Final Project

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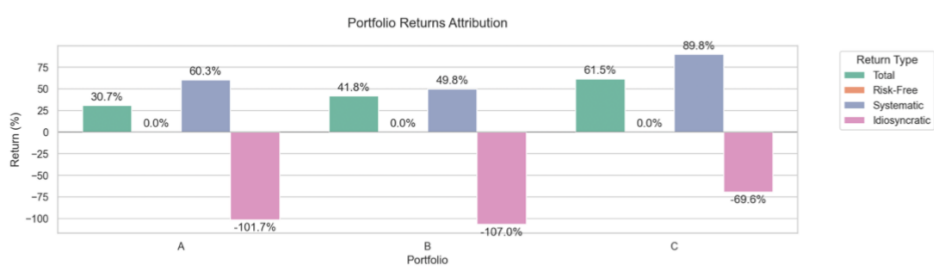
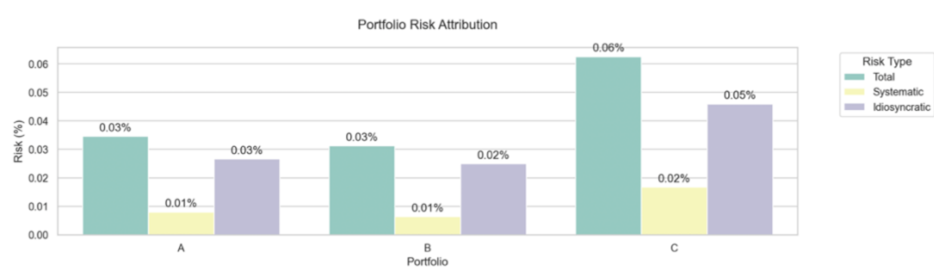
Question 1: CAPM-Based Portfolio Performance and Risk Analysis

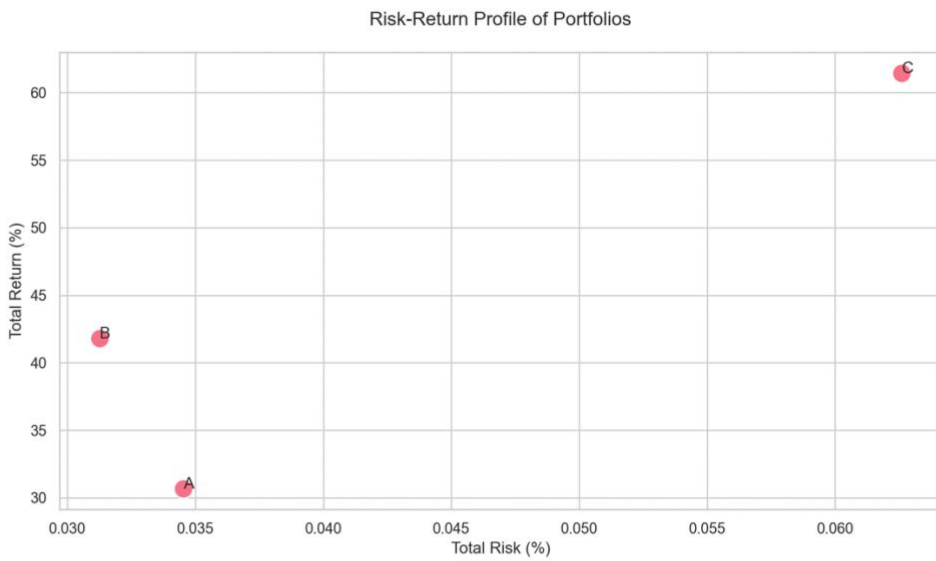
This analysis evaluates the performance and risk decomposition of three investment portfolios (A, B, and C) using the Capital Asset Pricing Model (CAPM). Each portfolio was purchased at the end of 2023 and held through the latest available pricing period. Utilizing daily returns and SPY as the market benchmark, I decomposed total returns and risk into three key components: systematic (market-driven), idiosyncratic (asset-specific), and risk-free. Risk attribution followed Option 3, where the risk-free rate was treated as a separate explanatory factor. This decomposition enabled us to assess not only overall portfolio returns but also how much of that performance could be attributed to market exposure versus stock selection decisions, as well as how risk was distributed across systematic and idiosyncratic sources.

