{r\ell})&=\partial_i\left(\frac{1}{2}g_{rs}(\partial_jg_{ks}+\partial_kg_{js}-\partial_sg_{jk})g_{r\ell}\right)\\
&=\partial_i\left(\frac{1}{2}\delta_{\ell}^{s}(\partial_jg_{ks}+\partial_kg_{js}-\partial_sg_{jk})\right)\\
&=\frac{1}{2}\partial_i(\partial_jg_{k\ell}+\partial_kg_{j\ell}-\partial_\ell g_{jk}).
\end{aligned}$$

Thus we have part I is

$$\partial_i(\Gamma_{jk}^{r}g_{r\ell})-\partial_j(\Gamma_{ik}^{r}g_{r\ell})=\frac{1}{2}(\partial_i\partial_kg_{j\ell}+\partial_j\partial_\ell g_{ik}-\partial_i\partial_\ell g_{jk}-\partial_j\partial_kg_{i\ell}).$$

So we have an explicit expression for $R_{ijk\ell}$

$$R_{ijk\ell}=\frac{1}{2}(\partial_i\partial_kg_{j\ell}+\partial_j\partial_\ell g_{ik}-\partial_i\partial_\ell g_{jk}-\partial_j\partial_kg_{i\ell})+g_{rs}(\Gamma_{ik}^{r}\Gamma_{j\ell}^{s}-\Gamma_{jk}^{r}\Gamma_{i\ell}^{s}).$$

From this expression, we can see in general curvature depends on second order partial derivatives of metric. Furthermore, there are some skew symmetries and symmetries of $R_{ijk\ell}$.

(1) $R_{ijk\ell}=-R_{jik\ell}$;
(2) $R_{ijk\ell}=-R_{ij\ell k}$;
(3) $R_{ijk\ell}=R_{k\ell ij}$.

Proposition 5.1.1. In normal coordinate, we have

$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{ik\ell j}(0)x^k x^\ell + O(|x|^3).$$

Proof. Firstly, let's prove that under normal coordinate, we have

$$\frac{\partial \Gamma^k_{ij}}{\partial x^\ell}(0) + \frac{\partial \Gamma^k_{\ell i}}{\partial x^j}(0) + \frac{\partial \Gamma^k_{j\ell}}{\partial x^i}(0) = 0$$

Indeed, in normal coordinate we have

$$0 = \Gamma^k_{ij}(tx)x^i x^j.$$

Then take differential with respect to t and evaluate at $t = 0$, we have

$$0 = \frac{\partial \Gamma^k_{ij}}{\partial x^\ell}(0)x^i x^j x^\ell,$$

which implies

$$\sum_{\sigma \in S_3} \frac{\partial \Gamma^k_{\sigma(i)\sigma(j)}}{\partial x^{\sigma(\ell)}}(0) = 0.$$

Then the claim holds from the symmetry of Christoffel symbol in term i, j.

Now let's back to the Taylor expansion of Riemannian metric at $x = 0$. It suffices to compute the second order partial derivatives of g_{ij}. By relations between Riemannian metric and Christoffel symbol of Levi-Civita connection, we have

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^m_{ki} g_{mj} + \Gamma^m_{kj} g_{mi}.$$

Then take partial differential with respect to x^ℓ, evaluate at $x = 0$ and use the fact that Christoffel symbol vanishes at $x = 0$, we have

$$\frac{\partial^2 g_{ij}}{\partial x^\ell \partial x^k}(0) = \frac{\partial \Gamma^m_{ki}}{\partial x^\ell}(0) g_{mj}(0) + \frac{\partial \Gamma^m_{kj}}{\partial x^\ell}(0) g_{mi}(0).$$

On the other hand, we have

$$\begin{aligned}
R_{ijk\ell}(0) &= \left(\frac{\partial \Gamma^m_{jk}}{\partial x^i}(0) - \frac{\partial \Gamma^m_{ik}}{\partial x^j}(0)\right) g_{m\ell}(0) \\
&= -\left(\frac{\partial \Gamma^m_{ij}}{\partial x^k}(0) + \frac{\partial \Gamma^m_{ki}}{\partial x^j}(0) + \frac{\partial \Gamma^m_{ik}}{\partial x^j}(0)\right) g_{m\ell}(0) \\
&= -\left(\frac{\partial \Gamma^m_{ij}}{\partial x^k}(0) + 2\frac{\partial \Gamma^m_{ki}}{\partial x^j}(0)\right) g_{m\ell}(0).
\end{aligned}$$

Thus we have

$$\begin{aligned}
2R_{ikj\ell}(0)x^k x^\ell &= -\left(R_{ik\ell j}(0) + R_{j\ell ki}(0)\right) x^k x^\ell \\
&= \left(\frac{\partial \Gamma^m_{ik}}{\partial x^\ell}(0) + 2\frac{\partial \Gamma^m_{i\ell}}{\partial x^k}(0)\right) g_{mj}(0)x^k x^\ell + \left(\frac{\partial \Gamma^m_{j\ell}}{\partial x^k}(0) + 2\frac{\partial \Gamma^m_{jk}}{\partial x^\ell}(0)\right) g_{mi}(0)x^k x^\ell \\
&= 3\frac{\partial^2 g_{ij}}{\partial x^k \partial x^\ell}(0)x^k x^\ell.
\end{aligned}$$

This completes the proof of the Taylor expansion of Riemannian metric in normal coordinate
$$g_{ij}=\delta_{ij}-\frac{1}{3}R_{ik\ell j}(0)x^kx^\ell+O(|x|^3).$$
□

Corollary 5.1.1 (Riemannian normal coordinate)**.** In Riemannian normal coordinate we have
(1) $g^{ij}=\delta_{ij}+\frac{1}{3}R_{ik\ell j}(0)x^kx^\ell+O(|x|^3)$.
(2) $\det(g_{ij})=1-\frac{1}{3}R_{k\ell}x^kx^\ell+O(|x|^3)$.
(3) $\sqrt{\det(g_{ij})}=1-\frac{1}{6}R_{k\ell}x^kx^\ell+O(|x|^3)$.

Proof. For (1). Note that g^{ij} gives a Riemannian metric on T^*M, and Levi-Civita connection ∇ on T^*M with respect to g^{ij} is exactly the induced connection from the one on TM. Note that
$$\nabla \mathrm{d}x^k=-\Gamma_{ij}^k\mathrm{d}x^i\otimes \mathrm{d}x^j,$$
where Γ_{ij}^k is the Christoffel symbol for Levi-Civita connection on TM, we have curvature form in this case differs a sign since
$$R_{ijk}^\ell(0)=\frac{\partial\Gamma_{jk}^\ell}{\partial x^i}-\frac{\partial\Gamma_{ik}^\ell}{\partial x^j}.$$
Thus all computations are same as proof above, but the result differs a sign in curvature.

For (2). By Jacobi's formula, we have
$$\frac{\partial\det(g_{ij})}{\partial x^k}=\det(g_{ij})g^{ij}\frac{\partial g_{ij}}{\partial x^k}.$$
Thus $\frac{\partial\det(g_{ij})}{\partial x^k}(0)=0$, since first-order partial derivatives of g_{ij} vanishes. Furthermore, since first-order partial derivatives of g^{ij} also vanishes, we have
$$\begin{aligned}\frac{1}{2}\frac{\partial^2\det(g_{ij})}{\partial x^\ell\partial x^k}&=\det(g_{ij})g^{ij}\frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^\ell\partial x^k}\\&=\det(g_{ij})g^{ij}\left(-\frac{1}{3}R_{ik\ell j}x^kx^\ell\right)\\&=-\frac{1}{3}\det(g_{ij})R_{k\ell}x^kx^\ell,\end{aligned}$$
which implies
$$\det(g_{ij})=1-\frac{1}{3}R_{k\ell}x^kx^\ell+O(|x|^3).$$
For (3). It follows from (2) directly. □

Theorem 5.1.1 (Ricci identity)**.** Let (M,g) be a Riemannian manifold and T be an (s,r)-tensor, locally written as $T^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r}$. Then
$$\nabla^2_{k,i}T^{j_1\dots j_s}_{i_1\dots i_r}-\nabla^2_{i,k}T^{j_1\dots j_s}_{i_1\dots i_r}=\sum_{l=1}^s R^{j_\ell}_{kiq}T^{j_1\dots j_{\ell-1}qj_{\ell+1}\dots j_s}_{i_1\dots i_r}-\sum_{m=1}^r R^q_{kii_m}T^{j_1\dots j_s}_{i_1\dots i_{m-1}qi_{m+1}\dots i_r}.$$

Proof. Without lose of generality, we may choose normal coordinate, by Proposition 3.1.1 and Remark 3.1.1, we have

$$\begin{aligned}\nabla^2_{k,i} T^{j_1\dots j_s}_{i_1\dots i_r} &= \nabla_k \nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}\\ &= \nabla_k \left(\frac{\partial T^{j_1\dots j_s}_{i_1\dots i_r}}{\partial x^i} + \sum_{l=1}^{s} \Gamma^{j_\ell}_{iq} T^{j_1\dots j_{\ell-1} q j_{\ell+1}\dots j_s}_{i_1\dots i_r} - \sum_{m=1}^{r} \Gamma^{q}_{i i_m} T^{j_1\dots j_s}_{i_1\dots i_{m-1} q i_{m+1}\dots i_r} \right)\\ &= \frac{\partial^2 T^{j_1\dots j_s}_{i_1\dots i_r}}{\partial x^k \partial x^i} + \sum_{l=1}^{s} \frac{\partial \Gamma^{j_\ell}_{iq}}{\partial x^k} T^{j_1\dots j_{\ell-1} q j_{\ell+1}\dots j_s}_{i_1\dots i_r} - \sum_{m=1}^{r} \frac{\partial \Gamma^{q}_{i i_m}}{\partial x^k} T^{j_1\dots j_s}_{i_1\dots i_{m-1} q i_{m+1}\dots i_r}.\end{aligned}$$

This completes the proof, since in normal coordinate one has

$$R^{\ell}_{ijk} = \frac{\partial \Gamma^{\ell}_{jk}}{\partial x^i} - \frac{\partial \Gamma^{\ell}_{ik}}{\partial x^j}.$$

□

5.2. **Bianchi identities.**

5.2.1. *First Bianchi.*

Proposition 5.2.1 (first Bianchi identity)**.** Let (M,g) be a Riemannian manifold and R be the curvature tensor of Levi-Civita connection. Let X,Y,Z,W be vector fields on M. Then

$$R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)=0.$$

Proof. It suffices to prove it locally. Suppose $X=\frac{\partial}{\partial x^i}, Y=\frac{\partial}{\partial x^j}, Z=\frac{\partial}{\partial x^j}, W=\frac{\partial}{\partial x^\ell}$. Then locally the first Bianchi identity is stated as

$$R_{ijk\ell}+R_{jki\ell}+R_{kij\ell}=0.$$

In other words, it suffices to prove

$$R^r_{ijk}+R^r_{jki}+R^r_{kij}=0.$$

Since we have

$$R^r_{ijk} = \underbrace{\partial_i \Gamma^r_{jk} - \partial_j \Gamma^r_{ik}}_{\text{part I}} + \underbrace{\Gamma^s_{jk}\Gamma^r_{is} - \Gamma^s_{ik}\Gamma^r_{js}}_{\text{part II}}.$$

(1) For the first part, if we permuting i,j,k, we have

$$\partial_i \Gamma^r_{jk} - \partial_j \Gamma^r_{ik} + \partial_j \Gamma^r_{ki} - \partial_k \Gamma^r_{ji} + \partial_k \Gamma^r_{ij} - \partial_i \Gamma^r_{kj} = 0,$$

since $\Gamma^r_{ij} = \Gamma^r_{ji}$ by torsion-free.

(2) For the second part, if we permuting i,j,k, we have

$$\Gamma^s_{jk}\Gamma^r_{is} - \Gamma^s_{ik}\Gamma^r_{js} + \Gamma^s_{ki}\Gamma^r_{js} - \Gamma^s_{ji}\Gamma^r_{ks} + \Gamma^s_{ij}\Gamma^r_{ks} - \Gamma^s_{kj}\Gamma^r_{is} = 0$$

by the same reason.

This completes the proof of first Bianchi identity. □

Remark 5.2.1. If we consider connection on arbitrary vector bundle E, there is no first Bianchi identity, since e_α is just a section of E, not a section of TM, and thus $R(e_\alpha,\cdot)$ or $R(\cdot,e_\alpha)$ is nonsense.

5.2.2. *Second Bianchi.* The second Bianchi identity for arbitrary vector bundle E is stated as follows

$$\nabla_X R(Y,Z,s,t)+\nabla_Y R(Z,X,s,t)+\nabla_Z R(X,Y,s,t)=0,$$

where $s,t\in C^\infty(M,E), X,Y,Z\in C^\infty(M,TM)$.

The first approach. In order to prove it, we use the normal coordinate, that is, $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=0$. Then

$$\nabla_{\frac{\partial}{\partial x^i}} g\left(\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^\ell}-\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^\ell},\frac{\partial}{\partial x^m}\right)=g\left(\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^\ell}-\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^\ell},\frac{\partial}{\partial x^m}\right).$$

By permuting i,j,k, we have

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^\ell}-\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^\ell}\\
&+\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^\ell}-\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^\ell}\\
&+\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^\ell}-\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^\ell}\\
=&R\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)\nabla_{\frac{\partial}{\partial x_k}}\frac{\partial}{\partial x_\ell}+R\left(\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_k}\right)\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_\ell}+R\left(\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_i}\right)\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_\ell}\\
=&0.
\end{aligned}$$

This completes proof of the second Bianchi identity. □

The second approach. Recall that curvature form Ω is a section of $\Omega^2_M\otimes\mathrm{End}E$. Let $\nabla^{\mathrm{End}E}$ be the induced connection on $\mathrm{End}(E)$. Then we have

$$\nabla^{\mathrm{End}E}\Omega=\mathrm{d}\Omega+\Omega\wedge\omega-\omega\wedge\Omega.$$

However, $\nabla^{\mathrm{End}E}\Omega=0$, since

$$\begin{aligned}
\nabla^{\mathrm{End}E}\Omega&=\mathrm{d}\Omega+\Omega\wedge\omega-\omega\wedge\Omega\\
&=\mathrm{d}(\mathrm{d}\omega-\omega\wedge\omega)+(\mathrm{d}\omega-\omega\wedge\omega)\wedge\omega-\omega\wedge(\mathrm{d}\omega-\omega\wedge\omega)\\
&=\mathrm{d}^2\omega-\mathrm{d}\omega\wedge\omega+\omega\wedge\mathrm{d}\omega+\mathrm{d}\omega\wedge\omega-\omega\wedge\omega\wedge\omega-\omega\wedge\mathrm{d}\omega+\omega\wedge\omega\wedge\omega\\
&=0.
\end{aligned}$$

If we back to local form, we have

$$(\partial_k\Omega^\beta_{ij\alpha}+\Omega^\gamma_{ij\alpha}\Gamma^\beta_{k\gamma}-\Gamma^\gamma_{k\alpha}\Omega^\beta_{ij\gamma})\mathrm{d}x^k\wedge\mathrm{d}x^i\wedge\mathrm{d}x^j=0.$$

In other words,

$$\partial_k\Omega^\beta_{ij\alpha}+\Omega^\gamma_{ij\alpha}\Gamma^\beta_{k\gamma}-\Gamma^\gamma_{k\alpha}\Omega^\beta_{ij\gamma}+\partial_i\Omega^\beta_{jk\alpha}+\Omega^\gamma_{jk\alpha}\Gamma^\beta_{i\gamma}-\Gamma^\gamma_{i\alpha}\Omega^\beta_{jk\gamma}+\partial_j\Omega^\beta_{ki\alpha}+\Omega^\gamma_{ki\alpha}\Gamma^\beta_{j\gamma}-\Gamma^\gamma_{j\alpha}\Omega^\beta_{ki\gamma}=0$$

Since $\Omega^\beta_{ij\alpha}-\Omega^\beta_{ji\alpha}=R^\beta_{ij\alpha}$ and

$$\nabla_k R^\beta_{ij\alpha}=\partial_k R^\beta_{ij\alpha}+\Gamma^\beta_{k\gamma}R^\gamma_{ij\alpha}-\Gamma^\gamma_{k\alpha}R^\beta_{ij\gamma},$$

we have $\nabla^{\mathrm{End}E}\Omega=0$ is equivalent to the second Bianchi identity. □

6. OTHER CURVATURES

6.1. **Sectional curvature.**

In this section we're going to introduce sectional curvature, which is used to characterize a two dimensional subspace of tangent space.

Fix $p \in M$ and let x, y are two linearly independent tangent vectors in T_pM, then sectional curvature for these two vectors are defined as

$$K_p(x,y) = \frac{R(x,y,y,x)}{g(x,x)g(y,y) - g(x,y)^2}.$$

In order to show it's a invariant defined for a two dimensional subspace, we need to check if $\mathrm{span}_{\mathbb{R}}\{x,y\} = \mathrm{span}_{\mathbb{R}}\{z,w\}$, then

$$K_p(x,y) = K_p(z,w).$$

Indeed, if we write

$$\begin{cases} z = ax + by \\ w = cx + dy. \end{cases}$$

Then by symmetry and skew symmetry properties of R we have

$$\begin{aligned} R(z,w,w,z) &= R(ax+by, cx+dy, cx+dy, ax+by) \\ &= R(ax,dy,dy,ax) + R(ax,dy,cx,by) + R(by,cx,dy,ax) + R(by,cx,cx,by) \\ &= a^2d^2R(x,y,y,x) - abcdR(x,y,y,x) - abcdR(x,y,y,x) + b^2c^2R(x,y,y,x) \\ &= (ad-bc)^2R(x,y,y,x). \end{aligned}$$

And by the same computations we have

$$g(z,z)g(w,w) - g(z,w)^2 = (ad-bc)^2\{g(x,x)g(y,y) - g(x,y)^2\}$$

Thus

$$K_p(x,y) = K_p(z,w).$$

Definition 6.1.1 (sectional curvature)**.** The sectional curvature $K_p(\sigma)$ for two dimensional subspace $\sigma \subseteq T_pM$ is defined as

$$K_p(\sigma) := K_p(x,y),$$

where $\{x,y\}$ is a basis of σ.

Definition 6.1.2 (isotropic)**.** A Riemannian manifold (M,g) is called isotropic, if for each point $p \in M$, the sectional curvature $K_p(\sigma)$ is independent of σ.

Definition 6.1.3 (constant sectional curvature)**.** A Riemannian manifold (M,g) has constant sectional curvature, if $K_p(\sigma)$ is constant for arbitrary $\sigma \subset T_pM, p \in M$.

Remark 6.1.1. By definition, we can see if a Riemannian manifold has constant sectional curvature, then it must be isotropic; Conversely, if the dimension of a Riemannian manifold ≥ 3, then isotropic is equivalent to constant sectional curvature, see Corollary 7.1.1

Lemma 6.1.1.

$$-6R(X,Y,Z,W)=\frac{\partial^2}{\partial s\partial t}\bigg|_{s=t=0}\{R(X+sZ,Y+tW,Y+tW,X+sZ)-R(X+sW,Y+tZ,Y+tZ,X+sW)\},$$

where X,Y,Z,W are vector fields.

Proof. It suffices to compute coefficients of st of $R(X+sZ,Y+tW,Y+tW,X+sZ)$ and exchange Z with W to obtain coefficients of st of $R(X+sW,Y+tZ,Y+tZ,X+sW)$.

It's easy to see coefficients of st of $R(X+sZ,Y+tW,Y+tW,X+sZ)$ is

$$R(Z,W,Y,X)+R(Z,Y,W,X)+R(X,W,Y,Z)+R(X,Y,W,Z),$$

so coefficients of st of $R(X+sW,Y+tZ,Y+tZ,X+sW)$ is

$$R(W,Z,Y,X)+R(W,Y,Z,X)+R(X,Z,Y,W)+R(X,Y,Z,W).$$

Thus the right hand of our desired identity is

$$-4R(X,Y,Z,W)-(R(Y,Z,W,X)+R(W,Y,Z,X))-(R(W,X,Y,Z)+R(W,Y,Z,X)).$$

By first Bianchi identity we have

$$\begin{aligned}R(Y,Z,W,X)+R(W,Y,Z,X)&=R(Y,Z,W,X)+R(Z,X,W,Y)\\&=R(X,Y,Z,W)\\R(W,X,Y,Z)+R(W,Y,Z,X)&=R(Y,Z,W,X)+R(Z,X,W,Y)\\&=R(X,Y,Z,W)\end{aligned}$$

This completes the proof. □

Notation 6.1.1. For convenience, we use $R_0(X,Y,Z,W)$ to denote

$$R_0(X,Y,Z,W)=g(X,W)g(Y,Z)-g(X,Z)g(Y,W),$$

where X,Y,Z,W are vector fields. Then we can write sectional curvature as

$$K_p(\sigma)=\frac{R(x,y,y,x)}{R_0(x,y,y,x)}.$$

where $\sigma\subset T_pM$ is spanned by x,y.

Proposition 6.1.1. A Riemannian manifold has constant sectional curvature K_p at point $p\in M$ if and only if the Riemannian curvature tensor $R=K_pR_0$, where K_p is a constant (depending on p).

Proof. If $R=K_pR_0$, then for a arbitrary x,y, we have

$$K_p(x,y)=\frac{R(x,y,y,x)}{R_0(x,y,y,x)}=K_p.$$

Conversely, if $K(\sigma)$ is constant at point $p\in M$, that is for arbitrary x,y we have

$$R(x,y,y,x)=K_pR_0(x,y,y,x).$$

If we denote

$$\begin{aligned}F(s,t)&=R(x+sz,y+tw,y+tw,x+sz)-R(x+sw,y+tz,y+tz,x+sw)\\F_0(s,t)&=R_0(x+sz,y+tw,y+tw,x+sz)-R_0(x+sw,y+tz,y+tz,x+sw),\end{aligned}$$

we still have $F(s,t) = K_p F_0(s,t)$. By Lemma 6.1.1, we have

$$R(x,y,z,w) = -\frac{1}{6}\left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} F(s,t),$$

and it's easy to see

$$R_0(x,y,z,w) = -\frac{1}{6}\left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} F_0(s,t).$$

This completes the proof. □

Corollary 6.1.1. A Riemannian manifold is isotropic if and only if $R = KR_0$, where K is a smooth function.

Corollary 6.1.2. A Riemannian manifold has constant sectional curvature K if and only if $R = KR_0$, where K is a constant.

Remark 6.1.2. An important corsub is that curvature tensor of Riemannian manifold with constant sectional curvature K is quite simple, since

$$R_{ijk\ell} = K(g_{i\ell}g_{jk} - g_{ik}g_{j\ell}),$$

that is, curvature is completely determined by zero order partial derivatives of metric, not the second order in general.

Remark 6.1.3. Let (M,g) be a Riemannian manifold of dimension two and $\{e_1,e_2\}$ be a basis of T_pM. Then

$$K_p = K_p(e_1,e_2) = \frac{R(e_1,e_2,e_2,e_1)}{|e_1|^2|e_2|^2 - |g(e_1,e_2)|^2}$$

is called Gauss curvature in the theory of surface.

6.2. **Ricci curvature and scalar curvature.**

Definition 6.2.1 (Ricci curvature)**.** For a Riemannian manifold (M,g), the Ricci curvature is defined to be

$$\mathrm{Ric}(X,Y) := \mathrm{tr}_g(Z \mapsto R(Z,X)Y),$$

where X,Y are vector fields.

Remark 6.2.1 (local form). The trace of above endomorphism is exactly R_{ijk}^i, and it can be written as

$$g^{i\ell}R_{ijk\ell}.$$

In other words, Ricci curvature tensor is the contracted tensor of curvature with respect to the first and fourth index.

Definition 6.2.2 (Ricci curvature in one direction)**.** For a point $p \in M$, and $x \in T_pM$, Ricci curvature in the direction x is defined as

$$\mathrm{Ric}_p(x) := \mathrm{Ric}(x,x).$$

Remark 6.2.2. For $x \in T_pM$, we can write it as $x = x^i e_i$, where $\{e_1,\dots,e_n\}$ is a basis of T_pM, then

$$\mathrm{Ric}_p(x) = R_{jk}x^jx^k.$$

Definition 6.2.3 (scalar curvature)**.** For a Riemannian manifold (M,g), the scalar curvature S at $p\in M$ is defined as

$$S(p):=\sum_{i=1}^n \operatorname{Ric}_p(e_i),$$

where $\{e_1,\dots,e_n\}$ is an orthonormal basis of T_pM.

Remark 6.2.3 (local form). Locally we have

$$S=g^{jk}R_{jk}.$$

Proposition 6.2.1 (contracted Bianchi identity)**.**

$$g^{jk}\nabla_k R_{ij}=\frac{1}{2}\nabla_i S,$$

where R_{ij} is Ricci curvature and S is scalar curvature.

Proof. Direct computation shows

$$\begin{aligned}
g^{jk}\nabla_k R_{ij} &= g^{jk}\nabla_k g^{pq}R_{pijq}\\
&= g^{jk}g^{pq}\nabla_k R_{pijq}\\
&= g^{jk}g^{pq}(-\nabla_p R_{ikjq}-\nabla_i R_{kpjq})\\
&= -g^{pq}\nabla_p R_{iq}+\nabla_i S\\
&= -g^{jk}\nabla_k R_{ij}+\nabla_i S.
\end{aligned}$$

This completes the proof. □

Proposition 6.2.2. The scalar curvature S at $p\in M$ is given by

$$S(p)=\frac{1}{\alpha_n}\int_{\mathbb{S}^{n-1}}\operatorname{Ric}_p(x)\mathrm{d}\mathbb{S}^{n-1}$$

where α_n is the volume of n-dimension unit ball in $\mathbb{R}^n$ and $\mathrm{d}\mathbb{S}^{n-1}$ is the area elements in $\mathbb{S}^{n-1}$.

Proof. Choose an orthonormal basis $\{e_1,\dots,e_n\}$ in T_pM and write $x=x^ie_i$, then

$$\operatorname{Ric}_p(x)=\operatorname{Ric}_p(x^ie_i)=(x^i)^2\operatorname{Ric}_p(e_i)$$

Since $|x|=1$, then the vector $\mu=(x^1,\dots,x^n)$ is a unit normal vector on $\mathbb{S}^{n-1}$. Denoting $V=(x^1\operatorname{Ric}_p(e_1),\dots,x^n\operatorname{Ric}_p(e_n))$, then Stokes theorem implies

$$\begin{aligned}
\frac{1}{\alpha_n}\int_{\mathbb{S}^{n-1}}(x^i)^2\operatorname{Ric}_p(e_i)\mathrm{d}\mathbb{S}^{n-1} &= \frac{1}{\alpha_n}\int_{\mathbb{S}^{n-1}}\langle V,\mu\rangle\mathrm{d}\mathbb{S}^{n-1}\\
&= \frac{1}{\alpha_n}\int_{B^n}\operatorname{div}V\mathrm{d}B^n\\
&= \operatorname{div}V\\
&= \sum_{i=1}^n\operatorname{Ric}_p(e_i)\\
&= S(p)
\end{aligned}$$

where B^n is unit ball in T_pM with $\partial B^n = \mathbb{S}^{n-1}$. □

Theorem 6.2.1. Let (M,g) be a Riemannian manifold, then for all $p \in M$ and r sufficiently small, the volume of the geodesic ball $B(p,r)$ is

$$\mathrm{vol}(B(p,r)) = \alpha_n r^n \left(1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3)\right)$$

where α_n is the volume of n-dimension unit ball in $\mathbb{R}^n$.

Proof. Note that we have

$$\sqrt{\det(g_{ij})} = \delta_{ij} - \frac{1}{6} R_{jk}(p) x^j x^k + O(|x^3|)$$

Directly computation shows

$$\begin{aligned}
\mathrm{Vol}(B(p,r)) &= \int_0^r \int_{\mathbb{S}^{n-1}(t)} \sqrt{\det g}\, \mathrm{d}S \mathrm{d}t \\
&= \int_0^r \int_{\mathbb{S}^{n-1}(t)} \left(1 - \frac{1}{6} \mathrm{Ric}_p(x) + O(|x|^3)\right) \mathrm{d}S \mathrm{d}t \\
&= \alpha_n r^n - \frac{\alpha_n}{6} \int_0^r t^{n+1} \mathrm{d}t + O(r^{n+3}) \\
&= \alpha_n r^n - \frac{\alpha_n S(p) r^{n+2}}{6(n+2)} + O(r^{n+3}) \\
&= \alpha_n r^n \left(1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3)\right)
\end{aligned}$$

where we use the fact $\alpha_n = \omega_{n-1}/n$. □

7. Basic models

7.1. **Einstein manifold.**

Definition 7.1.1 (Einstein manifold)**.** A Riemannian manifold (M,g) is called Einstein manifold, if its Ricci curvature satisfies $R_{ij} = \lambda g_{ij}$ for some $\lambda \in \mathbb{R}$.

Lemma 7.1.1 (Schur's lemma)**.** Let (M,g) be a Riemannian manifold with $\dim M \geq 3$, suppose $R_{ij} = f g_{ij}$, where f is a smooth function, then (M,g) is an Einstein manifold.

Proof. If $R_{ij} = f g_{ij}$, then contracted Bianchi identity shows

$$\frac{n}{2} \nabla_i f = g^{jk} \nabla_k f g_{ij} = \nabla_i f$$

for arbitrary i, which implies f is constant, since $n \geq 3$. □

Corollary 7.1.1. For a Riemannian manifold (M,g) with $\dim M \geq 3$, it is isotropic if and only if it has constant sectional curvature.

Proof. By Remark 6.1.1, it suffices to show if M is isotropic then it has constant sectional curvature. If M is isotropic, then there exists a smooth function K such that

$$R_{ijk\ell} = K(g_{i\ell} g_{jk} - g_{ik} g_{j\ell}).$$

Consider its Ricci curvature, that is

$$R_{jk} = (n-1)Kg_{jk}.$$

Then Schur's lemma implies $(n-1)K$ is constant, that is, K is constant. $\square$

Proposition 7.1.1. Let (M,g) be an Einstein 3-manifold, then (M,g) has constant sectional curvature.

Proof. For arbitrary point $p \in M$, without lose of generality we consider normal coordinate, that is $g_{ij} = \delta_{ij}$. Then

$$R_{11} = g^{ij}R_{i11j} = R_{2112} + R_{3113} = \lambda.$$

Similarly we have

$$R_{1221} + R_{3223} = \lambda, \quad R_{1331} + R_{2332} = \lambda.$$

Thus we can conclude

$$R_{1221} = R_{1331} = R_{2332} = \frac{\lambda}{2},$$

that is,

$$R_{ijk\ell} = \frac{\lambda}{2}(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}).$$

This shows (M,g) has constant sectional curvature $\lambda/2$. $\square$

Remark 7.1.1. In fact, it's a special case of Ricci curvature controls curvature. For a n-dimensional Riemannian manifold, it's easy to see R_{jk} has $n(n+1)/2$ independent components. But for $R_{ijk\ell}$, this counting problem becomes a little bit complicated, it has

$$\frac{n^2(n^2-1)}{12}$$

independent components. Indeed, since $R_{ijk\ell}$ is skew symmetric in ij and $k\ell$, this means that these pair of indices can take

$$m = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Also, $R_{ijk\ell}$ is symmetric when you swap ij with $k\ell$, which means there would be

$$\frac{m(m+1)}{2} = \frac{n^4 - 2n^3 + 3n^2 - 2n}{8}$$

choices. However, these are not independent, since there is first Bianchi identity

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0,$$

and it provides

$$\binom{n}{4} = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}$$

relations between these components, thus the number of independent components of $R_{ijk\ell}$ is

$$\frac{n^4-2n^3+3n^2-2n}{8}-\frac{n^4-6n^3+11n^2-6n}{24}=\frac{n^4-n^2}{12}=\frac{n^2(n^2-1)}{12}.$$

Therefore curvature is fully determined by the Ricci curvature if and only if

$$\frac{n^2(n^2-1)}{12}\le\frac{n(n+1)}{2},$$

or in other words, $n\le 3$.

7.2. **Sphere.**

Example 7.2.1 (Sphere)**.** *Let $\mathbb{S}^n(R)$ denote n-dimensional sphere with radius R. There is a natural inclusion $f:\mathbb{S}^n(R)\hookrightarrow(\mathbb{R}^{n+1},g_{can})$, and we can use f to pullback g_{can} to obtain a metric on $\mathbb{S}^n(R)$, denoted by g. Given a local chart (U,φ,x^i), we can write*

$$f(x^1,\dots,x^n)=\left(x^1,\dots,x^n,\sqrt{R^2-\sum_{i=1}^n(x^i)^2}\right).$$

As shown in Example 1.2.1, the Riemannian metric is given by

$$g_{ij}=\delta_{ij}+\frac{x^ix^j}{T^2},\quad T^2=R^2-\sum(x^i)^2,$$

and thus we have

$$g^{ij}=\delta^{ij}-\frac{x^ix^j}{R^2},\quad \frac{\partial g_{ij}}{\partial x^k}=\frac{\delta_{ki}x^j+\delta_{kj}x^i}{T^2}+\frac{2x^ix^jx^k}{T^4}.$$

Then Christoffel symbol can be computed as

$$\begin{aligned}
\Gamma^k_{ij}&=\frac{1}{2}g^{k\ell}\left(\frac{\partial g_{j\ell}}{\partial x^i}+\frac{\partial g_{i\ell}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^\ell}\right)\\
&=\sum_\ell\frac{1}{2}\left(\delta^{k\ell}-\frac{x^kx^\ell}{R^2}\right)\left(\frac{\delta_{ij}x^\ell+\delta_{i\ell}x^j}{T^2}+\frac{2x^ix^jx^\ell}{T^4}+\frac{\delta_{ji}x^\ell+\delta_{j\ell}x^i}{T^2}+\frac{2x^ix^jx^\ell}{T^4}-\frac{\delta_{\ell i}x^j+\delta_{\ell j}x^i}{T^2}-\frac{2x^ix^jx^\ell}{T^4}\right)\\
&=\sum_\ell\frac{x^\ell}{T^2}\left(\delta_{ij}+\frac{x^ix^j}{T^2}\right)\left(\delta^{k\ell}-\frac{x^kx^\ell}{R^2}\right)\\
&=\frac{g_{ij}}{T^2}x^k\left(1-\frac{\sum_{l=1}^n(x^\ell)^2}{R^2}\right)\\
&=\frac{x^k}{R^2}g_{ij}.
\end{aligned}$$

A direct computation shows[1]

$$\begin{aligned} R_{ijk\ell} &= \frac{1}{2}(\partial_i\partial_k g_{j\ell} + \partial_j\partial_\ell g_{ik} - \partial_i\partial_\ell g_{jk} - \partial_j\partial_k g_{i\ell}) + g_{rs}(\Gamma^r_{ik}\Gamma^s_{j\ell} - \Gamma^r_{jk}\Gamma^s_{i\ell}) \\ &= \frac{1}{R^2}(g_{i\ell}g_{jk} - g_{ik}g_{j\ell}). \end{aligned}$$

The Ricci curvature and scalar curvature can be computed as follows

$$\begin{aligned} R_{jk} &= g^{i\ell}R_{ijkl} = \frac{1}{R^2}g^{i\ell}(g_{il}g_{jk} - g_{ik}g_{jl}) = \frac{1}{R^2}(ng_{jk} - \delta^\ell_k g_{jl}) = \frac{n-1}{R^2}g_{jk}, \\ S &= g^{jk}R_{jk} = \frac{n(n-1)}{R^2}. \end{aligned}$$

7.3. **Lie group with invariant metric.**

Lemma 7.3.1. Let G be a Lie group equipped with left-invariant metric $\langle\cdot,\cdot\rangle$, and ∇ the Levi-Civita connection with respect to it. Then for all left-invariant vector fields X,Y,Z,

$$\langle X, \nabla_Y Z\rangle = \frac{1}{2}(\langle[X,Y],Z\rangle + \langle[X,Z],Y\rangle + \langle[Y,Z],X\rangle).$$

Proof. Recall that

$$\langle X, \nabla_Y Z\rangle = \frac{1}{2}(Y\langle Z,X\rangle + Z\langle X,Y\rangle - X\langle Y,Z\rangle - \langle[Y,X],Z\rangle - \langle[Z,X],Y\rangle - \langle[Y,Z],X\rangle).$$

But $Y\langle Z,X\rangle = Z\langle X,Y\rangle = X\langle Y,Z\rangle = 0$ since both metric and X,Y,Z are left-invariant, that is,

$$\langle X, \nabla_Y Z\rangle = \frac{1}{2}\{\langle[X,Y],Z\rangle + \langle[X,Z],Y\rangle + \langle[Y,Z],X\rangle\}.$$

Now set $Y = Z$ to conclude. □

Proposition 7.3.1. Let G be a Lie group equipped with bi-invariant metric $\langle\cdot,\cdot\rangle$, and ∇ the Levi-Civita connection with respect to it. Then for all left-invariant vector fields X,Y,Z,

$$\langle[X,Y],Z\rangle = \langle X,[Y,Z]\rangle.$$

Proof. Let y_t be the flow of Y, then

$$[X,Y] = \lim_{t\to 0}\frac{1}{t}((\mathrm{d}y_t)(X) - X).$$

On the other hand, since Y is left-invariant, that is $L_g \circ y_t = y_t \circ L_g$, giving

$$y_t(g) = y_t(L_g(e)) = L_g y_t(e) = g y_t(e) = R_{y_t(e)}(g).$$

Thus $\mathrm{d}y_t = \mathrm{d}R_{y_t(e)}$ and

$$[X,Y] = \lim_{t\to 0}\frac{1}{t}\left((\mathrm{d}R_{y_t(e)})(X) - X\right).$$

[1]Here I omit a huge computation, and I suggest you compute it by yourself. Maybe first it's quite tough for you to do this in first time, but you should try.

Note that the metric is bi-invariant, thus

$$\langle X,Z\rangle = \langle (\mathrm{d}R_{y_t(e)})\circ(\mathrm{d}L_{y_t^{-1}(e)})X,(\mathrm{d}R_{y_t(e)})\circ(\mathrm{d}L_{y_t^{-1}(e)})Z\rangle = \langle (\mathrm{d}R_{y_t(e)})X,(\mathrm{d}R_{y_t(e)})Z\rangle.$$

Differentiating the expression above with respect to t and setting $t=0$ we conclude

$$0 = \langle [X,Y],Z\rangle + \langle X,[Z,Y]\rangle.$$

This completes the proof. □

7.3.1. *Levi-Civita connection of bi-invariant metric.*

Theorem 7.3.1. Let G be a Lie group equipped with bi-invariant metric $\langle\cdot,\cdot\rangle$, and ∇ the Levi-Civita connection with respect to it. Then for every left-invariant vector field X on G, then $\nabla_X X = 0$.

Proof. From Lemma 7.3.1, we have

$$\langle Y,\nabla_X X\rangle = \langle X,[Y,X]\rangle,$$

From Proposition 7.3.1, we have

$$\langle X,[Y,X]\rangle = \langle [X,Y],X\rangle = -\langle X,[Y,X]\rangle,$$

that is, $\langle Y,\nabla_X X\rangle = 0$ for arbitrary vector field Y, which implies $\nabla_X X = 0$. □

Corollary 7.3.1. The assumptions are as above. If X,Y are left-invariant vector fields, then $\nabla_X Y = \frac{1}{2}[X,Y]$.

Proof. Note that

$$\begin{aligned} 0 &= \nabla_{X+Y}(X+Y)\\ &= \nabla_X Y + \nabla_Y X + \nabla_X X + \nabla_Y Y\\ &= \nabla_X Y + \nabla_Y X\\ &= 2\nabla_X Y - [X,Y]. \end{aligned}$$

Division by two finally yields

$$\nabla_X Y = \frac{1}{2}[X,Y].$$

□

Corollary 7.3.2. The assumptions are as above. If X,Y,Z are left-invariant vector fields, then $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$.

Proof. Directly from $\nabla_X Y = \frac{1}{2}[X,Y]$ and Jacobi's identity. □

Corollary 7.3.3. The assumptions are as above. If X,Y are left-invariant vector fields which are orthogonal, and σ is the plane generated by X and Y. Then

$$K(\sigma) = \frac{1}{4}\|[X,Y]\|^2.$$

Proof.

$$
\begin{aligned}
K(\sigma) &= \frac{\langle R(X,Y)Y,X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2} \\
&= \frac{\langle -\frac{1}{4}[[X,Y],Y],X\rangle}{\|X\|^2\|Y\|^2} \\
&= -\frac{1}{4}\langle [[X,Y],Y],X\rangle \\
&= -\frac{1}{4}\langle [X,Y],[Y,X]\rangle \\
&= \frac{1}{4}\|[X,Y]\|^2
\end{aligned}
$$

□

Remark 7.3.1. Therefore, sectional curvature of a Lie group with bi-invariant metric is non-negative. Furthermore, if the center of Lie algebra $\mathfrak{g}$ is trivial, then the sectional curvature is positive.

Theorem 7.3.2. Let G be a Lie group equipped with bi-invariant metric, the geodesics on G are precisely the integral curves of left-invariant vector fields.

Proof. Let $X \in \mathfrak{g}$ be a left-invariant vector field, and $\gamma : \mathbb{R} \to G$ its integral curve. Then

$$
\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \nabla_{\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})}\gamma_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \nabla_X X = 0
$$

which implies integral curves of left-invariant vector fields are geodesics. Furthermore, since geodesics are unique, we have geodesics are precisely integral curves of left-invariant vector fields. □

Corollary 7.3.4. The exponential map for the Lie group coincides with the exponential map of the Levi-Civita connection with respect to bi-invariant metric.

Part 3. Bochner's technique

8. Hodge theory on Riemannian manifold

In this section, for convenience, we assume (M,g) is a compact oriented Riemannian n-manifold, since we need to consider integrations on M.

8.1. Inner product on differential forms.

Before we introduce the Hodge theory on (M,g), let's recall some basic facts about differential k-forms. For a k-form φ, locally it can be written as

$$\varphi = \sum_{1\leq i_1<\cdots<i_k\leq n} \varphi_{i_1\dots i_k} \mathrm{d}x^{i_1}\wedge\cdots\wedge \mathrm{d}x^{i_k},$$

where $\varphi_{i_1\dots i_k} := \varphi(\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_k}})$ is skew-symmetric function. Equivalently, we have

$$\varphi = \frac{1}{k!}\varphi_{i_1\dots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge \mathrm{d}x^{i_k},$$

where the summation runs over arbitrary different k indices.

Notation 8.1.1. For convenience, we use $\varphi_I \mathrm{d}x^I$ to denote the summation $\varphi_{i_1\dots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge \mathrm{d}x^{i_k}$.

Since there is an induced metric g on $\bigotimes^k T^*M$, and Ω^k_M can be regarded as a subbundle of $\bigotimes^k T^*M$ naturally, we can define a metric on Ω^k_M as follows:

Definition 8.1.1. Let φ,ψ be two k-forms. The metric pairing $\langle$-,-$\rangle$ on Ω^k_M is defined as

$$\langle\varphi,\psi\rangle := \frac{1}{k!}g(\varphi,\psi),$$

where g is induced metric on $\bigotimes^k T^*M$.

Lemma 8.1.1. For $\varphi=\varphi_I\mathrm{d}x^I,\psi=\psi_J\mathrm{d}x^J$, we have

$$\langle\varphi,\psi\rangle = \varphi_I\psi_J g^{IJ},$$

where

$$g^{IJ} = \frac{1}{k!}g(\mathrm{d}x^I,\mathrm{d}x^J) = \det\begin{pmatrix} g^{i_1j_1} & \cdots & g^{i_1j_k} \\ \cdots & \cdots & \cdots \\ g^{i_kj_1} & \cdots & g^{i_kj_k}\end{pmatrix}.$$

Proof. It suffices to show

$$g(\mathrm{d}x^I,\mathrm{d}x^J) = k!\det\begin{pmatrix} g^{i_1j_1} & \cdots & g^{i_1j_k} \\ \cdots & \cdots & \cdots \\ g^{i_kj_1} & \cdots & g^{i_kj_k}\end{pmatrix}.$$

By definition one has

$$\mathrm{d}x^I = \mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k} = \sum_{\sigma\in S_k}(-1)^{|\sigma|}\mathrm{d}x^{i_{\sigma(1)}}\otimes\cdots\otimes\mathrm{d}x^{i_{\sigma(k)}}.$$

Then

$$\begin{aligned}
g(\mathrm{d}x^I,\mathrm{d}x^J) &= \sum_{\sigma,\tau}(-1)^{|\sigma|}(-1)^{|\tau|}g(\mathrm{d}x^{i_{\sigma(1)}}\otimes\cdots\otimes\mathrm{d}x^{i_{\sigma(k)}},\mathrm{d}x^{j_{\tau(1)}}\otimes\cdots\otimes\mathrm{d}x^{j_{\tau(k)}})\\
&= \sum_{\sigma,\tau}(-1)^{|\sigma|}(-1)^{|\tau|}g^{i_{\sigma(1)}j_{\tau(1)}}\dots g^{i_{\sigma(k)}j_{\tau(k)}}\\
&= \sum_{\sigma,\tau}(-1)^{|\sigma\tau^{-1}|}g^{i_{\sigma\tau^{-1}(1)}j_1}\dots g^{i_{\sigma\tau^{-1}(k)}j_k}\\
&= \sum_{\sigma}\sum_{\rho}(-1)^{|\rho|}g^{i_{\rho(1)}j_1}\dots g^{i_{\rho(k)}j_k}\\
&= \sum_{\sigma}\det(g^{i_pj_q})\\
&= k!\det(g^{i_pj_q}).
\end{aligned}$$

□

Corollary 8.1.1. For two k-forms φ,ψ, locally written as

$$\begin{aligned}
\varphi &= \sum_{1\le i_1<\cdots<i_k\le n}\varphi_{i_1\dots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
\psi &= \sum_{1\le j_1<\cdots<j_k\le n}\varphi_{j_1\dots j_k}\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_k}
\end{aligned}$$

with φ_I,ψ_J is skew-symmetric. Then

$$\langle\varphi,\psi\rangle = \sum_{\substack{1\le i_1<\dots i_k\le n\\ 1\le j_1<\dots j_k\le n}}\varphi_{i_1\dots i_k}\psi_{j_1\dots j_k}\det\begin{pmatrix} g^{i_1j_1} & \dots & g^{i_1j_k}\\ \dots & \dots & \dots\\ g^{i_kj_1} & \dots & g^{i_kj_k}\end{pmatrix}.$$

Example 8.1.1. *Let φ,ψ be two 2-forms, locally written as*

$$\begin{aligned}
\varphi &= \varphi_{i_1i_2}\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\\
\psi &= \psi_{j_1j_2}\mathrm{d}x^{j_1}\wedge\mathrm{d}x^{j_2},
\end{aligned}$$

where $i_1<i_2,j_1<j_2$. Then

$$\begin{aligned}
\langle\varphi,\psi\rangle &= \frac{1}{2}\varphi_{i_1i_2}\psi_{j_1j_2}g(\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2},\mathrm{d}x^{j_1}\wedge\mathrm{d}x^{j_2})\\
&= \frac{1}{2}\varphi_{i_1i_2}\psi_{j_1j_2}g(\mathrm{d}x^{i_1}\otimes\mathrm{d}x^{i_2}-\mathrm{d}x^{i_2}\otimes\mathrm{d}x^{i_1},\mathrm{d}x^{j_1}\otimes\mathrm{d}x^{j_2}-\mathrm{d}x^{j_2}\otimes\mathrm{d}x^{j_1})\\
&= \frac{1}{2}\varphi_{i_1i_2}\psi_{j_1j_2}(g^{i_1j_1}g^{i_2j_2}-g^{i_1j_2}g^{i_2j_1}-g^{i_2j_1}g^{i_1j_2}+g^{i_2j_2}g^{i_1j_1})\\
&= \varphi_{i_1i_2}\psi_{j_1j_2}(g^{i_1j_1}g^{i_2j_2}-g^{i_1j_2}g^{i_2j_1})\\
&= \varphi_{i_1i_2}\psi_{j_1j_2}\det\begin{pmatrix} g^{i_1j_1} & g^{i_1j_2}\\ g^{i_2j_1} & g^{i_2j_2}\end{pmatrix}.
\end{aligned}$$

Definition 8.1.2 (volume form)**.** The volume form vol is an n-form such that $\langle\mathrm{vol},\mathrm{vol}\rangle=1$.

Remark 8.1.1 (local form)*.* The volume form is locally given by $\sqrt{\det g}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$.

Definition 8.1.3 (inner product on Ω^k_M)**.** For two k-forms φ, ψ, the inner product is defined as

$$(\varphi, \psi) := \int_M \langle \varphi, \psi \rangle \operatorname{vol}.$$

Definition 8.1.4 (formal adjoint)**.** For a k-form φ and a $(k+1)$-form ψ, the formal adjoint of d is an operator $\mathrm{d}^* : C^\infty(M, \Omega^{k+1}_M) \to C^\infty(M, \Omega^k_M)$ such that

$$(\mathrm{d}\varphi, \psi) = (\varphi, \mathrm{d}^* \psi).$$

Remark 8.1.2. There is no guarantee for existence, but later we will see such d^* do exist, and give an explicit formula.

Definition 8.1.5 (Laplace-Beltrami operator)**.** The Laplace-Beltrami operator $\Delta_g : C^\infty(M, \Omega^k_M) \to C^\infty(M, \Omega^k_M)$ is defined as

$$\Delta_g = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}.$$

Definition 8.1.6 (harmonic)**.** A k-form α is called harmonic, if $\Delta_g \alpha = 0$.

Notation 8.1.2. The space of all harmonic forms is denoted by $\mathcal{H}^k(M)$.

Lemma 8.1.2. A k-form α is harmonic if and only if $\mathrm{d}\alpha = 0$ and $\mathrm{d}^*\alpha = 0$.

Proof. Note that

$$\begin{aligned}(\alpha, \Delta\alpha) &= (\alpha, \mathrm{d}\mathrm{d}^*\alpha) + (\alpha, \mathrm{d}^*\mathrm{d}\alpha) \\ &= \|\mathrm{d}^*\alpha\|^2 + \|\mathrm{d}\alpha\|^2.\end{aligned}$$

□

8.2. **Hodge star operator.** Although we have defined an inner product on Ω^k_M, it is still quite complicated to compute. However, inner product on Ω^k_M is independent of the choice of local frame, so we can use normal coordinate to give a local frame, and define the Hodge star operator on it, which will give us an effective method to compute.

8.2.1. *Baby case.* Let V be an $\mathbb{F}$-vector space with inner product $\langle -,- \rangle$, and $\{e_1, \dots, e_n\}$ is a basis of V. For any $0 \le k \le n$, there is a natural basis of $\bigwedge^k V$, consisting of $\{e_I := e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$.

With respect to this basis, the induced metric on $\bigwedge^k V$ can be written as

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k} \rangle = \det \begin{pmatrix} \langle e_{i_1}, e_{j_1} \rangle & \dots & \langle e_{i_1}, e_{j_k} \rangle \\ \vdots & & \vdots \\ \langle e_{i_k}, e_{j_1} \rangle & \dots & \langle e_{i_k}, e_{j_k} \rangle \end{pmatrix}.$$

it is clear if $\{e_1, \dots, e_n\}$ is an orthonormal basis of V, then $\{e_I\}$ is an orthonormal basis of $\bigwedge^k V$. From now on, we assume $\{e_I\}$ is an orthonormal basis of $\bigwedge^k V$ with respect to inner product induced from the inner product on V.

Definition 8.2.1 (Hodge star)**.** The Hodge star operator is a $\mathbb{F}$-linear operator defined as

$$\begin{aligned}\star : \bigwedge^k V &\to \bigwedge^{n-k} V \\ e_I &\mapsto \operatorname{sign}(I, I^c) e_{I^c},\end{aligned}$$

where I^c is $\{1,2,\dots,n\}\setminus I = \{i'_1,\dots,i'_{n-k}\}$ and $\operatorname{sign}(I,I^c)$ is the sign of the permutation $(i_1,\dots,i_k,i'_1,\dots,i'_{n-k})$.

Example 8.2.1. *it is clear* $\star 1 = \mathrm{vol}$ *and* $\star\,\mathrm{vol} = 1$.

Proposition 8.2.1.
$$\star^2 = (-1)^{k(n-k)}\operatorname{id}$$
holds on $\bigwedge^k V$.

Proof. It suffices to check on basis e_I as follows
$$\begin{aligned}\star^2 e_I &= \star(\operatorname{sign}(I,I^c)e_{I^c})\\ &= \operatorname{sign}(I,I^c)\operatorname{sign}(I^c,I)e_I\\ &= (-1)^{k(n-k)}e_I.\end{aligned}$$
□

Proposition 8.2.2. For $u \in \bigwedge^k V, v \in \bigwedge^{n-k} V$, we have
$$\star(u\wedge v) = (-1)^{k(n-k)}\langle u, \star v\rangle.$$

Proof. It suffices to check on basis $e_I = e_{i_1}\wedge\cdots\wedge e_{i_k}, e_J = e_{j_1}\wedge\cdots\wedge e_{j_{n-k}}$. Without lose of generality, we may assume $J = I^c$, otherwise it is clear $e_I\wedge e_J = 0$. A direct computation shows
$$\begin{aligned}\star(e_I\wedge e_{I^c}) &= \star(\operatorname{sign}(I,I^c)\mathrm{vol})\\ &= \operatorname{sign}(I,I^c).\\ \langle e_I, \star e_{I^c}\rangle &= \langle e_I, \operatorname{sign}(I,I^c)e_I\rangle\\ &= \operatorname{sign}(I,I^c)\langle e_I, e_I\rangle\\ &= \operatorname{sign}(I,I^c).\end{aligned}$$
This completes the proof. □

Corollary 8.2.1. For $u,v \in \bigwedge^k V$, we have
(1) $u\wedge\star v = v\wedge\star u = \langle u,v\rangle\,\mathrm{vol}$.
(2) $\langle\star u, \star v\rangle = \langle u, v\rangle$.

Proof. For (1).
$$\star(u\wedge\star v) = (-1)^{k(n-k)}\langle u, \star^2 v\rangle = \langle u, v\rangle,$$
which implies $u\wedge\star v = \langle u,v\rangle\,\mathrm{vol}$. Since $\langle u,v\rangle = \langle v,u\rangle$, we obtain $u\wedge\star v = v\wedge\star u$.

For (2).
$$\begin{aligned}\langle\star u, \star v\rangle &= (-1)^{k(n-k)}\star(\star u\wedge v)\\ &= (-1)^{2k(n-k)}\star(v\wedge\star u)\\ &= (-1)^{3k(n-k)}\langle v\wedge\star^2 u\rangle\\ &= (-1)^{4k(n-k)}\langle v, u\rangle\\ &= \langle u, v\rangle.\end{aligned}$$
□

Remark 8.2.1. (2) implies that the Hodge star operator is a linear isometry between $\bigwedge^k V$ and $\bigwedge^{n-k} V$.

Corollary 8.2.2 (almost self-adjoint)**.** For $u \in \bigwedge^k V, v \in \bigwedge^{n-k} V$, we have

$$\langle u, \star v\rangle = (-1)^{k(n-k)}\langle \star u, v\rangle.$$

Proof.

$$\langle u, \star v\rangle = \langle \star u, \star^2 v\rangle = (-1)^{k(n-k)}\langle \star u, v\rangle.$$

□

Remark 8.2.2. This corollary implies the adjoint operator of $\star$ is $(-1)^{k(n-k)}\star$, so here I call it almost self-adjoint.

8.2.2. *General case.* Now we're going to define the Hodge star operator on a smooth manifold M, which is an operator from $C^\infty(M, \Omega^k_M)$ to $C^\infty(M, \Omega^{n-k}_M)$. It suffices to define the Hodge star operator pointwisely, and then everything reduces to the baby case by considering the normal coordinate.

For each point $p \in M$, consider the local frame $\{\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^n}\}$ of TM given by normal coordinate, with dual frame $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$. The Hodge star operator is defined as

$$\star(\mathrm{d}x^I) := \mathrm{sign}(I, I^c)\mathrm{d}x^{I^c}.$$

Theorem 8.2.1.

