Estimation of Average Causal Effects

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Potential Outcomes Framework

Causality with Potential Outcomes

- Definition: Treatment

  Z_i: Indicator of treatment assignment for *unit* i, where $i = 1, \ldots, N$

$$Z_i = \begin{cases} 1 & \text{if unit } i \text{ receives treatment} \\ 0 & \text{otherwise} \end{cases}$$

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- Definition: Treatment

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  $$Z_i = \begin{cases} 1 & \text{if unit } i \text{ receives treatment} \\ 0 & \text{otherwise} \end{cases}$$

- Definition: Potential Outcomes (assuming SUTVA)

  y_{Ti} or y_{Ci}: Fixed value of the outcome for unit i if it were to receive treatment or control

  E.g., y_{Ti}: voter turnout of person i if person i were to receive mail encouraging turnout

  E.g., y_{Ci}: voter turnout of person i if person i were *not* to receive mail encouraging turnout

Defining Causal Effects + Observed Outcomes

- Causal effect (treatment effect) for unit i

  Additive causal effect of the treatment on the outcome for unit i:

$$\tau_i \quad = \quad y_{Ti} - y_{Ci}$$

- Other functions of of individual potential outcomes possible, e.g., $\frac{y_{Ti}}{y_{Ci}}$

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  Additive causal effect of the treatment on the outcome for unit i:

$$\tau_i \quad = \quad y_{Ti} - y_{Ci}$$

- Other functions of of individual potential outcomes possible, e.g., $\frac{y_{Ti}}{y_{Ci}}$
- Fundamental Problem of Causal Inference (Holland 1986):

  We can never observe both y_{Ti} and y_{Ci} for the same i

- We can observe only one of the two potential outcomes:

$$Y_i = Z_i y_{Ti} + (1 - Z_i) y_{Ci}$$

- Therefore, τ_i is unobserved for every unit

The average treatment effect

Average treatment effect: An example

- Definition: Average Treatment Effect (ATE)

$$\tau = \frac{1}{N}\sum_{i=1}^{N}(y_{Ti} - y_{Ci})$$

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- Example: "Village heads" study (Gerber and Green 2012, Chapter 2):

Village	Budget share (%) y_C	y_T	τ
1	10	15	5
2	15	15	0
3	20	30	10
4	20	15	-5
5	10	20	10
6	15	15	0
7	15	30	15
Average	15	20	5

Randomized experiments

Basic Setup for Randomized Experiment

- Units: $i = 1, \dots, N$
- Treatment: $Z_i = 0$ or $Z_i = 1$ is randomly assigned
- Potential outcomes: y_{Ci} and y_{Ti}
- Observed outcome: $Y_i = Z_i y_{Ti} + (1 - Z_i) y_{Ci}$

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- Treatment Assignment Mechanism

  (1) Bernoulli (simple) randomization: Each unit is independently assigned to treatment with probability p

  (2) Complete randomization: Exactly n_T units are treated and $N - n_T = n_C$ units are untreated

  (3) In practice, (1) and (2) are equivalent when we fix n_T by conditioning on its observed value

- Under complete or simple (conditioning on observed n_T) randomization

$$\mathrm{E}\left[Z_i\right] = \frac{n_T}{N}$$

Difference-in-Means estimator

Unbiasedness of Difference-in-Means: Proof

- Difference-in-Means estimator

$$
\begin{aligned}
\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y}) &= n_T^{-1} \boldsymbol{Z}^\top \boldsymbol{Y} - n_C^{-1} (\mathbf{1} - \boldsymbol{Z})^\top \boldsymbol{Y} \\
&= \frac{1}{n_T} \sum_{i=1}^{N} Z_i Y_i - \frac{1}{n_C} \sum_{i=1}^{N} (1 - Z_i) Y_i
\end{aligned}
$$

Unbiasedness of Difference-in-Means: Proof

- Difference-in-Means estimator
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- Unbiased for the ATE under complete randomization

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- Difference-in-Means estimator

$$\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right) = n_T^{-1} \boldsymbol{Z}^\top \boldsymbol{Y} - n_C^{-1} \left(\boldsymbol{1} - \boldsymbol{Z}\right)^\top \boldsymbol{Y} = \frac{1}{n_T} \sum_{i=1}^{N} Z_i Y_i - \frac{1}{n_C} \sum_{i=1}^{N} (1 - Z_i) Y_i$$

