Review 1. Lem 1: $g_T \sim \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} \Leftrightarrow V = W_1 \oplus W_2$

W_i are invariant for T

Lem 2: $g_T \sim \begin{pmatrix} \boxed{A_1} & N \\ & A_2 \end{pmatrix} \Leftrightarrow V$ has $\boxed{W_1}$ invariant for T

Review 2: Def: $v \in V$, $v \neq 0$ is said to be an eigenvector for $T: V \to V$ with eigenvalue $\lambda \in \mathbb{C}$ if $\underline{Tv = \lambda \cdot v}$

Once we have a basis $\{e_i\}$ $\quad T(e_1 \cdots e_n) = (e_1 \cdots e_n)\, g_T$

By $\{e_i\}$ is basis [illegible]

Once we have a basis $\{e_i\}$ $T(e_1 \cdots e_n) = (e_1 \cdots e_n) g_T$

By $\{e_i\}$ is basis, $v = \sum v_i e_i$ $\Rightarrow$ $v = (e_1 \cdots e_n) \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ $\vec{v} \triangleq \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$$Tv = T(e_1 \cdots e_n)\vec{v} = (e_1 \cdots e_n) g_T \vec{v} = \lambda \cdot (e_1 \cdots e_n)\vec{v}$$

$$\Longleftrightarrow \quad g_T \vec{v} = \lambda \vec{v} = \lambda I \cdot \vec{v}$$

Δ $(*)$ $(g_T - \lambda_0 I)\vec{v} = 0$ This is a system of linear equations

Need to find nonzero solution for this equation

$$\exists \text{ nonzero solution} \Longleftrightarrow \det(g_T - \lambda I) = 0.$$

If $\det \neq 0$ then $g_T - \lambda I$ is invertible

$(g_T - \lambda I)^{-1} \times (*)$

$\Rightarrow$ $\vec{v} = (g_T - \lambda I)^{-1} \cdot 0 = 0$

Δ Definition The characteristic polynomial P_T

$$P_T(\lambda) = \det(\lambda I - g_T) \qquad \text{rmk: } \deg P_T = n$$

Corollary: λ_0 is a solution of $P_T(\lambda) = 0$ iff there exist eigenvector v. with eigenvalue λ_0

translation from Δ

Property of $P_T(\lambda)$: $P_T(\lambda)$ is independent of Basis

Pf: If we change basis How does $\det(\lambda I - g'_T)$ change?

Recall $(e'_1 \cdots e'_n) = (e_1 \cdots e_n) h$ $h \in GL_n(\mathbb{C})$

$$g'_T = h^{-1} \cdot g_T \cdot h$$

$$\det(\lambda I - g_T) \overset{?}{=} \det(\lambda I - g'_T) = \det(\lambda I - h^{-1} g_T h)$$

very good $I = h^{-1} \cdot I \cdot h$

$$= \det(h^{-1}(\lambda I) h - h^{-1} g_T h)$$

$$\det(h^{-1} \cdot [\lambda I - g_T] \cdot h) = \det(h^{-1}) \det(\lambda I - g_T) \det(h)$$

$\det(h^{-1}h) = 1$

$\det \left(h^{-1} \cdot [\lambda I - g_T] \cdot h \right) = \det(h^{-1}) \det(\lambda I - g_T) \det(h)$

Recall: $\det(AB) = \det(A) \cdot \det(B)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\text{LHS}} \qquad \#$

$\triangle \quad (\lambda I - g_T)\vec{v} = 0$

Recall $\quad \dim\{\text{Solutions } \vec{v}\} = n - \operatorname{rank}(\lambda I - g_T)$

$\dim \ker(\lambda I - g_T) = n - \dim \operatorname{Image}(\lambda I - g_T)$

$\ker \widehat{\lambda I - T} = \{ v \mid \widehat{\lambda I - T}\, v = 0 \}$

$\triangle$ Lemma 1 before implies

Lemma: T has $\overset{\dim V}{n}$ linear independent eigenvectors $\{v_1 \cdots v_n\}$ eigenvalues $\{\lambda_1 \cdots \lambda_n\}$ $\Leftrightarrow g_T \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

Pf: "$\Rightarrow$" each eigenvector v_i span 1 dim invariant subspace $\mathbb{C}v_i$

(invariant : $Tv_i = \lambda_i v_i \in \mathbb{C}v_i$)

Lem 1 $\Rightarrow$ when $V = \bigoplus \underbrace{\mathbb{C}v_i}_{\text{each dim} = 1}$ we have $g_T \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

"$\Leftarrow$" for some basis $\{e_1 \cdots e_n\}$ $\quad T(e_1 \cdots e_n) = (e_1 \cdots e_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\Rightarrow \quad T e_i = \lambda_i e_i$

