Self-dual Equations on Riemann surface

Bowen Liu[1]

Qiliang Luo[2]

Baiting Xie[3]

[1]liubw22@mails.tsinghua.edu.cn
[2]lql23@mails.tsinghua.edu.cn
[3]xbt23@mails.tsinghua.edu.cn

Contents

To readers

It's a note for a reading seminar about self-dual equations on principal bundles, and the main reference is Hitchin's celebrated paper [Hit87]

1. NOTATION

(1) Σ Riemann surface
(2) G Lie group
(3) P Principle bundle

2. THE HITCHIN EUQATION (**QILIANG LUO**)

2.1. **Higgs field on principle bundle and vector bundle.** Let Σ be a Riemann surface, G a fixed Lie group, P a principle G-bundle. Firstly, we introduce the concept on Higgs field on principle bundle P.

A point on P can be written as (x,p). The left action of G on P can be written as $g(x,p)=(x,pg^{-1})$. For any G representation

$$\rho : G \to GL(V)$$

We have an associate vector bundle $E = P \times_\rho V$ defined by equivalent relation in $P \times V$

$$(x,p,v) \sim (x,pg^{-1},gv)$$

By adjoint vector bundle $ad(P)$ we denote the associate bundle of P of the canonical representation $ad : G \to Lie(G)$ where $Lie(G)$ is the Lie algebra of G.

Definition 2.1. A Higgs field ϕ on the principle bundle P is a section in $\Omega^{1,0}(\Sigma, ad(P))$.

Any structure on the principle bundle can induce the corresponding structure on the associated vector bundle by the representation ρ and the bundle maps

$$P \to GL(E) \qquad ad(P) \to End(E)$$

also denoted by ρ. For example, the connection A, the curvature $F(A)$ and the Higgs field ϕ on P. We claim $F(d_A) = \rho(F(A))$.

Let E be a vector bundle on the Riemann surface Σ. Its structure can be reduced to the principle bundle P, if E is an associated bundle of P. Its connection can be reduced to the principle bundle P, if E is an associated bundle of P and the connection on E is come from a connection on P, and so on.

2.2. **The Hitchin equation on principle bundle.** Let Σ be a Riemann surface, G a fixed Lie group, P a principle G-bundle. The Hitchin equation on the principle bundle P is an equation about connection A and Higgs field ϕ

Definition 2.2. A pair of connection and Higgs field (A,ϕ) satisfies the Hitchin equation if

$$\begin{cases} F(A) = -[\phi, \phi^*] \\ d_A'' \phi = 0 \end{cases} \tag{2.1}$$

Explain the equation.....(what is [,] and $*$)

Lemma 2.1. Assume $H < G$ is a subgroup. If the principle G-bundle P_G can be reduced to the principle H-bundle P_H, and the connection and the Higgs field can also be reduced to P_H. Then (A,ϕ) satisfies the Hitchin equation in P_G if and only is they satisfies the Hitchin equation in P_H.

3. Stability (Bowen Liu)

3.1. **General theory of slope stability.** Let X be a smooth projective variety and ω be an ample divisor on X.

Definition 3.1.1. Let $\mathcal{E}$ be a torsion-free sheaf over X. The *ω-slope* of $\mathcal{E}$ is defined as

$$\mu_\omega(\mathcal{E}) = \frac{c_1(\mathcal{E})\cdot\omega^{n-1}}{\operatorname{rk}\mathcal{E}}.$$

Definition 3.1.2. A torsion-free sheaf $\mathcal{E}$ on X is called ω-semistable (resp. stable), if $\mu_\omega(\mathcal{F}) \leq \mu_\omega(\mathcal{E})$ (resp. $\mu_\omega(\mathcal{F}) < \mu_\omega(\mathcal{E})$) for all subsheaves $\mathcal{F}$ of $\mathcal{E}$.

Lemma 3.1.1. Let C be a smooth projective curve and $\mathcal{E}$ be a locally free sheaf on C. Then $\mathcal{E}$ is semistable (resp. stable) if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) < \mu(\mathcal{E})$) for all subbundles $\mathcal{F}$ of $\mathcal{E}$.

Proof. For any subsheaf $\mathcal{F} \subseteq \mathcal{E}$, suppose $\widetilde{\mathcal{F}}$ is the saturation of $\mathcal{F}$ in $\mathcal{E}$, that is, $\mathcal{E}/\widetilde{\mathcal{F}}$ is torsion-free. Then we must have $\mu(\widetilde{\mathcal{F}}) \geq \mu(\mathcal{F})$, since $\widetilde{\mathcal{F}}$ is the extension of $\mathcal{F}$ by a torsion sheaf supported on a zero-dimensional subvariety. Thus in order to test the stability of $\mathcal{E}$, it suffices to test saturated subsheaves of $\mathcal{E}$.

