Uncertainty, consistency and hypothesis testing

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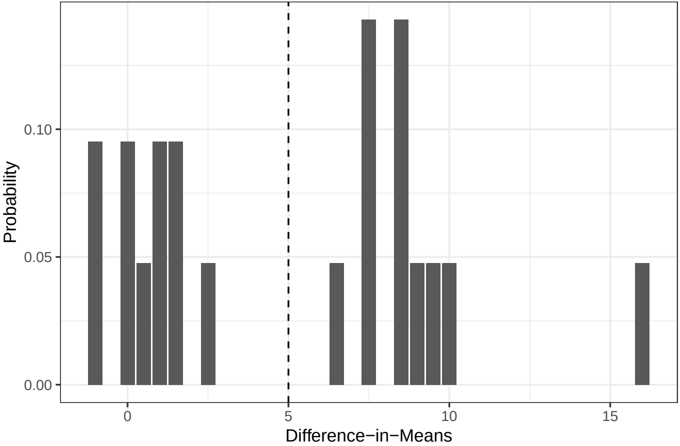
Variance of Difference-in-Means

Variance of Difference-in-Means estimator

- Yesterday we showed that Difference-in-Means estimator in "village heads" example is



- What is the variance of this estimator?
- Why do we care about variance of an estimator?

Variance of Difference-in-Means estimator

- Variance is average squared distance of estimator from its expected value:

$$\underbrace{\mathrm{E}\left[\underbrace{\left(\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)-\underbrace{\mathrm{E}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]}_{\text{Expected value}}\right)^2}_{\text{Squared distance from expected value}}\right]}_{\text{Average (or expected) squared distance}}$$

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$$\mathrm{E}\left[\left(\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)-\tau\right)^2\right]$$

- In "village heads" example with 21 possible assignments

$$\left(\hat{\tau}\left(\boldsymbol{z}_1,\boldsymbol{y}_1\right)-\tau\right)^2\Pr\left(\boldsymbol{Z}=\boldsymbol{z}_1\right)+\ldots+\left(\hat{\tau}\left(\boldsymbol{z}_{21},\boldsymbol{y}_{21}\right)-\tau\right)^2\Pr\left(\boldsymbol{Z}=\boldsymbol{z}_{21}\right)$$

Variance is ≈ 21.19

Variance of Difference-in-Means estimator

- Neyman (1923) derived expression for variance of Diff-in-Means under complete random assignment

$$\text{Var}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right] = \frac{1}{N-1}\left(\frac{n_T \sigma^2_{y_C}}{n_C} + \frac{n_C \sigma^2_{y_T}}{n_T} + 2\sigma_{y_C, y_T}\right)$$

where

$\sigma^2_{y_C} = \frac{1}{N}\sum_{i=1}^{N}\left(y_{Ci} - \frac{1}{N}\sum_{i=1}^{N} y_{Ci}\right)^2$ is var of control POs

$\sigma^2_{y_T} = \frac{1}{N}\sum_{i=1}^{N}\left(y_{Ti} - \frac{1}{N}\sum_{i=1}^{N} y_{Ti}\right)^2$ is var of treated POs

$\sigma_{y_C, y_T} = \frac{1}{N}\sum_{i=1}^{N}\left(y_{Ci} - \frac{1}{N}\sum_{i=1}^{N} y_{Ci}\right)\left(y_{Ti} - \frac{1}{N}\sum_{i=1}^{N} y_{Ti}\right)$ is cov of POs

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⋆ Note that $\sigma^2_{y_C}$, $\sigma^2_{y_T}$ and σ_{y_C,y_T} depend on unknown potential outcomes

Variance of Difference-in-Means estimator

- Sometimes you might see equivalent expression (Imbens and Rubin 2015)

$$\frac{S_{y_C}}{n_C} + \frac{S_{y_T}}{n_T} - \frac{S_{\tau}}{N},$$

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- Example: "Village heads" study (Gerber and Green 2012, Chapter 2):