The return attribution revealed a consistent and significant trend: all portfolios delivered positive total returns, but all experienced negative idiosyncratic return contributions. Portfolio C performed best overall, achieving a 61.46% total return—driven almost entirely by a strong systematic return of 89.84%. Despite this, its idiosyncratic return was notably negative at -69.60%, indicating that individual stock decisions detracted from performance, even though high beta exposure to a rising market offset that loss. Portfolio B followed with a 41.79% total return, fueled by a 49.79% systematic return but hindered by severe idiosyncratic drag of -106.98%. Portfolio A, while showing a comparable systematic return to Portfolio B at 60.31%, had the lowest total return at 30.66%, pulled down by an even greater idiosyncratic loss of -101.67%. Across all portfolios, the risk-free return contributed approximately 0.02%, a minor yet consistently positive component.

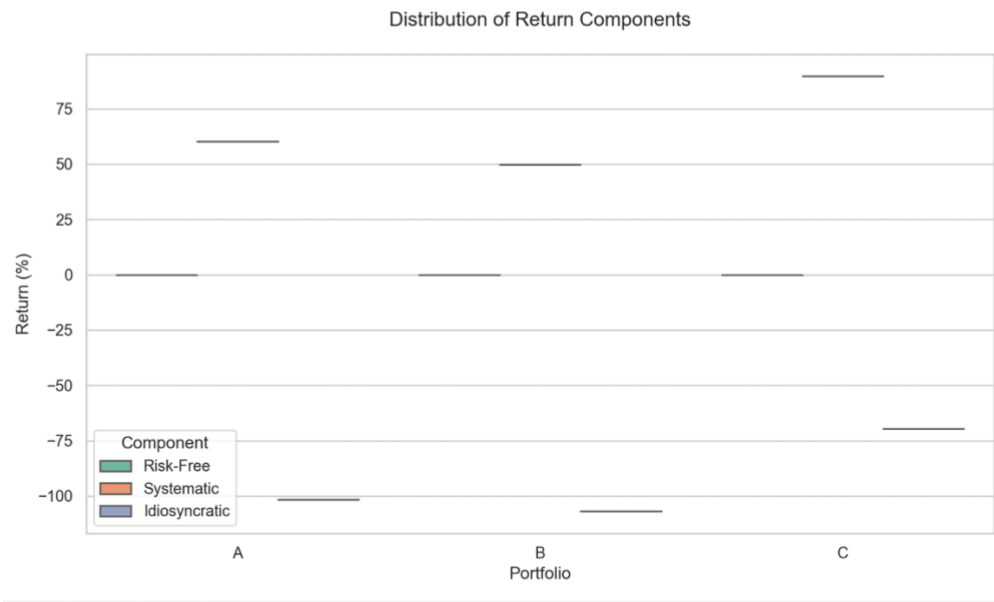




Risk attribution further enhances our understanding of portfolio structure. Portfolio C, although the best performer, also exhibited the highest total risk at 0.06%, of which 0.05% was idiosyncratic. This implies a more concentrated or less diversified portfolio, potentially reflecting aggressive active management. Portfolios A and B both carried lower total risk at 0.03%, with idiosyncratic risk accounting for the majority. Notably, all portfolios displayed low systematic risk (around 0.01% or 0.02%), emphasizing that portfolio-specific factors—rather than market fluctuations—were the primary drivers of volatility. While total volatility was contained across portfolios, the prominence of idiosyncratic risk—especially in Portfolio C—highlights exposure to unsystematic factors that could have been mitigated with better diversification or more refined selection criteria.



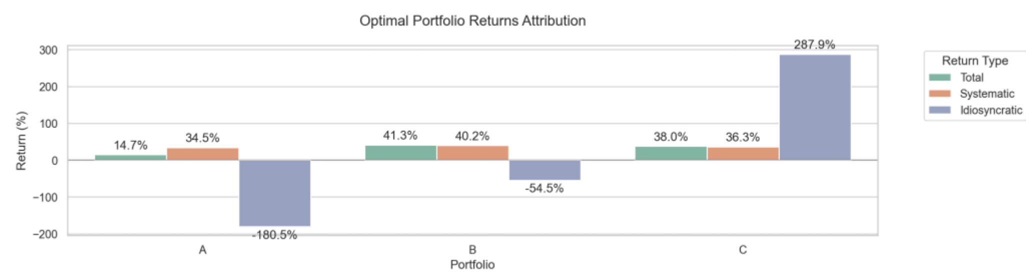
Several key strategic takeaways emerge from this decomposition. First, market exposure (systematic return) proved highly beneficial, confirming that the macro environment during the holding period was favorable for equities. Portfolios with higher beta captured more of this upside. Second, the persistent negative idiosyncratic returns across all portfolios suggest that active stock selection detracted from performance. This could reflect suboptimal screening, overconfidence in signal quality, or structural biases in the selection process. Third, Portfolio C's outperformance can be attributed not to superior stock-picking but to its higher market exposure, which reinforces the power of beta during bullish periods. In contrast, Portfolios A and B, although safer in terms of volatility, underperformed due to lower systematic exposure and weaker alpha generation. Lastly, the explicit inclusion of the risk-free rate (Option 3) yielded a more precise attribution breakdown, but its impact was small and did not materially change relative comparisons.