(1) $\star 1 = \mathrm{vol}, \star\,\mathrm{vol} = 1$.
(2) $\star^2 = (-1)^{k(n-k)}$ on k-forms.
(3) If u is a k-form and v a $(n-k)$-form. Then

$$\star(u \wedge v) = (-1)^{k(n-k)}\langle u, \star v\rangle,$$
$$\langle u, \star v\rangle = (-1)^{k(n-k)}\langle \star u, v\rangle.$$

(4) For any two k-forms u, v. Then

$$u \wedge \star v = v \wedge \star u = \langle u, v\rangle\,\mathrm{vol} = \langle v, u\rangle\,\mathrm{vol},$$
$$\langle \star u, \star v\rangle = \langle u, v\rangle.$$

(5) $\mathrm{d}^* = (-1)^{nk+n+1} \star \mathrm{d} \star$ on k-forms.

Proof. It suffices to check (5), as we have already proved other cases in baby case. Take any $(k-1)$-form α and k-form β, we need to show

$$(\mathrm{d}\alpha, \beta) = (\alpha, \mathrm{d}^*\beta),$$

that is, to show

$$\int_M \mathrm{d}\alpha \wedge \star\beta = \int_M \alpha \wedge \star \mathrm{d}^*\beta.$$

By Stokes theorem and Leibniz rule we have

$$0 = \int_M \mathrm{d}(\alpha \wedge \star\beta) = \int_M \mathrm{d}\alpha \wedge \star\beta + (-1)^{k-1}\int_M \alpha \wedge \mathrm{d} \star \beta.$$

Since $\star^2 = (-1)^{(n-k+1)(k-1)}$ on $(n-k+1)$-forms, we have

$$(-1)^{k-1}\int_M \alpha\wedge \mathrm{d}\star\beta = (-1)^{k-1+(n-k+1)(k-1)}\int_M \alpha\wedge\star^2\mathrm{d}\star\beta.$$

Therefore

$$\begin{aligned}(\mathrm{d}\alpha,\beta) &= \int_M \mathrm{d}\alpha\wedge\star\beta\\ &= (-1)^{k+(n-k+1)(k-1)}\int_M \alpha\wedge\star\star\mathrm{d}\star\beta\\ &= (-1)^{nk+k+1}\int_M \alpha\wedge\star(\star\mathrm{d}\star\beta),\end{aligned}$$

which implies

$$\mathrm{d}^*\beta = (-1)^{nk+k+1}\star\mathrm{d}\star\beta.$$

□

Remark 8.2.3. For differential k-forms φ and ψ, (4) allows us to give a new expression for inner product by

$$(\varphi,\psi) := \int_M \langle\varphi,\psi\rangle\,\mathrm{vol} = \int_M \varphi\wedge\star\psi.$$

In some literature, the Hodge star operator is defined in this way.

8.3. **Computations of adjoint operator.**

Lemma 8.3.1 (Jacobi's formula)**.** For a differentiable function $(a_{ij}(t))$ which is valued in $\mathrm{GL}(n,\mathbb{R})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(a_{ij}(t)) = \det(a_{ij}(t))a^{ij}(t)\frac{\mathrm{d}a_{ij}(t)}{\mathrm{d}t},$$

where $(a^{ij}(t))$ is the inverse matrix of $(a_{ij}(t))$.

Lemma 8.3.2. Let (M,g) be a Riemannian manifold. For any two vector fields X,Y, one has

$$\nabla_Y\circ\iota_X = \iota_X\circ\nabla_Y + \iota_{\nabla_Y X}.$$

Proof. Let ω be a $(k+1)$-form. Then for vector fields $Y_1,\dots,Y_k$, a direct computation shows

$$\begin{aligned}\nabla_Y\circ\iota_X\omega(Y_1,\dots,Y_k) =& \nabla_Y\omega(X,Y_1,\dots,Y_k)\\ =& Y\omega(X,Y_1,\dots,Y_k) - \omega(\nabla_X Y,Y_1,\dots,Y_k)\\ &- \sum_{i=1}^k \omega(X,Y_1,\dots,Y_{i-1},\nabla_X Y_i,Y_{i+1},\dots,Y_k)\\ &= (\iota_X\circ\nabla_Y\omega + \iota_{\nabla_X Y}\omega)(Y_1,\dots,Y_k).\end{aligned}$$

□

Lemma 8.3.3. Let (M,g) be a compact Riemannian manifold. Then

$$\langle \mathrm{d}x^i \wedge \alpha, \beta\rangle = \langle \alpha, g^{ij}\iota_{\frac{\partial}{\partial x^j}}\beta\rangle,$$

where $\{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\}$ is a local frame of TM, and α,β are forms with appropriate degrees.

Proof. It suffices to check with respect to normal coordinate. If we locally write $\alpha = \alpha_I \mathrm{d}x^I$ and $\beta = \beta_J \mathrm{d}x^J$, it suffices to check the case there is no $\mathrm{d}x^i$ in $\mathrm{d}x^I$ and there is $\mathrm{d}x^i$ in $\mathrm{d}x^J$, since other cases are trivial. Suppose $|I| = k$ and $|J| = k+1$, and $\mathrm{d}x^i$ in the m-th position of $\mathrm{d}x^J$. Then

$$\langle \alpha, \iota_{\frac{\partial}{\partial x^i}}\beta\rangle = (-1)^{m+2}\alpha_I\beta_J \det G,$$

where G is a $k\times k$ matrix. By definition if we write

$$\langle \mathrm{d}x^i \wedge \alpha, \beta\rangle = \alpha_I\beta_J \det G',$$

where G' is a $(k+1)\times(k+1)$ matrix. it is clear $\det G' = (-1)^{m+2}\det G$ by expansion of $\det G'$ by the first row. □

Lemma 8.3.4. Let (M,g) be a Riemannian manifold with volume form vol. Then

$$\begin{aligned}\mathcal{L}_X \mathrm{vol} &= \Big(\frac{\partial X^i}{\partial x^i} + \frac{1}{2}X^i g^{pq}\frac{\partial g_{pq}}{\partial x^i}\Big)\mathrm{vol}\\ &= \Big(\frac{\partial X^i}{\partial x^i} + \Gamma^j_{ij}X^i\Big)\mathrm{vol}.\end{aligned}$$

Proof. By Cartan's magic formula we have

$$\mathcal{L}_X = \iota_X\circ \mathrm{d} + \mathrm{d}\circ\iota_X,$$

and thus

$$\begin{aligned}\mathcal{L}_X \mathrm{vol} &= (\iota_X\circ \mathrm{d} + \mathrm{d}\circ\iota_X)\mathrm{vol}\\ &= \mathrm{d}\circ\iota_X \mathrm{vol}\\ &= \mathrm{d}\{(-1)^{i-1}X^i\sqrt{\det g}\,\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge \mathrm{d}x^n\}\\ &= \frac{1}{\sqrt{\det g}}\frac{\partial(X^i\sqrt{\det g})}{\partial x^i}\mathrm{vol}\\ &= \frac{1}{\sqrt{\det g}}\Big(\frac{\partial X^i}{\partial x^i}\sqrt{\det g} + X^i\frac{\partial\sqrt{\det g}}{\partial x^i}\Big)\mathrm{vol}\\ &= \Big(\frac{\partial X^i}{\partial x^i} + X^i\frac{\partial\log\sqrt{\det g}}{\partial x^i}\Big)\mathrm{vol}\\ &= \Big(\frac{\partial X^i}{\partial x^i} + \frac{1}{2}X^i\frac{\partial\log\det g}{\partial x^i}\Big)\mathrm{vol}.\end{aligned}$$

Then the following Jacobi's formula shows the first equality

$$\frac{\partial\log\det g}{\partial x^i} = \frac{1}{\det g}\frac{\partial\det g}{\partial x^i} = g^{pq}\frac{\partial g_{pq}}{\partial x^i},$$

and the second equality holds from the formula of Christoffel in terms of metric, that is $g^{jk}(\Gamma^{\ell}_{ij}g_{\ell k}+\Gamma^{\ell}_{ik}g_{j\ell})=2\Gamma^{j}_{ij}$. □

Proposition 8.3.1. Let (M,g) be a compact Riemannian manifold equipped with Levi-Civita connection ∇. Then

$$\mathrm{d}^* = -g^{ij}\iota_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}},$$

where $\{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\}$ is a local frame of TM.

Proof. A direct computation shows

$$\begin{aligned}
0 &= \int_M \mathrm{d}(\alpha\wedge\star\beta)\\
&= \int_M \mathcal{L}_{\frac{\partial}{\partial x^i}}(\mathrm{d}x^i\wedge\alpha\wedge\star\beta)\\
&= \int_M \mathcal{L}_{\frac{\partial}{\partial x^i}}(\langle \mathrm{d}x^i\wedge\alpha,\beta\rangle\,\mathrm{vol})\\
&\overset{(1)}{=} \int_M \mathcal{L}_{\frac{\partial}{\partial x^i}}(\langle\alpha, g^{ij}\iota_{\frac{\partial}{\partial x^j}}\beta\rangle\,\mathrm{vol})\\
&\overset{(2)}{=} \int_M (\langle\nabla_i\alpha, g^{ij}\iota_{\frac{\partial}{\partial x^j}}\beta\rangle+\langle\alpha,\nabla_i(g^{ij}\iota_{\frac{\partial}{\partial x^j}}\beta)\rangle+\frac{1}{2}g^{pq}\frac{\partial g_{pq}}{\partial x^i}\langle\alpha,g^{ij}\iota_{\frac{\partial}{\partial x^j}}\beta\rangle)\mathrm{vol}\\
&= \int_M (\langle \mathrm{d}x^i\wedge\nabla_i\alpha,\beta\rangle+\langle\alpha,\frac{\partial g^{ij}}{\partial x^i}\iota_{\frac{\partial}{\partial x^j}}\beta\rangle+\langle\alpha,g^{ij}\nabla_i(\iota_{\frac{\partial}{\partial x^j}}\beta)\rangle+\frac{1}{2}g^{pq}\frac{\partial g_{pq}}{\partial x^i}\langle\alpha,g^{ij}\iota_{\frac{\partial}{\partial x^j}}\beta\rangle)\mathrm{vol}\\
&\overset{(3)}{=} \int_M (\langle \mathrm{d}x^i\wedge\nabla_i\alpha,\beta\rangle+\langle\alpha,\frac{\partial g^{ij}}{\partial x^i}\iota_{\frac{\partial}{\partial x^j}}\beta\rangle+\langle\alpha,g^{ij}\iota_{\frac{\partial}{\partial x^j}}(\nabla_i\beta)\rangle+\langle\alpha,g^{i\ell}\Gamma^j_{i\ell}\iota_{\frac{\partial}{\partial x^j}}\rangle\\
&\quad+\frac{1}{2}g^{pq}\frac{\partial g_{pq}}{\partial x^i}\langle\alpha,g^{ij}\iota_{\frac{\partial}{\partial x^j}}\beta\rangle)\mathrm{vol},
\end{aligned}$$

where

(1) holds from Lemma 8.3.3.
(2) holds from Lemma 8.3.4.
(3) holds from Lemma 8.3.2.

it is easy to see

$$\frac{\partial g^{ij}}{\partial x^i}+g^{i\ell}\Gamma^j_{i\ell}+\frac{1}{2}g^{ij}g^{pq}\frac{g^{pq}}{\partial x^i}=0,$$

since

$$\Gamma^{\ell}_{ij}=\frac{1}{2}g^{k\ell}(\frac{\partial g_{kj}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}).$$

□

In the following examples, we always compute with respect to normal coordinate.

Example 8.3.1. *For a* 1*-form* ω *locally written as* $\omega_i\mathrm{d}x^i$*. Then*

$$\mathrm{d}^*\omega=-\sum_{i=1}^n\frac{\partial\omega_i}{\partial x^i}$$

Example 8.3.2. *For a smooth function f. Then*

$$\begin{aligned}\Delta_g f &= (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})f\\ &= \mathrm{d}^*\mathrm{d}f\\ &= \mathrm{d}^*(\frac{\partial f}{\partial x^i}\mathrm{d}x^i)\\ &= -\sum_{i=1}^n \frac{\partial^2 f}{\partial x^i \partial x^i}.\end{aligned}$$

This shows the Laplace-Beltrami operator differs a sign with scalar Laplacian.

Remark 8.3.1. Unless otherwise specified, we use Δ to denote the scalar Laplacian and Δ_g to denote the Laplace-Beltrami operator.

Example 8.3.3. *For an n-form ω written as $f\,\mathrm{vol}$, where f is a smooth function. Then*

$$\begin{aligned}\mathrm{d}^*\omega &= (-1)^n \star \mathrm{d} \star (f\,\mathrm{vol})\\ &= (-1)^n \star \mathrm{d}f\\ &= (-1)^n \star (\frac{\partial f}{\partial x^i}\mathrm{d}x^i)\\ &= \sum_{i=1}^n (-1)^{n+i-1}\frac{\partial f}{\partial x^i}\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n.\end{aligned}$$

8.4. **Divergence.**

Definition 8.4.1 (divergence)**.** For any vector field X, its divergence $\mathrm{div}X$ is defined as $\mathrm{tr}\nabla X$.

Remark 8.4.1 (local form). If we locally write X as $X^i\frac{\partial}{\partial x^i}$, then

$$\nabla X = \nabla_i X^j \mathrm{d}x^i \otimes \frac{\partial}{\partial x^j},$$

and thus

$$\mathrm{div}X = \nabla_i X^i.$$

Proposition 8.4.1.

$$\mathrm{div}X\,\mathrm{vol} = \mathcal{L}_X\,\mathrm{vol}.$$

Proof. Cartan's magic formula shows that

$$\mathcal{L}_X = \iota_X \circ \mathrm{d} + \mathrm{d}\circ\iota_X.$$

So

$$\begin{aligned}\mathcal{L}_X \mathrm{vol} &= (\iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X)\mathrm{vol} \\ &= \mathrm{d} \circ \iota_X \mathrm{vol} \\ &= \mathrm{d}((-1)^{i-1} X^i \sqrt{\det g} \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n) \\ &= \frac{1}{\sqrt{\det g}} \frac{\partial (X^i \sqrt{\det g})}{\partial x^i} \mathrm{vol} \\ &= \frac{1}{\sqrt{\det g}} (\frac{\partial X^i}{\partial x^i} \sqrt{\det g} + X^i \frac{\partial \sqrt{\det g}}{\partial x^i}) \mathrm{vol} \\ &= (\frac{\partial X^i}{\partial x^i} + X^i \frac{\partial \log \sqrt{\det g}}{\partial x^i}) \mathrm{vol} \\ &= (\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i \frac{\partial \log \det g}{\partial x^i}) \mathrm{vol}.\end{aligned}$$

Note that Jacobi's formula says

$$\frac{\partial \log \det g}{\partial x^i} = \frac{1}{\det g} \frac{\partial \det g}{\partial x^i} = g^{jk} \frac{\partial g_{jk}}{\partial x^i} = g^{jk}(\Gamma_{ij}^{\ell} g_{\ell k} + \Gamma_{ik}^{\ell} g_{j\ell}) = 2\Gamma_{ij}^{j}$$

Thus

$$\begin{aligned}\mathcal{L}_X \mathrm{vol} &= (\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i \frac{\partial \log \det g}{\partial x^i}) \mathrm{vol} \\ &= (\frac{\partial X^i}{\partial x^i} + \Gamma_{ij}^{j} X^i) \mathrm{vol} \\ &= (\frac{\partial X^i}{\partial x^i} + \Gamma_{ij}^{i} X^j) \mathrm{vol} \\ &= \nabla_i X^i \mathrm{vol}.\end{aligned}$$

□

Remark 8.4.2. From the proof, we can say there is the following formula for divergence of a vector field X written as $X^i \frac{\partial}{\partial x^i}$, one has

$$\mathrm{div} X = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (\sqrt{\det g} X^i).$$

Corollary 8.4.1 (divergence theorem)**.**

$$\int_M \mathrm{div} X \, \mathrm{vol} = 0.$$

Proposition 8.1. Let f be a smooth function on M. Then for any smooth function φ, one has

$$-\int_M \langle \nabla \varphi, \nabla f \rangle \mathrm{vol} = \int_M \Delta \varphi \cdot f \, \mathrm{vol}.$$

Proof. Direct computation shows

$$\begin{aligned}\operatorname{div}(f\nabla\varphi) &= \nabla_k(f\nabla\varphi)^k \\ &= \frac{\partial(f\nabla\varphi)^k}{\partial x^k} + \Gamma^k_{ks}(f\nabla\varphi)^s \\ &= \frac{\partial(fg^{ik}\frac{\partial\varphi}{\partial x^i})}{\partial x^k} + \Gamma^k_{ks} f g^{is}\frac{\partial\varphi}{\partial x^i} \\ &= \underbrace{g^{ik}\frac{\partial f}{\partial x^k}\frac{\partial\varphi}{\partial x^i}}_{\text{part I}} + \underbrace{f(\frac{\partial g^{ik}}{\partial x^k}\frac{\partial\varphi}{\partial x^i} + g^{ik}\frac{\partial^2\varphi}{\partial x^k\partial x^i} + g^{is}\Gamma^k_{ks}\frac{\partial\varphi}{\partial x^i})}_{\text{part II}}.\end{aligned}$$

We have the following observations:

(1) Part I equals

$$\begin{aligned}g^{ik}\frac{\partial f}{\partial x^k}\frac{\partial\varphi}{\partial x^i} &= g_{lj}g^{\ell k}\frac{\partial f}{\partial x^k}g^{ji}\frac{\partial\varphi}{\partial x^i} \\ &= \langle g^{\ell k}\frac{\partial f}{\partial x^k}\frac{\partial}{\partial x^\ell}, g^{ji}\frac{\partial\varphi}{\partial x^i}\frac{\partial}{\partial x^j}\rangle \\ &= \langle\nabla f, \nabla\varphi\rangle.\end{aligned}$$

(2) Note

$$\begin{aligned}\frac{\partial g^{ik}}{\partial x^k} + g^{is}\Gamma^k_{ks}\frac{\partial\varphi}{\partial x^i} &= -g^{is}g^{kt}\frac{\partial g_{st}}{\partial x^k} + \frac{1}{2}g^{is}g^{kt}(\frac{\partial g_{kt}}{\partial x^s} + \frac{\partial g_{st}}{\partial x^k} - \frac{\partial g_{ks}}{\partial x^t}) \\ &= -\frac{1}{2}g^{is}g^{kt}(\frac{\partial g_{ks}}{\partial x^t} + \frac{\partial g_{st}}{\partial x^k} - \frac{\partial g_{kt}}{\partial x^s}) \\ &= -g^{kt}\Gamma^i_{kt},\end{aligned}$$

where $\frac{\partial g^{ik}}{\partial x^k} = -g^{is}g^{kt}\frac{\partial g_{st}}{\partial x^k}$ holds from the fact $g^{ik}g_{kt} = \delta^i_t$. Then take partial derivative with respect to x^k to conclude.

(3) From (2) and local expression of Δ, it is clear part II equals $f\Delta\varphi$.

Thus we have

$$\operatorname{div}(f\nabla\varphi) = \langle\nabla\varphi, \nabla f\rangle + f\Delta\varphi.$$

Then divergence theorem completes the proof. □

Proposition 8.4.2. Let ω be a 1-form. Then

$$\mathrm{d}^*\omega = -\operatorname{div}(\omega^\sharp).$$

Proof. It suffices to check with respect to normal coordinate:

(1) Remark 8.4.2 or direct computation shows

$$\operatorname{div}\omega^\sharp = \sum_{i=1}^n \frac{\partial\omega_i}{\partial x^i}.$$

(2) Example 8.3.1 implies

$$\mathrm{d}^*\omega = -\sum_{i=1}^n \frac{\partial\omega_i}{\partial x^i}.$$

This completes the proof. □

8.5. **Conformal Laplacian.** For a smooth function u, according to Proposition 8.4.2 and Remark 8.4.2, we can write $\Delta_g u$ as follows

$$\begin{aligned}\Delta_g u &= \mathrm{d}^*\mathrm{d}u\\ &= -\mathrm{div}(g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j})\\ &= -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}g^{ij}\frac{\partial u}{\partial x^i}).\end{aligned}$$

Thus the Laplace-Beltrami Δ_g with respect to g is

$$\Delta_g = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i}).$$

If we consider conformal transformation of metric given by $\widetilde{g} = e^{2f}g$ for some smooth function f, we have

$$\begin{aligned}\widetilde{g}_{ij} &= e^{2f}g_{ij}\\ \widetilde{g}^{ij} &= e^{-2f}g^{ij}\\ \det\widetilde{g} &= e^{2nf}\det g\\ \sqrt{\det\widetilde{g}} &= e^{nf}\sqrt{\det g}.\end{aligned}$$

Thus

$$\begin{aligned}\Delta_{\widetilde{g}} &= -\frac{1}{e^{nf}\sqrt{\det g}}\frac{\partial}{\partial x^j}(e^{nf}\sqrt{\det g}e^{-2f}g^{ij}\frac{\partial}{\partial x^i})\\ &= -\frac{e^{-nf}}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(e^{(n-2)f}\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i})\\ &= -\frac{e^{-2f}}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i}) - \frac{(n-2)e^{-2f}}{\sqrt{\det g}}\frac{\partial f}{\partial x^j}\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i}\\ &= -e^{-2f}\Delta_g - (n-2)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}.\end{aligned}$$

This shows

$$\Delta_{\widetilde{g}} = -e^{-2f}\Delta_g$$

when $n = 2$. In this case, the Laplace-Beltrami Laplacian is called conformal invariant. However, it fails in higher dimensional cases.

In the following section, we assume $n > 3$ and introduce the so-called conformal Laplacian, which is defined as

$$\begin{aligned}L\colon C^\infty(M) &\to C^\infty(M)\\ u &\mapsto -\frac{4(n-1)}{n-2}\Delta_g u + Su,\end{aligned}$$

where S is the scalar curvature. Let's show

$$\widetilde{L}u = e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f}u),$$

where $\widetilde{L}$ is the conformal Laplacian after conformal transformation. In order to show this, we divide computations into several parts:

(1)

$$\begin{aligned}\nabla^2(e^{\frac{n-2}{2}f}u) =&\nabla(\frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}u\,\mathrm{d}x^i + e^{\frac{n-2}{2}f}\frac{\partial u}{\partial x^i}\mathrm{d}x^i)\\ =&e^{\frac{n-2}{2}f}\nabla^2 u + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}\mathrm{d}x^i\otimes\mathrm{d}x^j\\ &+(\frac{(n-2)^2}{4}e^{\frac{n-2}{2}f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}\frac{\partial u}{\partial x^j})\mathrm{d}x^i\otimes\mathrm{d}x^j + \frac{n-2}{2}e^{\frac{n-2}{2}f}u\nabla^2 f.\end{aligned}$$

(2)

$$\begin{aligned}\Delta_g(e^{\frac{n-2}{2}f}u) =& \mathrm{tr}_g\nabla^2(e^{\frac{n-2}{2}f}u)\\ =&e^{\frac{n-2}{2}f}\Delta_g u + \frac{n-2}{2}e^{\frac{n-2}{2}f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}\\ &+g^{ij}(\frac{(n-2)^2}{4}e^{\frac{n-2}{2}f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}\frac{\partial u}{\partial x^j}) + \frac{n-2}{2}e^{\frac{n-2}{2}f}u\Delta_g f.\end{aligned}$$

(3)

$$\begin{aligned}e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f}u) =& -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}\\ &-g^{ij}(n-2)(n-1)e^{-2f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} - 2(n-1)e^{-2f}u\Delta_g f + e^{-2f}Su\\ =& -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}\\ &-(n-2)(n-1)e^{-2f}u|\mathrm{d}f|^2 - 2(n-1)e^{-2f}u\Delta_g f + e^{-2f}Su.\end{aligned}$$

(4)

$$-\frac{4(n-1)}{n-2}\Delta_{\widetilde{g}}u = -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}.$$

(5) Note that

$$\widetilde{S} = e^{-2f}S - 2(n-1)e^{-2f}\Delta_g f - (n-2)(n-1)e^{-2f}|\mathrm{d}f|^2.$$

This completes the computation. In particular, in (2) if we take $u = 1$ we have

$$-\frac{4(n-1)}{n-2}\Delta_g(e^{\frac{n-2}{2}f}) = -(n-2)(n-1)e^{\frac{n-2}{2}f}|\mathrm{d}f|^2 - 2(n-1)e^{\frac{n-2}{2}f}\Delta_g f.$$

Thus we have

$$\widetilde{S} = e^{-\frac{n+2}{2}f}(-\frac{4(n-1)}{n-2}\Delta_g e^{\frac{n-2}{2}f} + Se^{\frac{n-2}{2}f}) = e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f}).$$

So if we put $e^{2f} = \varphi^{\frac{4}{n-2}}$, we have

$$\widetilde{S} = \varphi^{-\frac{n+2}{n-2}}L\varphi.$$

So it is clear g is conformal to $\widetilde{g}$ with constant scalar curvature λ if and only if φ is a smooth positive solution to the Yamabe equation

$$L\varphi = \lambda\varphi^{\frac{n+2}{n-2}}.$$

8.6. **Hodge theorem and corollaries.**

Theorem 8.6.1 (Hodge theorem)**.** Let $\Delta_g: C^\infty(M,\Omega_M^k)\to C^\infty(M,\Omega_M^k)$ be the Laplace operator. Then

(1) $\dim_{\mathbb{R}}\mathcal{H}^k(M)<\infty$.
(2) There is an orthogonal decomposition

$$C^\infty(M,\Omega_M^k)=\mathcal{H}^k(M)\perp \operatorname{im}\Delta_g.$$

Corollary 8.6.1. There is the following more explicit orthogonal decomposition

$$C^\infty(M,\Omega_M^k)=\mathcal{H}^k(M)\oplus \mathrm{d}(C^\infty(M,\Omega_M^{k-1}))\oplus \mathrm{d}^*(C^\infty(M,\Omega_M^{k+1})).$$

Proof. It suffices to check $\mathrm{d}(C^\infty(M,\Omega_M^{k-1}))$ is orthogonal to $\mathrm{d}^*(C^\infty(M,\Omega_M^{k+1}))$. For $\mathrm{d}\alpha$ and $\mathrm{d}^*\beta$, where α is a $(k-1)$-form and β is a $(k+1)$-form, we have

$$(\mathrm{d}\alpha,\mathrm{d}^*\beta)=(\mathrm{d}^2\alpha,\beta)=0.$$

□

Corollary 8.6.2.

$$\ker\mathrm{d}=\mathcal{H}^k(M)\oplus\mathrm{d}(C^\infty(M,\Omega_M^{k-1})),$$
$$\ker\mathrm{d}^*=\mathcal{H}^k(M)\oplus\mathrm{d}^*(C^\infty(M,\Omega_M^{k+1})).$$

Proof. It is clear from above corollary. □

Corollary 8.6.3. The natural map $\mathcal{H}^k(M)\to H^k(M,\mathbb{R})$ is an isomorphism. In other words, every element in $H^k(M,\mathbb{R})$ is represented by a unique harmonic form.

Proof. It is clear from above corollary. □

Corollary 8.6.4. $\star:\mathcal{H}^k(M)\to\mathcal{H}^{n-k}(M)$ is an isomorphism.

Proof. It suffices to show the Hodge star $\star$ maps harmonic forms to harmonic forms, since $\star$ maps k-forms to $n-k$-forms and $\star^2=(-1)^{k(n-k)}$ on k-forms. By Lemma 8.1.2, we just need to show $\mathrm{d}\star\alpha=\mathrm{d}^*\star\alpha=0$ for a harmonic form α. A direct computation shows

$$\mathrm{d}\star\alpha=(-1)^{\bullet_1}\star\star\mathrm{d}\star\alpha=(-1)^{\bullet_2}\star\mathrm{d}^\star\alpha=0,$$
$$\mathrm{d}^*\star\alpha=(-1)^{\bullet_3}\star\mathrm{d}\star\star\alpha=(-1)^{\bullet_4}\star\mathrm{d}\alpha=0.$$

Here we use $\bullet,\bullet'$ to denote the power of (-1), since it is not necessary for us to know what exactly it is. □

Remark 8.6.1. In fact, above corollary follows from the following identity

$$\Delta_g\circ\star=\star\circ\Delta_g,$$

which can be checked directly.

Corollary 8.6.5 (Poincaré duality)**.** $H^k(M,\mathbb{R})\cong H^{n-k}(M,\mathbb{R})$.

Proof. It follows from Corollary 8.6.3 and Corollary 8.6.4. □

9. Bochner's technique

9.1. **Bochner formula.** Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. In Example 3.1.2 and Remark 3.4.1 we have already seen the Hessian of a smooth function f and its scalar Laplacian locally can be written as follows:

$$\begin{aligned}\operatorname{Hess} f &= \nabla^2_{i,j} f \, \mathrm{d}x^i \otimes \mathrm{d}x^j \\ \Delta f &= g^{ij}\nabla^2_{i,j} f,\end{aligned}$$

where

$$\nabla^2_{i,j} f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}.$$

Proposition 9.1.1 (Bochner formula)**.** Let $f : (M,g) \to \mathbb{R}$ be a smooth function. Then

$$\frac{1}{2}\Delta|\nabla f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f) + g(\nabla\Delta f, \nabla f).$$

Proof. Since the norm of a tensor is independent of its type, so we may write $\nabla f = g^{ij}\nabla_i f \frac{\partial}{\partial x^j}$. Then

$$\begin{aligned}|\nabla f|^2 &= g(\nabla f, \nabla f) \\ &= g\Big(g^{ij}\nabla_i f \frac{\partial}{\partial x^j}, g^{k\ell}\nabla_k f \frac{\partial}{\partial x^\ell}\Big) \\ &= g^{ij}g^{k\ell}g_{j\ell}\nabla_i f \nabla_k f \\ &= g^{ij}\nabla_i f \nabla_j f.\end{aligned}$$

In the following computation we may use the normal coordinate. Then in this case

$$\begin{aligned}\frac{1}{2}\Delta|\nabla f|^2 &\overset{(1)}{=} \frac{1}{2}g^{k\ell}\nabla_k\nabla_\ell(g^{ij}\nabla_i f\nabla_j f) \\ &\overset{(2)}{=} \frac{1}{2}g^{k\ell}g^{ij}\nabla_k\nabla_\ell(\nabla_i f\nabla_j f) \\ &\overset{(3)}{=} g^{k\ell}g^{ij}\nabla_\ell\nabla_i f\cdot\nabla_k\nabla_j f + g^{k\ell}g^{ij}\nabla_k\nabla_\ell\nabla_i f\cdot\nabla_j f \\ &= |\operatorname{Hess} f|^2 + g^{k\ell}g^{ij}\nabla_k\nabla_\ell\nabla_i f\cdot\nabla_j f,\end{aligned}$$

where

(1) holds from in normal coordinate $\Delta f = g^{ij}\nabla_i\nabla_j f$.
(2) holds from Proposition 3.4.1, that is magic formula.
(3) holds from the following direct computation

$$\begin{aligned}\nabla_k\nabla_\ell(\nabla_i f\nabla_j f) =& \nabla_k(\nabla_\ell\nabla_i f\cdot\nabla_j f + \nabla_i f\cdot\nabla_\ell\nabla_j f) \\ =& \nabla_k\nabla_\ell\nabla_i f\cdot\nabla_j f + \nabla_\ell\nabla_i f\cdot\nabla_k\nabla_j f \\ &+ \nabla_k\nabla_i f\cdot\nabla_\ell\nabla_j f + \nabla_i f\cdot\nabla_k\nabla_\ell\nabla_j f \\ =& 2\nabla_\ell\nabla_i f\cdot\nabla_k\nabla_j f + 2\nabla_k\nabla_\ell\nabla_i f\cdot\nabla_j f.\end{aligned}$$

Then the following computation completes the proof:

$$
\begin{aligned}
g^{k\ell}g^{ij}\nabla_k\nabla_\ell\nabla_i f\cdot\nabla_j f &\overset{(4)}{=} g^{k\ell}g^{ij}\nabla_k\nabla_i\nabla_\ell f\cdot\nabla_j f\\
&\overset{(5)}{=} g^{k\ell}g^{ij}(\nabla_i\nabla_k\nabla_\ell f - R^s_{ki\ell}\nabla_s f)\cdot\nabla_j f\\
&= g^{ij}\nabla_i(g^{k\ell}\nabla_k\nabla_\ell f)\cdot\nabla_j f + g^{ij}R^s_i\nabla_s f\cdot\nabla_j f\\
&= g^{ij}\nabla_i(\Delta f)\cdot\nabla_j f + \mathrm{Ric}(\nabla f,\nabla f)\\
&= g(\nabla\Delta f,\nabla f)+\mathrm{Ric}(\nabla f,\nabla f),
\end{aligned}
$$

where

(4) holds from symmetry of Hessian.

(5) holds from Theorem 5.1.1, that is, Ricci identity.

□

9.2. **Obstruction to the existence of Killing fields.**

Definition 9.2.1 (Killing field)**.** A vector field X on a Riemannian manifold (M,g) is called a Killing field, if $\mathcal{L}_X g=0$.

Proposition 9.2.1. X is a Killing field if and only if flows generated by X acts as isometries.

Proof. If ϕ_t is the flow generated by X, then by definition of Lie derivative one has

$$\mathcal{L}_X g = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\phi_t)^* g.$$

Thus X is a Killing vector field if and only if its flow acts as isometries. □

Theorem 9.2.1. The following statements are equivalent:

(1) X is a Killing field.

(2) For any two vector fields Y,Z, we have

$$\langle\nabla_Y X,Z\rangle+\langle Y,\nabla_Z X\rangle=0.$$

Proof. Note that

$$
\begin{aligned}
\mathcal{L}_X\langle Y,Z\rangle &= X\langle Y,Z\rangle-\langle\mathcal{L}_X Y,Z\rangle-\langle Y,\mathcal{L}_X Z\rangle\\
&=\langle\nabla_X Y,Z\rangle+\langle Y,\nabla_X Z\rangle-\langle[X,Y],Z\rangle-\langle Y,[X,Z]\rangle\\
&=\langle\nabla_Y X,Z\rangle+\langle Y,\nabla_Z X\rangle.
\end{aligned}
$$

□

Remark 9.2.1. For (2) locally we have

$$g_{kj}\nabla_i X^j=-g_{ij}\nabla_k X^j.$$

Thus X is a Killing vector if and only if ∇X is a skew-symmetric $(1,1)$-tensor, that is $\nabla_i X^j$ is skew-symmetric in i,j.

Corollary 9.2.1. If X is a Killing field, then for arbitrary vector field Y one has

$$\langle\nabla_Y X,Y\rangle=0.$$

Proof. Set $Y = Z$ in $\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0$ to conclude. □

Corollary 9.2.2. If X is parallel, then X is Killing.

Proof. A zero matrix must be skew-symmetric. □

Corollary 9.2.3. If X is Killing, then $\mathrm{div}X = \nabla_i X^i = 0$.

Proof. The trace of a skew-symmetric matrix is zero. □

Lemma 9.2.1 (Bochner formula for Killing field)**.** Let X be a Killing field, and $f = \frac{1}{2}|X|^2$. Then
(1) $\nabla f = -\nabla_X X$.
(2) $\mathrm{Hess} f(Y,Y) = \langle \nabla_Y X, \nabla_Y X\rangle - R(Y,X,X,Y)$ holds for any vector field Y.
(3) $\Delta f = |\nabla X|^2 - \mathrm{Ric}(X,X)$.

Proof. For (1). By direct computation we have

$$\begin{aligned}
\nabla f &= \langle \nabla X, X\rangle \\
&= \langle \nabla_k X^i \mathrm{d}x^k \otimes \frac{\partial}{\partial x^i}, X^j \frac{\partial}{\partial x^j}\rangle \\
&= g_{ij} X^j \nabla_k X^i \mathrm{d}x^k \\
&\overset{\mathrm{I}}{=} -g_{ik} X^j \nabla_j X^i \mathrm{d}x^k. \\
\nabla_X X &= X^j \nabla_j X^i \frac{\partial}{\partial x^i} \\
&= g_{ik} X^j \nabla_j X^i \mathrm{d}x^k,
\end{aligned}$$

where I holds from skew-symmetry of ∇X.

For (2). By direct computation we have

$$\begin{aligned}
\mathrm{Hess} f(Y,Y) &= \frac{1}{2} Y^i Y^j \nabla_i \nabla_j (g_{k\ell} X^k X^\ell) \\
&= Y^i Y^j g_{k\ell} (\nabla_i X^k \nabla_j X^\ell + \nabla_i \nabla_j X^k \cdot X^\ell) \\
&= \langle \nabla_Y X, \nabla_Y X\rangle + Y^i Y^j g_{k\ell} \nabla_i \nabla_j X^k \cdot X^\ell,
\end{aligned}$$

and

$$\begin{aligned}
Y^i Y^j g_{k\ell} \nabla_i \nabla_j X^k \cdot X^\ell &= -Y^i Y^j g_{kj} \nabla_i \nabla_\ell X^k \cdot X^\ell \\
&\overset{\mathrm{II}}{=} -Y^i Y^j g_{kj} X^\ell (\nabla_\ell \nabla_i X^k + R^k_{ilm} X^m) \\
&= -Y^i Y^j X^\ell X^m R_{i\ell mj} \\
&= -R(Y,X,X,Y),
\end{aligned}$$

where (II) holds from $g_{kj} X^\ell \nabla_\ell \nabla_i X^k = 0$, since this expression is skew symmetric in i,j.

(3) holds from (2) directly. □

Theorem 9.2.2 (Bochner)**.** Let (M,g) be a compact, oriented Riemannian manifold.

(1) If $\mathrm{Ric}(g) \le 0$, then every Killing field is parallel.

(2) If $\mathrm{Ric}(g) \leq 0$ and $\mathrm{Ric}(g) < 0$ at some point, then there is no non-trivial Killing field.

Proof. For (1). Let X be a Killing field and set $f = \frac{1}{2}|X|^2$. Then

$$\begin{aligned} 0 &= \int_M \Delta f \,\mathrm{vol} \\ &= \int_M (|\nabla X|^2 - \mathrm{Ric}(X,X))\mathrm{vol} \\ &\geq \int_M |\nabla X|^2 \,\mathrm{vol} \\ &\geq 0. \end{aligned}$$

Thus $|\nabla X| \equiv 0$, that is X is parallel.

For (2). From proof of (1) one can see if $\mathrm{Ric}(g) \leq 0$ and X is a Killing field, then

$$\int_M \mathrm{Ric}(X,X) = 0,$$

which implies $\mathrm{Ric}(X,X) \equiv 0$. If $\mathrm{Ric}(g) < 0$ at some point $p \in M$, then $X_p = 0$, and thus $X \equiv 0$, since it is parallel. □

9.3. **Obstruction to the existence of harmonic 1-forms.** In some sense, the Killing field is dual to harmonic 1-form. Let's explain this in more detail.

Lemma 9.3.1. For a harmonic 1-form α, locally written as $\alpha_i \mathrm{d}x^i$, we have

$$\begin{aligned} \nabla_i \alpha_j &= \nabla_j \alpha_i, \\ g^{ij}\nabla_j \alpha_i &= 0. \end{aligned}$$

Proof. Recall α is harmonic if and only if

$$\begin{aligned} \mathrm{d}\alpha &= 0, \\ \mathrm{d}^*\alpha &= 0. \end{aligned}$$

it is clear

$$\mathrm{d}(\alpha_j \mathrm{d}x^j) = \nabla_i \alpha_j \mathrm{d}x^i \wedge \mathrm{d}x^j = 0$$

implies $\nabla_i \alpha_j = \nabla_j \alpha_i$. Similarly, explicit expression for d^* implies the second identity. □

Remark 9.3.1. Recall that the Killing field condition implies $g_{ij}\nabla_k X^j$ is skew-symmetric in i,k, we can see both Killing field and harmonic 1-form implies some (skew)symmetries.

Lemma 9.3.2. If α is a harmonic 1-form, then

$$\frac{1}{2}\Delta|\alpha|^2 = |\nabla\alpha|^2 + \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp).$$

Proof. Routine computation as follows:

$$\begin{aligned}
\frac{1}{2}\Delta|\alpha|_g^2 &= \frac{1}{2}g^{k\ell}\nabla_k\nabla_\ell(g^{ij}\alpha_i\alpha_j) \\
&= |\nabla\alpha|^2 + g^{k\ell}g^{ij}\nabla_k\nabla_\ell\alpha_i\cdot\alpha_j \\
&= |\nabla\alpha|^2 + g^{k\ell}g^{ij}\nabla_k\nabla_i\alpha_l\cdot\alpha_j \\
&= |\nabla\alpha|^2 + g^{k\ell}g^{ij}(\nabla_i\nabla_k\alpha_l - R^s_{ki\ell}\alpha_s)\alpha_j \\
&= |\Delta\alpha|^2 - g^{k\ell}g^{ij}R^s_{ki\ell}\alpha_s\cdot\alpha_j \\
&= |\Delta\alpha|^2 + \mathrm{Ric}(\alpha^\sharp,\alpha^\sharp).
\end{aligned}$$

□

Theorem 9.3.1 (Bochner)**.** Let (M,g) be a compact, oriented Riemannian manifold.

(1) If $\mathrm{Ric}(g) \geq 0$, then every harmonic 1-form is parallel.
(2) If $\mathrm{Ric}(g) \geq 0$ and $\mathrm{Ric}(g) > 0$ at some point, then there is no non-trivial harmonic 1-form.

Proof. The same as before. □

Corollary 9.3.1. Let (M,g) be a compact, oriented Riemannian manifold with $\mathrm{Ric}(g) \geq 0$ and $\mathrm{Ric}(g) > 0$ at some point. Then $b_1(M) = 0$.

Proof. it is clear from above theorem and Corollary 8.6.3. □

Remark 9.3.2. it is a kind of vanishing theorem. In geometry, "positivity" may cause "vanishing", which is an important philosophy.

Part 4. Minimal length curve problem

10. Pullback connection

10.1. Pullback and pushforward.

Definition 10.1.1 (pullback vector bundle)**.** Let $f: M \to N$ be a smooth map between manifolds, E a vector bundle over N. The pullback vector bundle f^*E over M is defined by the set

$$\widehat{E} = f^*E := \{(p,v) \in M \times E \mid f(p) = \pi(v)\}$$

endowed with subspace topology.

Remark 10.1.1 (local form). A local frame of $\widehat{E}$ can be written as

$$\widehat{e}_\alpha(x) := f^* e_\alpha(x) = e_\alpha \circ f(x)$$

where $x \in M$ and $\{e_\alpha\}$ is a local frame of E.

Let $f: M \to N$ be a smooth map between manifolds, and $\mathrm{d}f: TM \to TN$ its differential. There is another viewpoint to see it, consider

$$\begin{aligned} \mathrm{d}f: TM &\to f^*TN \subseteq M \times TN \\ X_p &\mapsto (p, \mathrm{d}f_p(X_p)) \end{aligned}$$

that is one can regard $\mathrm{d}f$ as a section of $T^*M \otimes f^*TN$.

Definition 10.1.2 (pushforward)**.** For vector field X over M, the pushforward of X is defined as $f_*X = \mathrm{d}f \circ X \in C^\infty(M, f^*TN)$.

Remark 10.1.2. In a word, pushforward of a vector field is no longer a vector field, but a section of pullback bundle.

Remark 10.1.3 (local form). Let $\{x^i\}$ and $\{y^m\}$ be local coordinates of M,N respectively, one has

$$f_*(\frac{\partial}{\partial x^i}) = \frac{\partial f^m}{\partial x^i} f^*(\frac{\partial}{\partial y^m})$$

10.2. **Pullback connection.** Let $f: M \to N$ be a smooth map between manifolds, E a vector bundle over N with connection ∇. Now we want to give a connection $\widehat{\nabla}$ on pullback bundle $\widehat{E}$ induced by ∇. Let $\{e_\alpha\}$ be a local frame of E. Then $\{\widehat{e}_\alpha := f^*e_\alpha\}$ is a local frame of $\widehat{E}$. Now we define

$$\widehat{\nabla}(\widehat{e}_\alpha) := f^*(\nabla e_\alpha)$$

and

$$\widehat{\nabla}(f\widehat{s}) := \mathrm{d}f \otimes \widehat{s} + f\widehat{\nabla}\widehat{s} \tag{10.1}$$

where $\widehat{s} = f^*s$, and s is a section of E. Suppose ∇ is given by Christoffel symbol $\Gamma^\beta_{m\alpha}$. Then by definition

$$\widehat{\nabla}(\widehat{e}_\alpha) = f^*(\Gamma^\beta_{m\alpha}\mathrm{d}y^m \otimes e_\beta) = \frac{\partial f^m}{\partial x^i}\Gamma^\beta_{m\alpha}(f) \cdot \mathrm{d}x^i \otimes \widehat{e}_\beta \tag{10.2}$$

Then here comes a natural question, we need to check our definition is independent of the choice of local frame, that is to check if $\{\widehat{e}'_\beta\}$ is another local frame with $\widehat{e}_\alpha = g^\beta_\alpha \widehat{e}'_\beta$. Then

$$\widehat{\nabla}(\widehat{e}_\alpha) = \widehat{\nabla}(g^\beta_\alpha \widehat{e}'_\beta)$$

It's straightforward computation, since the left hand is computed by (10.2), and right hand can be computed by (10.1). Thus we obtain a linear operator

$$\widehat{\nabla}\colon C^\infty(M,\widehat{E}) \to C^\infty(M,T^*M\otimes\widehat{E})$$

Now it remains to show it's an affine connection, that is to check for any smooth function $f\in C^\infty(M)$ and $s\in C^\infty(M,\widehat{E})$, one has

$$\widehat{\nabla}(fs) = \mathrm{d}f\otimes s + f\widehat{\nabla}s$$

Note that it doesn't follow from (10.1), since here locally if we write $s = s^\alpha \widehat{e}_\alpha$, and by definition we only obtain

$$\widehat{\nabla}(fs) = \widehat{\nabla}(fs^\alpha \widehat{e}_\alpha) = \mathrm{d}(fs^\alpha)\otimes\widehat{e}_\alpha + fs^\alpha\widehat{\nabla}\widehat{e}_\alpha$$

and it's necessary to do a stepforward computation to show above equation equals to the following one

$$\mathrm{d}f\otimes s^\alpha e_\alpha + f\widehat{\nabla}(s^\alpha e_\alpha)$$

Definition 10.2.1 (pullback metric)**.** Let g be a metric on E, the pullback metric on f^*E is $\widehat{g} = f^*g$.

Remark 10.2.1 (local form)*.* On local frames one has

$$\begin{aligned}\widehat{g}_{\alpha\beta}\widehat{e}^\alpha\otimes\widehat{e}^\beta :=& f^*(g_{\alpha\beta}e^\alpha\otimes e^\beta)\\ =& g_{\alpha\beta}(f)\cdot\widehat{e}^\alpha\otimes\widehat{e}^\beta\end{aligned}$$

that is $\widehat{g}_{\alpha\beta} = g_{\alpha\beta}(f)$.

Proposition 10.2.1. If connection ∇ is compatible with metric g. Then pullback connection $\widehat{\nabla}$ is compatible with $\widehat{g}$, that is for any vector field X of M and section s,t of $\widehat{E}$, we have

$$X\widehat{g}(s,t) = \widehat{g}(\widehat{\nabla}_X s,t) + \widehat{g}(s,\widehat{\nabla}_X t)$$

Proof. It suffices to check on local frames, consider $X = \frac{\partial}{\partial x^i}, s = \widehat{e}_\alpha, t = \widehat{e}_\beta$. Then

$$\begin{aligned}\frac{\partial}{\partial x^i}\widehat{g}_{\alpha\beta} &= \frac{\partial}{\partial x^i}g_{\alpha\beta}(f)\\ &= \frac{\partial f^m}{\partial x^i}\frac{\partial}{\partial y^m}g_{\alpha\beta}(f)\\ &\overset{(1)}{=} \frac{\partial f^m}{\partial x^i}(\Gamma^\gamma_{m\alpha}(f)\cdot g_{\gamma\beta}(f) + \Gamma^\gamma_{m\beta}(f)\cdot g_{\alpha\gamma}(f))\\ \widehat{g}(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\alpha,\widehat{e}_\beta) &= \Gamma^\gamma_{m\alpha}(f)\cdot\frac{\partial f^m}{\partial x^i}\cdot g_{\gamma\beta}(f)\\ \widehat{g}(\widehat{e}_\alpha,\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\beta) &= \Gamma^\gamma_{m\beta}(f)\cdot\frac{\partial f^m}{\partial x^i}\cdot g_{\alpha\gamma}(f)\end{aligned}$$

where (1) holds from ∇ is compatible with g. □

10.3. Pullback curvature.

Definition 10.3.1 (pullback curvature)**.** Let $f: M \to N$ be a smooth map between manifolds, E a vector bundle over N with connection ∇. The curvature tensor $\widehat{R}$ of pullback connection $\widehat{\nabla}$ on vector bundle $\widehat{E} \to M$ is given by

$$\widehat{R}(X,Y,s,t) = \widehat{g}(\widehat{\nabla}_X \widehat{\nabla}_Y s - \widehat{\nabla}_Y \widehat{\nabla}_X s, t)$$

where X, Y are vector fields on M and s, t are sections of $\widehat{E}$.

Remark 10.3.1 (local form).

$$\widehat{R}_{ij\alpha\beta} = R_{mn\alpha\beta} \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j}$$

where $R_{mn\alpha\beta}$ is curvature of ∇.

Proof.

$$\begin{aligned}
\widehat{R}_{ij\alpha\beta} &= \widehat{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \widehat{e}_\alpha, \widehat{e}_\beta) \\
&= \widehat{g}(\widehat{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\widehat{e}_\alpha, \widehat{e}_\beta) \\
&= \widehat{g}(\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{e}_\alpha - \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_\alpha, \widehat{e}_\beta)
\end{aligned}$$

The first term can be computed as follows

$$\begin{aligned}
\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{e}_\alpha &= \widehat{\nabla}_{\frac{\partial}{\partial x^i}} (\Gamma^\gamma_{m\alpha}(f) \cdot \frac{\partial f^m}{\partial x^j} \widehat{e}_\gamma) \\
&= \frac{\partial}{\partial x^i}(\Gamma^\gamma_{m\alpha}(f) \cdot \frac{\partial f^m}{\partial x^j})\widehat{e}_\gamma + \Gamma^\gamma_{m\alpha}(f) \cdot \frac{\partial f^m}{\partial x^j} \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_\gamma \\
&= (\frac{\partial \Gamma^\gamma_{m\alpha}}{\partial y^n} \frac{\partial f^n}{\partial x^i} \frac{\partial f^m}{\partial x^j} + \Gamma^\gamma_{m\alpha}(f) \cdot \frac{\partial^2 f^m}{\partial x^i \partial x^j})\widehat{e}_\gamma + \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} \Gamma^\gamma_{m\alpha} \Gamma^\delta_{n\gamma} \widehat{e}_\delta \\
&= \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} (\frac{\partial \Gamma^\gamma_{m\alpha}}{\partial y^n} + \Gamma^\delta_{m\alpha} \Gamma^\gamma_{n\delta})\widehat{e}_\gamma + \Gamma^\gamma_{m\alpha} \frac{\partial^2 f^m}{\partial x^i \partial x^j} \widehat{e}_\gamma
\end{aligned}$$

Thus

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{e}_\alpha - \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_\alpha = \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} R^\gamma_{mn\alpha} \widehat{e}_\gamma$$

that is

$$\begin{aligned}
\widehat{R}_{ij\alpha\beta} &= \widehat{g}(\frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} R^\gamma_{mn\alpha} \widehat{e}_\gamma, \widehat{e}_\beta) \\
&= \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} R^\gamma_{mn\alpha} g_{\gamma\beta} \\
&= \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} R_{mn\alpha\beta}
\end{aligned}$$

□

10.4. **Parallel transport.** Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇, $\gamma\colon I\to M$ a smooth curve, E a vector bundle over M and γ^*E endowed with pullback connection $\widehat{\nabla}$.

Definition 10.4.1 (parallel)**.** Let s be a section of γ^*E, it's called parallel along γ, if $\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}s=0$.

From local form we can see $\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}s=0$ is a system of ODEs locally, which can always be solved uniquely in a sufficiently short interval if an initial value is given, that's how we define parallel transport.

Definition 10.4.2 (parallel transport)**.** For $t_0,t\in I$, parallel transport $P_{t_0,t;\gamma}$ is an isomorphism[2] between vector spaces defined by

$$\begin{aligned}P_{t_0,t;\gamma}\colon E_{\gamma(t_0)}&\to E_{\gamma(t)}\\ s_0&\mapsto s(t)\end{aligned}$$

where s is the unique parallel section along γ satisfying $s(t_0)=s_0$.

Definition 10.4.3 (parallel orthonormal frame along curve)**.** Suppose $\{e_\alpha\}$ is an orthonormal basis of $E_{\gamma(t_0)}$, then there is a local frame $\{e_\alpha(t)\}$ of $E_{\gamma(t)}$ along γ obtained by parallel transport, such that $e_\alpha(0)=e_\alpha$.

Proposition 10.4.1. A connection ∇ is compatible with metric if and only if for arbitrary curve $\gamma\colon I\to M$ and two parallel sections s_1,s_2 along γ we have $g(s_1,s_2)$ is constant.

Proof. It's clear if ∇ is compatible with metric g, then for any two sections s,t which are parallel along γ, one has

$$\mathrm{d}g(s_1,s_2)=g(\nabla s_1,s_2)+g(s_1,\nabla s_2)=0$$

which implies $g(s,t)$ is constant. Conversely, suppose $\{e_\alpha(t)\}$ is a parallel orthonormal frame with respect to g along γ and write

$$\begin{aligned}s_1(t)&=s_1^\alpha(t)e_\alpha(t)\\ s_2(t)&=s_2^\alpha(t)e_\alpha(t)\end{aligned}$$

Direct computation shows

$$\begin{aligned}g(\nabla s_1,s_2)+g(s_1,\nabla s_2)&=\sum_\alpha \frac{\mathrm{d}s_1^\alpha}{\mathrm{d}t}s_2^\alpha+s_1^\alpha\frac{\mathrm{d}s_2^\alpha}{\mathrm{d}t}\\ &=\frac{\mathrm{d}}{\mathrm{d}t}(\sum_\alpha s_1^\alpha s_2^\alpha)\\ &=\frac{\mathrm{d}}{\mathrm{d}t}g(s_1,s_2)\end{aligned}$$

□

[2]Its inverse is $P_{t,t_0;\gamma}$.

Proposition 10.4.2. Let (M,g) be a Riemannian manifold, $\gamma\colon I\to M$ a smooth curve and $P_{s,t;\gamma}\colon T_{\gamma(s)}M\to T_{\gamma(t)}M$ is the parallel transport along γ. For any $s\in I$ with $v=\gamma'(s)$, one has

$$\nabla_v R=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s}(P_{s,t;\gamma})^*R_{\gamma(t)}$$

In particular, if $\nabla R=0$ then

$$(P_{s,t;\gamma})^*R_{\gamma(t)}=R_{\gamma(s)}$$

holds for arbitrary $t,s\in I$.

Proof. Let $v_1=v$ and choose $v_2,\dots,v_5\in T_{\gamma(s)}M$. For each $1\le i\le 5$, we define vector fields along $\gamma(t)$ by $X_i(t)=P^{\gamma}_{s,t}(v_i)$. In particular, one has $X_1(t)=\gamma'(t)$. Direct computation shows

$$\begin{aligned}(\nabla R)_p(v_1,\dots,v_5)&=\lim_{t\to s}\nabla R(X_1,\dots,X_5)\\&=\lim_{t\to s}X_1R(X_2,\dots,X_5)-\sum_{i=2}^{5}R(X_2,\dots,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}X_i,\dots,X_5)\\&\overset{(1)}{=}\lim_{t\to s}X_1R(X_2,\dots,X_5)\\&=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s}R_{\gamma(t)}(X_2(t),\dots,X_5(t))\\&=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s}(P_{s,t;\gamma})^*R_{\gamma(t)}(v_2,\dots,v_5)\end{aligned}$$

where (1) holds from $X_i(t)$ are parallel vector fields along $\gamma(t)$. □

10.5. **Second fundamental form.** In this section, we fix the following notations:

(1) Let $(M,g),(N,g')$ be Riemannian manifolds with Levi-Civita connection ∇ and ∇' respectively.
(2) $f:M\to N$ is a smooth map between manifolds.
(3) Γ^k_{ij} is used to denote Christoffel symbol of ∇ and Γ^{ℓ}_{mn} is used to denote Christoffel symbol of ∇'.
(4) $\widehat{\nabla}$ is the connection on f^*TN induced by ∇'.

