Boolean Algebra

Adapted from slides by Jared Moore

Boolean Algebra

Equations that we have previously discussed are not necessarily the most efficient!

$$Y = AB + AC$$

Key: Not always a straightforward process to simplify.

Axioms

- Rules that are assumed to be correct.
- Unprovable as they are definitions.
- Use them to prove all theorems of Boolean Algebra

Duality: If symbols 0 and 1 and the operators AND and OR are interchanged, the statement will still be correct.

	Axiom		Dual	Name
A1	$B = 0$ if $B \neq 1$	A1′	$B = 1$ if $B \neq 0$	Binary field
A2	$\overline{0} = 1$	A2′	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	A3′	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	A4′	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	A5′	$1 + 0 = 0 + 1 = 1$	AND/OR

Working in binary.

NOT Behavior

	Axiom		Dual	Name
A1	$B = 0$ if $B \neq 1$	A1′	$B = 1$ if $B \neq 0$	Binary field
A2	$\overline{0} = 1$	A2′	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	A3′	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	A4′	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	A5′	$1 + 0 = 0 + 1 = 1$	AND/OR

AND Behavior

OR Behavior

Theorems

Use to simplify equations and circuits

Key: Remove Gates!

	Theorem		Dual	Name
T1	$B \bullet 1 = B$	T1′	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	T2′	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	T3′	$B + B = B$	Idempotency
T4		$\overline{\overline{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	T5′	$B + \overline{B} = 1$	Complements

(a)

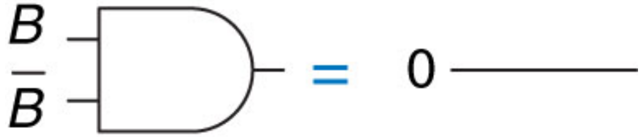
(a)

(a)

(b)

(b)

(b)



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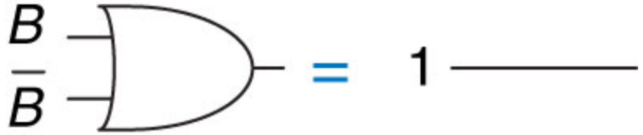

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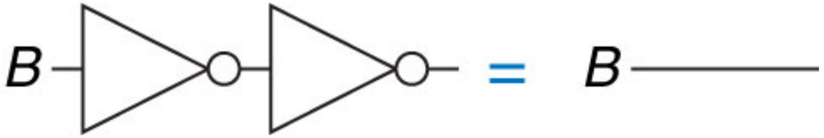


	Theorem		Dual	Name
T6	$B \bullet C = C \bullet B$	T6′	$B + C = C + B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	T7′	$(B + C) + D = B + (C + D)$	Associativity
T8	$(B \bullet C) + (B \bullet D) = B \bullet (C + D)$	T8′	$(B + C) \bullet (B + D) = B + (C \bullet D)$	Distributivity
T9	$B \bullet (B + C) = B$	T9′	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	T10′	$(B + C) \bullet (B + \overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D)$ $= B \bullet C + \overline{B} \bullet D$	T11′	$(B + C) \bullet (\overline{B} + D) \bullet (C + D)$ $= (B + C) \bullet (\overline{B} + D)$	Consensus
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \ldots}$ $= (\overline{B_0} + \overline{B_1} + \overline{B_2} \ldots)$	T12′	$\overline{B_0 + B_1 + B_2 \ldots}$ $= (\overline{B_0} \bullet \overline{B_1} \bullet \overline{B_2} \ldots)$	De Morgan's Theorem

Some are not similar to algebra

For example, in standard algebra, addition does not distribute over multiplication.

T8	$(B \bullet C) + (B \bullet D) = B \bullet (C + D)$	T8′	$(B + C) \bullet (B + D) = B + (C \bullet D)$	Distributivity

Eliminate Redundant Variables

T9 – T11 allow us to eliminate redundant variables.

(We'll give this a try later in examples.)

