L3

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Recall vectorspace $V/\mathbb{R}$, $V/\mathbb{C}$, V/F

You can $\begin{cases} \text{add two vectors} & v_1+v_2 \in V \quad v \quad -v \in V, \ 0 \in V \\ \text{scalar multiplication} & v \in V, \ c \in \mathbb{R}, \mathbb{C}, F \quad c v \in V \end{cases}$

$c\cdot(v_1+v_2) = c\cdot v_1 + c v_2$

$(c_1 \cdot c_2)\cdot v = c_1 \cdot (c_2 v)$

Linear operator $T: V \to W$ $\quad V, W / \mathbb{C}$

$\begin{cases} T(v_1+v_2) = Tv_1 + Tv_2 \\ T(c\cdot v) = c\cdot Tv \end{cases}$

△ $\mathbb{C} = \mathbb{R} \oplus \mathbb{R} i \simeq \mathbb{R}^2$ as a $V/\mathbb{R}$ $\quad \dim_{\mathbb{R}} \mathbb{C} = 2$

In general Question: $V/\mathbb{C}$ What is it like when viewed as $V/\mathbb{R}$

For example: $\dim_{\mathbb{C}} V = n$ what is $\dim_{\mathbb{R}} V$? $2n$

△ Lemma: If $\dim_{\mathbb{C}} V = n$, with basis $B_{\mathbb{C}} = \{e_1, e_2, \cdots e_n\}$

Then as $V/\mathbb{R}$ we have $\dim_{\mathbb{R}} V = 2n$ with basis $B_{\mathbb{R}} = \{e_1, ie_1, e_2, ie_2, \cdots\}$

Pf: Just need to show $B_{\mathbb{R}}$ is actually a basis $\begin{cases} ① \text{ linearly independent} \\ ② \text{ any } v \in V, \ v = \sum a_k e_k + \sum b_j \, i e_j \end{cases}$

$B_{\mathbb{C}} = \{e_1\}$

For simplicity check $\dim_{\mathbb{C}} V = 1$ $\quad B_{\mathbb{R}} = \{e_1, ie_1\}$ $\quad a_k, b_j \in \mathbb{R}$

To check ①: If $a e_1 + b i e_1 = 0$ $(\ast)$, $a, b \in \mathbb{R}$, Need to prove $a = b = 0$

Multiply i to your equation $\quad a \cdot ie_1 - b e_1 = 0$ (Δ)

$(\ast) \times a - (\Delta) \times b$

$(a^2+b^2)e_1 = a^2 e_1 - (-b^2 e_1) = 0 \qquad e_1 \neq 0$

$\Rightarrow a^2 + b^2 = 0 \qquad a, b \in \mathbb{R} \Rightarrow a = b = 0$

② Need: $\forall v \in V$, $v = a\cdot e_1 + b \cdot ie_1$ for some $a, b \in \mathbb{R}$.

Use that $B_{\mathbb{C}} = \{e_1\}$ is a basis for $V/\mathbb{C}$

$\Rightarrow v = z \cdot e_1 \qquad z = x + iy$

$= (x+iy)e_1 = x e_1 + y \, i e_1$ #

△ $V/\mathbb{C} \to V/\mathbb{R}$ think about i means for $V/\mathbb{R}$

$\begin{cases} i\cdot(v_1+v_2) = i\cdot v_1 + i \cdot v_2 \\ i\cdot (c v_1) = c\cdot(i v_1) \end{cases}$ $\quad i$ is a linear operator $/\mathbb{R}$

This gives an identification between i & $I_V \in GL_{2n}(\mathbb{R})$

such that $i \cdot v = I_V \cdot v$ for any $v \in V$.

$\{ g \in M_{2n\times 2n}(\mathbb{R}), \det g \neq 0\}$

General Linear group

Lemma: $I_V^2 = -I$

Pf: $i^2 = -1 \qquad i \cdot i \cdot v = I_V(I_V v)$

$-v = I_V^2 \cdot v$

$\Rightarrow I_V^2 = -I$ #

△ Question: When a vector space $V/\mathbb{R}$ can be viewed as a vectorspace $/\mathbb{C}$?

Necessary that $\dim_{\mathbb{R}} V = 2n$ is even

Lemma: $V/\mathbb{R}$ is $V/\mathbb{C}$ if and only if $\dim_{\mathbb{R}} V = 2n$.