Now we claim that every saturated subsheaf of a locally free sheaf on a smooth projective curve is a subbundle[4]. Suppose $\mathcal{F} \subseteq \mathcal{E}$ is a saturated subsheaf. Then we have the following exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is torsion-free and thus locally free, since every torsion-free sheaf is a locally free sheaf on a codimensional two locus. To prove the locally freeness of $\mathcal{F}$, note that on for each $p \in X$, there is the exact sequence

$$0 \to \mathcal{F}_p \to \mathcal{O}_{C,p}^{\oplus m} \to \mathcal{O}_{C,p}^{\oplus n} \to 0,$$

which implies $\mathcal{F}_p$ is free, since the local ring $\mathcal{O}_{C,p}$ is PID and submodule of a free module over a PID is free. This shows $\mathcal{F}$ is locally free since $\mathcal{F}_p$ is free for every $p \in C$. $\square$

Theorem 3.1.1 ([HN75])**.** Let $\mathcal{E}$ be a torsion-free sheaf on X. Then there exists a unique filtration Σ_ω,

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_s = \mathcal{E},$$

which is called the *Harder-Narasimhan filtration*, such that

(1) $\operatorname{Gr}_i(\Sigma_\omega) = \mathcal{E}_i/\mathcal{E}_{i+1}$ is a torsion-free ω-semistable sheaf;
(2) $\mu_\omega(\operatorname{Gr}_i(\Sigma_\omega))$ is a strictly decreasing function in i.

Sketch of existence. Put $\mu_\omega^{\max}(\mathcal{E}) := \sup\{\mu_\omega(\mathcal{F}) \mid 0 \neq \mathcal{F} \subseteq \mathcal{E} \text{ a coherent subsheaf}\}$. Then we need to prove that

(1) $\mu_\omega^{\max}(\mathcal{E}) < \infty$;
(2) There exists a saturated subsheaf $\mathcal{F}_1 \subseteq \mathcal{E}$ with maximal slope.

[4]A subbundle $\mathcal{F} \subseteq \mathcal{E}$ means $\mathcal{F}$ is locally free and the quotient $\mathcal{E}/\mathcal{F}$ is again locally free.

After that, suppose both $\mathcal{F}_1$ and $\mathcal{F}_2$ coherent subsheaves of rank r_1 and r_2 with maximal slope. By the following exact sequence

$$0 \to \mathcal{F}_1 \cap \mathcal{F}_2 \to \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_1 + \mathcal{F}_2 \to 0,$$

one has

$$\begin{aligned} c_1(\mathcal{F}_1 + \mathcal{F}_2) &= c_1(\mathcal{F}_1) + c_1(\mathcal{F}_2) - c_1(\mathcal{F}_1 \cap \mathcal{F}_2) \\ \mathrm{rk}(\mathcal{F}_1 + \mathcal{F}_2) &= \mathrm{rk}(\mathcal{F}_1) + \mathrm{rk}(\mathcal{F}_2) - \mathrm{rk}(\mathcal{F}_1 \cap \mathcal{F}_2). \end{aligned}$$

Then

$$\begin{aligned} \mathrm{rk}(\mathcal{F}_1 + \mathcal{F}_2)\mu_\omega(\mathcal{F}_1 + \mathcal{F}_2) &= r_1\mu_\omega(\mathcal{F}_1) + r_2\mu_\omega(\mathcal{F}_2) - \mathrm{rk}(\mathcal{F}_1 \cap \mathcal{F}_2)\mu_\omega(\mathcal{F}_1 \cap \mathcal{F}_2) \\ &\geq (r_1 + r_2)\mu_\omega^{\max}(\mathcal{E}) - \mathrm{rk}(\mathcal{F}_1 \cap \mathcal{F}_2)\mu_\omega^{\max}(\mathcal{E}) \\ &= \mathrm{rk}(\mathcal{F}_1 + \mathcal{F}_2)\mu_\omega^{\max}(\mathcal{E}). \end{aligned}$$

This shows $\mathcal{F}_1 + \mathcal{F}_2$ also has the maximal slope. By adding all these subsheaves together, this gives the *maximal ω-destabilizing subsheaf* $\mathcal{E}_1$. Repeat above process to obtain the maximal ω-destabilizing subsheaf of $\mathcal{E}/\mathcal{E}_1$, and consider its preimage to obtain $\mathcal{E}_2$, that is, $\mathcal{E}_2/\mathcal{E}_1 = (\mathcal{E}/\mathcal{E}_1)_1$. Then $\mu_\omega(\mathcal{E}_1) > \mu_\omega(\mathcal{E}_2/\mathcal{E}_1)$, otherwise we would have $\mu_\omega(\mathcal{E}_1) \leq \mu_\omega(\mathcal{E}_2)$, a contradiction. □