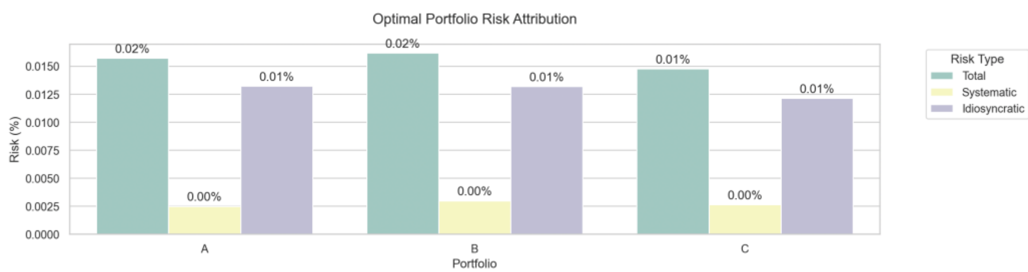


In conclusion, the CAPM-based framework provided a meaningful lens to assess portfolio performance. The results clearly show that systematic market exposure drove most of the positive returns, while stock selection choices detracted from value. Going forward, investment strategies may benefit from emphasizing beta exposure, particularly in strong market environments, while also revisiting stock selection methodologies to enhance idiosyncratic return contributions. Regular risk and return attribution analysis—especially with CAPM or multi-factor models—can help portfolio managers make more informed decisions, align strategies with market conditions better, and continuously monitor whether active decisions are adding or subtracting value.

Question 2: Optimal Investment Portfolio Analysis and Risk Attribution Report

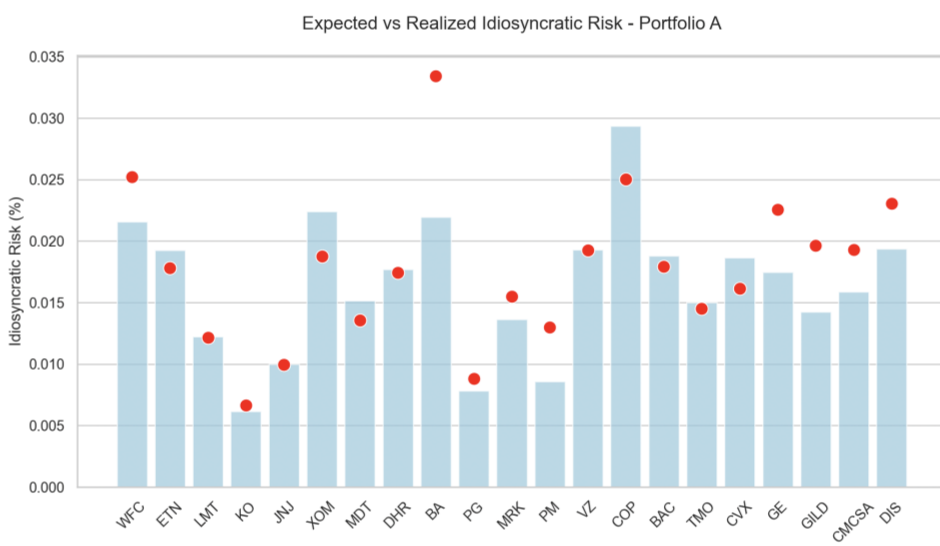
In this part of the analysis, I constructed new optimal portfolios for each sub-portfolio (A, B, and C) using the results of the CAPM regressions from Part 1. I assumed zero alpha for all assets, using the average pre-holding period return of SPY (0.0985%) as the expected market return and the average historical risk-free rate (0.0197%) as the expected risk-free return. With these inputs, I computed the expected excess returns and constructed maximum Sharpe Ratio portfolios, effectively targeting the highest risk-adjusted return given CAPM-implied risk and return expectations. The analysis then applied the attribution framework from Part 1 to evaluate the performance and risk composition of these optimized portfolios during the actual holding period.