So e_i is eigenvector with eigenvalue $\lambda_i \qquad \#$

Example: $\quad g_T = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \in M_2(\mathbb{C}) \qquad T(e_1\, e_2) = (e_1\, e_2)\, g_T$

Goal: find $h \in GL_2(\mathbb{C})$ s.t. $h^{-1} g_T h$ is diagonal

Want to find eigenvectors

Step 1 Compute $P(\lambda) = \det(\lambda I - g_T) = \begin{vmatrix} \lambda - 1 & 3 \\ & \end{vmatrix}$

Step 1: Compute $P_T(\lambda) = \det(\lambda I - g_T) = \begin{vmatrix} \lambda-1 & 3 \\ -3 & \lambda-1 \end{vmatrix}$

$= (\lambda-1)(\lambda-1) - (3)(-3) = (\lambda-1)^2 + 9$

Step 2: Find eigenvalues $(\lambda-1)^2 + 9 = 0$

$(\lambda-1)^2 = -9$

taking square root $\lambda - 1 = \pm 3i$

$\lambda = 1 \pm 3i$ $\begin{cases} \lambda_1 = 1+3i \\ \lambda_2 = 1-3i \end{cases}$

Step 3: Find eigenvectors $(\lambda_i I - g_T)\vec{V}_i = 0$

λ_1: $\left[(1+3i) I - \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$

$\begin{pmatrix} 3i & 3 \\ -3 & 3i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$x_1 = i \quad x_2 = 1$ $\qquad V_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = ie_1 + 1e_2$

λ_2 $\quad 1-3i \quad \begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$ $\quad \begin{cases} y_1 = -i \\ y_2 = 1 \end{cases}$

$V_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

Step 4: Find h. new basis $= \{ \underbrace{\vec{V}_1 \ \vec{U}_2}_{\text{should be eigenvector}} \} = (e_1\ e_2) \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$

old basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$h = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$

Step 5: Check $h^{-1} g_T h$ is diagonal matrix $\begin{pmatrix} 1+3i & \\ & 1-3i \end{pmatrix}$

lemma. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ det ... $g^{-1} = \frac{1}{\ldots} \begin{pmatrix} d & -b \\ \ldots \end{pmatrix}$

Lemma: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det g \neq 0$, $g^{-1} = \frac{1}{\det g}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Continue:

$\det(h) = i \cdot 1 - (1)(-i) = i + i = 2i$ $\quad h^{-1} = \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$

Leave to you to check indeed $h^{-1} S_T h$ is diagonal #

$\triangle$ Lemma 2 implies

Lemma: $\forall g \in M_n(\mathbb{C})$ is conjugate

$$g' = \begin{pmatrix} \overbrace{\lambda_1 \ \ddots \ \lambda_1}^{n_1} & \overbrace{}^{n_2} & * & \\ & \lambda_2 \ \ddots \ \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_k \end{pmatrix}$$

"very weak Jordan form"

Rmk: λ_i are solutions with matching multiplicity to $P_T(\lambda) = 0$

$P_T(\lambda) = (\lambda - \lambda_1)^{n_1} \cdot (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$

This is because $\det(\lambda I - g) = \det(\lambda I - g')$

$$= \det \begin{pmatrix} \lambda - \lambda_1 & & & * \\ & \lambda - \lambda_1 & & \\ & & \lambda - \lambda_2 & \\ 0 & & & \lambda - \lambda_2 \ \ddots \end{pmatrix}$$

$g = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_2 \ \ddots \end{pmatrix}$

$$= \underbrace{(\lambda - \lambda_1)(\lambda - \lambda_1) \cdots}_{n_1} \underbrace{(\lambda - \lambda_2)}_{n_2} \cdots \underbrace{(\lambda - \lambda_k)}_{n_k}$$

Pf of Lemma above:

Step 1: Need to a 1-dim invariant subspace

This is easy because we just compute $P_T(\lambda) = 0$

By FTA find λ_1 solution to $\uparrow$

$\det(\lambda_1 I - g) = 0 \Rightarrow$ find nonzero v_1 eigenvector

$\mathrm{Span}\{v_1\} = \mathbb{C} v_1$ is 1-dim invariant subspace

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Lemma 2 $\Rightarrow$ $g \sim \left(\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & g_{n-1} \end{array}\right)$

Step 2: We do step 1 for g_{n-1} (doing same step T)

Compute $P_{g_{n-1}}(\lambda)$ See if λ_1 is still a solution for $P_{g_{n-1}}(\lambda) = 0$