Remark 3.1.1. The maximal ω-destabilizing subsheaf of $\mathcal{E}$ is characterized by the following properties:

(1) $\mu_\omega(\mathcal{E}_1) \geq \mu_\omega(\mathcal{F})$ for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$;
(2) If $\mu_\omega(\mathcal{E}_1) = \mu_\omega(\mathcal{F})$ for $\mathcal{F} \subset \mathcal{E}$, then $\mathcal{F} \subset \mathcal{E}_1$.

Remark 3.1.2. The ω-semistable filtration of the dual sheaf $\mathcal{E}^*$ is essentially the same as that of $\mathcal{E}$, with each entry substituted by the duals of the quotient $\mathcal{E}/\mathcal{E}_{s-i}$.

3.2. **Stability of rank two Higgs bundle on Riemann surfaces.** Let Σ be a compact Riemann surface.

Definition 3.2.1. A Higgs bundle $(\mathcal{E}, \theta)$ on Σ is defined to be *stable* if, for every θ-invariant subbundle $\mathcal{F} \subseteq \mathcal{E}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$.

If the Higgs field $\theta = 0$, then it reduces to the stability of a vector bundle. However, there exists a stable Higgs bundle $(\mathcal{E}, \theta)$, but $\mathcal{E}$ is not stable as a vector bundle.

Example 3.2.1. *Suppose Σ is a compact Riemann surface with genus $g > 1$. Then consider the uniformalizing bundle $\mathcal{V} = K_\Sigma^{\frac{1}{2}} \oplus K_\Sigma^{-\frac{1}{2}}$ and the Higgs field θ is given by*

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since $K_\Sigma^{-\frac{1}{2}}$ is the only θ-invariant subbundle, and it is of negative degree, so $(\mathcal{V}, \theta)$ is stable. However, it is clear that $\mathcal{V}$ is not stable as a vector bundle.

An interesting fact is that above phenomenon only happens on compact Riemann surface with genus $g > 1$.

Lemma 3.2.1. Let Σ be a compact Riemann surface of genus $g \le 1$ and $(\mathcal{E},\theta)$ be a semistable Higgs bundle. Then $\mathcal{E}$ is semistable as a vector bundle.

Proof. Suppose $\mathcal{E}$ is not semistable and its Harder-Narasimhan filtration is given by

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{s-1} \subset \mathcal{E}_s = \mathcal{E},$$

where the $\mathcal{G}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ are semistable bundles with $\mu(\mathcal{G}_i) > \mu\left(\mathcal{G}_j\right)$ if $i < j$.

The maximal destabilizer $\mathcal{E}_1 = \mathcal{G}_1$ has $\deg(\mathcal{E}_1) > \deg(\mathcal{E})$. Suppose $\theta(\mathcal{E}_1)$ is non-zero and take $\mathcal{E}_\ell$ such that $\theta(\mathcal{E}_1) \subseteq \mathcal{E}_\ell \otimes K_\Sigma$ but $\theta(\mathcal{E}_1) \nsubseteq \mathcal{E}_{\ell-1} \otimes K_\Sigma$. Then θ induces a non-zero morphism $\mathcal{G}_1 \to \mathcal{G}_\ell \otimes K_\Sigma$.

Since $g \le 1$, we have $\mu(\mathcal{G}_\ell \otimes K_\Sigma) = \mu(\mathcal{G}_\ell) + 2g - 2 \le \mu(\mathcal{G}_\ell) < \mu(\mathcal{G}_1)$, and thus there are no non-zero morphisms unless $\ell = 1$, since $\mathcal{G}_i$ are semistable. This shows $\mathcal{E}_1$ is θ-invariant and $(\mathcal{E},\theta)$ is not θ-semistable, a contradiction. $\square$

3.2.1. *Rank two stable Higgs bundle on projective line and elliptic curve.*

Example 3.2.2. *There are no stable Higgs bundle of rank two on $\mathbb{P}^1$. Suppose $(\mathcal{E},\theta)$ is a Higgs bundle of rank two. By Grothendieck's classification of vector bundle on $\mathbb{P}^1$, $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)$, where $m,n \in \mathbb{Z}$. Suppose the Higgs field θ is given by*

$$\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix},$$

where $\theta_{11},\theta_{22} \in H^0(\mathbb{P}^1,K_{\mathbb{P}^1}) = H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, $\theta_{12} \in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(m-n-2))$ and $\theta_{21} \in H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(n-m-2))$. Without lose of generality, we may assume $m \ge n$, hence $\theta_{21} = 0$ and $\mathcal{O}_{\mathbb{P}^1}(m)$ is θ-invariant. However,

$$\mu(\mathcal{O}_{\mathbb{P}^1}(m)) = m > \frac{1}{2}(m+n) = \mu(\mathcal{E}).$$

This shows $(\mathcal{E},\theta)$ is not stable.

