Linear Algebra – Fall 2020
Midterm Exam

NAME:

STUDENT ID:

Instructions:

- Show answers and arguments in the space provided. You may use the back of the pages also, but indicate clearly any such material that you want graded. Answers given without supporting work may receive zero credit.
- This is a closed book exam: no calculators, notes, or formula sheets.

QUESTION	POINTS	SCORE
1	14	
2	10	
3	12	
4	14	
5	14	
6	12	
7	14	
8	10	
TOTAL	100	

1. (a) (10 points) Find all solutions of the system of linear equations:

$$\begin{array}{rcrcrcrcr} x_1 & + & & & 2x_3 & + & 4x_4 & = & -8 \\ & & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - & x_2 & + & 3x_3 & + & 4x_4 & = & -12 \end{array}$$

(b) (4 points) Identify the reduced row echelon form R of the coefficient matrix of the system.

2. (10 points) Find all 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that commute with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, that is,

$$A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A.$$

Show that every 2×2 matrix A that commutes with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ can be written as a linear combination of two particular 2×2 matrices.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}.$$

(a) (6 points) Find the LU decomposition of A.

(b) (6 points) Use the LU decomposition to solve the linear system of equations $A\mathbf{x} = (1, 0, 0)$.

4. (a) (12 points) Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(b) (2 points) Use A^{-1} to solve the system of equations $A\mathbf{x} = (0, 1, 0, 0)$.

5. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \end{bmatrix}.$$

(a) (6 points) Find a linear relation on b_1, b_2, b_3 that guarantees that $\mathbf{b} = (b_1, b_2, b_3)$ is a vector in the column space $\mathbf{C}(A)$.

(b) (8 points) Find a spanning set (the special solutions) for the null space $\mathbf{N}(A)$.

6. Given the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) (2 points) Without doing any calculations, explain why $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are not linearly independent.

(b) (2 points) Without doing any calculations, explain why $\mathbf{v}_1$ and $\mathbf{v}_2$ do not span $\mathbf{R}^3$.

(c) (4 points) Determine whether $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbf{R}^3$.

(d) (4 points) Determine whether $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ form a basis for $\mathbf{R}^3$.

7. (a) (8 points) Show that the set of vectors (b_1, b_2, b_3) with $b_1 = b_2$ is a subspace of $\mathbf{R}^3$. (Verify all three properties of a subspace.)

   (b) (6 points) Show that the set of vectors (b_1, b_2, b_3) with $b_1 b_2 b_3 = 0$ is *not* a subspace of $\mathbf{R}^3$. (Show that at least one property of a subspace fails.)

8. (a) (6 points) How long is the vector $\mathbf{v} = (1, 1, \ldots, 1)$ in 9 dimensions? Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ and a unit vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

   (b) (4 points) Pick any numbers x, y, z such that $x + y + z = 0$. Find the angle between your vector $\mathbf{v} = (x, y, z)$ and the vector $\mathbf{w} = (z, x, y)$.

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