If so we use λ_1 again as in Step 1

Step 1 $\Rightarrow$ $h_{n-1}^{-1} \underline{g_{n-1}} h_{n-1} = \left(\begin{array}{c|c} \lambda_1 & * \\ \hline & g_{n-2} \end{array}\right)$

$$\underset{\text{Step}}{h_n^{-1} g h_n} = \left(\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & g_{n-1} \end{array}\right), \quad \begin{pmatrix} 1 & \\ & h_{n-1}^{-1} \end{pmatrix} \underbrace{h_n^{-1} g h_n}_{\begin{pmatrix} \lambda_1 & * \\ 0 & g_{n-1} \end{pmatrix}} \begin{pmatrix} 1 & \\ & h_{n-1} \end{pmatrix} \overset{\text{step }1+2}{=} \begin{pmatrix} \lambda_1 & * \\ 0 & h_{n-1}^{-1} g_{n-1} h_n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & * & * \\ 0 & \lambda_1 & \\ 0 & 0 & g_{n-2} \end{pmatrix}$$

Δ If λ_1 is no longer a solution, just pick any other solution λ_2

Continue similarly (Induction on size n)

#

Δ When we compute eigenvectors for λ_1, we are looking at

$$\underline{(\lambda_1 I - g)} \vec{v} = 0$$

g in very weak Jordan form

$$\lambda_1 I - g = \left(\begin{array}{c|c} \begin{matrix} 0 & \ddots & * \\ & & 0 \end{matrix} & * \\ \hline 0 & \underbrace{\begin{matrix} \lambda_1 - \lambda_2 & & \\ & \ddots & \\ & & \lambda_1 - \lambda_k \end{matrix}} \end{array}\right)$$

is invertible

as $\underline{\lambda_1 - \lambda_i} \neq 0 \quad \forall i \neq 1$

Upper left part has shape $\begin{pmatrix} 0 & \ddots & * \\ 0 & & 0 \end{pmatrix}$

Δ Definition: a matrix g is called nilpotent (upper triangular) if $g = \begin{pmatrix} 0 & \ddots & * \\ 0 & \ddots & 0 \end{pmatrix}$

Lemma: if $g \in M_{n \times n}(\mathbb{C})$ is nilpotent, then $g^n = 0$

Pf: $n=3$. In general $g = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$

$$g \cdot g = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \cdot z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$g^3 = \begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = 0$$

In general informal:
- g has zero line on diagonal
- $g \cdot g$ zero line shift / propagate by 1 position
- $g^i = \begin{pmatrix} \overbrace{0 \cdots 0}^{i} & * \\ & 0 \end{pmatrix}$

Δ Lemma above $\Rightarrow$ $(\lambda_1 I - g)^{n_1} = \left(\begin{array}{c|c} 0 & * \\ \hline 0 & \begin{matrix} (\lambda_1-\lambda_2)^{n_1} & * \\ & \ddots \\ 0 & (\lambda_1-\lambda_k)^{n_1} \end{matrix} \end{array}\right)$

$\ker(\lambda_1 I - g)^{n_1} = \{ \vec{v} \mid (\lambda_1 I - g)^{n_1} v = 0 \}$

is $\dim = n - \operatorname{rank}((\lambda_1 I - g)^{n_1}) = n - (n - n_1) = n_1$

Easy to see solutions are of shape $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_{n_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Δ Definition: The generalized eigenspace V_{λ_1} associated to eigenvalue λ_1 is $V_{\lambda_1} = \ker(\lambda_1 I - T)^i$ for large enough i

Rmk: Pick $i = n_1$ $\quad V_{\lambda_1} = \ker(\lambda_1 I - T)^{n_1}$

Implicit here is the fact: $\forall i > n_1$, $\ker(\lambda_1 I - T)^i = \ker(\lambda_1 I - T)^{n_1}$

Previous $(\lambda I - g)^{n_1}$ $\quad \left(\begin{array}{c|c} 0 & * \\ \hline & (\lambda_1-\lambda_2)^{n_1} \end{array}\right)$ — by row operation

Previous Computation $(\lambda I - g)^{n_1} = \begin{pmatrix} 0 & * & \\ 0 & (\lambda_1-\lambda_2)^{n_1} & * \\ 0 & 0 & \ddots \quad (\lambda_1-\lambda_k)^{n_1} \end{pmatrix}$ 0 by row operation

$(\lambda I - g)^{i}$ for $i > n_1$ $= \begin{pmatrix} 0 & * & \\ 0 & (\lambda_1-\lambda_2)^{i} & 0 \\ & 0 & (\lambda_1-\lambda_k)^{i} \end{pmatrix}$ $\leftarrow$ rank keeps same $= n - n_1$

Cor: $\dim V_\lambda$ = multiplicity of λ for $P_T(\lambda) = 0$