Example 3.2.3. *Let $(\mathcal{E},\theta)$ be a rank two stable Higgs bundle on elliptic curve E. Since K_E is trivial, the Higgs field θ are endomorphisms of $\mathcal{E}$, so without lose of generality we may assume $\mathcal{E}$ is indecomposable, otherwise if $\mathcal{E}$ is decomposable, it cannot be stable.*

By Atiyah's classification of vector bundles on elliptic curve ([Ati57, Theorem 5, Theorem 6]*), we know that, after tensoring with a line bundle, $\mathcal{E}$ is equivalent to the non-trivial extension*

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{O}_E \to 0$$

defined by $H^1(E,\mathcal{O}_E) = H^0(E,\mathcal{O}_E) \cong \mathbb{C}$, or

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{O}_E(1) \to 0$$

defined by $H^1(E,\mathcal{O}_E(-1)) = H^0(E,\mathcal{O}_E(1)) \cong \mathbb{C}$.

In the first case, the distinguished trivial bundle $\mathcal{L} \cong \mathcal{O}$ is invariant by each endomorphism, but $\mu(\mathcal{L}) = 0 = \mu(\mathcal{E})$, which contradicts to the assumption $(\mathcal{E},\theta)$ is stable. In the second case, $\mathcal{E}$ is a stable vector bundle, hence the only endomorphisms are scalars.

Thus the only stable rank two Higgs bundles on elliptic curve are $(\mathcal{E},\theta)$, where $\mathcal{E}$ is the unique indecomposable vector bundle of odd degree and θ is a scalar.

In the case of compact Riemann surface of general type, stable Higgs bundles occur with more frequency, such as the uniformalizing Higgs bundle, but there are still restrictions on the holomorphic structure of the underlying vector bundle $\mathcal{E}$.

3.2.2. *Rank two stable Higgs bundle on higher genus curve.*

Proposition 3.2.1. Let Σ be a compact Riemann surface of genus $g>1$ and $\mathcal{E}$ be a rank two vector bundle. Then $\mathcal{E}$ occurs as a stable Higgs bundle $(\mathcal{E},\theta)$ if and only if one of the following holds:

(1) $\mathcal{E}$ is stable;
(2) $\mathcal{E}$ is semistable and $g>2$;
(3) If $\mathcal{E}$ is semistable and $g=2$, then $\mathcal{E}\cong\mathcal{U}\otimes\mathcal{L}$, where $\mathcal{U}$ is either decomposable or an extension of the trivial bundle by itself;
(4) $\mathcal{E}$ is unstable with maximal destabilizer $\mathcal{L}$, and $\dim H^0(\Sigma,\mathcal{L}^{-2}\otimes\det\mathcal{E}\otimes K_\Sigma)\geq 2$.
(5) $\mathcal{E}$ is unstable and decomposable as
$$\mathcal{E}=\mathcal{L}\oplus(\mathcal{L}^*\otimes\det\mathcal{E}),$$
where $\mathcal{L}$ is the maximal destabilizer and $\dim H^0(\Sigma,\mathcal{L}^{-2}\otimes\det\mathcal{E}\otimes K_\Sigma)=1$.

Proof. Firstly, we prove the following lemma:

Lemma 3.2.2. The vector bundles $\mathcal{E}$ for which a generic Higgs field θ leaves no subbundle invariant are thoes which have no subbundle invariant by all θ.

Proof. It is clear that a vector bundle $\mathcal{E}$ for which a generic Higgs field leaves no subbundle invariant have no subbundle invariant by all Higgs fields, so it suffices to prove the converse statement.

