18. $\sum_{n=1}^{\infty} np_n = \sum_{n=1}^{\infty} n(r_n - r_{n+1}) = \sum_{n=1}^{\infty} nr_n - \sum_{n=1}^{\infty} nr_{n+1} = r_1 + \sum_{n=2}^{\infty} nr_n - \sum_{n=2}^{\infty}(n-1)r_n = \sum_{n=1}^{\infty} r_n$

$E(X) = \sum_{n=1}^{\infty} np_n = \sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} p_k\right) = \sum_{n=1}^{\infty} P(X \geq n)$.

20. 记 $X = \min(X_1, X_2, \cdots, X_m)$ 也为随机变量，

则 $E\{\min(X_1, X_2, \cdots, X_m)\} = E(X) = \sum_{n=1}^{\infty} P(X \geq n)$ (由18) $= \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} p_k\right)^m = \sum_{n=1}^{\infty} r_n^m$

21. $E(X) = \int_0^{\infty} uf(u)du = \int_0^{\infty} f(u)du \cdot \int_0^u dt = \int_0^{\infty} dt \int_t^{\infty} f(u)du = \int_0^{\infty} r(t)dt = \int_0^{\infty} r(u)du$
(交换积分次序)

由定义，$P(X \geq u) = \int_u^{\infty} f(t)dt = r(u)$，故 $E(X) = \int_0^{\infty} P(X \geq u)du = \int_0^{\infty} r(u)du$.

22. $E(X) = \int_0^{\infty} r(u)du = \int_0^{\infty}\int_u^{\infty} \lambda e^{-\lambda t} dt\, du = \int_0^{\infty}\left[-e^{-\lambda t}\Big|_u^{\infty}\right]du = \int_0^{\infty} e^{-\lambda u} du = -\frac{e^{-\lambda u}}{\lambda}\Big|_0^{\infty} = \frac{1}{\lambda}$

23. $E(T) = \int_0^{\infty} P(T \geq t)dt = \int_0^{\infty} P(T > t)dt = \int_0^{\infty} ae^{-\lambda t} + (1-a)e^{-\mu t} dt$

$= \frac{-ae^{-\lambda t}}{\lambda}\Big|_0^{\infty} - \frac{(1-a)e^{-\mu t}}{\mu}\Big|_0^{\infty} = \frac{a}{\lambda} + \frac{1-a}{\mu}$

$E(T^2) = \int_0^{\infty} P(T^2 \geq t)dt = \int_0^{\infty} P(T > \sqrt{t})dt = \int_0^{\infty} ae^{-\lambda\sqrt{t}} + (1-a)e^{-\mu\sqrt{t}} dt$

$= \int_0^{\infty} \frac{-2a\lambda\sqrt{t}}{\lambda^2} \cdot e^{-\lambda\sqrt{t}} + \frac{-2(1-a)\mu\sqrt{t}}{\mu^2} \cdot e^{-\lambda\sqrt{t}}\, d(-\lambda\sqrt{t}) = \frac{2a}{\lambda^2} + \frac{2(1-a)}{\mu^2}$

故 $\sigma^2(T) = E(T^2) - [E(T)]^2 = \frac{2a}{\lambda^2} + \frac{2(1-a)}{\mu^2} - \left(\frac{a}{\lambda} + \frac{1-a}{\mu}\right)^2$

24. 由 $f(t) = \lambda e^{-\lambda t}$，$t > 0$ 知 $P(T > t) = \int_t^{\infty} f(u)du = \int_t^{\infty} \lambda e^{-\lambda u} du = e^{-\lambda t}$

故 $P(T > n+t \mid T > n) = \frac{P(T > n+t, T > n)}{P(T > n)} = \frac{P(T > n+t)}{P(T > n)} = \frac{e^{-\lambda(n+t)}}{e^{-\lambda n}} = e^{-\lambda t}$

因此，$E(T \mid T > n) = \int_n^{\infty} P(T > u \mid T > n)du = \int_0^{\infty} P(T > n+t \mid T > n)dt = \int_0^{\infty} e^{-\lambda t}dt = \frac{1}{\lambda}$

29. $E(\min) = \int_0^1 P(\min \geq u)\, du = \int_0^1 (1-u)^n \cdot du = -\frac{(1-u)^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$

$E(\max) = \int_0^1 P(\max \geq u)du = \int_0^1 1 - P(\max < u)\, du = \int_0^1 1 - u^n du = \left(u - \frac{u^{n+1}}{n+1}\right)\Big|_0^1 = \frac{n}{n+1}$

$E(\text{range} = \max - \min) = E(\max) - E(\min) = \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}$

33. X_j 的生成函数 $g(z) = \frac{z}{4} + \frac{1}{2} + \frac{1}{4z}$，故 $S = \sum_{j=1}^{n} X_j$ 的生成函数为 $h(z) = [g(z)]^n = \left(\frac{z}{4} + \frac{1}{2} + \frac{1}{4z}\right)^n$

化简得，$h(z) = \frac{(z+1)^{2n}}{(4z)^n} = \frac{\sum_{x=0}^{2n}\binom{2n}{x}z^x}{(4z)^n}$，故 $P(S=0) = \frac{\binom{2n}{n} \cdot z^n}{(4z)^n} = \binom{2n}{n} \cdot \frac{1}{4^n}$ ①

记 A 第 j 次投掷的随机变量为 $A_j = \begin{cases} 1, & \frac{1}{2}\text{机率,面} \\ 0, & \frac{1}{2}\text{机率,底} \end{cases}$，B 则为 $B_j = \begin{cases} -1, & \frac{1}{2}\text{机率,面} \\ 0, & \frac{1}{2}\text{机率,底} \end{cases}$.

