SVD

- "NETFLIX CHALLENGE PROBLEM"

EQUIVALENT TO A "MATRIX COMPLETION" PROBLEM

PREDICT WHICH USERS WILL LIKE CERTAIN MOVIES

MOVIES

GIANT MATRIX

ROWS ARE PEOPLE

NETFLIX SUBSCRIBERS

KNOWN

ROW i
COLUMN j

KNOWN

THE RATING THAT PERSON i WOULD GIVE MOVIE j

HIGH LEVEL

$$\begin{pmatrix} 1 & ? & ? \\ ? & 2 & ? \\ ? & 6 & 9 \\ ? & ? & 3 \\ 4 & 4 & ? \end{pmatrix}$$

SOME ENTRIES KNOWN; OTHERS ARE MISSING / '?'S

GOAL: REPLACE '?' WITH NUMBERS THAT REPRESENT TRUE PREFERENCES.

ADDITIONAL INFORMATION: EACH ROW IS A MULTIPLE OF OTHER ROWS

HIGH LEVEL

$$\begin{pmatrix} \underline{1} & 1 & \frac{3}{2} \\ 2 & 2 & 3 \\ 6 & \underline{6} & \underline{9} \\ 2 & 2 & \underline{3} \\ \underline{4} & \underline{4} & 6 \end{pmatrix}$$

SOME ENTRIES KNOWN; OTHERS ARE MISSING / ?'S

GOAL: REPLACE '?' WITH NUMBERS THAT REPRESENT TRUE PREFERENCES.

ADDITIONAL INFORMATION: EACH ROW IS A MULTIPLE OF OTHER ROWS

$\equiv$ THE MATRIX HAS RANK-1

RANK-0 MATRIX $\equiv$ ALL ZEROS MATRIX

RANK-1 MATRIX $\equiv$ ALL ROWS ARE MULTIPLES OF EACH OTHER

COLUMNS ARE MULTIPLES OF EACH OTHER

EQUIVALENTLY IF WE HAVE A RANK-1 MATRIX

$A = u \cdot v^T$ u and v are vectors

$m \times n$ MATRIX

OUTER PRODUCT

$m \times 1$ VECTOR

$n \times 1$ VECTOR

ij^{th} entry of A

$\equiv u_i \cdot v_j$

$A = u \cdot v^T$

$$A = \begin{bmatrix} u_1 \cdot v^T \\ u_2 \cdot v^T \\ \vdots \\ u_m \cdot v^T \end{bmatrix} \quad \left[\begin{array}{c|c|c|c} v_1 \cdot u & v_2 \cdot u & & v_n \cdot u \end{array}\right]$$

NOW CONSIDER CASE WHERE A IS A RANK-2 MATRIX

THIS MEANS A IS THE SUM OF 2 RANK-1 MATRICES

(AND A IS NOT RANK-1)

$$A = u \cdot v^T + w z^T$$

$$\begin{bmatrix} u_1 v^T + w_1 \cdot z^T \\ \vdots \\ u_m v^T + w_m \cdot z^T \end{bmatrix} \qquad \begin{bmatrix} v_1 \cdot u + z_1 \cdot w & \cdots & v_n \cdot u + z_n \cdot w \end{bmatrix}$$

ROWS

$$\begin{array}{cc} & 2 \\ m & \begin{bmatrix} | & | \\ u & w \\ | & | \end{bmatrix} \end{array} \cdot \begin{array}{cc} & n \\ 2 & \begin{bmatrix} v^T \\ z^T \end{bmatrix} \end{array}$$

MATRIX

$u_1 v_1 + w_1 z_1$

DEFINE THE SINGULAR VALUE DECOMPOSITION OF A MATRIX

EVERY MATRIX $A = U \cdot S \cdot V^T$

U: $m \times m$ ORTHOGONAL MATRIX

S: $m \times n$ DIAGONAL MATRIX

V^T: $n \times n$ ORTHOGONAL MATRIX

COLUMNS OF U ARE LEFT SINGULAR VECTORS

ROWS OF V^T ARE "RIGHT SINGULAR VECTORS"

ENTRIES OF S

$s_1 \geq s_2 \geq \cdots \geq 0$

SINGULAR VALUES ARE UNIQUE

SINGULAR VECTORS ARE NOT UNIQUE

$$A = \sum_{i=1}^{\min(n,m)} \underline{s_i} \, u_i \cdot v_i^T$$

OUTER PRODUCT

SVD CAN BE COMPUTED IN TIME

$O(m^2 n)$ or $O(n^2 m)$ (WHICHEVER IS SMALLER)

ONE WAY WE COULD re-WRITE A / represent A

$$A = U S V^T$$

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix} \end{array}$$

ZERO OUT SOME ENTRIES OF S.

