1. (a) (10 points) Find all solutions of the system of linear equations:

$$\begin{array}{rcrcrcrcr} x_1 & + & & & 2x_3 & + & 4x_4 & = & -8 \\ & & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - & x_2 & + & 3x_3 & + & 4x_4 & = & -12 \end{array}$$

(b) (4 points) Identify the reduced row echelon form R of the coefficient matrix of the system.

Augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right] \xrightarrow[-3\text{Row }1]{\text{Row }3} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right] \xrightarrow[\text{Row }4+\text{Row }2]{\text{Row }3-4\text{Row }2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right] \xrightarrow[\text{Then: Row }3 \leftrightarrow \text{Row }4]{\frac{1}{3}\text{Row }4} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{Row }2+\text{Row }3]{\text{Row }1-4\text{Row }3}$$

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{This is } R} \rightsquigarrow \begin{array}{l} x_1 + 2x_3 = 0 \\ x_2 - 3x_3 = 4 \\ x_3 \text{ free} \\ x_4 = -2 \end{array}$$

All solutions: $\vec{x} = \begin{bmatrix} -2x_3 \\ 4+3x_3 \\ x_3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. (10 points) Find all 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that commute with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, that is,

$$A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A.$$

Show that every 2×2 matrix A that commutes with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ can be written as a linear combination of two particular 2×2 matrices.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$:

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}}$$

Need: $a+b = a+c$

$a+b = b+d$

$c+d = a+c$

$c+d = b+d$

$\downarrow$

$b = c, \; a = d$

The matrices that commute with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ look like:

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Every one of these matrices is a linear combination of these 2.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}.$$

(a) (6 points) Find the LU decomposition of A.

(b) (6 points) Use the LU decomposition to solve the linear system of equations $A\mathbf{x} = (1, 0, 0)$.

(a) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \xrightarrow[\text{Row 3} - \overset{1}{\text{Row 1}}]{\text{Row 2} - \overset{1}{\text{Row 1}}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} \xrightarrow{\text{Row 3} - \boxed{3}\,\text{Row 2}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

So $$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) $$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow L(\underbrace{U\vec{x}}_{\vec{y}}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solve $L\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ first, then $U\vec{x} = \vec{y}$:

$$\begin{aligned} y_1 \qquad\qquad &= 1 \\ y_1 + y_2 \qquad &= 0 \\ y_1 + 3y_2 + y_3 &= 0 \end{aligned} \longrightarrow \begin{aligned} y_1 &= 1 \\ y_2 &= -1 \\ y_3 &= -1 - 3(-1) = 2 \end{aligned}$$

Now solve $U\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= -1 \\ 2x_3 &= 2 \end{aligned} \longrightarrow \begin{aligned} x_1 &= 1 - (-3) - 1 = 3 \\ x_2 &= -1 - 2(1) = -3 \\ x_3 &= 1 \end{aligned}$$

So $\vec{x} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$.

4. (a) (12 points) Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(b) (2 points) Use A^{-1} to solve the system of equations $A\mathbf{x} = (0, 1, 0, 0)$.

Elimination method:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row 4 - Row 1}]{\text{Row 3 - Row 1}} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Row 3+Row 2}} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row 3 + Row 4}]{\text{Row 2 + 2 Row 3}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row 2+Row 3}} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Row 1 - Row 2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{array} \underbrace{\begin{array}{cccc} 5 & -2 & -1 & -3 \\ -4 & 2 & 1 & 3 \\ -2 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{array}}_{\text{This is } A^{-1}}\right]$$

(b) $A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \vec{x} = A^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -1 & -3 \\ -4 & 2 & 1 & 3 \\ -2 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

5. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \end{bmatrix}.$$

(a) (6 points) Find a linear relation on b_1, b_2, b_3 that guarantees that $\mathbf{b} = (b_1, b_2, b_3)$ is a vector in the column space $\mathbf{C}(A)$.

(b) (8 points) Find a spanning set (the special solutions) for the null space $\mathbf{N}(A)$.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & b_1 \\ -1 & -2 & -1 & 1 & -1 & b_2 \\ 4 & 8 & 5 & -8 & 9 & b_3 \end{array}\right] \xrightarrow[\text{Row 3} - 4\text{ Row 1}]{\text{Row 2} + \text{Row 1}}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & b_1 \\ 0 & 0 & 1 & -4 & 5 & b_1+b_2 \\ 0 & 0 & -3 & 12 & -15 & -4b_1+b_3 \end{array}\right] \xrightarrow{\text{Row 3} + 3\text{ Row 2}}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & b_1 \\ 0 & 0 & 1 & -4 & 5 & b_1+b_2 \\ 0 & 0 & 0 & 0 & 0 & \underbrace{-b_1+3b_2+b_3} \end{array}\right]$$

(a) Solutions will exist <u>only</u> if

$$-b_1+3b_2+b_3=0$$

Row 1 −2 Row 2

$$\underbrace{\begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{This is } R}$$

(b) Need to solve:

$$x_1 + 2x_2 + 3x_4 - 4x_5 = 0$$
$$x_3 - 4x_4 + 5x_5 = 0$$

The spanning set (special solutions)

$$N(A) = \text{all vectors like } \vec{x} = \begin{bmatrix} -2x_2 - 3x_4 + 4x_5 \\ x_2 \\ 4x_4 - 5x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

6. Given the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) (2 points) Without doing any calculations, explain why $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are not linearly independent.