Definition 10.5.1 (second fundamental form)**.** The second fundamental form $B\in C^\infty(M,T^*M\otimes T^*M\otimes f^*TN)$ of f is defined as

$$B(X,Y):=\widehat{\nabla}_X(f_*Y)-f_*(\nabla_XY)\in C^\infty(M,f^*TN)$$

where $X,Y\in C^\infty(M,TM)$.

Remark 10.5.1 (local form). Note that

$$f_*(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}) = \Gamma_{ij}^k f_*(\frac{\partial}{\partial x^k}) = \Gamma_{ij}^k \frac{\partial f^m}{\partial x^k} f^*(\frac{\partial}{\partial y^m})$$

$$\begin{aligned}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}(\frac{\partial f^m}{\partial x^j} f^*(\frac{\partial}{\partial y^m})) &= \frac{\partial^2 f^m}{\partial x^i \partial x^j} f^*(\frac{\partial}{\partial y^m}) + \frac{\partial f^m}{\partial x^j}\widehat{\nabla}_{\frac{\partial}{\partial x^i}} f^*(\frac{\partial}{\partial y^m}) \\ &= (\frac{\partial^2 f^\ell}{\partial x^i \partial x^j} + \frac{\partial f^n}{\partial x^i}\frac{\partial f^m}{\partial x^j}\Gamma_{nm}^\ell(f)) f^*(\frac{\partial}{\partial y^\ell})\end{aligned}$$

Therefore

$$\begin{aligned}B_{ij} &:= B(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\ &= (\frac{\partial^2 f^\ell}{\partial x^i \partial x^j} + \frac{\partial f^n}{\partial x^i}\frac{\partial f^m}{\partial x^j}\Gamma_{mn}^\ell(f) - \Gamma_{ij}^k\frac{\partial f^\ell}{\partial x^k}) f^*(\frac{\partial}{\partial y^\ell})\end{aligned}$$

that is

$$B = (\frac{\partial^2 f^\ell}{\partial x^i \partial x^j} + \frac{\partial f^n}{\partial x^i}\frac{\partial f^m}{\partial x^j}\Gamma_{mn}^\ell(f) - \Gamma_{ij}^k\frac{\partial f^\ell}{\partial x^k}) \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes f^*(\frac{\partial}{\partial y^\ell})$$

Proposition 10.5.1. The second fundamental form B is symmetric.

Proof. It's clear from local expression. □

Corollary 10.5.1 (torsion-free)**.** For $X, Y \in C^\infty(M, TM)$, one has

$$\widehat{\nabla}_X(f_*Y) - \widehat{\nabla}_Y(f_*X) = f_*(\nabla_X Y - \nabla_Y X) = f_*([X,Y])$$

Proof. The first equality holds from the symmetry of second fundamental form and the second equality holds since ∇ is torsion-free. □

Example 10.5.1 (geodesic)**.** *A smooth curve $\gamma\colon I \to M$ can be regarded as $\gamma\colon (I, g_{\mathrm{can}}) \to (M, g)$, thus second fundamental form in this case is*

$$B = (\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\Gamma_{ij}^k \circ \gamma)\mathrm{d}t \otimes \mathrm{d}t \otimes \gamma^*(\frac{\partial}{\partial x^k})$$

since canonical metrics on I has vanishing Christoffel symbol. In this viewpoint, a smooth curve is a geodesic, if it has vanishing second fundamental form as smooth maps between Riemannian manifolds.

Example 10.5.2 (Hessian)**.** *A smooth function f can be regarded as $f\colon (M, g) \to (\mathbb{R}, g_{\mathrm{can}})$, thus second fundamental form in this case is*

$$B = (\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k})\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes f^*(\frac{\partial}{\partial y})$$

since canonical metrics on $\mathbb{R}$ has vanishing Christoffel symbol. Recall Hessian of f is

$$\operatorname{Hess} f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\mathrm{d}x^i \otimes \mathrm{d}x^j$$

So second fundamental form generalizes Hessian of smooth function.

Proposition 10.5.2. Let B be fundamental form of f. Then
$$B=\widetilde{\nabla}\mathrm{d}f$$
where $\widetilde{\nabla}$ is the connection on $T^*M\otimes f^*TN$ induced by ∇ together with pull-back connection $\widehat{\nabla}$ on f^*TN.

Proof. Suppose $\mathrm{d}f$ is locally written as
$$\mathrm{d}f=\frac{\partial f^m}{\partial x^i}\mathrm{d}x^i\otimes f^*(\frac{\partial}{\partial y^m})$$
Then direct computation shows
$$\begin{aligned}\widetilde{\nabla}\mathrm{d}f=&\widetilde{\nabla}(\frac{\partial f^m}{\partial x^i}\mathrm{d}x^i\otimes f^*(\frac{\partial}{\partial y^m}))\\ =&\frac{\partial^2 f^m}{\partial x^j\partial x^i}\mathrm{d}x^j\otimes\mathrm{d}x^i\otimes f^*(\frac{\partial}{\partial y^m})-\frac{\partial f^m}{\partial x^i}\Gamma^i_{jk}\mathrm{d}x^j\otimes\mathrm{d}x^k\otimes f^*(\frac{\partial}{\partial y^m})\\ &+\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\Gamma^\ell_{mn}(f)\cdot\mathrm{d}x^i\otimes\mathrm{d}x^j\otimes f^*(\frac{\partial}{\partial y^\ell})\\ =&(\frac{\partial^2 f^\ell}{\partial x^i\partial x^j}-\frac{\partial f^\ell}{\partial x^k}\Gamma^k_{ij}+\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\Gamma^\ell_{mn}(f))\mathrm{d}x^i\otimes\mathrm{d}x^j\otimes f^*(\frac{\partial}{\partial y^\ell})\\ =&B\end{aligned}$$
This completes the proof. $\square$

11. Variation formulas of curves

In this section and Section 12, we fix the following notations:

(1) $I = [a,b] \subseteq \mathbb{R}$ is a closed interval.
(2) (M,g) is a Riemannian manifold equipped with Levi-Civita connection ∇.
(3) For two different points $p,q \in M$, the space of piecewise smooth curves from p to q is denoted as $\mathcal{L}_{p,q}$.
(4) For $\gamma \in \mathcal{L}_{p,q}$, $\gamma'(t)$ denotes $\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})$, which is a piecewise smooth vector field along γ.
(5) The arc-length functional and energy functional on $\mathcal{L}_{p,q}$ are defined as follows

$$L(\gamma) = \int_a^b |\gamma'(t)|_{\widehat{g}}\mathrm{d}t$$
$$E(\gamma) = \frac{1}{2}\int_a^b |\gamma'(t)|^2_{\widehat{g}}\mathrm{d}t$$

where $\widehat{g}$ is pullback metric on γ^*TM.

11.1. **First variation formula.**

Definition 11.1.1 (variation)**.** Given $\gamma \in \mathcal{L}_{p,q}$, a variation (fixing endpoints) of γ is a map

$$\alpha\colon [a,b] \times (-\varepsilon,\varepsilon) \to M$$

such that

(1) $\alpha(\text{-},s) \in \mathcal{L}_{p,q}$ for any $s \in (-\varepsilon,\varepsilon)$.
(2) There is a subdivision $a = t_0 < t_1 < \cdots < t_k = b$ of I such that α is smooth on each strip $(t_{i-1},t_i] \times (-\varepsilon,\varepsilon)$ for $i = 1,\ldots,k$.
(3) $\alpha(t,0) = \gamma(t)$ for any $t \in [a,b]$.

Remark 11.1.1. In general, we can consider variations of γ without fixing endpoints, but in this section we only consider variations fixing endpoints.

Notation 11.1.1. For pullback bundle α^*TM, $\overline{\nabla}$ and $\overline{g}$ denote connection and metric pulled back from the ones on TM respectively. By definition the restriction of $\overline{\nabla}$ on γ^*TM is exactly $\widehat{\nabla}$, and the restriction of $\overline{g}$ on γ^*TM is $\widehat{g}$.

Definition 11.1.2 (variation vector field)**.** For a variation α of $\gamma \in \mathcal{L}_{p,q}$, $\alpha_*(\frac{\partial}{\partial s})\big|_{s=0}$ is called variation vector field of variation α, which is a piecewise smooth vector field along γ.

Remark 11.1.2. Note that for a variation

$$\begin{cases} \alpha(a,s) = p \\ \alpha(b,s) = q \end{cases}$$

holds for any $s \in (-\varepsilon,\varepsilon)$. Thus we have

$$\begin{cases} \alpha_*(\frac{\partial}{\partial s})(a,s) = 0 \\ \alpha_*(\frac{\partial}{\partial s})(b,s) = 0 \end{cases}$$

holds for any $s \in (-\varepsilon, \varepsilon)$. In particular, it holds for $s = 0$. In other words, variation vector field vanishes at endpoints.

Lemma 11.1.1. Let $\gamma \in \mathcal{L}_{p,q}$ and X a piecewise smooth vector field along γ which vanishes at endpoints. Then there exists a variation α of γ such that the variation vector field is exactly X, that is

$$\alpha_*(\frac{\partial}{\partial s})\Big|_{s=0} = X$$

Proof. See Proposition 2.2 in Page193 of [Car92]. □

Theorem 11.1.1 (first variation formula of smooth version)**.** Let $\gamma\colon I \to (M, g)$ be a unit-speed smooth curve, α a variation of γ with variation vector fields V. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-}, s)) \overset{(1)}{=} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-}, s)) &\overset{(2)}{=} \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t \\ &\overset{(3)}{=} -\int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t\end{aligned}$$

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-}, s)) = \int_a^b \frac{1}{2|\gamma'(t)|} \frac{\partial}{\partial s}\Big|_{s=0} |\alpha_*(\frac{\partial}{\partial t})|^2 \mathrm{d}t = \frac{1}{|\gamma'(t)|} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-}, s))$$

since γ is unit-speed, This show equation marked by (1). Since $\gamma(t)$ is smooth, integration by parts shows

$$0 = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\langle V, \gamma'(t)\rangle \mathrm{d}t = \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t + \int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t$$

This shows equation marked by (3). For equation marked by (2), direct computation shows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} E(\alpha(\text{-}, s)) &= \frac{\mathrm{d}}{\mathrm{d}s} \frac{1}{2} \int_a^b |\alpha_*(\frac{\partial}{\partial t})|^2 \mathrm{d}t \\ &= \frac{1}{2} \int_a^b \frac{\partial}{\partial s} |\alpha_*(\frac{\partial}{\partial t})|^2 \mathrm{d}t \\ &= \frac{1}{2} \int_a^b 2\langle \overline{\nabla}_{\frac{\partial}{\partial s}} \alpha_*(\frac{\partial}{\partial t}), \alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}} \mathrm{d}t \\ &\overset{(4)}{=} \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial t}} \alpha_*(\frac{\partial}{\partial s}), \alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}} \mathrm{d}t\end{aligned}$$

The hallmark of above computation is the equality marked by (4), which can be seen from Corollary 10.5.1. Thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-}, s)) = \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t$$

since $\alpha_*(\frac{\partial}{\partial s})\big|_{s=0} = V$ and $\alpha_*(\frac{\partial}{\partial t})\big|_{s=0} = \gamma'(t)$. □

Corollary 11.1.1 (first variation formula of piecewise smooth version)**.** Let $\gamma: I \to (M,g)$ be a unit-speed piecewise smooth curve with breakpoints $a = t_0 < t_1 < \cdots < t_k = b$, α a variation of γ with the variation vector field V. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-},s)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-},s)) = \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t$$
$$= -\int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t - \sum_{i=1}^{k-1} \langle V_{t_i}, \Delta_{t_i}\gamma'\rangle$$

where $\Delta_t\gamma' = \gamma'(t_+) - \gamma'(t_-)$.

Proof. Note that $\gamma(t)$ is smooth on each $(t_{i-1}, t_i]$. Then by Theorem 11.1.1 one has

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-},s)) = \int_{t_{i-1}}^{t_i} \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t$$

and integration by parts shows

$$\langle V, \gamma'(t)\rangle\Big|_{t_{i-1}}^{t_i} = \int_{t_{i-1}}^{t_i} \frac{\mathrm{d}}{\mathrm{d}t}\langle V, \gamma'(t)\rangle \mathrm{d}t = \int_{t_{i-1}}^{t_i} \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t + \int_{t_{i-1}}^{t_i} \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t$$

Then we add these equations together to obtain desired equation. □

Remark 11.1.3. Suppose α is a variation of γ without fixing endpoints with variation vector field V, from the proof of equality marked by (3), it's clear to see its first variation formula is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{s=0} L(\alpha(\text{-},s)) = \langle V(b), \gamma'(b)\rangle - \langle V(a), \gamma'(a)\rangle - \int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t - \sum_{i=1}^{k-1} \langle V_{t_i}, \Delta_{t_i}\gamma'\rangle$$

Corollary 11.1.2. Given $\gamma \in \mathcal{L}_{p,q}$. The following statements are equivalent.

(1) γ is a critical point of energy functional $E: \mathcal{L}_{p,q} \to \mathbb{R}$.
(2) γ has constant speed $|\gamma'(t)| = c > 0$ and γ is a critical point of arc-length functional $L: \mathcal{L}_{p,q} \to \mathbb{R}$.
(3) γ is a geodesic. In particular, it's smooth.

Proof. From (3) to (2). Firstly a geodesic must have constant speed c, and $c > 0$ since p, q are distinct points. It's also a critical point of L since first variation formula implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-},s)) = -\int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t = 0$$

Note that there is only one term in first variation formula, since geodesic is a smooth curve.

From (2) to (1). It's clear, since from above proof we have already seen for constant speed curve, the first variation of arc-length functional and energy functional only differs a scalar.

From (1) to (3). In order to show $\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t) = 0$ and it's smooth, it suffices to choose appropriate variation vector fields to conclude. □

11.2. **Second variation formula.** We already know a geodesic γ is a critical point for energy functional or arc-length functional, so it suffices to compute second variation of geodesics to determine whether it's local minimum or not. To see this, we need to consider the following 2-dimensional variation

$$\alpha\colon [a,b]\times(-\varepsilon_1,\varepsilon_1)\times(-\varepsilon_2,\varepsilon_2)$$

such that

(1) $\alpha(t,0,0)=\gamma(t)$
(2) $\alpha(\text{-},s_1,s_2)$ is a smooth curve connecting p and q.

11.2.1. *Second variation formula for energy.*

Theorem 11.2.1 (second variation formula for energy)**.** Let $\gamma\colon[a,b]\to(M,g)$ be a smooth curve. If α is a 2-dimensional variation of γ with variation fields V,W. Then

$$\begin{aligned}\frac{\partial^2}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0}E(\alpha(\text{-},s_1,s_2))&=\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W\rangle\mathrm{d}t\\&-\int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t-\int_a^b\langle\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial s_2})\Big|_{s_1=s_2=0},\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle\mathrm{d}t\end{aligned}$$

Proof. By first variation formula we have

$$\frac{\partial}{\partial s_2}E(\alpha(\text{-},s_1,s_2))=-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t$$

Thus

$$\frac{\partial^2}{\partial s_1\partial s_2}E(\alpha(\text{-},s_1,s_2))=\underbrace{-\int_a^b\langle\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t}_{\text{part I}}\underbrace{-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial s_1}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t}_{\text{part II}}$$

For part II, we have

$$\overline{\nabla}_{\frac{\partial}{\partial s_1}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})=R(\alpha_*(\frac{\partial}{\partial s_1}),\alpha_*(\frac{\partial}{\partial t}))\alpha_*(\frac{\partial}{\partial t})+\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial t})$$

Thus we can write part II as

$$-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),R(\frac{\partial}{\partial s_1},\frac{\partial}{\partial t})\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t\underbrace{-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t}_{\text{part III}}$$

For part III, we have

$$\begin{aligned}-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t&=-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s_1})\rangle_{\overline{g}}\mathrm{d}t\\&=\int_a^b\langle\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s_1})\rangle_{\overline{g}}\mathrm{d}t\end{aligned}$$

Now let's evaluate at $s_1=s_2=0$. Then we have

(1) Part I

$$-\int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_*(\frac{\partial}{\partial s_2})\Big|_{s_1=s_2=0}, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle \mathrm{d}t$$

(2) Part II

$$\int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle \mathrm{d}t - \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t$$

This completes the proof. □

Corollary 11.2.1. Let $\gamma\colon [a,b]\to(M,g)$ be a geodesic. Then

$$\frac{\partial^2}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0} E(\alpha(\text{-},s_1,s_2)) = \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle \mathrm{d}t - \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t$$

11.2.2. *Second variation formula for arc-length.*

Theorem 11.2.2 (second variation formula for arc-length)**.** Let $\gamma\colon [a,b]\to(M,g)$ be a unit-speed curve. If α is a 2-dimensional variation of γ with variation fields V,W. Then

$$\begin{aligned}\frac{\partial^2}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0} L(\alpha(\text{-},s_1,s_2)) &= \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle \mathrm{d}t - \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t \\ &\quad - \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_*(\frac{\partial}{\partial s_2})\Big|_{s_1=s_2=0}, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle \mathrm{d}t \\ &\quad - \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'\rangle\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W,\gamma'\rangle \mathrm{d}t\end{aligned}$$

Corollary 11.2.2. Let $\gamma\colon [a,b]\to(M,g)$ be a unit-speed geodesic. If α is a 2-dimensional variation of γ with variation fields V,W. Then

$$\begin{aligned}\frac{\partial^2}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0} L(\alpha(\text{-},s_1,s_2)) &= \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle \mathrm{d}t - \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t \\ &\quad - \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'\rangle\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W,\gamma'\rangle \mathrm{d}t \\ &= \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V^\perp, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W^\perp\rangle \mathrm{d}t - \int_a^b R(V^\perp,\gamma',\gamma',W^\perp)\mathrm{d}t\end{aligned}$$

where

$$V^\perp = V - \langle V,\gamma'\rangle\gamma', \quad W^\perp = W - \langle W,\gamma'\rangle\gamma'$$

Proof. It suffices to check the second equality. Direct computation shows:

$$\begin{aligned}\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle &= \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}(V^\perp + \langle V,\gamma'\rangle\gamma'), \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}(W^\perp + \langle W,\gamma'\rangle\gamma')\rangle \\ &= \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V^\perp, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W^\perp\rangle + \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V,\gamma'\rangle\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W,\gamma'\rangle\end{aligned}$$

Thus

$$\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle - \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V,\gamma'\rangle\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W,\gamma'\rangle = \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V^\perp, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W^\perp\rangle$$

since

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}(\langle V,\gamma'\rangle\gamma') = \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V,\gamma'\rangle\gamma'$$

and it's clear

$$R(V,\gamma',\gamma',W) = R(V^{\perp},\gamma',\gamma',W^{\perp})$$

$\square$

So if we want to show a geodesic γ is a (locally) minimal geodesic, it suffices to show for any 2-dimensional variation α with variation vector fields, we have

$$\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} L(\alpha(-,s_1,s_2)) \geq 0$$

This motivates us to consider the following bilinear form defined on the space of variation vector fields.

Definition 11.2.1 (index form)**.** Let $\gamma\colon [a,b] \to (M,g)$ be a unit-speed geodesic. The index form I_γ is defined as

$$I_\gamma(V,W) = \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W \rangle \mathrm{d}t - \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t$$

where V,W are vector fields along γ.

By Corollary 11.2.2, a geodesic γ is locally minimal if and only if index form defined on the space of normal[3] variation fields are semi-positive definite. In the following section, we will study when the index form defined on the normal variation vector fields along γ is positive definite, semi-positive definite or not.

[3]A vector field V along γ is called normal, if V is perpendicular to γ'.

12. JACOBI FIELD I: AS THE NULL SPACE

12.1. First properties.

Definition 12.1.1 (Jacobi field)**.** A vector field J along geodesic γ is called a Jacobi field, if it satisfies

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J + R(J,\gamma')\gamma' = 0$$

Notation 12.1.1. For convenience,

$$J' = \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J$$
$$J'' = \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J$$

Remark 12.1.1 (local form). Suppose $\{e_1,\dots,e_n\}$ is a parallel orthonormal frame along γ, and $J(t) = J^i(t)e_i(t)$, the condition for Jacobi fields becomes

$$\frac{\mathrm{d}^2 J^k}{\mathrm{d}t^2} + \langle J^j R(e_j,\gamma')\gamma', e_k\rangle = 0$$

Thus by standard results in ODEs, a Jacobi field J is completely determined by its initial conditions

$$J(0), J'(0) \in T_{\gamma(0)}M$$

Consequently, the set of Jacobi fields is a vector space with dimension $2n$.

Example 12.1.1. *There is always a trivial Jacobi field along geodesic $\gamma\colon [a,b]\to M$, that is $J(t) = (at+b)\gamma'(t)$.*

Proposition 12.1.1. Let $J(t)$ be a Jacobi field along geodesic γ.

(1) If $J(t)\not\equiv 0$. Then the set consisting of zeros of $J(t)$ is discrete.
(2) There exist constants λ,μ such that

$$J(t) = J^{\perp}(t) + (\lambda t+\mu)\gamma'(t)$$

where $\langle J^{\perp}(t),\gamma'\rangle \equiv 0$.
(3) $J(t)\perp\gamma'$ if and only if there exist $t_1\neq t_2$ such that

$$\langle J(t_1),\gamma'(t_1)\rangle = \langle J(t_2),\gamma'(t_2)\rangle = 0$$

(4) If there exists t_0 such that

$$\langle J(t_0),\gamma'(t_0)\rangle = \langle J'(t_0),\gamma'(t_0)\rangle = 0$$

then $J(t)\perp\gamma'(t)$.

Proof. For (3). Note that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J(t),\gamma'(t)\rangle = \langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J,\gamma'\rangle = \langle R(J,\gamma')\gamma',\gamma'\rangle = 0$$

Thus $\langle J(t),\gamma'(t)\rangle = \lambda t+\mu$. Note that $\langle J_i(t_1),\gamma'(t_1)\rangle = \langle J_i(t_2),\gamma'(t_2)\rangle = 0$, which implies $\langle J(t),\gamma'(t)\rangle\equiv 0$.

□

Proposition 12.1.2. Let $\gamma\colon [a,b]\to(M,g)$ be a geodesic and $\alpha\colon [a,b]\times(-\varepsilon,\varepsilon)\to(M,g)$ a variation of γ consisting of geodesics. Then

$$J=\alpha_*(\frac{\partial}{\partial s})\Big|_{s=0}$$

is a Jacobi field.

Proof. Direct computation shows

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s})&\overset{(1)}{=}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s}}\alpha_*(\frac{\partial}{\partial t})\\&=R(\frac{\partial}{\partial t},\frac{\partial}{\partial s})\alpha_*(\frac{\partial}{\partial t})+\overline{\nabla}_{\frac{\partial}{\partial s}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\\&\overset{(2)}{=}R(\frac{\partial}{\partial t},\frac{\partial}{\partial s})\alpha_*(\frac{\partial}{\partial t})\end{aligned}$$

where

(1) holds from Corollary 10.5.1.
(2) holds from α is a variation consisting of geodesics.

Set $s=0$ one has

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J=R(\gamma',J)\gamma'=-R(J,\gamma')\gamma'$$

which implies J is a Jacobi field. □

Remark 12.1.2. In fact, all Jacobi fields can be obtained by above construction.

Corollary 12.1.1. Let (M,g) be a Riemannian manifold, and (x^i,U,p) is a normal coordinate centered at $p\in M$. For each $q\in U\setminus\{p\}$, and $w\in T_qM$, there exists a Jacobi field J along geodesic connecting p,q such that $J(0)=0$, $J(0)'=w$ and $J(1)=w$.

Proof. For $q\in U\setminus\{p\}$, there exists a unique $v\in T_pM$ such that $\exp_p(v)=q$, and $\gamma(t)=\exp_p(tv)$. Consider the following variation of $\gamma(t)$ consisting of geodesics

$$\alpha(t,s)=\exp_p(t(v+sw))$$

where $w\in T_pM$. Its variation vector field $J(t)=\alpha_*(\frac{\partial}{\partial s})\big|_{s=0}$ is a Jacobi field along γ. In normal coordinate (x^i,U,p), $\alpha(t,s)$ can be written explicitly as

$$\alpha(t,s)=(t(v^1+sw^1),\dots,t(v^n+sw^n))$$

where $v=(v^1,\dots,v^n),w=(w^1,\dots,w^n)$. Thus $J(t)$ is given by the formula

$$J(t)=tw^i\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

and it's clear $J(0)=0$, $J(0)'=w$ and $J(1)=w$. □

12.2. **Conjugate points.**

Definition 12.2.1 (conjugate point)**.** Let $p \neq q$ be two endpoints of a geodesic γ. p and q are called conjugate along γ if there exists a non-zero Jacobi field J along γ which vanishes at endpoints.

Notation 12.2.1. The conjugate locus of p is defined as

$$\mathrm{conj}(p) := \{q \in M \mid p \text{ and } q \text{ are conjugate along some geodesic.}\}$$

Proposition 12.2.1. There are at most $n-1$ linearly independent Jacobi fields $J(t)$ along γ such that $J(a) = J(b) = 0$.

Proof. By Remark 12.1.1, there are at most n linearly independent Jacobi fields such that $J(a) = 0$, and the trivial Jacobi field $J(t) = (t-a)\gamma'(t)$ never vanishes at $t = b$. □

Theorem 12.2.1. Let (M, g) be a Riemannian manifold, $p \in M$ and $v \in V_p \subseteq T_pM$. Then $(\mathrm{d}\exp_p)_v$ is not injective if and only if $q = \exp_p(v)$ is conjugate to p along $\exp_p(tv)$.

Proof. For any $0 \neq w \in T_pM$, if $(\mathrm{d}\exp_p)_v(w) = 0$, then $J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$ is a Jacobi field such that $J(0) = J(1) = 0$, which implies p is conjugate to q. Conversely, if p and q are conjugate along $\exp_p(tv)$, then there exists a Jacobi field J such that $J(0) = J(1) = 0$. Note that $J(t)$ can be written as

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

where $0 \neq w = J'(0) \in T_pM$, since $J(t)$ is determined by $J(0)$ and $J'(0)$. In particular, one has

$$(\mathrm{d}\exp_p)_v(w) = J(1) = 0$$

which implies $(\mathrm{d}\exp_p)_v$ is not injective. □

Corollary 12.2.1. Let (M, g) be a complete Riemannian manifold, $p \in M$. If the conjugate locus $\mathrm{conj}(p) = \varnothing$, then $\exp_p \colon T_pM \to M$ is a local diffeomorphism.

Proof. The exponential map $\exp_p \colon T_pM \to M$ is surjective since M is complete. On the other hand, for arbitrary $v \in T_pM$, $(\mathrm{d}\exp_p)_v$ is injective since the conjugate locus $\mathrm{conj}(p) = \varnothing$. Thus $\exp_p$ is a local diffeomorphism. □

Example 12.2.1. *For $p \in \mathbb{S}^n$,* $\mathrm{conj}(p) = \{-p\}$.

Example 12.2.2. *For $p \in \mathbb{S}^1 \times \mathbb{R}$,* $\mathrm{conj}(p) = \varnothing$.

12.3. **Locally minimal geodesic.**

Lemma 12.3.1. Let $\gamma \colon [a, b] \to (M, g)$ be a unit-speed geodesic with no conjugate points, there exist Jacobi fields $J_2, \dots, J_n$ along γ such that

(1) $J_i(a) = 0, i \geq 2$ and $\{\gamma'(b), J_2(b), \dots, J_n(b)\}$ is an orthonormal basis of $T_{\gamma(b)}M$.
(2) $\langle J_i(t), \gamma'(t)\rangle \equiv 0$ for any $t \in [a, b]$.
(3) $\{\gamma'(t), J_2(t), \dots, J_n(t)\}$ are linearly independent for $t \in (a, b]$.

Proof. Suppose $\{\gamma'(b), e_2, \ldots, e_n\}$ is an orthonormal basis of $T_{\gamma(b)}M$, since there is no conjugate points along γ, there exists a unique Jacobi field J_i such that

$$J_i(a) = 0, J_i(b) = e_i$$

for each $i = 2, \ldots, n$. Now it suffices to show Jacobi fields $J_i(t)$ satisfy properties (2) and (3).

For (2). Note that $\langle J_i(a), \gamma'(a)\rangle = \langle J_i(b), \gamma'(b)\rangle = 0$. Then by (3) of Proposition 12.1.1 one $\langle J_i(t), \gamma'(t)\rangle \equiv 0$.

For (3). Suppose there exists $c \in (a, b]$ and $\lambda_i \in \mathbb{R}$ such that

$$\sum_{i=2}^{n} \lambda_i J_i(c) = 0$$

which implies

$$W(t) = \sum_{i=2}^{n} \lambda_i J_i(t) \equiv 0$$

on $(a, c]$ since there is no conjugate points. By (1) of Proposition 12.1.1 one has $W(t) \equiv 0$ on $(a, b]$, thus we have $\lambda_i = 0, i = 2, \ldots, n$, since $\{\gamma'(b), J_2(b), \ldots, J_n(b)\}$ is linearly independent. □

Theorem 12.3.1. Let $\gamma\colon [a, b] \to (M, g)$ be a unit-speed geodesic. Then

(1) If γ has no conjugate points. Then index form I_γ is **positive definite** on vector space consisting of normal variation fields.
(2) If γ only has conjugate points as endpoints. Then index form is **semi-positive definite** on vector space consisting of normal variation fields. Furthermore, Jacobi field is null space.
(3) If γ has an interior conjugate point. Then index form is **not positive definite** on vector space consisting of normal variation fields.

Proof. For (1). Suppose $\gamma(t)$ has no conjugate points, choose Jacobi fields $\{J_1(t) = \gamma'(t), J_2(t), \ldots, J_n(t)\}$ in Lemma 12.3.1. Then for any normal variation vector V along γ we write it as

$$V = \sum_{i=2}^{n} V^i(t) J_i(t)$$

Then it's clear $V^i(b) = 0$ since $V(b) = 0$ and $\{J_2(b), \ldots, J_n(b)\}$ is linearly independent. Direct computation shows

$$\begin{aligned} I_\gamma(V, V) = \sum_{i,j=2}^{n} & \underbrace{\int_a^b V^i V^j \langle J_i', J_j'\rangle + \frac{\mathrm{d}V^i}{\mathrm{d}t} V^j \langle J_i, J_j'\rangle + V^i \frac{\mathrm{d}V^j}{\mathrm{d}t}\langle J_i', J_j\rangle \mathrm{d}t}_{\text{Part I}} \\ & + \underbrace{\int_a^b \left\{\frac{\mathrm{d}V^i}{\mathrm{d}t}\frac{\mathrm{d}V^j}{\mathrm{d}t}\langle J_i, J_j\rangle - V^i V^j R(J_i, \gamma', \gamma', J_j)\right\} \mathrm{d}t}_{\text{Part II}} \end{aligned}$$

Note that

$$\langle J_i', J_j\rangle = \langle J_i, J_j'\rangle$$

Then Part I is

$$\int_a^b \left\{ (V^i V^j \langle J_i', J_j \rangle)' - V^i V^j \langle J_i'', J_j \rangle \right\} \mathrm{d}t$$

Thus

$$\begin{aligned} I_\gamma(V,V) &= \sum_{i,j=2}^n V^i V^j \langle J_i', J_j \rangle \Big|_a^b + \sum_{i,j=2}^n \int_a^b \frac{\mathrm{d}V^i}{\mathrm{d}t} \frac{\mathrm{d}V^j}{\mathrm{d}t} \langle J_i, J_j \rangle \mathrm{d}t \\ &= \sum_{i,j=2}^n \int_a^b \frac{\mathrm{d}V^i}{\mathrm{d}t} \frac{\mathrm{d}V^j}{\mathrm{d}t} \langle J_i, J_j \rangle \mathrm{d}t \\ &\geq 0 \end{aligned}$$

Furthermore, $I_\gamma(V,V) = 0$ if and only if $\sum_{i=2}^n \frac{\mathrm{d}V^i}{\mathrm{d}t} J_i(t) = 0$ if and only if $\frac{\mathrm{d}V^i}{\mathrm{d}t}(t) = 0, t \in [a,b]$, thus $V^i(t) \equiv 0$, that is $V = 0$.

For (2). For any $c \in (a,b)$, consider geodesic $\gamma_c \colon [a,c] \to (M,g)$. By (1) one has I_{γ_c} is positive definite on the vector space consisting of normal variation fields along γ_c. Then a standard approximation argument shows I_γ is semi-positive definite.

To see its null space: It's clear a normal variation Jacobi field V satisfies $I_\gamma(V,V) = 0$. Conversely, if a normal variation field V satisfies $I_\gamma(V,V) = 0$, then by a variation argument we have for arbitrary W we have

$$I_\gamma(V,W) = 0$$

Take appropriate W to see V satisfies the equation for Jacobi fields.

For (3). If $\gamma(a)$ is conjugate to $\gamma(c)$ for some $c \in (a,b)$, then there exists a non-zero normal Jacobi field $\widetilde{J}(t)$ along γ such that $\widetilde{J}(a) = \widetilde{J}(c) = 0$. Consider

$$J(t) = \begin{cases} \widetilde{J}(t) & t \in [a,c] \\ 0 & t \in [c,b] \end{cases}$$

Although $J(t)$ may not be a smooth vector field along γ, one still has $I_\gamma(J,J) = 0$ by integrating piecewisely. Let W be a smooth normal variation vector field along γ such that

$$W(c) = -\lim_{t \to c^-} \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} \widetilde{J}(t)$$

It' clear $W(c) \neq 0$, otherwise $\widetilde{J}(t) \equiv 0$. If we define $J_\varepsilon = J + \varepsilon W$, then one has

$$I_\gamma(J_\varepsilon, J_\varepsilon) = 2\varepsilon I_\gamma(J,W) + \varepsilon^2 I_\gamma(W,W)$$

And integration by parts implies

$$I_\gamma(J,W) = \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \widetilde{J}, W \rangle \Big|_a^c = -W(c)^2 < 0$$

So for sufficiently small ε we have $I_\gamma(J_\varepsilon, J_\varepsilon) < 0$, and by approximation argument we can show there exists a smooth normal variation field such that $I_\gamma(V,V) < 0$, a contradiction. □

Corollary 12.3.1. Let $\gamma \colon [a,b] \to (M,g)$ be a unit-speed geodesic with no conjugate points, and V, W are normal vector fields satisfying $V(a) = W(a), V(b) = W(b)$. If V is a Jacobi field. Then $I_\gamma(V,V) \leq I_\gamma(W,W)$, and the equality holds if and only if $V = W$.

Proof. Since V, W agree at end points, then $V - W$ is a normal variation field. Thus we have

$$0 \leq I_\gamma(V - W, V - W) = I_\gamma(V, V) + I_\gamma(W, W) - 2I_\gamma(V, W)$$

Since V is a Jacobi field, then integration by parts shows

$$I_\gamma(V, V) = \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, V \rangle \Big|_a^b = \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, W \rangle \Big|_a^b = I_\gamma(V, W)$$

Hence we get $I_\gamma(V, V) \leq I_\gamma(W, W)$, and the equality holds if and only if $V = W$. □

Remark 12.3.1. From second variation formula, we can conclude that a geodesic γ is a **locally minimal geodesic** if and only if it has no interior conjugate points. However, it may not be **globally minimal geodesic**. For example, consider $M = \mathbb{S}^1 \times \mathbb{R}$. It's clear there is no conjugate points for any geodesic on M, and thus for geodesic $\gamma \colon [a, b] \to M$ starting at $(x, y) \in M$, it's locally minimal. But if there exists $c \in (a, b)$ such that $\gamma(c) \in \{-x\} \times \mathbb{R}$, then γ is not a minimal curve.

13. Cut locus and injective radius

13.1. Cut locus.

Definition 13.1.1 (cut time/point/locus)**.** Let (M,g) be a complete Riemannian manifold, $p \in M$ and $v \in T_pM$.

(1) The cut time of (p,v) is defined as
$$t_{\mathrm{cut}}(p,v) = \sup\{c > 0 \mid \gamma_v|_{[0,c]} \text{ is a minimal geodesic}\}$$
(2) If $t_{\mathrm{cut}}(p,v) < \infty$, the cut point of p along γ_v is defined as $\gamma_v(t_{\mathrm{cut}}(p,v))$.
(3) The cut locus of p is defined as
$$\mathrm{cut}(p) = \{q \in M \mid \exists v \in T_pM \text{ such that } q \text{ is a cut point of } p \text{ along } \gamma_v.\}$$

Remark 13.1.1.

(1) It's possible for $t_{\mathrm{cut}}(p,v) = +\infty$. For example, consider $(M,g) = (\mathbb{R}^n, g_{\mathrm{can}})$.
(2) If cut point exists, it occurs at or before the first conjugate point.
(3) It's clear that $t_{\mathrm{cut}}(p,v)$ depends on the $|v|$, but $\gamma_v(t_{\mathrm{cut}}(p,v))$ is independent of $|v|$.

Theorem 13.1.1. Let (M,g) be a complete Riemannian manifold, $p \in M, v \in T_pM$ with $|v| = 1$. Let $c = t_{\mathrm{cut}}(p,v) \in (0,\infty]$.

(1) If $0 < b < c$ and b is finite, then $\gamma_v|_{[0,b]}$ has no conjugate point and it is the unique minimal unit-speed geodesic between endpoints.
(2) If $c < \infty$, then $\gamma_v|_{[0,c]}$ is a minimal geodesic.
(3) In the case of (2), one or both of the following holds.
    (a) $\gamma_v(c)$ is conjugate to p along γ_v.
    (b) There are two or more different unit-speed minimal geodesic connecting p and $\gamma_v(c)$.

Proof. For (1). It's clear $\gamma_v|_{[0,b]}$ has no conjugate point since cut point occurs at or before the first conjugate point, and it's minimal by definition. To see it's unique, suppose $\sigma\colon [0,b] \to M$ is another minimal unit-speed geodesic. Note that $\gamma_v'(b) \neq \sigma'(b)$, otherwise by uniqueness one has $\gamma_v(t) = \sigma(t)$ in $t \in [0,b]$. Now we take $b' \in (b,c)$, and construct a new unit-speed curve as follows
$$\widetilde{\gamma}(t) = \begin{cases} \sigma(t), & t \in [0,b] \\ \gamma_v(t), & t \in (b,b'] \end{cases}$$
Then $\widetilde{\gamma}$ has length b', so it's also a minimal curve from p to $\gamma_v(b')$, since $\mathrm{dist}(p,\gamma_v(b')) = b'$. However, $\widetilde{\gamma}(t)$ is not smooth at $t = b$, which contradicts to the fact that minimal curve are smooth geodesics.

For (2). By definition, there exists a sequence $\{c_i\}$ increasing to c such that $\gamma_v|_{[0,c_i]}$ is minimal. By continuity of distance function, one has
$$\mathrm{dist}(p,\gamma_v(c)) = \lim_{i\to\infty} \mathrm{dist}(p,\gamma_v(c_i)) = \lim_{i\to\infty} c_i = c$$
which implies γ_v is minimal on $[0,c]$.

For (3). Assume $\gamma_v(c)$ is not conjugate to p along γ_v, we shall prove the existence of another unit-speed minimal geodesic from p to $\gamma_v(c)$. Let $\{c_i\}$ be a sequence descending to c. By definition $\gamma_v\colon[0,c_i]\to M$ is not a minimal geodesic, so there exists a unit-speed minimal geodesic $\gamma_i\colon[0,a_i]\to M$ connecting p and $\gamma_v(c_i)$. By construction one has $\gamma_i(a_i)=\gamma_v(c_i)$ and $a_i<c_i$. If we denote $\omega_i=\gamma_i'(0)\in T_pM$, by compactness of unit sphere in T_pM and the fact $\{a_i\}$ is bounded, by taking subsequence twice we can find a subsequence of $\{\gamma_i(t)\}$, still denoted by $\{\gamma_i(t)\}$, such that ω_i converging to some $w\in T_pM$ with $|w|=1$, and $\lim_{i\to\infty}a_i=a$. In fact $a=c$, since

$$\begin{aligned} c&=\mathrm{dist}(p,\gamma_v(c))\\ &=\lim_{i\to\infty}\mathrm{dist}(p,\gamma_v(c_i))\\ &=\lim_{i\to\infty}\mathrm{dist}(p,\gamma_i(a_i))\\ &=\lim_{i\to\infty}a_i=a \end{aligned}$$

Since $\gamma_v(c)$ is not conjugate to p, $(\mathrm{d}\exp_p)_{cv}$ is non-degenerated, which implies $\exp_p$ is injective in $B_\varepsilon(cv)$ for sufficiently small $\varepsilon>0$. On one hand we have $a_iw_i\neq c_iv$ since $a_i<c_i$, but on the other hand we have

$$\exp_p(c_iv)=\gamma_v(c_i)=\gamma_i(a_i)=\exp_p(a_iw_i)$$

Thus injectivity implies $a_iw_i\notin B_\varepsilon(cv)$ for sufficiently large i, since $c_iv\in B_\varepsilon(cv)$. By taking limits one has

$$aw\neq cv$$

This shows $w\neq v$ since $a=c$, and thus $\exp_p(tw)$ gives another unit-speed minimal geodesic connecting p and $\gamma_v(c)$, since $\exp_p(cw)=\exp_p(aw)=\exp_p(av)$. □

Corollary 13.1.1. If $\gamma\colon[0,b]\to M$ is a minimal geodesic connecting $\gamma(0)$ and $\gamma(b)$, then it's the unique minimal geodesic connecting any two points strictly between $\gamma(0)$ and $\gamma(b)$.

Proof. It follows from the proof of (1). □

Corollary 13.1.2. Let (M,g) be a complete Riemannian manifold with $p,q\in M$.

(1) If $q\in\mathrm{cut}(p)$, then $p\in\mathrm{cut}(q)$.
(2) If $q\notin\mathrm{cut}(p)$, then there exists a unique minimal geodesic connecting p and q.

Proof. For (1). If q is cut point of p along geodesic γ, then γ is a minimal geodesic connecting p and q, and by Theorem 13.1.1, there are the following two cases.

(a) q is conjugate to p along γ.
(b) There are two more different unit-speed geodesic connecting p and q.

Note that we already have γ^{-1} is a minimal geodesic connecting q and p, so if we want to show $p\in\mathrm{cut}(q)$, it suffices to show γ^{-1} is no longer minimal after p. Let's discuss case by case.

(a) It's clear γ^{-1} is no longer minimal after p in the first case, since if q is conjugate to p, then p is also conjugate to q.
(b) In the second case, if γ^{-1} is still minimal after p, then by Corollary 13.1.1, one has γ^{-1} is the unique minimal geodesic connecting q and p, which contradicts to the second case.

For (2). If there exist two or more minimal geodesic connecting p and q. Then for any minimal geodesic γ connecting p and q, it's no longer minimal after q by Corollary 13.1.1, which contradicts to $q \notin \mathrm{cut}(p)$. □

Example 13.1.1.

(1) *For* $p \in \mathbb{S}^n$, $\mathrm{cut}(p) = \mathrm{conj}(p) = \{-p\}$. *In this case both* ($a$),($b$) *hold in Theorem 13.1.1.*
(2) *For* $p \in \mathbb{S}^1 \times \mathbb{R}$, $\mathrm{cut}(p) = \{-p\} \times \mathbb{R}$ *and there is no conjugate point. In this case* (a) *fails and* (b) *holds in Theorem 13.1.1.*

Definition 13.1.2 (tangent cut locus and injectivity domain)**.** Let (M,g) be a complete Riemannian manifold, given $p \in M$, we define

(1) the tangent cut locus
$$\mathrm{TCL}(p) := \{v \in T_pM : |v| = t_{\mathrm{cut}}(p, v/|v|)\}$$
(2) the injectivity domain
$$\mathrm{ID}(p) := \{v \in T_pM : |v| < t_{\mathrm{cut}}(p, v/|v|)\}$$

It's clear that $\mathrm{TCL}(p) = \partial\,\mathrm{ID}(p)$ and $\mathrm{cut}(p) = \exp_p(\mathrm{TCL}(p))$. Furthermore, we have the following properties.

Proposition 13.1.1. Let (M,g) be a complete Riemannian manifold and $p \in M$. Then

(1) The cut locus of p is a closed subset of M of measure zero.
(2) The restriction of $\exp_p$ to $\mathrm{ID}(p)$ is a diffeomorphism onto $M \setminus \mathrm{cut}(p)$.

Proof. See Theorem 10.34 of Page311 of [Lee18]. □

13.2. **Injective radius.**

Definition 13.2.1 (injective radius)**.** Let (M,g) be a Riemannian manifold, $p \in M$. The injective radius of p is defined as
$$\mathrm{inj}(p) := \sup\{\rho > 0 : \exp_p \text{ is defined on } B(0,\rho) \subseteq T_pM \text{ and injective}\}$$
The injectivity radius of M is
$$\mathrm{inj}(M) := \inf_{p \in M} \mathrm{inj}(p)$$

Theorem 13.2.1. Let (M,g) be a complete Riemannian manifold. Then
$$\mathrm{inj}(p) = \begin{cases} \mathrm{dist}(p, \mathrm{cut}(p)) & \mathrm{cut}(p) \neq \varnothing \\ \infty & \mathrm{cut}(p) = \varnothing \end{cases}$$

Proof. See [Lee18, Proposition 10.36]. □

Proposition 13.2.1. Let (M,g) be a complete Riemannian manifold and $p \in M$. Suppose there exists some point $q \in \mathrm{cut}(p)$ such that $\mathrm{dist}(p,q) = \mathrm{dist}(p, \mathrm{cut}(p))$. Then

(1) Either q is a conjugate point of p along some minimizing geodesic from p to q, or there are exactly two minimizing geodesics from p to q, say $\gamma_1, \gamma_2 \colon [0,b] \to M$, such that $\gamma_1'(b) = -\gamma_2'(b)$.
(2) If in addition that $\mathrm{inj}(p) = \mathrm{inj}(M)$, and q is not conjugate to p along any minimizing geodesic. Then there is a closed unit-speed geodesic $\gamma \colon [0,2b] \to M$ such that $\gamma(0) = \gamma(2b) = p$ and $\gamma(b) = q$ where $b = \mathrm{dist}(p,q)$.

Proof. For (1). Suppose q is not conjugate to p along any minimizing geodesic. Then by Theorem 13.1.1 there are at least two unit-speed minimal geodesics $\gamma_1(t), \gamma_2(t)$ such that $\gamma_1(b) = \gamma_2(b) = q$. Suppose $\gamma_1'(b) \neq -\gamma_2'(b)$. Then there exists a unit vector $v \in T_qM$ such that

$$\langle v, \gamma_1'(b) \rangle < 0, \quad \langle v, \gamma_2'(b) \rangle < 0$$

Since q is not conjugate to p along γ_1, there exists a neighborhood U_1 of $b\gamma_1'(0)$ in T_pM such that $\exp_p|_{U_1}$ is diffeomorphism. Now choose a sufficiently small s and let

$$\xi_1(s) = (\exp_p|_{U_1})^{-1} \exp_q(sv)$$

Consider the following variation of γ_1 consisting of geodesics:

$$\alpha_1(t,s) = \exp(\frac{t}{b}\xi_1(s))$$

It's clear $\alpha_1(t,0) = \gamma_1(t)$, since $\xi_1(0) = (\exp_p|_{U_1})^{-1}\exp_q(0) = (\exp_p|_{U_1})^{-1}(q) = b\gamma_1'(0)$. Then by Remark 11.1.3, that is the first variation formula of general variation, one has

$$\left.\frac{\mathrm{d}L(\gamma_s)}{\mathrm{d}s}\right|_{s=0} = \langle v, \gamma_1'(b) \rangle < 0$$

which implies for sufficiently small s we have $L(\alpha_1(t,s)) < L(\gamma_1(t))$. For γ_2 we can do the same construction and the same argument implies for sufficiently small s we have $L(\alpha_2(t,s)) < L(\gamma_2(t))$. Thus for each sufficiently small s we have two geodesics $\alpha_1(t,s), \alpha_2(t,s)$ from p to $\exp_q(sX_q)$. However,

$$\mathrm{d}(p, \exp_q(sv)) \le L(\alpha_1(t,s)) < L(\gamma_1(t)) = \mathrm{dist}(p,q) = \mathrm{inj}(p) \tag{13.1}$$

A contradiction to the definition of injective radius. So any two different minimizing geodesics γ_1, γ_2 from p to q satisfy $\gamma_1'(b) = -\gamma_2'(b)$, which implies there are exactly two minimizing geodesics from p to q.

For (2). By (1) we know that there exists exactly two geodesics γ_1, γ_2 such that $\gamma_1(b) = \gamma_2(b) = q$ with $\gamma_1'(b) = \gamma_2'(b)$. Consider the loop $\gamma = \gamma_1 \circ \gamma_2^{-1}$. Then it's a unit-speed geodesic such that $\gamma(0) = \gamma(2b) = p, \gamma(b) = q$, where $b = \mathrm{dist}(p,q)$, since we have already shown $\gamma_1'(b) = -\gamma_2'(b)$. To show γ is a closed geodesic, it suffices to show $\gamma'(2b) = \gamma'(0)$, that is equivalent to show $(\gamma_1^{-1})'(b) = (\gamma_2^{-1})'(b)$. Note that in the proof of (1), condition of $\mathrm{dist}(p,q) =$

$\text{dist}(p, \text{cut}(p)) = \text{inj}(p)$ is used in inequality (13.1), and in fact we only need $\text{dist}(p,q) \le \text{inj}(p)$, strict equality is not necessary. So if $\text{inj}(p) = \text{inj}(M)$, thus

$$\text{dist}(q,p) = \text{dist}(p,q) = \text{inj}(p) \le \text{inj}(q)$$

Then (1) implies $(\gamma_1^{-1})'(b) = (\gamma_2^{-1})'(b)$. $\square$

14. JACOBI FIELD II: A USEFUL TOOL

14.1. Taylor expansion of Jacobi field.

Proposition 14.1.1. Let (M,g) be a Riemannian manifold, and $\gamma\colon[0,1]\to M$ is a geodesic with $\gamma(0)=p,\gamma'(0)=v$, where $p\in M,v\in T_pM$. For any $w\in T_pM$ with $|w|=1$, suppose $J(t)$ is the Jacobi field along γ given by

$$J(t)=(\mathrm{d}\exp_p)_{tv}(tw)$$

Then

$$\begin{aligned}|J(t)|^2&=t^2-\frac{1}{3}R(J',\gamma',\gamma',J')(0)t^4+O(t^4)\\|J(t)|&=t-\frac{1}{6}R(J',\gamma',\gamma',J')(0)t^3+O(t^3)\end{aligned}$$

Proof. It suffices to prove the first equality, and the second follows from the first one. Note that $J(0)=0,J'(0)=w$, direct computation shows

$$\begin{aligned}\langle J,J\rangle(0)&=0\\\langle J,J\rangle'(0)&=2\langle J,J'\rangle(0)=0\\\langle J,J\rangle''(0)&=2\langle J',J'\rangle(0)+2\langle J'',J\rangle(0)=2\\\langle J,J\rangle'''(0)&=6\langle J',J''\rangle(0)+2\langle J''',J\rangle(0)\\&=6\langle J',R(J,\gamma')\gamma'\rangle(0)=0\\\langle J,J\rangle''''(0)&=8\langle J',J'''\rangle(0)+6\langle J'',J''\rangle(0)+2\langle J'''',J\rangle(0)\\&=8\langle J',J'''\rangle(0)+6\langle R(J,\gamma')\gamma',R(J,\gamma')\gamma'\rangle(0)\\&=8\langle J',J'''\rangle(0)\end{aligned}$$

Now it remains to compute J'''. For arbitrary vector field W along γ, direct computation shows

$$\begin{aligned}\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}R(J,\gamma')\gamma',W\rangle&=\frac{\mathrm{d}}{\mathrm{d}t}\langle R(J,\gamma')\gamma',W\rangle-\langle R(J,\gamma')\gamma,W'\rangle\\&=\frac{\mathrm{d}}{\mathrm{d}t}\langle R(W,\gamma')\gamma',J\rangle-\langle R(J,\gamma')\gamma,W'\rangle\\&=\langle R(W,\gamma')\gamma',J'\rangle-\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}R(W,\gamma')\gamma',J\rangle-\langle R(J,\gamma')\gamma,W'\rangle\\&=\langle R(J',\gamma')\gamma',W\rangle-\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}R(W,\gamma')\gamma',J\rangle-\langle R(J,\gamma')\gamma,W'\rangle\end{aligned}$$

By setting $t=0$ we obtain

$$\langle J',J'''\rangle(0)=-\langle J'(0),\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}R(J,\gamma')\gamma'\Big|_{t=0}\rangle=-R(J',\gamma',\gamma',J')(0)$$

As a consequence

$$|J(t)|^2=t^2-\frac{1}{3}R(J',\gamma',\gamma',J')(0)t^4+O(t^4)$$

□

Corollary 14.1.1. In normal coordinate (x^i, U, p), one has

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iklj}(0) x^k x^\ell + O(|x|^3)$$

Corollary 14.1.2. In normal coordinate (x^i, U, p), one has
(1) $g^{ij} = \delta_{ij} + \frac{1}{3} R_{iklj}(0) x^k x^\ell + O(|x|^3)$
(2) $\det(g_{ij}) = 1 - \frac{1}{3} R_{kl}(0) x^k x^\ell + O(|x|^3)$
(3) $\sqrt{\det(g_{ij})} = 1 - \frac{1}{6} R_{kl}(0) x^k x^\ell + O(|x|^3)$

Proof. For (1). Note that g^{ij} gives a Riemannian metric on T^*M, and Levi-Civita connection ∇ on T^*M with respect to g^{ij} is exactly the induced connection from the one on TM. Note that

$$\nabla \mathrm{d}x^k = -\Gamma_{ij}^k \mathrm{d}x^i \otimes \mathrm{d}x^j$$

where Γ_{ij}^k is the Christoffel symbol for Levi-Civita connection on TM, we have curvature form in this case differs a sign since

$$R_{ijk}^\ell(0) = \frac{\partial \Gamma_{jk}^\ell}{\partial x^i} - \frac{\partial \Gamma_{ik}^\ell}{\partial x^j}$$

Thus all computations are same as proof above, but result differs a sign in curvature.

For (2). By Jacobi's formula, we have

$$\frac{\partial \det(g_{ij})}{\partial x^k} = \det(g_{ij}) g^{ij} \frac{\partial g_{ij}}{\partial x^k}$$

Thus $\frac{\partial \det(g_{ij})}{\partial x^k}(0) = 0$, since first-order partial derivatives of g_{ij} vanishes. Furthermore, since first-order partial derivatives of g^{ij} also vanishes, we have

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 \det(g_{ij})}{\partial x^\ell \partial x^k} &= \det(g_{ij}) g^{ij} \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^\ell \partial x^k} \\ &= \det(g_{ij}) g^{ij} \left(-\frac{1}{3} R_{iklj} x^k x^\ell\right) \\ &= -\frac{1}{3} \det(g_{ij}) R_{kl} x^k x^\ell \end{aligned}$$

which implies

$$\det(g_{ij}) = 1 - \frac{1}{3} R_{kl}(0) x^k x^\ell + O(|x|^3)$$

For (3). It follows from (2). □

Theorem 14.1.1. Let (M, g) be a Riemannian manifold. For all $p \in M$ and r sufficiently small, the volume of $B(p, r)$ is

$$\mathrm{Vol}(B(p, r)) = \alpha_n r^n \left(1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3)\right)$$

where α_n is the volume of n-dimension unit ball in $\mathbb{R}^{n+1}$.