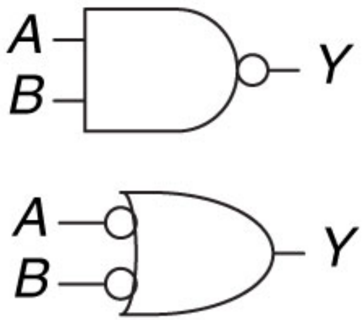
T9	$B \bullet (B + C) = B$	T9′	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	T10′	$(B + C) \bullet (B + \overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D)$ $= B \bullet C + \overline{B} \bullet D$	T11′	$(B + C) \bullet (\overline{B} + D) \bullet (C + D)$ $= (B + C) \bullet (\overline{B} + D)$	Consensus

De Morgan's Theorem

The complement of the product of all the terms is equal to the sum of the complement of each term.

Powerful tool for circuit design

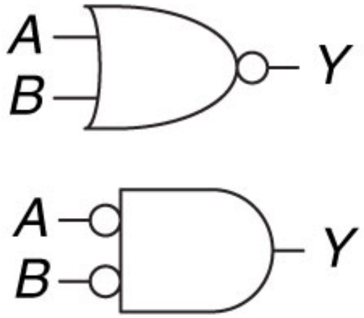
NAND



$$Y = \overline{AB} = \overline{A} + \overline{B}$$

A	B	Y
0	0	1
0	1	1
1	0	1
1	1	0

NOR



$$Y = \overline{A + B} = \overline{A}\,\overline{B}$$

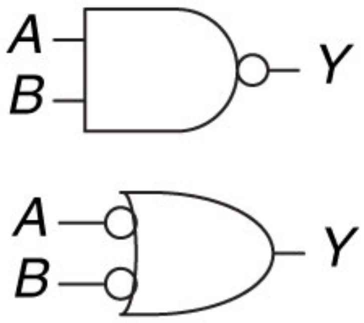
A	B	Y
0	0	1
0	1	0
1	0	0
1	1	0

De Morgan’s Theorem

This is why you *need to be careful with your NOTs*

Note that (AB)′ is *not* the same as A′B′

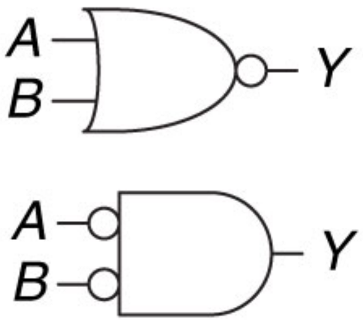
NAND



$$Y = \overline{AB} = \overline{A} + \overline{B}$$

A	B	Y
0	0	1
0	1	1
1	0	1
1	1	0

NOR



$$Y = \overline{A + B} = \overline{A}\,\overline{B}$$

A	B	Y
0	0	1
0	1	0
1	0	0
1	1	0

Putting it into practice

Simplify Y = (A′ B′) + (A′B)

	Theorem		Dual	Name
T6	$B \bullet C = C \bullet B$	T6′	$B + C = C + B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	T7′	$(B + C) + D = B + (C + D)$	Associativity
T8	$(B \bullet C) + (B \bullet D) = B \bullet (C + D)$	T8′	$(B + C) \bullet (B + D) = B + (C \bullet D)$	Distributivity
T9	$B \bullet (B + C) = B$	T9′	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	T10′	$(B + C) \bullet (B + \overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D)$ $= B \bullet C + \overline{B} \bullet D$	T11′	$(B + C) \bullet (\overline{B} + D) \bullet (C + D)$ $= (B + C) \bullet (\overline{B} + D)$	Consensus
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \ldots}$ $= (\overline{B_0} + \overline{B_1} + \overline{B_2} \ldots)$	T12′	$\overline{B_0 + B_1 + B_2 \ldots}$ $= (\overline{B_0} \bullet \overline{B_1} \bullet \overline{B_2} \ldots)$	De Morgan's Theorem

Simplification

From Sum-of-Products form, typically look for PA + P'A = A

Implicant: Product of one or more literals

Minimized: uses the fewest possible implicants

(If several with same number of implicants, minimal one has the fewest literals.)