	Budget share (%)		
Village	y_C	y_T	τ
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3	20	30	10
4	20	15	-5
5	10	20	10
6	15	15	0
7	15	30	15
Average	15	20	5

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- With access to true POs, we can directly calculate $\mathrm{Var}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right]$:

  $\sigma^2_{y_C} \approx 14.29$, $\sigma^2_{y_T} \approx 42.86$, $\sigma_{y_C, y_T} \approx 7.14$

- So, $\mathrm{Var}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right] = \frac{1}{N-1}\left(\frac{n_T \sigma^2_{y_C}}{n_C} + \frac{n_C \sigma^2_{y_T}}{n_T} + 2\sigma_{y_C, y_T}\right) \approx 21.19$

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- In practice, $\sigma^2_{y_C}$, $\sigma^2_{y_T}$ and σ_{y_C, y_T} unknown, so we estimate $\mathrm{Var}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right]$

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We showed that the Diference-in-Means estimator's variance is

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- We use **conservative** "plug-in" estimator (Neyman 1923)
- **Conservative** means that

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- Potential outcomes will be perfectly positively correlated if and only if τ_i is the same for all $i=1,\ldots,N$ units

Conservative variance estimator

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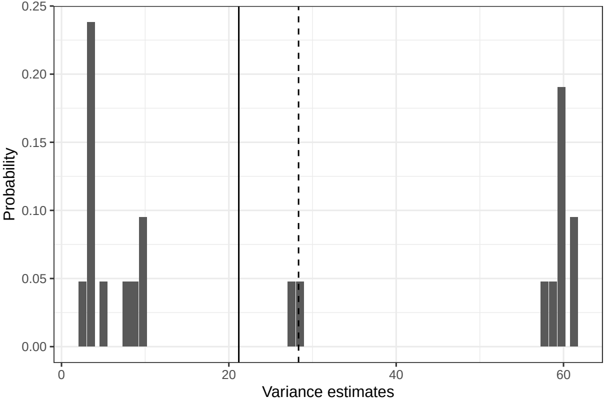
- Now all quantities can be estimated!
- So, just "plug-in" estimators $\hat{\sigma}^2_{y_C}$ and $\hat{\sigma}^2_{y_T}$ for $\sigma^2_{y_C}$ and $\sigma^2_{y_T}$

$$\widehat{\mathrm{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]=\frac{N}{N-1}\left(\frac{\hat{\sigma}^2_{y_C}}{n_C}+\frac{\hat{\sigma}^2_{y_T}}{n_T}\right)\tag{1}$$

- For exact expressions of $\hat{\sigma}^2_{y_C}$ and $\hat{\sigma}^2_{y_T}$, see Variance estimators

Variance estimation

- Here is the conservative variance estimator in "village heads" example:



Solid line is true variance of Difference-in-Means estimator, $\mathrm{Var}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right]$

Dashed line is expected value of conservative estimator, $\mathrm{E}\left[\widehat{\mathrm{Var}}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right]\right]$

Asymptotic properties

Asymptotics

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  N never goes to ∞ in an actual experiment

  But properties as $N \to \infty$ **approximate** properties with fixed, but large N

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- In words:

Pick any ϵ you want, no matter how small. There will be some value N^* such that, if size of experiment is greater than N^*, the probability of an estimate within a distance of ϵ from the truth will be arbitrarily close to 1.

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- Intuitively, with large experiment, our estimate will be close to true ATE!