Proof. Direct computation shows

$$\begin{aligned}
\operatorname{Vol}(B(p,r)) &= \int_0^r \int_{\mathbb{S}^{n-1}(t)} \sqrt{\det g}\,\mathrm{d}S\,\mathrm{d}t \\
&\overset{(1)}{=} \int_0^r \int_{\mathbb{S}^{n-1}(t)} (1 - \frac{1}{6}\operatorname{Ric}_p(x) + O(|x|^3))\mathrm{d}S\,\mathrm{d}t \\
&\overset{(2)}{=} \alpha_n r^n - \frac{\alpha_n}{6}\int_0^r S(p)t^{n+1}\mathrm{d}t + O(r^{n+3}) \\
&= \alpha_n r^n - \frac{\alpha_n S(p) r^{n+2}}{6(n+2)} + O(r^{n+3}) \\
&= \alpha_n r^n (1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3))
\end{aligned}$$

where

(1) holds from Corollary 14.1.2.
(2) holds from Proposition **??**.

□

14.2. **Gauss lemma.** Let (M,g) be a Riemannian manifold, and (x^i,U,p) is the normal coordinate centered at $p \in M$.

Definition 14.2.1 (radial distance function)**.** The radial distance function r defined on U is given by

$$r(q) := \sqrt{\sum_{i=1}^n (q^i)^2}$$

where $q = (q^1,\dots,q^n)$ in normal coordinate (x^i,U,p).

Definition 14.2.2 (radial vector field)**.** The radial vector field in $U \setminus \{p\}$ is defined as

$$\partial_r = \frac{x^i}{r}\frac{\partial}{\partial x^i}$$

Proposition 14.2.1. The geodesic starting at p with unit-speed is the integral curve of radial vector field ∂_r over $U \setminus \{p\}$.

Proof. Let $\gamma\colon I \to U$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v$, where $|v| = 1$, by definition we need to show

$$\gamma'(b) = \partial_r|_{\gamma(b)}$$

where I is an open interval and $b \in I$. In normal coordinate γ looks like $\gamma(t) = (tv^1,\dots,tv^n)$. If we denote $\gamma(b) = q = (q^1,\dots,q^n)$. Then it's clear $v^i = q^i/b$, and $r(q) = b$, since $|v| = 1$. Then

$$\gamma'(b) = v^i \frac{\partial}{\partial x^i}\Big|_q = \frac{q^i}{b}\frac{\partial}{\partial x^i}\Big|_q = \frac{q^i}{r(q)}\frac{\partial}{\partial x^i}\Big|_q = \partial_r|_q$$

□

Lemma 14.2.1. Let $f\colon M \to \mathbb{R}$ be a smooth function and X a vector field over M, if

(1) $Xf = |X|^2$.
(2) X is perpendicular to the level set of f.

Then $X = \nabla f$.

Theorem 14.2.1 (Gauss lemma)**.** The radial vector field ∂_r is perpendicular to the level set of radial distance function r on $U \setminus \{p\}$.

Proof. For any $q \in U \setminus \{p\}$ written as $q = (q^1, \dots, q^n)$ in normal coordinate with $b = r(q)$. Given $w \in T_qM$ which is tangent to the level set of r, we need to show $\langle \partial_r|_q, w\rangle = 0$. By definition there exists a curve $c(s)\colon (-\varepsilon, \varepsilon) \to M$ such that $c(0) = q, c'(0) = w$ with $\sum_{i=1}^n (c^i(s))^2 = b$, where c^i is the coordinates of c in normal coordinate. Taking derivative with respect to s one has

$$\sum_{i=1}^n 2c^i(s)(c^i(s))' = 0$$

In particular, one has $\sum_{i=1}^n c^i(0)(c^i)'(0) = 0$. By Corollary 12.1.1 there is a Jacobi field $J(t)$ along geodesic connecting p, q such that $J(0) = 0, J'(0) = w$ and $J(1) = w$. Note that the metric at T_pM is standard metric, thus $\langle J'(0), \gamma'(0)\rangle = 0$, and by construction $\langle J(0), \gamma'(0)\rangle = 0$. Then by (4) of Proposition 12.1.1 one has $\langle J(t), \gamma'(t)\rangle \equiv 0$. In particular, one has $\langle J(1), \gamma'(1)\rangle$, which complete the proof since $\gamma'(t)$ is integral curve of ∂_r. □

Corollary 14.2.1.

(1) $|\partial_r|^2 = 1$.
(2) $g^{ij}\frac{\partial r}{\partial x^i}\frac{\partial}{\partial x^j} = \nabla r = \partial_r$.

Proof. For (1). It's clear, since we have already shown geodesic with unit-speed is integral curve of ∂_r.

For (2). By Lemma 14.2.1 and Theorem 14.2.1, it suffices to show $Xr = |\partial_r|^2$, which can be seen from

$$Xr = \frac{x^i}{r}\frac{\partial r}{\partial x^i} = \sum_{i=1}^n \frac{(x^i)^2}{r^2} = 1 = |\partial_r|^2$$

□

Corollary 14.2.2. The following identities hold in (x^i, U, p):

(1) $g_{ij}x^j = x^i$.
(2) $g_{im} = \delta_{im} - \frac{\partial g_{ij}}{\partial x^m}x^j$.
(3) $\frac{\partial g_{ij}}{\partial x^m}x^j = \frac{\partial g_{mj}}{\partial x^i}x^j$.
(4) $\frac{\partial g_{ij}}{\partial x^m}x^j x^i = \frac{\partial g_{mj}}{\partial x^i}x^j x^i = 0$.
(5) $\Gamma_{ij}^k x^i x^j = 0$.

Proof. For (1). On one hand by Corollary 14.2.1 we have $\partial_r = \nabla r = g^{ij}\frac{x^i}{r}\frac{\partial}{\partial x^j}$. On the other hand by definition of ∂_r we have $\partial_r = \frac{x^j}{r}\frac{\partial}{\partial x^j}$, which implies

$$g^{ij}x^i = x^j$$

This shows (1).

For (2). Take partial derivatives of (1) with respect to x^m, we have

$$\frac{\partial g_{ij}}{\partial x^m}x^j + g_{ij}\delta_{jm} = \delta_{im}$$

This shows (2).

For (3). It follows from (2), since g_{im}, δ_{im} are symmetric in i, m.

For (4). It follows from (1) and (2), since

$$\frac{\partial g_{ij}}{\partial x^m}x^j x^i \overset{(2)}{=} (\delta_{im} - g_{im})x^i = x^m - g_{im}x^i \overset{(1)}{=} 0$$
$$\frac{\partial g_{mj}}{\partial x^i}x^j x^i \overset{(2)}{=} (\delta_{mi} - g_{mi})x^i = x^m - g_{im}x^i \overset{(1)}{=} 0$$

For (5). It follows from (4) and

$$\Gamma_{ij}^k = \frac{1}{2}g^{mk}(\frac{\partial g_{mj}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m})$$

□

Corollary 14.2.3.

$$\nabla_{\partial_r}\partial_r = 0$$

holds in $U \setminus \{p\}$.

Proof. Direct computation shows

$$\begin{aligned}\nabla_{\partial_r}\partial_r &= \frac{x^k}{r}\nabla_{\frac{\partial}{\partial x^k}}(g^{ij}\frac{x^i}{r}\frac{\partial}{\partial x^j}) \\ &= g^{ij}\frac{x^k}{r}\{\underbrace{(\frac{\delta_{ki}}{r} - \frac{x^k x^i}{r^3})\frac{\partial}{\partial x^j}}_{\text{part I}} + \underbrace{\frac{x^i}{r}\Gamma_{kj}^m\frac{\partial}{\partial x^m}}_{\text{part II}}\}\end{aligned}$$

By (1) and (5) of Corollary 14.2.2 one has

$$g^{ij}\frac{x^k x^i}{r}\Gamma_{kj}^m = \frac{1}{r}\Gamma_{kj}^m x^k x^j = 0$$

which implies part II is zero. For part I, we have

$$\frac{1}{r^2}(g^{ij}x^k\delta_{ki} - \frac{(x^k)^2}{r^2}g^{ij}x^i) = \frac{1}{r^2}(g^{ij}x^i - g^{ij}x^i) = 0$$

□

14.3. **Jacobi fields on constant sectional curvature manifold.**

Proposition 14.3.1. Let (M, g) be a Riemannian manifold with constant sectional curvature k and $\gamma: [0, b] \to M$ a unit-speed geodesic. Then the normal Jacobi field with $J(0) = 0$ is of the form

$$J(t) = m\,\mathrm{sn}_k(t)E(t)$$

where

(1) The constant m is determined by $J'(0) = mE(0)$.

(2)
$$\mathrm{sn}_k(t) = \begin{cases} t, & k = 0 \\ \frac{\sin(\sqrt{k}t)}{\sqrt{k}}, & k > 0 \\ \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}}, & k < 0 \end{cases}$$

(3) $E(t)$ is a normal parallel vector field along γ with $|E(t)| = 1$

Proof. Since (M,g) has constant sectional curvature k, thus $R_{ijkl} = k(g_{il}g_{jk} - g_{ik}g_{jl})$, so for any normal vector field J along γ we have
$$\begin{aligned} R(J,\gamma',\gamma',W) &= k(\langle J,W\rangle\langle\gamma',\gamma'\rangle - \langle J,\gamma'\rangle\langle\gamma',W\rangle) \\ &= k\langle J,W\rangle \end{aligned}$$
which implies
$$R(J,\gamma')\gamma' = kJ$$
since γ is unit-speed and J is normal. Thus equation for Jacobi field can be written as
$$0 = J'' + kJ$$
Assume $J = u(t)E(t)$. Then
$$(u''(t) + ku(t))E(t) = 0$$
So if we want to find normal Jacobi fields J, it suffices to solve
$$\begin{cases} u''(t) + ku = 0 \\ u(0) = 0 \end{cases}$$
and it's clear $\mathrm{sn}_k(t)$ is the solution of this ODE. □

14.4. **Polar decomposition of metric with constant sectional curvature.** Let $\pi\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ given by $\pi(x) = x/|x|$. We can use π to pullback canonical metric on $\mathbb{S}^{n-1}$, and still use $g_{\mathbb{S}^{n-1}}$ to denote it.

Lemma 14.4.1. Let $\overline{g}$ be the Euclidean metric on $\mathbb{R}^n \setminus \{0\}$. Then
$$\overline{g} = \mathrm{d}r \otimes \mathrm{d}r + r^2 g_{\mathbb{S}^{n-1}}$$
where $r(x) = |x|$.

Theorem 14.4.1 (polar decomposition)**.** Let (x^i, U, p) be a normal coordinate centered at $p \in S(n,k)$. Then in U the metric g can be written as
$$g = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r) g_{\mathbb{S}^{n-1}}$$
where r is radial distance function.

Proof. We use g_c to denote metric $\mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r) g_{\mathbb{S}^{n-1}}$ and $\overline{g}$ to denote standard metric on Euclidean space. By Corollary 14.2.1, we have
$$g(\partial_r, \partial_r) = 1 = g_c(\partial_r, \partial_r)$$
So it remains to show for each $q \in U \setminus \{p\}$ and $w_1, w_2 \in T_qM$ such that $g(w_i, \partial_r|_q) = 0, i = 1,2$, we have
$$g(w_1, w_2) = g_c(w_1, w_2)$$

By polarization it suffices to show that $g(w,w)=g_c(w,w)$ for every such vector w.

Suppose $\mathrm{dist}(p,q)=b$, on one hand we have

$$|w|^2_{g_c}\overset{(1)}{=}\mathrm{sn}_k^2(b)|w|^2_{g_{\mathbb{S}^{n-1}}}\overset{(2)}{=}\frac{\mathrm{sn}_k(b)}{b^2}|w|^2_g$$

where

(1) holds from definition of g_c.
(2) holds from polar decomposition of standard metric of Euclidean space, that is Lemma 14.4.1.

On the other hand, let $\gamma\colon[0,b]\to U$ be a unit-speed geodesic connecting p,q, and we can write it with respect to normal coordinate U as

$$\gamma(t)=(\frac{tq^1}{b},\dots,\frac{tq^n}{b})$$

where $q=(q^1,\dots,q^n)$ in normal coordinate U. Let J be a Jacobi field such that $J(0)=0,J(b)=w$. Then we have

$$|w|^2_g=|J(b)|^2_g\overset{(3)}{=}\mathrm{sn}_k^2(b)|J'(0)|^2_g\overset{(4)}{=}\mathrm{sn}_k^2(b)|J'(0)|^2_g$$

where

(3) holds from the fact Jacobi field on constant sectional curvature space is of form $J(t)=|J'(0)|\,\mathrm{sn}_k(t)E(t)$.
(4) holds from the metric on T_pM is standard metric in normal coordinate.

Furthermore, suppose $J'(0)=a$. Then we can write it as $J(t)=\alpha_*(\frac{\partial}{\partial s})\big|_{s=0}$, where

$$\alpha(s,t)=\exp_p(t(\gamma'(0)+sJ'(0)))$$

In normal coordinate we can write $\alpha(s,t)$ explicitly as

$$\alpha(s,t)=(\frac{tq^1}{b}+tsa^1,\dots,\frac{tq^n}{b}+tsa^n)$$

thus $J(t)=ta^i\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$. We can conclude $a^i=\frac{w^i}{b}$ by setting $t=b$, in particular, we have $J'(0)=\frac{w^i}{b}\frac{\partial}{\partial x^i}\Big|_p$. Then

$$\mathrm{sn}_k^2(b)|J'(0)|^2_g=\mathrm{sn}_k^2(b)\frac{|w|^2_g}{b^2}=|w|^2_{g_c}$$

□

Remark 14.4.1. Note that there are four points we need in the proof of above theorem, and the **key point** is (3), that is Jacobi field of $S(n,k)$ has the form of

$$J(t)=m\,\mathrm{sn}_k(t)E(t)$$

So this motivates us that if on a normal neighborhood of some point, the Jacobi field has the above form. Then metric g can be written as

$$g=\mathrm{d}r\otimes\mathrm{d}r+\mathrm{sn}_k(r)^2g_{\mathbb{S}^{n-1}}$$

in U. In particular, it has constant sectional curvature k.

14.5. **A criterion for constant sectional curvature space.** Recall that for a smooth function $f\colon M\to\mathbb{R}$, $\operatorname{Hess} f$ is a $(0,2)$-tensor, we use $\mathscr{H}_f$ to denote its $(1,1)$-type, that is

$$g(\mathscr{H}_f(X),Y)=\operatorname{Hess} f(X,Y)$$

where X,Y are two vector fields.

In particular, if r is the radial distance function on a normal coordinate. Then Hessian r is a $(2,0)$-tensor, that is $\nabla^2 r$. Then we have

$$\mathscr{H}_r=\nabla\partial_r$$

since $(1,0)$-type of ∇r is ∂_r.

Proposition 14.5.1. Let (M,g) be a complete Riemannian manifold, (x^i,U,p) a normal coordinate centered at p and r the radial distance function on U. If $\gamma\colon[0,b]\to M$ is unit-speed geodesic with $\gamma(0)=p,\gamma'(0)=v\in T_pM$, and J is a normal Jacobi field along γ with $J(0)=0$. Then for all $t\in(0,b]$

$$\mathscr{H}_r(J(t))=J'(t)$$
$$\mathscr{H}_r(\gamma'(t))=0$$

Proof. Here we only prove the first identity, the second can be computed in the same method. Let $J'(0)=w$. Then $J(t)=tw^i\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$,

$$\begin{aligned}
J'(t)&=\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}(tw^i\frac{\partial}{\partial x^i}\Big|_{\gamma(t)})\\
&=w^i\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}+tw^i\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}\\
&=w^i\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}+tw^i\Gamma_{ij}^k(\gamma(t))\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\partial}{\partial x^k}\Big|_{\gamma(t)}\\
&=(w^k+tw^iv^j\Gamma_{ij}^k(\gamma(t)))\frac{\partial}{\partial x^k}\Big|_{\gamma(t)}\\
\mathscr{H}_r(J(t))&=\nabla_{J(t)}\partial_r\\
&=\nabla_{tw^i\frac{\partial}{\partial x^i}\big|_{\gamma(t)}}(\frac{x^j}{r}\frac{\partial}{\partial x^j})\\
&=tw^i\nabla_{\frac{\partial}{\partial x^i}\big|_{\gamma(t)}}(\frac{x^j}{r}\frac{\partial}{\partial x^j})\\
&=tw^i\frac{x^j}{r}\Gamma_{ij}^k(\gamma(t))\frac{\partial}{\partial x^k}\Big|_{\gamma(t)}+\sum_{i=1}^n tw^i(\frac{\delta_{ij}}{r}-\frac{x^ix^j}{r^3})\frac{\partial}{\partial x^j}\Big|_{\gamma(t)}
\end{aligned}$$

However, we have the following observations:

(1) $r(\gamma(t))=t$.
(2) $x^i=tv^i$.
(3) $\sum_{i=1}^n a^iv^i=0$

where the last equality holds since J is a normal vector field. Then

$$0=\langle J(t),\gamma'(t)\rangle=\langle J(0),\gamma'(0)\rangle+\langle J'(0),\gamma'(0)\rangle t$$

implies $\langle J'(0),\gamma'(0)\rangle=\sum_{i=1}^n a^i v^i=0$. □

Corollary 14.5.1. With the same assumption as above proposition, for any vector field W along γ with $W(0)=0$,

$$\begin{aligned}\operatorname{Hess} r(J(s),W(s)) &\overset{(1)}{=} g(\mathscr{H}_r(J(s),W(s)))\\ &\overset{(2)}{=} g(J'(t),W(s))\\ &\overset{(3)}{=} \int_0^s \langle J'(t),W(t)\rangle'\mathrm{d}t\\ &\overset{(4)}{=} \int_0^s \langle J'(t),W'(t)\rangle-R(J,\gamma',\gamma',W)\mathrm{d}t\end{aligned}$$

Proof. It's clear, since

(1) holds from definition of $\mathscr{H}_r$.
(2) holds from $\mathscr{H}_r(J(t))=J'(t)$.
(3) holds from $W(0)=0$.
(4) holds from J is a Jacobi field.

□

Corollary 14.5.2. Let $p\in U\subseteq S(n,k)$, where U is a normal neighborhood of p. Then the following holds in $U\setminus\{p\}$

$$\mathscr{H}_r=\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r$$

where r is the radial distance function on U, and for each $q\in U\setminus\{p\}$, $\pi_r: T_qM\to T_qM$ is the orthogonal projection onto the orthogonal complement of $\partial_r|_q$.

Proof. For $p\in U\setminus\{q\}$, it's clear

$$\mathscr{H}_r(\partial_r|_q)=0=\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r(\partial_r|_q)$$

For $w\in T_qM$ such that $g(w,\partial_r|_q)$, choose a unit-speed geodesic $\gamma\colon[0,b]\to M$ connecting p and q and $J(t)$ is the Jacobi field such that $J(0)=0, J(b)=w$. Then we must have

$$J(t)=m\,\mathrm{sn}_k(t)E(t)$$

where $E(t)$ is a normal parallel vector field along γ with $|E(t)|=1$. Then

$$\begin{aligned}m\,\mathrm{sn}_k'(t)E(t)&=J'(t)\\ &=\mathscr{H}_r(J(t))\\ &=\mathscr{H}_r(m\,\mathrm{sn}_k(t)E(t))\\ &=m\,\mathrm{sn}_k(t)\mathscr{H}_r(E(t))\end{aligned}$$

Setting $t=b$ and dividing by $\mathrm{sn}_k(b)$ one has
$$\mathscr{H}_r(E(b))=\frac{\mathrm{sn}_k'(b)}{\mathrm{sn}_k(b)}E(b)$$
Note that $w=m\,\mathrm{sn}_k(b)E(b)$, this completes the proof. □

Furthermore, the converse of above corollary still holds:

Proposition 14.5.2. Let (M,g) be a Riemannian manifold and U a normal neighborhood of $p\in M$, r radial distance function. If
$$\mathscr{H}_r=\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r$$
holds in $U\setminus\{p\}$. Then (M,g) has constant sectional curvature k in U.

Proof. Let $\gamma\colon[0,b]\to U$ be a unit-speed geodesic $r(0)=p$, J is a normal Jacobi vector field along γ with $J(0)=0$. Then $\mathscr{H}_r(J)=J'$ implies
$$J'(t)=\frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}J(t)$$
holds for $t\in(0,b]$, that is
$$(\frac{J(t)}{\mathrm{sn}_k(t)})'=0$$
holds for $t\in(0,b]$. So we can write every normal Jacobi fields as $J(t)=m\,\mathrm{sn}_k(t)E(t)$, where E is normal a parallel vector field with $|E|=1$ and $t\in[0,b]$. Thus by Remark 14.4.1, g has constant sectional curvature k in U. □

Remark 14.5.1. For convenience, we record the exact formulas for the quotient $\frac{\mathrm{sn}_k'}{\mathrm{sn}_k}$ as follows
$$\frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}=\begin{cases}\frac{1}{t}, & k=0\\ \frac{1}{\sqrt{k}}\cot\frac{t}{\sqrt{k}}, & k>0\\ \frac{1}{\sqrt{k}}\coth\frac{t}{\sqrt{k}}, & k<0\end{cases}$$
and we can draw the graph as follows.

Part 5. Harmonic maps

15. Harmonic map

In this section, let $f : (M,g) \to (N,h)$ be a smooth map between Riemannian manifolds with second fundamental form $B \in C^\infty(M, T^*M \otimes T^*M \otimes f^*TN)$.

15.1. Harmonic map and totally geodesic.

Definition 15.1.1 (scalar Laplacian)**.** The scalar Laplacian of f is defined as

$$\Delta f := \mathrm{tr}_g B \in C^\infty(M, f^*TN)$$

Definition 15.1.2 (harmonic map)**.** f is called a harmonic map if its scalar Laplacian $\Delta f = 0$.

Definition 15.1.3 (totally geodesic)**.** f is called totally geodesic, if its second fundamental form $B = 0$.

Example 15.1.1. *For a geodesic $\gamma : [a,b] \to (M,g)$, if $[a,b]$ is endowed with standard metric. Then γ is totally geodesic, thus it's harmonic.*

Example 15.1.2. *For a smooth function $f : (M,g) \to \mathbb{R}$, if $\mathbb{R}$ is endowed with standard metric. Then f is a harmonic map if and only if it's a harmonic function.*

Lemma 15.1.1. Let $\gamma : [a,b] \to M$ be a smooth curve and $\widetilde{\gamma} = f \circ \gamma$. Then

$$\widetilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \widetilde{\gamma}_*(\frac{\mathrm{d}}{\mathrm{d}t}) = f_*(\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})) + \gamma^* B$$

where $\widehat{\nabla}$ and $\widetilde{\nabla}$ are the induced connection on γ^*TM and $\widetilde{\gamma}^*TN$ respectively.

Proof. Direct computation shows

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \widetilde{\gamma}_*(\frac{\mathrm{d}}{\mathrm{d}t}) &= (\frac{\mathrm{d}^2\widetilde{\gamma}^l}{\mathrm{d}t^2} + \Gamma^l_{mn}(\widetilde{\gamma}) \cdot \frac{\mathrm{d}\widetilde{\gamma}^m}{\mathrm{d}t} \frac{\mathrm{d}\widetilde{\gamma}^n}{\mathrm{d}t}) \frac{\partial}{\partial y^l} \\
&= \left\{ \frac{\partial f^l}{\partial x^k} \frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + (\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \Gamma^l_{mn} \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j}) \frac{\partial \gamma^i}{\mathrm{d}t} \frac{\partial \gamma^j}{\mathrm{d}t} \right\} \frac{\partial}{\partial y^l} \\
&= \left\{ \frac{\partial f^l}{\partial x^k} (\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + \Gamma^k_{ij} \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t}) + (\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \Gamma^l_{mn} - \Gamma^k_{ij} \frac{\partial f^l}{\partial x^k}) \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \right\} \frac{\partial}{\partial y^l} \\
&= f_*(\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})) + \gamma^* B
\end{aligned}$$

□

Theorem 15.1.1. The following statements are equivalent.

(1) f is totally geodesic.
(2) f maps geodesics in M to geodesics in N.

Proof. It follows from above lemma. □

15.2. **First variation of smooth map.**

Definition 15.2.1 (energy functional)**.** The energy density of f is defined as $e(f) = |\mathrm{d}f|^2$, and the energy functional of f is

$$E(f) = \frac{1}{2}\int_M e(f)\mathrm{vol}$$

Remark 15.2.1 (local form). Suppose $\mathrm{d}f$ is locally given by $\frac{\partial f^m}{\partial x^i}\mathrm{d}x^i \otimes f^*(\frac{\partial}{\partial y^m})$. Then

$$\begin{aligned} e(f) &= \langle \frac{\partial f^m}{\partial x^i}\mathrm{d}x^i \otimes f^*(\frac{\partial}{\partial y^m}), \frac{\partial f^n}{\partial x^j}\mathrm{d}x^j \otimes f^*(\frac{\partial}{\partial y^n})\rangle \\ &= g^{ij}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}h_{mn}(f) \end{aligned}$$

Lemma 15.2.1. $\Delta f = -\widehat{\nabla}^*\mathrm{d}f \in C^\infty(M, f^*TN)$.

Theorem 15.2.1. The critical point of energy functional is harmonic maps.

Proof. We fix the following notations in the proof:

(1) Consider a smooth variation $f: M\times\mathbb{R} \to N$ of f, we also write $f_t(\text{-}) = F(\text{-}, t)$ for convenience.
(2) Set $\overline{M} = M\times\mathbb{R}$ and there is a natural metric $\overline{g} = g\times g_{\mathbb{R}}$ on $\overline{M}$.
(3) The pullback F^*TN bundle is denoted by W, and induced connection on W is denoted by ∇^W.
(4) Fix $t\in\mathbb{R}$, $f_t: M\to N$. Then $\mathrm{d}f_t$ is a section of $T^*M\otimes f_t^*TN$, and we can regard it as a section of $T^*\overline{M}\otimes W$.

Holding above notations, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(f_t) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\mathrm{d}f_t|^2\mathrm{vol} \\ &= \int_M \langle \nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}f_t, \mathrm{d}f_t\rangle\,\mathrm{vol} \end{aligned}$$

Here we claim

$$\langle \nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}f_t, \mathrm{d}f_t\rangle \overset{1}{=} \langle \nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}F, \mathrm{d}f_t\rangle \overset{2}{=} \langle \nabla^W F_*(\frac{\partial}{\partial t}), \mathrm{d}f_t\rangle$$

(1) For equation marked 1: Note that

$$\begin{aligned} \mathrm{d}F - \mathrm{d}f_t &= \frac{\partial F^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m} + \frac{\partial F^m}{\partial t}\mathrm{d}t\otimes\frac{\partial}{\partial y^m} - \frac{\partial f_t^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m} \\ &= \frac{\partial F^m}{\partial t}\mathrm{d}t\otimes\frac{\partial}{\partial y^m} \end{aligned}$$

since $\frac{\partial F^m}{\partial x^i} = \frac{\partial f_t^m}{\partial x^i}$. So we have

$$\begin{aligned} \nabla^{T^*\overline{M}\otimes W}(\mathrm{d}F - \mathrm{d}f_t) &= \frac{\partial^2 F^l}{\partial t^2}\mathrm{d}t\otimes\mathrm{d}t\otimes\frac{\partial}{\partial y^l} + \frac{\partial F^m}{\partial t}\mathrm{d}t\otimes(\frac{\partial F^n}{\partial t}\Gamma^l_{mn}\mathrm{d}t\otimes\frac{\partial}{\partial y^l} + \frac{\partial F^n}{\partial x^i}\Gamma^l_{mn}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l}) \\ &= (\frac{\partial^2 F^l}{\partial t^2} + \frac{\partial F^m}{\partial t}\frac{\partial F^n}{\partial t}\Gamma^l_{mn})\mathrm{d}t\otimes\mathrm{d}t\otimes\frac{\partial}{\partial y^l} + \frac{\partial F^m}{\partial t}\frac{\partial F^n}{\partial x^i}\Gamma^l_{mn}\mathrm{d}x^i\otimes\mathrm{d}t\otimes\frac{\partial}{\partial y^l} \end{aligned}$$

Thus we have

$$\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}(\mathrm{d}F-\mathrm{d}f_t)=(\frac{\partial^2 F^l}{\partial t^2}+\frac{\partial F^m}{\partial t}\frac{\partial F^n}{\partial t}\Gamma^l_{mn})\mathrm{d}t\otimes\frac{\partial}{\partial y^l}$$

From above expression it's clear

$$\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}(\mathrm{d}F-\mathrm{d}f_t),\mathrm{d}f_t\rangle=0$$

since there is no $\mathrm{d}t$ in $\mathrm{d}f_t$, which implies equation marked 1 holds.

(2) For equation marked 2: For arbitrary $X\in C^\infty(M,TM)\subseteq C^\infty(\overline{M},T^*\overline{M})$, since second fundamental form is symmetric, thus

$$\begin{aligned}(\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}F)(X)&=(\nabla^{T^*\overline{M}\otimes W}_{X}\mathrm{d}F)(\frac{\partial}{\partial t})\\&=\nabla^W_X F_*(\frac{\partial}{\partial t})-F_*(\nabla^{\overline{M}}_X\frac{\partial t}{\partial t})\\&=\nabla^W_X F_*(\frac{\partial}{\partial t})\end{aligned}$$

Now let v be an arbitrary variation vector field, that is

$$v=F_*(\frac{\partial}{\partial t})\Big|_{t=0}\in C^\infty(M,f^*TN)$$

Hence when $t=0$ we have

$$(\nabla^W F_*(\frac{\partial}{\partial t}))\Big|_{t=0}=\widehat{\nabla}v$$

where $\widehat{\nabla}$ is the induced connection on f^*TN. So we have first variation formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}E(f_t)&=\int_M\langle\widehat{\nabla}v,\mathrm{d}f\rangle\,\mathrm{vol}\\&=\int_M\langle v,\widehat{\nabla}^*\mathrm{d}f\rangle\,\mathrm{vol}=0\end{aligned}$$

where $\widehat{\nabla}^*$ is the formal adjoint operator of $\widehat{\nabla}$. Since v is arbitrary, we deduce $\widehat{\nabla}^*\mathrm{d}f=0$. □

15.3. **Second variation formula of harmonic map.** Consider the following variation map of f

$$f:M\times(-\varepsilon_1,\varepsilon_1)\times(-\varepsilon_2,\varepsilon_2)\to N$$

with variation fields

$$\begin{aligned}v&=F_*(\frac{\partial}{\partial t})\Big|_{s=t=0}\in C^\infty(M,f^*TN)\\w&=F_*(\frac{\partial}{\partial s})\Big|_{s=t=0}\in C^\infty(M,f^*TN)\end{aligned}$$

For convenience we denote $F(\text{-},s,t)=f_{s,t}(\text{-})$.

Theorem 15.3.1 (second variation formula)**.** If $f:(M,g)\to(N,h)$ is a harmonic map. Then the second variation of the harmonic map f along v,w is

$$\frac{\partial^2}{\partial s\partial t}\bigg|_{s=t=0}E(f_{s,t})=\int_M\langle\widehat{\nabla}v,\widehat{\nabla}w\rangle\,\mathrm{vol}-\int_M g^{ij}R_{pmnq}v^pw^q\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\,\mathrm{vol}$$

Proof. In this proof, we still use the notations in proof of first variation formula. By first variation formula, we have

$$\frac{\partial}{\partial t}E(f_{s,t})=\int_M\langle\nabla^W F_*(\frac{\partial}{\partial t}),\mathrm{d}f_{s,t}\rangle\,\mathrm{vol}$$

So

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}E(f_{s,t})=&\underbrace{\int_M\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial s}}\nabla^W F_*(\frac{\partial}{\partial t}),\mathrm{d}f_{s,t})\rangle\,\mathrm{vol}}_{\text{part I}}\\&+\underbrace{\int_M\langle\nabla^W F_*(\frac{\partial}{\partial t}),\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial s}}\mathrm{d}f_{s,t}\rangle\,\mathrm{vol}}_{\text{part II}}\end{aligned}$$

Note that

$$\begin{aligned}\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial s}}\mathrm{d}f_{s,t}&=\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial s}}(\frac{\partial F^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m})\\&=\frac{\partial^2F^m}{\partial s\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m}+\frac{\partial F^m}{\partial x^i}\frac{\partial F^n}{\partial s}\Gamma^l_{mn}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l}\\&=(\frac{\partial^2F^l}{\partial s\partial x^i}+\frac{\partial F^m}{\partial x^i}\frac{\partial F^n}{\partial s}\Gamma^l_{mn})\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l}\\\widehat{\nabla}w&=\widehat{\nabla}\frac{\partial}{\partial x^i}(\frac{\partial F^n}{\partial s}\bigg|_{t=s=0})\mathrm{d}x^i\otimes\frac{\partial}{\partial y^n}+\frac{\partial F^m}{\partial s}\frac{\partial F^n}{\partial x^i}\bigg|_{t=s=0}\Gamma^l_{mn}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l}\\&=(\frac{\partial^2F^l}{\partial s\partial x^i}+\frac{\partial F^m}{\partial x^i}\frac{\partial F^n}{\partial s}\bigg|_{t=s=0}\Gamma^l_{mn})\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l}\end{aligned}$$

which implies setting $t=s=0$ we have part II is

$$\int_M\langle\widehat{\nabla}v,\widehat{\nabla}w\rangle\,\mathrm{vol}$$

For part I, take arbitrary $X\in C^\infty(M,TM)\subseteq C^\infty(\overline{M},T^*\overline{M})$, we have
Hence we obtain

$$\nabla^{T*\overline{M}\otimes W}_{\frac{\partial}{\partial s}}\nabla^W F_*(\frac{\partial}{\partial t})(X)=(\nabla^{T*\overline{M}\otimes W}\nabla^W F_*(\frac{\partial}{\partial t})(X))(\frac{\partial}{\partial s},X)$$

Setting $t=s=0$ we have
Hence

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}\bigg|_{t=s=0}E\left(f_{s,t}\right)=&\int_M\langle\widehat{\nabla}(\nabla^W_{\frac{\partial}{\partial s}}F_*(\frac{\partial}{\partial t})\bigg|_{s=t=0}),\mathrm{d}f\rangle\,\mathrm{vol}\\&+\int_M g^{ij}R_{pmqn}v^pw^q\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\,\mathrm{vol}+\int_M\langle\widehat{\nabla}w,\widehat{\nabla}v\rangle\,\mathrm{vol}\end{aligned}$$

If f is harmonic, that is $\widehat{\nabla}^*\mathrm{d}f=0$, we obtain the desired formula. □

15.4. **Bochner formula for harmonic map.** Recall that for a smooth function $f:(M,g)\to\mathbb{R}$,

$$\frac{1}{2}\Delta|\mathrm{d}f|^2=|\mathrm{Hess}\, f|^2+\mathrm{Ric}(\nabla f,\nabla f)+g(\nabla\Delta f,\nabla f)$$

In this section we generalize this formula to smooth map $f:(M,g)\to(N,h)$ between Riemannian manifolds, to get similar Bochner's theorem we have proven before.

Theorem 15.4.1. Let $f:(M,g)\to(N,h)$ be a smooth map between Riemannian manifolds. Then

$$\frac{1}{2}\Delta|\mathrm{d}f|^2=|\widetilde{\nabla}\mathrm{d}f|^2+\langle\widehat{\nabla}(\mathrm{d}f),\mathrm{d}f\rangle+g^{ik}g^{jl}R_{ij}\frac{\partial f^m}{\partial x^k}\frac{\partial f^n}{\partial x^l}h_{mn}-g^{kl}g^{ij}R_{mnpq}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\frac{\partial f^p}{\partial x^k}\frac{\partial f^q}{\partial x^l}$$

Theorem 15.4.2. Let $f:(M,g)\to(N,h)$ be a harmonic map between Riemannian manifolds. If

(1) M is compact with positive Ricci curvature.
(2) N has non-positive sectional curvature.

Then f is constant.

Proof. Suppose $|\mathrm{d}f|^2$ attains its maximum at some point $p\in M$, we have

$$\Delta|\mathrm{d}f|^2(p)\leq 0$$

On the other hand,

$$\frac{1}{2}\Delta|\mathrm{d}f|^2\geq g^{ik}g^{jl}R_{ij}\frac{\partial f^m}{\partial x^k}\frac{\partial f^n}{\partial x^l}h_{mn}-g^{kl}g^{ij}R_{mnpq}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\frac{\partial f^p}{\partial x^k}\frac{\partial f^q}{\partial x^l}$$

since $|\widetilde{\nabla}\mathrm{d}f|^2+\langle\widehat{\nabla}(\mathrm{d}f),\mathrm{d}f\rangle\geq 0$.

Without lose of generality, we may assume $g_{ij}(p)=\delta_{ij}, h_{mn}(f(p))=\delta_{mn}$ by choosing normal coordinates. Then

$$\frac{1}{2}\Delta|\mathrm{d}f|^2\geq\sum_{i,j,m}R_{ij}\frac{\partial f^m}{\partial x^i}\frac{\partial f^m}{\partial x^j}-\sum_{i,j}R_{mnpq}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^i}\frac{\partial f^p}{\partial x^j}\frac{\partial f^q}{\partial x^j}\geq 0$$

since R_{ij} is positive, which implies $\mathrm{d}f\equiv 0$, thus f is constant since we always assume M is connected. □

Corollary 15.4.1. Let (M,g) be a compact Riemannian manifold with non-negative Ricci curvature, (N,h) a Riemannian manifold with non-positive sectional curvature, and $f:(M,g)\to(N,h)$ a harmonic map. Then

(1) f is totally geodesic.
(2) If $\mathrm{Ric}(g)$ is strictly positive at some point. Then f is constant.
(3) If sectional curvature of h is negative. Then either f is constant or its image is a closed geodesic.

Part 6. **Topology of Riemannian manifold**

16. ISOMETRY GROUP

16.1. **Isometry and local isometry.**

Definition 16.1.1 (local isometry)**.** A smooth map $f\colon (M,g_M)\to(N,g_N)$ between Riemannian manifolds is called a local isometry if for each point $p\in M$ the differential $(\mathrm{d}f)_p\colon T_pM\to T_{f(p)}N$ is a linear isometry.

Definition 16.1.2 (isometry)**.** A local isometry $f\colon (M,g_M)\to(N,g_N)$ between Riemannian manifolds is called an isometry if it's a diffeomorphism.

Proposition 16.1.1. Let $f\colon (M,g_M)\to(N,g_N)$ be smooth map between Riemannian manifolds. The following statements are equivalent.

(1) f is a local isometry.
(2) For each $p\in M$, there are open neighborhoods U of p and V of $f(p)$ such that $f|_U\colon U\to V$ is an isometry.

Proof. It's clear (2) implies (1), and the converse follows immediately from inverse function theorem. □

Definition 16.1.3 (isometry group)**.** The isometry group $\mathrm{Iso}(M,g)$ of Riemannian manifold (M,g) is the group consisting of all isometries from (M,g) to itself.

Proposition 16.1.2. Let $f\colon (M,g_M)\to(N,g_N)$ be a local isometry.

(1) f maps geodesics to geodesics.
(2) $f\circ\exp_p=\exp_{f(p)}\circ(\mathrm{d}f)_p$ holds on V_p.
(3) f is distance decreasing.
(4) If f is an isometry, then f is distance preserving.

Proof. See [Pet16, Proposition 5.6.1]. □

Remark 16.1.1. A non-trivial fact is that any bijective map which preserves distance is an isometry. See [MS39] or [Pal57].

Theorem 16.1.1. Let $\varphi,\psi\colon (M,g_M)\to(N,g_N)$ be two local isometries between Riemannian manifolds, and M is connected. If there exists $p\in M$ such that

$$\varphi(p)=\psi(p)$$
$$(\mathrm{d}\varphi)_p=(\mathrm{d}\psi)_p$$

then $\varphi=\psi$.

Proof. Consider the following set

$$A=\{p\in M\mid \psi(p)=\varphi(p),(\mathrm{d}\psi)_p=(\mathrm{d}\varphi)_p\}$$

Since M is connected, it suffices to show it's both open and closed.

(1) To see it's open: For $p \in A$, let $B(p,\delta)$ be a geodesic ball centered at p such that $\varphi|_{B(p,\delta)}, \psi|_{B(p,\delta)}$ are diffeomorphisms. Then

$$f := (\varphi^{-1} \circ \psi)|_{B(p,\delta)} : B(p,\delta) \to B(p,\delta)$$

satisfies $f(p) = p, (\mathrm{d}f)_p = \mathrm{id}$. For any $q \in B(p,\delta)$, there exists a unique $v \in T_pM$ such that $\exp_p(v) = q$. Then

$$\begin{aligned} f(q) &= f \circ \exp_p(v) \\ &= \exp_{f(p)} \circ (\mathrm{d}f)_p(v) \\ &= \exp_p(v) \\ &= q \end{aligned}$$

which implies $q \in A$, and thus A is open.

(2) To see it's closed: Suppose $\{p_i\}_{i=1}^n \subseteq A$, that is

$$\begin{aligned} \psi(p_i) &= \varphi(p_i) \\ (\mathrm{d}\psi)_{p_i} &= (\mathrm{d}\varphi)_{p_i} \end{aligned}$$

The desired result can be obtained from taking limits.

□

Theorem 16.1.2 (Myers-Steenrod)**.** Let (M,g) be a Riemannian manifold and $G = \mathrm{Iso}(M,g)$. Then

(1) G is a Lie group with respect to compact-open topology.
(2) For each $p \in M$, the isotropy group G_p is compact.
(3) G is compact if M is compact.

Proof. See [MS39]. □

16.2. **Properties of Killing field.**

Proposition 16.2.1. Let (M,g) be a Riemannian manifold and X be a Killing field.

(1) If γ is a geodesic, then $J(t) = X(\gamma(t))$ is a Jacobi field.
(2) For any two vector fields Y, Z,

$$\nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X,Y)Z = 0$$

Proof. For (1). Suppose φ_s is the flow generated by X. Then we obtain a variation $\alpha(s,t) = \varphi_s(\gamma(t))$ consisting of geodesics, and thus

$$X(\gamma(t)) = \left.\frac{\partial \varphi_s(\gamma(t))}{\partial s}\right|_{s=0}$$

is a Jacobi field.

For (2). It's an equation of tensors, so we check it pointwisely and use normal coordinate $\{x^i\}$ centered at p. Furthermore, we assume $X = X^i \frac{\partial}{\partial x^i}, Y =$

$\frac{\partial}{\partial x^j}, Z = \frac{\partial}{\partial x^k}$. Then

$$\nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X,Y)Z = \nabla_j \nabla_k X + X^i R_{ijk}^{\ell} \frac{\partial}{\partial x^{\ell}}$$
$$= (\frac{\partial^2 X^{\ell}}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{ki}^{\ell}}{\partial x^j} + X^i R_{ijk}^{\ell}) \frac{\partial}{\partial x^{\ell}}$$
$$= (\frac{\partial^2 X^{\ell}}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^{\ell}}{\partial x^i}) \frac{\partial}{\partial x^{\ell}}$$

since $R_{ijk}^{\ell} = \frac{\partial \Gamma_{jk}^{\ell}}{\partial x^i} - \frac{\partial \Gamma_{ik}^{\ell}}{\partial x^j} + \Gamma_{jk}^s \Gamma_{is}^{\ell} - \Gamma_{ik}^s \Gamma_{js}^{\ell}$. Now it suffices to show $\frac{\partial^2 X^{\ell}}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^{\ell}}{\partial x^i} \equiv 0$. In order to show this, for arbitrary $p \in M$, consider a geodesic γ starting at p and consider Jacobi field $J(t) = X(\gamma(t))$. Direct computation shows

$$J'(t) = (\frac{\partial X^i}{\partial x^k} \frac{\mathrm{d}\gamma^k}{\mathrm{d}t} + X^i \Gamma_{ki}^{\ell} \frac{\mathrm{d}\gamma^k}{\mathrm{d}t}) \frac{\partial}{\partial x"^{\ell}} \Big|_{\gamma(t)}$$
$$J''(0) = (\frac{\partial^2 X^{\ell}}{\partial x^j \partial x^k} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \frac{\mathrm{d}\gamma^k}{\mathrm{d}t} + X^i \frac{\partial \Gamma_{ki}^{\ell}}{\partial x^j} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \frac{\mathrm{d}\gamma^k}{\mathrm{d}t}) \frac{\partial}{\partial x^{\ell}} \Big|_p$$
$$= (\frac{\partial^2 X^{\ell}}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{ki}^{\ell}}{\partial x^j}) \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \frac{\mathrm{d}\gamma^k}{\mathrm{d}t} \frac{\partial}{\partial x^{\ell}} \Big|_p$$
$$= (\frac{\partial^2 X^{\ell}}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^{\ell}}{\partial x^i} + X^i \frac{\partial \Gamma_{ki}^{\ell}}{\partial x^j} - X^i \frac{\partial \Gamma_{jk}^{\ell}}{\partial x^i}) \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \frac{\mathrm{d}\gamma^k}{\mathrm{d}t} \frac{\partial}{\partial x^{\ell}} \Big|_p$$
$$= (\frac{\partial^2 X^{\ell}}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^{\ell}}{\partial x^i}) \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \frac{\mathrm{d}\gamma^k}{\mathrm{d}t} \frac{\partial}{\partial x^{\ell}} \Big|_p - R(X, \gamma')\gamma'$$

which implies

$$\frac{\partial^2 X^{\ell}}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^{\ell}}{\partial x^i} = 0$$

holds at point p, and since p is arbitrary, this completes the proof. □

Corollary 16.2.1. Let (M, g) be a complete Riemannian manifold and $p \in M$. Then a Killing field X is determined by the values X_p and $(\nabla X)_p$ for arbitrary $p \in M$.

Proof. The equation $\mathcal{L}_X g \equiv 0$ is linear in X, so the space of Killing fields is a vector space. Therefore, it suffices to show if $X_p = 0$ and $(\nabla X)_p = 0$, then $X \equiv 0$. For arbitrary $q \in M$, let $\gamma: [0,1] \to M$ be a geodesic connecting p and q with $\gamma'(0) = v$. Since $J(t) = X(\gamma(t))$ is a Jacobi field, and a direct computation shows

$$(\nabla_v X)_p = J'(0)$$

Thus $J(t) \equiv 0$, since Jacobi field is determined by two initial values. In particular, $X_q = J(1) = 0$, and since q is arbitrary, one has $X \equiv 0$. □

Corollary 16.2.2. The dimension of vector space consisting of Killing fields $\leq n(n+1)/2$.

Proof. Note that ∇X is skew-symmetric and the dimension of skew-symmetric matrices is $n(n-1)/2$. Thus, the dimension of vector space consisting of Killing fields $\leq n + n(n-1)/2 = n(n+1)/2$. □

Lemma 16.2.1. Killing field on a complete Riemannian manifold (M, g) is complete.

Proof. For a Killing field X, we need to show the flow $\varphi_t \colon M \to M$ generated by X is defined for $t \in \mathbb{R}$. Otherwise, we assume φ_t is defined on (a, b). Note that for each $p \in M$, curve $\varphi_t(p)$ is a curve defined on (a, b) having finite constant speed, since φ_t is isometry. Then we have $\varphi_t(p)$ can be extended to the one defined on $\mathbb{R}$, since M is complete. □

Theorem 16.2.1. Let (M, g) be a complete Riemannian manifold and $\mathfrak{g}$ the space of Killing fields. Then $\mathfrak{g}$ is isomorphic to the Lie algebra of $G = \mathrm{Iso}(M, g)$.

Proof. It's clear $\mathfrak{g}$ is a Lie algebra since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$. Now let's see it's isomorphic to Lie algebra consisting of Killing field as Lie algebra.

(1) Given a Killing field X, by Lemma 16.2.1, one deduces that the flow $\varphi \colon \mathbb{R} \times M \to M$ generated by X is a one parameter subgroup $\gamma \colon \mathbb{R} \to G$, and $\gamma'(0) \in T_e G$.
(2) Given $v \in T_e G$, consider the one-parameter subgroup $\gamma(t) = \exp(tv) \colon \mathbb{R} \to G$ which gives a flow by

$$\begin{aligned} \varphi \colon \mathbb{R} \times M &\to M \\ (t, p) &\mapsto \exp(tv) \cdot p \end{aligned}$$

Then the vector field X generated by this flow is a Killing field.

This gives a one to one correspondence between Killing fields and Lie algebra of G, and it's a Lie algebra isomorphism. □

Corollary 16.2.3 (Cartan decomposition)**.** Let (M, g) be a complete Riemannian manifold and $G = \mathrm{Iso}(M, g)$ with Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ of G has the following decomposition as vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} \mid X_p = 0\} \\ \mathfrak{m} &= \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}, \end{aligned}$$

and they satisfy

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$$

Proof. The decomposition follows from Corollary 16.2.1 and Theorem 16.2.1, and it's easy to see

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$

For arbitrary $X \in \mathfrak{k}, Y \in \mathfrak{m}$ and $v \in T_p M$, one has

$$\begin{aligned} \nabla_v [X, Y] &= \nabla_v \nabla_X Y - \nabla_v \nabla_Y X \\ &= -R(Y, v)X + \nabla_{\nabla_v X} Y + R(X, v)Y - \nabla_{\nabla_v Y} X \\ &= 0 \end{aligned}$$

since $X_p = 0$ and $(\nabla Y)_p = 0$. This shows $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$. □

16.3. **Cartan-Ambrose-Hicks theorem.**

Theorem 16.3.1 (Cartan-Ambrose-Hicks)**.** Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds, and $\Phi_0 \colon T_pM \to T_{\widetilde{p}}\widetilde{M}$ is a linear isometry, where $p \in M, \widetilde{p} \in \widetilde{M}$. For $0 < \delta < \min\{\mathrm{inj}_p(M), \mathrm{inj}_{\widetilde{p}}(\widetilde{M})\}$, The following statements are equivalent.

(1) There exists an isometry $\varphi \colon B(p, \delta) \to B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p = \Phi_0$.
(2) For $v \in T_pM, |v| < \delta, \gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_{\widetilde{p}}(t\Phi_0(v))$, if we define
$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma} \colon T_{\gamma(t)}M \to T_{\widetilde{\gamma}(t)}\widetilde{M}$$
then Φ_t preserves curvature, that is $(\Phi_t)^*R = R$.

Proof. From (1) to (2). If we can show $\Phi_t = (\mathrm{d}\varphi)_{\gamma(t)}$. Then it's clear that Φ_t preserves curvature, since φ is an isometry. By definition of Φ_t, it suffices to show the following diagram commutes

$$\begin{tikzcd} T_pM \arrow[r, "(\mathrm{d}\varphi)_p"] \arrow[d, "P^{\gamma}_{0,t}"] & T_{\widetilde{p}}\widetilde{M} \arrow[d, "P_{0,t;\widetilde{\gamma}}"] \\ T_{\gamma(t)}M \arrow[r, "(\mathrm{d}\varphi)_{\gamma(t)}"] & T_{\widetilde{\gamma}(t)}\widetilde{M} \end{tikzcd}$$

since $(\mathrm{d}\varphi)_p = \Phi_0$. Note that $\varphi(\gamma(t)) = \widetilde{\gamma}(t)$ since both of them are geodesics, and they and their derivatives agree at $t = 0$. So it's tautological that
$$P^{\varphi\circ\gamma}_{0,t} \circ (\mathrm{d}\varphi)_p(v) = (\mathrm{d}\varphi)_{\gamma(t)} \circ P^{\gamma}_{0,t}(v)$$
where $v = \gamma'(0)$, since
$$P^{\gamma}_{0,t}(v) = \gamma'(t)$$
$$(\mathrm{d}\varphi)_{\gamma(t)}(\gamma'(t)) = (\varphi \circ \gamma)'(t) = P^{\varphi\circ\gamma}_{0,t} \circ (\mathrm{d}\varphi)_p(v)$$
Now consider $w \in T_pM$ which is not parallel to $v = \gamma'(0)$. Since both $(\mathrm{d}\varphi)_{\gamma(t)}$ and parallel transport preserve angles, so $P^{\varphi\circ\gamma}_{0,t} \circ (\mathrm{d}\varphi)_p(w)$ and $(\mathrm{d}\varphi)_{\gamma(t)} \circ P^{\gamma}_{0,t}(w)$ has the same angle with $(\mathrm{d}\varphi)_{\gamma(t)}(\gamma'(t))$, and they have the same length, so they're equal.

From (2) to (1). Define
$$\varphi = \exp_{\widetilde{p}} \circ \Phi_0 \circ \exp_p^{-1}$$
It suffices to show for any $q \in B(p, \delta)$,
$$(\mathrm{d}\varphi)_q \colon T_qM \to T_{\varphi(q)}\widetilde{M}$$
is a linear isometry. For any $w \in T_qM$, by Corollary 12.1.1, there exists a geodesic $\gamma \colon [0,1] \to M$ with $\gamma(0) = p, \gamma(1) = q$ and a Jacobi field J such that $J(0) = 0, J(1) = w$ along γ. Now we claim:

(1) **Claim 1**: $\widetilde{J}(t) = \Phi_t(J(t))$ is a Jacobi field.
(2) **Claim 2**: $\widetilde{J}(1) = (\mathrm{d}\varphi)_q(J(1))$.

From claim 2 we have

$$|(\mathrm{d}\varphi)_q(w)| = |\widetilde{J}(1)| = |J(1)| = |w|$$

since Φ_t preserves length. This completes the proof. Now let's give proofs of these two claims.

(1) **Proof of Claim 1**: Given an orthonormal $\{e_1(0) = \frac{\gamma'(0)}{|\gamma'(0)|}, e_2(0), \dots, e_n(0)\}$ of T_pM and use parallel transport to obtain a parallel frame along γ. With respect to this frame we can write $J(t) = J^i(t)e_i(t)$. Then $\widetilde{J}(t) = J^i(t)\widetilde{e}_i(t)$, where $\widetilde{e}_i(t) = \Phi_t(e_i(t))$. Furthermore, $\widetilde{e}_i(t)$ is also a parallel frame by definition of Φ_t. Then $\widetilde{J}(t)$ is a Jacobi field, since

$$\begin{aligned}
&\frac{\mathrm{d}^2 J^j}{\mathrm{d}t^2} + J^i(t)|\widetilde{\gamma}(t)|^2\widetilde{R}(\widetilde{e}_i(t), \widetilde{e}_1(t), \widetilde{e}_1(t), \widetilde{e}_j(t))\\
=&\frac{\mathrm{d}^2 J^j}{\mathrm{d}t^2} + J^i(t)|\gamma(t)|^2 R(e_i(t), e_1(t), e_1(t), e_j(t))\\
=&0
\end{aligned}$$

holds for arbitrary j, where we use the fact Φ_t preserves the length and curvature, and $J(t)$ is a Jacobi field.

(2) **Proof of Claim 2**: Since $\widetilde{J}(t) = \Phi_t(J(t))$. Then $\widetilde{J}'(0) = \Phi_0 J'(0)$. On the other hand, by Corollary one has

$$\begin{aligned}
J(t) &= (\mathrm{d}\exp_p)_{t\gamma'(0)}(tJ'(0))\\
\widetilde{J}(t) &= (\mathrm{d}\exp_{\widetilde{p}})_{t\widetilde{\gamma}'(0)}(t\widetilde{J}'(0))
\end{aligned}$$

Therefore

$$\begin{aligned}
\widetilde{J}(1) &= (\mathrm{d}\exp_{\widetilde{p}})_{\widetilde{\gamma}'(0)} \circ \Phi_0(J'(0))\\
&= (\mathrm{d}\exp_{\widetilde{p}})_{\widetilde{\gamma}'(0)} \circ \Phi_0 \circ (\mathrm{d}\exp_p)^{-1}_{\gamma'(0)}(J(1))
\end{aligned}$$

which completes the proof of claim 2.

