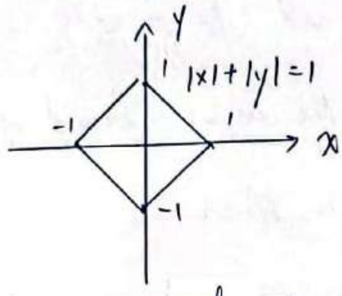
清華大學
Tsinghua University

Assigned exercise (hand in)

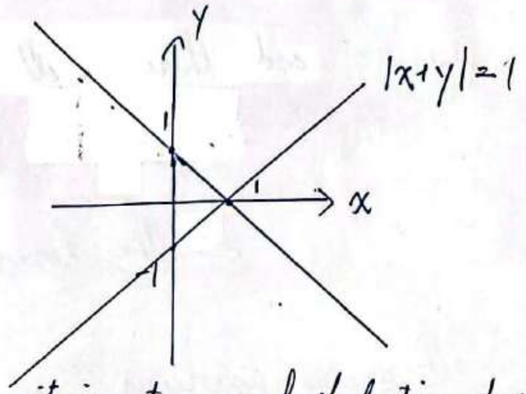
1. §1.3 Exercise 22.

a.



it is not a graph of function of x because there are two value f(x) when $x=0$

b.



it is not a graph of funtion of x because there are two value f(x) when $x=0$

§1.3 Exercise 27.

a. $f(x)=\begin{cases} x & \text{if } x\in[0,1) \\ -x+2 & \text{if } x\in[1,2] \end{cases}$

b. $f(x)=\begin{cases} 0 & \text{if } x\in[1,2)\cup[3,4] \\ 2 & \text{if } x\in[0,1)\cup[2,3) \end{cases}$

2. $\sqrt{x(x-3)}=\sqrt{3x-5} \iff \begin{cases} x(x-3)\geq 0 \\ 3x-5\geq 0 \\ x(x-3)=3x-5 \end{cases} \iff \begin{cases} x\leq 0 \text{ or } x\geq 3 \\ x\geq \frac{5}{3} \\ x=5 \text{ or } x=1 \end{cases} \iff x=5$

3. $|a-b|\geq\big||a|-|b|\big|$

$\Updownarrow$

$(a-b)^2\geq(|a|-|b|)^2$

$\Updownarrow$

$a^2-2ab+b^2\geq a^2-2|a||b|+b^2$

$\Updownarrow$

$ab\leq|a||b|$

$\Updownarrow$

$ab\leq|ab|$

because $ab\leq|ab|$, so $|a-b|\geq\big||a|-|b|\big|$

when $a=2$, $b=1$, $|a-b|=\big||a|-|b|\big|$

so this inequality is strict

4. the LUB is $M=1$

Prove: because $\frac{1}{\sqrt{n}} \geq 0 \Leftrightarrow -\frac{1}{\sqrt{n}} \leq 0 \Leftrightarrow 1-\frac{1}{\sqrt{n}} \leq 1$

so $M=1$ is the upper-bound of S

then, we assum that $M'<1$ is the upper-bound of S

we set $d = 1-M'$, and there exist $\sqrt{n^2} > \frac{1}{d}$. $(n \in \mathbb{N})$. and $d > \frac{1}{\sqrt{n^2}}$

then $M' = 1-d < 1-\frac{1}{\sqrt{n^2}}$, and $1-\frac{1}{\sqrt{n^2}} \in S$

if $M'<1$. then M' is not the upper-bound of S

so the lower upper-bound is $M=1$

Bonus exercises

1. $\forall a,b>0, a,b \in \mathbb{R}$: we have $(\sqrt{a}-\sqrt{b})^2 \geq 0$

$(\sqrt{a}-\sqrt{b})^2 \geq 0 \Leftrightarrow a+b \geq 2\sqrt{ab} \Leftrightarrow \frac{a+b}{2} \geq \sqrt{ab}$

so that for any $a,b \in \mathbb{R}, a,b>0$, we have $\frac{a+b}{2} \geq \sqrt{ab}$

2. we assume that $\sqrt{2} \in \mathbb{Q}$, and $\frac{p}{q} = \sqrt{2}$ $(p,q \in \mathbb{N}, \gcd(p,q)=1)$

then $\frac{p^2}{q^2} = 2 \Leftrightarrow p^2 = 2q^2$

because $p \in \mathbb{N}$. so $p = 2p'$ $(p' \in \mathbb{N})$

and $4p'^2 = 2q^2 \Leftrightarrow 2p' = q^2$, also $q = 2q'$ $(q' \in \mathbb{N})$

$\gcd(p,q)=1$, but p and q have a common divisor 2

so there doesn't exist $x \in \mathbb{Q}$ such that $x^2=2$

3. we assume that $d = b-a$, there exist $k \in \mathbb{N}$ s.t $10^k > \frac{1}{d} \Leftrightarrow 10^k d > 1$

and $b \cdot 10^k - a \cdot 10^k = (b-a)10^k = d \cdot 10^k > 1$, so there exist $n \in \mathbb{Z}$ that $a \cdot 10^k < n < b \cdot 10^k$

so we can let $x = \frac{n}{10^k}$, so that $a < \frac{n}{10^k} < b$