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Solutio of Homework 5.

1.: $g=\begin{pmatrix}2&1&0\\-1&2&0\\0&0&4\end{pmatrix}$ find h so that $h^{-1}gh$ is diagonal.

pf: $|\lambda I-g|=\begin{vmatrix}\lambda-2&-1&0\\1&\lambda-2&0\\0&0&\lambda-4\end{vmatrix}=(\lambda^2-4\lambda+5)(\lambda-4)$

$\Rightarrow \lambda_1=2+i,\ \lambda_2=2-i,\ \lambda_3=4$

For $\lambda_1=2+i$, $(g-\lambda_1 I)v=0 \Rightarrow \begin{pmatrix}-i&1&0\\-1&-i&0\\0&0&2-i\end{pmatrix}v=0 \Rightarrow v_1=\begin{pmatrix}-i\\1\\0\end{pmatrix}$

$\lambda_2=2-i$ $(g-\lambda_2 I)v=0 \Rightarrow \begin{pmatrix}i&1&0\\-1&i&0\\0&0&2+i\end{pmatrix}v=0 \Rightarrow v_2=\begin{pmatrix}i\\1\\0\end{pmatrix}$

$\lambda_3=4$ $(g-\lambda_3 I)v=0 \Rightarrow \begin{pmatrix}-2&1&0\\-1&-2&0\\0&0&0\end{pmatrix}v=0 \Rightarrow v_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$

Put $h=\begin{pmatrix}-i&i&0\\1&1&0\\0&0&1\end{pmatrix}$ then $h^{-1}gh=\begin{pmatrix}2+i&&\\&2-i&\\&&4\end{pmatrix}$

2. Assume we have a_1, a_2, a_3 so that

$$a_1v_1+a_2v_2+a_3v_3=0$$

then $a_1\lambda_1v_1+a_2\lambda_2v_2+a_3\lambda_3v_3=0$ by applying T

again, we have $a_1\lambda_1^2v_1+a_2\lambda_2^2v_2+a_3\lambda_3^2v_3=0$

$$\Rightarrow \begin{pmatrix}1&1&1\\\lambda_1&\lambda_2&\lambda_3\\\lambda_1^2&\lambda_2^2&\lambda_3^2\end{pmatrix}\begin{pmatrix}a_1v_1\\a_2v_2\\a_3v_3\end{pmatrix}$$

However, if $\{\lambda_i\}$ are pairwise different, we have $\begin{vmatrix}1&1&1\\\lambda_1&\lambda_2&\lambda_3\\\lambda_1^2&\lambda_2^2&\lambda_3^2\end{vmatrix}\neq 0$

$\Rightarrow a_1v_1=a_2v_2=a_3v_3=0 \Rightarrow a_1=a_2=a_3=0$

So v_1, v_2, v_3 are linearly independent

3. Consider $n_i = \begin{pmatrix} 0 & a_i & b_i \\ 0 & 0 & c_i \\ 0 & 0 & 0 \end{pmatrix}$ $i=1,2,3.$

then $n_1 n_2 n_3 = \begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix} n_3$

$$= \begin{pmatrix} 0 & 0 & a_1 c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a_3 & b_3 \\ 0 & 0 & c_3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4 (a) $\det(\lambda I - g) = \det \begin{pmatrix} \lambda & 0 & 0 & 0 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda-1 \end{pmatrix} = \lambda^3 (\lambda - 1)$

so $\lambda_1 = 0$ with multiplicity 3 and $\lambda_2 = 1$

(b) for $\lambda_1 = 0$, $(g - \lambda_1 I)v = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$

for $\lambda_2 = 1$ $(g - \lambda_2 I)v = 0 \Rightarrow \begin{pmatrix} -1 & & & \\ 1 & -1 & & \\ 0 & 1 & -1 & \\ 0 & 0 & 1 & 0 \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

(c). $(g - \lambda_1)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

$(g - \lambda_1)^2 v = 0 \Rightarrow$ $v_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ generlized eigenvector $\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

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$$(g-\lambda_1)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{So } (g-\lambda_1)^3 v = 0 \implies v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$\Rightarrow$ generalized vector : v_1, v_2, v_3)

$\Rightarrow$ Generalized eigenspace $\langle v_1, v_2, v_3 \rangle$

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Solution of Homework 6.

