Sample Partial Differential Equation

Suppose given the PDE (Diffusion)
$\partial_t u = D\partial_x^2 u,\ \ for\ x\ \in [0,1], and\, t > 0$

Exact Solution (can be found in multiple PDEs textbooks):

$$u(x,t) = \sum_{k=1}^{n} = \frac{4}{(k\pi)^2} sin(\frac{k\pi}{2}) sin(k\pi x) e^{-0.5(k\pi)^2 t}$$

We are going to compute the numerical solution with the given information below.

for $x \in [0,1]$ and t $> 0:$

$$the\ initial\ condition\ u(x,0) = \begin{cases} x, & \text{if } 0 \leq x \leq 1/2 \\ 1-x, & \text{if } 1/2 < x \leq 1 \end{cases}$$

the boundary condition $u(0,t) = u(1,t) = 0$

The Explicit Difference Schema:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}, where\ D = \frac{1}{2}$$

Take $\Delta x = 0.1, \Delta t = 1/100$ to proceed the calculations;

Solving Idea for a particular t

Use of the finite difference scheme for discretizing partial derivatives reduces the problem to vector-matrix multiplication.

Where the unknown vector $U^n = (U_1^n, U_2^n, \ldots U_N^n)^T$ refers to sample of grid values of the solution function $u(x,t)$ evaluated at $t = t_n$.

Then what we focuse on for $u_j^n = u(x_j, t_n)$ is in a vector of $U^n to\ U^{n+1}$

To calculated the U^{n+1}, we must use the explicit difference scheme

A Look of Explicit Difference Scheme Equation

As we said we interested the vector U^{n+1} = a set of $u_j^{n+1} j \in [1, 11]$,
for the case that $\Delta x = 0.1, x \in [0, 1]$

Hence we would like to rearrange the equation to:

$$u_j^{n+1} = \frac{1}{2}\frac{\Delta t}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1^n}) + u_j^n$$

$$= u(x_j, t_{n+1}) = \frac{1}{2}\frac{\Delta t}{(\Delta x)^2}[u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)] + u(x_j, t_n)$$

for easier programming purpose, we shift the index of t left by one

$$= u(x_j, t_n) = \frac{1}{2}\frac{\Delta t}{(\Delta x)^2}[u(x_{j+1}, t_{n-1}) - 2u(x_j, t_{n-1}) + u(x_{j-1}, t_{n-1})] + u(x_j, t_{n-1})$$

A Look of How Initial and Boundary Condition are Important

The expression of $u(t, x)$ in 2D matrix:

$$
\begin{array}{cccccc}
 & 1 & 2 & 3 & \cdots & n \\
1 & u(0,0) & u(0,1\Delta t) & u(0,2\Delta t) & \cdots & u(0,n\Delta t) \\
2 & u(1\Delta x,0) & \cdots & \cdots & \cdots & u(1\Delta x,n\Delta t) \\
3 & u(2\Delta x,0) & \cdots & \cdots & \cdots & u(2\Delta x,n\Delta t) \\
 & . & & & & \\
 & . & & & & \\
11 & u(1,0) & & & &
\end{array}
$$

The initial condition givien the cover the values for first columns
And the boundary condition cover the values for first and last rows

Now, the expression of $u(t, x)$ should be:

$$
\begin{array}{cccccc}
 & 1 & 2 & 3 & \cdots & n \\
1 & 0 & 0 & 0 & \cdots & 0 \\
2 & 0.1 & u(1\Delta x,1\Delta t) & \cdots & \cdots & u(1\Delta x,n\Delta t) \\
3 & 0.2 & u(2\Delta x,1\Delta t) & \cdots & \cdots & u(2\Delta x,n\Delta t) \\
 & . & & & & \\
5 & 0.5 & \cdots & \cdots & \cdots & \\
6 & 0.4 & \cdots & \cdots & \cdots & \\
7 & 0.3 & \cdots & \cdots & \cdots & \\
 & . & & & & \\
11 & 0 & 0 & 0 & \cdots & 0
\end{array}
$$

Now from this matrix, we see the pattern of how explicit difference help
us to calculate the $u(x_j, t_n)$;

For example: $u(x_2, t_2)$, we need $u(x_1, t_1)$, $u(x_2, t_1), u(x_3, t_1)'s$ values
to explicit difference equation:

$$u(x_j, t_n) = \frac{1}{2}\frac{\Delta t}{(\Delta x)^2}[u(x_{j+1}, t_{n-1}) - 2u(x_j, t_{n-1}) + u(x_{j-1}, t_{n-1})] + u(x_j, t_{n-1})$$

Hence we know we must calculated the current columns's val then proceed to
proceed to the row, U^n = each columns value, and U^{n+1} = next columns value

MATLAB Implementation

```
clear
close all

prompt = 'Enter the value of Δt: ';
% set t_max= x then coloumns needed = x / delta_t + 1
% row = 1/ (0.01) + 1 = 11 (by default x[0,1]
delta_t = input(prompt);

prompt1 = 'Enter the value of T(max): ';
column = input(prompt1) / delta_t + 1 ;
row = 11;

% 2-D array with all zeros for exact solution
arr1 = zeros(row,column);

% initial variables for exact solution
result = 0;
arr1_x = 0;
arr1_t = 0;

% exact solution
% taking 14 terms
for i = 1:row
    for j = 1:column
        for k = 1:14
            result = result +
4/(k*pi)^2*sin(k*pi/2)*sin(k*pi*arr1_x)*exp(-0.5*(k*pi)^2*arr1_t);
        end
        arr1(i,j) = result;
        arr1_t = arr1_t + delta_t;
        result = 0;
    end
    arr1_t = 0;
    arr1_x = arr1_x + 0.1;
end

% Numerical Computing Solution
% (Initial Condition)
arr2 = zeros(row,column);
x_num = 0;
for i = 1 : row
    if(x_num>0.5)
        arr2(i,1) = 1-x_num;
    else
        arr2(i,1) = x_num;
    end
    x_num = x_num + 0.1;
```

```
end

% applied explicit differences to calculate other points
% applied boundary condition as well
dt = delta_t;
dx = 0.1;
for j = 2: column
    for i = 1:row
        if (i == 1 || i == row)
            arr2(i,j) = 0;
        else
            arr2(i,j) =0.5 *dt/(dx)^2*(arr2(i+1, j-1)-2*arr2(i,j-
1)+arr2(i-1,j-1))+arr2(i,j-1);
        end
    end
end


% Error |Exact - Numerical|
error = zeros(row, column);
for i = 1 : row
    for j = 1 : column
        error(i,j) = abs(arr1(i,j)- arr2(i,j));
    end
end



%% plot Portion
x_ = 0:0.1:1;

prompt_1 = 'Enter the value of t to get the graph u(x,t) at t =: ' ;
t = input(prompt_1);
exctVal = arr1(:, t/delta_t+1 );
numVal = arr2(:, t/delta_t+1 );
plot (x_,numVal,x_,exctVal,'lineWidth', 2);
hold off
xlabel('x');
s = num2str(t);
txt = ['u(x,t), t= ', s];
ylabel(txt);
legend('Numerical','Exact');
```

Graphical output

```
Enter the value of Δt: 0.01
Enter the value of T(max): 5
Enter the value of t to get the graph u(x,t) at t =: 1
```