Formal Modeling with Z: An Introduction

Luca Viganò

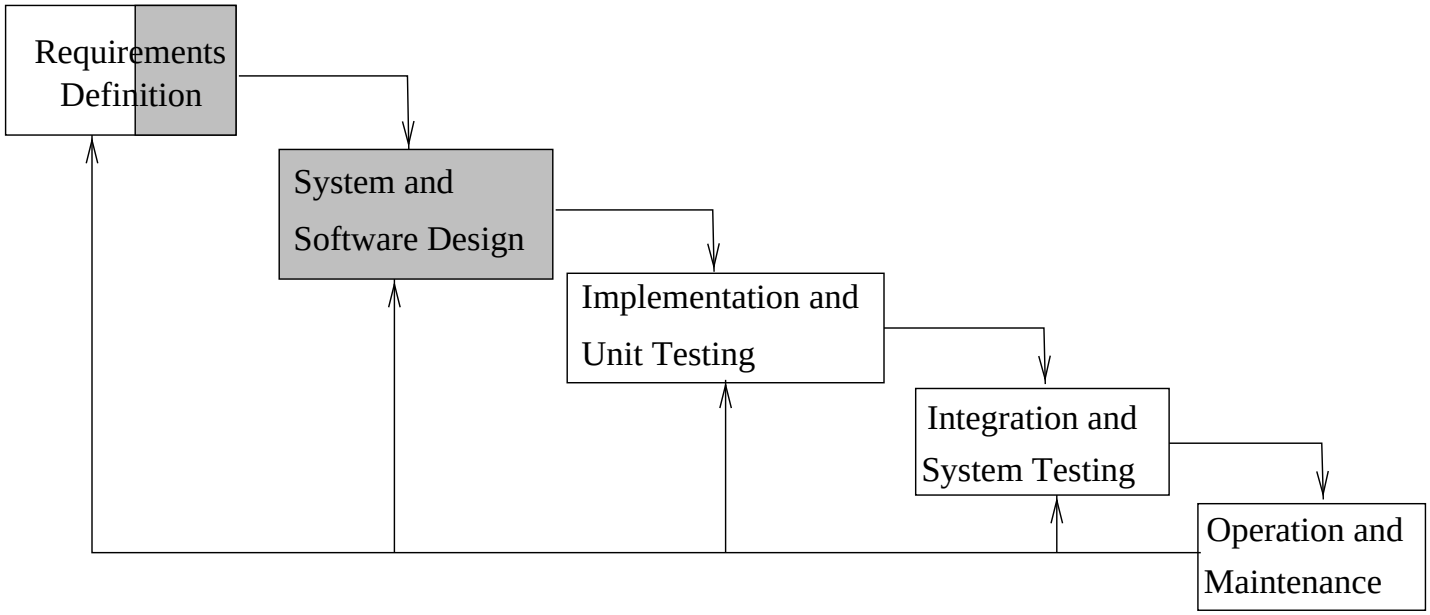
Institut für Informatik
Albert-Ludwigs-Universität Freiburg

Spring 2002

Overview

- Today:
  - Modeling with formal languages: why and how?
  - A standard introductory example.
  - The Z language — first definitions.
- Next classes: Z in detail, the mathematical toolkit, and applications.

Modeling



- Goal: specify the requirements as far as possible, but abstract as possible.
- Definition: A model is a construction or mathematical object that describes a system or its properties.

Which Modeling Language?

- There are hundreds! Differences include:

  System view: static, dynamic, functional, object-oriented,...
  Degree of abstraction: e.g. requirements versus system architecture.
  Formality: informal, semi-formal, formal.
  Religion: OO-school (OOA/OOD, OMT, Fusion, UML), algebraic specification, Oxford Z/CSP-Sect, Church of HOL,...

- Examples:

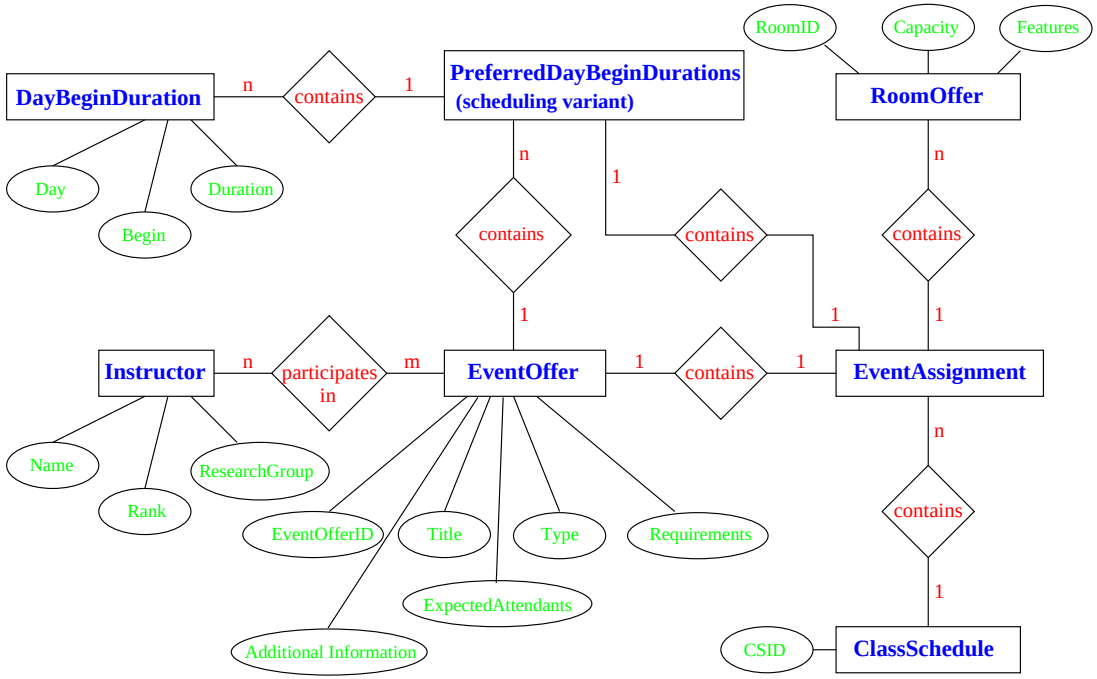
  Function trees, data-flow diagrams, E/R diagrams, syntax diagrams, data dictionaries, pseudo-code, rules, decision tables, (variants of) automata, Petri-nets, class diagrams, CRC-cards, message sequence charts,...

- Why are UML or other semi-formal languages not enough?

Disadvantages of Semi-Formal Languages

- Modeling is detailed and intuitive (and "simple", i.e. also for managers and laymen).
- Semantics of models/diagrams is often imprecise.
- Often only syntax.

Example: Problems with E/R-Diagrams



- Are the relations "directed"?
- Several properties cannot be specified graphically (e.g. constraints).
- etc.

We will employ Z to formalize semi-formal diagrams and models.

Formal Languages

- A language is formal if its syntax and semantics are defined formally (mathematically).

- Formal languages allow for the design, the development, the verification, the testing and the maintenance of a system:
  - remove ambiguities and introduce precision,
  - structure information at an appropriate abstraction level,
  - support the verification of design properties,
  - are supported by tools and systems.

- Using mathematics (and formal methods) may appear to be expensive, but in the long run it pays off (and how!).

Z ("zed")

- Is a very expressive formal language.
- Based on first-order logic with equality (PL1=) and typed set-theory.
- Has a mathematical toolkit: a library of mathematical definitions and abstract data-types (sets, lists, bags, ...).
- Supports the structured modeling of a system, both static and dynamic:
  - modeling/specification of data of the system,
  - functional description of the system (state transitions).
- Is supported by several tools and systems.

Z and other Formal Languages/Methods

- A number of successfully employed Formal Methods are based on PL1= with type-theory, e.g.
  - VDM ("Vienna Development Method", 80's),
  - B (applied extensively in France).
- Other formal languages:
  - Equational logic or Horn logic (in algebraic specifications),
  - PL1=,
  - Higher-order logic (HOL).
- Z:
  - Applied successfully since 1989 (Oxford University Computing Laboratory), e.g. British government requires Z-specifications for security-critical systems.
  - Is (will soon be) an ISO standard.

An Example (1)

A mathematical model that describes the intended behavior of a system is a formal specification.

Q: Why do we need such a specification if we can simply write a program? Why not directly implement the program?

A: A program can be quite cryptic, and therefore we need a specification that describes formally the intended behavior at the appropriate abstraction level.

An Example (2)

Example: What does the following simple SML-program do?

```
fun int_root a  =
(* integer square root *)
    let val i   = ref(0);
        val k   = ref(1);
        val sum = ref(1);
    in  while (!sum <= a) do
           (k   := !k+2;
            sum := !sum + !k;
            i   := !i+1);
        !i
    end;
```

An Example (3)

- The program is efficient, short and well-structured.
- The program name and the comment suggest that `int_root` simply computes the "integer square root" of the input, but is it really the case?
- Moreover: What happens in special input cases, e.g. when the input is 0 or -3?
- Such questions can be answered by code-review (or reverse-engineering), but this requires time and can be problematic for longer programs.
- The key is abstraction: understanding the code must be separated from understanding its "function".