Simplification

Key: Sometimes we might expand an implicant to minimize an equation.

Simplify: $Y = A'B'C' + AB'C' + AB'C$

Step	Equation	Justification
	$\overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$	
1	$\overline{B}\,\overline{C}(\overline{A} + A) + A\overline{B}C$	T8: Distributivity
2	$\overline{B}\,\overline{C}(1) + A\overline{B}C$	T5: Complements
3	$\overline{B}\,\overline{C} + A\overline{B}C$	T1: Identity

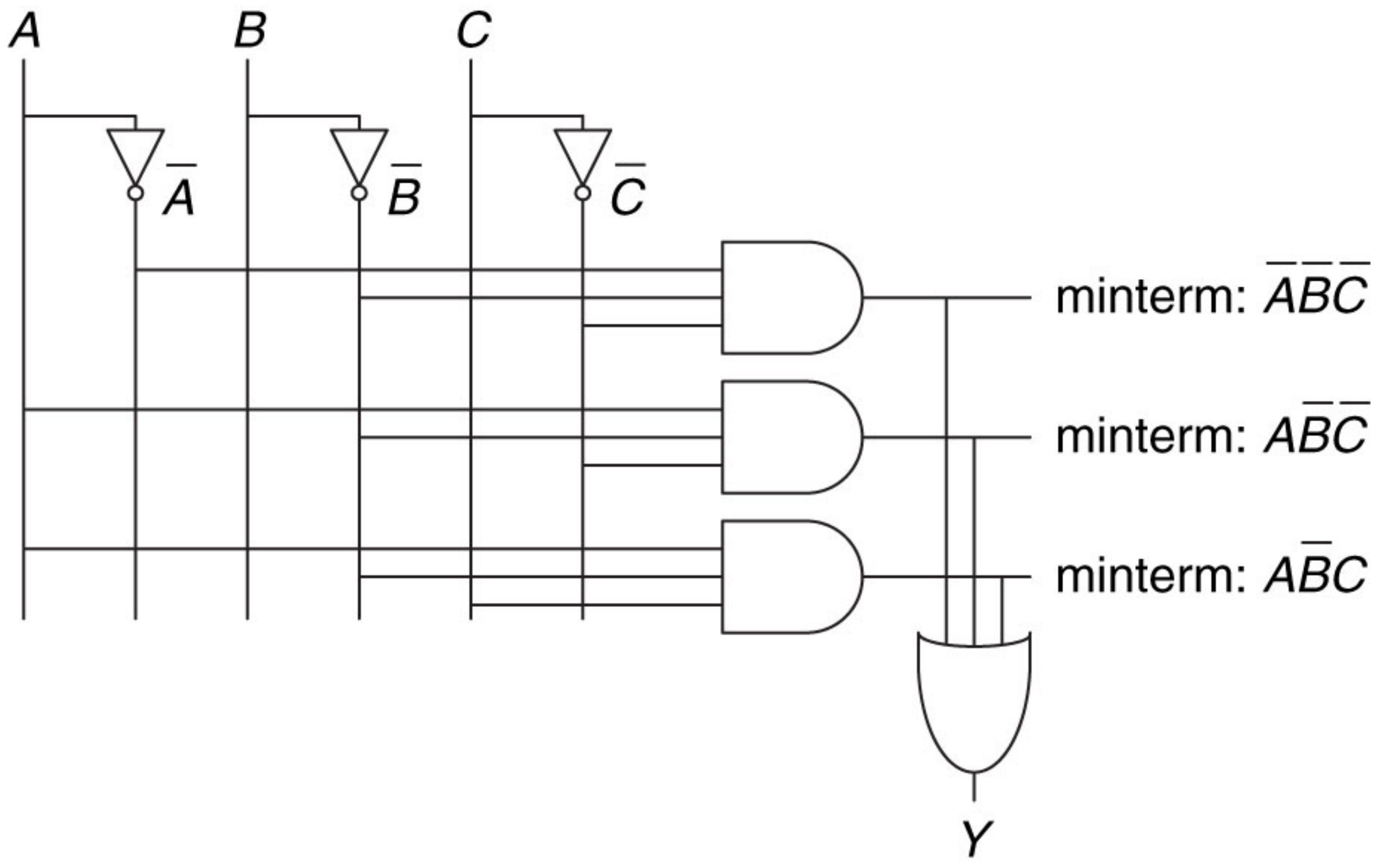
Simplification

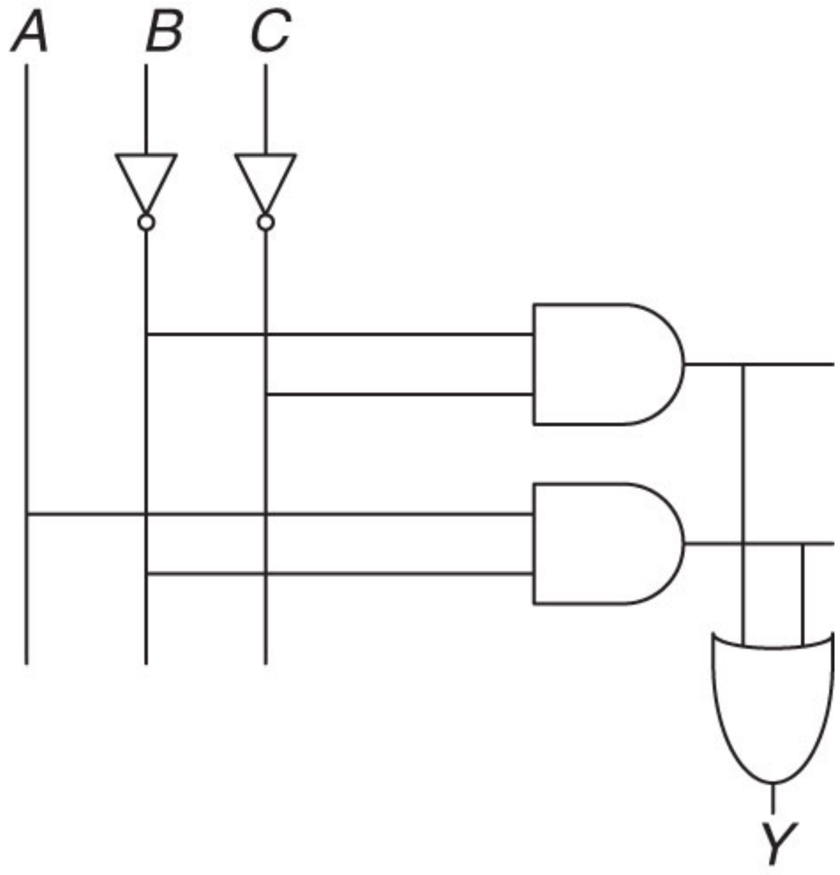
You have an A to remove from the first two terms, and you have a C to remove from the last two terms.

Do you have to choose one or the other?

$$\text{Simplify: } Y = A'B'C' + AB'C' + AB'C$$

Step	Equation	Justification
	$\overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$	
1	$\overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$	T3: Idempotency
2	$\overline{B}\,\overline{C}(\overline{A} + A) + A\overline{B}(\overline{C} + C)$	T8: Distributivity
3	$\overline{B}\,\overline{C}(1) + A\overline{B}(1)$	T5: Complements
4	$\overline{B}\,\overline{C} + A\overline{B}$	T1: Identity


A
B
C
$\overline{A}$
$\overline{B}$
$\overline{C}$
minterm: $\overline{A}\,\overline{B}\,\overline{C}$
minterm: $A\overline{B}\,\overline{C}$
minterm: $A\overline{B}C$
Y


A
B
C
Y

Why Simplify?

1. Reduce the number of gates.
2. Smaller/Cheaper Circuits
3. Maybe faster.

Yes, but this process seems *ad-hoc*.

