Causal Inference Assignment 1

Due noon Saturday 6/21

ICPSR Session 1 (June 16, 2025)

1. Refer to the the `acorn` data set that accompanies this assignment. Using the mean of turnout proportions in treatment group precincts, $n_1^{-1}\mathbf{Z}'\mathbf{y}$, as test statistic, simulate its rerandomization distribution under the null hypothesis of strictly no effect, reporting:
   (a) your simulation p-value;
   (b) your simulation approximation of $\mathrm{E}[n_1^{-1}\mathbf{Z}'\mathbf{y}]$, the null expected value of the test statistic;
   (c) your simulation approximation of $\mathrm{Var}[n_1^{-1}\mathbf{Z}'\mathbf{y}]$, this test statistic's variance under the null.

   Note that n_1 is the number of treated units, n is the total number of units, $\mathbf{Z}$ is the random assignment variable and $\mathbf{y}$ is the observed outcome variable
2. Calculate $\mathrm{E}[n_1^{-1}\mathbf{Z}'\mathbf{y}]$, the expected value of the sample mean under the strict null hypothesis of no effect, from first principles — i.e., without simulations — using data in $\mathbf{y}$.
3. In this setup, $\mathrm{Var}\left[\bar{y}_1\right] = \frac{1}{n_1}\frac{n_0}{n}\frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{n-1}$, where $\bar{y}_1 = n_1^{-1}\mathbf{Z}'\mathbf{y}$ is the mean of ys among the group assigned to treatment ($\{i : Z_i = 1\}$) while $\bar{y} = n^{-1}\mathbf{1}'\mathbf{y}$ is the mean of y over the full study population. If you've seen formulas for the sampling variance of the mean in earlier stats courses, you probably saw

$$\widehat{\mathrm{Var}}\left[\bar{y}_1\right] = \frac{s_y^2}{n_1} = \frac{1}{n_1}\frac{\sum_{i=1}^{n_1}(y_i-\bar{y}_1)^2}{n_1-1}, \text{ and/or } \mathrm{Var}\left[\bar{y}_1\right] = \frac{\sigma_y^2}{n_1} = \frac{1}{n_1}\frac{\sum_{i=1}^{N}(y_i-\bar{y})^2}{N}.$$

   Which gives a larger value, our new $\mathrm{Var}\left[\bar{y}_1\right]$-formula or the $\mathrm{Var}\left[\bar{y}_1\right]$ formula immediately above? Explain why it makes sense that the formula we've given would differ in the direction it does from this other formula.
4. Calculate $\mathrm{Var}\left[n_1^{-1}\mathbf{Z}'\mathbf{y}\right]$ using the appropriate formula. Determine the error of the simulation-based approximation to this quantity that you reported in question 1, expressing it as a percentage of $\mathrm{Var}\left[n_1^{-1}\mathbf{Z}'\mathbf{y}\right]$.
5. Determine the Normal theory approximation to $\Pr(n_1^{-1}\mathbf{Z}'\mathbf{y} \geq n_1^{-1}\mathbf{z}'\mathbf{y})$. (Hints: Use the variance and expected values calculated in 4 and 2 to transform your observed treatment group mean into a corresponding "z-score." To to determine Normal quantiles corresponding to z-scores in R, use `pnorm()`; type `?pnorm` for help. And $\mathbf{z}'$ refers to the observed Acorn data treatment assignment.)
6. A researcher plans to ask six subjects to donate time to an adult literacy program. Each subject will be asked to donate either 30 ($Z = 0$) or 60 ($Z = 1$) minutes. The researcher is considering three methods for randomizing the treatment. Method I is to make independent decisions for each subject, tossing a coin each time. Method C is to write "30" and "60" on three playing cards each, and then shuffle the six cards. Method P tosses one coin for each of the 3 pairs $(1, 2)$, $(3, 4)$, $(5, 6)$, asking for 30 (60) minutes from exactly one member of each pair.
   a Discuss strengths & weaknesses of each method.
   b How would your answers to (a) change if $n : 6 \mapsto 600$?
   c Determine $\mathrm{E}[Z_1]$ under each method.
   d Determine $\mathrm{E}\big[Z_1 + Z_2 + \cdots + Z_6\big]$ under each method.
   e Calculate $\mathrm{E}[\mathbf{Z}'\mathbf{1}]$ under each of the three methods.
   f For which of the methods does $\mathrm{E}\big[\big(\mathbf{Z}'\mathbf{1} - \mathrm{E}[\mathbf{Z}'\mathbf{1}]\big)^2\big] = 0$?[1]
   h For two of the three methods, the linearity property of expected value[2] entails that $\mathrm{E}[\frac{\mathbf{Z}'\mathbf{x}}{\mathbf{Z}'\mathbf{1}}] = (x_1 + x_2 + \ldots + x_6)/6$. Which two are these, and why doesn't the argument work for the third?

[1] I.e., for which does $\mathrm{Var}[\mathbf{Z}'\mathbf{1}] = 0$? (In general, $\mathrm{Var}[V] = \mathrm{E}\big[V - \mathrm{E}[V]\big]^2$.)

[2] For constant α and β, $\mathrm{E}(\alpha + \beta Y) = \alpha + \beta\mathrm{E}(X)$. Also $\mathrm{E}(\alpha X + \beta Y) = \alpha\mathrm{E}(Y) + \beta\mathrm{E}(X)$.