Let $\mathbb{P}(\mathcal{E})$ be the projective complex surface obtained by projectivizing the vector bundle $\mathcal{E}$ and $p:\mathbb{P}(\mathcal{E})\to\Sigma$ be the natural projection. There is a tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ such that
$$p_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k)=S^k\mathcal{E}^*,$$
where S^k means k-th symmetric power. For $A\in\mathscr{E}nd(\mathcal{E})$, we have $v\mapsto Av\wedge v$ defines a quadratic map from $\mathcal{E}$ to $\bigwedge^2\mathcal{E}=\det\mathcal{E}$, and the scalar endomorphisms are mapped to zero. This provides an isomorphism
$$S^2\mathcal{E}^*\otimes\det\mathcal{E}\cong\mathscr{E}nd_0(\mathcal{E}),$$
where $\mathscr{E}nd_0(\mathcal{E})$ is the sheaf of traceless endomorphisms. By the projection formula, there is an isomorphism of sections
$$s: H^0(\Sigma,\mathscr{E}nd_0(\mathcal{E})\otimes K_\Sigma)\cong H^0\left(\mathbb{P}(\mathcal{E}),\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\otimes p^*(\det\mathcal{E}\otimes K_\Sigma)\right).$$

In this framework, a subbundle $\mathcal{L}\subset\mathcal{E}$ defines a global section of $\mathcal{L}^*\otimes\mathcal{E}$, and hence a section on $\mathbb{P}(\mathcal{E})$ of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}\otimes p^*(\mathcal{L}^*\otimes\det\mathcal{E})$. Note that a Higgs field $\theta\in$

$H^0(\Sigma, \mathscr{E}nd_0(\mathcal{E})\otimes K_\Sigma)$ leaves $\mathcal{L}$ invariant if and only if

$$\theta v \wedge v = 0$$

for all $v \in \mathcal{L}$, equivalently, $s(\theta) \in H^0\left(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\otimes p^*(\det\mathcal{E}\otimes K_\Sigma)\right)$ vanishes on $D(\mathcal{L})$, the divisor of $\mathbb{P}(\mathcal{E})$ determined by the subbundle $\mathcal{L}$. This shows $D(\mathcal{L})$ is a proper component of the divisor of $s(\theta)$.

If the divisor of $s(\theta)$ is irreducible, then θ would leave invariant no subbundle. By Bertini's theorem, the generic divisor of a linear system of dimension at least two with no fixed component is irreducible.

Firstly let's show the linear system of divisors $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\otimes p^*(\det\mathcal{E}\otimes K_\Sigma)$ has dimension at least two. Indeed, by the projection formula it is equivalent to the dimension of $H^0(\Sigma, \mathscr{E}nd_0(\mathcal{E})\otimes K_\Sigma)$. However, by Riemann-Roch theorem

$$h^0(\Sigma, \mathscr{E}nd_0(\mathcal{E})\otimes K_\Sigma) - h^1(\Sigma, \mathscr{E}nd_0(\mathcal{E})\otimes K_\Sigma) = 3g-3 \geq 3$$

if $g \geq 1$.

If $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\otimes p^*(\det\mathcal{E}\otimes K_\Sigma)$ has no fixed point, then by Bertini's theorem, a generic Higgs field leaves invariant no subbundle. If the linear system has a fixed part $\mathcal{L}_0$, it it contained in the divisors of a line bundle of one of the following types:

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\otimes p^*\mathcal{L}, \quad \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\otimes p^*\mathcal{L}, \quad p^*\mathcal{L},$$

where $\mathcal{L}$ is a line bundle on Σ. Moreover, we have the following isomorphism

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{L}_0)\otimes H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\otimes p^*(\det\mathcal{E}\otimes K_\Sigma)\otimes\mathcal{L}_0^*) = H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\otimes p^*(\det\mathcal{E}\otimes K_\Sigma)).$$

For the first type, the global section of fixed part is

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\otimes p^*\mathcal{L}) = H^0(\Sigma, \mathscr{E}nd_0(\mathcal{E})\otimes\mathcal{L}\otimes\det\mathcal{E}^*).$$

Then every Higgs field $\theta \in H^0(\Sigma, \mathscr{E}nd_0(\mathcal{E})\otimes K_\Sigma)$ is of the form $\theta = \theta_0 s$, where $\theta_0 \in H^0(\Sigma, \mathscr{E}nd_0(\mathcal{E})\otimes\mathcal{L}\otimes\det\mathcal{E}^*)$ and $s \in H^0(\Sigma, \mathcal{L}^*\otimes\det\mathcal{E}\otimes K_\Sigma)$. If $\operatorname{tr}\theta_0^2 \neq 0$, then consider the linear map

$$\begin{aligned} \alpha\colon H^0(\Sigma, \mathcal{L}^*\otimes\det\mathcal{E}\otimes K_\Sigma) &\to H^0(\Sigma, K_\Sigma^{\otimes 2}) \\ s &\mapsto \operatorname{tr}\theta_0^2 s_0 s, \end{aligned}$$

where s_0 is some fixed non-zero section of $H^0(\Sigma, \mathcal{L}^*\otimes\det\mathcal{E}\otimes K_\Sigma)$. It is well-defined since

$$\operatorname{tr}\theta_0^2 \in H^0(\Sigma, \mathcal{L}^{\otimes 2}\otimes(\det\mathcal{E}^*)^{\otimes 2}).$$

