L11

2021年5月13日 19:06

△ Review inner product on $\mathbb{R}^3$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ $\underline{(\vec{x}, \vec{y})} = \vec{x} \cdot \vec{y} = \underline{x_1 y_1 + x_2 y_2 + x_3 y_3}$

it can be used to desribe when $\vec{x} \perp \vec{y} \Leftrightarrow (\vec{x}, \vec{y}) = 0$

it can give length / distance $\underline{\|\vec{x}\| = (\vec{x}, \vec{x})^{\frac{1}{2}}}$

properties for $(\ ,\ )$

- △ symmetric : $(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$
- △ linear in both variables : $((a_1 \vec{x}_1 + a_2 \vec{x}_2, \vec{y}) = \underline{a_1 (\vec{x}_1, \vec{y}) + a_2 (\vec{x}_2, \vec{y})}$

  similar in $\vec{y}$ position

△ (optional) positive definite : $(\vec{x}, \vec{x}) \geq 0$

with $(\vec{x}, \vec{x}) = 0$ iff $\vec{x} = 0$

$(\vec{x} + \vec{z}) \cdot \vec{y} = \vec{x} \cdot \vec{y} + \vec{z} \cdot \vec{y}$

linear in $\vec{x}$ $(\vec{x} + \vec{z}, \vec{y}) = \left(\begin{pmatrix} x_1 + z_1 \\ x_2 + z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)$

$\mathbb{R}^2$, $(\vec{x}, \vec{y}) \stackrel{\triangle}{=} x_1 y_1 + x_2 y_2$ $= (x_1 + z_1) \cdot y_1 + (x_2 + z_2) y_2$

$\vec{x}, \vec{y}, \vec{z} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdots \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ $= x_1 y_1 + z_1 y_1 + x_2 y_2 + z_2 y_2$

$= (\vec{x}, \vec{y}) + (\vec{z}, \vec{y})$

△ Review : transpose of a matrix : $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{T} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ (transpose)

$(x_1 \cdots x_n)^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^T = (x_1 \cdots x_n)$ $\begin{pmatrix} 1 & 2 & \boxed{3} \\ 4 & 5 & 6 \\ \boxed{7} & 8 & 9 \end{pmatrix}$

$g = (g_{ij})_{ij}$ $g^T = (g_{ji})$ ← i-row j-col element of g^T is $\underline{g_{ji}}$

Lem : $(AB)^T = B^T \cdot A^T$ $\left((1, 2) \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)^T = \begin{pmatrix} 3 \\ 4 \end{pmatrix}^T \cdot (1, 2)^T$

$1 \cdot 3 + 2 \cdot 4 = (3, 4) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Given V / C

Given $V/\mathbb{C}$

$\triangle$ Definition: a Complex inner product / Hermitian form is a map

$$\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}$$

$$v_1, v_2 \longmapsto \langle v_1, v_2\rangle$$

Strang Chap 9

Satisfying following properties

① Conjugate Symmetric: $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}$

② Sesquilinear: linear in 2nd Component $\langle v_1, a_1x_1+a_2x_2\rangle = a_1\langle v_1, x_1\rangle + a_2\langle v_1, x_2\rangle$

$a_1\ a_2 \in \mathbb{C}$

Conjugate linear in 1st Component $\langle a_1v_1+a_2v_2, x\rangle = \bar{a}_1\langle v_1, x\rangle + \bar{a}_2\langle v_2, x\rangle$

③ (optional) positive definite Hermitian form; $\langle v, v\rangle \geq 0$ is a real number

$\langle v, v\rangle = 0$ iff $v=0$

Example: $V=\mathbb{C}$ $\quad |z| = (\bar{z}\cdot z)^{\frac{1}{2}}$ $\quad \langle z_1, z_2\rangle = \bar{z}_1\cdot z_2$

Standard Hermitian form: $V=\mathbb{C}^n$ $\quad \vec{x} = (x_1 \cdots x_n)^T$ $\quad \vec{y} = (y_1 \cdots y_n)^T$ $\quad \boxed{\langle\vec{x}, \vec{y}\rangle \overset{\text{definition}}{=} \sum_i \bar{x}_i\cdot y_i}$

Check: $\langle\vec{y}, \vec{x}\rangle = \sum \bar{y}_i\cdot x_i = \overline{\left(\sum \bar{x}_i y_i\right)} = \overline{\langle\vec{x}, \vec{y}\rangle}$

