清華大學

Tsinghua University

Part 1

1. C   2. A   3. B   4. C   5. A

Part 2a.

$$\text{Volume} = \pi\int_1^{\sqrt{2}} f(x)^2\,dx = \pi\int_1^{\sqrt{2}} \frac{1}{(4-x^2)^{\frac{3}{2}}}dx = \pi\int_1^{\sqrt{2}}\left(\frac{1}{\sqrt{4-x^2}}\right)^3 dx$$

Let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$

$$\pi\int_1^{\sqrt{2}}\left(\frac{1}{\sqrt{4-x^2}}\right)^3 dx = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\frac{1}{\cos\theta}\right)^3\cdot 2\cos\theta\, d\theta = 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{1}{\cos^2\theta}\,d\theta$$

$$= 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec^2\theta\, d\theta = 2\pi\left[\tan\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = 2\pi\left(1-\frac{\sqrt{3}}{3}\right)$$

Part 2.b

$$f'(x) = \frac{4}{3}(6-x)\cdot(8-x)^{-\frac{2}{3}},\quad f''(x) = (8-x)^{-\frac{5}{3}}\left(\frac{4}{9}x-\frac{16}{3}\right)$$

(explanation is below)

$$f(x) = x(8-x)^{\frac{1}{3}} = \sqrt[3]{x^3(8-x)} = (8x^3-x^4)^{\frac{1}{3}}$$

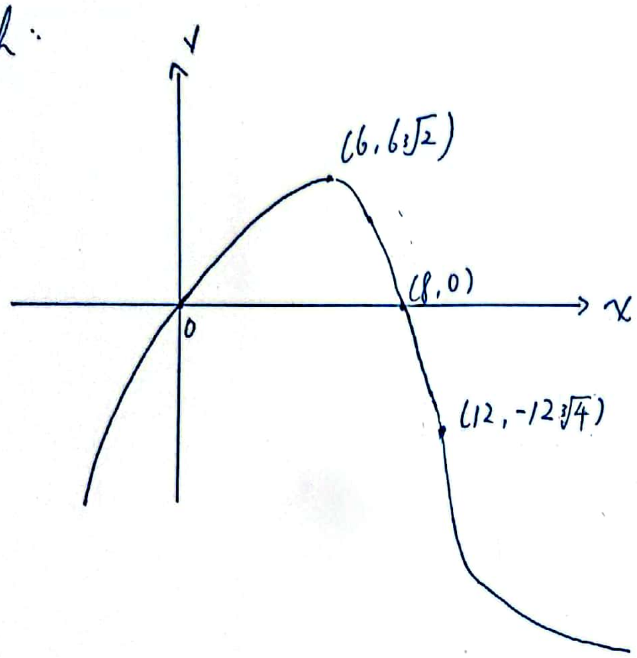
$$f'(x) = (24x^2-4x^3)\cdot\frac{1}{3}(8x^3-x^4)^{-\frac{2}{3}} = 4x^2(6-x)\cdot\frac{1}{3}\cdot\frac{1}{x^2}(8-x)^{-\frac{2}{3}} = \frac{4}{3}(6-x)\cdot(8-x)^{-\frac{2}{3}}$$

$$f''(x) = -\frac{4}{3}(8-x)^{-\frac{2}{3}} + \frac{4}{3}(6-x)\cdot(-1)\left(-\frac{2}{3}\right)(8-x)^{-\frac{5}{3}} = -\frac{4}{3}(8-x)^{-\frac{2}{3}} + \frac{8}{9}(6-x)(8-x)^{-\frac{5}{3}}$$

$$= (8-x)^{-\frac{5}{3}}\left(-\frac{4}{3}(8-x)+\frac{8}{9}(6-x)\right) = (8-x)^{-\frac{5}{3}}\left(\frac{4}{9}x-\frac{16}{3}\right)$$

x	$(-\infty,0)$	0	$(0,6)$	6	$(6,8)$	8	$(8,12)$	12	$(12,+\infty)$
$f(x)$	−	0	+	+	+	0	−	−	−
$f'(x)$	+	+	+	0	−	doesn't exist	−	−	−
$f''(x)$	−	−	−	−	−	doesn't exist	−	0	+

Graph:



Part 2c

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x)(\cos(x)-1)}{(\cos(x))^2-5\cos(x)+6}dx = \int_0^{\frac{\pi}{2}} \frac{\sin(x)(\cos(x)-1)}{(\cos(x)-3)(\cos(x)-2)}dx$$

Let $u = \cos x$, $du = -\sin x \, dx$

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x)(\cos(x)-1)}{(\cos(x)-3)(\cos(x)-2)}dx = -\int_1^0 \frac{u-1}{(u-3)(u-2)}du = \int_0^1 \frac{2}{u-3} - \frac{1}{u-2}\,du$$

$$= \Big[2\ln|u-3| - \ln|u-2|\Big]_0^1 = 2\ln 2 - 2\ln 3 + \ln 2 = 3\ln 2 - 2\ln 3$$