□

17. Riemannian covering

17.1. Riemannian covering.

Definition 17.1.1 (smooth covering). A smooth map $\pi\colon \widetilde{M} \to M$ between smooth manifolds is called a smooth covering if

(1) π is a covering.
(2) π is a local diffeomorphism.

Definition 17.1.2 (Riemannian covering). A smooth map $\pi\colon (\widetilde{M},\widetilde{g}) \to (M,g)$ between Riemannian manifolds is called a Riemannian covering if

(1) π is a smooth covering map.
(2) π is a local isometry.

Example 17.1.1. *Let (M,g) be a Riemannian manifold with smooth covering $\pi\colon \widetilde{M} \to M$. If $\widetilde{M}$ is equipped with pullback metric $\widetilde{g}$, then $\pi\colon (\widetilde{M},\widetilde{g}) \to (M,g)$ is a Riemannian covering.*

Proposition 17.1.1. Let $\pi\colon (\widetilde{M},\widetilde{g}) \to (M,g)$ be a Riemannian universal covering with deck transformation $\Gamma \subseteq \mathrm{Iso}(\widetilde{M},\widetilde{g})$. Then

(1) (M,g) is isometric to $(\widetilde{M}/\Gamma,\widetilde{g})$.
(2) Γ acts on $\widetilde{M}$ isometrically, transitively and properly discontinuous.

Proposition 17.1.2. If $\pi\colon (\widetilde{M},\widetilde{g}) \to (M,g)$ is a Riemannian covering, then M is complete if and only if $\widetilde{M}$ is.

Proposition 17.1.3. Let $(M,g_M),(N,g_N)$ be Riemannian manifolds with M is complete and let $f\colon M \to N$ be a local diffeomorphism such that for all $p \in M$, and for all $v \in T_pM$, one has $|(\mathrm{d}f)_p v| \geq |v|$. Then f is a Riemannian covering map.

Proof. It suffices to show f has path lifting property, that is, for a smooth curve $c\colon [0,1] \to N$ and $p \in M$ such that $f(p) = c(0)$, there exists a curve $\widetilde{c}\colon [0,1] \to M$ such that $f \circ \widetilde{c} = c$ and $\widetilde{c}(0) = p$. Let A be the set of values such that c can be lifted to a curve defined on A. It's clear $A \subseteq [0,1]$ and $A \neq \varnothing$. Now it suffices to show A is both open and closed.

(1) To see it's open: For $x \in A$, note that f is a local diffeomorphism at $c(x)$, so there exists an open interval I of x such that c can be lifted on U, and thus A is open.
(2) To see it's closed: It suffices to show if an increasing sequence $\{x_i\}_{i=1}^{\infty} \subseteq A$ converging to x, then one has $x \in A$. Firstly we claim $\{\widetilde{c}(x_i)\}_{i=1}^{\infty}$ is contained in a compact set K. If not, the distance from $\widetilde{c}(x_i)$ to $\widetilde{c}(0)$ can be

arbitrary large since M is complete. However,

$$\begin{aligned} l_{0,x_i}(c) &= \int_0^{x_i} |\frac{\mathrm{d}c}{\mathrm{d}t}|\mathrm{d}t \\ &= \int_0^{x_i} |(\mathrm{d}f)_{\overline{c}(t)}\frac{\mathrm{d}\overline{c}}{\mathrm{d}t}|\mathrm{d}t \\ &\geq \int_0^{x_i} |\frac{\mathrm{d}\overline{c}}{\mathrm{d}t}|\mathrm{d}t \\ &\geq \mathrm{dist}(\widetilde{c}(x_i), \widetilde{c}(0)) \end{aligned}$$

implies the length of c between $\widetilde{c}(x_i)$ and $\widetilde{c}(0)$ is bounded, a contradiction. Since $\{\widetilde{c}(x_i)\}_{i=1}^{\infty}$ is contained in a compact set K, there exists a converging subsequence, still denoted by $\{\widetilde{c}(x_i)\}_{i=1}^{\infty}$ which converges to $r \in M$, and by continuity one has $f(r) = c(x)$. Let V be a neighborhood of r such that $f|_V$ is a diffeomorphism. Then one has $c(x) \in f(V)$, and by continuity there exists an open interval I of x such that $c(I) \subseteq f(V)$. Pick an index n such that $\widetilde{c}(x_n) \in V$, there is a lifting of segment $c\colon [x_n, x] \to M$ since $f|_V$ is a diffeomorphism, which implies c can be lifted to a curve defined on $[0, x]$, that is, $x \in A$.

□

Corollary 17.1.1. Let (M, g_M) be a complete Riemannian manifold and $f\colon (M, g_M) \to (N, g_N)$ be a local isometry. Then f is a Riemannian covering map.

18. Topology of non-positive sectional curvature manifold

18.1. **Cartan-Hadamard manifold.**

Definition 18.1.1 (Cartan-Hadamard manifold)**.** A simply-connected, complete Riemannian manifold with non-positive sectional curvature is called Cartan-Hadamard manifold.

18.1.1. *Expansion property of exponential map.*

Theorem 18.1.1. Let (M,g) be a simply-connected complete Riemannian manifold. The following statements are equivalent.

(1) M is Cartan-Hadamard manifold.
(2) For any $p \in M$ and $v,w \in T_pM$, we have
$$|(\mathrm{d}\exp_p)_v w| \geq |w|$$
(3) For any $p \in M, T > 0$ and $v,w \in T_pM$, we have
$$|v-w| \leq \frac{\operatorname{dist}(\exp_p(tv), \exp_p(tw))}{t}$$
holds for arbitrary $t > 0$.

Proof. From (1) to (2). For all $p \in M$ and $v,w \in T_pM$, $J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$ is a Jacobi field along $\exp_p(tv)$. If M has non-positive sectional curvature, direct computation shows
$$|J(t)|'' = \frac{|J^2||J'|^2 - \langle J, J'\rangle^2}{|J|^3} - \frac{R(J,\gamma',\gamma',J)}{|J|} \geq 0$$
for all $t > 0$. Thus consider
$$f(t) = |J(t)| - t|w|$$
It's clear $f''(t) \geq 0$ and $f'(0) = 0$, and thus $f(t) \geq 0$ for all $t > 0$ since $f(0) = 0$. In particular, set $t = 1$ we have
$$|(\mathrm{d}\exp_p)_v(w)| - |w| \geq 0$$

From (2) to (1). If M has sectional curvature $K(\sigma) > 0$ at $p \in M$, where σ is the plane spanned by v,w with $|v| = |w| = 1$. Then consider geodesic $\exp_p(tv)$ and Jacobi field
$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$
along it. Then by Proposition 14.1.1 we have $|J(t)|'' < 0$ for sufficiently small t. If we set $f(t) = |J(t)| - t|w|$. Then we can see $f(0) = 0, f'(0) = 0$ and $f''(0) < 0$ for sufficiently small t. In particular, we have
$$|(\mathrm{d}\exp_p)_{\varepsilon v}(\varepsilon w)| - |\varepsilon w| = f(\varepsilon) < 0$$
where $\varepsilon > 0$ is sufficiently small. This leads to a contradiction.

From (2) to (3). For arbitrary $t > 0$. Let $\gamma(s) : [0,1] \to M$ be a geodesic connecting $\exp_p(tv), \exp_p(tw)$ and choose a curve $v(s) \in T_pM$ such that
$$\exp_p(v(s)) = \gamma(s)$$

for all $s \in [0,1]$. Hence, $v(0) = tv, v(1) = tw$. Then

$$\begin{aligned}
\mathrm{dist}(\exp_p(tv), \exp_p(tw)) &= \int_0^1 |\gamma'(s)| \mathrm{d}s \\
&= \int_0^1 |(\mathrm{d}\exp_p)_{v(s)}(v'(s))| \mathrm{d}s \\
&\geq \left| \int_0^1 v'(s) \mathrm{d}s \right| \\
&= t|v - w|
\end{aligned}$$

This shows

$$|v - w| \leq \frac{\mathrm{dist}(\exp_p(tv), \exp_p(tw))}{t}$$

holds for arbitrary $t > 0$.

From (3) to (2). Note that

$$\begin{aligned}
|(\mathrm{d}\exp_p)_v(w)| &= \lim_{t\to 0} \frac{\mathrm{dist}(\exp_p(v + tw), \exp_p(v))}{t} \\
&= \lim_{t\to 0} \frac{\mathrm{dist}(\exp_p(tv' + tw), \exp_p(tv'))}{t} \\
&\geq |v' + w - v'| \\
&= |w|
\end{aligned}$$

□

Remark 18.1.1. This shows the exponential map of simply-connected complete Riemannian manifold with non-positive sectional curvature has expansion" property.

Corollary 18.1.1. Let (M, g) be a Cartan-Hadamard manifold with $a, b, c \in M$. Such points determine a unique geodesic triangle T with vertices a, b, c. Let α, β, γ be the angles of the vertices a, b, c respectively, and let A, B, C be the lengths of the side opposite the vertices a, b, c respectively. Then

(1) $A^2 + B^2 - 2AB\cos\gamma \leq C^2 (< C^2, \text{if } K < 0)$.
(2) $\alpha + \beta + \gamma \leq \pi (< \pi, \text{if } K < 0)$

Proof. See Lemma 3.1 in Page259 of [Car92]. □

18.1.2. *Complete Riemannian manifold with non-positive sectional curvature is* $K(G,1)$.

Lemma 18.1.1. If (M, g) is a complete Riemannian manifold with sectional curvature $K \leq 0$, then for any $p \in M$, $\mathrm{conj}(p) = \varnothing$. In particular, $\exp_p : T_pM \to M$ is a local diffeomorphism.

Proof. Suppose q is conjugate to p along $\gamma : [0,1] \to M$, and without lose of generality we may assume there is no conjugate point for $t \in (0,1)$. Let J be a

Jacobi field along γ with $J(0) = J(1) = 0$. Then

$$\begin{aligned}(\frac{1}{2}|J|^2)' &= (g(J',J))' \\ &= g(J'',J) + g(J',J') \\ &= -R(J,\gamma',\gamma',J) + |J'|^2 \\ &\geq |J'|^2\end{aligned}$$

Since $J'(0) \neq 0$, we have

$$\begin{aligned}g(J',J)(t) &\geq \int_0^t |J'|^2 + g(J'(0),J(0)) \\ &= \int_0^t |J'|^2 \\ &> 0\end{aligned}$$

which implies $(\frac{1}{2}|J|^2)' = g(J',J) > 0$, a contradiction to $J(1) = 0$. □

Theorem 18.1.2 (Cartan-Hadamard)**.** If (M,g) is a complete Riemannian manifold with sectional curvature $K \leq 0$, then $\exp_p : T_pM \to M$ is a covering map.

Proof. Lemma 18.1.1, together with Proposition 17.1.3 and Theorem 18.1.1 completes the proof. □

Corollary 18.1.2. Cartan-Hadamard manifold is diffeomorphic to $\mathbb{R}^n$.

Corollary 18.1.3. If (M,g) is a complete Riemannian manifold with $K \leq 0$, then $\pi_k(M) = 0, k \geq 2$, that is M is $K(\pi_1(M),1)$.

Remark 18.1.2. Theory in topology says if a finite dimensional CW-complex is a $K(G,1)$ space, then its fundamental group is torsion-free. In particular, if M is a complete Riemannian manifold with $K \leq 0$, then $\pi(M)$ is torsion-free. This fact can be proved later by tools of Riemannian manifold, called Cartan's torsion-free theorem.

Corollary 18.1.4. If M and N are two compact Riemannian manifold and one of them is simply-connected, then $M \times N$ has no metric with non-positive sectional curvature.

Proof. If both of M and N are simply-connected, and $M \times N$ admits a metric with non-positive sectional curvature, then it's diffeomorphic to $\mathbb{R}^n$ for some positive integer n, a contradiction to compactness.

Without lose of generality, we assume M is simply-connected and N is not simply-connected with universal covering $\widetilde{N}$. Then there is a universal covering

$$\pi : M \times \widetilde{N} \to M \times N$$

If $M \times N$ admits a Riemannian metric g with non-positive sectional curvature, then $\pi^* g$ is a complete metric of non-positive sectional curvature on $M \times \widetilde{N}$, so we have $M\times\widetilde{N}$ is diffeomorphic to $\mathbb{R}^n$ for some n. M is orientable since it's simply-connected, which implies $H^m(M)=\mathbb{Z}$, where $m=\dim M$. By Künneth formula $H^m(M\times\widetilde{N})\neq 0$, a contradiction to Poincaré lemma. □

Remark 18.1.3. The condition simply-connected is crucial, for example $S^1\times S^1$ admits a metric with flat curvature.

18.2. **Cartan's torsion-free theorem.**

Lemma 18.2.1. Let (M,g) be a Cartan-Hadamard manifold and $p\in M$, $v\in T_pM$. For all $q\in M$, one has

$$2\operatorname{dist}(p,q)^2+\operatorname{dist}(p_0,p)^2+\operatorname{dist}(p_1,p)^2\leq \operatorname{dist}(p_0,q)^2+\operatorname{dist}(p_1,q)^2$$

where $p_0=\exp_p(-v)$ and $p_1=\exp_p(v)$.

Proof. Since $\exp_p: T_pM\to M$ is a diffeomorphism, there exists $w\in T_pM$ such that $q=\exp_p(w)$ with $\operatorname{dist}(p,q)=|w|$. Direct computation shows

$$\begin{aligned}
&\operatorname{dist}(p_0,q)=\operatorname{dist}(\exp_p(-v),\exp_p(w))\geq|w+v|\\
&\operatorname{dist}(p_1,q)=|w-v|\\
&\operatorname{dist}(p,q)^2=|w|^2\\
&\qquad\qquad\quad=\frac{|w+v|^2+|w-v|^2}{2}-|v|^2\\
&\qquad\qquad\quad\leq\frac{\operatorname{dist}(p_0,q)^2+\operatorname{dist}(p_1,q)^2}{2}-\frac{\operatorname{dist}(p_0,p)^2+\operatorname{dist}(p_1,p)^2}{2}
\end{aligned}$$

□

Lemma 18.2.2 (Serre)**.** Let (M,g) be a Cartan-Hadamard manifold, $p\in M$ and $B(p,r)$ is the closed ball of radius r. If $\varnothing\neq\Omega\subseteq M$ is a bounded set with

$$r_\Omega=\inf\{r>0\mid \Omega\subseteq B(p,r),p\in M\}$$

then there exists a unique $p_\Omega\in M$ such that $\Omega\subseteq B(p_\Omega,r_\Omega)$.

Proof. Choose a bounded sequence $\{r_i>r_\Omega\}$ converging to r_Ω and $p_i\in M$ such that

$$\Omega\subseteq B(p_i,r_i)$$

For fixed $q\in\Omega$, one has $\operatorname{dist}(q,p_i)\leq r_i$ for each i, and thus $\{p_i\}$ is bounded since $\{r_i\}$ is bounded. Since M is complete, then $\{p_i\}$ has a convergent subsequence, and the limit of this convergent subsequence is p_Ω, which gives the existence of p_Ω. Suppose $p_1,p_2\in M$ are two points such that

$$\Omega\subseteq B(p_0,r_\Omega)\cap B(p_1,r_\Omega)$$

Since $\exp_{p_0}$ is a diffeomorphism, there exists a unique v_0 such that $p_1=\exp_{p_0}(v_0)$. By Lemma 18.2.1, if we denote $p=\exp_{p_0}(v_0/2)$, then for all $q\in\Omega$ one has

$$\begin{aligned}
\operatorname{dist}(p,q)^2&\leq\frac{\operatorname{dist}(p_0,q)^2+\operatorname{dist}(p_1,q)^2}{2}-\frac{\operatorname{dist}(p_0,p)^2+\operatorname{dist}(p_1,p)}{2}\\
&\leq r_\Omega^2-\frac{\operatorname{dist}(p_0,p_1)^2}{4}
\end{aligned}$$

since $\mathrm{dist}(p_0,p)=\mathrm{dist}(p_1,p)=\mathrm{dist}(p_0,p_1)/2$. By the definition of r_Ω, one has $\mathrm{dist}(p_0,p_1)=0$, and thus $p_0=p_1$. $\square$

Theorem 18.2.1 (Cartan's fixed point theorem)**.** Let (M,g) be a Cartan-Hadamard manifold and G is a compact Lie group acting on M isometrically. Then G has a fixed point.

Proof. For $p\in M$, suppose Ω is the orbit of p, that is

$$\Omega=\{gp\mid g\in G\}$$

It's a bounded since M is compact, and

$$\Omega=g\Omega\subseteq B(gp_\Omega,r_\Omega)$$

since Ω is the orbit. Then by uniqueness of p_Ω, one has p_Ω is a fixed point of G. $\square$

Corollary 18.2.1. If (M,g) is a complete Riemannian manifold with $K\le 0$, then $\pi_1(M)$ is torsion-free.

Proof. Let $(\widetilde{M},\widetilde{g})$ be the universal covering of M with pullback metric. Then $(\widetilde{M},\widetilde{g})$ is a Cartan-Hadamard manifold, and (M,g) is isometric to $(\widetilde{M}/\Gamma,\widetilde{g})$, where $\Gamma\subseteq\mathrm{Iso}(\widetilde{M},\widetilde{g})$ is isomorphic to $\pi_1(M)$ which acts on $\widetilde{M}$ freely.

Now it suffices to show Γ has no torsion element. If not, suppose φ is a torsion element, then consider the finite group G generated by φ, which is a 0-dimension Lie group if it's equipped with discrete topology. By Cartan's fixed point theorem there exists a fixed point of G, which implies φ is identity, since Γ acts on $\widetilde{M}$ freely. $\square$

18.3. **Preissmann's Theorem.**

Definition 18.3.1 (axis)**.** Let (M,g) be a complete Riemannian manifold and $\varphi\colon M\to M$ be an isometry. A non-trivial geodesic $\gamma\colon\mathbb{R}\to M$ is called an axis of φ if there exists $0\neq c\in\mathbb{R}$ such that

$$\varphi(\gamma(t))=\gamma(t+c)$$

Definition 18.3.2 (axial)**.** An isometry without fixed points which has an axis is said to be axial.

Lemma 18.3.1. Let (M,g) be a complete Riemannian manifold and $\varphi\colon M\to M$ be an isometry. If $\delta_\varphi(p)=\mathrm{dist}(p,\varphi(p))$ has a positive minimum, then φ has an axis.

Proof. Suppose δ_φ attains its minimum at $p\in M$ and $\gamma(t):[0,1]\to M$ is a minimum geodesic connecting p and $\varphi(p)$. Then $\varphi\circ\gamma\colon[0,1]\to M$ is also a geodesic connecting $\varphi(p)$ and $\varphi^2(p)$, since φ is an isometry. We claim these two geodesics form an angle π at point $\varphi(p)$ and thus fit together an extension of γ to $[0,2]$. Indeed, for any $t\in[0,1]$, one has

$$\begin{aligned}\delta_\varphi(p)&=\mathrm{dist}(p,\varphi(p))\\&\leq\delta_\varphi(\gamma(t))\\&=\mathrm{dist}(\gamma(t),\varphi\circ\gamma(t))\\&\leq\mathrm{dist}(\gamma(t),\gamma(1))+\mathrm{dist}(\gamma(1),\varphi\circ\gamma(t))\\&=\mathrm{dist}(\gamma(t),\gamma(1))+\mathrm{dist}(\varphi\circ\gamma(0),\varphi\circ\gamma(t))\\&=\mathrm{dist}(\gamma(t),\gamma(1))+\mathrm{dist}(\gamma(0),\gamma(t))\\&=\delta_\varphi(p)\end{aligned}$$

Thus we have $\varphi(\gamma(t))=\gamma(1+t)$ for $0\leq t\leq 1$. Repeating this argument to obtain a geodesic $\gamma\colon\mathbb{R}\to M$ with period 1, and it's an axis for φ. □

Lemma 18.3.2. Let $\pi\colon\widetilde{M}\to M$ be the universal covering and φ be a non-trivial deck transformation.

(1) Let $\widetilde{\gamma}_0,\widetilde{\gamma}_1\colon[0,1]\to\widetilde{M}$ be curves connecting $\widetilde{x}_0,\varphi(\widetilde{x}_0)$ and $\widetilde{x}_1,\varphi(\widetilde{x}_1)$ respectively. Then $\pi\circ\widetilde{\gamma}_1$ is free homotopic to $\pi\circ\widetilde{\gamma}_2$.
(2) Let $\widetilde{\gamma}_0\colon[0,1]\to\widetilde{M}$ be a path connecting $\widetilde{x}_0$ and $\varphi(\widetilde{x}_0)$ and $\gamma_0=\pi\circ\widetilde{\gamma}_0$. Then any loop γ_1 that is freely homotopic to γ_0 must be of the form $\gamma_1=\pi\circ\widetilde{\gamma}_1$, where $\widetilde{\gamma}_1\colon[0,1]\to\widetilde{M}$ has property $\varphi(\widetilde{\gamma}_1(0))=\widetilde{\gamma}_1(1)$.

Proof. For (1). Consider the path $\widetilde{H}(s,0)\colon[0,1]\to\widetilde{M}$ with $\widetilde{H}(0,0)=\widetilde{x}_0$ and $\widetilde{H}(0,1)=\widetilde{x}_1$ and define

$$\begin{aligned}\widetilde{H}(s,1)&=\varphi(\widetilde{H}(s,0))\\\widetilde{H}(0,t)&=\widetilde{\gamma}_0(t)\\\widetilde{H}(1,t)&=\widetilde{\gamma}_1(t)\end{aligned}$$

This defines $\widetilde{H}$ on the boundary of $[0,1]\times[0,1]$, and it can be extended a continuous $\widetilde{H}\colon[0,1]\times[0,1]\to\widetilde{M}$ since $\widetilde{M}$ is simply-connected. Therefore, $H=\pi\circ\widetilde{H}$ gives a free homotopy between $\pi\circ\widetilde{\gamma}_1$ and $\pi\circ\widetilde{\gamma}_2$.

For (2). Let $H\colon[0,1]\times[0,1]\to M$ be a free homotopy from γ_0 to γ_1 and $\widetilde{H}\colon[0,1]\times[0,1]$ be its lift such that $\widetilde{H}(0,0)=\widetilde{x}_0$. The unique homotopy lifting property implies $\widetilde{\gamma}_0(t)=\widetilde{H}(0,t)$. Note that both $\widetilde{H}(s,0)$ and $\widetilde{H}(s,1)$ are lifts of the same curve $H(s,0)$, but on the other hand, $\varphi\circ\widetilde{H}(s,0)$ is also a lift of $H(s,0)$ and $\varphi\circ\widetilde{H}(0,0)=\widetilde{H}(0,1)$. Then by unique homotopy lifting property again one has $\varphi\circ\widetilde{H}(s,0)=\widetilde{H}(s,1)$. Now we define $\widetilde{\gamma}_1(t)=\widetilde{H}(1,t)$. Then

$$\begin{aligned}\widetilde{\gamma}_1(0)&=\widetilde{H}(1,0)\\\widetilde{\gamma}_1(1)&=\widetilde{H}(1,1)=\varphi\circ\widetilde{H}(1,0)=\varphi(\widetilde{\gamma}_1(0))\end{aligned}$$

This completes the proof. □

Proposition 18.3.1. Let (M,g) be a compact Riemannian manifold with universal covering $\widetilde{M}$. If φ is a non-trivial deck transformation, then

(1) δ_φ has a positive minimum $\geq 2\,\mathrm{inj}(M)$, and thus φ has an axis $\gamma\colon\mathbb{R}\to\widetilde{M}$.
(2) $\pi\circ\gamma$ is a closed geodesic in M whose length is minimal in its homotopy class.

Proof. For (1). By (1) of Lemma 18.3.2, if φ is a non-trivial deck transformation, then none of loops obtained by projecting path connecting $\widetilde{x}_0$ and $\varphi(\widetilde{x}_0)$ can be homotopically trivial. This implies $\delta_\varphi(\widetilde{x}) \geq 2\operatorname{inj}_{\pi(\widetilde{x})} M$, as otherwise the minimal geodesic from $\widetilde{x}_0$ to $\varphi(\widetilde{x}_0)$ would generate a loop of length $< 2\operatorname{inj}_{\pi(\widetilde{x})} M$, which are contractible as they lie in geodesical ball.

For (2). Let $\{\widetilde{x}_i\}$ be a sequence in $\widetilde{M}$ such that $\lim \delta_\varphi(\widetilde{x}_i) = \inf \delta_\varphi$ and $\{\widetilde{\gamma}_i\}$ be a sequence of minimal geodesics such that $\widetilde{\gamma}_i(0) = \widetilde{x}_i$ and $\widetilde{\gamma}_i(1) = \varphi(\widetilde{x}_i)$. Then $\{\gamma_i := \pi \circ \widetilde{\gamma}_i\}$ is a sequence of loops in M. Since $|\widetilde{\gamma}_i'| = |\gamma_i'| = \delta_\varphi(\widetilde{x}_i)$, the compactness of M implies that we may assume $x = \lim \gamma_i(0)$ and $v = \lim \gamma_i'(0) \in T_q M$ with $|v| = \inf \delta_\varphi$ by passing subsequences. □

Lemma 18.3.3. Let (M, g) be a Cartan-Hadamard manifold with negative sectional curvature. If isometry $\varphi: M \to M$ has an axis, then it's unique up to reparametrization.

Proof. Suppose $\gamma_1, \gamma_2 : \mathbb{R} \to M$ are two axes of φ, without lose of generality we may assume

$$\varphi(\gamma_1(t)) = \gamma_1(t+1)$$
$$\varphi(\gamma_2(t)) = \gamma_2(t+1)$$

If γ_1, γ_2 do not intersect, then points $A = \gamma_1(0), B = \gamma_1(1) = \varphi(A), C = \gamma_2(0)$ and $D = \gamma_2(1) = \varphi(C)$ are all distinct. Let γ be a geodesic from A to C. Then $\varphi \circ \gamma$ is the geodesic from B to D. Furthermore, the geodesic quadrilateral $ABCD$ has angle sum 2π, since φ preserves angles. However, according to Corollary 18.1.1, triangle $\triangle ABC$ and $\triangle BCD$ have angle sum strictly less than π, and

$$\angle ACD \leq \angle ACB + \angle BCD$$
$$\angle ABD \leq \angle ABC + \angle CBD$$

thus the angle sum of $ABCD$ is strictly less than 2π, a contradiction. Hence, γ_1 and γ_2 must intersect at some point $p = \gamma_1(t_1) = \gamma_2(t_2)$. Then

$$\begin{aligned} \varphi(p) = \varphi(\gamma_1(t_1)) &= \gamma_1(t_1+1) \\ = \varphi(\gamma_2(t_2)) &= \gamma_2(t_2+1) \end{aligned}$$

is another intersection point. Since (M, g) is a Cartan-Hadamard manifold, any two points are joined by a unique geodesic, and thus γ_1 is a reparametrization of γ_2. □

Lemma 18.3.4. If H is an additive subgroup of $\mathbb{R}$, then either H is dense in $\mathbb{R}$ or $H \cong \mathbb{Z}$.

Proof. Let H be an additive subgroup of $\mathbb{R}$. It's clear $H \cap \mathbb{R}_{>0} \neq \varnothing$, and we consider

$$b := \inf\{h \in H \cap \mathbb{R}_{>0}\}$$

(1) If $b > 0$: Let $h \in H$ and $k \in \mathbb{Z}$ such that

$$kb \leq |h| < (k+1)b$$

then we have $|h| - kb \in H$, and $0 \leq |h| - kb < (k+1)b - kb = b$. By the choice of b, we have $|h| - kb$, which implies $h = \pm kb$. In this case $H = b\,\mathbb{Z}$.

(2) If $b = 0$: For arbitrary $r \in \mathbb{R}_{\geq 0}$ and $\varepsilon > 0$, there exists $h \in H \cap (0, \varepsilon]$ since $b = 0$ and $k \in \mathbb{N}$ such that

$$kh \leq r \leq (k+1)h$$

Thus

$$0 \leq r - kh \leq (k+1)h - kh = h \leq \varepsilon$$

which implies $|r - kh| \leq \varepsilon$, that is H is dense in $\mathbb{R}_{\geq 0}$. For the same argument you can show H is also dense in $\mathbb{R}_{\leq 0}$.

□

Theorem 18.3.1 (Preissmann)**.** If (M, g) is a compact Riemannian manifold with negative sectional curvature, then any non-trivial Abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of M equipped with pullback metric. Then it's a Cartan-Hadamard manifold with negative sectional curvature. Now it suffices to show every non-trivial Abelian subgroup H of group consisting of deck transformations is isomorphic to $\mathbb{Z}$. Let φ be a non-trivial deck transformation in H. Then Proposition 18.3.1, φ has an axis $\gamma\colon \mathbb{R} \to \widetilde{M}$, that is there exists $c \neq 0$ such that

$$\varphi \circ \gamma(t) = \gamma(t + c)$$

for all $t \in \mathbb{R}$. If ψ is another non-trivial element of H, then for any $t \in \mathbb{R}$ we have

$$\varphi \circ \psi(\gamma(t)) = \psi \circ \varphi(\gamma(t)) = \psi \circ \gamma(t + c)$$

which implies $\psi \circ \gamma$ is also an axis of φ. So by Lemma 18.3.3 we have $\psi \circ \gamma$ is a reparametrization of γ. Furthermore, $\psi \circ \gamma$ and γ have the same speed since ψ is an isometry, and thus there are two cases:

(1) $\psi \circ \gamma(t) = \gamma(t + a)$.
(2) $\psi \circ \gamma(t) = \gamma(-t + a)$

(2) can't happen, otherwise $\psi \circ \gamma(\frac{a}{2}) = \gamma(\frac{a}{2})$, contradicts to deck transformation acts on $\widetilde{M}$ freely. Consider

$$f\colon H \to \mathbb{R}$$
$$\psi \mapsto a$$

where a is determined $\psi \circ \gamma(t) = \gamma(t + a)$. It's easy to see F is a group homomorphism with trivial kernel thus $F(H)$ is an additive subgroup of $\mathbb{R}$. Consider

$$b := \inf\{h \in F(H) \cap \mathbb{R}_{>0}\}$$

By Lemma 18.3.4, it suffices to show $b > 0$. If $b = 0$, then there exist $a \in (0, \mathrm{inj}(M))$ and $\psi \in H$ such that $a = F(\psi)$, that is

$$\psi \circ \gamma(t) = \gamma(t + a)$$

Since $\pi \circ \psi = \pi$, we have $\pi \circ \gamma(t) = \pi \circ \gamma(t + a)$. Set $t = 0$ one has

$$\pi \circ \gamma(a) = \pi \circ \gamma(0)$$

A contradiction to $0 < a < \mathrm{inj}(M)$ since $\pi \circ \gamma$ is a geodesic. □

Corollary 18.3.1. Suppose M and N are compact smooth manifolds. Then $M \times N$ doesn't admit a Riemannian metric with negative sectional curvature.

Proof. If $M \times N$ admits a Riemannian metric with negative sectional curvature, Cartan's torsion-free theorem implies $\pi_1(M \times N)$ is torsion-free, and thus for arbitrary $\alpha \in \pi_1(M), \beta \in \pi_1(N)$, unless either M or N is simply-connected, $\pi_1(M \times N)$ will contain an Abelian subgroup $\mathbb{Z} \times \mathbb{Z}$ generated by α, β, which contradicts to Preissmann's theorem.

Without lose of generality, we assume M is simply-connected. Then consider the universal covering $M \times \widetilde{N}$ of $M \times N$, Cartan-Hadamard's theorem implies it's diffeomorphic to $\mathbb{R}^n$ for $n \in \mathbb{Z}_{>0}$, but M is orientable since it's simply-connected, so $H^m(M) = \mathbb{Z}$ where $m = \dim M$. So by Künneth formula $H^n(M \times \widetilde{N}) \neq 0$, a contradiction to Poincaré lemma. □

Lemma 18.3.5. Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature and $\widetilde{M}$ be the universal covering. If $\gamma\colon \mathbb{R} \to \widetilde{M}$ is a common axis for all deck transformations, then M is not compact.

Proof. For any point $\widetilde{x} = \widetilde{\gamma}(s_0) \in \widetilde{M}$ and a real number $k > 0$, consider the unit-speed geodesic $\widetilde{\beta}\colon [0,k] \to \widetilde{M}$ such that

$$\widetilde{\beta}(0) = \widetilde{x}$$
$$\langle \widetilde{\beta}'(0), \widetilde{\gamma}'(s_0) \rangle = 0$$

Let $\beta = \pi \circ \beta, \gamma = \pi \circ \widetilde{\gamma}$ and $x = \pi(\widetilde{x})$. If α_k is a minimizing geodesic in M connecting $\beta(k)$ and $\beta(0)$, then $\ell(\alpha_k) \leq \ell(\beta) = \ell(\widetilde{\beta}) = k$. Now we're going to show $\ell(\alpha_k) = k$, and since k can be arbitrarily large, one has M is unbounded, and thus non-compact.

Let $\widetilde{\alpha}_k$ be the lift of α_k starting from $\widetilde{\beta}(k)$ and φ be the deck transformation given by the loop $\alpha_k^{-1}\beta$. Since $\widetilde{\gamma}$ is a common axis, one has

$$\varphi(\widetilde{x}) = \varphi(\widetilde{\gamma}(s_0)) = \widetilde{\gamma}(s_0 + c)$$

Since the sectional curvature is non-positive, by Corollary 18.1.1 one has

$$\ell(\widetilde{\alpha}_k) \geq \ell(\widetilde{\beta})$$

On the other hand, one has

$$\ell(\widetilde{\alpha}_k) = \ell(\alpha_k) \leq k$$

Hence we deduce $\ell(\widetilde{\alpha}_k) = k$, which completes the proof. □

Theorem 18.3.2 (Preissmann)**.** If (M, g) is a compact Riemannian manifold with negative sectional curvature, then $\pi_1(M)$ is not Abelian.

Proof. Suppose $\pi_1(M)$ is Abelian and γ is the axis of some deck transformation. Then it's the axis of all deck transformations since $\pi_1(M)$ is Abelian, which implies M is non-compact by Lemma 18.3.5, a contradiction. □

18.4. **Other facts.**

Theorem 18.4.1 (Byers)**.** If (M,g) is a compact Riemannian manifold with negative sectional curvature, then any non-trivial solvable subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

Theorem 18.4.2 (Yau)**.** Let (M,g) be a compact Riemannian manifold with non-positive sectional curvature. If $\pi_1(M)$ is solvable, then M is flat.

Theorem 18.4.3 (Farrell-Jones)**.** Let $(M_i,g_i), i=1,2$ be two compact Riemannian manifolds with non-positive sectional curvature. If $\pi_1(M_1)=\pi_1(M_2)$, then M_1 and M_2 are homeomorphic.

19. Topology of positive curvature manifold

19.1. Myers' theorem.

Theorem 19.1.1 (Myers)**.** Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g) \geq \frac{n-1}{R^2} g$. Then $\mathrm{diam}(M) \leq \pi R$, and thus M is compact.

Proof. If $\mathrm{diam}(M) > \pi R$, then there exists $b > \pi R$ and a minimal geodesic $\gamma\colon [0,b] \to M$ of unit-speed, since M is complete. Let $\{e_1(t), \dots, e_n(t)\}$ be a parallel orthonormal frame along γ with $e_1(t) = \gamma'(t)$, and for each $i = 2, \dots, n$, we define

$$V_i(t) = \sin(\frac{\pi t}{b}) e_i(t)$$

It's clear $V_i(0) = V_i(b) = 0$ for $2 \leq i \leq n$. Note that

$$\begin{aligned} I_\gamma(V_i, V_i) &= \int_0^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i \rangle \mathrm{d}t - \int_0^b R(V_i, \gamma', \gamma', V_i) \mathrm{d}t \\ &= \int_0^b \sin^2(\frac{\pi t}{b}) \Big\{ (\frac{\pi}{b})^2 - R(e_i, e_1, e_1, e_i) \Big\} \mathrm{d}t \end{aligned}$$

Thus

$$\begin{aligned} I_\gamma(\sum_{i=2}^n V_i, \sum_{i=2}^n V_i) &= \int_0^b \sin^2(\frac{\pi t}{b}) \left\{ (n-1)(\frac{\pi}{b})^2 - \sum_{i=2}^n R(e_i, e_1, e_1, e_i) \right\} \mathrm{d}t \\ &\leq \int_0^b \sin^2(\frac{\pi t}{b}) \left\{ (n-1)(\frac{\pi}{b})^2 - \frac{(n-1)}{R^2} \right\} \mathrm{d}t \end{aligned}$$

Since γ is a minimal geodesic, one has I_γ is semi-positive definite, which implies $b \leq \pi R$. □

Remark 19.1.1. The estimate for the diameter given by Myers' theorem can't be improved. Indeed, the unit sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ has constant sectional curvature $K = 1$ and $\mathrm{diam}(\mathbb{S}^n) = \pi$. Furthermore, there is a rigidity theorem: Let (M,g) be a complete Riemannian n-manifold, $\mathrm{Ric}(g) \geq \frac{n-1}{R^2} g$ and $\mathrm{diam}(M) = \pi R$. Then (M,g) is isometric to sphere $\mathbb{S}^n(R)$ with standard metric, that's Cheng's theorem, we will see it in Theorem 22.4.2.

Corollary 19.1.1. Let M be a complete Riemannian manifold with positive Ricci curvature. Then the fundamental group $\pi_1(M)$ is finite.

Proof. Let $\widetilde{M}$ be the universal covering of M equipped with pullback metric $\widetilde{g}$. It's clear $(\widetilde{M}, \widetilde{g})$ is a complete Riemannian manifold with positive Ricci curvature. By Myers' theorem $\widetilde{M}$ is compact, which implies $\pi\colon \widetilde{M} \to M$ is a finite covering, that is $\pi_1(M)$ is finite, since $|\pi_1(M)|$ equals the number of sheets of covering. □

19.2. Synge's theorem.

Lemma 19.2.1. Let A be an orthogonal linear transformation of $\mathbb{R}^{n-1}$ and suppose $\det A = (-1)^n$. Then 1 is an eigenvalue of A.

Proof.

(1) If n is even, then $\det(\lambda I - A)$ is a polynomial of odd degree, therefore A has at least a real eigenvalue, and it must be ± 1 since A is orthogonal. Furthermore, since $\det A = 1$ and the product of complex eigenvalue is positive, there is at least a real eigenvalue which equals 1.
(2) If n is odd, then $\det A = -1$. Because the product of complex eigenvalue is positive, there are at least two real eigenvalues, and one of them is 1.

□

Theorem 19.2.1 (Synge)**.** Let (M,g) be a compact Riemannian manifold with positive sectional curvature. Then

(1) If $\dim M$ is even and orientable, then M is simply-connected.
(2) If $\dim M$ is odd, then M is orientable.

Proof. Let $(\widetilde{M},\widetilde{g})$ be the universal covering of M equipped with pullback metric, and $\widetilde{M}$ is equipped with pullback orientation if $\dim M$ is even, otherwise equipped with arbitrary orientation. Suppose the conclusions are not correct, and thus $\pi_1(M)$ is non-trivial. Choose a non-trivial deck transformation $f\colon \widetilde{M} \to \widetilde{M}$ such that

(1) If $\dim M$ is even, F is orientation preserving.
(2) If $\dim M$ is odd, F is orientation reserving.

By Lemma, there exists an axis $\widetilde{\gamma}\colon \mathbb{R} \to \widetilde{M}$ for F and $\gamma = \pi \circ \widetilde{\gamma}$ is a closed geodesic in M that minimizes the length in $[\gamma]$,

$$F(\widetilde{\gamma}(t)) = \widetilde{\gamma}(t+1)$$

□

Corollary 19.2.1. Let (M,g) be a compact Riemannian manifold with even dimension and positive sectional curvature. If M is non-orientable, then $\pi_1(M) = \mathbb{Z}_2$.

Proof. Let $\widetilde{M}$ be the orientable double covering of M equipped with pullback metric. Synge's theorem implies $\widetilde{M}$ is simply-connected, and thus it's the universal covering of M. This shows $\pi_1(M) = \mathbb{Z}_2$. □

Example 19.1. $\mathbb{RP}^n \times \mathbb{RP}^n$ admits no Riemannian metric with positive sectional curvature, since its fundamental group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Conjecture 19.2.1 (Hopf conjecture)**.** Does $S^2 \times S^2$ admit a Riemannian metric with positive sectional curvature?

19.3. **Other facts.**

Theorem 19.3.1. Let (M,g) be a compact, simply-connected Riemannian n-manifold.

(1) (Hamilton) If $n = 3$, $\mathrm{Ric}(g) > 0$, then M is diffeomorphism to S^3.
(2) (Hamilton) If $n = 4$ with curvature operator > 0, then M is diffeomorphism to S^4.
(3) (Böhm-Wilking) If curvature operator > 0, then M is diffeomorphism to S^n.

Theorem 19.3.2 (soul theorem)**.** Let (M,g) be a complete, non-compact Riemannian n-manifold.

(1) If M has non-negative sectional curvature, then there exists a compact totally geodesic submanifold $S \subseteq M$ (called a soul of M) such that M is diffeomorphic to the normal bundle of S in M.
(2) If M has positive sectional curvature, then its soul is a point and M is diffeomorphic to $\mathbb{R}^n$.

Theorem 19.3.3 (differentiable sphere theorem)**.** Let (M,g) be a compact, simply-connected Riemannian n-manifold with $n \geq 4$. If sectional curvature satisfies $\frac{1}{4} < K \leq 1$, then M is diffeomorphism to S^n.

20. Topology of constant sectional curvature manifold

20.1. **Hopf's theorem.**

Theorem 20.1.1 (Hopf)**.** Let (M,g) be a simply-connected complete Riemannian manifold with constant sectional curvature K. Then (M,g) is isometric to $(\widetilde{M},g_{\mathrm{can}})$, where

$$(\widetilde{M},g_{\mathrm{can}})=\begin{cases}(\mathbb{S}^n(\frac{1}{\sqrt{K}}),g_{\mathrm{can}}) & K>0\\ (\mathbb{R}^n,g_{\mathrm{can}}) & K=0\\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}),g_{\mathrm{can}}) & K<0\end{cases}$$

Proof. Let M be a simply-connected complete Riemannian manifold with constant sectional curvature K.

(1) If $K\leq 0$, let $\widetilde{M}=\mathbb{R}^n$ or $\mathbb{H}^n(\frac{1}{\sqrt{-K}})$. Fix $p\in M,\widetilde{p}\in\widetilde{M}$ and a linear isometry $\Phi_0\colon T_{\widetilde{p}}\widetilde{M}\to T_pM$, Cartan-Ambrose-Hicks's theorem implies

$$\varphi=\exp_p\circ\Phi_0\circ\exp_{\widetilde{p}}^{-1}:\widetilde{M}\to M$$

is a local isometry. Furthermore, Cartan-Hadamard's theorem implies φ is a diffeomorphism, since $M,\widetilde{M}$ are simply-connected with non-positive sectional curvature. This completes the proof of this part.

(2) If $K>0$, let $\widetilde{M}=\mathbb{S}^n(\frac{1}{\sqrt{K}})$. Fix $p\in M,\widetilde{p}\in\widetilde{M}$ and a linear isometry $\Phi_0\colon T_{\widetilde{p}}\widetilde{M}\to T_pM$. Consider the following smooth map

$$\varphi_1=\exp_p\circ\Phi_0\circ\exp_{\widetilde{p}}^{-1}:\widetilde{M}\setminus\{-\widetilde{p}\}\to M$$

it's well-defined since the only cut point of $\widetilde{p}$ is its antipodal point $-\widetilde{p}$. Then Cartan-Ambrose-Hicks's theorem implies φ_1 is a local isometry. Choose $\widetilde{q}\in\widetilde{M}\setminus\{\widetilde{p},-\widetilde{p}\}$, $q=\varphi_1(\widetilde{q})$ and $\Psi_0=(\mathrm{d}\varphi_1)_{\widetilde{q}}\colon T_{\widetilde{q}}\widetilde{M}\to T_qM$. Then the same argument shows

$$\varphi_2=\exp_q\circ\Phi_0\circ\exp_{\widetilde{q}}^{-1}:\widetilde{M}\setminus\{-q\}\to M$$

is a well-defined local isometry defined on $\widetilde{M}\setminus\{-\widetilde{q}\}$. Note that

$$\varphi_2(\widetilde{q})=q=\varphi_1(\widetilde{q})$$
$$(\mathrm{d}\varphi_2)_{\widetilde{q}}=\Psi_0=(\mathrm{d}\varphi_1)_{\widetilde{q}}$$

So by Theorem 16.1.1, we have the φ_1 agrees with φ_2 on $\widetilde{M}\setminus\{-\widetilde{p},-\widetilde{q}\}$. Thus,

$$\varphi(x)=\begin{cases}\varphi_1(x), & x\in\widetilde{M}\setminus\{-\widetilde{p}\}\\ \varphi_2(x), & x\in\widetilde{M}\setminus\{-\widetilde{q}\}\end{cases}$$

is a well-defined local isometry from $\widetilde{M}\to M$. In particular, φ is a local diffeomorphism. Then by Proposition **??** we have φ is a diffeomorphism, since $\mathbb{S}^n$ is compact and simply-connected, and thus φ is an isometry.

□

Corollary 20.1.1. Let (M,g) be a Riemannian manifold with constant sectional curvature K. Then (M,g) is isometric to $\widetilde{M}/\Gamma$, where $\Gamma \subseteq \mathrm{Iso}(\widetilde{M},\widetilde{g})$ is isomorphic to $\pi_1(M)$ and

$$(\widetilde{M},\widetilde{g}) = \begin{cases} (\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{\mathrm{can}}) & K > 0 \\ (\mathbb{R}^n, g_{\mathrm{can}}) & K = 0 \\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{\mathrm{can}}) & K < 0 \end{cases}$$

Proof. Let $(\widetilde{M},\widetilde{g})$ be the universal covering of M with pullback metric. Then M is isometric to $\widetilde{M}/\Gamma$, where $\Gamma \subseteq \mathrm{Iso}(\widetilde{M},\widetilde{g})$ is isomorphic to $\pi_1(M)$. Since $(\widetilde{M},\widetilde{g})$ is a simply-connected Riemannian manifold with constant sectional curvature k, the Hopf's theorem completes the proof. □

Definition 20.1.1 (space form)**.** A complete, simply-connected Riemannian n-manifold with constant sectional curvature k is called space form, and is denoted by $S(n,k)$.

Example 20.1.1. *Let (M,g) be a complete Riemannian manifold with constant sectional curvature $K=1$. If $\dim M = 2n$. Then (M,g) is isometric to the sphere $(\mathbb{S}^{2n}, g_{\mathrm{can}})$ or the real projective space $(\mathbb{RP}^{2n}, g_{\mathrm{can}})$.*

Proof. Note that Hopf's theorem implies (M,g) is isometric to $(\mathbb{S}^{2n}/\Gamma, g_{\mathrm{can}})$, where Γ is isomorphic to $\pi_1(M)$, and Synge's theorem implies if $\dim M$ is even and $K>0$. Then $\pi_1(M)=\{e\}$ or $\pi_1(M)=\mathbb{Z}_2$.

(1) If $\pi_1(M)=\{e\}$. Then (M,g) is isometric to $(\mathbb{S}^{2n}, g_{\mathrm{can}})$.
(2) If $\pi_1(M)=\{e,\varphi\}$, to show (M,g) is isometric to $(\mathbb{RP}^{2n}, g_{\mathrm{can}})$, it suffices to show φ is antipodal map. Note that only possible eigenvalues of φ is ± 1, and if 1 is an eigenvalue of φ. Then it exists a fixed point, which implies $\varphi = e$, since $\pi_1(M)$ acts on $\mathbb{S}^{2n}$ freely.

□

Remark 20.1.1. In general, we have no ideal about what does $\pi_1(M)$ look like.

Part 7. Comparison theorems

21. Comparison theorems based on sectional curvature

In this section, we will see the following philosophy: The larger curvature is, the smaller the distance is."

21.1. Rauch comparison.

Rauch comparison theorem is one of the most important comparison theorems, which gives bounds on the sizes of Jacobi fields based on sectional curvature bounds. Recall that Jacobi field is a quite useful tool, based on the following observations:

(1) Corollary 12.1.1 implies that in a normal neighborhood of p, every vector field can be represented as the value of Jacobi field that vanishes at p.
(2) The zeros of Jacobi fields corresponds to conjugate points, beyond which geodesics can't be minimal.

Theorem 21.1.1 (Rauch comparison)**.** Let (M,g) and $(\widetilde{M},\widetilde{g})$ be two Riemannian manifold with $\dim M \leq \dim \widetilde{M}$. Suppose $\gamma\colon [0,b] \to M$ and $\widetilde{\gamma} : [0,b] \to \widetilde{M}$ are two unit-speed geodesics such that

(1) For all $t \in [0,b]$, and any planes $\Sigma \subseteq T_{\gamma(t)}M, \gamma'(t) \in \Sigma, \widetilde{\Sigma} \subseteq T_{\widetilde{\gamma}(t)}\widetilde{M}, \widetilde{\gamma}'(t) \in \widetilde{\Sigma}$, we have $K_{\gamma(t)}(\Sigma) \leq K_{\widetilde{\gamma}(t)}(\widetilde{\Sigma})$.
(2) $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}|_{[0,b]}$.

Then for any Jacobi fields $J(t)$ and $\widetilde{J}(t)$ with

(1)
$$\begin{cases} J(0) = c\gamma'(0) \\ \widetilde{J}(0) = c\widetilde{\gamma}'(0) \end{cases}$$
(2) $|J'(0)| = |\widetilde{J}'(0)|$.
(3) $\langle J'(0), \gamma'(0)\rangle = \langle \widetilde{J}'(0), \widetilde{\gamma}'(0)\rangle$.

One has $|J(t)| \geq |\widetilde{J}(t)|$ for all $t \in [0,b]$.

Proof. Firstly we consider the following simple case:

(1) $J(0) = \widetilde{J}(0) = 0$.
(2) $|J'(0)| = |\widetilde{J}'(0)|$.
(3) $\langle J'(0), \gamma'(0)\rangle = \langle \widetilde{J}'(0), \widetilde{\gamma}'(0)\rangle = 0$.

Since $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}|_{[0,b]}$. Then $\frac{|J(t)|^2}{|\widetilde{J}(t)|^2}$ is well-defined for all $t \in (0,b]$, and standard calculus implies

$$\lim_{t\to 0} \frac{|J|^2}{|\widetilde{J}^2|} = \lim_{t\to 0} \frac{\langle J'(t), J(t)\rangle}{\langle \widetilde{J}'(t), \widetilde{J}(t)\rangle} = \lim_{t\to 0} \frac{|J'|^2}{|\widetilde{J}'|^2} = 1$$

So it suffices to show in $(0,b]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{|J|^2}{|\widetilde{J}|^2}) \geq 0$$

Direct computation shows above inequality is equivalent to:

$$\frac{\langle J'(t), J(t)\rangle}{|J(t)|^2} \geq \frac{\langle \widetilde{J}'(t), \widetilde{J}(t)\rangle}{|\widetilde{J}(t)|^2}$$

holds for arbitrary $t \in (0,b]$. For arbitrary $s \in (0,b]$, we can define the following Jacobi fields by scaling $J(t)$:

$$W_s(t) = \frac{J(t)}{|J(s)|}, \quad \widetilde{W}_s(t) = \frac{\widetilde{J}(t)}{|\widetilde{J}(s)|}$$

Then

$$\frac{\langle J'(s), J(s)\rangle}{|J(s)|^2} = \langle W_s'(s), W_s(s)\rangle$$

So it suffices to show

$$\langle W_s'(s), W_s(s)\rangle \geq \langle \widetilde{W}_s'(s), \widetilde{W}_s(s)\rangle$$

holds for arbitrary $s \in (0,b]$. Direct computation shows:

$$\begin{aligned}
\langle W_s'(s), W_s(s)\rangle &= \int_0^s (\langle W_s(t), W_s(t)\rangle)' \mathrm{d}t \\
&= \int_0^s \langle W_s'(t), W_s'(t)\rangle \mathrm{d}t + \int_0^s \langle W_s''(t), W_s(t)\rangle \mathrm{d}t \\
&= \int_0^s \langle W_s'(t), W_s'(t)\rangle \mathrm{d}t - \int_0^s R(W_s(t), \gamma'(t), \gamma'(t), W(t)) \mathrm{d}t
\end{aligned}$$

Choose a parallel orthonormal frame $\{e_1(t), \dots, e_n(t)\}$ with $e_1(t) = \gamma'(t), e_2(t) = W_s(t)$. With respect to this frame we write

$$W_s(t) = \lambda^i(t) e_i(t)$$

Similarly, we choose a parallel orthogonal frame $\{\widetilde{e}_1(t), \dots, \widetilde{e}_n(t)\}$ and construct the following vector field

$$\widetilde{V}(t) = \lambda^i(t) \widetilde{e}_i(t)$$

Then it's clear we have

$$\int_0^s \langle W_s'(t), W_s'(t)\rangle \mathrm{d}t = \int_0^s \langle \widetilde{V}'(t), \widetilde{V}'(t)\rangle \mathrm{d}t$$

and our curvature condition implies

$$\int_0^s R(W_s(t), \gamma'(t), \gamma'(t), W_s(t)) \mathrm{d}t \leq \int_0^s \widetilde{R}(\widetilde{V}(t), \gamma'(t), \gamma'(t), \widetilde{V}(t)) \mathrm{d}t$$

Thus we have

$$\begin{aligned}
\langle W_s'(s), W_s(s)\rangle &\leq \int_0^s \langle \widetilde{V}'(t), \widetilde{V}'(t)\rangle \mathrm{d}t - \int_0^s R(\widetilde{V}(t), \gamma'(t), \gamma'(t), \widetilde{V}(t)) \mathrm{d}t \\
&= \widetilde{I}(\widetilde{V}, \widetilde{V})
\end{aligned}$$

where $\widetilde{I}$ is index form on $\widetilde{M}$. According to Corollary 12.3.1, we have

$$\widetilde{I}(\widetilde{V}, \widetilde{V}) \geq \widetilde{I}(\widetilde{W}_s, \widetilde{W}_s)$$

since $\widetilde{W}_s$ is a Jacobi field. This shows the desired result.

For general case, we consider the following decomposition

$$J(t)=J_1(t)+\langle J(t),\gamma'(t)\rangle\gamma'(t)$$
$$\widetilde{J}(t)=\widetilde{J}_1(t)+\langle\widetilde{J}(t),\widetilde{\gamma}'(t)\rangle\widetilde{\gamma}'(t)$$

Then it's clear $J_1(t)$ and $\widetilde{J}_1(t)$ satisfy requirement of our simple case, that is for $t\in[0,1]$ we have

$$|J_1(t)|\geq|\widetilde{J}_1(t)|$$

Furthermore,

$$\langle J(t),\gamma'(t)\rangle=\langle\widetilde{J}(t),\widetilde{\gamma}'(t)\rangle$$

always holds, since

$$\begin{aligned}\langle J(t),\gamma'(t)\rangle&=\langle J(0),\gamma'(0)\rangle+\langle J'(0),\gamma'(0)\rangle t\\&\overset{(1)}{=}\langle\widetilde{J}(0),\widetilde{\gamma}'(0)\rangle+\langle\widetilde{J}'(0),\widetilde{\gamma}'(0)\rangle t\\&=\langle\widetilde{J}(t),\widetilde{\gamma}'(t)\rangle\end{aligned}$$

where (1) holds from our assumption. □

Corollary 21.1.1. Let (M,g) be a Riemannian manifold, U a normal neighborhood of $p\in M$, $\gamma\colon[0,b]\to U$ a unit-speed geodesic with $\gamma(0)=p$ and J a Jacobi field along γ with $J(0)=0$.

(1) If the sectional curvature $K\leq k$ in U. Then $|J(t)|\geq\mathrm{sn}_k(t)|J'(0)|$, for all $t\in[0,b_0]$, where

$$b_0=\begin{cases}b, & k\leq 0\\ \min\{b,\pi R\}, & k=\frac{1}{R^2}>0\end{cases}$$

(2) If the sectional curvature $K\geq k$ in U. Then

$$|J(t)|\leq\mathrm{sn}_k(t)|J'(0)|$$

for all $t\in[0,b]$.