"$\Rightarrow$" is already done

Pf: "$\Leftarrow$" Claim: If we are given a matrix J s.t. $J^2 = -I$

By definition

$i \cdot (e_1, ie_1, e_2, ie_2) = (e_1, ie_1, e_2, ie_2) I_V$

$(ie_1, -e_1, ie_2, -e_2) = (e_1, ie_1, e_2, ie_2) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

In general $I_V = \begin{pmatrix} \begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix} & & \\ & \begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix} & \\ & & \ddots \end{pmatrix}$ $\quad$ blockwisely diagonal, each block $= \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

Check: $I_V^2 = -I$ $\quad$ only need to check $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

△ Another way to organize $B_{\mathbb{R}} = \{e_1, e_2, \cdots e_n, ie_1, ie_2, \cdots ie_n\}$

$i(e_1, e_2, ie_1, ie_2) = (e_1, e_2, ie_1, ie_2) \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$(ie_1, ie_2, -e_1, -e_2)$

$I_V = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ in general $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$

I_n = $n \times n$ identity matrix

Lemma: $V/\mathbb{R}$ is $V/\mathbb{C}$ if and only if $\dim_{\mathbb{R}} V = 2n$.

"$\Rightarrow$" is already done

Pf: "$\Leftarrow$" Claim: If we are given a matrix J s.t. $J^2 = -I$ and $\dim_{\mathbb{R}} V = 2n$, then V can be viewed as $V/\mathbb{C}$

existence of J given here

Pf of claim: Just need to define scalar multiplication by $\mathbb{C}$

$$(a+bi)\cdot v := a\cdot v + b\cdot Jv$$

Check: it satisfies all required properties for scalar multiplication

$$(i\cdot i)v \overset{?}{=} i(iv)$$

$(i\cdot i)v = -1\cdot v$, $i(iv) = J\cdot(Jv) = -Iv$

Δ $T: V/\mathbb{C} \to V/\mathbb{C} \Rightarrow T_{\mathbb{R}}: V/\mathbb{R} \to V/\mathbb{R}$

Lem: Given $T_{\mathbb{R}}: V/\mathbb{R} \to V/\mathbb{R}$, $V/\mathbb{R}$ viewed as $V/\mathbb{C}$ ($\dim_{\mathbb{R}} V = 2n$, Given I_v $I_v^2 = -I$). Then $T_{\mathbb{R}}$ can be viewed as a linear operator over $\mathbb{C}$ iff $T_{\mathbb{R}} \circ I_v = I_v \circ T_{\mathbb{R}}$

Rmk: $\forall v \in V$ $T_{\mathbb{R}}(I_v v) = I_v(T_{\mathbb{R}} v)$ meaning

Recall $T/\mathbb{C}$ satisfies $\boxed{T(i\cdot v) = i\cdot Tv}$

$\Updownarrow$ Direct translation

$$T_{\mathbb{R}}(I_v v) = I_v(T_{\mathbb{R}} v)$$

Δ Motivation: $V/\mathbb{R}$ why care about viewing it as $V/\mathbb{C}$?

Δ halve the dimensions Δ richer structures

. Example: given $V/\mathbb{R}$ $\dim_{\mathbb{R}} V = 2$. V = xy-plane $v \in V$ $v = (x, y)$

$|v| = \sqrt{x^2+y^2}$

$$T = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

Problem: Want to show, for any $v \in V$, $v \neq 0$, $\dfrac{|Tv|}{|v|}$ is constant.

Solution: $\dim_{\mathbb{R}} V = 2$, V can viewed as 1-dim $\mathbb{C}$-space

We can just say $V = \mathbb{C}$ $v \leftrightarrow z = x+iy$

$|v| = |z|$

$V/\mathbb{C}$ comes with $I_v = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

We rewrite $T = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} = \underline{3I + 1I_v}$

Check $T \circ I_v = I_v \circ T$ So T is a linear operator $/\mathbb{C}$

T corresponds to scalar multiplication by $3+i$

$$\frac{|Tv|}{|v|} = \frac{|(3+i)\cdot v|}{|v|} = \frac{|3+i|\cdot|v|}{|v|} = |3+i| = \sqrt{3^2+1^2} = \sqrt{10}$$

Δ We have done: (i) As long as we have $\dim_{\mathbb{R}} V = 2n$, $J^2 = -I$ $\Rightarrow$ Complex structure on V

(ii) Constructed explicitly I_v $I_v^2 = -I$

Question: Whether all such J is conjugate to I_v?

Lemma: Yes J is to I_v

Pf: J conjugate to I_v $\Leftrightarrow$ After a change of basis J acts like I_v

$\{e_1, \ldots e_n, ie_1, \ldots ie_n\}$ $\begin{pmatrix} & -I \\ I & \end{pmatrix}$

Need to show for an abstract linear operator J, we can choose a basis

St. J acts exactly like I_v