2. (i) $P_\lambda(g) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & 3-\lambda & 2 \\ 0 & -1 & -\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = (3-\lambda)(\lambda-1)(\lambda-2)$

Let $P_\lambda(g)=0$, g have eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

(ii) Then find eigenvectors for each eigenvalue:

For $\lambda_1 = 1$, $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix} \cdot V_1 = 0$, just pick $V_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 2$, $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \cdot V_2 = 0$, pick $V_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

For $\lambda_3 = 3$, $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & -3 \end{bmatrix} V_3 = 0$, pick $V_3 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

so, let $h = \begin{bmatrix} 0 & 0 & -2 \\ -1 & -2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, so $h^{-1} = \begin{bmatrix} -1/2 & 1 & 2 \\ 1 & -1 & -1 \\ -1/2 & 0 & 0 \end{bmatrix}$

check $h^{-1} g h = \begin{bmatrix} -1/2 & 1 & 2 \\ 1 & -1 & -1 \\ -1/2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ -1 & -2 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonal matrix.

3. Since g is a upper triangular matrix, eigenvalues are the entries of the diagonal,

for g, g have eigenvalue $\lambda=3$ with multiplicity 4.

then calculate $g_\lambda = g-\lambda I = \begin{bmatrix} 0&1&1&0\\0&0&1&1\\0&0&0&1\\0&0&0&0 \end{bmatrix}$ $g_\lambda^2 = \begin{bmatrix} 0&0&1&2\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix}$ $g_\lambda^3 = \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$ $g_\lambda^4 = 0$

So $d_{3,1} = 4-3 = 1$, $d_{3,2} = 4-2 = 2$, $d_{3,3} = 4-1 = 3$, $d_{3,4} = 4-0 = 4$

which gives $0 < C_4 = C_3 = C_2 = C_1 = 1$

there exists 1 jump at $j=4$, gives 1 Jordan block $J_{3,4}$

So $g \sim (J_{3,4}) = \begin{bmatrix} 3&1&0&0\\0&3&1&0\\0&0&3&1\\0&0&0&3 \end{bmatrix}$

4. (i) $g_\mathbb{R}$ can be viewed as block matrix $g = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ where $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

we can pick $I_V = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

check $I_V^2 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} = -I$

then check $g \cdot I_V = I_V \cdot g$:

$$g \cdot I_V = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} B & A \\ -A & B \end{bmatrix} \text{ and } I_V \cdot g = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \begin{bmatrix} B & A \\ -A & B \end{bmatrix}$$

As $g \cdot I_V = I_V \cdot g$ and $I_V^2 = -I$, we can say T can be viewed as a linear operator for complex vector space $V_\mathbb{C}$.

(ii) For linear operator T, $T(e_1, e_2, ie_1, ie_2) = (e_1, e_2, ie_1, ie_2)\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & -1 & 3 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$

so $T(e_1) = 3e_1 + 2ie_2$ and $T(e_2) = -ie_1 + 2e_2$

because T can be viewed as a linear operator for complex vector space,

so $T_\mathbb{C}(e_1\, e_2) = (e_1\, e_2) \cdot g_\mathbb{C}$, as $T(e_1) = 3e_1 + 2ie_2$ and $T(e_2) = -ie_1 + 2e_2$

gives $g_\mathbb{C} = \begin{bmatrix} 3 & -i \\ 2i & 2 \end{bmatrix}$, $P_\lambda(g_\mathbb{C}) = \begin{vmatrix} 3-\lambda & -i \\ 2i & 2-\lambda \end{vmatrix} = (\lambda-1)(\lambda-4)$, eigenvalues are 1 and 4.

(both multiplicity 1).

(iii)
$$P_\lambda(g_\mathbb{R}) = \begin{vmatrix} 3-\lambda & 0 & 0 & 1 \\ 0 & 2-\lambda & -2 & 0 \\ 0 & -1 & 3-\lambda & 0 \\ 2 & 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 2-\lambda & -2 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 2-\lambda & -2 \\ 0 & -1 & 3-\lambda \\ 2 & 0 & 0 \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ -1 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2-\lambda & -2 \\ -1 & 3-\lambda \end{vmatrix}$$

$$= (\lambda^2 - 5\lambda + 6 - 2)\cdot(\lambda^2 - 5\lambda + 6 - 2)$$

$$= (\lambda^2 - 5\lambda + 4)^2$$

$$= (\lambda-1)^2(\lambda-4)^2$$

So eigenvalues for $g_\mathbb{R}$ are $\lambda_1 = 1$ (multiplicity 2) and $\lambda_2 = 4$ (multiplicity 2)