It is clearly that α is injective, and

$$h^0(\Sigma, \mathcal{L}^*\otimes\det\mathcal{E}\otimes K_\Sigma) \geq 3g-3 = h^0(\Sigma, K_\Sigma^{\otimes 2}).$$

Thus α is an isomorphism. However, $\alpha(s)$ vanishes at zeros of s_1 but $K_\Sigma^{\otimes 2}$ has no basepoints, a contradiction. This shows $\operatorname{tr}\theta_0^2 = \det\theta_0 = 0$, and the kernel of θ_0 then defines a subbundle invariant by all Higgs fields.

For the second type, the global section of the fixed part is

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\otimes p^*\mathcal{L}) = H^0(\Sigma, \mathcal{E}^*\otimes\mathcal{L}).$$

Then the subbundle defined by kernel of the fixed section $\theta_0 \in H^0(\Sigma, \mathcal{E}^* \otimes \mathcal{L})$ is by definition invariant for all Higgs fields, a contradiction to the assumption that there is no subbundle invariant by all Higgs fields.

For the third type, the divisor of the bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes p^*(\det \mathcal{E} \otimes K_\Sigma \otimes \mathcal{L}^*)$ has no fixed component. Then by Bertini's theorem, a generic $\theta \in H^0(\Sigma, \mathscr{E}nd_0(\mathcal{E}) \otimes K_\Sigma \otimes \mathcal{L}^*)$ leaves invariant no subbundle. Multiplying by the fixed section of $\mathcal{L}$ on Σ, we find that a generic section of $\mathscr{E}nd_0(\mathcal{E}) \otimes K_\Sigma$ leaves invariant no subbundle.

□

Now let's return to the proof of Proposition 3.2.1. If $\mathcal{E}$ is a stable bundle, then $(\mathcal{E}, \theta)$ is a stable Higgs bundle for any Higgs field θ, so without lose of generality we may assume therefore that $\mathcal{E}$ has a subbundle $\mathcal{L}$ with $\deg \mathcal{L} \geq \frac{1}{2} \deg \mathcal{E}$ and write $\mathcal{E}$ as an extension

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^* \otimes \det \mathcal{E} \to 0. \tag{3.1}$$

Note that there exists a subbundle

$$\mathcal{L}^2 \otimes \det \mathcal{E}^* \otimes K_\Sigma \subset \mathscr{E}nd_0(\mathcal{E}) \otimes K_\Sigma$$

which leaves only $\mathcal{L}$ invariant. Since

$$\deg(\mathcal{L}^2 \otimes \det \mathcal{E}^* \otimes K_\Sigma) \geq 2g - 2,$$

this bundle has sections. This shows that if $\mathcal{E}$ has a subbundle invariant by all Higgs fields, then this subbundle must be $\mathcal{L}$.

By Lemma 3.2.2, if $\mathcal{L}$ is not invariant for some Higgs fields θ, then a generic Higgs field on $\mathcal{E}$ leaves no subbundle invariant and thus $\mathcal{E}$ occurs in some stable pair. Conversely, if $\mathcal{E}$ occurs in some stable pair, then there exists a Higgs field θ such that $\theta(\mathcal{L}) \subsetneq \mathcal{L}$.

This shows $\mathcal{E}$ can occur in a stable pair if and only if there exists $\theta \in H^0(\Sigma, \mathscr{E}nd_0(\mathcal{E}) \otimes K_\Sigma)$ such that $\theta(\mathcal{L}) \subsetneq \mathcal{L}$. Note that sections of $\mathscr{E}nd_0(\mathcal{E}) \otimes K_\Sigma$ which leave $\mathcal{L}$ invariant are sections of $\mathcal{E}^* \otimes \mathcal{L} \otimes K_\Sigma$, and there is the following exact sequence

$$0 \to \mathcal{E}^* \otimes \mathcal{L} \otimes K_\Sigma \to \mathscr{E}nd_0(\mathcal{E}) \otimes K_\Sigma \to \mathcal{L}^{-\otimes 2} \otimes \det \mathcal{E} \otimes K_\Sigma \to 0,$$

which leads to a long exact sequence of cohomology groups

$$0 \to H^0(\Sigma, \mathcal{E}^* \otimes \mathcal{L} \otimes K_\Sigma) \to H^0(\Sigma, \mathscr{E}nd_0(\mathcal{E}) \otimes K_\Sigma) \to H^0(\Sigma, \mathcal{L}^{-\otimes 2} \otimes \det \mathcal{E} \otimes K_\Sigma) \xrightarrow{\delta} H^1(\Sigma, \mathcal{E}^* \otimes \mathcal{L} \otimes K_\Sigma) \to \dots$$