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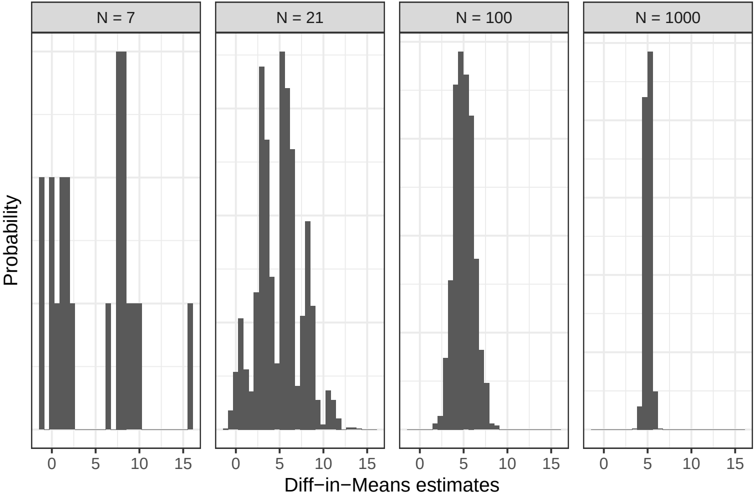

Hypothesis testing

Hypothesis tests of the weak null

- The finite population CLT tells us that

$$\frac{\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right) - \mathrm{E}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right]}{\sqrt{\mathrm{Var}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right]}} \xrightarrow{d} \mathcal{N}\left(0, 1\right)$$

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- This justifies use of standard Normal distribution for hypothesis tests

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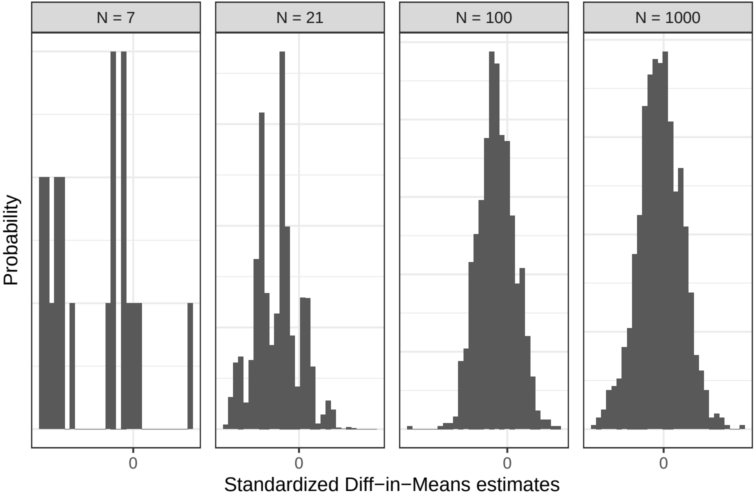

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$$H_0 : \tau = \tau_0 \text{ versus either}$$
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- Calculate upper(u), lower(l) or two-sided(t) p-value as

$$p_u = 1 - \Phi\left(\frac{\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y}) - \tau_0}{\sqrt{\widehat{\mathrm{Var}}\left[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})\right]}}\right)$$

$$p_l = \Phi\left(\frac{\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y}) - \tau_0}{\sqrt{\widehat{\mathrm{Var}}\left[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})\right]}}\right)$$

$$p_t = 2\left(1 - \Phi\left(\frac{\left|\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y}) - \tau_0\right|}{\sqrt{\widehat{\mathrm{Var}}\left[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})\right]}}\right)\right)$$

Hypothesis tests of the weak null

- To test null hypothesis relative to alternative

$$H_0 : \tau = \tau_0 \text{ versus either}$$
$$H_A : \tau > \tau_0,\ H_A : \tau < \tau_0 \text{ or } H_A : |\tau| > |\tau_0|$$

- Calculate upper(u), lower(l) or two-sided(t) p-value as

$$p_u = 1 - \Phi\left(\frac{\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y}) - \tau_0}{\sqrt{\widehat{\mathrm{Var}}\left[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})\right]}}\right)$$
$$p_l = \Phi\left(\frac{\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y}) - \tau_0}{\sqrt{\widehat{\mathrm{Var}}\left[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})\right]}}\right)$$
$$p_t = 2\left(1 - \Phi\left(\frac{|\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y}) - \tau_0|}{\sqrt{\widehat{\mathrm{Var}}\left[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})\right]}}\right)\right)$$