Proof. Apply Rauch comparison between M and space form $\widetilde{M}=S(n,k)$ to conclude. However, in order to avoid geodesic $\widetilde{\gamma}$ of $\widetilde{M}$ from having conjugate points, we need to let $b_0<\min\{b,\pi R\}$, when $k=\frac{1}{R^2}>0$. □

Remark 21.1.1. In particular, from above corollary, we immediately have the following corollary when $K\leq k$:

(1) If $k\leq 0$, we have already known M has no conjugate point along any geodesic.
(2) If $k=\frac{1}{R^2}>0$. Then there is no conjugate point along any geodesic with length $<\pi R$. Or in other words, the distance between two consecutive conjugate points is $\geq\pi R$.

Corollary 21.1.2 (metric comparison)**.** Let (M,g) be a Riemannian n-manifold, U a normal neighborhood of $p\in M$. For all $k\in\mathbb{R}$, we use g_k to denote the metric $\mathrm{d}r\otimes\mathrm{d}r+\mathrm{sn}_k(r)g_{\mathbb{S}^{n-1}}$ in $U\setminus\{p\}$.

(1) If $K \leq k$ holds for all $q \in U \setminus \{p\}$. Then for $w \in T_qM$ we have

$$g(w,w) \geq g_k(w,w)$$

holds in $U_0 \setminus \{p\}$, where

$$U_0 = \begin{cases} U, & k \leq 0 \\ U \cap B(p, \pi R), & k = \frac{1}{R^2} > 0 \end{cases}$$

(2) If $K \geq k$ holds for all $q \in U \setminus \{p\}$. Then for $w \in T_qM$ we have

$$g(w,w) \leq g_k(w,w)$$

holds in $U \setminus \{p\}$.

Proof. If $w = \partial_r|_q$, it's clear

$$g(\partial_r|_q, \partial_r|_q) = 1 = g_k(\partial_r|_q, \partial_r|_q)$$

by Gauss lemma. Then it suffices to check for $w \in T_qM$ such that $g(w, \partial_r|_q) = 0$, we have

$$g(w,w) \geq g_k(w,w)$$

Let $\gamma\colon [0,b] \to M$ be a unit-speed geodesic connecting p and q, and J a Jacobi field such that $J(0) = 0, J(b) = w$. In normal coordinate $J(t)$ can be written as $ta^i \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$ for some a^i.

Since (x^i, U, p) is both normal coordinate for metric g and g_k, thus γ is also a radial geodesic for g_c, and $J(t)$ is also a Jacobi field with respect to g_c along γ. Thus we have

$$g(w,w) = |J(b)|_g^2$$
$$g_k(w,w) = |J(b)|_{g_k}^2$$

Then by Corollary 21.1.1, this completes the proof. □

Remark 21.1.2. The **ideal** of this proof and the proof of Theorem 14.4.1 is almost the same, that is by using Corollary 12.1.1 to construct a Jacobi field valued a given vector, and then one can use Rauch comparison to compare length of given vectors.

Corollary 21.1.3. Let (M,g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds with $K \leq \widetilde{K}$. Fix $p \in M, \widetilde{p} \in \widetilde{M}$, linear isometry $\Phi_0\colon T_pM \to T_{\widetilde{p}}\widetilde{M}$ and $0 \leq \delta < \min(\mathrm{inj}(p), \mathrm{inj}(\widetilde{p}))$. Then for any smooth curve $\gamma\colon [0,1] \to \exp_p(B(0,\delta))$ and $\widetilde{\gamma}(t) = \exp_{\widetilde{p}} \circ \Phi_0 \circ \exp_p^{-1}(\gamma(t))$, we have

$$L(\gamma) \geq L(\widetilde{\gamma})$$

Proof. Let $c(s) = \exp_p^{-1} \circ \gamma(s)$ and $\widetilde{c}(s) = \exp_{\widetilde{p}}^{-1} \circ \widetilde{\gamma}(s)$. Then $\widetilde{c}(s) = \Phi_0(c(s))$. Consider the following variations

$$\alpha(t,s) = \exp_p(tc(s))$$
$$\widetilde{\alpha}(t,s) = \exp_{\widetilde{p}}(t\widetilde{c}(s))$$

and Jacobi fields

$$J_s(t) = \alpha_*(\frac{\partial}{\partial s})(t,s)$$
$$\widetilde{J}_s(t) = \widetilde{\alpha}_*(\frac{\partial}{\partial s})(t,s)$$

A crucial observation is for arbitrary $s_0 \in [0,1]$, we have

$$J_{s_0}(1) = \gamma'(s_0)$$
$$\widetilde{J}_{s_0}(1) = \widetilde{\gamma}'(s_0)$$

So it suffices to prove $|J_{s_0}(1)| \geq |\widetilde{J}_{s_0}(1)|$ holds for arbitrary $s_0 \in [0,1]$, that is we need to use Rauch comparison to Jacobi fields $J_{s_0}(t), \widetilde{J}_{s_0}(t)$ along γ_{s_0} and $\widetilde{\gamma}_{s_0}$, where $\gamma_{s_0}(t) = \alpha(t,s_0)$ and $\widetilde{\gamma}_{s_0}(t) = \widetilde{\alpha}(t,s_0)$. Check requirements as follows:

(1) $J_{s_0}(0) = \widetilde{J}_{s_0}(0) = 0$.
(2) $J'_{s_0}(0) = c'(s_0), \widetilde{J}'_{s_0}(0) = \widetilde{c}'(s_0)$, and $\widetilde{c}(s_0) = \Phi_0(c(s_0))$ implies $|J'_{s_0}(0)| = |\widetilde{J}'_{s_0}(0)|$, since Φ_0 is linear isometry.
(3) $\langle \widetilde{J}'_{s_0}(0), \widetilde{\gamma}'_{s_0}(0)\rangle = \langle \Phi_0(c'(s_0)), \Phi_0(c(s_0))\rangle = \langle c'(s_0), c(s_0)\rangle = \langle J'_{s_0}(0), \gamma'_{s_0}(0)\rangle$.

□

Corollary 21.1.4. Let (M,g) be a Riemannian n-manifold, $0 < k_1 \leq K \leq k_2$. Let γ be any geodesic in M and b the distance along γ between two consecutive conjugate points. Then

$$\frac{\pi}{\sqrt{k_2}} \leq b \leq \frac{\pi}{\sqrt{k_1}}$$

Proof. Without lose of generality, we assume $\gamma\colon [0,b] \to M$ is a unit-speed geodesic with $\gamma(0) = p, \gamma(b) = q$ and p,q are two consecutive conjugate points along γ.

(1) By Remark 21.1.1, we have already seen $b \geq \frac{\pi}{\sqrt{k_2}}$.
(2) Apply Rauch comparison to (M,g) and $(\mathbb{S}^n(\frac{\pi}{\sqrt{k_1}}), g_{\mathrm{can}})$, we have

$$|J(t)| \leq |\widetilde{J}(t)|$$

for $t \in [0,b]$, where $J(t), \widetilde{J}(t)$ are defined the same as before. Suppose $b > \frac{\pi}{\sqrt{k_1}}$. Then take $t = \frac{\pi}{\sqrt{k_1}}$, we have

$$0 < |J(t)| \leq |\widetilde{J}(t)| = 0$$

A contradiction.

□

Theorem 21.1.2. Let (M,g) be a compact Riemannian manifold with sectional curvature $K \leq k, k > 0$. If we define

$$l(M) := \inf\{L(\gamma) \mid \gamma \text{ is a closed geodesic in } M\}$$

Then either $\mathrm{inj}(M) \geq \frac{\pi}{\sqrt{k}}$ or $\mathrm{inj}(M) = \frac{l(M)}{2}$.

Proof. By compactness of M, there exists $p,q \in M, q \in \mathrm{cut}(p)$ such that $\mathrm{dist}(p,q) = \mathrm{inj}(M) = \mathrm{inj}(p)$. Let $\gamma\colon [0,b] \to M$ be a minimal geodesic connecting p and q, that is $b = \mathrm{dist}(p,q) = \mathrm{inj}(M)$. Then

(1) If p and q are conjugate along γ. Then by Corollary 21.1.4 we have $\mathrm{inj}(M) = b \geq \frac{\pi}{\sqrt{k}}$.
(2) If p and q are not conjugate along γ. Then by Proposition 13.2.1 there exists a unit-speed closed geodesic $\gamma\colon [0,2b] \to M$ with $\gamma(0) = p, \gamma(b) = q$, where $b = \mathrm{dist}(p,q) = \mathrm{inj}(M)$. On one hand by definition of $l(M,g)$ one has $2b \geq l(M)$. On the other hand, $l(M) \geq 2b$, since $\mathrm{dist}(p,q) = q$. Thus in this case $\mathrm{inj}(M) = \frac{l(M)}{2}$.

□

21.2. **Hessian comparison.**

Theorem 21.2.1 (Hessian comparison)**.** Let (M,g) and $(\widetilde{M},\widetilde{g})$ be two Riemannian manifolds with the same dimension, $U \subseteq M, \widetilde{U} \subseteq \widetilde{M}$ normal neighborhoods around $p \in M$ and $\widetilde{p} \in \widetilde{M}$ respectively. Suppose

$$\gamma\colon [0,b] \to U, \gamma(0) = p, \gamma(b) = q$$
$$\widetilde{\gamma}\colon [0,b] \to \widetilde{U}, \widetilde{\gamma}(0) = \widetilde{p}, \widetilde{\gamma}(b) = \widetilde{q}$$

are two unit-speed geodesics such that

For all $t \in [0,b]$, and any planes $\Sigma \subseteq T_{\gamma(t)}M, \gamma'(t) \in \Sigma, \widetilde{\Sigma} \subseteq T_{\widetilde{\gamma}(t)}\widetilde{M}, \widetilde{\gamma}'(t) \in \widetilde{\Sigma}$, we have $K_{\gamma(t)}(\Sigma) \leq K_{\widetilde{\gamma}(t)}(\widetilde{\Sigma})$.

Then for any $v \in T_qM, \widetilde{v} \in T_{\widetilde{q}}\widetilde{M}$ with unit length and $v \perp \gamma'(b), \widetilde{v} \perp \widetilde{\gamma}'(b)$, we have

(1) $\mathrm{Hess}\, r(v,v) \geq \mathrm{Hess}\, \widetilde{r}(\widetilde{v},\widetilde{v})$.
(2) $\Delta r(\gamma(t)) \geq \widetilde{\Delta}\widetilde{r}(\widetilde{\gamma}(t))$ for all $t \in (0,b]$.
(3) Moreover, the equality holds if and only if $K_\Sigma(\gamma(t)) = \widetilde{K}_{\widetilde{\Sigma}}(\widetilde{\gamma}(t))$.

Proof. For (1). Let $\{e_1(t),\dots,e_n(t)\}$ be a parallel orthonormal basis along γ such that $e_n(t) = \gamma'(t)$ and $\{\widetilde{e}_1(t),\dots,\widetilde{e}_n(t)\}$ a parallel orthonormal basis along $\widetilde{\gamma}$ such that $\widetilde{e}_n(t) = \widetilde{\gamma}'(t)$. Without lose of generality we may assume $\langle v, e_i(b)\rangle_g = \langle \widetilde{v}, \widetilde{e}_i(b)\rangle_{\widetilde{g}}$ for $i = 1,\dots,n-1$, it's just a trick of linear algebra.

Use Corollary 12.1.1 to construct Jacobi fields

$$\begin{cases} J(0) = 0, J(b) = v \\ \widetilde{J}(0) = 0, \widetilde{J}(b) = \widetilde{v} \end{cases}$$

With respect to $\{\widetilde{e}_i(t)\}$ we can write $\widetilde{J}(t)$ as $\widetilde{J}(t)=\lambda^i(t)\widetilde{e}_i(t)$, and construct $V(t)=\lambda^i(t)e_i(t)$. Then

$$
\begin{aligned}
\operatorname{Hess} r(v,v) &= \operatorname{Hess} r(J(b),J(b)) \\
&\overset{\mathrm{I}}{=} \int_0^b \langle J'(t),J'(t)\rangle - R(J,\gamma',\gamma',J)\mathrm{d}t \\
&\overset{\mathrm{II}}{\geq} \int_0^b \langle V'(t),V'(t)\rangle - R(V,\gamma',\gamma',V)\mathrm{d}t \\
&\overset{\mathrm{III}}{\geq} \int_0^b \langle \widetilde{J}'(t),\widetilde{J}'(t)\rangle - \widetilde{R}(\widetilde{J},\widetilde{\gamma},\widetilde{\gamma},\widetilde{J})\mathrm{d}t \\
&= \operatorname{Hess}\widetilde{r}(\widetilde{J}(b),\widetilde{J}(b)) \\
&= \operatorname{Hess}\widetilde{r}(\widetilde{v},\widetilde{v})
\end{aligned}
$$

where

I holds from Corollary 14.5.1.
II holds from Corollary 12.3.1.
III holds from our assumption on curvature and the choice of V.

For (2) and (3). They directly follow from (1) and proof of (1).

□

Corollary 21.2.1 (Hessian and Laplacian comparison)**.** Let (M,g) be a Riemannian n-manifold and U a normal neighborhood of $p\in M$.

(1) If sectional curvature $K\leq k$ in $U\setminus\{p\}$. Then

$$
\mathscr{H}_r \geq \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r,\quad \Delta r\geq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}
$$

holds in $U_0\setminus\{p\}$, where

$$
U_0=\begin{cases} U, & k\leq 0\\ U\cap B(p,\pi R), & k=\frac{1}{R^2}>0\end{cases}
$$

(2) If sectional curvature $K\geq k$ in $U\setminus\{p\}$. Then

$$
\mathscr{H}_r \leq \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r,\quad \Delta r\leq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}
$$

holds in $U\setminus\{p\}$.

(3) Moreover, if equality holds, g has constant sectional curvature k in U_0 or U.

Proof. For (1). Apply Hessian comparison to (M,g) and space form $S(n,k)$. Then we directly have

$$
\operatorname{Hess} r(v,v)\geq \operatorname{Hess}\widetilde{r}(\widetilde{v},\widetilde{v})
$$

for any $v\in T_qM,\widetilde{v}\in T_qS(n,k)$ with unit length and $v\perp\gamma'(b),\widetilde{v}\perp\widetilde{\gamma}'(b)$, where

$$
\begin{aligned}
\gamma&:[0,b]\to U,\gamma(0)=p,\gamma(b)=q\\
\widetilde{\gamma}&:[0,b]\to\widetilde{U},\widetilde{\gamma}(0)=\widetilde{p},\widetilde{\gamma}(b)=\widetilde{q}
\end{aligned}
$$

are two unit-speed geodesics, and $U,\widetilde{U}$ are normal neighborhoods of $p,\widetilde{p}$ respectively. However, we must be careful here, since if sectional curvature of M is ≤ 0. Then b can be infinite, and in this case if $k>0$, the diameter of $\widetilde{U}$ is $<\frac{\pi}{\sqrt{k}}$. Thus we only have

$$\operatorname{Hess} r(v,v) \geq \operatorname{Hess}\widetilde{r}(\widetilde{v},\widetilde{v})$$

for $0<b<\frac{\pi}{\sqrt{k}}$ if $k>0$, and there is no restriction for b if $k\le 0$. Thus by taking different geodesics and different Jacobi fields, we can show this holds for arbitrary $v\in T_qM, \widetilde{v}\in T_qS(n,k)$, where $q\in U_0\setminus\{p\}$, that is we have

$$\mathscr{H}_r \geq \frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}\pi_r$$

holds in $U_0\setminus\{p\}$. By taking trace we obtain $\Delta r\ge (n-1)\frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}$ holds in $U_0\setminus\{p\}$, since π_r is a projection onto a subspace with codimension 1.

For (2), the same as (1).

For (3), if

$$\mathscr{H}_r = \frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}\pi_r$$

holds in $U\setminus\{p\}$. Then it's directly from Proposition 14.5.2. If

$$\Delta r \geq (n-1)\frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}$$

holds in $U\setminus\{p\}$, that is the trace of $\mathscr{H}_r-\frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}\pi_r$ vanishes identically in $U\setminus\{p\}$. Then $\mathscr{H}_r-\frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}\pi_r$ vanishes identically, since it's semi-positive definite. □

22. Comparison theorems based on Ricci curvature

22.1. Local Laplacian comparison.

Theorem 22.1.1 (local Laplacian comparison). Let (M,g) be a Riemannian n-manifold and U a normal coordinate of $p\in M$. If there exists $k\in\mathbb{R}$ such that $\mathrm{Ric}(g)\geq(n-1)kg$. Then

$$\Delta r\leq(n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$

holds in $U_0\setminus\{p\}$, where

$$U_0=\begin{cases}U, & k\leq 0\\ U\cap B(p,\pi R), & k=\frac{1}{R^2}>0\end{cases}$$

Moreover, if equality holds. Then g has constant sectional curvature in U_0.

22.1.1. *Proof by using Jacobi fields.*

Proof of Theorem 22.1.1 via Jacobi fields. For arbitrary $q\in U_0\setminus\{p\}$, choose a unit-speed geodesic $\gamma\colon[0,b]\to M$ with $\gamma(0)=p,\gamma(b)=q$, and $\{e_1(t),\dots,e_n(t)\}$ is a parallel orthonormal frame along γ with $e_n(t)=\gamma'(t)$. Then by definition $\Delta r=\sum_{i=1}^n\mathrm{Hess}\,r(e_i,e_i)$.

By Corollary 12.1.1 one can construct Jacobi fields $J_i(t),i=1,\dots,n$ such that $J_i(0)=0,J_i(b)=e_i(b)$. Then we have

$$\Delta r=\sum_{i=1}^{n-1}\mathrm{Hess}\,r(J_i(b),J_i(b))\overset{(1)}{=}\sum_{i=1}^{n-1}I(J_i,J_i)$$

where (1) holds from Corollary 14.5.1. Now let $\widetilde{M}$ be the space form $S(n,k)$ and $\widetilde{U}$ a normal coordinate of $\widetilde{p}\in\widetilde{M}$. Repeat the same process as above we have

$$(n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}=\widetilde{\Delta}\widetilde{r}=\sum_{i=1}^{n-1}\widetilde{I}(\widetilde{J}_i,\widetilde{J}_i)$$

If we denote $V_i(t)=f(t)e_i(t)$, routine computation shows

$$\begin{aligned}\Delta r&=\sum_{i=1}^{n-1}I(J_i,J_i)\\&\leq\sum_{i=1}^{n-1}I(V_i,V_i)\\&=\sum_{i=1}^{n-1}\int_0^b\langle V_i'(t),V_i'(t)\rangle-R(V_i,\gamma',\gamma',V_i)\mathrm{d}t\end{aligned}$$

Until now, all computations are the same as what we have done in Hessian comparison based on sectional curvature. A crucial observation is that $\widetilde{J}_i(t)=$

$f(t)\widetilde{e}_i(t)$, and the **key point** is that $f(t)$ is independent of i. Then

$$\begin{aligned}\Delta r &\overset{(2)}{=} \sum_{i=1}^{n-1}\int_0^b \langle V_i'(t), V_i'(t)\rangle - f^2(t)R(e_i,e_n,e_n,e_i)\mathrm{d}t\\ &= \sum_{i=1}^{n-1}\int_0^b \langle V_i'(t), V_i'(t)\rangle - \int_0^b f^2(t)\mathrm{Ric}(e_n,e_n)\mathrm{d}t\\ &\leq \sum_{i=1}^{n-1}\int_0^b \langle \widetilde{J}_i(t), \widetilde{J}_i(t)\rangle - \int_0^b (n-1)kf^2(t)\mathrm{d}t\\ &= \sum_{i=1}^{n-1}\widetilde{I}(\widetilde{J}_i,\widetilde{J}_i)\\ &= (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\end{aligned}$$

the key point is used in equality marked by (2), and others are routines. □

22.1.2. *Proof by using Bochner's technique.*

Lemma 22.1.1. Let (M,g) be a Riemannian manifold, (x^i,U,p) a normal coordinate centered at p. Then

$$\Delta r = \partial_r \log(r^{n-1}\sqrt{\det g})$$

in $U\setminus\{p\}$. Moreover, along any unit-speed geodesic $\gamma\colon[0,b]\to U$ with $\gamma(0)=p$, if we define $f(t):=\Delta r(\gamma(t))$. Then

$$f(t) = \frac{n-1}{t} + O(1)$$

Proof. Direct computation shows

$$\begin{aligned}\Delta r &= \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}(g^{ij}\sqrt{\det g}\frac{\partial r}{\partial x^j})\\ &= \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}(g^{ij}\sqrt{\det g}\frac{x^j}{r})\\ &= \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}(\frac{x^i}{r}\sqrt{\det g})\\ &= \frac{\partial}{\partial x^i}(\frac{x^i}{r}) + \frac{1}{\sqrt{\det g}}\frac{x^i}{r}\frac{\partial}{\partial x^i}(\sqrt{\det g})\\ &= \frac{n-1}{r} + \frac{1}{\sqrt{\det g}}\partial_r(\sqrt{\det g})\\ &= \partial_r\log(r^{n-1}\sqrt{\det g})\end{aligned}$$

Moreover, for unit-speed geodesic $\gamma\colon[0,b]\to U$, we have

$$f(t) = \frac{n-1}{r(\gamma(t))} + \partial_r(\log\sqrt{\det g})\Big|_{\gamma(t)}$$

Then note that

(1) $r(\gamma(t))=t$, since γ is unit-speed geodesic.
(2) Jacobi's formula implies

$$\partial_r(\log\sqrt{\det g})\Big|_{\gamma(t)}=\frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}=O(1)$$

we obtain the desired results. $\square$

Lemma 22.1.2 (Riccati comparison theorem)**.** If $f\colon(0,b)\to\mathbb{R}$ is a smooth function satisfying
(1) $f(t)=\frac{1}{t}+O(1)$.
(2) $f'+f^2+k\leq 0$.
Then

$$f(t)\leq\frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}$$

for all $t\in(0,b)$, where $k>0, b\leq\frac{\pi}{\sqrt{k}}$.

Proof. Consider $f_k(t)=\frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}$, it's a smooth function defined on $(0,b)$ satisfying
(1) $f_k(t)=\frac{1}{t}+O(1)$
(2) $f_k'+f_k^2+k=0$
Choose a smooth function $f\colon(0,b)\to\mathbb{R}$ satisfying
(1) $F(t)=2\log t+O(1)$.
(2) $F'(t)=f+f_k$
Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^F(f-f_k))=e^F(f^2-f_k^2+f'-f_k')\leq 0$$
$$\lim_{t\to 0}e^F(f-f_k)=0$$

Then we have $f(t)\leq f_k(t)$ holds for all $t\in(0,b)$. $\square$

Lemma 22.1.3.

$$|\operatorname{Hess} r|^2\geq\frac{(\Delta r)^2}{n-1}$$

Proof. Let $\{e_1,\dots,e_n\}$ be an orthonormal frame with $e_1=\partial_r$. Then

$$\begin{aligned}|\operatorname{Hess} r|^2&=\sum_{i,j=1}^n(\langle\nabla_{e_i}\partial_r,e_j\rangle)^2\\&=\sum_{i,j=2}^n(\langle\nabla_{e_i}\partial_r,e_j\rangle)^2\\&\geq\frac{1}{n-1}\sum_{i=2}^n(\langle\nabla_{e_i}\partial_r,e_i\rangle)^2\\&=\frac{1}{n-1}(\Delta r)^2\end{aligned}$$

The inequality

$$|A|^2\geq\frac{1}{k}|\operatorname{tr}(A)|^2$$

for a $k\times k$ matrix A is a direct consequence of the Cauchy-Schwarz inequality. $\square$

Proof of Theorem 22.1.1 by using Bochner's technique. Recall Bochner's technique says

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess} f|^2 + \mathrm{Ric}(\nabla f,\nabla f) + g(\nabla\Delta f,\nabla f)$$

Set $f = r$ we have

$$\begin{aligned} 0 &= |\mathrm{Hess}\, r|^2 + \mathrm{Ric}(\nabla r,\nabla r) + g(\nabla\Delta r,\nabla r) \\ &\overset{(1)}{\geq} |\mathrm{Hess}\, r|^2 + \partial_r(\Delta r) + (n-1)k \\ &\overset{(2)}{\geq} (\frac{\Delta r}{n-1})^2 + \partial_r(\frac{\Delta r}{n-1}) + k \end{aligned}$$

where

(1) holds from $\partial_r = \nabla_r$ and lower bounded of Ricci.
(2) holds from Lemma 22.1.3 and divided by $n-1$.

Thanks to Lemma 22.1.1, we can apply Riccati comparison to $f(r) = \frac{\Delta r}{n-1}$. Then we have

$$\frac{\Delta r}{n-1} \leq \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$

This shows desired comparison.

Furthermore, if equality holds

$$\frac{\Delta r}{n-1} = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$

then direct computation shows

$$(\frac{\Delta r}{n-1})^2 + \partial_r(\frac{\Delta r}{n-1}) + k = 0$$

which implies inequalities in (1) and (2) are in fact equalities. In particular, one has

$$|\mathrm{Hess}\, r|^2 = \frac{(\Delta r)^2}{n-1}$$

that is inequality in Cauchy-Schwarz inequality holds, which implies

$$\mathscr{H}_r = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r$$

Then g has constant sectional curvature k in U_0 by Proposition 14.5.2. $\square$

22.2. **Maximal principle.**

Proposition 22.2.1. Let (M,g) be a Riemannian manifold and f,h be two smooth functions on M. If there is a point p such that $f(p) = h(p)$ and $f(x) \geq h(x)$ for all x near p. Then

$$\nabla f(p) = \nabla h(p), \quad \mathrm{Hess}\, f|_p \geq \mathrm{Hess}\, h|_p, \quad \Delta f(p) \geq \Delta h(p)$$

Proof. Firstly let's consider the case $(M,g)\subseteq(\mathbb{R}^n,g_{\mathrm{can}})$, it's a simple calculus since we can use Taylor expansion. To be explicit, for all x near p, we have

$$f(x)=f(p)+\nabla f(p)^T(x-p)+\frac{1}{2}(x-p)^T\operatorname{Hess}f|_p(x-p)+O(|x|^3)$$

where ∇f is an n column vector and $\operatorname{Hess}f$ is an $n\times n$ matrix in this case. Similarly, we have

$$h(x)=h(p)+\nabla h(p)^T(x-p)+\frac{1}{2}(x-p)^T\operatorname{Hess}h|_p(x-p)+O(|x|^3)$$

Then consider

$$f(x)-h(x)=(\nabla f-\nabla h)(p)^T(x-p)+\frac{1}{2}(x-p)^T\operatorname{Hess}(f-h)|_p(x-p)+O(|x|^3)$$

Since $f(x)-h(x)\geq 0$ for all x near p. Then we must have

$$\begin{aligned}\nabla f(p)&=\nabla h(p)\\ \operatorname{Hess}f|_p&\geq\operatorname{Hess}h|_p\end{aligned}$$

By taking trace we have

$$\Delta f(p)\geq\Delta h(p)$$

For general case, take $\gamma:(-\varepsilon,\varepsilon)\to M$ to be a geodesic with $\gamma(0)=p$. Then use previous case on $f\circ\gamma,h\circ\gamma$ to obtain

$$\begin{aligned}\nabla_{\gamma'(0)}f(p)&=\nabla_{\gamma'(0)}h(p)\\ \operatorname{Hess}f_p(\gamma'(0),\gamma'(0))&\geq\operatorname{Hess}h_p(\gamma'(0),\gamma'(0))\end{aligned}$$

Then it's clear this proposition holds if we let $v=\gamma'(0)$ run over all $v\in T_pM$. □

Definition 22.2.1 (barrier sense)**.** Let (M,g) be a Riemannian manifold and $f\in C(M)$. Suppose f_q is a C^2 function defined in a neighborhood of U of $q\in M$.

(1) f_q is called a lower barrier function of f at q if

$$f_q(q)=f(q),\quad f_q(x)\leq f(x),\quad x\in U$$

(2)

$$\Delta f(q)\geq c$$

in the barrier sense if for all $\varepsilon>0$, there exists a lower barrier function $f_{q,\varepsilon}$ of f at q such that

$$\Delta f_{q,\varepsilon}(q)\geq c-\varepsilon$$

(3)

$$\Delta f(q)\leq c$$

in the barrier sense if for all $\varepsilon>0$, there exists an upper barrier function $f_{q,\varepsilon}$ of f at q such that

$$\Delta f_{q,\varepsilon}(q)\leq c+\varepsilon$$

Definition 22.2.2 (distribution sense)**.** Let (M,g) be an orientable Riemannian manifold and $f\in C(M)$.

$$\Delta f\le h$$

in distribution sense, if

$$\int_M f\Delta\varphi\le\int_M h\varphi$$

holds for all $\varphi\ge 0\in C_c^\infty(M)$

Theorem 22.2.1 (maximal principle)**.** Let (M,g) be a Riemannian manifold and $f\in C(M)$.

(1) If $\Delta f\ge 0$ in the barrier sense or distribution sense. Then if f has a local(global) maximum. Then it's local(global) constant.
(2) If $\Delta f\le 0$ in the barrier sense or distribution sense. Then if f has a local(global) minimal. Then it's local(global) constant.
(3) $\Delta f=0$ implies $f\in C^\infty(M)$.

Proof. See Theorem 66 in Page280 of [Pet16]. □

22.3. **Global Laplacian comparison.**

22.3.1. *In the barrier sense.*

Proposition 22.3.1. Let (M,g) be a complete Riemannian manifold and $p,q\in M$. Let $\gamma:[0,b]\to M$ be a unit-speed minimal geodesic with $\gamma(0)=p$ and $\gamma(b)=q$. For any small $\varepsilon>0$,

$$r_\varepsilon(x)=\varepsilon+\operatorname{dist}(\gamma(\varepsilon),x)$$

where $x\in M$. Then

(1) $q\notin\operatorname{cut}(\gamma(\varepsilon))$ and in particular, r_ε is smooth at q.
(2) r_ε is an upper barrier function of $r(x)=\operatorname{dist}(p,x)$ at point q.

Proof. For (1). If $q\in\operatorname{cut}(\gamma(\varepsilon))$, by Corollary 13.1.2, one has $\gamma(\varepsilon)\in\operatorname{cut}(q)$, a contradiction to γ is a minimal geodesic connecting p and q.

For (2). Firstly note that $\gamma(b)=q$. Then

$$r(q)=\operatorname{dist}(p,q)=\operatorname{dist}(\gamma(0),\gamma(b))\overset{\mathrm{I}}{=}\operatorname{dist}(\gamma(0),\gamma(\varepsilon))+\operatorname{dist}(\gamma(\varepsilon),\gamma(b))\overset{\mathrm{II}}{=}r_\varepsilon(q)$$

where

I holds since γ is a minimal geodesic.
II holds since γ is unit-speed minimal geodesic. Then $\operatorname{dist}(\gamma(0),\gamma(\varepsilon))=\varepsilon$.

By triangle inequality, one has

$$r(q')=\operatorname{dist}(p,q')\le\varepsilon+\operatorname{dist}(\gamma(\varepsilon),q)=r_\varepsilon(q')$$

for all q' near q. Combining these two facts together we have r_ε is an upper barrier function of r. □

Theorem 22.3.1 (global Laplacian comparison)**.** Let (M,g) be a complete Riemannian manifold with

$$\mathrm{Ric}(g) \geq (n-1)kg$$

Then for $q \in M$

$$\Delta r(q) \leq (n-1)\frac{\mathrm{sn}_k'(r(q))}{\mathrm{sn}_k(r(q))}$$

in the barrier sense.

Proof. We consider the following three cases:

(1) If $q \in M \setminus \{p\} \cup \mathrm{cut}(p)$, it's exactly smooth case we have proven.
(2) If $q = p$, it's clear, since the right hand is infinite.
(3) For arbitrary $q \in \mathrm{cut}(p)$, there exists a unit-speed $\gamma\colon [0,b] \to M$ with $\gamma(0) = p, \gamma(b) = q$. Then for each $\gamma > 0$, define

$$\gamma_\varepsilon(x) = \varepsilon + \mathrm{dist}(\gamma(\varepsilon), x)$$

Then by Proposition 22.3.1 we have $\gamma_\varepsilon(x)$ is an upper barrier of $r(x)$ and γ_ε is smooth at q. Thus we have

$$\begin{aligned}\Delta\gamma_\varepsilon(q) &= \Delta\,\mathrm{dist}(\gamma(\varepsilon), q)\\ &\leq (n-1)\frac{\mathrm{sn}_k'(\gamma_\varepsilon(q)-\varepsilon)}{\mathrm{sn}_k(\gamma_\varepsilon(q)-\varepsilon)}\\ &= (n-1)\frac{\mathrm{sn}_k'(\gamma(q)-\varepsilon)}{\mathrm{sn}_k(\gamma(q)-\varepsilon)}\end{aligned}$$

which descends to $(n-1)\frac{\mathrm{sn}_k'(\gamma(q))}{\mathrm{sn}_k(\gamma(q))}$ as $\varepsilon \to 0$ by monotonicity. This completes the proof.

□

22.3.2. *In the distribution sense.*

Proposition 22.1. Let (M,g) be an orientable Riemannian manifold and $f\colon M \to \mathbb{R}$ a Lipschitz function. Then for any $\varphi \in C_0^\infty(M,\mathbb{R})$, one has

$$-\int_M \langle \nabla\varphi, \nabla f\rangle d\mathrm{vol}_g = \int_M \Delta\varphi \cdot f\, d\mathrm{vol}_g$$

Theorem 22.3.2 (global Laplacian comparison II)**.** Let (M,g) be a complete Riemannian manifold with

$$\mathrm{Ric}(g) \geq (n-1)kg$$

Then for $x \in M$

$$\Delta r(x) \leq (n-1)\frac{\mathrm{sn}_k'(r(x))}{\mathrm{sn}_k(r(x))}$$

in the distribution sense.

Proof. For fixed $p \in M$, the domain $\Sigma(p)$ of injective radius $\mathrm{inj}(p)$ is a star-shaped open subset of T_pM and $M = \exp_p(\Sigma(p)) \cup \mathrm{cut}(p)$. The boundary of $\Sigma(p)$ is locally a graph of continuous function and so there exists a family of star-shaped domains $\{U_j\}$ with smooth boundaries such that

$$U_j \subseteq U_{j+1} \subseteq \cdots \subseteq \Sigma(p), \quad \Sigma(p) = \bigcup U_j$$

If we set $\Omega = \exp_p(\Sigma(p))$. Then $\Omega = \bigcup \Omega_j$, where $\Omega_j = \exp_p(U_j)$. Since each U_j is star-shaped, by Gauss lemma, on each boundary $\partial\Omega_j$, one has $\frac{\partial r}{\partial v} = g(\nabla r, v) \geq 0$ where v is the outer normal vector on $\partial\Omega_j$.

Therefore for each $\varphi \in C_c^\infty(M)$ with $\varphi \geq 0$, one has

$$\begin{aligned}
\int_M r\Delta\varphi \,\mathrm{vol} &\overset{(1)}{=} -\int_M \langle \nabla r, \nabla\varphi\rangle \,\mathrm{vol} \\
&\overset{(2)}{=} -\lim_j \int_{\Omega_j\setminus\{p\}} \langle \nabla r, \nabla\varphi\rangle \\
&\overset{(3)}{=} \lim_j \Big(\int_{\Omega_j\setminus\{p\}} \Delta r\varphi \,\mathrm{vol} - \int_{\partial\Omega_j} \varphi\frac{\partial r}{\partial v}\Big) \\
&\overset{(4)}{\leq} \lim_j \int_{\Omega_j\setminus\{p\}} \Delta r\varphi \,\mathrm{vol} \\
&\overset{(5)}{\leq} \lim_j \int_{\Omega_j\setminus\{p\}} (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\varphi \,\mathrm{vol} \\
&\overset{(6)}{=} \int_{\Omega\setminus\{p\}} (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k'(r)} \,\mathrm{vol} \\
&\overset{(7)}{=} \int_M (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\varphi \,\mathrm{vol}
\end{aligned}$$

where

(1) holds from the fact r is Lipschitz and Proposition 22.1.
(2) and (6) holds from dominated convergence theorem.
(3) holds from Stokes theorem.
(4) holds from $\varphi \geq 0$ and $\frac{\partial r}{\partial v} \geq 0$.
(5) holds from Local Laplacian comparison theorem, that is Theorem 22.1.1.
(7) holds from the fact $\mathrm{cut}(p)$ is zero-measure.

□

22.4. **Volume comparison.**

Lemma 22.4.1. Let (M,g) be a complete, connected Riemannian manifold and $p \in M$. For any $\delta \in \mathbb{R}^+$

$$\exp_p(B(0,\delta) \cap \Sigma(p)) \subseteq B(p,\delta) \subseteq \exp_p(B(0,\delta) \cap \Sigma(p)) \cup \mathrm{cut}(p)$$

In particular, under the map $\Phi\colon \mathbb{R}^+\times\mathbb{S}^{n-1}\to T_pM\setminus\{0\}$ given by $\Phi(\rho,\omega)=\rho\omega$

$$\begin{aligned}\mathrm{Vol}(B(p,\delta)) &= \mathrm{Vol}(\exp_p(B(0,\delta))\cap\Sigma(p))\\ &= \int_{B(0,\delta)\cap\Sigma(p)}\exp_p^*\mathrm{vol}\\ &= \int_{B(0,\delta)}\chi_{\Sigma(p)}\exp_p^*\mathrm{vol}\\ &= \int_{\mathbb{S}^{n-1}}\int_0^\delta \chi_{\Sigma(p)}\sqrt{\det g}\circ\Phi(\rho,\omega)\rho^{n-1}\mathrm{d}\rho\,\mathrm{vol}_{\mathbb{S}^{n-1}}\end{aligned}$$

Corollary 22.4.1. Let $p\in S(n,k)$

(1) If $k\leq 0$. Then for any $\delta\in\mathbb{R}^+$

$$\mathrm{Vol}(B(p,\delta))=\int_{\mathbb{S}^{n-1}}\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\,\mathrm{vol}_{\mathbb{S}^{n-1}}$$

(2) If $k=\frac{1}{R^2}\geq 0$. Then for any $\delta\in\mathbb{R}^+$

$$\mathrm{Vol}(B(p,\delta))=\int_{\mathbb{S}^{n-1}}\int_0^\delta \chi_{B(0,\pi R)}\,\mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\,\mathrm{vol}_{\mathbb{S}^{n-1}}$$

Lemma 22.4.2. Let (M,g) be a Riemannian manifold, and (x^i,U,p) be a geodesic ball chart of radius b around $p\in M$.

(1) If $K\leq k$. Then for each fixed $\omega\in\mathbb{S}^{n-1}$ the volume density ratio

$$\lambda(\rho,\omega)=\frac{\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)}{\mathrm{sn}_k^{n-1}(\rho)}$$

is non-decreasing in $\rho\in(0,b_0)$ where

$$b_0=\begin{cases}b, & k\leq 0\\ \min\{b,\pi R\}, & k=\frac{1}{R^2}\end{cases}$$

Moreover, $\lim_{\rho\to 0}\lambda(\rho,\omega)=1$.

(2) If $K\geq k$ or $\mathrm{Ric}(g)\geq(n-1)kg$. Then for each fixed $\omega\in\mathbb{S}^{n-1}$ the volume density ratio $\lambda(\rho,\omega)$ is non-increasing in $\rho\in(0,b)$ and $\lim_{\rho\to 0}\lambda(\rho,\omega)=1$.

Proof. By Corollary 21.2.1 and Lemma 22.1.1

$$\partial_r\log(r^{n-1}\sqrt{\det g})=\Delta r\geq(n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}=\partial_r\log(\mathrm{sn}_k^{n-1}(r))$$

Hence $\log\left(\frac{r^{n-1}\sqrt{\det g}}{\mathrm{sn}_k^{n-1}(r)}\right)$ is a non-decreasing function of r along each radial geodesic γ, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(\frac{r^{n-1}\sqrt{\det g}}{\mathrm{sn}_k^{n-1}(r)}\right)\circ\gamma(t)\right)\geq 0$$

Hence, $f(r) = \frac{r^{n-1}\sqrt{\det g}}{\mathrm{sn}_k^{n-1}(r)}$ is a non-decreasing function of r along each radial geodesic γ. It is easy to see that $r \circ \Phi = \rho$ (the exponential map is used in normal coordinate). Hence,

$$\lambda(\rho, \omega) = f \circ \Phi(\rho, \omega)$$

is nondecreasing in ρ for any fixed $\omega \in \mathbb{S}^{n-1}$. It is obvious that

$$\lim_{\rho \to 0} \sqrt{\det g} = \lim_{\rho \to 0} \frac{\rho^{n-1}}{\mathrm{sn}_k^{n-1}(\rho)} = 1$$

The proof of (2) is similar. □

Lemma 22.4.3. Let $f : [0, +\infty) \to [0, +\infty), g : [0, +\infty) \to (0, +\infty)$ be two integrable functions. If

$$\lambda(t) = \frac{f(t)}{g(t)} : [0, +\infty) \to [0, +\infty)$$

is non-increasing. Then

$$F(t) = \frac{\int_0^t f(\tau)\mathrm{d}\tau}{\int_0^t g(\tau)\mathrm{d}\tau} : [0, +\infty) \to [0, +\infty)$$

is non-increasing. Moreover, if there exists $0 < t_1 < t_2$ such that

$$F(t_1) = F(t_2),$$

then $\lambda(t) \equiv \lambda(t_1)$ for almost all $t \in [0, t_2]$.

Proof. We can assume $f(t) > 0$ for all $t \in [0, +\infty)$, otherwise we replace it by $f(t) + \varepsilon g(t)$ for some $\varepsilon > 0$. Given $0 < t_1 < t_2$, we need to show

$$\int_0^{t_1} f(\tau)d\tau \int_0^{t_2} g(\tau)d\tau - \int_0^{t_2} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau \geq 0$$

Indeed,

$$\begin{aligned}
&\int_0^{t_1} f(\tau)d\tau \int_0^{t_2} g(\tau)d\tau - \int_0^{t_2} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau \\
=&\int_0^{t_1} f(\tau)d\tau \int_0^{t_2} g(\tau)d\tau - \int_0^{t_1} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau - \int_{t_1}^{t_2} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau \\
=&\int_0^{t_1} f(\tau)d\tau \int_{t_1}^{t_2} g(\tau)d\tau - \int_{t_1}^{t_2} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau \\
\overset{(1)}{\geq}&\int_0^{t_1} \frac{f(t_1)}{g(t_1)} g(\tau)d\tau \int_{t_1}^{t_2} \frac{g(t_1)}{f(t_1)} f(\tau)d\tau - \int_{t_1}^{t_2} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau \\
=&0
\end{aligned}$$

where (1) holds from $\lambda(t)$ is non-increasing. It is clear that if $F(t_1) = F(t_2)$. Then the inequality marked by (1) is an equality, which implies for almost all $t \in [0, t_2], \lambda(t) \equiv \lambda(t_1)$. □

Remark 22.4.1. For any $0 \le \delta_1 < \delta_2 \le \delta_3 < \delta_4$, we can slightly adapt above proof to show

$$\frac{\int_{\delta_3}^{\delta_4} f(\tau)\mathrm{d}\tau}{\int_{\delta_3}^{\delta_4} g(\tau)\mathrm{d}\tau} \le \frac{\int_{\delta_1}^{\delta_2} f(\tau)\mathrm{d}\tau}{\int_{\delta_1}^{\delta_2} g(\tau)\mathrm{d}\tau}$$

Indeed, just note that

$$\begin{aligned}
&\int_{\delta_3}^{\delta_4} f(\tau)\mathrm{d}\tau \int_{\delta_2}^{\delta_1} g(\tau)\mathrm{d}\tau - \int_{\delta_1}^{\delta_2} f(\tau)\mathrm{d}\tau \int_{\delta_3}^{\delta_4} g(\tau)\mathrm{d}\tau \\
\le & \int_{\delta_3}^{\delta_4} \frac{f(\delta_3)}{g(\delta_3)} g(\tau)\mathrm{d}\tau \int_{\delta_2}^{\delta_1} g(\tau)\mathrm{d}\tau - \int_{\delta_1}^{\delta_2} \frac{f(\delta_2)}{g(\delta_2)} g(\tau)\mathrm{d}\tau \int_{\delta_3}^{\delta_4} g(\tau)\mathrm{d}\tau \\
= & (\frac{f(\delta_3)}{g(\delta_3)} - \frac{f(\delta_2)}{g(\delta_2)}) \int_{\delta_2}^{\delta_1} g(\tau)\mathrm{d}\tau \int_{\delta_3}^{\delta_4} g(\tau)\mathrm{d}\tau \\
\le & 0
\end{aligned}$$

Theorem 22.4.1 (Bishop-Gromov)**.** Let (M,g) be a complete Riemannian manifold and $p \in M$. Let $B(p,\delta)$ be the metric ball centered at p with radius δ and g_k be the metric with constant sectional curvature k on $B(p,\delta) \setminus \{p\}$.

(1) Suppose $K \le k$. Then the volume ratio $\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))}$ is non-decreasing for any $0 < \delta \le \delta_0$ where $\delta_0 = \mathrm{inj}(p)$ if $k \le 0$, and $\delta_0 = \min\{\mathrm{inj}(p), \pi/\sqrt{k}\}$ if $k > 0$. Moreover,

$$\lim_{\delta \to 0} \frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} = 1$$

In particular,

$$\mathrm{Vol}_g(B(p,\delta)) \ge \mathrm{Vol}_{g_k}(B(p,\delta)),$$

(2) If $K \ge k$ or $\mathrm{Ric}(g) \ge (n-1)kg$. Then the volume ratio $\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))}$ is non-increasing for $\delta \in \mathbb{R}^+$. Moreover,

$$\lim_{\delta \to 0} \frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} = 1$$

In particular,

$$\mathrm{Vol}_g(B(p,\delta)) \le \mathrm{Vol}_{g_k}(B(p,\delta)),$$

(3) Furthermore, if there exists $\delta_1 < \delta_2$ such that

$$\frac{\mathrm{Vol}_g(B(p,\delta_1))}{\mathrm{Vol}_{g_k}(B(p,\delta_1))} = \frac{\mathrm{Vol}_g(B(p,\delta_2))}{\mathrm{Vol}_{g_k}(B(p,\delta_2))}$$

then $\mathrm{Vol}_g(B(p,\delta)) = \mathrm{Vol}_{g_k}(B(p,\delta))$ for any $\delta \in [0,\delta_2]$ and g has constant sectional curvature k on $B(p,\delta_2)$.

Proof. For (1). By the assumption, we know the metric ball $B(p,\delta)$ is actually a geodesic ball. We have the expression

$$
\begin{aligned}
\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} &\overset{\mathrm{I}}{=} \frac{\int_{\mathbb{S}^{n-1}}\int_0^\delta \rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}{\int_{\mathbb{S}^{n-1}}\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}\\
&\overset{\mathrm{II}}{=} \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\left(\frac{\int_0^\delta \rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho}{\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho}\right)\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}
\end{aligned}
$$

where

I holds from Lemma 22.4.1.
II holds from Fubini's theorem.

By Lemma 22.4.2, one has $\lambda(\rho,\omega)=\frac{\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)}{\mathrm{sn}_k^{n-1}(\rho)}$ is non-decreasing in ρ. Then by Lemma 22.4.3 we have $\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))}$ is non-decreasing in ρ. On the other hand,

$$\lim_{\delta\to 0}\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))}=1$$

Hence, for any $0<\delta\le\delta_0, \mathrm{Vol}_g(B(p,\delta))\ge \mathrm{Vol}_{g_k}(B(p,\delta))$

For (2). Let's divide into the following two cases:

(a) If $k\le 0$, for any $\delta\in\mathbb{R}^+$, we get

$$
\begin{aligned}
\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} &= \frac{\int_{\mathbb{S}^{n-1}}\int_0^\delta \chi_{\Sigma(p)}\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}{\int_{\mathbb{S}^{n-1}}\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}\\
&= \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\left(\frac{\int_0^\delta \chi_{\Sigma(p)}\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho}{\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho}\right)\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}
\end{aligned}
$$

where these two equalities hold from the same reasons. So in this case we consider

$$\widetilde{\lambda}(\rho,\omega):=\chi_{\Sigma(p)}\lambda(\rho,\omega)$$

It's clear $\widetilde{\lambda}$ is also non-increasing in ρ, since $\chi_{\Sigma(p)}$ is just a cut-off function. Then the same argument implies for arbitrary $\delta\in\mathbb{R}^+$, one has $\mathrm{Vol}_g(B(p,\delta))\le\mathrm{Vol}_{g_k}(B(p,\delta))$.

(b) If $k=\frac{1}{R^2}>0$, for any $\delta\in\mathbb{R}^+$, we get

$$
\begin{aligned}
\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} &= \frac{\int_{\mathbb{S}^{n-1}}\int_0^\delta \chi_{\Sigma(p)}\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}{\int_{\mathbb{S}^{n-1}}\int_0^\delta \chi_{B(0,\pi R)}\mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}\\
&= \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\left(\frac{\int_0^\delta \chi_{\Sigma(p)}\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho}{\int_0^\delta \chi_{B(0,\pi R)}\mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho}\right)\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}
\end{aligned}
$$

So in this case we consider[4]

$$\widetilde{\lambda}(\rho,\omega) := \frac{\chi_{\Sigma(p)}}{\chi_{B(0,\pi R)}}\lambda(\rho,\omega)$$

Then the same argument shows the result.

For (3).

□

Corollary 22.4.2. Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g)\geq 0$. Then the volume growth of (M,g) satisfies

$$\mathrm{Vol}_g(B(p,r)) \leq c_n r^n$$

where c_n is a constant >0 depending only on n.

Proof. Consider $k=0$ and use Theorem 22.4.1, one has

$$\mathrm{Vol}_g(B(p,r)) \leq \mathrm{Vol}_{g_0}(B(p,r)) = \frac{\mathrm{Vol}_{g_1}(\mathbb{S}^{n-1})r^n}{n}$$

where $\mathbb{S}^{n-1}$ is the unit sphere. Thus we just set $c_n = \mathrm{Vol}_{g_1}(\mathbb{S}^{n-1})/n$ to conclude. □

Corollary 22.4.3. Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g)\geq 0$. If

$$\lim_{r\to\infty}\frac{\mathrm{Vol}_g(B(p,r))}{r^n} \geq \frac{\mathrm{Vol}_{g_1}(\mathbb{S}^{n-1})}{n}$$

where $\mathbb{S}^{n-1}$ is unit sphere. Then (M,g) is isometric to $(\mathbb{R}^n, g_{\mathrm{can}})$.

Proof. Note that $\mathrm{Vol}_{g_0}(B(p,r)) = \frac{\mathrm{Vol}_{g_1}(\mathbb{S}^{n-1})r^n}{n}$. Then our assumption is equivalent to say

$$\lim_{r\to\infty}\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))} = 1$$

However, by Theorem 22.4.1 we know volume ratio $\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))}$ is non-increasing, with

$$\lim_{r\to 0}\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))} = 1$$

which implies $\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))} = 1$ holds for arbitrary $r>0$. By rigidity of volume comparison, we conclude g has constant sectional curvature 0 on $B(p,r)$ for arbitrary $r>0$. Since $\overline{B(p,\infty)} = M$, we deduce (M,g) has constant sectional curvature 0.

[4]Be careful, our notation here is a little ambiguous, since it's nonsense if $\chi_{B(0,\pi R)} = 0$. However, Myers' theorem implies $\mathrm{diam}(M,g)\leq \pi R$, hence $\Sigma(p)\subseteq B(0,\pi R)$, so here the explicit means of $\frac{\chi_{\Sigma(p)}}{\chi_{B(0,\pi R)}}$ is as follows

$$\frac{\chi_{\Sigma(p)}}{\chi_{B(0,\pi R)}} = \begin{cases} 1, & \delta\in\Sigma(p) \\ 0, & \text{otherwise}\end{cases}$$

Thanks to Hopf's theorem, now it suffices to show M is simply-connected, suppose $\pi\colon \mathbb{R}^n \to M$ is the universal covering, one deduces that

$$|\pi_1(M)| = \frac{\mathrm{Vol}_{g_0}(\mathbb{R}^n)}{\mathrm{Vol}_g(M)} = 1$$

which implies M is simply-connected. □

Corollary 22.4.4. Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g) \geq (n-1)kg$ for some constant $k > 0$. Then

$$\mathrm{Vol}_g(M) \leq \mathrm{Vol}_{g_k}(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$$

If the equality holds. Then (M,g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$ with standard metric.

Proof. Let $k = 1/R^2$. Then Myers' theorem implies $\mathrm{diam}(M,g) \leq \pi R$, thus compact. Hence, for any $p \in M$ one has $\Sigma(p) \subseteq B(0,\pi R)$. Therefore

$$\mathrm{Vol}_g(B(p,\pi R)) = \mathrm{Vol}_g(M)$$

where $B(p,\pi R)$ is a metric ball in M. On the other hand, it is obvious that

$$\mathrm{Vol}_{g_k}(B(p,\pi R)) = \mathrm{Vol}_{g_k}(\mathbb{S}^n(R))$$

Hence by Theorem 22.4.1, one has

$$\mathrm{Vol}_g(M) \leq \mathrm{Vol}_{g_k}(\mathbb{S}^n(R))$$

Furthermore, if the equality holds, g has constant sectional curvature on $B(p,\pi R)$. Then use the argument in Corollary 22.4.3 completes the proof. □

Corollary 22.4.5. Let (M,g) be a complete Riemannian manifold and $p \in M$. Let $B(p,\delta)$ be the metric ball centered at p with radius δ and g_k be the metric with constant sectional curvature k on $B(p,\delta)\setminus\{p\}$. If $\mathrm{Ric}(g) \geq (n-1)kg$. Then for any $0 \leq \delta_1 < \delta_2 \leq \delta_3 < \delta_4$

$$\frac{\mathrm{Vol}_g(B(p,\delta_4)) - \mathrm{Vol}_g(B(p,\delta_3))}{\mathrm{Vol}_g(B(p,\delta_2)) - \mathrm{Vol}_g(B(p,\delta_1))} \leq \frac{\mathrm{Vol}_{g_k}(B(p,\delta_4)) - \mathrm{Vol}_{g_k}(B(p,\delta_3))}{\mathrm{Vol}_{g_k}(B(p,\delta_2)) - \mathrm{Vol}_{g_k}(B(p,\delta_1))}$$

Proof. Just note that volume density ratio is non-decreasing. Then by Remark 22.4.1, one has

$$\frac{\mathrm{Vol}_g(B(p,\delta_4)) - \mathrm{Vol}_g(B(p,\delta_3))}{\mathrm{Vol}_{g_k}(B(p,\delta_4)) - \mathrm{Vol}_{g_k}(B(p,\delta_3))} \leq \frac{\mathrm{Vol}_g(B(p,\delta_2)) - \mathrm{Vol}_g(B(p,\delta_1))}{\mathrm{Vol}_{g_k}(B(p,\delta_2)) - \mathrm{Vol}_{g_k}(B(p,\delta_1))}$$

This gives desired result. □

Theorem 22.4.2 (Cheng)**.** Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g) \geq (n-1)kg$ for some constant $k > 0$. If $\mathrm{diam}(M) = \pi/\sqrt{k}$. Then (M,g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$ with standard metric.