By the exactness, we have

$$\ker \delta = \frac{H^0(\Sigma, \mathscr{E}nd_0(\mathcal{E}) \otimes K_\Sigma)}{H^0(\Sigma, \mathcal{E}^* \otimes \mathcal{L} \otimes K_\Sigma)}.$$

Thus there exists a Higgs field θ which does not preserve $\mathcal{L}$ if and only if δ is not injective.

Consider the exact sequence of bundles

$$0 \to \mathcal{L}^{\otimes 2} \otimes \det \mathcal{E}^* \otimes K_\Sigma \to \mathcal{E}^* \otimes \mathcal{L} \otimes K_\Sigma \xrightarrow{\pi} K_\Sigma \to 0$$

and its cohomology sequence

(3.2) $H^0(\Sigma,K_\Sigma)\to H^1(\Sigma,\mathcal{L}^{\otimes 2}\otimes\det\mathcal{E}^*\otimes K_\Sigma)\to H^1(\Sigma,\mathcal{E}^*\otimes\mathcal{L}\otimes K_\Sigma)\xrightarrow{\pi} H^1(\Sigma,K_\Sigma)\to 0.$

If $\deg\mathcal{L}>\frac{1}{2}\deg\mathcal{E}$, then

$$H^1(\Sigma,\mathcal{L}^{\otimes 2}\otimes\det\mathcal{E}^*\otimes K_\Sigma)=H^0(\Sigma,\mathcal{L}^{-\otimes 2}\otimes\det\mathcal{E})=0,$$

so

$$\pi\colon H^1(\Sigma,\mathcal{E}^*\otimes\mathcal{L}\otimes K_\Sigma)\xrightarrow{\cong} H^1(\Sigma,K_\Sigma)=\mathbb{C}.$$

Thus if $\dim H^0(\Sigma,\mathcal{L}^{-\otimes 2}\otimes\det\mathcal{E}\otimes K_\Sigma)\geq 2$, then δ can never be injective. This provides the case (4).

The map

$$\pi\circ\delta\colon H^0(\Sigma,\mathcal{L}^{-\otimes 2}\otimes\det\mathcal{E}\otimes K_\Sigma)\to H^1(\Sigma,K_\Sigma)$$

is given by the cup product with the extension class e of (3.1). Thus if the extension class $e=0$, then δ can never be injective. This provides the case (5). It is clear that if $H^0(\Sigma,\mathcal{L}^{-\otimes 2}\otimes\det\mathcal{E}\otimes K_\Sigma)=0$, then δ is always injective.

It remains to consider the semistable case with $\deg\mathcal{L}=\frac{1}{2}\deg\mathcal{E}$. In this case, the bundle $\mathcal{L}^{\otimes 2}\otimes\det\mathcal{E}^*$ is of degree zero.

(1) If $\mathcal{L}^{\otimes 2}\otimes\det\mathcal{E}^*$ is not trivial, then $H^1(\Sigma,\mathcal{L}^{\otimes 2}\otimes\det\mathcal{E}^*\otimes K_\Sigma)=H^0(\Sigma,\mathcal{L}^{-\otimes 2}\otimes\det\mathcal{E})=0$ since a degree zero line bundle has global section if and only if it is trivial. In this case,

$$H^1(\Sigma,\mathcal{E}^*\otimes\mathcal{L}\otimes K_\Sigma)\cong H^1(\Sigma,K_\Sigma)=\mathbb{C}$$

as above. By Riemann-Roch theorem,

$$\dim H^0(\mathcal{L}^{\otimes -2}\otimes\det\mathcal{E}\otimes K_\Sigma)=g-1,$$

thus δ can be injective if and only if $g=2$, and the extension class of (3.1) is non-trivial.