(b) (2 points) Without doing any calculations, explain why $\mathbf{v}_1$ and $\mathbf{v}_2$ do not span $\mathbf{R}^3$.

(c) (4 points) Determine whether $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbf{R}^3$.

(d) (4 points) Determine whether $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ form a basis for $\mathbf{R}^3$.

(a) More than 3 vectors in $\mathbb{R}^3$ have to be dependent.

(b) Fewer than 3 vectors cannot span $\mathbb{R}^3$

(c) $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{bmatrix} \xrightarrow[\text{Row } 3 - 2\text{Row } 1]{\text{Row } 2 - 2\text{Row } 1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\text{Row } 3 - \text{Row } 2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Row of 0's can't span.

Free variable, not independent

Not a basis

(d) $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \end{bmatrix} \xrightarrow[\text{Row } 3 - 2\text{Row } 1]{\text{Row } 2 - 2\text{Row } 1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{Row } 3 - \text{Row } 2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

No free variable, no row of 0's

Is a basis.

7. (a) (8 points) Show that the set of vectors (b_1, b_2, b_3) with $b_1 = b_2$ is a subspace of $\mathbf{R}^3$. (Verify all three properties of a subspace.)

(b) (6 points) Show that the set of vectors (b_1, b_2, b_3) with $b_1 b_2 b_3 = 0$ is *not* a subspace of $\mathbf{R}^3$. (Show that at least one property of a subspace fails.)

(a) $S =$ all $\vec{b}$ with $b_1 = b_2$

1. Is $\vec{0}$ in S? $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow b_1 = b_2 = 0$ ✓

2. If $\vec{b}, \vec{c}$ in S, what about $\vec{b} + \vec{c}$?

$\vec{b} + \vec{c} = \begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix}$ Know $b_1 = b_2$ and $c_1 = c_2$, so $b_1 + c_1 = b_2 + c_2$. ✓

3. If $\vec{b}$ in S, what about $c\vec{b}$?

$c\vec{b} = \begin{bmatrix} cb_1 \\ cb_2 \\ cb_3 \end{bmatrix}$ Since $b_1 = b_2$, then $cb_1 = cb_2$ also ✓

(b) $T =$ all $\vec{b}$ with $b_1 b_2 b_3 = 0$

T is not closed under vector addition.

~~$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in T, but $(-1)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not because $-1 > 2 > 3$~~

$\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ are in T, but $\begin{bmatrix}1\\1\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$

is <u>not</u>, because $(1)(1)(1) \neq 0$.

8. (a) (6 points) How long is the vector $\mathbf{v} = (1, 1, \ldots, 1)$ in 9 dimensions? Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ and a unit vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

(b) (4 points) Pick any numbers x, y, z such that $x + y + z = 0$. Find the angle between your vector $\mathbf{v} = (x, y, z)$ and the vector $\mathbf{w} = (z, x, y)$.

(a) $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\underbrace{1+1+\ldots+1}_{9 \text{ times}}} = \sqrt{9} = 3$

Scale $\vec{v}$ by its length to get a unit vector:

$$\vec{u} = \underbrace{\left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)}_{9 \text{ times}}$$

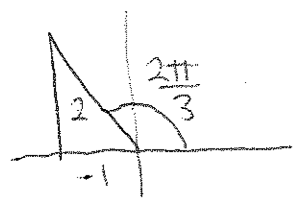
$\vec{w}$ perpendicular to $\vec{v}$: $\vec{v}\cdot\vec{w} = 0$.

Could try $(1, -1, \underbrace{0, \ldots, 0}_{7 \text{ times}})$, but this is not a unit vector.

Scale by length: $\vec{w} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \underbrace{0, \ldots, 0}_{7 \text{ times}}\right)$ works.

(b) $x = 1$, $y = 2$, $z = -3$ works.

Angle: $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \dfrac{9(1,2,-3)\cdot(-3,1,2)}{\sqrt{1+4+9}\sqrt{9+1+4}} = \dfrac{-7}{14} = -\dfrac{1}{2}$



Angle θ between (x, y, z) and (z, x, y) is $\frac{2\pi}{3}$