- If p-value is less than size α-level of test, reject. Otherwise, don't
- Note that, since we don't know $\mathrm{Var}\left[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})\right]$,
  we have used its conservative estimator instead, $\widehat{\mathrm{Var}}\left[\hat{\tau}(\boldsymbol{Z}, \boldsymbol{Y})\right]$

Hypothesis tests of the weak null

Hypothesis tests susceptible to two errors:

- Type I error: Rejecting null hypothesis when it is true
- Type II error: *Failing* to reject null hypothesis when it is false

Hypothesis tests of the weak null

Hypothesis tests susceptible to two errors:

- Type I error: Rejecting null hypothesis when it is true
- Type II error: *Failing* to reject null hypothesis when it is false

A good test controls these errors:

1. Type I error probability is less than or equal to size (α-level) of test
2. Power (1 - type II error probability) is at least as great as α-level
3. Power tends to 1 as $N \to \infty$

Hypothesis tests of the weak null

- We can prove that tests of weak null satisfy (1) – (3) as $N \to \infty$
- Thus, when experiments are large, we can often safely use such tests
- But (1) – (3) may not always be satisfied when experiments are small, have skewed outcome distributions or extreme outliers

Confidence intervals

- Equivalence between hypothesis testing and confidence intervals
- Confidence interval is set of null hypotheses we fail to reject

Consider two-sided confidence interval, $\mathcal{C}_t$:

$$
\begin{aligned}
\mathcal{C}_t &= \left\{\tau_0 : \left|\frac{\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)-\tau_0}{\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]}}\right| \leq z_{1-\alpha/2}\right\} \\
&= \left\{\tau_0 : -z_{1-\alpha/2} \leq \frac{\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)-\tau_0}{\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]}} \leq z_{1-\alpha/2}\right\} \\
&= \left\{\tau_0 : -z_{1-\alpha/2}\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]} \leq \hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)-\tau_0 \leq z_{1-\alpha/2}\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]}\right\} \\
&= \left\{\tau_0 : -\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)-z_{1-\alpha/2}\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]} \leq -\tau_0 \leq -\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)+z_{1-\alpha/2}\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]}\right\} \\
&= \left\{\tau_0 : \hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)+z_{1-\alpha/2}\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]} \geq \tau_0 \geq \hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)-z_{1-\alpha/2}\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]}\right\} \\
&= \left\{\tau_0 : \hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)-z_{1-\alpha/2}\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]} \leq \tau_0 \leq \hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)+z_{1-\alpha/2}\sqrt{\widehat{\operatorname{Var}}\left[\hat{\tau}\left(\boldsymbol{Z},\boldsymbol{Y}\right)\right]}\right\}
\end{aligned}
$$

Appendix: Estimator of Difference-in-Means estimator's variance

Neyman's conservative estimator of the Difference-in-Means estimator's variance is

$$\widehat{\mathrm{Var}}\left[\hat{\tau}\left(\boldsymbol{Z}, \boldsymbol{Y}\right)\right] = \frac{N}{N-1}\left(\frac{\hat{\sigma}^2_{y_C}}{n_C} + \frac{\hat{\sigma}^2_{y_T}}{n_T}\right),$$

where

$$\hat{\sigma}^2_{y_C} = \left(\frac{N-1}{n\left(n_C - 1\right)}\right) \sum_{i:Z_i=0}^{N} \left(y_{Ci} - \hat{\mu}_{y_C}\right)^2$$

$$\hat{\sigma}^2_{y_T} = \left(\frac{N-1}{n\left(n_T - 1\right)}\right) \sum_{i:Z_i=1}^{n} \left(y_{Ti} - \hat{\mu}_{y_T}\right)^2$$

$$\hat{\mu}_{y_C} = \left(\frac{1}{n_C}\right) \sum_{i=1}^{N} \left(1 - Z_i\right) y_{Ci}$$

$$\hat{\mu}_{y_T} = \left(\frac{1}{n_T}\right) Z_i y_{Ti}$$