Finals project:

Recall that in your previous homework, you've shown that the Hamiltonian with coloumb potential

$$H = \frac{1}{2} P_r^2 + \frac{\ell(\ell+1)}{2r^2} - \frac{1}{r}$$

where $[P_r, r] = -i$ and we've set $m, \hbar$ to 1.

You found that

<r^-1> = -2E

$$(\star) \quad 8E_\ell n \langle r^{n-1} \rangle = (1-n)[n(n-2) - 4\ell(\ell+1)]\langle r^{n-3} \rangle$$

$$-4(2n-1)\langle r^{n-2} \rangle$$

where E_ℓ is the energy of energy eigenstate Φ_E and

$\langle r^a \rangle$ is the expectation value of r^a.

with respect to Φ_E. i.e

$$\langle r^a \rangle = \int \Phi_E^* r^a \Phi_E.$$

Note that eq. $(\star)$ can be solved iteratively such that we have $\langle r^a \rangle$ as a function of E.

The project consists of the following steps

1. Find the function

$$\langle r^a \rangle (E)$$

for various values of a.

2. Recall that the Hankel matrix

$$\begin{pmatrix} \langle r^0 \rangle & \langle r^1 \rangle & \langle r^2 \rangle & \cdots \\ \langle r^1 \rangle & \langle r^2 \rangle & \langle r^3 \rangle & \cdots \\ \langle r^2 \rangle & \langle r^3 \rangle & \cdots & \end{pmatrix}$$

is a positive semi-definite matrix from our discussion in class. Please explain why

$$\begin{pmatrix} \langle r^1 \rangle & \langle r^2 \rangle & \langle r^3 \rangle & \cdots \\ \langle r^2 \rangle & \langle r^3 \rangle & \langle r^4 \rangle & \cdot \\ \cdots & & & \end{pmatrix}$$

the "shifted" Hankel matrix must also be a positive semi-definite matrix.

3. Use the above conditions, determine the allowed values for E_e. For certain values

of l.

Discuss the emerging pattern of degeneracy in your analysis and the convergence property of your result as you increase the matrix size.

4. Redo the analysis with the hamiltonian

$$H = \frac{1}{2} P_r^2 + \frac{l(l+1)}{2r^2} - \frac{1}{r} + \frac{2}{r^2}$$

Do you find any interesting qualitative change in your spectrum?