A1. $\lim\limits_{x\to 0} 8x-4\sin(2x) = \lim\limits_{x\to 0} x^3=0$. so $\lim\limits_{x\to 0}\frac{8x-4\sin(2x)}{x^3} = \lim\limits_{x\to 0}\frac{8-8\cos(2x)}{3x^2}$

also $\lim\limits_{x\to 0} 8-8\cos(2x) = \lim\limits_{x\to 0} 3x^2 = 0$. so $\lim\limits_{x\to 0}\frac{8-8\cos(2x)}{3x^2} = \lim\limits_{x\to 0}\frac{16\sin(2x)}{6x}$

$\lim\limits_{x\to 0} 16\sin(2x) = \lim\limits_{x\to 0} 6x=0$. so $\lim\limits_{x\to 0}\frac{16\sin(2x)}{6x} = \lim\limits_{x\to 0}\frac{32\cos(x)}{6} = \frac{32}{6} = \frac{16}{3}$

$\lim\limits_{x\to 0}\frac{8x-4\sin(2x)}{x^3} = \frac{16}{3}$

A2 $f''(x) = \frac{d(x-2)^3}{dx}x^2(x-1) + \frac{d(x^2(x-1))}{dx}(x-2)^3 = 3(x-2)^2x^2(x-1) + (x-2)^3(2x(x-1)+$

$x^2) = (x-2)^2(3x^2(x-1)+(x-2)2x(x-1)+(x-2)x^2) = (x-2)^2x(6x^2-11x+4)$

① $f'(0)=0$, $f''(0)=0$. so $x=0$ is not local extrema

② $f'(1)=0$, $f''(1)<0$. so $x=1$ is local maxima

③ $f'(2)=0$. $f''(2)=0$. so $x=2$ is not local extrema

in sum, $x=1$ is local maxima

A3 we know that $f'(c)=0$ because it have local minimum at $x=c$

and $\forall x, x<c \Rightarrow f'(x)<f'(c)=0$ and $\forall x, x>0 \Rightarrow f'(x)>f'(c)=0$ because f is convex

① if $x_1, x_2 \leq c$:

Suppose that $x_1<x_2$ and $f(x_1)\leq f(x_2)$, we can know that there exist $x_3\in(x_1,x_2)$

such that $f'(x_3)=\frac{f(x_2)-f(x_1)}{x_2-x_1}\geq 0$, but $\forall x . x<c \Rightarrow f'(x)<0$.

so we can know that $\forall x_1, x_2<c$, if $x_1<x_2 \Rightarrow f(x_1)>f(x_2)$

② also we can use the same method to prove that $\forall x_1, x_2>c$, if $x_1<x_2 \Rightarrow f(x_1)<f(x_2)$

so we know that $\forall \delta>0$, $f(c-\delta)>f(c)$ and $f(c+\delta)>f(c)$

and $\forall x\neq c$. $\exists \delta>0$ such that $f(x-\delta)<f(x)$ and $f(x+\delta)>f(x)$

or $f(x-\delta)>f(x)$ and $f(x+\delta)<f(x)$

so the global minimum is attained only at $x=c$

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A4. because $\forall \varepsilon > 0$, $\exists M > 0$ such that $\forall x > M \Rightarrow |f(x) - 0| < \varepsilon$

and $\forall n > M$, $n \in \mathbb{N}$. we get $\left|\frac{f(n+1) - f(n)}{n+1-n}\right| < 2\varepsilon$

and there exist $c \in (n, n+1)$ such that $|f'(c) - 0| < 2\varepsilon$ (MVT)

so if $\lim\limits_{x \to +\infty} f'(x)$ exist, we know that $\lim\limits_{x \to +\infty} f'(x) = 0$

A5. $L_a(x) = f'(a)(x-a) + f(a)$

① if $x > a$:

we know that $\frac{f(x) - f(a)}{x-a} > f'(a)$ because $\forall c \in (a, x)$, $f'(c) > f'(a)$

so $f(x) > f'(a)(x-a) + f(a) = L_a(x)$

② if $x < a$:

we know that $\frac{f(x) - f(a)}{x-a} < f'(a)$ because $\forall c \in (x, a)$, $f'(c) < f'(a)$

so $f(x) > f'(a)(x-a) + f(a) = L_a(x)$

so we know that $\forall x \neq a$, we have $f(x) > L_a(x)$

B_1 we can prove the contrapositive :

if f isn't convex . we know that $\exists\, x_1, x_2$ such that $f'(x)$ is decreasing on (x_1, x_2) . and it is concave

we can use the same way on A_5 to prove that the graph of f is below the Tangent line on (x_1, x_2)

so the graph of f is not above all it tangents if f is not convex.

the the contrapositive is proved.

B_2 Let $g(x) = f(x) - \frac{(x-a)(x-b)}{(d-a)(d-b)} f(d)$. and $g(a) = g(d) = g(b) = 0$

so there exist $x_1 \in (a, d)$ and $x_2 \in (d, b)$ such that

$$g'(x_1) = \frac{g(a) - g(d)}{a-d} = 0 \quad \text{and} \quad g'(x_2) = \frac{g(d) - g(b)}{d-b} = 0$$

also , we know that there exist $c \in (x_1, x_2)$ such that

$$g''(c) = \frac{g'(x_1) - g'(x_2)}{x_1 - x_2} = 0$$

and $g''(x) = f''(x) - \frac{2f(d)}{(d-a)(d-b)}$

is $x = c$, we get $0 = f''(c) - 2\frac{f(d)}{(d-a)(d-b)}$

$$\Downarrow$$

$$f(d) = \frac{f''(c)}{2} \cdot (d-a)(d-b)$$