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Tsinghua University

§ 5.3 Exercise 77

(a) $U=(max_1+max_2+max_3+\dots+max_n)\Delta x$. where $max_1=f(x_1)$, $max_2=f(x_2)$ .., $max_n=f(x_n)$ cause f is increasing in $[a,b]$.

$L=(min_1+min_2+min_3+\dots+min_n)\Delta x$. where $min_1=f(x_0)$, $min_2=f(x_1)$ ..., $min_n=f(x_{n-1})$ cause f is increasing in $[a,b]$.

$$
\begin{aligned}
U-L &= ((max_1-min_1)+(max_2-min_2)+\dots+(max_n-min_n))\Delta x \\
&= ((f(x_1)-f(x_0))+(f(x_2)-f(x_1))+\dots+(f(x_n)-f(x_{n-1})))\Delta x \\
&= (f(x_n)-f(x_0))\Delta x = (f(b)-f(a))\Delta x
\end{aligned}
$$

(b) $U=max_1\Delta x_1+max_2\Delta x_2+max_3\Delta x_3+\dots+max_n\Delta x_n$. where $max_1=f(x_1)$, $max_2=f(x_2)$ ..., $max_n=f(x_n)$ cause f is increasing in $[a,b]$

$L=min_1\Delta x_1+min_2\Delta x_2+min_3\Delta x_3+\dots+min_n\Delta x_n$, where $min_1=f(x_0)$, $min_2=f(x_1)$, ..., $min_n=f(x_{n-1})$ cause f is increasing in $[a,b]$,

$$
\begin{aligned}
U-L &= (max_1-min_1)\Delta x_1+(max_2-min_2)\Delta x_2+\dots+(max_n-min_n)\Delta x_n \\
&= (f(x_1)-f(x_0))\Delta x_1+(f(x_2)-f(x_1))\Delta x_2+\dots+(f(x_n)-f(x_{n-1}))\Delta x_n \\
&\le (f(x_1)-f(x_0))\Delta x_{max}+(f(x_2)-f(x_1))\Delta x_{max}+\dots+(f(x_n)-f(x_{n-1}))\Delta x_{max} \\
&= (f(b)-f(a))\Delta x_{max} = |f(b)-f(a)|\Delta x_{max} \text{ cause } f(b)>f(a)
\end{aligned}
$$

so $U-L\le|f(b)-f(a)|\Delta x_{max}$

Then $\lim\limits_{\|P\|\to 0}(U-L)=\lim\limits_{\|P\|\to 0}|f(b)-f(a)|\Delta x_{max}=0$ cause $\Delta x_{max}=\|P\|$

§ 5.4 Exercise 68

a. True

$h(x) = \int_0^x f(t)\,dt \Rightarrow h'(x) = f(x)$ (FTC part 1)

since f is differentiable for all x, h has a second derivative at x

b. True

since $h'(x)$ and $h(x)$ are differentiable, they are both continue

c. True

since $h'(1) = f(1) = 0$

d. True

since $h'(1) = f(1) = 0$ and $h''(1) = f'(1) < 0$

e. False

since $h''(1) = f'(1) < 0$

f. False

since $h''(1) = f'(1) \neq 0$, the sign of h doesn't change at 1

g. True

since $h'(1) = f(1) = 0$, and $h'(x)$ is decreasing function (because $f'(x) < 0$)

§ 5.5 Exercise 52

Let $u = \sqrt{\theta}$, $du = \frac{1}{2\sqrt{\theta}} d\theta$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta \cos^3\sqrt{\theta}}} d\theta = 2\int \frac{\sin u}{\sqrt{\cos^3 u}} du$$

Let $t = \cos u$. $dt = -\sin u \, du$

$$2\int \frac{\sin u}{\cos^3 u} du = -2\int \frac{1}{t^{\frac{3}{2}}} dt = -2 \cdot (-2 \frac{1}{\sqrt{t}}) + c = \frac{4}{\sqrt{t}} + c = \frac{4}{\sqrt{\cos\sqrt{\theta}}} + c$$

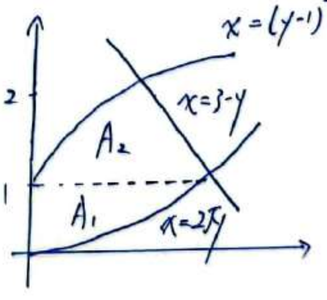
§ 5.6 Exercise 78

Limit of the integration:

1. $(y-1)^2 = 3-y \Rightarrow y=2$ or $y=-1 \Rightarrow y=2$ cause $y>0$
2. $2\sqrt{y} = 3-y \Rightarrow y=1$

so we can separate the graph in two part:

$$A_1 = \int_0^1 2\sqrt{y}\, dy = 2\left[\frac{2y^{\frac{3}{2}}}{3}\right]_0^1 = \frac{4}{3}$$



$$A_2 = \int_1^2 ((3-y)-(y-1)^2)\, dy = \int_1^2 (-y^2+y+2)\, dy$$

$$= \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y\right]_1^2 = \left(-\frac{8}{3} + 2 + 4\right) - \left(-\frac{1}{3} + \frac{1}{2} + 2\right)$$

$$= \frac{7}{6}$$

Therefore, the total area is $\frac{4}{3} + \frac{7}{6} = \frac{5}{2}$

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§ 5.6 Exercise 87

Let $u = a - x$, $du = -dx$; if $x = 0$, $u = a$, if $x = a$, and $u = 0$

Hence. $\int_0^a \frac{f(x)dx}{f(x)+f(a-x)} = \int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du) = \int_0^a \frac{f(a-u)du}{f(a-u)+f(u)} = \int_0^a \left(1 - \frac{f(u)\,du}{f(a-u)+f(u)}\right)$

$= a - I$ #

so $I = a - I$, $I = \frac{a}{2}$

Bonus exercise

B1.

left side: $\frac{d}{dx}\int_0^x \left(\int_0^u f(t)dt\right)du = \int_0^x f(t)dt$

right side: $\frac{d}{dx}\int_0^x f(u)(x-u)du = \frac{d}{dx}\int_0^x f(u)x\,du - \frac{d}{dx}\int_0^x f(u)u$

$= \frac{d}{dx}\left[x\int_0^x f(u)du\right] - xf(x) = \int_0^x f(u)du + x\cdot\left(\frac{d}{dx}\int_0^x f(u)du\right) - xf(x)$

$= \int_0^x f(u)du + xf(x) - xf(x) = \int_0^x f(u)du$

Since each side has the same derivative, and we know that both side equal to 0 when $x = 0$, so the constant must be 0,

Therefore. $\int_0^x \left(\int_0^u f(t)dt\right)du = \int_0^x f(u)(x-u)du$