We will examine another technique, Karnaugh Maps, in a future lecture to make the process more direct.

Hardware Reduction

Sum-of-Products is two-level logic.

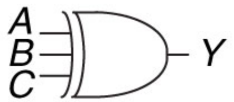
Might have circuits with three levels instead.

- Possibly simplifies a circuit.
- More complex logic available.

3-Input XOR

$$Y = \overline{A}\,\overline{B}C + \overline{A}B\overline{C} + A\overline{B}\,\overline{C} + ABC$$

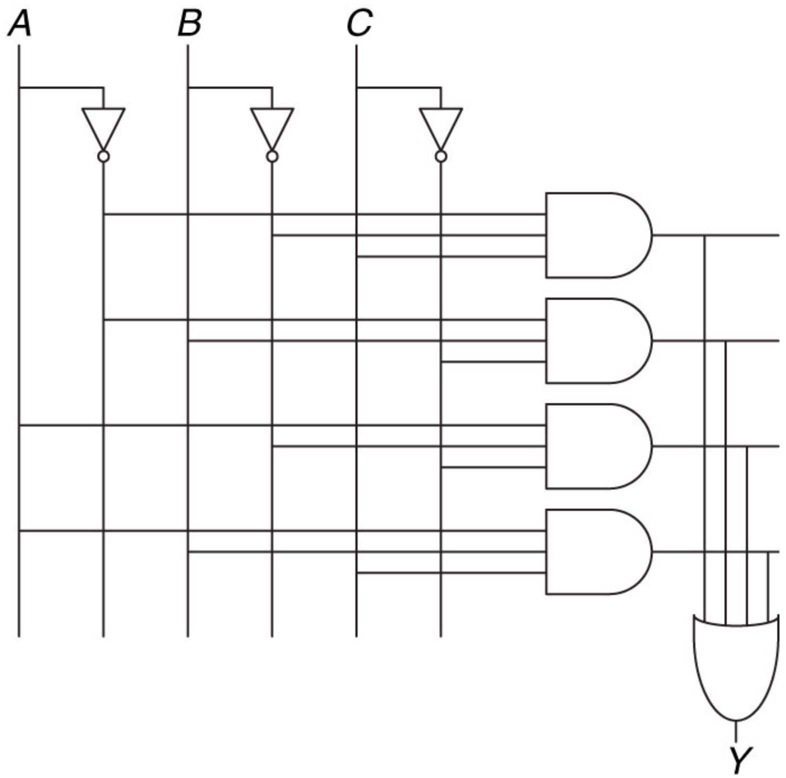
XOR3



$Y = A \oplus B \oplus C$

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

(a)



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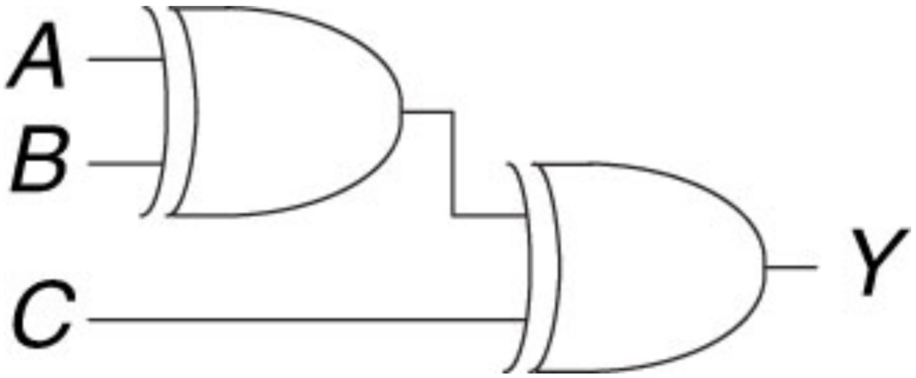

8-Input XOR

8 input XOR needs 128 eight-input AND gates and 1 128-input OR gate.

Can do the same with 7 two-input XOR gates.