- Unbiased for the ATE under complete randomization

$$\begin{aligned}
\mathrm{E}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right] &= \mathrm{E}\left[\frac{1}{n_T} \sum_{i=1}^{N} Z_i Y_i - \frac{1}{n_C} \sum_{i=1}^{N} (1 - Z_i) Y_i\right] \\
&= \frac{1}{n_T} \sum_{i=1}^{N} \mathrm{E}\left[Z_i Y_i\right] - \frac{1}{n_C} \sum_{i=1}^{N} \mathrm{E}\left[(1 - Z_i) Y_i\right] \quad (\because \text{Linearity of } \mathrm{E}) \\
&= \frac{1}{n_T} \sum_{i=1}^{N} \mathrm{E}\left[Z_i y_{Ti}\right] - \frac{1}{n_C} \sum_{i=1}^{N} \mathrm{E}\left[(1 - Z_i) y_{Ci}\right] \quad (\because \text{Definition of POs}) \\
&= \frac{1}{n_T} \sum_{i=1}^{N} y_{Ti}\, \mathrm{E}\left[Z_i\right] - \frac{1}{n_C} \sum_{i=1}^{N} y_{Ci}\, \mathrm{E}\left[1 - Z_i\right] \quad (\because \text{POs are fixed}) \\
&= \frac{1}{n_T} \sum_{i=1}^{N} y_{Ti} \left(\frac{n_T}{N}\right) - \frac{1}{n_C} \sum_{i=1}^{N} y_{Ci} \left(\frac{n_C}{N}\right) \quad (\because \text{Complete randomization}) \\
&= \frac{1}{N} \sum_{i=1}^{N} y_{Ti} - \frac{1}{N} \sum_{i=1}^{N} y_{Ci} = \frac{1}{N} \sum_{i=1}^{N} \left(y_{Ti} - y_{Ci}\right) = \tau
\end{aligned}$$

Unbiasedness of Difference-in-Means: Example

	Budget share (%)		
Village	y_C	y_T	τ
1	10	15	5
2	15	15	0
3	20	30	10
4	20	15	-5
5	10	20	10
6	15	15	0
7	15	30	15
Average	15	20	5

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7	15	30	15
Average	15	20	5

2 treated and 5 control villages

$\implies \binom{7}{2} = 21$ assignments

$$\Omega = \left\{ \begin{bmatrix}1\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\\0\\0\\0\end{bmatrix}, \cdots, \begin{bmatrix}0\\0\\0\\0\\0\\1\\1\end{bmatrix} \right\}$$

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Average	15	20	5

2 treated and 5 control villages

$\Longrightarrow \binom{7}{2} = 21$ assignments

$$\Omega = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Therefore, there are $\binom{7}{2} = 21$ possible realizations of data:

$\boldsymbol{z}_1$	$\boldsymbol{y_C}$	$\boldsymbol{y_T}$	$\boldsymbol{y}_1$
1	?	15	15
1	?	15	15
0	20	?	20
0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

$\boldsymbol{z}_2$	$\boldsymbol{y_C}$	$\boldsymbol{y_T}$	$\boldsymbol{y}_2$
1	?	15	15
0	15	?	15
1	?	30	30
0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

. . .

$\boldsymbol{z}_{21}$	$\boldsymbol{y_C}$	$\boldsymbol{y_T}$	$\boldsymbol{y}_{21}$
0	10	?	10
0	15	?	15
0	20	?	20
0	20	?	20
0	10	?	10
1	?	15	15
1	?	30	30

Unbiasedness of Difference-in-Means: Example

z_1	y_C	y_T	y_1
1	?	15	15
1	?	15	15
0	20	?	20
0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

z_2	y_C	y_T	y_2
1	?	15	15
0	15	?	15
1	?	30	30
0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

$\cdots$

z_{21}	y_C	y_T	y_{21}
0	10	?	10
0	15	?	15
0	20	?	20
0	20	?	20
0	10	?	10
1	?	15	15
1	?	30	30

- Random vectors $\boldsymbol{Z}$ and $\boldsymbol{Y}$ can take on any $(\boldsymbol{z}_1, \boldsymbol{y}_1), \cdots, (\boldsymbol{z}_{21}, \boldsymbol{y}_{21})$

Unbiasedness of Difference-in-Means: Example

z_1	y_C	y_T	y_1
1	?	15	15
1	?	15	15
0	20	?	20
0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

z_2	y_C	y_T	y_2
1	?	15	15
0	15	?	15
1	?	30	30
0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

. . .

z_{21}	y_C	y_T	y_{21}
0	10	?	10
0	15	?	15
0	20	?	20
0	20	?	20
0	10	?	10
1	?	15	15
1	?	30	30

- Random vectors $\boldsymbol{Z}$ and $\boldsymbol{Y}$ can take on any $(\boldsymbol{z}_1, \boldsymbol{y}_1), \cdots, (\boldsymbol{z}_{21}, \boldsymbol{y}_{21})$