  For example, consider a VCR whose only documentation is the blue-print of its electronic.

A Example (4)

- Solution: we can specify the program in Z.

  Formalize what the system must do without specifying/prescribing how.

- We specify `int_root` in Z by means of a so-called axiomatic definition (or axiomatic description):

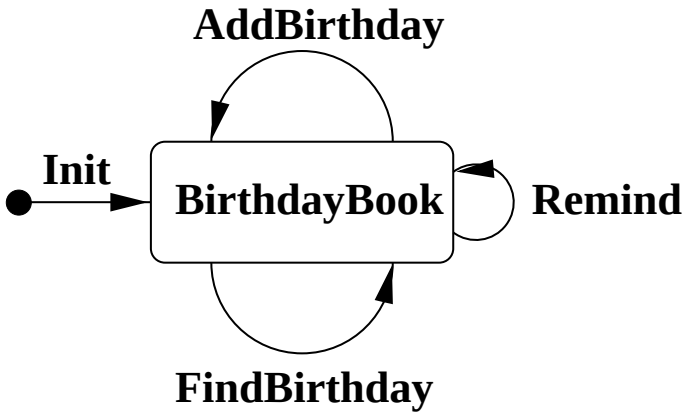
$int\_root : \mathbb{Z} \rightarrow \mathbb{N}$	declaration
$\forall\, a : \mathbb{N} \bullet \mathbf{let}\ y = int\_root(a) \bullet$ $\quad y * y \leq a < (y+1) * (y+1)$ $\forall\, a : \mathbb{N} \setminus \{0\} \bullet int\_root(-a) = 0$	predicate ... predicate

More about Z $\rightsquigarrow$ implementation in a few weeks.

An Introductory Example: The Birthdaybook

- The Birthdaybook is a small database containing peoples' names and their birthdays.

- A simple event-model:



- A structured Z-specification in 3 steps:

  1. Define the (auxiliary) functions and types of the system.
  2. Define the state-space of the system.
  3. Define the operations of the system (based on the relations of the state-space).

The Birthdaybook: Z-Specification (1)

Step 1. Define the (auxiliary) functions and types of the system:

- Basic types

$$[NAME, DATE]$$

The precise form of names and dates is not important (e.g. strings, 06/03, 03/06, 6.3, 06.03, March 6, 6.Mar, or ...).

The Birthdaybook: Z-Specification (2)

Step 2. Define the state-space of the system using a Z-schema:

Birthdaybook

$known : \mathbb{P}\, NAME$

$birthday : NAME \pfun DATE$

$known = \mathrm{dom}\ birthday$

Name of schema

declaration of typed variables
(represent observations of the state)

---

relationships between values of vars
(are true in all states of the system
and are maintained by every
operation on it)

Notation and remarks:

- *known* is the set (symbol $\mathbb{P}$) of names with stored birthdays,
- *birthday* is a partial function (symbol $\pfun$), which maps some names to the corresponding birthdays,
- The relation between known and birthday is the *invariant* of the system:

  the set *known* corresponds to the domain (dom) of the function *birthday*.

The Birthdaybook: Z-Specification (3)

- Example of a possible state of the system:

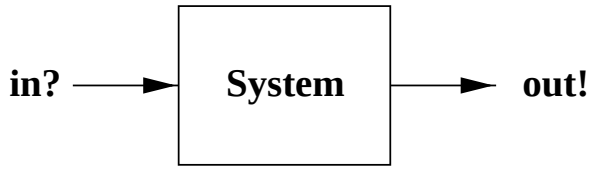
$$\begin{array}{lcl} known & = & \{\text{Susy},\ \text{Mike},\ \text{John}\} \\ birthday & = & \{\text{John} \mapsto 25.\text{Mar}\,, \\ & & \ \ \text{Susy} \mapsto 20.\text{Dec}\,, \\ & & \ \ \text{Mike} \mapsto 20.\text{Dec}\} \end{array}$$

- Invariant $known = \text{dom}\ birthday$ is satisfied:
  - *birthday* stores a date for exactly the three names in *known*.
- N.B.:
  - no limit on stored birthdays,
  - no particular (prescribed) order of the entries,
  - each person has only one birthday ($birthday$ is a function),
  - two persons can have the same birthday.

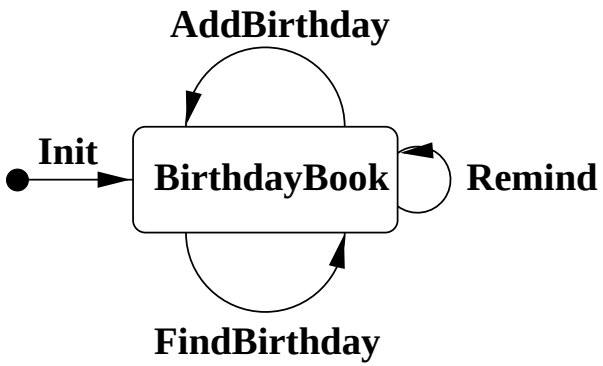
The Birthdaybook: Z-Specification (4)

Step 3. Define the operations of the system (based on the relations of the state-space).

