Last time introduced complex numbered $\mathbb{C} = \{x + iy, \ x, y \in \mathbb{R}, i^2 = -1\}$

What is 'good' function? $P(z)$; $\frac{1}{P(z)}$ $P(z) \neq 0$

△ Introduce another example: $f(z) = e^z$ $f(x) = e^x$ $x \in \mathbb{R}$

Taylor: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \frac{x^n}{n!} + \cdots$ $\to 0$

Define $e^z = 1 + z + \frac{z^2}{2} + \cdots \frac{z^n}{n!}$ Converges for any z

$z^2 = z \cdot z$ $|z|$ $|z^n| = |z|^n$

$e^{x_1 + x_2} = e^{x_1} \cdot e^{x_2}$

Expect $\boxed{e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}}$

$i^3 = i^2 \cdot i = -i$ $i^4 = 1$

△ $z = i\theta$ $\theta \in \mathbb{R}$

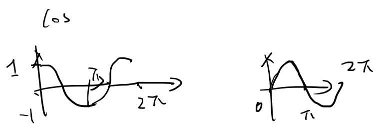
$$e^{i\theta} = 1 + i\theta + \frac{(-1)\theta^2}{2} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

Recall $\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$

$\cdot i$ $\cdot i$

$e^{i\theta} = \cos\theta + i \sin\theta$



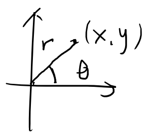
Both periodic
period $= 2\pi$

take $\theta = \pi$ $e^{i\pi} = \cos(\pi) + i\sin(\pi)$

$= -1 + i\,0 = -1$

Euler's identity $\boxed{e^{i\pi} = -1}$

△ $z = x + iy \longrightarrow (x, y)$ in xy-plane



polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$z = x + iy = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta) = re^{i\theta}$ $r = |z|$

$z_i = r_i \cdot e^{i\theta_i}$

$$z_1 \cdot z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) \overset{\text{not proved}}{=} r_1 r_2 \, e^{i(\theta_1+\theta_2)}$$

Geometric meaning $z_1 \cdot z_2$: $\begin{cases} \text{multiply lengths} \\ \text{add angles} \end{cases}$

△ Lemma $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

Pf: $e^{i\theta} = \cos\theta + i\sin\theta$

$$\text{LHS} = (\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2)$$

$$= (\underbrace{\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \sin\theta_2}_{\shortparallel}) + i(\underbrace{\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}_{\shortparallel})$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

$$= e^{i(\theta_1+\theta_2)}$$

#

△ Definition: $f: U \to \mathbb{C}$ $U \subset \mathbb{C} = \mathbb{R}^2$ function

Say f is holomorphic at a point $z_0 \in U$ if

$$\left.\frac{\partial f}{\partial \bar z}\right|_{z_0} = 0$$

More explicitly (equivalently) $\left.\frac{\partial f}{\partial x} + i \cdot \frac{\partial f}{\partial y}\right|_{z_0} = 0$

Chain law of partial derivative

Roughly speaking: no $\bar z$ part in $f(z)$

$f(z) = 3z^2 + 2z + 1$ ✓ $f(z) = z^2 + \bar z^2$ ✗

$f(z) = \dfrac{1}{3z^2+3z+1}$ ✓

e^z ✓

$f(z) = |z| = \sqrt{z \cdot \bar z}$ ✗

$\begin{cases} z = x+iy \\ \bar z = x - iy \end{cases}$ $(z, \bar z) \longleftrightarrow (x, y)$ $\begin{cases} x = \dfrac{z+\bar z}{2} \\ y = \dfrac{z-\bar z}{2i} \end{cases}$

$$\begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases} \qquad (z, \bar{z}) \longleftrightarrow (x, y) \qquad \begin{cases} x = \frac{z+\bar{z}}{2} \\ y = \frac{z - \bar{z}}{2i} \end{cases}$$

$$f(x,y) = x^2 + y^2 = z \cdot \bar{z} \quad \times \qquad \frac{\partial f}{\partial x} + i \frac{\partial}{\partial y} f \stackrel{?}{=} 0$$

$$= 2x + i\, 2y \neq 0$$

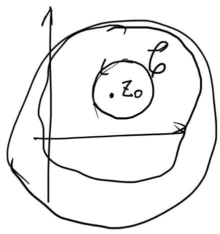
$$f(x,y) = x + iy = z \quad \checkmark \qquad \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \stackrel{?}{=} 0$$

$$= 1 + i(i) = 1 - 1 = 0$$

△ Properties for holomorphic functions

'good'

$f : U \to \mathbb{C}$ holomorphic function on U



Cauchy integration formula: $f(z_0)$ = average of $f(z)$ on $\mathcal{C}$

Corollary (Maximal modulus principle) $\quad |f(z_0)| \le \max\limits_{z \in \mathcal{C}} |f(z)| \qquad U$

△ Thm (FTA): $P(z) \in \mathbb{C}[z]$ polynomial of complex coefficients. Then $P(z) = 0$ always has a solution $z \in \mathbb{C}$.

