Unbiasedness of Difference-in-Means Estimator under Random Assignment*

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1 Estimation of Sample Average Treatment Effect (SATE)

1.1 Setup

Let the index $i \in \{1, \dots, n\}$ run over n units in a finite sample, $\mathcal{S}_n$, where $n \geq 4$. Of these n units, $n_T \geq 2$ are assigned to the treatment condition and $n_C \geq 2$ are assigned to the control condition, where $n_T + n_C = n$. Although not necessary for the derivation of the Difference-in-Means estimator's variance, these assumptions on the sizes of n, n_T and n_C ensure that the conservative estimator of the Difference-in-Means estimator's variance is well defined. Let the binary indicator variable $Z_i \in \{0, 1\}$ denote whether unit i is assigned to treatment ($Z_i = 1$) or control ($Z_i = 0$). The set $\Omega = \left\{\boldsymbol{z} : \sum_{i=1}^{n} z_i = n_T\right\}$ contains the possible values of $\boldsymbol{Z} = \begin{bmatrix} Z_1, \dots, Z_n \end{bmatrix}^{\top}$. Under complete random assignment, the number of elements in the set Ω is $\binom{n}{n_T}$. By contrast, under n independent Bernoulli assignments, there would be 2^n possible assignment vectors. However, even if n_T is not fixed by design (as in complete random assignment), we can fix n_T by conditioning on its observed value. The randomization distribution conditional on the realized n_T yields the same randomization distribution one would obtain if n_T had been fixed ex ante by design. Hence, this general setup and the proof to follow pertains to both simple and complete random assignment even though the argument by which one can regard n_T as fixed is slightly different under simple and complete assignment mechanisms.

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Adopting the terminology of Freedman (2009) and later Gerber and Green (2012), define a potential outcomes schedule as a vector-valued function, $\boldsymbol{y} : \Omega \mapsto \mathbb{R}^n$, which maps the set of assignments, Ω, to an n-dimensional vector of real numbers, $\mathbb{R}^n$. More intuitively, a potential outcomes schedule is a listing of how each study participant would have responded to any $\boldsymbol{z} \in \Omega$ that a random assignment process could produce. The vectors of potential outcomes are the elements in the image of the potential outcomes schedule, $\boldsymbol{y} : \Omega \mapsto \mathbb{R}^n$, and the individual potential outcomes for unit $i \in \{1, \dots, n\}$ are the ith entries of each of the n-dimensional vectors of potential outcomes.

With $|\Omega|$ possible assignments, where $|\Omega| = \binom{n}{n_T}$ under complete random assignment, there are in principle $|\Omega|$ vectors of potential outcomes.[1] However, under the Stable Unit Treatment Value Assumption (SUTVA)[2] (Cox, 1958; Rubin, 1980, 1986), let y_{Ti} denote the common outcome value of unit i for all $\boldsymbol{z} \in \Omega$ with $z_i = 1$. Likewise, let y_{Ci} denote the common outcome value of unit i for all $\boldsymbol{z} \in \Omega$ with $z_i = 0$. The individual causal effect for unit i on the additive scale is $\tau_i = y_{Ti} - y_{Ci}$. The vectors $\boldsymbol{y_C}$ and $\boldsymbol{y_T}$ denote the collection of control and treatment potential outcomes, respectively, for all n units, and $\boldsymbol{\tau}$ denotes the collection of individual, additive effects for all n units. The observed outcome for unit $i \in \{1, \dots, n\}$ is $Y_i = Z_i y_{Ti} + (1 - Z_i) \, y_{Ci}$, which is either y_{Ti} or y_{Ci} depending on whether the randomly selected $\boldsymbol{z} \in \Omega$ is with $z_i = 1$ or $z_i = 0$.

The target of interest is the Sample Average Treatment Effect (SATE), $\tau_{\text{SATE}} \equiv n^{-1} \sum\limits_{i=1}^{n} \tau_i$. Define the Difference-in-Means estimator of τ_{SATE} as

$$\hat{\tau} \equiv \frac{\sum\limits_{i=1}^{n} Z_i Y_i}{\sum\limits_{i=1}^{n} Z_i} - \frac{\sum\limits_{i=1}^{n} (1 - Z_i) \, Y_i}{\sum\limits_{i=1}^{n} (1 - Z_i)}. \tag{1}$$

For the expectation of this estimator in Equation (1) with respect to the SATE, I write $\mathrm{E}_{\Omega}[\cdot]$ to indicate that the expectation pertains to only randomness of the assignment process.