- Some operations modify the state and some leave it unchanged.
- Some operations have input and/or output:



- Examples of operations: AddBirthday, FindBirthday, Remind (and Init).



The Birthdaybook: Z-Specification (5)

Add the birthday of a person, who is not yet known to the system:

AddBirthday

$\Delta BirthdayBook$

$name? : NAME$

$date? : DATE$

$name? \notin known$

$birthday' =$

$birthday \cup \{name? \mapsto date?\}$

Name of operation (schema)

structured import (symbol Δ)

input of operation (symbol ?)

input of operation (symbol ?)

---

precondition for success of operation

extend the birthday function

(if precondition is satisfied)

- This schema modifies the state:
  - it describes the state before (variables without '),
  - and that after the operation (variables with ').
- Note that we do not specify what happens when the precondition is not satisfied.
- It is possible to extend (refine) the specification so that an error message is generated.

The Birthdaybook: Z-Specification (6)

- We expect that $AddBirthday$ extends the set of known names with the new name:

$$known' = known \cup \{name?\}$$

- We can use the specification of $AddBirthday$ to prove this, by exploiting the invariant of the state before and after the operation:

$$\begin{array}{lcll} known' & = & \mathsf{dom}\ birthday' & [\text{invariant after}] \\ & = & \mathsf{dom}(birthday \cup \{name? \mapsto date?\}) & [\text{spec of } Addbirthday] \\ & = & (\mathsf{dom}\ birthday) \cup (\mathsf{dom}\{name? \mapsto date?\}) & [\text{fact about } \mathsf{dom}\,] \\ & = & (\mathsf{dom}\ birthday) \cup \{name?\} & [\text{fact about } \mathsf{dom}\,] \\ & = & known \cup \{name?\} & [\text{invariant before}] \end{array}$$

- Proving such properties ensures that the specification is correct:

  We can analyze the behavior of the system without having to implement it!

The Birthdaybook: Z-Specification (7)

Find the birthday of a person known to the system:

FindBirthday

$\Xi BirthdayBook$

$name? : NAME$

$date! : DATE$

$name? \in known$

$date! = birthday(name?)$

Name of operation (schema)

structured import (symbol Ξ)

input of operation (symbol ?)

output of operation (symbol !)

---

precondition for success of operation

output of operation (if successfull)

This schema leaves the state unchanged and is equivalent to:

FindBirthday

$\Delta BirthdayBook$

$name? : NAME$

$date! : DATE$

$known' = known$

$birthday' = birthday$

$name? \in known$

$date! = birthday(name?)$

The Birthdaybook: Z-Specification (8)

- Find out who has his birthday at some particular date:

$$
\begin{array}{|l}
\hline
\textit{Remind} \\
\Xi BirthdayBook \\
today? : DATE \\
cards! : \mathbb{P}\, NAME \\
\hline
cards! = \{\, n \in known \mid birthday(n) = today? \,\} \\
\hline
\end{array}
$$

$cards!$ is a set of names, to whom "birthday-cards" should be sent.

- Initial state of the system:

$$
\begin{array}{|l}
\hline
\textit{InitBirthdayBook} \\
BirthdayBook \\
\hline
known = \varnothing \\
\hline
\end{array}
$$

$known = \varnothing$ implies $birthday$ is also empty.

The Birthdaybook: Z-Specification (9)

- What does the Z-specification tell us about the implementation?
- It describes what the system does without specifying/prescribing how.
- For example, the Z-specification identifies legal and illegal data and operations. Illegal operations are for instance:
  - simultaneous addition of the birthdays of two persons,
  - addition of the birthday of a person who is already known to the system ($name? \in known$).
    An operation $ChangeBirthday$ is not specified and could be added, or only realized in the implementation.
- More in the next classes.

Summary

- Z is an expressive language (PL1= and typed set-theory).
- Z supports structured, static and dynamic, modeling.
- More about Z:

  http://archive.comlab.ox.ac.uk/z.html

- Tools and systems: see the course webpage.
  - ZETA (an open environment, including a type-checker; emacs Zeta-Mode)
    /usr/local/zeta
  - HOL-Z tool (an embedding of Z in the theorem prover Isabelle).
  - Object-Z (an object-oriented extension of Z).
  - Books about Z and LaTeX style-file.