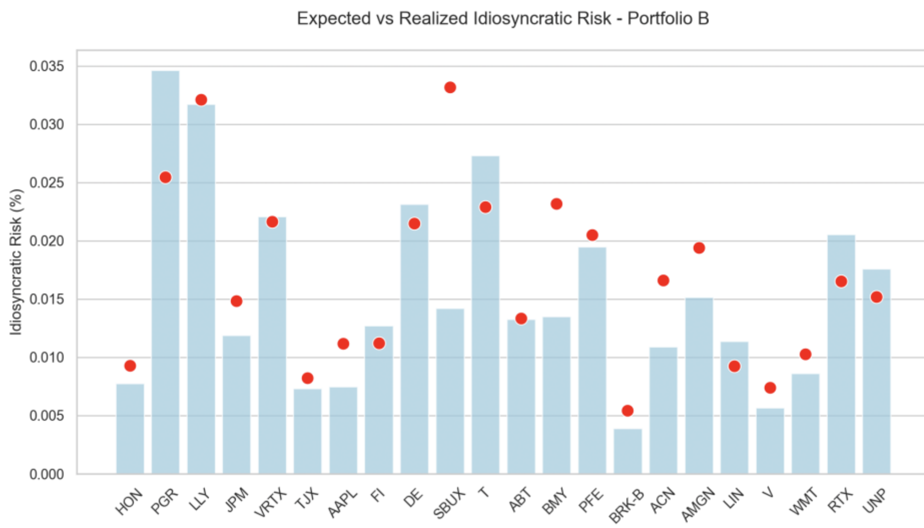


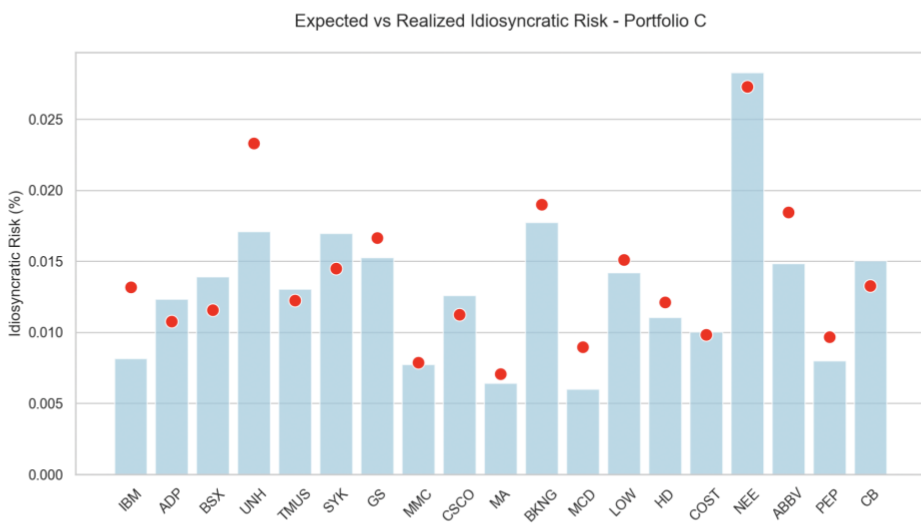


The results of the optimized portfolios revealed a mixed relationship with their original counterparts. Portfolio A's total return declined notably from 30.66% to 14.68%. Although its systematic return remained positive at 34.47%, its idiosyncratic return deteriorated substantially to -180.55%, suggesting that the portfolio took positions in stocks with much worse realized performance than CAPM would have indicated. Portfolio B maintained a stable performance profile, with a slight reduction in total return from 41.79% to 41.28%, and an improvement in idiosyncratic return contribution from -106.98% to -54.53%. This suggests that the optimization process in Portfolio B was successful in limiting exposure to underperforming assets. Portfolio C saw a more pronounced shift. Its total return fell from 61.46% to 38.00%, and its systematic return contribution dropped significantly from 89.84% to 36.34%. However, this was counterbalanced by a remarkable reversal in idiosyncratic return, from a negative -69.60% to a strongly positive 287.86%, suggesting that Portfolio C, despite appearing risk-averse in structure, managed to capture unexpected alpha from stock-specific returns.