Pf: prove by contradiction. Assume $P(z) \neq 0 \quad \forall z \in \mathbb{C}$

Suppose $\deg P(z) = n \qquad P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$

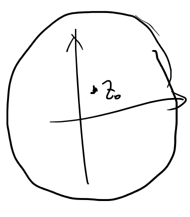
$\frac{1}{P(z)}$ is holomorphic function for $\forall z \in \mathbb{C}$.

△ note that when $|z| \to +\infty$ Claim $P(z) \sim a_n z^n (1 + \text{errors})$

$$\lim_{|z| \to +\infty} \frac{P(z)}{z^n} = a_n$$

$$\left| \frac{1}{P(z)} \right| \sim \frac{1}{|a_n z^n (1 + \text{error})|} \to 0$$

△ $\quad \left| \frac{1}{P(z_0)} \right| \le \max\limits_{z \in \mathcal{C}} \left| \frac{1}{P(z)} \right| \to 0 \qquad \mathcal{C}$ = circle



maximal

But if we let the radius of $\mathcal{C} \to +\infty$ $\quad [\ \ldots\ |z| \to +\infty]$

$z \in \mathcal{C}$ $\quad$ $\mathcal{C}$ = Circle

maximal modulus principle for $\frac{1}{p(z)}$

But if we let the radius of $\mathcal{C} \to +\infty$ $\quad$ ($z \in \mathcal{C}$ $|z| \to +\infty$)

$\mathcal{C}$ is a circle of radius N, center 0 $\quad$ N large

$$\left|\frac{1}{p(z_0)}\right| \le \max_{z \in \mathcal{C}} \left|\frac{1}{p(z)}\right| = \max_{|z|=N} \left|\frac{1}{a_n z^n}\right|$$

$$\le \frac{1}{a_n N^n} \quad \text{for any large } N$$

$\downarrow$

$\mathbb{R}_{\ge 0}$

$$|\gamma| \le \frac{1}{a_n N^n} \quad \text{for } \forall N$$

Only possible if $\gamma = 0$

But $\gamma = \frac{1}{p(z_0)}$ $\quad$ $\gamma = 0$ not possible $\quad$ #

▷ Corollary: any $p(z) \in \mathbb{C}[z]$ can be factorized as $p(z) = a_n (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$

Here λ_i can be the same.

Pf: (FTA) $\Rightarrow$ $p(z) = 0$ has a solution $z = \lambda_1$

$\Rightarrow$ $z - \lambda_1 \mid p(z)$

because we can do long division

Example $p(z) = z^2 - 3z + 2$ has a solution $z = 1$

$$\begin{array}{r|l} & z \; - 2 \\ \hline z - 1 & z^2 - 3z + 2 \\ & z^2 - z \\ \hline & -2z + 2 \\ & -2z + 2 \\ \hline & 0 \end{array}$$

$z^2 - 3z + 2 = (z-1) \times (z-2)$

We can write $p(z) = (z - \lambda_1) P_{n-1}(z)$ $\quad$ $\deg P_{n-1} = n-1$

Apply FTA to $P_{n-1}(z)$ find 2nd solution $z = \lambda_2$

$$P_{n-1}(z) = (z-\lambda_2)\cdot P_{n-2}(z)$$

For finite steps we get $P(z) = (z-\lambda_1)(z-\lambda_2)\cdots \underbrace{P_1(z)}_{= a_n(z-\lambda_n)}$ #

§ Block matrix (Organize your matrix)

$$g = \left(\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \quad B = 0,\ C = 0$$

Def g is blockwisely diagonal

" uppertriangular if $C=0$ $B\neq 0$

△ $$\begin{array}{c} \\ n_1\{ \\ n_2\{ \end{array}\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \end{array} \begin{array}{c} n_1\{ \\ n_2\{ \end{array}\left(\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right) = \left(\begin{array}{c|c} AE+BG & AF+BH \\ \hline CE+DG & CF+DH \end{array}\right)$$

Careful: Can not swap order in $A\cdot E$, BG

$EA + BG$ ✗

△ $$\begin{array}{c} n_1\{ \\ n_2\{ \end{array}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$ make sure A, D are square matrices

Recall: If g_1, g_2 both diagonal / upper ◺

then $g_1 \cdot g_2$ is still diag / upper ◺

Lemma If g_1, g_2 are blockwisely diag / upper ◺

then: $g_1 \cdot g_2$ is also " " / "

△ D ‖ a (a_1 a_2 *) | (a)

Δ : Recall: $g = \begin{pmatrix} a_1 & & * \\ & a_2 & \\ & & \ddots \\ 0 & & & a_n \end{pmatrix}$ $\det(g) = a_1 \cdot a_2 \cdots a_n$

Lemma: If $g = \begin{pmatrix} A_1 & * & * \\ 0 & A_2 & * \\ 0 & 0 & A_3 \end{pmatrix}$ A_i are square matrices

$$\det(g) = \det(A_1) \cdot \det(A_2) \cdot \det(A_3)$$