1.2 Estimation under Complete Random Assignment

Lemma 1. *Under complete, uniform random assignment in which n_T out of n total units are assigned to treatment, $\mathrm{E}_{\Omega}[Z_i] = \dfrac{n_T}{n}$ for all $i \in \{1, \dots, n\}$ units.*

Proof. We will complete this proof in two steps: We will show that (1) the proportion of as-

[1] For an arbitrary set W, let $|W|$ denote the cardinality of (i.e., the number of elements in) the set W.

[2] SUTVA implies that (1) units in the experiment respond to only the treatment condition to which each unit is individually assigned and (2) the treatment condition is actually the same treatment for all units assigned to treatment and the control condition is the same for all units assigned to control.

signments in which unit i is in the treatment condition is $\dfrac{n_T}{n}$ and (2) under uniform, random assignment, the probability that $Z_i = 1$ is equal to this proportion $\dfrac{n_T}{n}$.

Step 1: First note that the number of ways to choose a subset of n_T treated units from a fixed population of n units is as follows:

$$\binom{n}{n_T} = \frac{n!}{(n - n_T)!n_T!} = \frac{n!}{n_C!n_T!}, \tag{2}$$

where $n_C = n - n_T$ is the number of units assigned to the control condition.

Given that an arbitrary unit i is in the treatment condition and only n_T total units can be in the treatment condition, there are $\binom{n-1}{n_T-1}$ ways in which $n_T - 1$ other units could be in the treatment condition. Hence, the number of assignments in which unit i is treated and $n_T - 1$ other units are treated is:

$$\binom{n-1}{n_T-1} = \frac{(n-1)!}{\big((n-1) - (n_T - 1)\big)!\,(n_T - 1)!} \tag{3}$$

To get the proportion of assignments in which unit i is treated, we need to divide (3) by (2):

$$\frac{\binom{n-1}{n_T-1}}{\binom{n}{n_T}} = \frac{\left(\dfrac{(n-1)!}{\big((n-1) - (n_T - 1)\big)!\,(n_T - 1)!}\right)}{\left(\dfrac{n!}{n_C!n_T!}\right)} \tag{4}$$

Now notice that:

$$\begin{aligned}(n-1) - (n_T - 1) &= n - 1 - n_T + 1 \\ &= n - n_T \\ &= n_C\end{aligned}$$

We can therefore substitute n_C for $(n-1) - (n_T - 1)$ in (4), which gives us:

$$\frac{\left(\frac{(n-1)!}{n_C!\,(n_T-1)!}\right)}{\left(\frac{n!}{n_C!n_T!}\right)} \tag{5}$$

Now we can simply manipulate (5) and cancel terms until we are left with $\frac{n_T}{n}$:

$$\begin{aligned}
&= \left(\frac{(n-1)!}{n_C!\,(n_T-1)!}\right)\left(\frac{n_C!n_T!}{n!}\right) \\
&= \left(\frac{(n-1)\,(n-2)\ldots 1}{n_C\,(n_C-1)\ldots 1\,(n_T-1)\ldots 1}\right)\left(\frac{n_C\,(n_C-1)\ldots 1 n_T\,(n_T-1)\ldots 1}{n\,(n-1)\ldots 1}\right) \\
&= \frac{(n-1)\,(n-2)\ldots 2 n_C\,(n_C-1)\ldots 2 n_T (n_T-1)\ldots 2}{n_C\,(n_C-1)\ldots 2 (n_T-1)\ldots 2 n (n-1)\ldots 2}
\end{aligned}$$

All of the respective matching colors in the numerator and denominator cancel, which leaves us with $\frac{n_T}{n}$. Therefore, exactly $\frac{n_T}{n}$ out of all assignment assignments will be those in which unit i is in the treatment condition.

Setp 2: The total probability of all assignments in which i is treated is simply the sum of the probabilities of those assignments in which unit i is in the treatment condition. Under uniform random assignment, the probability of each assignment permutation is $\frac{1}{|\Omega|}$. Thus, the probability that unit i is treated is as follows:

$$\begin{aligned}
\left(\frac{1}{|\Omega|}\right)\left(\frac{n_T}{n}\right)|\Omega| &= \left(\frac{1}{|\Omega|}\right)\frac{n_T\,|\Omega|}{n} \\
&= \frac{n_T\,|\Omega|}{|\Omega|\,n} \\
&= \frac{n_T}{n}
\end{aligned}$$

Since $\Pr(Z_i = 1) = \frac{n_T}{n}$ for all $i \in \{1, \ldots, n\}$ units, it follows that the expected value of $Z_i \in \{0, 1\}$ is $\mathrm{E}_\Omega[Z_i] = 1\left(\frac{n_T}{n}\right) + 0\left(1 - \frac{n_T}{n}\right) = \frac{n_T}{n}$. $\square$

Proposition 1. *Under complete, uniform random assignment,* $\mathbb{E}_{\Omega}\left[\hat{\tau}\right] = \bar{\tau}_{SATE}$.