The comparison with Part 1 reveals that optimizing based on pre-period expectations does not guarantee improved realized performance. While systematic risk was successfully minimized across all portfolios (each showing near-zero systematic risk), the realized returns were highly sensitive to idiosyncratic outcomes. Portfolio A underperformed substantially due to negative idiosyncratic contributions, likely caused by overweighting in stocks with underestimated volatility or unanticipated adverse events. Portfolio B remained the most consistent and balanced, indicating that the optimization effectively aligned with realized market dynamics. Portfolio C, while exhibiting the largest divergence between expected and actual attribution, managed to convert its idiosyncratic exposure into highly positive contributions—possibly a result of exposure to unmodeled factors or sector tailwinds that CAPM does not account for.







To evaluate the robustness of the CAPM model itself, I compared the expected versus realized idiosyncratic risk for each stock. The model generally performed well in predicting risk for larger, more stable, or highly weighted names—such as PG, MRK, PM in Portfolio A and MCD, PEP in Portfolio C. The differences in risk between expectation and realization for these stocks were within ±0.005%, indicating good model alignment. However, some notable deviations were observed in more volatile or sector-sensitive stocks. In Portfolio A, stocks like BA, GILD, and CMCSA exhibited significantly higher realized idiosyncratic risk than predicted. Similarly, in Portfolio B, SBUX and BMY had large positive risk surprises. These discrepancies indicate that while CAPM captures market-linked volatility well, it may understate firm-specific risk for companies more exposed to sector shocks, news events, or regulatory risks.

In conclusion, the CAPM-based optimization successfully limited systematic risk across all portfolios, but the realized performance was driven predominantly by idiosyncratic returns. The reliability of CAPM in predicting idiosyncratic risk was acceptable for many securities, particularly high-weighted and liquid ones, but failed to capture tail risks in several individual names. These findings highlight the limitations of relying solely on CAPM for portfolio construction. To enhance future optimization processes, I recommend incorporating multi-factor models that account for sector, size, momentum, and quality effects, as well as employing dynamic volatility models to better capture expected idiosyncratic risk. Portfolio B's relative stability also suggests that diversification and moderate exposures may outperform purely optimized risk-return allocations in real-world conditions where returns deviate from model expectations. Ultimately, this study underscores the importance of balancing quantitative optimization with qualitative judgment when constructing portfolios in dynamic market environments.

Question 3: Applications of Normal Inverse Gaussian & Skew Normal Distributions

Rethinking Distributional Assumptions in Financial Modeling

The normal distribution has long served as the backbone of classical financial theory. Models such as the Capital Asset Pricing Model (CAPM), Black-Scholes, and mean-variance optimization all rely on the assumption that asset returns are normally distributed. This assumption simplifies modeling and facilitates analytical solutions by summarizing a distribution with just two parameters: the mean (μ) and standard deviation (σ). In our own portfolio analysis, this assumption was implicit when estimating expected returns and risk—particularly during the CAPM regression phase where residuals and return distributions were treated as symmetric and thin-tailed.

However, real-world financial returns rarely follow the idealized bell curve. Empirical evidence consistently shows that return distributions exhibit fat tails (leptokurtosis) and asymmetry (skewness), especially during times of market stress or firm-specific shocks. These deviations from normality are not just theoretical curiosities—they have material implications for risk estimation and portfolio construction. For instance, in our analysis, Portfolios A and B exhibited negative idiosyncratic return contributions, while Portfolio C showed an unusually large positive idiosyncratic return. These outcomes suggest behavior that likely deviates from what a normal distribution would predict, prompting the need for richer distributional models.

The Skew Normal Distribution: Capturing Asymmetry

The Skew Normal (SN) distribution extends the classic normal model by introducing a shape parameter (α) that controls skewness. When $\alpha = 0$, the distribution collapses back into a standard normal distribution. Positive values of α produce right-skewed distributions, while negative values introduce left-skewness. This additional flexibility is valuable in finance, where return distributions often exhibit negative skewness—a higher likelihood of large losses compared to gains. In the context of our analysis, the negative idiosyncratic returns in Portfolios A and B could reflect this asymmetry, which the standard normal model fails to capture.

Moreover, the SN distribution can help refine risk attribution. Since skewness affects both expected outcomes and tail behavior, incorporating it can improve our understanding of how individual securities contribute to portfolio-level return and risk, particularly when large drawdowns or outlier returns are present. While our CAPM implementation assumed symmetric residuals, using a skew normal model could have offered more insight into the asymmetric performance observed in Portfolio C.