$\langle\vec{x}, a\vec{y}\rangle = \sum \bar{x}_i\cdot(a y_i) = a\left(\sum \bar{x}_i\cdot y_i\right)$

$\langle a\vec{x}, \vec{y}\rangle = \sum \overline{(ax_i)}\, y_i = \bar{a}\left(\sum \bar{x}_i y_i\right)$

Check $\langle\vec{x}, \vec{x}\rangle = \sum \underbrace{\bar{x}_i\cdot x_i}_{\geq 0} \geq 0$ with $=0$ iff $x_i=0$ $\forall i$

$\triangle$ Fix a basis $B_V = \{e_1 \cdots e_n\}$, then Hermitian form can described in terms of matrix $H = (\langle e_i, e_j\rangle)_{ij}$

As B_V is basis $v = \sum_i a_i e_i$ $\quad \vec{w} = \sum_j b_j e_j$

$$\langle v, w\rangle \overset{\text{sesquilinearity}}{=} \sum_{i,j} \bar{a}_i b_j \langle e_i, e_j\rangle = (\bar{a}_1 \cdots \bar{a}_n) H \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\text{RHS} = \left((\bar{a}_1 \cdots \bar{a}_n) H\right)\cdot\begin{pmatrix} b_1 \\ \vdots \\ \vdots \end{pmatrix} = \sum_j \left(\sum_i \bar{a}_i\langle e_i, e_j\rangle\right) b_j$$

$$RHS = \left(\underbrace{\overline{(a_1 \cdots a_n)}}\, H\right) \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_j \left(\sum_i \bar{a}_i \langle e_i, e_j \rangle \right) b_j$$

j - column of it $= \sum \bar{a}_i \langle e_i, e_j \rangle$

△ Lemma: $v = (e_1 \cdots e_n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\quad w = (e_1 \cdots e_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$H = (\langle e_i, e_j \rangle)_{ij}$. then $\langle v, w \rangle = \overline{(a_1 \cdots a_n)}\, H \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$\shortparallel \quad \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}^*$

△ Definition: Given g, $g^* = \overline{(g^T)}$

Lemma: $H^* = H$

Pf: by def, $\underline{(H^*)_{ij}} = \left(\overline{H^T}\right)_{ij} = \overline{(H^T)_{ij}} = \overline{(H)_{ji}}$

$= \overline{\langle e_j, e_i \rangle} = \langle e_i, e_j \rangle = \underline{(H)_{ij}} \quad \Rightarrow H^* = H$ #

△ Def alternatively: Hermitian matrix is any H s.t. $\underline{H^* = H}$ △

Hermitian form $\xleftrightarrow{\text{basis}}$ Hermitian matrix

△ Def: $v_1 \perp v_2$ iff $\langle v_1, v_2 \rangle = 0$ iff $\langle v_2, v_1 \rangle = 0$

Def: ONB = $\boxed{\text{Orthonormal basis}}$ for $\langle\, , \rangle$ is $\{e_1 \cdots e_n\}$ s.t.

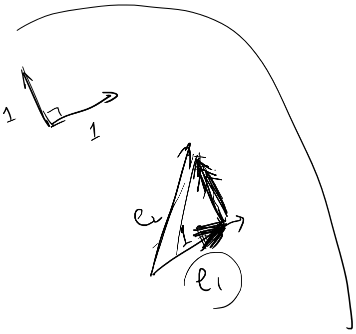
$\langle e_i, e_i \rangle = 1 \qquad \langle e_i, e_j \rangle = 0 \quad i \neq j$

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise } (i \neq j) \end{cases}$$

△ Lem: ONB always exists of positive definite Hermitian forms.

Pf: Gram-schmidt algorithm.

Given any basis $\{e_1 \cdots e_n\} \longrightarrow$ ONB



Step 1: $e_1^\circ = \dfrac{e_1}{\langle e_1, e_1\rangle^{\frac12}}$ $\quad \langle e_1^\circ, e_1^\circ\rangle = \dfrac{1}{\langle e_1, e_1\rangle}\langle e_1, e_1\rangle = 1$

Step 2: $e_2' = e_2 - \langle e_2, e_1^\circ\rangle e_1^\circ$

Claim: $e_2' \perp e_1^\circ$ indeed $\langle e_2', e_1^\circ\rangle$

$= \langle e_2, e_1^\circ\rangle - \langle e_2, e_1^\circ\rangle \underbrace{\langle e_1^\circ, e_1^\circ\rangle}_{1} = 0$