- Applying Diff-in-Means estimator to all 21 possible realizations of data $\implies$ 21 possible outputs of estimator:

$$\hat{\tau}(\boldsymbol{z}_1, \boldsymbol{y}_1) = -1,\ \hat{\tau}(\boldsymbol{z}_2, \boldsymbol{y}_2) = 7.5,\ \cdots,\ \hat{\tau}(\boldsymbol{z}_{21}, \boldsymbol{y}_{21}) = 7.5$$

Unbiasedness of Difference-in-Means: Example

$\boldsymbol{z}_1$	$\boldsymbol{y}_C$	$\boldsymbol{y}_T$	$\boldsymbol{y}_1$
1	?	15	15
1	?	15	15
0	20	?	20
0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

$\boldsymbol{z}_2$	$\boldsymbol{y}_C$	$\boldsymbol{y}_T$	$\boldsymbol{y}_2$
1	?	15	15
0	15	?	15
1	?	30	30
0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

$\ldots$

$\boldsymbol{z}_{21}$	$\boldsymbol{y}_C$	$\boldsymbol{y}_T$	$\boldsymbol{y}_{21}$
0	10	?	10
0	15	?	15
0	20	?	20
0	20	?	20
0	10	?	10
1	?	15	15
1	?	30	30

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- Expected value of Diff-in-Means estimator:

$$\mathrm{E}\left[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})\right] = \hat{\tau}(\boldsymbol{z}_1, \boldsymbol{y}_1)\Pr(\boldsymbol{Z} = \boldsymbol{z}_1) + \ldots + \hat{\tau}(\boldsymbol{z}_{21}, \boldsymbol{y}_{21})\Pr(\boldsymbol{Z} = \boldsymbol{z}_{21})$$

Unbiasedness of Difference-in-Means: Example

$\boldsymbol{z}_1$	$\boldsymbol{y}_C$	$\boldsymbol{y}_T$	$\boldsymbol{y}_1$
1	?	15	15
1	?	15	15
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0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

$\boldsymbol{z}_2$	$\boldsymbol{y}_C$	$\boldsymbol{y}_T$	$\boldsymbol{y}_2$
1	?	15	15
0	15	?	15
1	?	30	30
0	20	?	20
0	10	?	10
0	15	?	15
0	15	?	15

$\ldots$

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0	15	?	15
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0	10	?	10
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- Expected value of Diff-in-Means estimator:

$$\mathrm{E}[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})] = \hat{\tau}(\boldsymbol{z}_1, \boldsymbol{y}_1)\Pr(\boldsymbol{Z} = \boldsymbol{z}_1) + \ldots + \hat{\tau}(\boldsymbol{z}_{21}, \boldsymbol{y}_{21})\Pr(\boldsymbol{Z} = \boldsymbol{z}_{21})$$

- So, in "village heads" example

$$\mathrm{E}[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})] = (-1)(1/21) + (7.5)(1/21) + \ldots + (7.5)(1/21) = 5$$

Unbiasedness of Difference-in-Means: Example

- Diff-in-Means estimator under complete random assignment

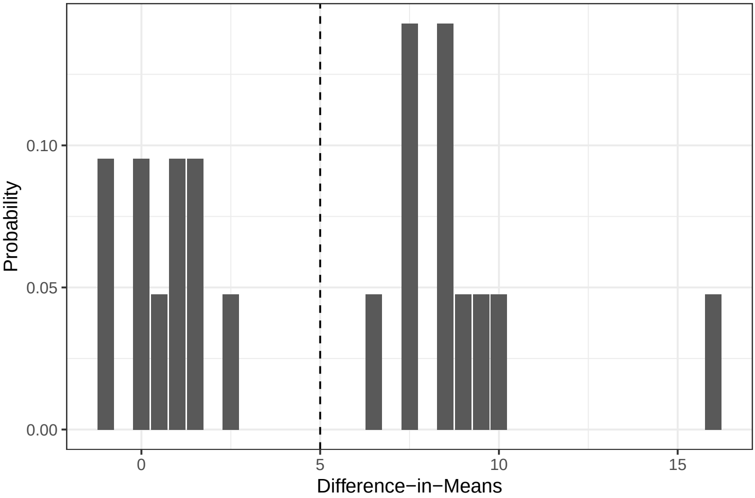

Figure 1: Difference-in-Means estimator in "Village heads" example

Next steps

- What happens when the size of our experiment grows large?
- Consistency of Difference-in-Means estimator for ATE
- Asymptotic validity of hypothesis tests about ATE