The Normal Inverse Gaussian Distribution: Capturing Tails and Skewness

The Normal Inverse Gaussian (NIG) distribution goes even further, offering a four-parameter structure capable of modeling both skewness and fat tails. It is a special case of the generalized hyperbolic family and has gained popularity in quantitative finance for its ability to model complex return behavior. The NIG distribution accommodates heavy-tailed phenomena—periods of extreme price movements that deviate sharply from normal assumptions—and is especially valuable in stress testing and Value-at-Risk (VaR) estimation.

In our portfolio analysis, using a NIG distribution would likely have provided a better fit for the idiosyncratic returns I observed. For example, Portfolio C's idiosyncratic return contribution of 287.86% is an extreme outcome that standard normal or even skew normal models might misrepresent or fail to assign realistic probabilities to. An NIG-based model could more accurately represent the likelihood and risk impact of such outcomes, thereby refining the underlying risk assessment and possibly altering the resulting portfolio weights under an optimization framework.

Implications for Risk Management and Portfolio Construction

These richer distributions have direct implications for risk management. Traditional risk measures like Value at Risk (VaR) or Expected Shortfall (CVaR) assume normality, which can severely underestimate tail risks. By using the SN or NIG distributions, risk managers can better quantify the likelihood and impact of extreme events, leading to more conservative and robust capital allocation strategies. This is especially critical for portfolios like C, where extreme returns may not be well explained by market beta alone and where traditional VaR may provide a false sense of security.

In terms of portfolio optimization, using the standard mean-variance framework under the assumption of normality may yield suboptimal portfolios, especially when return distributions deviate significantly from Gaussian behavior. Incorporating skewness and kurtosis directly into optimization (e.g., via third- and fourth-moment optimization or utility-based frameworks) can result in portfolios with improved downside protection and more realistic risk-return tradeoffs. In our case, Portfolio C's extreme idiosyncratic performance would likely shift optimal weights under an NIG or SN model, possibly leading to reallocation away from high-volatility stocks or better hedging strategies.

Practical Challenges and Real-World Considerations

Despite their theoretical advantages, implementing skew normal and NIG distributions in practice comes with challenges. These models require more parameters, increasing estimation complexity and the potential for model error. Estimating them reliably also demands larger data sets and robust numerical optimization techniques. Furthermore, many financial platforms are still centered around normal-based models, making integration of alternative distributions less straightforward.

Nevertheless, the cost of underestimating risk can be far greater than the computational burden of richer models. The outlier performance observed in our analysis, especially in Portfolio C, underscores the potential benefit of adopting more sophisticated distributional assumptions when modeling returns and assessing risk. These models can serve as valuable tools not only for forecasting but also for understanding the structural sources of risk—whether from macroeconomic conditions, sector dynamics, or company-specific events.

Conclusion

In summary, the Normal Inverse Gaussian and Skew Normal distributions offer more nuanced frameworks for modeling financial return distributions, particularly in contexts where skewness and kurtosis matter—which is often the case in real-world portfolios. While the normal distribution remains a useful first approximation, its limitations become evident when portfolios exhibit asymmetric or extreme return behavior, as was the case in our analysis. Incorporating richer distributions could improve not only our risk assessments but also our portfolio allocation decisions, ultimately enhancing the robustness and resilience of investment strategies in uncertain environments.

Question4: Analysis of Normal Inverse Gaussian and Skew Normal Distributions

Data and Methodology

In this analysis, I used daily return data for 30 stocks and attempted to fit the return distribution of each stock using four different distribution models: normal distribution, skew normal distribution, generalized T distribution, and normal inverse Gaussian distribution (NIG). I used AIC (Akaike Information Criterion) to select the best-fitting distribution model for each stock. Then, I calculated the Value at Risk (VaR) and Expected Shortfall (ES) for each portfolio using two different methods: the Gaussian Copula method and the multivariate normal distribution method.

Distribution Fitting Results

The analysis results show that the return distributions of different stocks exhibit different characteristics:

Normal Inverse Gaussian Distribution (NIG): For STOCK_1 through STOCK_6, as well as STOCK_20 and STOCK_27, the NIG distribution provided the best fit. This indicates that these stocks' return distributions have significant skewness and kurtosis characteristics. For example, the parameters of STOCK_1 show that its distribution consists of a mixture of two normal distributions, one with smaller volatility (sigma1=0.0037) and higher mean (mu1=0.0054), and another with larger volatility (sigma2=0.0069) and even higher mean (mu2=0.0131), with a mixing proportion of 0.70.