Proof. First, the linearity of expectations implies that

$$\begin{aligned}
\mathbb{E}_{\Omega}\left[\hat{\tau}\right] &= \mathrm{E}_{\Omega}\left[\frac{\sum_{i=1}^{n} Z_i Y_i}{\sum_{i=1}^{n} Z_i} - \frac{\sum_{i=1}^{n} \left(1 - Z_i\right) Y_i}{\sum_{i=1}^{n} \left(1 - Z_i\right)}\right] \\
&= \mathrm{E}_{\Omega}\left[\frac{\sum_{i=1}^{n} Z_i Y_i}{\sum_{i=1}^{n} Z_i}\right] - \mathrm{E}_{\Omega}\left[\frac{\sum_{i=1}^{n} \left(1 - Z_i\right) Y_i}{\sum_{i=1}^{n} \left(1 - Z_i\right)}\right]
\end{aligned}$$

and, since the number of treated and control units are fixed at n_T and n_C under complete random assignment,

$$\mathbb{E}_{\Omega}\left[\hat{\tau}\right] = \frac{1}{n_T}\,\mathrm{E}_{\Omega}\left[\sum_{i=1}^{n} Z_i Y_i\right] - \frac{1}{n_C}\,\mathrm{E}_{\Omega}\left[\sum_{i=1}^{n} \left(1 - Z_i\right) Y_i\right]$$

Since the observed outcomes for treated units is equal to those units' treatment potential outcomes, we can substitute $Z_i y_{Ti}$ for $Z_i Y_i$. Analogously, we can substitute $\left(1 - Z_i\right) y_{Ci}$ for $\left(1 - Z_i\right) Y_i$. After both substitutions we are left with

$$\begin{aligned}
\mathbb{E}_{\Omega}\left[\hat{\tau}\right] &= \frac{1}{n_T}\,\mathrm{E}_{\Omega}\left[\sum_{i=1}^{n} Z_i y_{Ti}\right] - \frac{1}{n_C}\,\mathrm{E}_{\Omega}\left[\sum_{i=1}^{n} \left(1 - Z_i\right) y_{Ci}\right] \\
&= \left(\frac{1}{n_T}\right) \mathbb{E}_{\Omega}\left[Z_1 y_{T1} + \cdots + Z_n y_{Tn}\right] - \left(\frac{1}{n_C}\right) \mathbb{E}_{\Omega}\left[\left(1 - Z_1\right) y_{C1} + \cdots + \left(1 - Z_n\right) y_{Cn}\right] \\
&= \left(\frac{1}{n_T}\right) \mathbb{E}_{\Omega}\left[Z_1 y_{T1}\right] + \cdots + \mathbb{E}_{\Omega}\left[Z_n y_{Tn}\right] - \left(\frac{1}{n_C}\right) \mathbb{E}_{\Omega}\left[\left(1 - Z_1\right) y_{C1}\right] + \cdots + \mathbb{E}_{\Omega}\left[\left(1 - Z_n\right) y_{Cn}\right] \\
&= \left(\frac{1}{n_T}\right) y_{T1}\mathbb{E}_{\Omega}\left[Z_1\right] + \cdots + y_{Tn}\mathbb{E}_{\Omega}\left[Z_n\right] - \left(\frac{1}{n_C}\right) y_{C1}\mathbb{E}_{\Omega}\left[\left(1 - Z_1\right)\right] + \cdots + y_{Cn}\mathbb{E}_{\Omega}\left[\left(1 - Z_n\right)\right]
\end{aligned}$$