ONE WAY WE COULD re-WRITE A / represent A

$$A = U S V^T$$

$m \times n$

$$\begin{array}{cc} & \quad n \\ m & \begin{pmatrix} s_1 & & 0 \\ & \ddots \; s_k & 0 \\ 0 & & \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

ZERO OUT SOME ENTRIES OF S.

$$S \quad \begin{array}{cc} & k \quad n \\ k & \\ m & \begin{pmatrix} \quad \\ 0 \end{pmatrix} \end{array} \text{MATRIX} \longrightarrow \begin{array}{cc} & k \\ k & \begin{pmatrix} s_1 & 0 \\ 0 & \ddots \; s_k \end{pmatrix} \end{array}$$

k MATRIX, DIAGONAL

$$A \approx U' \cdot S' \cdot V'^T$$

$m \times n$

U': $m \times k$ TAKEN ONLY k COLUMNS FROM U

S': $k \times k$

V'^T: $k \times n$ — k ROWS FROM V^T

DEFINE FROBENIUS NORM OF A MATRIX TO BE

$$\sqrt{\sum_{i,j} A_{ij}^2}$$

GIVEN: A MATRIX A

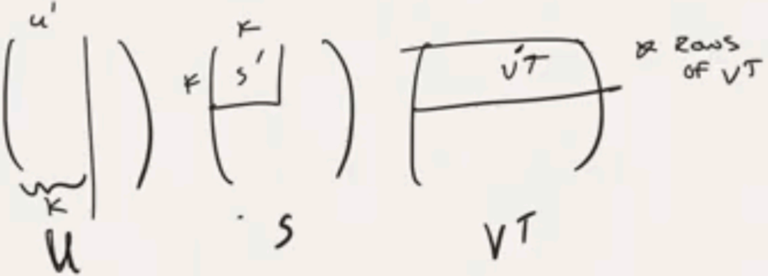
GOAL: FIND A MATRIX A' SUCH THAT A' HAS RANK K

AND MINIMIZES $\underline{\|A - A'\|_F}$ OVER ALL RANK K MATRICES.

ANSWER: COMPUTE SVD OF A AND TAKE TOP K SINGULAR VECTORS AND VALUES.

$$A = USV^T$$

Answer is $A' = U'S'V^{T'}$



A' is still $m \times n$

MATRIX COMPLETION

A

- REPLACE '?' WITH EITHER 0
  AVG VALUE OF KNOWN ENTRIES
  AVG VALUE IN THAT COLUMN OR ROW.

- FIND BEST RANK K APPROXIMATION TO A AFTER FILLING IN THE '?'s.

- OUTPUT THIS BEST RANK K APPROXIMATION.

HOW TO CHOOSE K? K IS A HYPERPARAMETER.

ONE TYPICAL HEURISTIC FOR CHOOSING k

IS TO TAKE ENOUGH SINGULAR VALUES SO THAT THE SUM OF REMAINING VALUES $\leq \frac{1}{10}$ OF VALUES YOU DID TAKE.

APPLICATION: LINEAR REGRESSION

$$\boxed{\min_{x \in \mathbb{R}^n} \|Ax - b\|^2 \qquad b \in \mathbb{R}^m \qquad A \text{ is } m \times n \text{ MATRIX}}$$

SOLUTION IS

$$\begin{pmatrix} \frac{1}{d_1} & & & \\ & \frac{1}{d_2} & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \cdot \overset{b}{\begin{pmatrix} \quad \end{pmatrix}}$$

$D^{\dagger} \rightarrow$ PSEUDOINVERSE OF D.

EASY CASE

$A = D$ (DIAGONAL)

$$D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$x_1 = \frac{b_1}{d_1} \quad x_2 = \frac{b_2}{d_2}$

$d_j = 0 \Rightarrow x_j = 0$

SUMMARIZE: $x = D^{\dagger} \cdot b$

$$\|Ax-b\|^2 \equiv \min_x \|USV^Tx-b\|^2$$

$$\|Ux\| = \|x\| \qquad \equiv \min_x \|SV^Tx-U^Tb\|^2$$

$$\underline{y} = \underline{V}^Tx \equiv \underline{Vy} = \underline{x}$$

$$\equiv \min_x \|Sy-U^Tb\|^2$$

$$\Rightarrow y = S^{+}\cdot U^Tb$$

$$\Rightarrow \boxed{x = VS^{+}U^Tb}$$

RECALL PCA: FIND EIGENDECOMPOSITION OF COVARIANCE MATRIX

$$X^TX = \underline{(USV^T)^T} \cdot USV^T$$

$$VS\underline{U^T \cdot U} \cdot SV^T$$

$$\underset{\text{ORTHO}}{V} S^2 \underset{\text{ORTHO}}{V^T}$$

EIGENDECOMPOSITION

SINGULAR VALUES VS EIGENVALUES?

RIGHT SINGULAR VECTORS OF X (ROWS OF V^T) <u>ARE</u> THE PRINCIPAL COMPONENTS (TOP EIGENVECTORS OF X^TX).

SINGULAR VALUES ARE THE <u>SQUARE ROOT</u> OF EIGENVALUES OF X^TX

ONE FURTHER APPLICATION:

IMAGE COMPRESSION.

IMAGE BLACK AND WHITE IMAGE

MATRIX

$$\begin{array}{cc} & n \\ m & \left(\quad A \quad \right) \end{array}$$

EACH ENTRY IS $\{0, 1\}$

FOR COMPRESSION

COMPUTE A' = LOW-RANK APPROXIMATION OF A FOR SOME VALUE K.

ENTRIES OF A'?

NOT NECESSARILY

COULD BE NUMBERS BTWN 0 and 1