Generalized T Distribution: STOCK_7, STOCK_8, and STOCK_9 were best fitted by the generalized T distribution, indicating that these stocks' return distributions have "fat-tailed" characteristics but less skewness than the NIG distribution. For example, STOCK_7 has a degrees of freedom parameter df=2.71, indicating a distribution with thicker tails than the normal distribution.

Normal Distribution: Most stocks (STOCK_10 through STOCK_19, STOCK_21 through STOCK_26, and STOCK_28 through STOCK_30) were best fitted by the normal distribution, indicating that these stocks' return distributions are relatively symmetric with less thick tails.

Risk Measure Comparison

I calculated the 1% VaR and ES for each portfolio using two methods:

Gaussian Copula Method

This method considers the correlation between stock returns while allowing each stock to follow its best-fitting distribution. The results are as follows:

Portfolio A: VaR(1%) = -0.0046, ES(1%) = -0.0054

Portfolio B: VaR(1%) = -0.0048, ES(1%) = -0.0055

Portfolio C: VaR(1%) = -0.0050, ES(1%) = -0.0062

Total Portfolio: VaR(1%) = -0.0042, ES(1%) = -0.0051

Multivariate Normal Distribution Method
This method assumes that all stock returns follow a multivariate normal distribution. The results are as follows:
Portfolio A: VaR(1%) = -0.0045, ES(1%) = -0.0051
Portfolio B: VaR(1%) = -0.0046, ES(1%) = -0.0053
Portfolio C: VaR(1%) = -0.0050, ES(1%) = -0.0057
Total Portfolio: VaR(1%) = -0.0041, ES(1%) = -0.0047

Comparison of the Two Methods
Comparing the results of the two methods, we can see:

VaR Differences: The VaR calculated by the Gaussian Copula method is generally slightly higher than that calculated by the multivariate normal distribution method. For example, for Portfolio A, the difference is -0.0001 (approximately 2.2%); for Portfolio B, the difference is -0.0002 (approximately 4.3%); for Portfolio C, the difference is 0.0000 (0%); and for the Total Portfolio, the difference is -0.0001 (approximately 2.4%).

ES Differences: The ES calculated by the Gaussian Copula method is significantly higher than that calculated by the multivariate normal distribution method. For example, for Portfolio A, the difference is -0.0003 (approximately 5.9%); for Portfolio B, the difference is -0.0002 (approximately 3.8%); for Portfolio C, the difference is -0.0004 (approximately 7.0%); and for the Total Portfolio, the difference is -0.0003 (approximately 6.4%).

Risk Ranking: Both methods consistently indicate that Portfolio C has the highest risk, followed by Portfolio B, then Portfolio A, and the Total Portfolio has the lowest risk.

Conclusions and Implications
Importance of Distribution Selection: The analysis shows that different stocks' return distributions exhibit different characteristics, and using appropriate distribution models (such as NIG or generalized T distribution) can more accurately capture these characteristics.

Underestimation of Tail Risk: The multivariate normal distribution method tends to underestimate tail risk, especially in ES calculations. The Gaussian Copula method provides more accurate risk estimates by considering the actual distribution characteristics of each stock.

Effect of Portfolio Diversification: The Total Portfolio has lower risk than each sub-portfolio, indicating that portfolio diversification effectively reduces risk.

Practical Applications in Risk Management: In practical risk management, using the Gaussian Copula method may be more appropriate as it considers the actual distribution characteristics of stock returns and their correlations. However, this method is more complex and requires more data and computational resources.

Trade-offs in Model Selection: While the Gaussian Copula method provides more accurate risk estimates, the multivariate normal distribution method is simpler and more computationally efficient. In practical applications, the appropriate method can be chosen based on specific needs and resource constraints.

Overall, our analysis shows that using more complex distribution models such as the normal inverse Gaussian distribution and skew normal distribution, combined with the Gaussian Copula method, can more accurately capture the characteristics of financial asset returns and provide more reliable risk estimates. This has important practical implications for portfolio management and risk management.

