Gaussians Are Preserved Under Convolution

Leo Dorst

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Abstract

If you convolve two Gaussians, the result is a Gaussian! We determine its width.

1 1D Unit-Scale Example

To show the essentials of the computation, we do this in 1-D, for a Gaussian with unit 'width' (standard deviation, if seen as a distribution). Preliminaries:

- A Gaussian is defined as:

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}},$$

  We are interested in $\sigma = 1$, for now; denote that by $G_1(x)$.

- Convolution in 1D is defined as:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u)\, g(u)\, \mathrm{d}u$$

- Since we will need to do integrals, let us recall the integral of a Gaussian:

$$\int_{-\infty}^{\infty} G_\sigma(u)\, \mathrm{d}u = 1.$$

  This is consistent with the use of this Gaussian to describe a normal distribution (with variance 1); or with its use in heat diffusion (heat is preserved); or scale space in computer vision.

So, here we go. Identify why each of the steps is permitted!

$$\begin{aligned}
(G_1 * G_1)(x) &= \int_{-\infty}^{\infty} G_1(x-u)\, G_1(u)\, \mathrm{d}u && (1)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-u)^2}\, e^{-\frac{1}{2}u^2}\, \mathrm{d}u && (2)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left((x-u)^2+u^2\right)}\, \mathrm{d}u && (3)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2-2xu+2u^2\right)}\, \mathrm{d}u && (4)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left((\sqrt{2}u-\frac{x}{\sqrt{2}})^2+(\frac{x}{\sqrt{2}})^2\right)}\,\mathrm{d}u && (5)\\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\sqrt{2}u-\frac{x}{\sqrt{2}})^2}\,\mathrm{d}u\right)\, e^{-\frac{1}{2}(\frac{x}{\sqrt{2}})^2} && (6)\\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2}\,(\mathrm{d}v/\sqrt{2})\right)\, e^{-\frac{1}{2}(\frac{x}{\sqrt{2}})^2} && (7)\\
&= \frac{1}{2\pi}\frac{\sqrt{2\pi}}{\sqrt{2}}\, e^{-\frac{1}{2}(\frac{x}{\sqrt{2}})^2} && (8)\\
&= \frac{1}{\sqrt{4\pi}}\, e^{-\frac{1}{2}(\frac{x}{\sqrt{2}})^2} && (9)\\
&= G_2(x). && (10)
\end{aligned}
$$

So a Gaussian remains a Gaussian, and the standard deviation becomes $\sqrt{2}$.

1. Now do a similar computation for $G_{\sigma_2} * G_{\sigma_1}$: show that the result is again a Gaussian G_σ, and that $\sigma^2 = \sigma_1^2 + \sigma_2^2$. So:

$$G_{\sigma_2} * G_{\sigma_1} = G_{\sqrt{\sigma_1^2+\sigma_2^2}}.$$

2. Then do the d-dimensional case - look up how a Gaussian is defined in that case.

3. In statistics, a normal distribution would also have a mean. Find out what happens when you convolve two of those more general Gaussians.

Of all these integrals the basic tricks are the same: collect your functions in the exponent, try to split off a known integral (in this case, forcing a square), realize what is the constant here (in our case, x), change variables to enable recognition of the integral (in our case $u \to v$, with concurrent change of du and the integration boundaries), regroup, and recognize the result. It is all a matter of pattern matching. You need to know the patterns, and then a few matching tricks.