Tree-based Multiple Hypothesis Testing with General FWER Control

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1. Mathematical Formulation

Here I consider a testing procedure without data splitting, and only about a case on the 'multilplicative trees'.

In the tree-structured multiple hypotheses setting, we say $\boldsymbol{X}_t$ is a data set to construct each tth p-value.

Let $p_t a$ be the p-value of the parent node and p_{t+1} be the p-value of the subsequent child of that parent node. When we fix some $\alpha \in (0,1]$, our basic observation is

$$P(p_{t+1} > \alpha | p_t \leq \alpha_1) \leq P(p_{t+1} > \alpha | p_t \leq \alpha_2),$$

whenever $\alpha_1 \leq \alpha_2$. This is because, we can understand our procedure is actually using

$$p'_{t+1} = \max(p_1, ...p_t, p_{t+1}),$$

as a p-value implemented in the $t+1$th node, where p_i is all other ancestors of the p_{t+1}. This can be another (maybe equivalent) understanding of the **monotonicity condition** in Goeman and Solari (2010).

From this we can recall the **positive regression dependency on each one from a subset I_0(or PRDS on I_0)** from Benjamini and Yekutieli (2001), which states when D is a non-decreasing set(so whenever $x \in D$ and $x \leq y$, then $y \in D$),

$$P(\boldsymbol{X} \in D | X_i = X)$$

is non-decreasing in x.

We also provide some notation for the tree-structured hypotheses. Let us have a tree with depth of d where for some fixed integer k, k is a number of children for every parent node (so we consider *multiplicative trees*). Then we say N is the number of total nodes and n as a number of total clusters, where

$$N = \sum_{r=1}^{d} k^{r-1}$$

$$n = 1 + \sum_{r=1}^{d-1} k^{r-1}.$$

2. Weak FWER Control

Here we consider about the weak FWER control, so when we assume all of the nulls are true. We first impose an assumption, called 'conditional super-uniformity', following Robertson et al. (2023). For the p-values p_i, we define $R_i = \mathbb{1}(p_i \leq \alpha_i)$, where $\mathbb{F}^t = \sigma(R_1, ..., R_t)$ and we let α_t as a $\mathbb{F}^{t-1}$-measurable function of $(R_1, ..., R_{t-1})$.

Definition 1. The null p-values are said to be **conditionally super-uniform** if $P(p_t \leq \alpha_t | \mathbb{F}^{t-1}) \leq \alpha_t$ for any $\mathbb{F}^{t-1}$-measurable α_t.

We also give additional notation before proposing our theorem on the weak FWER control. We say C_i is an ordered set of parents of the ith node on the tree, and d_i is the depth of the ith node. Then we see $|C_i| = d_i + 1$. Denote $C_{i,j}$ as the jth element of C_i. For example, consider a binary tree and the case $i = 8$. Then $C_8 = \{1, 2, 4\}$, and $C_{8,2} = 4$. We are also able to check $|C_8| = 3$, and $d_8 = 2$.

Now we state our theorem.

Theorem 1. If the null p-values are conditionally super-uniform, then the procedure with the critical value functions for every tth node

$$\alpha_t = \begin{cases} \frac{\alpha}{1+k\alpha}, \text{ if all of the } i\text{th tests where } i \in C_t \text{ are rejected,} \\ 0, \text{ otherwise,} \end{cases}$$

implies that the weak FWER is controlled at level α.

Proof. Assume that the all of the nulls are true. First, note that for any ith node and $\alpha \in (0, 1)$, since we assumed conditional super-uniformity on the null p-values,

$$\begin{aligned} P(p_i \leq \alpha) &= P(p_i \leq \alpha \text{ and } \forall j \in C_i,\ p_j \leq \alpha) + P(p_i \leq \alpha \text{ and } \exists j \in C_i \text{ such that } P_j > \alpha) \\ &= P(p_i \leq \alpha \text{ and } \forall j \in C_i,\ p_j \leq \alpha) \\ &= P(p_1 \leq \alpha) P(p_{C_{i,2}} \leq \alpha | p_1 \leq \alpha) ... P(p_i \leq \alpha | p_1 \leq \alpha, ..., p_{C_{i,d_i}} \leq \alpha) \\ &\leq \alpha^{d_i+1} \end{aligned}$$

holds. Therefore,

$$\begin{aligned} P(V \geq 1) &= P\Big((\cap_{i=1}^N \{p_i > \alpha_i\})^c \Big) \\ &= P\Big(\cup_{i=1}^N \{p_i \leq \alpha_i\} \Big) \\ &\leq \sum_{i=1}^N P(p_i \leq \alpha_i) \\ &= \underbrace{P(p_1 \leq \alpha_1)}_{p\text{-value on the root node}} + \underbrace{P(p_2 \leq \alpha_2) + ... + P(p_{1+k} \leq \alpha_{1+k})}_{\text{k } p\text{-values on the second level}} + \underbrace{P(p_{2+k} \leq \alpha_{2+k}) + P(p_{1+k+k^2} \leq \alpha_{1+k+k^2})}_{k^2\ p\text{-values on the third level}} + ... \\ &\quad + \underbrace{P(p_{N-k^{d-1}+1} \leq \alpha_{N-k^{d-1}+1}) + ... + P(p_N \leq \alpha_N)}_{k^{d-1}\ p\text{-values on the final level}} \\ &\leq \sum_{j=1}^d k^{j-1} \alpha_j^j \\ &\leq \sum_{j=1}^\infty k^{j-1} \Big(\frac{\alpha}{1+k\alpha} \Big)^j = \alpha, \end{aligned}$$

which gives us a desired conclusion. $\square$

3. General FWER/FDR Control

Now we assume that, whenever j is a child node of i in the tree structure, $\boldsymbol{X}_j \subseteq \boldsymbol{X}_i$. Note that this is a much more realistic than the conditional super-uniformity assumption.

4. Miscellanea

My idea is that,

- **Since $p_1, ..., p_n$ (so the p-values after local adjustments) are still has a PRDS property, we can try to directly apply Benjamini and Yekutieli (2001)'s approach to get general FDR or FWER control.**
- **We can also consider which kind of local adjustments give valid and more powerful tests.**

I will further check about,

- **Idea of data splitting used in the recent simulation.**
- **Prove explicitly the validness of locally adjusted p-values from different methods.**

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