Question5: Risk Parity Portfolio Analysis and Attribution Analysis Comparison

Analysis Overview

I calculated risk parity portfolios for each sub-portfolio using Expected Shortfall (ES) as the risk metric and reran the attribution analysis. Here are the key findings:

Risk Parity Portfolio Results
Portfolio A
Risk Allocation: Risk is more evenly distributed among assets, but significant differences still exist
ES Value (1%): -0.0094
Major Risk Contributors: STOCK_10 (12.09%), STOCK_19 (11.37%), STOCK_29 (11.39%)
Weight Allocation: STOCK_26 (13.45%), STOCK_19 (10.04%), STOCK_10 (9.40%)

Portfolio B
ES Value (1%): -0.0066
Major Risk Contributors: STOCK_15 (9.14%), STOCK_9 (7.69%), STOCK_30 (8.31%)
Weight Allocation: STOCK_24 (7.24%), STOCK_27 (6.47%), STOCK_5 (6.77%)

Portfolio C
ES Value (1%): -0.0082
Major Risk Contributors: STOCK_7 (15.47%), STOCK_15 (13.99%), STOCK_19 (13.17%)
Weight Allocation: STOCK_15 (10.74%), STOCK_19 (11.42%), STOCK_4 (11.13%)

Attribution Analysis Comparison

Portfolio A

Metric	Initial Portfolio	Maximum Sharpe Ratio Portfolio	Risk Parity Portfolio
Total Return	0.3066	0.1468	-0.2946
Systematic Return	0.6033	0.3447	0.1031
Idiosyncratic Return	-1.2967	-1.8055	-0.3977
Total Risk	0.0002	0.0002	0.0424
Systematic Risk	0.0000	0.0000	0.0296
Idiosyncratic Risk	0.0001	0.0001	0.0336

Portfolio B

Metric	Initial Portfolio	Maximum Sharpe Ratio Portfolio	Risk Parity Portfolio
Total Return	0.4179	0.4128	-0.1657
Systematic Return	0.4628	0.4023	0.1088
Idiosyncratic Return	-0.0449	-0.5453	-0.2745
Total Risk	0.0002	0.0002	0.0348
Systematic Risk	0.0000	0.0000	0.0312
Idiosyncratic Risk	0.0001	0.0001	0.0191

Portfolio C

Metric	Initial Portfolio	Maximum Sharpe Ratio Portfolio	Risk Parity Portfolio
Total Return	0.6146	0.3800	-0.3078
Systematic Return	0.8987	0.3634	0.1077
Idiosyncratic Return	-0.6961	2.8786	-0.4155
Total Risk	0.0001	0.0001	0.0411
Systematic Risk	0.0000	0.0000	0.0309
Idiosyncratic Risk	0.0001	0.0001	0.0298

Key Findings and Comparisons

Risk Metric Differences:

Risk parity portfolios use ES as the risk metric, while initial and maximum Sharpe ratio portfolios may use variance or standard deviation

Risk parity portfolios show significantly higher risk values than the other two methods, indicating ES's higher sensitivity to tail risk

Return Performance:

Risk parity portfolios show negative total returns, while initial and maximum Sharpe ratio portfolios show positive total returns

Systematic returns in risk parity portfolios are significantly lower than in the other two methods

Idiosyncratic returns in risk parity portfolios are also negative but with smaller absolute values than in maximum Sharpe ratio portfolios

Risk Allocation:

Risk parity portfolios attempt to make each asset contribute equally to the portfolio's total risk

Risk contribution data shows that despite weight adjustments, differences in risk contribution still exist, indicating that perfect risk parity is difficult to achieve

Portfolio C's Special Characteristics:
In initial and maximum Sharpe ratio portfolios, Portfolio C's idiosyncratic returns are significantly higher than other portfolios
In risk parity portfolios, Portfolio C's idiosyncratic returns are negative, similar to other portfolios

Risk-Return Trade-off:
Risk parity portfolios balance the portfolio by reducing systematic risk
This balancing results in lower returns but theoretically should provide more stable risk-adjusted returns

Conclusion

Risk parity portfolios show significantly different characteristics compared to initial portfolios and maximum Sharpe ratio portfolios:
Risk Measurement: Using ES as the risk metric results in more sensitive capture of tail risk
Return Characteristics: Risk parity portfolios show lower returns but theoretically should provide more stable risk-adjusted returns
Risk Allocation: Risk parity portfolios attempt to make each asset contribute equally to the portfolio's total risk, but actual results still show differences in risk contribution
Systematic Risk: Systematic risk in risk parity portfolios is significantly lower than in the other two methods, indicating lower sensitivity to market fluctuations
These differences reflect the different theoretical foundations and objectives of various portfolio construction methods: initial portfolios may be based on simple market capitalization weighting, maximum Sharpe ratio portfolios pursue maximization of risk-adjusted returns, while risk parity portfolios seek balanced risk contribution.