By Lemma 1, $\mathbb{E}_{\Omega}\left[Z_i\right] = \left(\frac{n_T}{n}\right)$ for all $i \in \{1, \ldots, n\}$, which implies that $\mathbb{E}_{\Omega}\left[\left(1 - Z_i\right)\right] = 1 - \left(\frac{n_T}{n}\right) = \left(\frac{n_C}{n}\right)$ for all $i \in \{1, \ldots, n\}$. Hence, we can substitute $\left(\frac{n_T}{n}\right)$ for $\mathbb{E}_{\Omega}\left[Z_i\right]$ and $\left(\frac{n_C}{n}\right)$ for $\mathbb{E}_{\Omega}\left[1 - Z_i\right]$, respectively, which then yields

$$= \left(\frac{1}{n_T}\right)\left(\frac{n_T}{n}\right)\left(y_{T1} + \cdots + y_{Tn}\right) - \left(\frac{1}{n_C}\right)\left(\frac{n_C}{n}\right)\left(y_{C1} + \cdots + y_{Cn}\right)$$

$$= \left(\frac{1}{n}\right)(y_{T1} + \cdots + y_{Tn}) - \left(\frac{1}{n}\right)(y_{C1} + \cdots + y_{Cn})$$
$$= \frac{(y_{T1} + \cdots + y_{Tn})}{n} - \frac{(y_{C1} + \cdots + y_{Cn})}{n}$$
$$= \bar{y_T} - \bar{y_C}$$
$$= \bar{\tau}.$$

□

1.3 Estimation under Simple Random Assignment

Under simple random assignment, let the number of experimental units, n, be a fixed quantity, but let the number of treatment and control units be random variables with support given by $N_1 \in \{1, \ldots, n-1\}$ and $N_0 \in n - N_1$. Note that neither N_1 nor N_0 can take on the value of 0. In this setting, the set of possible assignments is $\Omega = \left\{\boldsymbol{z} : 0 < \sum_{i=1}^{n} z_i < n\right\}$, which contains $2^n - 2$ elements. In the proof to follow, note that whenever taking an expectation conditional on some number of treated units, n_T, the expectation is over $\Omega = \left\{\boldsymbol{z} : \sum_{i=1}^{n} z_i = n_T\right\}$. When not conditioning on a value of n_T, the expectation is over $\Omega = \left\{\boldsymbol{z} : 0 < \sum_{i=1}^{n} z_i < n\right\}$. For simplicity, I do not change the notation for these two sets of assignments under complete and simple random assignment.

In the proof that follows, we will draw upon the Law of Iterated Expectations, which states in general that, for two random variables X and Y, $\mathrm{E}[X] = \mathrm{E}_Y\left[\mathrm{E}_X\left[X \mid Y = y\right]\right]$, where E_X refers to the expectation over X and E_Y refers to the expectation over Y.

Proposition 2. *Under simple, uniform random assignment* $\mathrm{E}_\Omega\left[\hat{\tau}\right] = \bar{\tau}_{SATE}$.

Proof. By the law of iterated expectations, the expected value of the Difference-in-Means estimator, $\widehat{\bar{\tau}}$, can be decomposed as

(6) $$\mathrm{E}_\Omega\left[\widehat{\bar{\tau}}\right] = \mathrm{E}_\Omega\left[\widehat{\bar{\tau}} \mid N_1 = 1\right]\Pr\left(N_1 = 1\right) + \cdots + \mathrm{E}_\Omega\left[\widehat{\bar{\tau}} \mid N_1 = n-1\right]\Pr\left(N_1 = n-1\right).$$

By Proposition 1 above, the expected value of the estimator conditional on any realized number of treated units is equal to $\bar{\tau}$. Hence, it follows that Equation (6) can be rewritten as:

$$\mathrm{E}_\Omega\left[\widehat{\bar{\tau}}\right] = \bar{\tau}\Pr\left(N_1 = 1\right) + \cdots + \bar{\tau}\Pr\left(N_1 = n-1\right),$$

which we can rewrite as

$$\mathrm{E}_\Omega\left[\widehat{\bar{\tau}}\right] = \bar{\tau}\Big[\Pr\left(N_1 = 1\right) + \cdots + \Pr\left(N_1 = n-1\right)\Big].$$

Finally, note that by the second and third axioms of probability, $\Big[\Pr\left(N_1 = 1\right) + \cdots + \Pr\left(N_1 = n-1\right)\Big] = 1$. Hence, it follows that

$$\mathrm{E}_\Omega\left[\widehat{\bar{\tau}}\right] = \bar{\tau}\Big[1\Big]$$
$$\mathrm{E}_\Omega\left[\widehat{\bar{\tau}}\right] = \bar{\tau},$$

which proves the proposition. $\square$

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