$e_2^\circ = \dfrac{e_2'}{\langle e_2', e_2\rangle^{\frac12}}$

Step 3: $e_3' = e_3 - \langle e_3, e_1^\circ\rangle e_1^\circ - \langle e_3, e_2^\circ\rangle e_2^\circ$

$e_3^\circ = \dfrac{e_3'}{\langle e_3, e_3\rangle^{\frac12}}$ $\qquad$ Finish in finite steps #

△ Change basis. $(e_1 \cdots e_n)$ $\quad \langle\, ,\, \rangle \to H = (\langle e_i, e_j\rangle)_{ij}$

new basis $(f_1 \cdots f_n) = (e_1 \cdots e_n)\, h$ $\quad \searrow \quad H' = (\langle f_i, f_j\rangle)_{ij}$

Lem: $H' = h^* \cdot H \cdot h$ $\qquad f_i = \sum e_k h_{ki}$ $\quad f_j = \sum e_l h_{lj}$

Pf (Sketch): $(H')_{ij} = (h^* H \cdot h)_{ij}$

LHS: $\langle f_i, f_j\rangle = \sum\sum \overline{h_{ki}}\, h_{lj} \langle e_k, e_l\rangle$

$= \text{RHS}$

△ Fix the standard Hermitian form $\langle\, ,\, \rangle$

Def: a unitary matrix g, is such that $g^* \cdot g = I$

Lemma: g is a unitary matrix iff $\langle v, w\rangle = \langle gv, gw\rangle$ for any $v, w \in V$

Pf: For the standard $\langle\, ,\, \rangle$ and standard basis $(e_1 \cdots e_n)$

on ONB

H associated to the $\{e_i\}$ $\left(\langle e_i, e_j\rangle\right)_{ij} = (\delta_{ij}) = I$

Previous lemma $\Rightarrow$ $\langle v, w\rangle = \underline{H} \cdot \vec{w} = \vec{v}^* \cdot \vec{w}$

$\langle gv, gw\rangle = (g\vec{v})^* \cdot (g\vec{w}) = \vec{v}^* \cdot g^* \cdot g \cdot \vec{w}$

$(g\vec{v})^* = \overline{(g\vec{v})^T} = \overline{\vec{v}}^T \cdot \overline{g}^T$

$\langle gv, gw\rangle = \langle v, w\rangle$ iff $\underline{\vec{v}}^* \underline{g^* g} \vec{w} = \vec{v}^* \cdot \vec{w}$ $\forall v, w$

iff $g^* g = I$

$\triangle$ Example $g = \begin{pmatrix} e^{i\theta_1} & \\ & e^{i\theta_2} \end{pmatrix}$ is unitary as $g^* = \begin{pmatrix} e^{-i\theta_1} & \\ & e^{-i\theta_2} \end{pmatrix}$

$\parallel$

$= \begin{pmatrix} \cos\theta_1 + i\sin\theta_1 & \\ & \cos\theta_2 + i\sin\theta_2 \end{pmatrix}^* \longrightarrow \begin{pmatrix} \cos\theta_1 - i\sin\theta_1 & \\ & \cos\theta_2 - i\sin\theta_2 \end{pmatrix}$

$\triangle$ Corollary: $\{e_1, \cdots e_n\}$ is ONB for $\overset{\text{standard}}{\langle\, ,\rangle}$, and g is unitary then $\{\underline{ge_1}, \cdots \underline{ge_n}\}$ is still ONB

$\triangle$ Thm: Fix standard $\langle\, ,\rangle$, given an Hermitian matrix H

① $\exists$ unitary $\underline{h}$, $\quad h^* h = I \quad h^{-1} = h^*$ s.t. $\underline{h^* H h}$ $\overset{h^{-1}Hh}{=}$ is diagonal

with diagonal entries being real numbers $\left(\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}^* = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \right.$

$\left. \overline{a_i} = a_i \Rightarrow \text{real } \# \right)$

$\quad \overset{\shortparallel}{x + iy}$

rephrasing ② $\exists$ another $\underline{\text{ONB}}$ $\{f_1, \cdots f_n\}$ s.t.

f_i are eigenvectors for H

δ Definition: For any subspace $W \subset V$, V with $\langle\, ,\rangle$

define $W^{\perp} = \{ v \in V \mid \langle v, w \rangle = 0 \quad \forall w \in W \}$