Proof. Let $k = 1/R^2$. Since M is complete, there exist points $p,q \in M$ and $\operatorname{dist}(p,q) = \pi R$, thus for any $\delta \in (0,\pi R)$

$$B(p,\delta) \cap B(q,\pi R - \delta) = \varnothing$$

Then

$$\begin{aligned}
\operatorname{Vol}_g(M) &\overset{(1)}{\geq} \operatorname{Vol}_g(B(p,\delta)) + \operatorname{Vol}_g(B(q,\pi R - \delta)) \\
&\overset{(2)}{\geq} \operatorname{Vol}_{g_k}(B(p,\delta)) \frac{\operatorname{Vol}_g(B(p,\pi R))}{\operatorname{Vol}_{g_k}(B(p,\pi R))} + \operatorname{Vol}_{g_k}(B(q,\pi R - \delta)) \frac{\operatorname{Vol}_g(B(q,\pi R))}{\operatorname{Vol}_{g_k}(B(q,\pi R))} \\
&\overset{(3)}{=} \operatorname{Vol}_g(M)
\end{aligned}$$

where

(1) holds from $B(p,\delta) \cap B(q,\pi R - \delta) = \varnothing$.
(2) holds from Theorem 22.4.1.
(3) holds since for any $x, y \in M$, $\operatorname{Vol}_g(B(x,\pi R)) = \operatorname{Vol}_g(M)$ and

$$\begin{aligned}
&\operatorname{Vol}_{g_k}(B(x,\pi R)) = \operatorname{Vol}_{g_k}(\mathbb{S}^n(R)) \\
&\operatorname{Vol}_{g_k}(B(x,\delta)) + \operatorname{Vol}_{g_k}(B(y,\pi R - \delta)) = \operatorname{Vol}_{g_k}(\mathbb{S}^n(R))
\end{aligned}$$

Hence, for any $0 < \delta < \pi R$.

$$\frac{\operatorname{Vol}_g(B(p,\delta))}{\operatorname{Vol}_{g_k}(B(p,\delta))} = \frac{\operatorname{Vol}_g(B(p,\pi R))}{\operatorname{Vol}_{g_k}(B(p,\pi R))} = \frac{\operatorname{Vol}_g(M)}{\operatorname{Vol}_{g_k}(\mathbb{S}^n(R))}$$

Let $\delta \to 0$, and we deduce $\operatorname{Vol}_g(M) = \operatorname{Vol}_{g_k}(\mathbb{S}^n(R))$. By Proposition 22.4.4, (M,g) is isometric to $\mathbb{S}^n(R)$ with standard metric. □

Theorem 22.4.3 (Bishop-Yau)**.** Let (M,g) be a complete non-compact Riemannian n-manifold with $\operatorname{Ric}(g) \geq 0$. Then the volume growth of (M,g) satisfies

$$c_n \operatorname{Vol}_g(B(p,1)) \cdot r \leq \operatorname{Vol}_g(B(p,r))$$

for $r \geq 1$, where c_n is a positive constant depending only on n.

Proof. Let $x \in \partial B(p,1+r)$. Then

$$B(p,1) \subseteq B(x,2+r) \setminus B(x,r), \quad B(x,r) \subseteq B(p,1+2r)$$

By Corollary 22.4.5, one has

$$\begin{aligned}
\operatorname{Vol}_g(B(p,1)) &\leq \operatorname{Vol}_g(B(x,2+r)) - \operatorname{Vol}_g(B(x,r)) \\
&\leq \operatorname{Vol}_g(B(x,r)) \cdot \frac{\operatorname{Vol}(B(x,2+r)) - \operatorname{Vol}(B(x,r))}{\operatorname{Vol}(B(x,r))} \\
&\leq \operatorname{Vol}_g(B(p,1+2r)) \cdot \frac{(2+r)^n - r^n}{r^n} \\
&\leq \operatorname{Vol}_g(B(p,1+2r)) \cdot \frac{1}{r} c_n
\end{aligned}$$

where $r \geq 1$. By changing variable, we obtain the lower bound. □

Proposition 22.4.1. Let (M,g) be a Cartan-Hadamard manifold with $\mathrm{Ric}(g) \leq -kg$ for some $k > 0$. Then for any $p \in M$

$$\mathrm{Vol}_g(B(p,r)) \geq c_n e^{\sqrt{k}r}$$

where c_n is a positive constant depending only on n.

Proposition 22.4.2 (Cheeger-Colding)**.** For each integer $n \geq 2$, there exists a real number $\delta(n) \in (0,1)$ with the following property: if (M,g) is a compact Riemannian manifold of dimension n with $\mathrm{Ric}(g) \geq (n-1)g$ and

$$\mathrm{Vol}(M,g) \geq (1-\delta(n))\mathrm{Vol}(\mathbb{S}^n)$$

then M is diffeomorphic to $\mathbb{S}^n$.

23. Splitting theorem

23.1. **Geodesic rays.**

Definition 23.1.1 (geodesic ray). A geodesic ray is a unit-speed geodesic $\gamma\colon [0,\infty) \to M$ such that for any $s,t \geq 0$,

$$\operatorname{dist}(\gamma(s),\gamma(t)) = |s-t|$$

Lemma 23.1.1. Let (M,g) be a complete Riemannian manifold. Then the following statements are equivalent.

(1) M is non-compact.
(2) For any $p \in M$, there exists a geodesic ray $\gamma\colon [0,\infty) \to M$ starting from p.

Proof. From (1) to (2). If M is non-compact, for any $p \in M$, there is a sequence of points $\{p_i\}$ such that $\operatorname{dist}(p,p_i) = i$. Let $\gamma_i(t) = \exp_p(tv_i)$ be a unit-speed minimal geodesic connecting p and p_i, that is $\gamma_i(0) = p$ and $\gamma_i(i) = p_i$. By possibly passing to a subsequence, we may assume $v_i \to v \in T_pM$. Then

$$\gamma(t) = \exp_p(tv), \quad t \in [0,+\infty)$$

is a unit-speed geodesic ray. Indeed, for any $s,t \geq 0$, and for any $k > \max\{s,t\}$, one has

$$\operatorname{dist}(\gamma_k(s),\gamma_k(t)) = |s-t|$$

By continuity of exponential map $\exp_p$, one obtains

$$\operatorname{dist}(\gamma(s),\gamma(t)) = \lim_{k\to+\infty} \operatorname{dist}(\gamma_k(s),\gamma_k(t)) = |s-t|$$

Hence γ is a geodesic ray.

From (2) to (1). It's trivial. □

23.2. **Buseman function.**

Definition 23.2.1. Let (M,g) be a complete Riemannian manifold, $p \in M$ and $\gamma\colon [0,\infty) \to M$ be a geodesic ray starting from p. For any $t \geq 0$, $b^t_\gamma : M \to \mathbb{R}$ as

$$b^t_\gamma(x) := \operatorname{dist}(x,\gamma(t)) - t$$

Proposition 23.2.1. Let (M,g) be a complete non-compact Riemannian manifold, $p \in M$ and γ be a geodesic ray starting from p. The function $b^t_\gamma(x) : M \to \mathbb{R}$ has the following properties:

(1) For any fixed $x \in M, b^t_\gamma(x)$ is non-increasing in t.
(2) For any $x \in M$ and $t \geq 0, |b^t_\gamma(x)| \leq \operatorname{dist}(x,\gamma(0))$.
(3) For any $x,y \in M$ and $t \geq 0, |b^t_\gamma(x) - b^t_\gamma(y)| \leq \operatorname{dist}(x,y)$.

Proof. For (1). Note that for $t > s > 0$, one has

$$\begin{aligned}
b^t_\gamma(x) - b^s_\gamma(t) &= \operatorname{dist}(x,\gamma(t)) - \operatorname{dist}(x,\gamma(s)) + s - t \\
&\leq \operatorname{dist}(\gamma(t),\gamma(s)) + s - t \\
&= |t-s| + s - t \\
&= 0
\end{aligned}$$

For (2), (3). Directly from triangle inequality. □

Definition 23.2.2 (Buseman function)**.** The Buseman function with respect to the geodesic ray is defined as

$$b_\gamma := \lim_{t\to\infty} b^t_\gamma(x)$$

Example 23.2.1 (Buseman function on hyperbolic plane)**.** *Note that geodesics on* $\mathbb{H} = \{(x,y)\in\mathbb{R}^2 \mid y \geq 0\}$ *are*

(1) *Semicircles centered on* $\mathbb{R}$.
(2) *Straight lines perpendicular to* $\mathbb{R}$.

Given $x\in\mathbb{H}$*, in order to compute Buseman function*

$$b_\gamma(x) = \lim_{t\to\infty} \operatorname{dist}(x,\gamma(t)) - \operatorname{dist}(\gamma(0),\gamma(t))$$

It suffices to solve the following calculus: Fix $z_1, z_2 \in \mathbb{H}$ *and* $\alpha \in \partial\mathbb{H} = \mathbb{R}\cup\{\infty\}$*, solve*

$$\lim_{q\to\alpha} \operatorname{dist}(q,z_1) - \operatorname{dist}(q,z_2) = ? \tag{23.1}$$

then we can set $q = \gamma(t), \alpha = \gamma(\infty), z_1 = x, z_2 = \gamma(0)$ *to conclude. Let's divide into several steps:*

Step one: *For arbitrary* $r > s > 0$*, the distance between* ri, si *in* $\mathbb{H}$ *is* $\ln\frac{r}{s}$*, where* i *is imaginary number. Indeed, since metric on this line is exactly* $\frac{\mathrm{d}y\otimes\mathrm{d}y}{y^2}$.

Step two: *In hyperbolic planes, it's possible to use isometry to translate any two points to the positive imaginary axis. To be explicit, consider the Möbius transformation* V *mapping Poincaré disk* $\mathbb{D}$ *to* $\mathbb{H}$ *with inverse* V^{-1}*, given by*

$$\begin{aligned} z &= V(w) = \frac{-iw+1}{w-i} \\ w &= V^{-1}(z) = \frac{iz+1}{z+i} \end{aligned}$$

Now for arbitrary $z_1, z_2 \in \mathbb{H}$*, firstly use* V^{-1} *to send* z_1, z_2 *to* $w_1, w_2 \in \mathbb{D}$ *respectively. Then let* $S(w) = e^{i\theta}\frac{w-w_1}{1-\overline{w_1}w}$ *be transformation in* $\mathbb{D}$ *that send* w_1 *to* 0*, with* θ *chosen carefully so that* w_2 *get sent to the positive imaginary axis, that is,* w_2 *get sent to the point* ki*, where* $k = |S(w_2)|$*. Finally apply* V *to this situation,* 0 *gets sent to* i *and* ki *get sents to* $\frac{1+k}{1-k}i$.

Step three: *Combine step one and two, one can conclude that for arbitrary* $z_1, z_2 \in \mathbb{H}$*, the distance between them are*

$$\operatorname{dist}(z_1,z_2) = \ln\frac{1+k}{1-k}$$

If we express k *in terms of* z_1, z_2*, one has*

$$\operatorname{dist}(z_1,z_2) = \ln\frac{|z_1+z_2|+|z_1-z_2|}{|z_1+z_2|-|z_1-z_2|}$$

Step four: *Consider a special case of (23.1), that is we assume* $z_1 = ri, z_2 = i$*, where* $\ln r = \operatorname{dist}(z_1,z_2)$*. Now we choose a sequence* $q_n = u_n + iv_n$ *such that*

$u_n \to \alpha$ and $v_n \to v$, where $v = 0$ as $n \to \infty$. Then

$$\begin{aligned}
\lim_{q\to\alpha}(\mathrm{dist}(q,ri)-\mathrm{dist}(q,i)) &= \lim_{n\to\infty}(\ln(\frac{|q_n+ri|+|q_n-ri|}{|q_n+ri|-|q_n-ri|})-\ln(\frac{|q_n+i|+|q_n-i|}{|q_n+i|-|q_n-i|}))\\
&= \lim_{n\to\infty}(\ln(\frac{|q_n+ri|+|q_n-ri|}{|q_n+i|+|q_n-i|})+\ln(\frac{|q_n+i|-|q_n-i|}{|q_n+ri|-|q_n-ri|}))\\
&= \lim_{n\to\infty}\ln\frac{\sqrt{u_n^2+(v_n+r)^2}+\sqrt{u_n^2+(v_n-r)^2}}{\sqrt{u_n^2+(v_n+1)^2}+\sqrt{u_n^2+(v_n-1)^2}}\\
&\quad+\lim_{n\to\infty}\ln\frac{\sqrt{u_n^2+(v_n+1)^2}-\sqrt{u_n^2+(v_n-1)^2}}{\sqrt{u_n^2+(v_n+r)^2}-\sqrt{u_n^2+(v_n-r)^2}}\\
&= \lim_{v_n\to 0}\ln\underbrace{\frac{\sqrt{\alpha^2+(v_n+r)^2}+\sqrt{\alpha^2+(v_n-r)^2}}{\sqrt{\alpha^2+(v_n+1)^2}+\sqrt{\alpha^2+(v_n-1)^2}}}_{part\ I}\\
&\quad+\lim_{v_n\to 0}\ln\underbrace{\frac{\sqrt{\alpha^2+(v_n+1)^2}-\sqrt{\alpha^2+(v_n-1)^2}}{\sqrt{\alpha^2+(v_n+r)^2}-\sqrt{\alpha^2+(v_n-r)^2}}}_{part\ II}
\end{aligned}$$

It's clear Part I is $\frac{\sqrt{\alpha^2+r^2}}{\sqrt{\alpha^2+1}}$, and apply L'Hospital's rule to Part II one has

$$\lim_{v_n\to 0}\frac{\sqrt{\alpha^2+(v_n+1)^2}-\sqrt{\alpha^2+(v_n-1)^2}}{\sqrt{\alpha^2+(v_n+r)^2}-\sqrt{\alpha^2+(v_n-r)^2}} = \lim_{v_n\to 0}\frac{\frac{v_n+1}{\sqrt{\alpha^2+(v_n+1)^2}}-\frac{v_n-1}{\sqrt{\alpha^2+(v_n-1)^2}}}{\frac{v_n+r}{\sqrt{\alpha^2+(v_n+r)^2}}-\frac{v_n-r}{\sqrt{\alpha^2+(v_n-r)^2}}} = \frac{\sqrt{\alpha^2+r^2}}{r\sqrt{\alpha^2+1}}$$

which implies

$$\lim_{q\to\alpha}\mathrm{dist}(q,ri)-\mathrm{dist}(q,i) = \ln\frac{\alpha^2+r^2}{\alpha^2+1}-\ln r$$

Step five: *In order to solve general case of (23.1), we can use processes in step two to translate z_1, z_2 to the positive imaginary axis. However, α is also translated into a new point α', that is*

$$\alpha' = V\circ S\circ V^{-1}(\alpha)$$

where V, V^{-1} and S are defined in step two. Thus from step four one has

$$\lim_{q\to\alpha}\mathrm{dist}(q,z_1)-\mathrm{dist}(q,z_2) = \ln\frac{(\alpha')^2+r^2}{(\alpha')^2+1}-\ln r$$

where $\ln r = \mathrm{dist}(z_1,z_2)$.

Proposition 23.2.2. Let (M,g) be a complete non-compact Riemannian manifold, $p\in M$ and γ be a geodesic ray starting from p. The Busemann function $b_\gamma\colon M\to\mathbb{R}$ is Lipschitz continuous with $\mathrm{Lip}(b_\gamma)\le 1$

Proof. It follows from Arezla-Ascoli lemma. □

Proposition 23.2.3. Let (M,g) be a complete non-compact Riemannian manifold, and γ be a geodesic ray starting from $p \in M$. If $\mathrm{Ric}(g) \geq 0$. Then

$$\Delta b_\gamma \leq 0$$

in the sense of distribution.

Proof. For any non-negative smooth function $\varphi \in C_0^\infty(M)$, one has

$$\begin{aligned}\int_M \Delta\varphi b_\gamma^t \,\mathrm{vol} &= \int_M \Delta\varphi(\mathrm{dist}(x,\gamma(t)) - t)\,\mathrm{vol} \\ &\overset{(1)}{=} \int_M \Delta\varphi\,\mathrm{dist}(x,\gamma(t))\,\mathrm{vol} \\ &\overset{(2)}{\leq} \int_M \frac{(n-1)\varphi}{\mathrm{dist}(x,\gamma(t))}\,\mathrm{vol}\end{aligned}$$

where

(1) holds from Stokes' theorem.
(2) holds from Theorem 22.3.2.

Then Lebesgue's dominated convergence implies

$$\int_M \Delta\varphi b_\gamma \,\mathrm{vol} \leq 0$$

□

Definition 23.2.3 (geodesic line)**.** A geodesic line is a unit-speed geodesic $\gamma\colon (-\infty,\infty) \to M$ such that for any $s,t \in \mathbb{R}$,

$$\mathrm{dist}(\gamma(s),\gamma(t)) = |s-t|$$

Lemma 23.2.1. Let (M,g) be a connected, non-compact Riemannian manifold. If M contains a compact subset K such that $M \setminus K$ has at least two unbounded components[5]. Then there is a geodesic line passing through K.

Proof. Since $M \setminus K$ has at least two unbounded components, there are two unbounded sequences of points $\{p_i\}$ and $\{q_i\}$ such that any curve from p_i to q_i passes through K. Let $\gamma_i\colon [-a_i,b_i] \to M$ be minimal geodesics connecting p_i and q_i with $\gamma_i(-a_i) = p_i$, $\gamma_i(b_i) = q_i$ and $\gamma_i(0) \in K$. Hence, $a_i \to +\infty$ and $b_i \to +\infty$. By possibly passing to subsequences, $\{\gamma_i\}$ converges to a geodesic line $\gamma_\infty\colon (-\infty,+\infty) \to M$. □

Proposition 23.2.4. Let (M,g) be a complete non-compact Riemannian manifold with $\mathrm{Ric}(g) \geq 0$. If (M,g) contains a geodesic line γ. Then $b_{\gamma_\pm}\colon M \to \mathbb{R}$ are smooth harmonic functions with

$$|\nabla b_{\gamma_\pm}| = 1, \quad \mathrm{Hess}\, b_{\gamma_\pm} = 0$$

where $\gamma_\pm(t) = \gamma(\pm t)\colon [0,+\infty) \to M$.

---

[5] Some authors use ends" to call such unbounded components.

Proof. Let $b(x) = b_{\gamma_+}(x) + b_{\gamma_-}(x)$. By the triangle inequality

$$\begin{aligned} b(x) &= \lim_{s\to+\infty} \operatorname{dist}(x,\gamma_+(s)) + \operatorname{dist}(x,\gamma_-(s)) - 2s \\ &= \lim_{s\to+\infty} \operatorname{dist}(x,\gamma(s)) + \operatorname{dist}(x,\gamma(-s)) - 2s \\ &\geq 0 \end{aligned}$$

By Proposition 23.2.3, $\Delta b \leq 0$ in the sense of distributions. On the other hand,

$$b(\gamma(t)) = \lim_{s\to+\infty} \operatorname{dist}(\gamma(t),\gamma(s)) + \operatorname{dist}(\gamma(t),\gamma(-s)) - 2s = 0$$

Hence the subharmonic function b attains its absolute minimum, by Theorem 22.2.1, $b \equiv 0$, that is $b_{\gamma_+} = -b_{\gamma_-}$. Hence $\Delta b_{\gamma_+} = \Delta b_{\gamma_-} = 0$, and by Wely's lemma one has $b_{\gamma_\pm}$ are smooth.

Bochner's formula says

$$\frac{1}{2}\Delta|\nabla f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f,\nabla f) + g(\nabla\Delta f,\nabla f)$$

Let $f = b_{\gamma_+}$. Then

$$\frac{1}{2}\Delta|\nabla b_{\gamma_+}|^2 \geq |\operatorname{Hess} b_{\gamma_+}|^2 \geq 0$$

since b_{γ_+} is harmonic and $\operatorname{Ric}(g) \geq 0$, thus $|\nabla b_{\gamma_+}|^2$ is superharmonic. On the other hand, by Proposition 23.2.2, $\operatorname{Lip}(b_{\gamma_+}) \leq 1$, and so $|\nabla b_{\gamma_+}| \leq 1$. Note that

$$b_{\gamma_+}(\gamma_+(t)) = \lim_{s\to+\infty} \operatorname{dist}(\gamma_+(t),\gamma_+(s)) - s = \lim_{s\to+\infty} |t-s| - s = -t$$

For any $x = \gamma_+(t_0)$

$$|\nabla b_{\gamma_+}|(x) \overset{(1)}{=} |\nabla b_{\gamma_+}||\gamma_+'(t_0)| \overset{(2)}{\geq} |\langle \nabla b_{\gamma_+}(x), \gamma_+'(t_0)\rangle| = 1$$

where

(1) holds from the trivial fact γ_+ is unit-speed.
(2) holds from Cauchy-Schwarz inequality.

Hence, the superharmonic function $|\nabla b_{\gamma_+}|^2$ attains its absolute maximum in M, hence $|\nabla b_{\gamma_+}|^2 \equiv 1$ on M. Again by the Bochner formula, one has Hess $b_{\gamma_+} = 0$. The same argument holds for b_{γ_-}, this completes the proof. □

Lemma 23.2.2. Let (M,g) be a complete Riemannian manifold, and V a smooth vector field with $|V|_g \leq C$ for some constant C. Then V is a complete vector field.

Proof. We need to show the integral curve of V is globally defined, that is defined on $\mathbb{R}$. Suppose $\gamma\colon (a,b) \to M$ is an integral curve of M and $b < \infty$. For arbitrary $t,s \in (a,b)$, we have

$$\gamma(t) = \gamma(s) + \int_s^t V(\gamma(\tau))\mathrm{d}\tau$$

By using the boundedness of V, we can conclude that

$$|\gamma(t) - \gamma(s)| \leq C|t-s|$$

which implies $\gamma(t)$ is uniformly continuous on (a,b), thus it's possible to extend γ to $(a,b]$ since $b < \infty$, a contradiction. □

Proposition 23.2.5. Let (M, g) be a complete Riemannian manifold. Suppose $f \in C^\infty(M, \mathbb{R})$ satisfies

$$|\nabla f| = 1 \quad \text{and} \quad \operatorname{Hess} f = 0$$

Let Σ denote $f^{-1}(0)$, with induced metric $h := g|_\Sigma$.

(1) (Σ, h) is a totally geodesic submanifold of (M, g).

(2) The map

$$f : (\mathbb{R} \times \Sigma, g_{\mathbb{R}} \oplus h) \to (M, g), \quad F(t, p) = \exp_p(t \nabla_p f)$$

is an isometry.

Proof. For (1). Recall that (Σ, h) is a totally geodesic submanifold of (M, g) if the second fundamental form of Σ vanishes, and facts in basic differential geometry says the second fundamental form of a hyperplane Σ with induced metric is given by

$$\mathbf{II}(v, w) := \langle \nabla_v n, w \rangle$$

where n is the normal vector of Σ. In this case, if we consider $\Sigma = f^{-1}(0)$. Then the normal vector of Σ is exactly ∇f, and thus

$$\mathbf{II}(v, w) := \langle \nabla_v \nabla f, w \rangle$$

Then $\operatorname{Hess} f = \nabla^2 f = 0$ implies the second fundamental form of Σ vanishes, that is Σ is a totally geodesic submanifold of (M, g).

For (2). For a fixed p, let $X = \nabla f$, and consider $\gamma(t) = \exp_p(tX_p)$. Since $\nabla X = 0$, we have $E(t) = X(\gamma(t))$ and $\gamma'(t)$ are two parallel vector fields along γ with the same initial value. Hence

$$\gamma'(t) = X(\gamma(t))$$

that is γ is exactly the integral curve of X. Furthermore, since $|X| = 1$, by Lemma 23.2.2 one has γ is globally defined, and one can deduce F is a global flow of X, thus it's a diffeomorphism.

Now it remains to prove that F is an isometry. For $v \in T_p\Sigma$, let J be the Jacobi field along γ with $J(0) = 0$ and $J'(0) = v$. By the radial curvature equation

$$R(\text{-}, \nabla f, \nabla f, \text{-}) = \operatorname{Hess}(\frac{1}{2}|\nabla f|^2)(\text{-}, \text{-}) - (\nabla_{\nabla f} \operatorname{Hess} f)(\text{-}, \text{-}) - \operatorname{Hess} f(\nabla_{\text{-}} \nabla f, \text{-})$$

one has $R(\text{-}, \nabla f, \nabla f, \text{-}) = 0$, thus Jacobi equation

$$J''(t) + R(J, \gamma')\gamma' = 0$$

reduces to $J''(t) = 0$. It implies that $J'(t)$ is a parallel vector field and in particular, $|J'(t)| \equiv |J'(0)| = |v|$. By uniqueness of Jacobi fields, we deduce

$$J(t) = tJ'(t)$$

Then F is an isometry holds as follows:

(a) It is easy to see that $(\mathrm{d}F)_{(1,p)}v = J(1)$, thus $|(\mathrm{d}F)_{(1,p)}v| = |J(1)| = |J'(1)| = |v|$.

(b) $|(\mathrm{d}F)_{(0,p)}\partial_t| = |\nabla f| = 1 = |\partial_t|$.

□

23.3. **Splitting theorem and its corollaries.**

Theorem 23.3.1 (splitting theorem)**.** Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g)\geq 0$. If there is a geodesic line in M. Then (M,g) is isometric to $(\mathbb{R}\times N, g_{\mathbb{R}}\oplus g_N)$, where $\mathrm{Ric}(g_N)\geq 0$.

Proof. Directly from Proposition 23.2.4 and Proposition 23.2.5. □

Corollary 23.3.1. Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g)\geq 0$

(1) (M,g) is isometric to $(\mathbb{R}^k\times N, g_{\mathbb{R}^k}\oplus g_N)$, where N does not contain a geodesic line and $\mathrm{Ric}(g_N)\geq 0$.
(2) The isometry group splits

$$\mathrm{Iso}(M,g)\cong \mathrm{Iso}(\mathbb{R}^k)\times \mathrm{Iso}(N,g_N)$$

Theorem 23.3.2 (structure theorem for manifold with $\mathrm{Ric}\geq 0$)**.** Let (M,g) be a compact Riemannian manifold with $\mathrm{Ric}(g)\geq 0$, and $\pi\colon(\widetilde{M},\widetilde{g})\to(M,g)$ is its universal covering with the pullback metric.

(1) There exists some integer $k\geq 0$ and a compact Riemannian manifold (N,g_N) with $\mathrm{Ric}(g_N)\geq 0$ such that $(\widetilde{M},\widetilde{g})$ is isometric to $(\mathbb{R}^k\times N, g_{\mathrm{can}}\oplus g_N)$.
(2) The isometry group splits

$$\mathrm{Iso}(\tilde{M},\widetilde{g})\cong \mathrm{Iso}(\mathbb{R}^k)\times \mathrm{Iso}(N,g_N)$$

Proof. For (1). Suppose to the contrary that N is non-compact. Then fix a point $x_0\in N$, there exists a geodesic ray $\gamma\colon[0,\infty)\to N$ starting from x_0. Since M is compact, there exists a compact subset $\widetilde{K}\subseteq\widetilde{M}$ such that

$$\mathrm{Aut}_\pi(\widetilde{M})\widetilde{K}=\widetilde{M}$$

□

Corollary 23.3.2. $\mathbb{S}^n\times\mathbb{S}^1$ doesn't admit any Ricci flat metrics when $n=2,3$.

Proof. If $\mathbb{S}^n\times\mathbb{S}^1$ admits a Ricci flat metric, after splitting its universal covering we obtain a Ricci flat metric on $\mathbb{S}^n$. However, $\mathbb{S}^n$ doesn't admit such a metric when $n=2,3$. Indeed, since any Einstein manifold with dimension 2 or 3 has constant sectional curvature, thus if $\mathbb{S}^n, n=2,3$ admit a Ricci flat metric. Then it has constant sectional curvature 0, and it's also simply-connected, so Hopf's theorem implies it's diffeomorphic to $\mathbb{R}^n$, a contradiction. □

Remark 23.3.1. It's clear $\mathbb{S}^1\times\mathbb{S}^1$ admits a Ricci flat metric, and when $n\geq 4$, we don't know whether $\mathbb{S}^n$ admit a Ricci flat metric or not.

Corollary 23.3.3. Let (M,g) be a compact Riemannian manifold with $\mathrm{Ric}(g)\geq 0$, and $(\widetilde{M},\widetilde{g})$ is its universal covering equipped with pullback metric.

(1) If $\widetilde{M}$ is contractible, then $(\widetilde{M},\widetilde{g})$ is isometric to $(\mathbb{R}^n,g_{\mathrm{can}})$ and (M,g) is flat.
(2) If $(\widetilde{M},\widetilde{g})$ doesn't contain a geodesic line, then $\pi_1(M)$ is finite and $b_1(M)=0$.

Proof. For (1). If $\widetilde{M} \cong N \times \mathbb{R}^k$ is contractible, we must have N is just a point, since it's compact,

For (2). If $\widetilde{M}$ doesn't contain a geodesic line. Then $\widetilde{M}$ is compact, which implies $|\pi_1(M)|$ is finite. Furthermore, since there is a natural Hurwicz surjective

$$h : \pi_1(M) \to H_1(M.\mathbb{Z})$$

thus $H_1(M.\mathbb{Z})$ can't have free part, otherwise h can't be surjective, since there is no surjective map from a finite group to an infinite one. So we have $b_1(M) = 0$. □

Corollary 23.3.4. Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric}(g) \geq 0$. If there exists a point $p \in M$ such that $\mathrm{Ric}(g) > 0$ on T_pM. Then $\pi_1(M)$ is finite and $b_1(M) = 0$.

Proof. Since $\mathrm{Ric}(g) > 0$ on the whole tangent space T_pM, the universal covering $(\widetilde{M}, \widetilde{g})$ can't split into a product $(\mathbb{R}^k \times N, g_{\mathrm{can}} \oplus g_N)$, since metric on $\widetilde{M}$ is pullback metric, and g_{can} on $\mathbb{R}^k$ has vanishing Ricci curvature. Thus $\widetilde{M}$ is compact, consequently, we have $|\pi_1(M)|$ is finite and $b_1(M) = 0$. □

Remark 23.3.2. We have already seen this in Bochner's technique.

Part 8. **Appendix**

Appendix A. Fundamental group

A.1. **Homotopy.** In this section we assume $I = [0,1]$, a path in a topological space X is a continuous map $\gamma\colon I \to X$ and a loop is a path γ such that $\gamma(0) = \gamma(1)$.

Definition A.1.1 (homotopy)**.** Let X and Y be topological spaces and $f,g\colon X \to Y$ be continuous maps. A homotopy from f to g is a continuous map $F\colon X \times I \to Y$ such that for all $x \in X$, one has

$$F(x,0) = f(x)$$
$$F(x,1) = g(x)$$

If there exists a homotopy from f to g, then we say f and g are homotopic, and write $f \simeq g$.

Definition A.1.2 (stationary homotopy)**.** Let X and Y be topological spaces and $A \subseteq X$ an arbitrary subset. A homotopy F between continuous maps $f,g\colon X \to Y$ is said to be stationary on A if

$$F(x,t) = f(x)$$

for all $x \in A$ and $t \in I$. If there exists such a homotopy, then we say f and g are homotopic relative to A.

Remark A.1.1. If f and g are homotopic relative to A, then f must agree with g on A.

Definition A.1.3 (path homotopy)**.** Let X be a topological space and γ_1,γ_2 be two paths in X. They are said to be path homotopic if they are homotopic relative on $\{0,1\}$, and write $\gamma_1 \simeq \gamma_2$.

Definition A.1.4 (loop homotopy)**.** Let X be a topological space and γ_1,γ_2 be two loops in X. They're called loop homotopic if they are homotopic relative on $\{0\}$, and write $\gamma_1 \simeq \gamma_2$.

Remark A.1.2. For convenience, if γ_1,γ_2 are paths (or loops), then when we say γ_1 is homotopic to γ_2, we mean γ_1 is path (or loop) homotopic to γ_2.

Definition A.1.5 (free homotopy)**.** Let X be a topological space and γ_1,γ_2 be two loops in X. They are said to be free (loop) homotopic if they're homotopic through loops (but not necessarily preserving the base point), that is, there exists a homotopy $F(s,t)\colon [0,1]\times[0,1] \to X$ such that

$$F(s,0) = \gamma_1(s)$$
$$F(s,1) = \gamma_2(s)$$
$$F(0,t) = F(1,t) \text{ holds for all } t \in [0,1]$$

A.2. **Fundamental group.**

Proposition A.2.1. Let X be a topological space. For any $p,q \in X$, path homotopy is an equivalence relation on the set of all paths in X from p to q. For any path γ in X, the path homotopy class is denoted by $[\gamma]$.

Proof. For path $\gamma\colon I \to X$, γ is homotopic to itself by $F(s,t) = \gamma(s)$. If γ_1 is homotopic to γ_2 by F, then γ_2 is homotopic to γ_1 by $G(s,t) = F(s,1-t)$. Finally, suppose γ_1 is homotopic to γ_2 by F, γ_2 is homotopic to γ_3 by G. Then consider

$$H = \begin{cases} F(s,2t) & 0 \le t \le \frac{1}{2} \\ G(s,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

which is a homotopy from γ_1 to γ_3. This shows path homotopy is an equivalence relation. □

Definition A.2.1 (reparametrization)**.** A reparametrization of a path $f: I \to X$ is of the form $f \circ \varphi$ for some continuous map $\varphi\colon I \to I$ fixing 0 and 1.

Lemma A.2.1. Any reparametrization of a path f is homotopic to f.

Proof. Suppose $f \circ \varphi$ is a reparametrization of f, and let $F\colon I \times I \to I$ denote the straight-line homotopy from the identity map to φ, that is, $F(s,t) = t\varphi(s) + (1-t)s$. Then $f \circ F$ is a path homotopy from f to $f \circ \varphi$. □

Definition A.2.2 (product of path)**.** Let X be a topological space and f,g be paths. f and g are composable if $f(1) = g(0)$. If f and g are composable, their product $f \cdot g\colon I \to X$ is defined by

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Proposition A.2.2. Let X be a topological space and f,g be paths in X such that $f \simeq g$. If $\overline{f}$ is the path obtained by reversing f, that is $\overline{f}(s) := f(1-s)$, then $\overline{f} \simeq \overline{g}$.

Proof. Suppose f is homotopic to g by homotopy F. Then $G(s,t) := F(1-s,t)$ is a homotopy from $\overline{f}$ to $\overline{g}$ since

$$\begin{aligned} G(s,0) &= F(1-s,0) = f(1-s) = \overline{f}(s) \\ G(s,1) &= F(1-s,1) = g(1-s) = \overline{g}(s) \end{aligned}$$

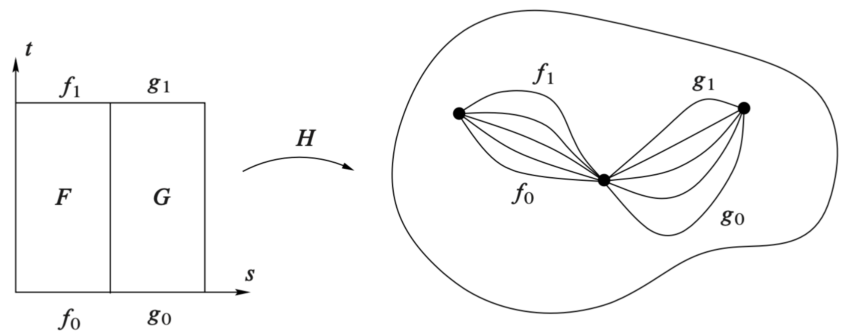
□

Proposition A.2.3. Let X be a topological space and f_0, f_1, g_0, g_1 be paths in X such that f_0, g_0 are composable and f_1, g_1 are composable. If $f_0 \simeq g_0$, $f_1 \simeq g_1$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.

Proof. Suppose the homotopy from f_0 to f_1 is given by F and the homotopy from g_0 to g_1 is given by G. Then the required homotopy H from $f_0 \cdot g_0$ to $f_1 \cdot g_1$ is given by

$$H(s,t) = \begin{cases} F(2s,t) & 0 \le s \le \frac{1}{2}, 0 \le t \le 1 \\ G(2s-1,t) & \frac{1}{2} \le s \le 1, 0 \le t \le 1 \end{cases}$$

□



Remark A.2.1. With above propositions, it makes sense to define the composition of path homotopy classes by setting $[\gamma_1]\cdot[\gamma_2] := [\gamma_1\cdot\gamma_2]$, and use the notation $[\overline{\gamma}]$.

Proposition A.2.4. Let X be a topological space and $[f],[g],[h]$ be homotopy classes of loops based at $p\in X$.

(1) $[c_p]\cdot[f]=[f]\cdot[c_p]=[f]$, where c_p is constant loop based at p.
(2) $[f]\cdot[\overline{f}]=[c_p]$ and $[\overline{f}]\cdot[f]=[c_p]$.
(3) $[f]\cdot([g]\cdot[h])=([f]\cdot[g])\cdot[h]$.

Proof. For (1). Let us show that $c_p\cdot f\simeq f$, and the other case is similar. Define $H\colon I\times I\to X$ by

$$H(s,t)=\begin{cases} p & t\geq 2s\\ f(\frac{2s-t}{2-t}) & t\leq 2s\end{cases}$$

This map is continuous since $f(0)=p$, and it's clear to see $H(s,0)=f(s)$ and $H(s,1)=c_p\cdot f(s)$. Thus H gives the desired homotopy.

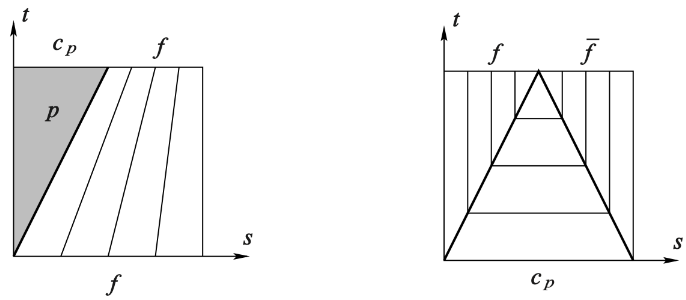
For (2). It suffices to show that $f\cdot\overline{f}\simeq c_p$, since the reverse path of $\overline{f}$ is f, the other relation follows by interchanging the roles of f and $\overline{f}$. Define

$$H(s,t)=\begin{cases} f(2s) & 0\leq s\leq \frac{t}{2}\\ f(t) & \frac{t}{2}\leq s\leq 1-\frac{t}{2}\\ f(2-2s) & 1-\frac{t}{2}\leq s\leq 1\end{cases}$$

It is easy to check that H is a homotopy from c_p to $f\cdot\overline{f}$.

For (3). It suffices to show $(f\cdot g)\cdot h\simeq f\cdot(g\cdot h)$. The first path follows f and then g at quadruple speed for $s\in[0,\frac{1}{2}]$, and then follows h at double speed for $s\in[\frac{1}{2},1]$, while the second follows f at double speed and then g and h at quadruple speed. The two paths are therefore reparametrizations of each other and thus homotopic by Lemma A.2.1.

□



Definition A.2.3 (fundamental group)**.** Let X be a topological space. The fundamental group of X based at $p \in X$, denoted by $\pi_1(X,p)$, is the set of path homotopy classes of loops based at p equipped with composition as its group structure.

Theorem A.2.1 (base point change)**.** Let X be a topological space, $p,q \in X$ and g is any path from p to q. The map

$$\begin{aligned}\Phi_g : \pi_1(X,p) &\to \pi_1(X,q)\\ [f] &\mapsto [\overline{g}]\cdot[f]\cdot[g]\end{aligned}$$

is a group isomorphism with inverse $\Phi_{\overline{g}}$.

Proof. It suffices to show Θ_g is a group homomorphism, since it's clear $\Phi_g \circ \Phi_{\overline{g}} = \Phi_{\overline{g}} \circ \Phi_g = \mathrm{id}$. For $[\gamma_1],[\gamma_2] \in \pi_1(X,p)$, one has

$$\begin{aligned}\Phi_g[\gamma_1]\cdot\Phi[\gamma_2] &= [\overline{g}]\cdot[\gamma_1]\cdot[g]\cdot[\overline{g}]\cdot[\gamma_2]\cdot[g]\\ &= [\overline{g}]\cdot[\gamma_1]\cdot[c_p]\cdot[\gamma_2]\cdot[g]\\ &= [\overline{g}]\cdot[\gamma_1]\cdot[\gamma_2]\cdot[g]\\ &= \Phi_g([\gamma_1]\cdot[\gamma_2])\end{aligned}$$

□

Corollary A.2.1. If X is a path-connected topological space, then its fundamental is independent of the choice of base point, and denoted by $\pi_1(X)$ for convenience.

Definition A.2.4. If X is a path-connected topological space with $\pi_1(X) = 0$, then it's called simply-connected.

Theorem A.2.2. The fundamental group of a topological manifold M is countable.

Proof. Since M is second countable, there exists a countable cover $\mathcal{U}$ of M consisting of coordinate balls, and for each $U,U' \in \mathcal{U}$ the intersection $U \cap U'$ has at most countably many components. We choose a point in each such component and let $\mathcal{X}$ denote the (countable) set consisting of all the chosen points as U,U' range over all the sets in $\mathcal{U}$. For each $U \in \mathcal{U}$ and $x,x' \in \mathcal{X}$ such that $x,x' \in U$, choose a definite path $h^U_{x,x'}$ from x to x' in U.

Now choose any point $p \in \mathcal{X}$ as base point. Let us say that a loop based at p is special if it is a finite product of paths of the form $h^{U}_{x,x'}$. Because both $\mathcal{U}$ and $\mathcal{X}$ are countable sets, there are only countably many special loops. Each special loop determines an element of $\pi_1(M,p)$. If we can show that every element of $\pi_1(M,p)$ is obtained in this way, we are done, because we will have exhibited a surjective map from a countable set onto $\pi_1(M,p)$.

So suppose f is any loop based at p. By the Lebesgue number lemma there is an integer n such that f maps each subinterval $[(k-1)/n, k/n]$ into one of the balls in $\mathcal{U}$, which is called U_k. Let $f_k = f|_{[(k-1)/n,k/n]}$ reparametrized on the unit interval, so that $[f] = [f_1]\cdot\cdots\cdot[f_n]$.

For each $k = 1,\dots,n-1$, the point $f(k/n)$ lies in $U_k \cap U_{k+1}$. Therefore, there is some $x_k \in \mathcal{X}$ that lies in the same component of $U_k \cap U_{k+1}$ as $f(k/n)$. Choose a path g_k in $U_k \cap U_{k+1}$ from x_k to $f(k/n)$, and set $\widetilde{f}_k = g_{k-1}\cdot f_k \cdot \overline{g}_k$ (taking $x_k = p$ and g_k to be the constant path c_p when $k = 0$ or n). It is immediate that $[f] = [\widetilde{f}_1]\cdot\cdots\cdot[\widetilde{f}_n]$, because all the g_k 's cancel out. But for each $k, \widetilde{f}_k$ is a path in U_k from x_{k-1} to x_k, and since U_k is simply connected, $\widetilde{f}_k$ is path-homotopic to $h^{U_k}_{x_{k-1}x_k}$. This shows that f is path-homotopic to a special loop and completes the proof. □

APPENDIX B. COVERING SPACE

In this section, we assume[6] all topological space are connected and locally path connected topological spaces, and all maps between them are continuous. References for this section are [Hat02] and [Lee10].

Definition B.0.1 (covering space)**.** A covering space of X is a map $\pi\colon \widetilde{X} \to X$ such that there exists a discrete space D and for each $x \in X$ an open neighborhood $U \subseteq X$, such that $\pi^{-1}(U) = \coprod_{d\in D} V_d$ and $\pi|_{V_d}\colon V_d \to U$ is a homeomorphism for each $d \in D$.

(1) Such a U is called evenly covered by $\{V_d\}$.
(2) The open sets $\{V_d\}$ are called sheets.
(3) For each $x \in X$, the discrete subset $\pi^{-1}(x)$ is called the fiber of x.
(4) The degree of the covering is the cardinality of the space D.

Definition B.0.2 (isomorphism between covering spaces)**.** Let $\pi_1\colon \widetilde{X}_1 \to X$ and $\pi_2\colon \widetilde{X}_2 \to X$ be two covering spaces. An isomorphism between covering spaces is a homeomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $\pi_1 = \pi_2 \circ f$.

B.1. **Proper map.**

Definition B.1.1 (proper)**.** Let $f\colon X \to Y$ be a continuous map between topological spaces. f is called proper if preimage of any compact set in Y is a compact subset in X.

Lemma B.1.1. Let $p\colon X \to Y$ be a proper map between topological spaces and Y be locally compact and Hausdorff. Then p is a closed map.

Proof. Let C be a closed subset of X. We need to prove that $p(C)$ is closed in Y, that is to prove $Y \setminus p(C)$ is open. Let $y \in Y \setminus p(C)$. Then y has an compact neighborhood V since Y is locally compact. Then $p^{-1}(V)$ is compact since f is proper. Let $E = C \cap p^{-1}(V)$. Then E is a compact and hence so is $p(E)$. Then $p(E)$ is closed since compact set in Hausdorff space is closed. Let $U = V \setminus p(E)$. Then U is an open neighborhood of y and disjoint from $p(C)$. This shows $Y \setminus p(C)$ is open as desired. □

Corollary B.1.1. Let $p\colon X \to Y$ be a proper map between topological spaces and Y be locally compact and Hausdorff. If $y \in Y$ and V is an open neighborhood of $p^{-1}(y)$, then there exists an open neighborhood U of y with $p^{-1}(U) \subseteq V$.

Proof. Since V is open, one has $X \setminus V$ is closed, and thus $A := p(X \setminus V)$ is also closed with $y \notin A$ since p is a closed map by Lemma B.1.1. Thus $U := Y \setminus A$ is an open neighborhood of y such that $p^{-1}(U) \subseteq V$. □

[6]We are including these hypotheses since most of the interesting results (such as lifting criterion) require them, and most of the interesting topological space (such as connected topological manifold) satisfy them. In fact, it's almost the strongest connected hypotheses, since if a topological space is connected and locally path-connected, then it's also path connected.

Theorem B.1.1. Let $p\colon X \to Y$ be a proper local homeomorphism between topological spaces and Y be locally compact and Hausdorff. Then p is a covering map.

Proof. For $y \in Y$, one has $\{y\}$ is a compact set since Y is locally compact and Hausdorff, and hence so is $p^{-1}(y)$ since p is proper. On the other hand, $p^{-1}(y)$ is a discrete set since p is a local homeomorphism. Then $p^{-1}(y)$ is a finite set, and we denote it by $\{x_1,\dots,x_n\}$. Since p is a local homeomorphism, for each $i = 1,\dots,n$, there exists an open neighborhood W_i of x_i and an open neighborhood U_i of y such that $p|_{W_i}$ is a homeomorphism. Without lose of generality we may assume W_i are pairwise disjoint. Now $W_1 \cup \cdots \cup W_n$ is an open neighborhood of $p^{-1}(y)$. Thus by Corollary B.1.1 there exists an open neighborhood $U \subseteq U_1 \cap \cdots \cap U_n$ of y with $p^{-1}(U) \subseteq W_1 \cup \cdots \cup W_n$. If we let $V_i = W_i \cap p^{-1}(U)$, then the V_i are disjoint open sets with

$$p^{-1}(U) = V_1 \cup \cdots \cup V_n$$

and all the mappings $p|_{V_i}$ are homeomorphisms. This shows p is a covering map. □

B.2. **The lifting theorems.**

Proposition B.2.1 (unique lifting property)**.** Let $\pi\colon \widetilde{X} \to X$ be a covering space and a map $f\colon Y \to X$. If two lifts $\widetilde{f}_1, \widetilde{f}_2\colon Y \to \widetilde{X}$ of f agree at one point of Y, then $\widetilde{f}_1$ and $\widetilde{f}_2$ agree on all of Y.

Proof. Let A be the set consisting of points of Y where $\widetilde{f}_1$ and $\widetilde{f}_2$ agree. If $\widetilde{f}_1$ agrees with $\widetilde{f}_2$ at some point of Y, then A is not empty, and we may assume $A \neq Y$, otherwise there is nothing to prove. For $y \notin A$, let $\widetilde{U}_1$ and $\widetilde{U}_2$ be the sheets containing $\widetilde{f}_1(y)$ and $\widetilde{f}_2(y)$ respectively. By continuity of $\widetilde{f}_1$ and $\widetilde{f}_2$, there exists a neighborhood N of y mapped into $\widetilde{U}_1$ by $\widetilde{f}_1$ and mapped into $\widetilde{U}_2$ by $\widetilde{f}_2$. Since $\widetilde{f}_1(y) \neq \widetilde{f}_2(y)$, then $\widetilde{U}_1 \cap \widetilde{U}_2 = \varnothing$. This shows $\widetilde{f}_2 \neq \widetilde{f}_2$ throughout the neighborhood N, and thus $Y \setminus A$ is open, that is A is closed. To see A is open, for $y \in A$ one has $\widetilde{f}_1(y) = \widetilde{f}_2(y)$, and thus $\widetilde{U}_1 = \widetilde{U}_2$. Since $\pi|_{\widetilde{U}_1}$ is a diffeomorphism, one has $\widetilde{f}_1 = \pi^{-1} \circ f = \widetilde{f}_2$ on $\widetilde{U}_i$. This shows the set A is open, and thus $A = Y$ since Y is connected. □

Theorem B.2.1 (homotopy lifting property)**.** Let $\pi\colon \widetilde{X} \to X$ be a covering space and $F\colon Y \times I \to X$ be a homotopy. If there exists a map $\widetilde{F}\colon Y \times \{0\} \to \widetilde{X}$ which lifts $F|_{Y\times\{0\}}$, then there exists a unique homotopy $\widetilde{F}\colon Y \times I \to \widetilde{X}$ which lifts F and restricting to the given $\widetilde{F}$ on $Y \times \{0\}$. Furthermore, if F is stationary on A, so is $\widetilde{F}$.

Proof. Firstly, let's construct a lift $\widetilde{F}\colon N \times I \to \widetilde{X}$ for some neighborhood N in Y of a given point $y_0 \in Y$. Since F is continuous, every point $(y_0, t) \in Y \times I$ has a product neighborhood $N_t \times (a_t, b_t)$ such that $F(N_t \times (a_t, b_t))$ is contained in an evenly covered neighborhood of $F(y_0, t)$. By compactness of $\{y_0\} \times I$, finitely many such products $N_t \times (a_t, b_t)$ cover $\{y_0\} \times I$. This implies that we can choose a single neighborhood N of y_0 and a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of I

such that for each i, one has $F(N\times[t_i,t_{i+1}])$ is contained in an evenly covered neighborhood U_i. Suppose $\widetilde{F}$ has been constructed on $N\times[0,t_i]$, starting with the given $\widetilde{F}$ on $N\times\{0\}$. Since U_i is evenly covered, there is an open set $\widetilde{U}_i$ of $\widetilde{X}$ projecting homeomorphically onto U_i by π and containing the point $\widetilde{F}(y_0,t_i)$. After replacing N by a smaller neighborhood of y_0 we may assume that $\widetilde{F}(N\times\{t_i\})$ is contained in $\widetilde{U}_i$. Now we can define $\widetilde{F}$ on $N\times[t_i,t_{i+1}]$ to be the composition of F with the homeomorphism $\pi^{-1}\colon U_i\to\widetilde{U}_i$ since $F(N\times[t_i,t_{i+1}])\subseteq U_i$, After a finite number of steps we eventually get a lift $\widetilde{F}\colon N\times I\to\widetilde{X}$ for some neighborhood N of y_0.

Next we show the uniqueness part in the special case that Y is a point, since in this case we can omit Y from the notation. Suppose $\widetilde{F}$ and $\widetilde{F}'$ are two lifts of $F\colon I\to X$ such that $\widetilde{F}(0)=\widetilde{F}'(0)$. As before, choose a partition $0=t_0<t_1<\cdots<t_m=1$ of I so that for each i, one has $F([t_i,t_{i+1}])$ is contained in some evenly covered neighborhood U_i. Assume inductively that $\widetilde{F}=\widetilde{F}'$ on $[0,t_i]$. Since $[t_i,t_{i+1}]$ is connected, so is $\widetilde{F}([t_i,t_{i+1}])$, which must therefore lie in a single one of the disjoint open sets $\widetilde{U}_i$ projecting homeomorphically to U_i. Similarly, $\widetilde{F}'([t_i,t_{i+1}])$ lies in a single $\widetilde{U}_i$, in fact in the same one that contains $\widetilde{F}([t_i,t_{i+1}])$ since $\widetilde{F}'(t_i)=\widetilde{F}(t_i)$. Because π is injective on $\widetilde{U}_i$ and $\pi\circ\widetilde{F}=\pi\circ\widetilde{F}'$, it follows that $\widetilde{F}=\widetilde{F}'$ on $[t_i,t_{i+1}]$, and the induction step is finished.

The last step in the proof of is to observe that since the $\widetilde{F}$ constructed above on sets of the form $N\times I$ are unique when restricted to each segment $\{y\}\times I$, they must agree whenever two such sets $N\times I$ overlap. So we obtain a well-defined lift $\widetilde{F}$ on all of $Y\times I$. This $\widetilde{F}$ is continuous since it is continuous on each $N\times I$, and $\widetilde{F}$ is unique since it is unique on each segment $\{y\}\times I$. □

Corollary B.2.1 (path lifting property)**.** Let $\pi\colon\widetilde{X}\to X$ be a covering space. Suppose $\gamma\colon I\to X$ is any path, and $\widetilde{x}\in\widetilde{X}$ is any point in the fiber of $\pi^{-1}(\gamma(0))$. Then there exists a unique lift $\widetilde{\gamma}\colon I\to\widetilde{X}$ of γ such that $\widetilde{\gamma}(0)=\widetilde{x}$.

Proof. Let Y be a point and F be the path γ in Theorem B.2.1. □

Corollary B.2.2 (monodromy theorem)**.** Let $\pi\colon\widetilde{X}\to X$ be a covering space. Suppose γ_1 and γ_2 are paths in X which are homotopic, and $\widetilde{\gamma}_1,\widetilde{\gamma}_2$ are their lifts with the same initial point. Then $\widetilde{\gamma}_1$ is homotopic to $\widetilde{\gamma}_2$.

Proof. Suppose $F\colon I\times I\to X$ is the homotopy from γ_1 to γ_2 which is stationary on $\{0,1\}$ and $\widetilde{\gamma}_1,\widetilde{\gamma}_2$ are lifts of γ_1,γ_2 with the same initial point. Then by Theorem B.2.1 there exists a homotopy $\widetilde{F}\colon I\times I\to\widetilde{X}$ from $\widetilde{\gamma}_1$ to $\widetilde{\gamma}_2$ which is also stationary on $\{0,1\}$, which shows $\widetilde{\gamma}_1$ is homotopic to $\widetilde{\gamma}_2$. □

Corollary B.2.3. Let $\pi\colon(\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ be a covering space. Then

(1) The map $\pi_*\colon\pi_1(\widetilde{X},\widetilde{x}_0)\to\pi_1(X,x_0)$ is injective.
(2) $\pi_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ consists of the homotopy class of loops in X whose lifts to $\widetilde{X}$ are still loops.
(3) The index of $\pi_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ in $\pi_1(X,x_0)$ is the degree of covering. In particular, the degree of universal covering equals $|\pi_1(X,x_0)|$.

Proof. For (1). An element of $\ker\pi_*$ is represented by a loop $\widetilde{\gamma}_0\colon I\to\widetilde{X}$ with a homotopy F of $\gamma_0=\pi\circ\widetilde{\gamma}_0$ to the trivial loop γ_1. By Theorem B.2.1 there is a lifted homotopy of loops $\widetilde{F}$ starting with $\widetilde{\gamma}_0$ and ending with a constant loop. Hence $[\widetilde{\gamma}_0] = 0$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$ and π_* is injective.

For (2). The loops at x_0 lifting to loops at $\widetilde{x}_0$ certainly represent elements of the image of $\pi_*: \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. Conversely, a loop representing an element of the image of π_* is homotopic to a loop having such a lift, so by Theorem B.2.1, the loop itself must have such a lift.