A_j 的生成函数 $g_1(z) = \frac{z}{2} + \frac{1}{2}$，$B_j$ 生成函数 $g_2(z) = \frac{1}{2z} + \frac{1}{2}$，故 $S = \sum_{k=0}^{n} A_k + \sum_{k=0}^{n} B_k$ 的生成函数为：

$g(z) = [g_1(z)]^n [g_2(z)]^n = \left(\frac{z}{2} + \frac{1}{2}\right)^n \cdot \left(\frac{1}{2z} + \frac{1}{2}\right)^n = \frac{(z+1)^{2n}}{(4z)^n} = \frac{\sum_{x=0}^{2n}\binom{2n}{x}z^x}{(4z)^n}$，现求面出现次数相等机率，即求 $P(S=0) = \binom{2n}{n}\frac{1}{4^n}$

(生成函数相同，故由①式可知)

37. (a) $E(e^{-\lambda X}) = \int_0^c e^{-\lambda u} f(u)\,du = \int_0^c e^{-\lambda u}\cdot\frac{1}{c}\,du = \left.\frac{e^{-\lambda u}}{-\lambda c}\right|_0^c = \frac{1-e^{-\lambda c}}{\lambda c}$, $\lambda>0$.

(b) $E(e^{-\lambda X}) = \int_0^c e^{-\lambda u}\cdot\frac{2u}{c^2}\,du = \left.\frac{2}{\lambda^2c^2}(-\lambda u-1)e^{-\lambda u}\right|_0^c = \frac{2(1-e^{-\lambda c}-\lambda c e^{-\lambda c})}{\lambda^2c^2}$, $\lambda>0$

(c) 先证 $f(x)$ 为密度函数.

则 $\int_0^\infty f(u)\,du = \int_0^\infty \frac{\lambda^n u^{n-1}}{(n-1)!}e^{-\lambda u}\,du = \frac{\lambda^n}{(n-1)!}\int_0^\infty u^{n-1}e^{-\lambda u}\,du$

$= \frac{\lambda^n}{(n-1)!}\left[\left.\frac{1}{-\lambda}u^{n-1}e^{-\lambda u}\right|_0^\infty + \frac{n-1}{\lambda}\int_0^\infty u^{n-2}e^{-\lambda u}\,du\right]$

$= \frac{\lambda^n}{(n-1)!}\cdot\frac{n-1}{\lambda}\int_0^\infty u^{n-2}e^{-\lambda u}\,du$

$= \frac{\lambda^{n-1}}{(n-2)!}\int_0^\infty u^{n-2}e^{-\lambda u}\,du$

同理，有 $\int_0^\infty u^{n-2}e^{-\lambda u}\,du = \frac{n-2}{\lambda}\int_0^\infty u^{n-3}e^{-\lambda u}\,du$，

不断降次，有 $\int_0^\infty f(u)\,du = \left.-e^{-\lambda u}\right|_0^\infty = 1$，为密度函数。

故 $L(\mu) = \int_0^\infty \frac{\lambda^n u^{n-1}}{(n-1)!}e^{-\lambda u}\cdot e^{-\mu u}\,du = \frac{\lambda^n}{(n-1)!}\int_0^\infty u^{n-1}e^{-(\lambda+\mu)u}\,du$

由 $\int_0^\infty f(u)\,du = \int_0^\infty \frac{\lambda^n u^{n-1}}{(n-1)!}e^{-\lambda u}\,du = 1$ 知 $\int_0^\infty u^{n-1}e^{-\lambda u} = \frac{(n-1)!}{\lambda^n}$ ①

故 $L(\mu) = \frac{\lambda^n}{(n-1)!}\int_0^\infty u^{n-1}e^{-(\lambda+\mu)u}\,du = \frac{\lambda^n}{(n-1)!}\cdot\frac{(n-1)!}{(\lambda+\mu)^n} = \frac{\lambda^n}{(\lambda+\mu)^n}$

38. 对于每个 T_j，其 Laplace 变换：$E(e^{-\mu T_j}) = \int_0^\infty e^{-\mu t}\cdot\lambda\cdot e^{-\lambda t}\,dt = \left.\frac{-\lambda e^{-(\mu+\lambda)t}}{\mu+\lambda}\right|_0^\infty = \frac{\lambda}{\mu+\lambda}$

由于 T_j 相互独立，故 S_n 的 Laplace 变换：$E(e^{-\mu S_n}) = E(e^{-\mu(T_1+\cdots+T_j)}) = E(e^{-\mu T_1})E(e^{-\mu T_2})\cdots E(e^{-\mu T_j})$

$= \left(\frac{\lambda}{\mu+\lambda}\right)^n$

设 S_n 的密度函数为 $f(t)$，则 $E(e^{-\mu S_n}) = \int_0^\infty e^{-\mu t}f(t)\,dt$.

故 $\int_0^\infty e^{-\mu t}f(t)\,dt = \frac{\lambda^n}{(\mu+\lambda)^n} = \lambda^n\cdot\frac{\int_0^\infty t^{n-1}e^{-(\lambda+\mu)t}\,dt}{(n-1)!}$ (由 37 ① 知) $= \int_0^\infty \frac{\lambda^n t^{n-1}}{(n-1)!}\cdot e^{-\lambda t}\cdot e^{-\mu t}\,dt$.

因此 $e^{-\mu t}f(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}\cdot e^{-\lambda t}\cdot e^{-\mu t} \Rightarrow f(t) = \frac{\lambda^n t^{n-1}e^{-\lambda t}}{(n-1)!}$.

所以 S_n 的分布函数 $P(a<x<b) = \int_a^b f(t)\,dt = \frac{\lambda^n}{(n-1)!}\cdot\int_a^b u^{n-1}\cdot e^{-\lambda u}\,du$.