(2) Suppose $\mathcal{L}^{\otimes 2}\otimes\det\mathcal{E}^*$ is trivial.
   (a) If the extension class of (3.1) is non-trivial,then the coboundary map $H^0(\Sigma,K_\Sigma)\to H^1(\Sigma,K_\Sigma)$ in exact sequence (3.2) is surjective, and thus $\dim H^1(\Sigma,\mathcal{E}^*\otimes\mathcal{L}\otimes K_\Sigma)=1$. However, in this case

$$\dim H^0(\Sigma,\mathcal{L}^{\otimes -2}\otimes\det\mathcal{E}\otimes K_\Sigma)=g$$

   and thus δ can never be injective for $g\geq 2$.
   (b) If the extension class of (3.1) is trivial, then

$$\mathcal{E}=\mathcal{L}\oplus(\mathcal{L}^*\otimes\det\mathcal{E})=\mathcal{L}^{\oplus 2}.$$

The above arguments provide the cases (2) and (3) and all cases occur in stable pairs. To prove these cases are the all possibilities, we need to look at the excluded bundles, and these were characterized as unstable bundles such that the maximal destabilizer $\mathcal{L}$ is invariant by all $H^0(\Sigma,\mathscr{E}nd_0(\mathcal{E})\otimes K_\Sigma)$, and thus cannot occur in stable pairs. □

Remark 3.2.1. Cases (4) and (5) of Proposition 3.2.1 show that when $\mathcal{E}$ is unstable, there is a constraint on the maximal destabilizer $\mathcal{L}$ in order for $\mathcal{E}$ to belong to a stable pair. In particular, since $\dim H^0(\Sigma, \mathcal{L}^{-2} \otimes K_\Sigma \otimes \det \mathcal{E}) \geq 1$, we must have

$$0 < \deg\left(\mathcal{L}^2 \otimes \det \mathcal{E}^*\right) \leq 2g - 2.$$

Note that if $\deg\left(\mathcal{L}^{\otimes 2} \otimes \det \mathcal{E}\right) = 2g - 2$, then the condition

$$\dim H^0(\Sigma, \mathcal{L}^{\otimes -2} \otimes \det \mathcal{E} \otimes K_\Sigma) = 1$$

implies that

$$\mathcal{L}^{\otimes 2} \otimes \det \mathcal{E} \cong K_\Sigma.$$

From case (5) of Proposition 3.2.1, this means that

$$\mathcal{E} \cong (K_\Sigma^{\frac{1}{2}} \oplus K_\Sigma^{-\frac{1}{2}}) \oplus \mathcal{L} \otimes K_\Sigma^{-\frac{1}{2}}.$$

In other words, $\mathcal{E}$ is exactly the uniformalizing bundle up to tensoring a line bundle.

The information we have derived so far is enough to provide an explicit description of rank two stable Higgs bundles for a compact Riemann surface of genus two. By tensoring a line bundle we may assume that $\det \mathcal{E}$ is trivial or $\mathcal{O}_\Sigma(x)$, where x is some fixed point of Σ.

Example 3.2.4. *If $g = 2$ and $\det \mathcal{E}$ is trivial. Then there are following possibilities for $\mathcal{E}$ to belong to a stable pair:*

(1) *$\mathcal{E}$ is stable;*
(2) *$\mathcal{E} \cong \mathcal{L} \oplus \mathcal{L}^*$, where $\mathcal{L}^{\otimes 2}$ is non-trivial;*
(3) *$\mathcal{E} \cong \mathcal{L}^{\oplus 2}$, where $\mathcal{L}^{\otimes 2}$ is trivial;*
(4) *$\mathcal{E}$ is a non-trivial extension of the trivial bundle by itself;*
(5) *$\mathcal{E} \cong K_\Sigma^{\frac{1}{2}} \oplus K_\Sigma^{-\frac{1}{2}}$.*

Example 3.2.5. *If $g = 2$ and $\det \mathcal{E} \cong \mathcal{O}_\Sigma(x)$ for some fixed point $x \in \Sigma$. Then there are following possibilities for $\mathcal{E}$ to belong to a stable pair:*

(1) *$\mathcal{E}$ is stable;*
(2) *$\mathcal{E} \cong \mathcal{L} \oplus \mathcal{L}^* \otimes \mathcal{O}_\Sigma(x)$, where $\mathcal{L}$ is a line bundle of degree one and*

$$\dim H^0(\Sigma, \mathcal{L}^{\otimes -2} \otimes \mathcal{O}_\Sigma(x) \otimes K_\Sigma) = 1.$$

3.2.3. *General criterion.*

Proposition 3.2.2. Let Σ be a compact Riemann surface of genus $g > 1$. A rank two vector bundle $\mathcal{E}$ occurs in a stable pair $(\mathcal{E}, \theta)$ if and only if there is a Zariski open subset $U \subseteq H^0(\Sigma, \mathscr{End}(\mathcal{E}) \otimes K_\Sigma)$ such that if $\theta \in U$, then θ leaves invariant no proper subbundle.

References

[Ati57] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[Hit87] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[HN75] G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Math. Ann.*, 212:215–248, 1974/75.