For (3). For a loop γ in X based at x_0, let $\widetilde{\gamma}$ be its lift to $\widetilde{X}$ starting at $\widetilde{x}_0$. A product $h \cdot \gamma$ with $[h] \in H = \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ has the lift $\widetilde{h} \cdot \widetilde{\gamma}$ ending at the same point as $\widetilde{\gamma}$ since $\widetilde{h}$ is a loop. Thus we may define a function Φ from cosets $H[\gamma]$ to $\pi^{-1}(x_0)$ by sending $H[\gamma]$ to $\widetilde{\gamma}(1)$. The path-connectedness of $\widetilde{X}$ implies that Φ is surjective since $\widetilde{x}_0$ can be joined to any point in $\pi^{-1}(x_0)$ by a path $\widetilde{\gamma}$ projecting to a loop γ at x_0. To see that Φ is injective, observe that $\Phi(H[\gamma_1]) = \Phi(H[\gamma_2])$ implies that $\gamma_1 \cdot \overline{\gamma}_2$ lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$, so $[\gamma_1][\gamma_2]^{-1} \in H$ and hence $H[\gamma_1] = H[\gamma_2]$. Thus the index of H is the same as $|\pi^{-1}(x_0)|$, which is the degree of the covering. □

Proposition B.2.2 (lifting criterion)**.** Let $\pi: (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space and $f: (Y, y_0) \to (X, x_0)$ be a map. A lift $\widetilde{f}: (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

Proof. The only if" statement is obvious since $f_* = \pi_* \circ f_*$. Conversely, let $y \in Y$ and let γ be a path in Y from y_0 to y. By Corollary B.2.1, the path $f\gamma$ in X starting at x_0 has a unique lift $\widetilde{f\gamma}$ starting at $\widetilde{x}_0$, and we define $\widetilde{f}(y) = \widetilde{f\gamma}(1)$.

To see it's well-defined, let γ' be another path from y_0 to y. Then $(f\gamma') \cdot (\overline{f\gamma})$ is a loop h_0 at x_0 with $[h_0] \in f_*(\pi_1(Y, y_0)) \subseteq \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. This means there is a homotopy H of h_0 to a loop h_1 that lifts to a loop $\widetilde{h}_1$ in $\widetilde{X}$ based at $\widetilde{x}_0$. Apply Theorem B.2.1 to H to get a lifting $\widetilde{H}$. Since $\widetilde{h}_1$ is a loop at $\widetilde{x}_0$, so is $\widetilde{h}_0$. By Proposition B.2.1, that is uniqueness of lifted paths, the first half of $\widetilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half is $\widetilde{f\gamma}$ traversed backwards, with the common midpoint $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. This shows $\widetilde{f}$ is well-defined.

To see $\widetilde{f}$ is continuous, let $U \subseteq X$ be an open neighborhood of $f(y)$ having a lift $\widetilde{U} \subseteq \widetilde{X}$ containing $\widetilde{f}(y)$ such that $\pi: \widetilde{U} \to U$ is a homeomorphism. Choose a path-connected open neighborhood V of y with $f(V) \subseteq V$. For paths from y_0 to points $y' \in V$, we can take a fixed path γ from y_0 to y followed by paths η in V from y to points y'. Then the paths $(f\gamma) \cdot (f\eta)$ in X have lifts $(\widetilde{f\gamma}) \cdot (\widetilde{f\eta})$ where $\widetilde{f\eta} = \pi^{-1} f\eta$. Thus $\widetilde{f}(V) \subseteq \widetilde{U}$ and $\widetilde{f}|_V = \pi^{-1} f$, so $\widetilde{f}$ is continuous at y. □

Corollary B.2.4. Let $\pi: \widetilde{X} \to X$ be a covering space and Y be a simply-connected space. Then every map $f: Y \to X$ has a lift.

Proof. It's clear $f_*(\pi_1(Y, y_0)) \subseteq \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ since $\pi_1(Y, y_0) = 0$. □

Theorem B.2.2. Suppose X is a topological manifold, E is a Hausdorff space and $\pi: E \to X$ is a local homeomorphism with the path lifting property. Then π is a covering space.

Proof. See Theorem 4.19 of [**?**]. □

B.3. **The classification of the covering spaces.**

Definition B.3.1 (universal covering)**.** A simply-connected covering space of X is called universal covering.

Definition B.3.2 (semilocally simply-connected)**.** A topological space X is called semilocally simply-connected if each $x \in X$ has a neighborhood U such that the inclusion induced map $\pi_1(U,x) \to \pi_1(X,x)$ is trivial.

Theorem B.3.1. If X is a semilocally simply-connected topological space, then X has a universal covering $\widetilde{X}$.

Proof. See construction in page 64 of [Hat02]. □

Proposition B.3.1. Suppose X is a semilocally simply-connected topological space. Then for every subgroup $H \subseteq \pi_1(X,x_0)$, there exists a covering space $\pi\colon X_H \to X$ such that $\pi_*(\pi_1(X_H,\widetilde{x}_0)) = H$ for a suitably chosen based point $\widetilde{x}_0 \in X_H$.

Proof. See Proposition 1.36 of [Hat02]. □

Lemma B.3.1. Let $\pi_1\colon \widetilde{X}_1 \to X$ and $\pi_2\colon \widetilde{X}_2 \to X$ be two coverings. There exists an isomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ taking a basepoint $\widetilde{x}_1 \in \pi_1^{-1}(x_0)$ to a basepoint $\widetilde{x}_2 \in \pi_2^{-1}(x_0)$ if and only if $\pi_{1*}(\pi_1(\widetilde{X}_1,\widetilde{x}_1)) = \pi_{2*}(\pi_1(\widetilde{X}_2,\widetilde{x}_2))$.

Proof. If there is an isomorphism $f\colon (\widetilde{X}_1,\widetilde{x}_1) \to (\widetilde{X}_2,\widetilde{x}_2)$, then from the two relations $\pi_1 = \pi_2 \circ f$ and $\pi_2 = \pi_1 \circ f^{-1}$ it follows that $\pi_{1*}(\pi_1(\widetilde{X}_1,\widetilde{x}_1)) = \pi_{2*}(\pi_1(\widetilde{X}_2,\widetilde{x}_2))$. Conversely, suppose that $\pi_{1*}(\pi_1(\widetilde{X}_1,\widetilde{x}_1)) = \pi_{2*}(\pi_1(\widetilde{X}_2,\widetilde{x}_2))$. By Proposition B.2.2, that is lifting criterion, we may lift π_1 to a map $\widetilde{\pi}_1\colon (\widetilde{X}_1,\widetilde{x}_1) \to (\widetilde{X}_2,\widetilde{x}_2)$ with $\pi_2 \circ \widetilde{\pi}_1 = \pi_1$. Similarly, one has $\widetilde{\pi}_2\colon (\widetilde{X}_2,\widetilde{x}_2) \to (\widetilde{X}_1,\widetilde{x}_1)$ with $\pi_1 \circ \widetilde{\pi}_2 = \pi_2$. Then Proposition B.2.1, that is the unique lifting property, $\widetilde{\pi}_1 \circ \widetilde{\pi}_2 = \mathrm{id}$ and $\widetilde{\pi}_2 \circ \widetilde{\pi}_1 = \mathrm{id}$ since these composed lifts fix the basepoints. □

Lemma B.3.2. For covering $\pi\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$, changing the basepoint $\widetilde{x}_0$ within $\pi^{-1}(x_0)$ corresponds exactly to changing $\pi_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ to a conjugate subgroup of $\pi_1(X,x_0)$.

Proof. Let $\widetilde{x}_1$ be another basepoint in $\pi^{-1}(x_0)$ and $\widetilde{\gamma}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}_1$. Then $\widetilde{\gamma}$ projects to a loop γ in X representing some element $g \in \pi_1(X,x_0)$. If we denote $H_i = \pi_*(\pi_1(\widetilde{X},\widetilde{x}_i))$ for $i = 0,1$, there is an inclusion $g^{-1}H_0 g \subseteq H_1$ since if $\widetilde{f}$ is a loop at $\widetilde{x}_0$, one has $\widetilde{\gamma}^{-1} \cdot \widetilde{f} \cdot \widetilde{\gamma}$ is a loop at $\widetilde{x}_1$. Similarly one has $gH_1g^{-1} \subseteq H_0$. This shows changing the basepoint from $\widetilde{x}_0$ to $\widetilde{x}_1$ changes H_0 to the conjugate subgroup $H_1 = g^{-1}H_0 g$. □

Theorem B.3.2. Let X be a semilocally simply-connected topological space. Then there is a bijection between the set of basepoint-preserving isomorphism classes of covering spaces $\pi\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ and the the set of subgroups of $\pi_1(X,x_0)$ obtained by associating the subgroup $\pi_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ to the covering space $(\widetilde{X},\widetilde{x}_0)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of covering spaces $\pi\colon \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X,x_0)$.

Proof. Proposition B.3.1 and Lemma B.3.1 completes the proof of the first half, and Lemma B.3.2 completes the proof of the last half. □

Corollary B.3.1. Let X be a semilocally simply-connected topological space. Then the universal covering of X is unique up to isomorphism.

B.4. The structure of the deck transformation group.

Definition B.4.1 (deck transformation)**.** Let $\pi\colon \widetilde{X} \to X$ be a covering space. The deck transformation group is following set

$$\mathrm{Aut}_\pi(\widetilde{X}) = \{f : \widetilde{X} \to \widetilde{X} \text{ is homeomorphism} \mid \pi \circ f = \pi\}$$

equipped with composition as group operation.

Proposition B.4.1. Let $\pi\colon \widetilde{X} \to X$ be a covering space. The deck transformation group $\mathrm{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ freely.

Proof. Suppose $f : \widetilde{X} \to \widetilde{X}$ is a deck transformation admitting a fixed point. Since $\pi \circ f = \pi$, we may regard f as a lift of π, and identity map of $\widetilde{X}$ is another lift of π. By Proposition B.2.1, that is unique lifting property, one has f is exactly identity map since it agrees with identity map at fixed point. □

Definition B.4.2 (normal)**.** A covering $\pi\colon \widetilde{X} \to X$ is called normal, if any deck transformation acts transitively on each fiber of $x \in X$.

Proposition B.4.2. Let $\pi\colon \widetilde{X} \to X$ be a normal covering. Then $\widetilde{X}/\mathrm{Aut}_\pi(\widetilde{X})$ is homeomorphic to X.

Proof. Let $\Phi\colon \widetilde{X}/\mathrm{Aut}_\pi(\widetilde{X}) \to X$ be the map sending the orbit $\mathcal{O}_{\widetilde{x}}$ to $\pi(\widetilde{x})$, where $\widetilde{x} \in \widetilde{X}$. It's clear Φ is well-defined bijection since $\mathrm{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ fiberwise transitive, and the following diagram commutes

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "p"'] \arrow[dr, "\pi"] & \\
\widetilde{X}/\mathrm{Aut}_\pi(\widetilde{X}) \arrow[r, "\Phi"'] & X
\end{tikzcd}$$

This diagram shows Φ is both continuous and open, since p is the quotient map and π is continuous and open, which shows $\widetilde{X}/\mathrm{Aut}_\pi(\widetilde{X})$ is homeomorphic to X. □

Proposition B.4.3. Let $\pi\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space and $H = \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$. Then

(1) π is a normal covering if and only if H is a normal subgroup of $\pi_1(X, x_0)$.
(2) $\mathrm{Aut}_\pi(\widetilde{X})$ is isomorphic to the quotient $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

In particular, $\mathrm{Aut}_\pi(\widetilde{X}) \cong \pi_1(X, x_0)$ if $\widetilde{X}$ is universal covering.

Proof. For (1). By proof of Lemma B.3.2 one has changing the basepoint $\widetilde{x}_0 \in \pi^{-1}(x_0)$ to $\widetilde{x}_1 \in \pi^{-1}(x_0)$ corresponds precisely to conjugating H by an element $[\gamma] \in \pi_1(X, x_0)$ where γ lifts to a path $\widetilde{\gamma}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. Thus $[\gamma]$ is in the normalizer $N(H)$ if and only if $\pi_*(\pi_1(\widetilde{x},\widetilde{x}_0)) = \pi_*(\pi_1(\widetilde{x},\widetilde{x}_1))$, which is equivalent to the existence of a deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$ by Lemma B.3.1. Thus the covering space is normal if and only if $N(H) = \pi_1(X,x_0)$, that is, $H \subseteq \pi_1(X,x_0)$ is a normal subgroup.

For (2). Define $\varphi\colon N(H) \to \mathrm{Aut}_\pi(\widetilde{X})$ by sending $[\gamma]$ to the deck transformation τ taking $\widetilde{x}_0$ to $\widetilde{x}_1$, in the notation above. Let's show φ is a homomorphism. If γ' is another loop corresponding to the deck transformation τ' taking $\widetilde{x}_0$ to $\widetilde{x}_1'$, then $\gamma\cdot\gamma'$ lifts to $\widetilde{\gamma}\cdot(\tau(\widetilde{\gamma}'))$, a path from $\widetilde{x}_0$ to $\tau(\widetilde{x}_1') = \tau\tau'(\widetilde{x}_0)$, so $\tau\tau'$ is the deck transformation corresponding to $[\gamma][\gamma']$. By the proof of (1) one has φ is surjective. The kernel of φ consists of classes $[\gamma]$ lifting to loops in $\widetilde{x}$, which are exactly the elements of $\pi_*(\pi_1(\widetilde{X},\widetilde{x}_0)) = H$. □

Corollary B.4.1. Let X be a topological space and $\pi\colon \widetilde{X} \to X$ be its universal covering space. Then the quotient space $\widetilde{X}/\pi_1(X)$ is homeomorphic X.

Proof. It follows from Proposition B.4.2 since $\pi_1(X) \cong \mathrm{Aut}_\pi(\widetilde{X})$ if $\widetilde{X}$ is the universal covering. □

B.5. **Covering of manifold.**

Lemma B.5.1. Let X be a topological space admitting a countable open covering $\{U_i\}$ such that each set U_i is second countable in the subspace topology. Then X is second countable.

Proof. Let $\mathcal{B}_\alpha$ be a countable base for U_α. Its members are by definition open in U_α, and as all U_α are open in X, these sets are also open in X. So $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$ is a countable family of open sets in X. Suppose that $x \in X$ and V is open in X with $x \in V$. Then $x \in U_\beta$ for some index β. Now apply the definition of a base to see that for some $B \in \mathcal{B}_\beta$ we have $x \in B \subseteq V \cap U_\beta$. This $B \in \mathcal{B}$ and $x \in B \subseteq V$. This shows that $\mathcal{B}$ is a countable base for X. □

Theorem B.5.1. Suppose M is a topological n-manifold and let $\pi\colon \widetilde{M} \to M$ be a covering map. Then $\widetilde{M}$ is a topological n-manifold.

Proof. Since π is a local diffeomorphism and M is locally Euclidean, one has $\widetilde{M}$ is also locally Euclidean. Now let's show $\widetilde{M}$ is Hausdorff, let $\widetilde{x}_1, \widetilde{x}_2$ be two distinct points in $\widetilde{M}$. If $\pi(\widetilde{x}_1) = \pi(\widetilde{x}_2)$ and $U \subseteq M$ is an evenly covered open subset containing $\pi(\widetilde{x}_1)$, then the component of $\pi^{-1}(U)$ containing $\widetilde{x}_1$ and $\widetilde{x}_2$ are disjoint open subsets of $\widetilde{M}$ that separate $\widetilde{x}_1$ and $\widetilde{x}_2$. If $\pi(\widetilde{x}_1) \neq \pi(\widetilde{x}_2)$, there are disjoint open subsets $U_1, U_2 \subseteq M$ containing $\pi(\widetilde{x}_1)$ and $\pi(\widetilde{x}_2)$ since M is Hausdorff, and then $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint open subsets of $\widetilde{M}$ containing $\widetilde{x}_1$ and $\widetilde{x}_2$, and thus $\widetilde{M}$ is Hausdorff.

To see $\widetilde{M}$ is second countable, firstly note that each fiber of π is countable since by Corollary B.2.3 one has the degree of covering less than or equal $|\pi(M,x)|$, and by Theorem A.2.2 one has the fundamental group of a topological manifold is countable.

The collection of all evenly covered open subsets is an open covering of M, and therefore has a countable subcover $\{U_i\}$. For any given i, each component of $\pi^{-1}(U_i)$ contains exactly one point in each fiber over U_i, so $\pi^{-1}(U_i)$ has countably many components. The collection of all components of all sets of the form $\pi^{-1}(U_i)$ is a countable open covering of $\widetilde{M}$. Since each such component is second countable, by Lemma B.5.1 one has $\widetilde{M}$ is also second countable. □

B.6. **Orientable double covering.**

B.6.1. *Topological and homological orientation.*

Definition B.6.1 (orientation preserving homeomorphism)**.** Let $U,V\subseteq\mathbb{R}^n$ be open subsets. A homeomorphism $f:U\to V$ is called orientation preserving if for each $x\in U$, the map

$$H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})\cong H_n(U,U\setminus\{x\})\xrightarrow{f_*}H_n(V,V\setminus\{f(x)\})\cong H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})$$

is the identity map.

Definition B.6.2 (topological orientation)**.** A topological orientation of a topological n-manifold M is the choice of a maximal oriented atlas, where an atlas $\{(U_i,\varphi_i)\}$ is called oriented if all coordinate changes $\varphi_i\circ\varphi_j^{-1}$ is orientation preserving.

Definition B.6.3 (oriented topological manifold)**.** A topological manifold M is called orientable if it has a topological orientation, and M is called oriented if M is equipped with an orientation.

Definition B.6.4 (local homological orientation)**.** A local homological orientation of a topological n-manifold M is the choice of a generator $[M]_x$ of the local homology group $H_n(M,M\setminus\{x\})$ for each $x\in M$.

Definition B.6.5 (homological orientation)**.** A local homological orientation of a topological n-manifold M is called a homological orientation if for each $x\in M$, there is an open neighborhood U and a class $\alpha\in H_n(M,M\setminus U)$ such that the map induced by $(M,M\setminus U)\to(M,M\setminus\{x\})$ maps α to $[M]_x$ for each $x\in U$.

Theorem B.6.1. The topological orientation and homological orientation are equivalent for topological manifold.

B.6.2. *Existence of orientable double covering.*

Theorem B.6.2. Every non-orientable topological manifold admits an orientable double covering.

APPENDIX C. GROUP ACTION

C.1. *G*-set.

Definition C.1.1 (group action)**.** Let G be a group and S be a set. A left G-action on S is a function

$$\theta : G \times S \to S$$

satisfying the following two axioms:

(1) $\theta(e,s) = s$, where $e \in G$ is the identity element.
(2) $\theta(g_1, \theta(g_2,s)) = \theta(g_1 g_2, s)$, where $g_1, g_2 \in G$.

For convenience we denote $\theta(g,s) = gs$ for $g \in G, s \in S$.

Definition C.1.2 (G-set)**.** Let G be a group. A set S endowed with a left (or right) G-action is called a left (or right) G-set.

Definition C.1.3 (orbit)**.** An orbit of a group action is the set of all images of a single element under the action by different group elements.

Definition C.1.4. Let G be a group and S be a left G-set.

(1) For $g \in G$, if $gs = s$ for some $s \in S$ implies $g = e$, then the group action is called free.
(2) For $g \in G$, if $gs = s$ for all $s \in S$ implies $g = e$, then the group action is called effective.
(3) If for arbitrary $s_1, s_2 \in S$, there exists $g \in G$ such that $gs_1 = s_2$, then the group action is called transitive.

Remark C.1.1. If a group action is free, then it's effective, but converse statement may not hold.

Definition C.1.5 (isotropy group)**.** Let G be a group and S be a right G-set. For any $s \in G$, the isotropy group of s, denoted by G_s, is the set of all elements of G that fix s, that is

$$G_s = \{g \in G \mid gs = s\}$$

Remark C.1.2. It's clear to see the action is free if and only if the isotropy group of every point is trivial.

C.2. Continuous action.

Definition C.2.1 (act by homeomorphisms)**.** Let Γ be a group and X be a topological space. The group Γ is calld acting X by homeomorphisms, if Γ acts on X, and for every $g \in \Gamma$, the map $x \mapsto gx$ is a homeomorphism.

Definition C.2.2 (topological group)**.** A group is called a topological group, if it's a topological space such that the multiplication and the inversion are continuous.

Definition C.2.3 (continuous action)**.** Let X be a topological space and G a topological group. A continuous G-action on X is given by the following data:

(1) G acts on X by homeomorphisms.

(2) The map $G \times X \to X$ given by $(g,x) \mapsto gx$ is continuous.

Lemma C.2.1. Let X be a topological space and Γ a group acting on X by homeomorphisms. Then the quotient map $\pi \colon X \to X/\Gamma$ is an open map.

Proof. For any $g \in \Gamma$ and any subset $U \subseteq X$, the set $gU \subseteq X$ is defined as

$$gU = \{gx \mid x \in U\}$$

If $U \subseteq X$ is open, then $\pi^{-1}(\pi(U))$ is the union of all sets of the form gU as g ranges over G. Since $p \mapsto gp$ is a homeomorphism, each set is open, and therefore $\pi^{-1}(\pi(U))$ is open in X. Since π is a quotient map, this implies $\pi(U)$ is open in X/Γ, and therefore π is an open map. □

C.2.1. *Proper action.*

Definition C.2.4 (proper)**.** Let X be a topological space and G a topological group. A continuous G-action on X is called proper if the continuous map

$$\begin{aligned}\Theta \colon G \times X &\to X \times X \\ (g,x) &\mapsto (gx,x)\end{aligned}$$

is proper, that is, the preimage of a compact set is compact.

Lemma C.2.2. Let X,Y be topological spaces and $\pi \colon X \to Y$ be an open quotient map. Then Y is Hausdorff if and only if the set $\mathcal{R} = \{(x_1,x_2) \mid \pi(x_1) = \pi(x_2)\}$ is closed in $X \times X$.

Proposition C.2.1. Let X be a topological space and G a topological group acting on X continuously. If the action is also proper, then the orbit space is Hausdorff.

Proof. Let $\Theta \colon G \times X \to X \times X$ be the proper map $\Theta(g,x) = (gx,x)$ and $\pi \colon X \to X/G$ be the quotient map. Define the orbit relation $\mathcal{O} \subseteq X \times X$ by

$$\mathcal{O} = \Theta(G \times X) = \{(gx,x) \mid x \in X, g \in G\}$$

Since proper continuous map is closed, it follows that $\mathcal{O}$ is closed in $X \times X$, and since π is open by Lemma C.2.1, one has X/G is Hausdorff by Lemma C.2.2. □

Proposition C.2.2. Let M be a topological manifold and G a topological group acting on M continuously. The following statements are equivalent.

(1) The action is proper.
(2) If $\{p_i\}$ is a sequence in M and $\{g_i\}$ is a sequence in G such that both $\{p_i\}$ and $\{g_i p_i\}$ converge, then a subsequence of $\{g_i\}$ converges.
(3) For every compact subset $K \subseteq M$, the set $G_K = \{g \in G \mid gK \cap K \neq \varnothing\}$ is compact.

Proof. Along the proof, let $\Theta \colon G \times M \to M \times M$ denote the map $(g,p) \mapsto (gp,p)$. For (1) to (2). Suppose Θ is proper, and $\{p_i\}$, $\{g_i\}$ are sequences satisfying the hypotheses of (2). Let U and V be precompact[7] neighborhoods of $p = \lim_i p_i$

[7] A set is called precompact, if its closure is compact.

and $q = \lim_i g_i p_i$. The assumption implies $\Theta(g_i, p_i)$ all lie in compact set $\overline{U} \times \overline{V}$ when i is sufficiently large, so there exists a subsequence of $\{(g_i, p_i)\}$ converges in $G \times M$ since Θ is proper. In particular, this means that a subsequence of $\{g_i\}$ converges in G.

For (2) to (3). Let K be a compact subset of M, and suppose $\{g_i\}$ is any sequence in G_K. This means for each i, there exists $p_i \in g_i K \cap K$, which is to say that $p_i \in K$ and $g_i^{-1} p_i \in K$. By passing to a subsequence twice, we may assume both $\{p_i\}$ and $\{g_i^{-1} p_i\}$ converge, and the assumption implies there exists a convergent subsequence of $\{g_i\}$. Since each sequence of G_K has a convergent subsequence, G_K is compact.

For (3) to (1). Suppose $L \subseteq M \times M$ is compact, and let $K = \pi_1(L) \cup \pi_2(L)$, where $\pi_1, \pi_2 \colon M \times M \to M$ are the projections onto the first and second factors, respectively. Then

$$\Theta^{-1}(L) \subseteq \Theta^{-1}(K \times K) = \{(g, p) \mid gp \in K, p \in K\} \subseteq G_K \times K$$

By assumption $G_K \times K$ is compact, and thus $\Theta^{-1}(L)$ is compact since it's a closed subset of a compact subset, which implies the action is proper. □

Corollary C.2.1. If G is a compact topological group, then every continuous G-action on M is proper.

Proof. Since G is compact, then every sequence in G admits a convergent subsequence, and thus the action is proper by (2) of Proposition C.2.2. □

C.3. **Properly discontinuous action.**

Definition C.3.1 (properly discontinuous)**.** Let Γ be a group acting on a topological space X by homeomorphisms. The action is called properly discontinuous, if every point $x \in X$ has a neighborhood U such that for each $g \in G$, $gU \cap U = \varnothing$ unless $g = e$.

Lemma C.3.1. Suppose Γ be a group acting properly discontinuous on a topological space X. Then every subgroup of Γ still acts properly discontinuous on X.

Lemma C.3.2. Let $\pi \colon \widetilde{X} \to X$ be a covering space. Then $\mathrm{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ properly discontinuous.

Proof. For $\widetilde{x} \in \widetilde{x}$, let $\widetilde{U} \subseteq \widetilde{X}$ be an open neighborhood of $\widetilde{x}$ projecting homeomorphically to $U \subseteq X$. If there exists $g \in \mathrm{Aut}_\pi(\widetilde{X})$ such that $g(\widetilde{U}) \cap \widetilde{U} \neq \varnothing$, then $g\widetilde{x}_1 = \widetilde{x}_2$ for some $\widetilde{x}_1, \widetilde{x}_2 \in U$. Since $\widetilde{x}_1$ and $\widetilde{x}_2$ lie in the same set $\pi^{-1}(x)$, which intersects $\widetilde{U}$ in only one point, we must have $\widetilde{x}_1 = \widetilde{x}_2 = \widetilde{x}$. Then $\widetilde{x}$ is a fixed point of g, which implies $g = e$ since deck transformation acts freely (Proposition B.4.1). □

Theorem C.3.1 (covering space quotient theorem)**.** Let E be a topological space and Γ be a group acting on E by homeomorphisms effectively . Then the quotient map $\pi \colon E \to E/\Gamma$ is a covering map if and only if Γ acts on E properly discontinuous. In this case, π is a normal covering and $\mathrm{Aut}_\pi(E) = \Gamma$.

Proof. Firstly, assume π is a covering map. Then the action of each $g \in \Gamma$ is an automorphism of the covering since it's a homeomorphism satisfying $\pi(ge) = \pi(e)$ for all $g \in \Gamma, e \in E$, so we can identify Γ with a subgroup of $\mathrm{Aut}_\pi(E)$. Then Γ acts on E properly discontinuous by Lemma C.3.1 and Lemma C.3.2.

Conversely, suppose the action is properly discontinuous. To show π is a covering map, suppose $x \in E/\Gamma$ is arbitrary. Choose $e \in \pi^{-1}(x)$, and let U be a neighborhood of e such that for each $g \in \Gamma$, $gU \cap U = \varnothing$ unless $g = 1$. Since E is locally path-connected, by passing to the component of U containing e, we may assume U is path-connected. Let $V = \pi(U)$, which is a path-connected neighborhood of x. Now $\pi^{-1}(V)$ is equal to the union of the disjoint connected open subsets gU for $g \in \Gamma$, so to show π is a covering space it remains to show π is a homeomorphism from each such set onto V. For each $g \in \Gamma$, the restriction map $g\colon U \to gU$ is a homeomorphism, and the diagram

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr, "\pi"'] & & gU \arrow[dl, "\pi"] \\ & V & \end{tikzcd}$$

commutes. Thus it suffices to show $\pi|_U\colon U \to V$ is a homeomorphism. It's surjective, continuous and open, and it's injective since $\pi(e) = \pi(e')$ for $e, e' \in U$ implies $e' = ge$ for some $g \in \Gamma$, so $e = e'$ by the choice of U. This shows π is a covering map.

To prove the final statement of the theorem, suppose the action is a covering space action. As noted above, each map $e \mapsto ge$ is a covering automorphism, so $\Gamma \subseteq \mathrm{Aut}_\pi(E)$. By construction, Γ acts transitively on each fiber, so $\mathrm{Aut}_\pi(E)$ does too, and thus π is a normal covering. If φ is any covering automorphism, choose $e \in E$ and let $e' = \varphi(e)$. Then there is some $g \in \Gamma$ such that $ge = e'$. Since φ and $x \mapsto gx$ are deck transformation that agree at a point, so they are equal. Thus $\Gamma = \mathrm{Aut}_\pi(E)$. □

Proposition C.3.1. Suppose G is a discrete topological group acting continuously and freely on a topological manifold M. The action is proper if and only if the following conditions both hold.

(1) G acts on M properly discontinuous.
(2) If $p, p' \in M$ are not in the same orbit, then there exist a neighborhood V of p and V' of p' such that $gV \cap V' = \varnothing$ for all $g \in G$.

Proof. Firstly, suppose that the action is free and proper and let $\pi\colon M \to M/G$ denote the quotient map. By Proposition C.2.1, the orbit space M/G is Hausdorff. If $p, p' \in M$ are not in the same orbit, we can choose disjoint neighborhoods W of $\pi(p)$ and W' of $\pi(p')$, and then $V = \pi^{-1}(W)$ and $V' = \pi^{-1}(W')$ satisfy the conclusion of condition (2). To show G acts on M properly discontinuous, we need to show for each $p \in M$, there exists an open neighborhood U of p such that $gU \cap U = \varnothing$ unless $g = e$. Let V be a precompact neighborhood of p. By Proposition C.2.2, the set $G_{\overline{V}}$ is a compact subset of G, and hence finite because G is discrete, so we write $G_{\overline{V}} = \{e, g_1, \ldots, g_m\}$. Shrinking

V if necessary, we may assume that $g_i^{-1}p \notin \overline{V}$ for $i = 1,\ldots,m$. Consider open subset

$$U = V \setminus (g_1\overline{V} \cup \cdots \cup g_m\overline{V})$$

It's clear $gU \cap U = \varnothing$ unless $g = e$.

Conversely, assume that (1) and (2) hold. Suppose $\{g_i\}$ is a sequence in G and $\{p_i\}$ is a sequence in M such that $p_i \to p$ and $g_i p_i \to p'$. If p and p' are in different orbits, there exist neighborhoods V of p and V' of p' as in (2), but for large enough i, we have $p_i \in V$ and $g_i p_i \in V'$, which contradicts the fact that $g_i V \cap V' = \varnothing$. This shows p and p' are in the same orbit, so there exists $g \in G$ such that $gp = p'$. This implies $g^{-1}g_i p_i \to p$. Since G acts on M properly discontinuous, there exists an open neighborhood U such that $gU \cap U = \varnothing$ unless $g = e$. For large enough i, one has p_i and $g^{-1}g_i p_i$ are both in U, and by the choice of U one has $g^{-1}g_i = e$. So $g_i = g$ when i is large enough, which certainly converges. By (2) of Proposition C.2.2, the action is proper. □

Proposition C.3.2. Let M be a topological manifold and $\pi\colon \widetilde{M} \to M$ be a normal covering space. If $\mathrm{Aut}_\pi(\widetilde{M})$ is equipped with the discrete topology, then it acts on $\widetilde{M}$ continuously, freely and properly.

Proof. By Proposition B.4.1 one has $\mathrm{Aut}_\pi(\widetilde{M})$ acts on $\widetilde{M}$ freely and the action is also continuously since $\mathrm{Aut}_\pi(\widetilde{M})$ is equipped with discrete topology. To see the action is properly, it suffices to show the action satisfies the two conditions in Proposition C.3.1.

(a) By Lemma C.3.2, one already has $\mathrm{Aut}_\pi(\widetilde{M})$ acts on $\widetilde{M}$ properly discontinuous.
(b) Since $\pi\colon \widetilde{M} \to M$ is a normal covering, one has the orbit space is homeomorphic to M by Proposition B.4.2 and thus orbit space is Hausdorff. If $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{M}$ are in different orbits, we can choose disjoint neighborhoods W of $\pi(\widetilde{x}_1)$ and W' of $\pi(\widetilde{x}_2)$ since orbit space is Hausdorff, and it follows that $V = \pi^{-1}(W)$ and $V' = \pi^{-1}(W')$ satisfy the second condition.

□

Appendix D. Review of smooth manifolds

In this section we give a quick review of facts in differential geometry we may use, and a good reference is [Lee03]

D.1. **Submersions, Immersions, and Embeddings.**

D.1.1. *Immersions and embeddings.*

Definition D.1.1 (immersion)**.** A smooth map $F\colon M \to N$ between smooth manifolds is called immersion if its differential is injective at each point.

Definition D.1.2 (embedding)**.** A smooth map $F\colon M \to N$ between smooth manifolds is called embedding if it's an immersion and a topological embedding, that is a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology.

Proposition D.1.1. Suppose M and N are smooth manifolds and $F\colon M \to N$ is an injective immersion. If any of the following holds, then F is a smooth embedding.

(1) F is an open or closed map.
(2) F is a proper map.
(3) M is compact.

Theorem D.1.1 (local embedding theorem)**.** Let $F\colon M \to N$ be a smooth map between smooth manifolds. Then F is an immersion if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U\colon U \to N$ is an embedding.

D.1.2. *Submersions.*

Definition D.1.3 (submersion)**.** A smooth map $F\colon M \to N$ between smooth manifolds is called submersion if its differential is surjective at each point.

Definition D.1.4 (local diffeomorphism)**.** A smooth map $F\colon M \to N$ between smooth manifolds is called a local diffeomorphism if every $p \in M$ has a neighborhood U such that $F(U)$ is open in N and $F|_U\colon U \to F(U)$ is a diffeomorphism.

Proposition D.1.2. Let M, N be smooth manifolds, and $\pi\colon M \to N$ is a submersion. Then π is an open map, and if it's surjective it's a quotient map.

Proposition D.1.3. Suppose M and N are smooth manifolds and $F\colon M \to N$ is a smooth map.

(1) F is a local diffeomorphism if and only if it's both immersion and submersion.
(2) If $\dim M = \dim N$ and F is either an immersion or a submersion, then it's a local diffeomorphism.

Theorem D.1.2 (characteristic property of surjective smooth submersion)**.** Suppose M and N are smooth manifolds, and $\pi\colon M \to N$ is a surjective smooth submersion. For any smooth manifold P, a map $F\colon N \to P$ is smooth if and only if $F \circ \pi$ is smooth.

$$\begin{tikzcd} M \arrow[d, "\pi"] \arrow[dr, "F\circ\pi"] & \\ N \arrow[r, "F"] & P \end{tikzcd}$$

Theorem D.1.3 (passing smoothly to the quotient)**.** Suppose M and N are smooth manifolds, and $\pi\colon M \to N$ is a surjective smooth submersion. If P is a smooth manifold and $F\colon M \to P$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\widetilde{F}\colon N \to P$ such that $\widetilde{F}\circ\pi = F$.

$$\begin{tikzcd} M \arrow[d, "\pi"] \arrow[dr, "F"] & \\ N \arrow[r, dashed, "\widetilde{F}"] & P \end{tikzcd}$$

D.1.3. *Rank theorem.*

Definition D.1.5 (rank)**.** Given a smooth map $F\colon M \to N$ between smooth manifolds and $p \in M$, rank of F at p is defined to be the rank of linear map $(\mathrm{d}F)_p\colon T_pM \to T_{f(p)}N$.

Definition D.1.6 (constant rank)**.** Given a smooth map $F\colon M \to N$ between smooth manifolds. F is called constant rank, if rank of F at any $p \in M$ is the same.

Theorem D.1.4 (global rank theorem)**.** Let $F\colon M \to N$ be a smooth map between smooth manifolds with constant rank.
(1) If F is surjective, then it's a submersion.
(2) If F is injective, then it's a immersion.
(3) If F is bijective, then it's a diffeomorphism.

D.2. **Submanifold.**

D.2.1. *Embedded submanifold.*

Definition D.2.1 (embedded submanifold)**.** Suppose M is a smooth manifold. An embedded submanifold of M is a subset $S \subseteq M$ that is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is an embedding.

Proposition D.2.1 (images of embeddings as submanifold)**.** Suppose M is a smooth manifold, N is a smooth, and $F\colon M \to N$ is an embedding. Let $S = F(N)$. With the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism onto its image.

Definition D.2.2 (properly embedded)**.** An embedded submanifold $S \subseteq M$ is said to be properly embedded if the inclusion $S \hookrightarrow M$ is a proper map.

Proposition D.2.2. Suppose M is a smooth manifold and $S \subseteq M$ is an embedding submanifold. Then S is properly embedded if and only if it's a closed subset of M.

Theorem D.2.1 (Whitney embedding theorem)**.** Every smooth n-manifold admits a proper embedding into $\mathbb{R}^{2n+1}$.

Theorem D.2.2 (strongly Whitney embedding theorem)**.** If $n > 0$, every smooth n-manifold admits a smooth embedding into $\mathbb{R}^{2n}$.

D.2.2. *Immersed submanifold.*

Definition D.2.3 (immersed submanifold)**.** Suppose M is a smooth manifold. An immersed submanifold of M is a subset $S \subseteq M$ that is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is an immersion.

Definition D.2.4 (weakly embedded)**.** If M is a smooth manifold and $S \subseteq M$ is an immersed submanifold, then S is said to be weakly embedded in M if every smooth manifold $F\colon N \to M$ whose image lie in S is smooth as a map from N to S.

D.2.3. *Regular value theorem.*

Theorem D.2.3 (constant rank level set theorem)**.** Let M and N be smooth manifolds, and $\Phi\colon M \to N$ be a smooth map with constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M.

Definition D.2.5 (regular/critical point)**.** If $\Phi\colon M \to N$ is a smooth map, a point $p \in M$ is said to be a regular point of Φ is $(\mathrm{d}\Phi)_p\colon T_pM \to T_{\Phi(p)}N$ is surjective; otherwise it's a critical point of Φ.

Definition D.2.6 (regular/critical value)**.** If $\Phi\colon M \to N$ is a smooth map, a point $c \in N$ is said to be a regular value of Φ if every point of $\Phi^{-1}(c)$ is a regular point; otherwise it's a critical value of Φ.

Theorem D.2.4 (Sard's theorem)**.** Suppose M and N are smooth manifolds and $F\colon M \to N$ is a smooth map. Then the set of critical values of F has measure zero in N.

Corollary D.2.1. Suppose M and N are smooth manifolds and $F\colon M \to N$ is a smooth map. If $\dim M < \dim N$, then $F(M)$ has measure zero in N.

D.3. **Lie group.**

Definition D.3.1 (Lie group)**.** A Lie group G is a smooth manifold which is also endowed with a group structure such that the multiplication map and the inverse map are smooth.

Proposition D.3.1. For any $g \in G$, L_g, R_g defined as follows

$$L_g(h) = gh$$
$$R_g(h) = hg$$

are diffeomorphisms.

Proof. It's clear they're smooth, since multiplication map is smooth. The inverse maps are given by $L_{g^{-1}}$ and $R_{g^{-1}}$, since inversion map of Lie group is also smooth. □

Definition D.3.2 (Lie subgroup)**.** A Lie subgroup of a Lie group G is a subgroup $H \subseteq G$ endowed with a smooth structure such that H is a Lie group and an immersed submanifold.

Theorem D.3.1 (Cartan's closed subgroup theorem)**.** Any closed subgroup of a Lie group is a Lie subgroup.

In the following statements of this section, G is a Lie group.

Definition D.3.3 (invariant vector field)**.** A vector field X on a G is called left-invariant, if

$$(L_g)_* X = X$$

for arbitrary $g \in G$.

Definition D.3.4 (Lie algebra)**.** The vector space consisting of left-invariant vector fields on G equipped with Lie bracket of vector fields is called Lie algebra of G, and it's denoted by $\mathfrak{g}$.

Remark D.3.1. There is an isomorphism between vector spaces

$$\begin{aligned} \mathfrak{g} = \{\text{left-invaraint vector fields on } G\} &\to T_eG \\ X &\mapsto X_e \end{aligned}$$

Thus T_eG equipped with Lie bracket from $\mathfrak{g}$ is also called Lie algebra of G.

Definition D.3.5 (adjoint representation)**.** The adjoint representation is defined as

$$\begin{aligned} \mathrm{Ad}\colon G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto (L_g)_*(R_{g^{-1}})_* \end{aligned}$$

Theorem D.3.2. Let $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{GL}(\mathfrak{g})$ be the differential of adjoint representation at identity element e. Then $\mathrm{ad}_X Y = [X, Y]$.

Definition D.3.6 (integral curve)**.** Let X be a vector field of G and $g \in G$. Then an integral curve of X through the point p is a smooth curve $\gamma\colon I \subseteq \mathbb{R} \to G$ such that

$$\begin{aligned} \gamma(0) &= g \\ \gamma'(t) &= X(\gamma(t)) \end{aligned}$$

Definition D.3.7 (complete vector field)**.** A vector field X is called complete, if its integral curve is defined for all $t \in \mathbb{R}$.

Proposition D.3.2. Every left-invariant vector field on G is complete.

Proof. Let X be a left-invariant vector field and γ the unique integral curve for X defined on $(-\varepsilon, \varepsilon)$ such that $\gamma(0) = e$. Then $\gamma_g(t) := L_g\gamma(t)$ is an integral curve for X such that $\gamma_g(0) = g$. Indeed,

$$\begin{aligned}
\gamma_g'(t) &= (\mathrm{d}L_g)_{\gamma(t)}(\gamma'(t)) \\
&= (\mathrm{d}L_g)_{\gamma(t)}(X(\gamma(t))) \\
&= X(L_g\gamma(t)) \\
&= X(\gamma_g(t))
\end{aligned}$$

In particular, for $t_0 \in (-\varepsilon,\varepsilon)$, the curve $t \mapsto \gamma(t_0)\gamma(t)$ is an integral curve for X starting at $\gamma(t_0)$. By uniqueness, this curve coincides with $\gamma(t_0+t)$ for all $t \in (-\varepsilon,\varepsilon)\cap(-\varepsilon-t_0,\varepsilon-t_0)$. Define

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(t) & t \in (-\varepsilon,\varepsilon) \\ \gamma(t_0)\gamma(t) & t \in (-\varepsilon-t_0,\varepsilon-t_0) \end{cases}$$

Repeat above operations to get our desired extension. □

Remark D.3.2. From this proof we can see integral curve of left-invariant vector fields through identity e is just a Lie group homomorphism $\gamma\colon \mathbb{R} \to G$, such homomorphism is called a one parameter subgroup.

D.4. **Killing form.**

Definition D.4.1 (Killing form)**.** Let $\mathfrak{g}$ be the Lie algebra of Lie group G. The Killing form $B\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ is a bilinear symmetric form defined as

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

Lemma D.4.1. Let B be the Killing form on Lie algebra $\mathfrak{g}$ of Lie group G. For any $g \in G$ and $X,Y,Z \in \mathfrak{g}$, one has
(1) $B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = B(X,Y)$.
(2) $B(\mathrm{ad}_Z X, Y) = -B(X, \mathrm{ad}_Z Y)$.

Proof. For (1). For any $X,Y \in \mathfrak{g}$, one has

$$\begin{aligned}
[\mathrm{Ad}_g X, Y] &= [\mathrm{Ad}_g X, \mathrm{Ad}_g \circ \mathrm{Ad}_{g^{-1}} Y] \\
&= \mathrm{Ad}_g([X, \mathrm{Ad}_{g^{-1}} Y]) \\
&= \mathrm{Ad}_g \circ \mathrm{ad}_X \circ (\mathrm{Ad}_g)^{-1}(Y)
\end{aligned}$$

If we use σ to denote Ad_g, then $\mathrm{ad}_{\sigma(X)} = \sigma\circ\mathrm{ad}_X \circ \sigma^{-1}$. Hence,

$$B(\sigma(X),\sigma(Y)) = \mathrm{tr}(\mathrm{ad}_{\sigma(X)}\circ\mathrm{ad}_{\sigma(Y)}) = \mathrm{tr}(\sigma\circ\mathrm{ad}_X\circ\mathrm{ad}_Y\circ\sigma^{-1}) = B(X,Y)$$

For (2). For $Z \in \mathfrak{g}$, from (1) one has

$$B(\mathrm{Ad}_{\exp(tZ)} X, \mathrm{Ad}_{\exp(tZ)} Y) = B(X,Y)$$

By taking derivative with respect to t and set $t = 0$, one has

$$B(\mathrm{ad}_Z X, Y) + B(X, \mathrm{ad}_Z Y) = 0$$

□

Proposition D.4.1. Let $\mathfrak{g}$ be a Lie algebra with Killing form B and $\mathfrak{h}$ an ideal of $\mathfrak{g}$. Then Killing form on $\mathfrak{h}$ is exactly the restriction of $B|_{\mathfrak{h}}$.

Proof. See Section 5.1 of [Hum12]. □

For convenience, we use ϕ to denote $\mathrm{ad}_X \circ \mathrm{ad}_Y$ in the following computations.

Example D.4.1. *Killing form $B(X,Y)$ on $\mathfrak{gl}(n)$ is $2n\operatorname{tr}(XY)-2\operatorname{tr}(X)\operatorname{tr}(Y)$.*

Proof. There is a canonical basis of $\mathfrak{gl}(n)$, that is $\{E_{ij}\}$, where E_{ij} is the matrix such that

$$(E_{ij})_{kl} = \begin{cases} 1, & (k,l)=(i,j) \\ 0, & \text{otherwise} \end{cases}$$

A direct computation shows

$$\phi(E_{ij}) = \sum_{k=1}^{n} (XY)_{jk}E_{ik} + (XY)_{ki}E_{kj} - \sum_{k,l=1}^{n} (X_{ki}Y_{jl} + Y_{ki}X_{jl})E_{kl}$$

which implies the trace of ϕ is

$$\sum_{i,j=1}^{n} (XY)_{jj} + (XY)_{ii} - X_{ii}Y_{jj} - Y_{ii}X_{jj} = 2n\operatorname{tr}(XY) - 2\operatorname{tr}(X)\operatorname{tr}(Y)$$

□

Example D.4.2. *Killing form $B(X,Y)$ on $\mathfrak{sl}(n)$ is $2n\operatorname{tr}(XY)$.*

Proof. Note that $\mathfrak{sl}(n)$ is an ideal of $\mathfrak{gl}(n)$, which implies the restriction of Killing form on $\mathfrak{gl}(n)$ to $\mathfrak{sl}(n)$ is exactly the one on $\mathfrak{sl}(n)$. Thus by Example D.4.1 one has Killing form on $\mathfrak{sl}(n)$ is $2n\operatorname{tr}(XY)$, since $\mathfrak{sl}(n)$ consisting of matrices with vanishing trace. □

Example D.4.3. *Killing form $B(X,Y)$ on $\mathfrak{so}(n)$ is $(n-2)\operatorname{tr}(XY)$.*

Proof. There is a natural basis of $\mathfrak{so}(n)$, that is $\{E_{ij}-E_{ji}\}_{i<j}$. If we denote

$$\phi(E_{ij}) = a_{ij,ij}E_{ij} + a_{ij,ji}E_{ji} + \dots$$

The computation in Example D.4.1 shows

$$\begin{aligned} a_{ij,ij} &= (XY)_{jj} + (XY)_{ii} - X_{ii}Y_{jj} - Y_{ii}X_{jj} \\ a_{ij,ji} &= \delta_{ij}((XY)_{jj} + (XY)_{ii}) - X_{ji}Y_{ji} - Y_{ji}X_{ji} \end{aligned}$$

Note that

$$\phi(E_{ij}-E_{ji}) = (a_{ij,ij}-a_{ij,ji})(E_{ij}-E_{ji}) + \dots$$

Thus the Killing form on $\mathfrak{so}(n)$ is

$$\begin{aligned}
B(X,Y) &= \sum_{i<j}((XY)_{jj}+(XY)_{ii}-X_{ii}Y_{jj}-X_{jj}Y_{ii}+X_{ji}Y_{ji}+Y_{ji}X_{ji}) \\
&= \frac{1}{2}\sum_{i\neq j}((XY)_{jj}+(XY)_{ii}-X_{ii}Y_{jj}-X_{jj}Y_{ii}+X_{ji}Y_{ji}+Y_{ji}X_{ji}) \\
&= (n-1)\operatorname{tr}(XY)+\frac{1}{2}\sum_{i\neq j}(-X_{ii}Y_{jj}-X_{jj}Y_{ii}+X_{ji}Y_{ji}+Y_{ji}X_{ji}) \\
&\overset{(1)}{=} (n-1)\operatorname{tr}(XY)-\operatorname{tr}(X)\operatorname{tr}(Y)-\frac{1}{2}\sum_{i\neq j}(X_{ji}Y_{ij}-Y_{ji}X_{ij}) \\
&\overset{(2)}{=} (n-1)\operatorname{tr}(XY)-\frac{1}{2}(\operatorname{tr}(XY)+\operatorname{tr}(YX)) \\
&= (n-2)\operatorname{tr}(XY)
\end{aligned}$$

where

(1) holds from X,Y are skew-symmetric.
(2) holds from skew-symmetry matrix has vanishing trace.

□

D.5. **Lie group action.** In this section we assume M is a smooth manifold and G is a Lie group.

Definition D.5.1 (smooth action)**.** A smooth G-action on M is given by the following data:

(1) For every $g\in G$, it gives a diffeomorphism θ_g of M.
(2) The map $G\times M\to M$ given by $(g,p)\mapsto gp$ is smooth.
(3) For $g_1,g_2\in G$ and $x\in M$, one has $(g_1g_2)p=g_1(g_2p)$.

Theorem D.5.1. Let M and N be smooth manifolds and G is a Lie group. Suppose $F\colon M\to N$ is a smooth map which is equivalent with transitive smooth G-action on M and any smooth G-action on N. Then F has constant rank.

Theorem D.5.2 (quotient manifold theorem)**.** Let G be a Lie group acting on M smoothly, freely and properly. Then topological manifold M/G admitting a unique smooth structure such that $\pi\colon M\to M/G$ is a submersion.

D.5.1. *Homogeneous space.*

Definition D.5.2 (G-homogeneous space)**.** A smooth manifold M equipped with a transitive smooth G-action is called a homogeneous G-space.

Theorem D.5.3. Let H be a closed subgroup of G. Then

(1) The left coset space G/H is a topological manifold of dimension $\dim G-\dim H$.
(2) G/H admits a unique smooth structure such that the quotient map $\pi\colon G\to G/H$ is a smooth submersion.

(3) The left action

$$\begin{aligned} G \times G/H &\to G/H \\ (g_1, g_2H) &\mapsto (g_1g_2)H \end{aligned}$$

turns G/H into a G-homogeneous space.

Proof. By Theorem D.5.2, it suffices to show H acting on G by left multiplication is smoothly, freely and properly, since it's clear G acting on G/H by left multiplication is transitive.

(1) It's smooth: By Theorem D.3.1 one has H is a Lie subgroup, and thus the action is smoothly, since multiplication of Lie group is smooth.
(2) It's free: If $hg = g$ for some $g \in G$, it's clear $h = e$.
(3) It's proper: Note that the inclusion $H \times G \hookrightarrow G \times G$ is proper, since H is closed and the intersection of a compact subset with a closed subset is compact. On the other hand,

$$\begin{aligned} G \times G &\to G \times G \\ (g_1, g_2) &\mapsto (g_1g_2, g_2) \end{aligned}$$

is proper, since it's a diffeomorphism. Thus

$$\begin{aligned} K \times G &\to G \times G \\ (h, g) &\mapsto (hg, g) \end{aligned}$$

is proper, since it's the composition of proper maps.

□

Theorem D.5.4. Let M be a G-homogeneous space and $p \in M$. Then the isotropy group G_p is a closed subgroup of G and the map

$$\begin{aligned} F \colon G/G_p &\to M \\ gG_p &\mapsto gp \end{aligned}$$

is a G-equivariant diffeomorphism.

Proof. Let's prove this by the following steps:

(1) For $p \in M$, consider the following map

$$\begin{aligned} \theta^{(g)} \colon G &\to M \\ g &\mapsto gp \end{aligned}$$

It's smooth since G acting on M smoothly, and $G_p = (\theta^{(g)})^{-1}(p)$ implies G_p is a closed subgroup of G.
(2) For $g_1, g_2 \in G$ with $g_1g_2^{-1} = g \in G_p$, it's clear $g_1p = g_2gp = g_2p$, which implies F is well-defined.
(3) If $\pi \colon G \to G/G_p$ is the projection map, then by Theorem D.5.3 there exists a unique smooth structure on G/G_p such that π is a surjective submersion. By Theorem D.1.2, F is smooth since $F \circ \pi = \theta^{(p)}$.

(4) It's clear F is G-equivariant, so Theorem D.5.1 shows F has constant rank. By Theorem D.1.4, it suffices to check F is bijective to conclude F is a diffeomorphism. Given any $q \in M$, there exists $g \in G$ such that $gp = q$ since M is a G-homogeneous space, and thus $F(gG_p) = q$. On the other hand, if $F(g_1G_p) = F(g_2G_p)$ for $g_1, g_2 \in G$, then $g_1p = g_2p$ implies $g_1g_2^{-1} \in G_p$, and thus $g_1G_p = g_2G_p$.

□

D.6. **Distributions and Foliations.**

D.6.1. *Distributions and Involutivity.* Let M be a smooth manifold.

Definition D.6.1 (distribution)**.** A (smooth) distribution on M of rank k is a rank-k (smooth) subbundle of TM.

Definition D.6.2. Suppose D is a distribution on M. D is called involutive if $C^\infty(M, D)$ is a Lie subalgebra of $\mathfrak{X}(M)$.

Definition D.6.3 (integral manifold)**.** Suppose D is a distribution on M. A non-empty immersed submanifold $N \subseteq M$ is called an integral manifold of D if $T_pN = D_p$ at each $p \in N$.

Definition D.6.4 (integrable)**.** A distribution D on M is integrable if each point of M is contained in an integral manifold of D.

Theorem D.6.1 (Frobenius)**.** A distribution is integrable if and only if it's involutive.

Theorem D.6.2. Every integral manifold of an involutive distribution is weakly embedded.

D.6.2. *Foliations.* Let M be a smooth manifold.

Definition D.6.5 (foliation)**.** A foliation of dimension k on M to be a collection $\mathcal{F}$ of disjoint, connected, non-empty, immersed k-dimensional submanifolds of M, whose union is M, and such that in a neighborhood of each point $p \in M$ there exists a flat chart for $\mathcal{F}$.

Theorem D.6.3 (global Frobenius theorem)**.** Let D be an involutive distribution on a smooth map M. The collection of all maximal connected integral manifolds of D forms a foliation of M.

Theorem D.6.4. Every Lie subgroup is an integral manifold of an involutive distribution, and therefore is a weakly embedded submanifold.

Theorem D.6.5. Suppose G is a Lie group and $\mathfrak{g}$ is its Lie algebra. If $\mathfrak{h}$ is any Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup of G whose Lie algebra is $\mathfrak{g}$.

REFERENCES

[Car92] Do Carmo. *Riemannian Geometry*. Springer Science, 1992.
[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.
[Hum12] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.
[Lee03] J.M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer, 2003.
[Lee10] John Lee. *Introduction to topological manifolds*, volume 202. Springer Science & Business Media, 2010.
[Lee18] John M. Lee. *Introduction to Riemannian Manifolds*. Springer Cham, 2018.
[MS39] S. B. Myers and N. E. Steenrod. The group of isometries of a riemannian manifold. *Annals of Mathematics*, 40(2):400–416, April 1939.
[Pal57] Richard S Palais. On the differentiability of isometries. *Proceedings of the American Mathematical Society*, 8(4):805–807, 1957.
[Pet16] Peter Petersen. *Riemannian Geometry*. Springer New York, NY, 2016.

MATHEMATICS DEPARTMENT, TSINGHUA UNIVERSITY, BEIJING, 100084, P.R. CHINA,
Email address: liubw22@mails.tsinghua.edu.cn