1. We have h_1, h_2 s.t. $g_1 = h_1^{-1} g_2 h_1 \quad g_3 = h^{-1} g_4 h$

$$\Rightarrow \begin{pmatrix} g_1 & \\ & g_3 \end{pmatrix} = \begin{pmatrix} h_1^{-1} & \\ & h_3^{-1} \end{pmatrix} \begin{pmatrix} g_2 & \\ & g_4 \end{pmatrix} \begin{pmatrix} h_1 & \\ & h_3 \end{pmatrix}$$
$$= \begin{pmatrix} h_1 & \\ & h_3 \end{pmatrix}^{-1} \begin{pmatrix} g_2 & \\ & g_2 \end{pmatrix} \begin{pmatrix} h_1 & \\ & h_3 \end{pmatrix}$$

Consider $H = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. then $H^{-1} = H$.

and $$H \begin{pmatrix} g_4 & \\ & g_2 \end{pmatrix} H = \begin{pmatrix} g_2 & \\ & g_4 \end{pmatrix}$$

so $\begin{pmatrix} g_1 & \\ & g_3 \end{pmatrix}$ is conjugate to $\begin{pmatrix} g_4 & \\ & g_2 \end{pmatrix}$

2. $g = \begin{pmatrix} 5 & i & 1 \\ 4i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then $\det(\lambda I - g) = \begin{vmatrix} \lambda-5 & -i & -1 \\ -4i & \lambda-1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$
$$= (\lambda-1)(\lambda-3)^2$$

$\Rightarrow \lambda_1 = 1. \quad \lambda_2 = \lambda_3 = 3$

$\lambda_1 = 1 \quad (g - \lambda_1 I)v = 0 \Rightarrow v_1 = \begin{pmatrix} 0 \\ -1 \\ i \end{pmatrix}$

$\lambda_2 = 3 \quad (g - \lambda_2 I)v = 0 \Rightarrow v_2 = \begin{pmatrix} i \\ -2 \\ 0 \end{pmatrix}$

$(g - \lambda_2 I)^2 v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ generalized eigenvector $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow h = \begin{pmatrix} 0 & i & 1 \\ -1 & -2 & 0 \\ i & 0 & 0 \end{pmatrix}$$

3. $\det(\lambda I - g) = (\lambda - 11)(\lambda - 9)(\lambda^2 - 7\lambda - 4)$

$\Rightarrow \lambda_1 = 11 \quad \lambda_2 = 9 \quad \lambda_3 = \frac{7+\sqrt{65}}{2} \quad \lambda_4 = \frac{7-\sqrt{65}}{2}$

$\Rightarrow$ λ_i are different. so Jordan normal form of g should by $\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}$

4 $g = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. eigenvalues of g are all zero,

and $\mathrm{rk}\, g = 2$. so Jordan normal form of g should by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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Solution of Homework 7.

1. $g = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $f(x) = e^x$. compute $f(g)$.

Pf: $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ $g_1 = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. $g_2 = 1$.

$\Rightarrow e^g = \begin{pmatrix} e^{g_1} & \\ & e^{g_2} \end{pmatrix} = \begin{pmatrix} e^{g_1} & 0 \\ 0 & e \end{pmatrix}$

$g_1 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow e^{g_1} = e^{\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}} \cdot e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$ since $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ commutes.

$= \begin{pmatrix} e^3 & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} e^3 & e^3 \\ 0 & e^3 \end{pmatrix}$

$\Rightarrow f(g) = \begin{pmatrix} e^3 & e^3 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e \end{pmatrix}$

2. (a) Minimal poly of g is to lowest common multiple of minimal poly of each Jordan block. So it is

$$m_g(\lambda) = (\lambda-1)^2(\lambda+1)^2$$

(b). $f(\lambda) = \lambda^6 - 3\lambda^4 + 3\lambda^2 - 1 = (\lambda-1)^2(\lambda+1)^2(\lambda^2-1)$

$= m_g(\lambda) \cdot (\lambda^2 - 1)$

So $f(g) = 0$.

3. (a). If $g = g_0^2$. $\Rightarrow$ $g_0^4 = g^2 = 0$.

So g_0 is nilpotent rk 2 matrix.

However, By Caley-Hamiltonian theorem, we see that if g_0 is rk 2 nilpotent. then $g_0^2 = 0 \Rightarrow g = 0$ contradiction

So we do not have such g_0

(b). Notice that if $g = g_0^2$ then $h^{-1} g h = h^{-1} g_0^2 h$

$$= h^{-1} g_0 h h^{-1} g_0 h$$
$$= (h^{-1} g_0 h)^2$$

So we may consider Jordan normal form of g

Case 1. Jordan normal form is $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

we may take $g_0 = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}$

Case 2 Jordan normal form is $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $\lambda \neq 0$

we may take $g_0 = \begin{pmatrix} \sqrt{\lambda} & \frac{1}{2\sqrt{\lambda}} \\ 0 & \sqrt{\lambda} \end{pmatrix}$

Solution of Homework 8.

1. $$\frac{df_1}{dt} = 2f_1(t) - f_2(t)$$

$$\frac{df_2}{dt} = -2f_1(t) + 3f_2(t)$$

Let $F = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$. then $\frac{dF}{dt} = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} F$

$$\Rightarrow F = e^{\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} t} \cdot F_0$$

Compute $e^{\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} t}$ $\quad A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$

Step 1. find h so that $h^{-1}Ah$ is Jordan normal form

$$\det(\lambda - A) = \begin{vmatrix} \lambda-2 & 1 \\ 2 & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda-3)-2$$
$$= \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$$

$$\lambda_1 = 1 \Rightarrow (A-\lambda_1 I)v = 0 \Rightarrow \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4 \Rightarrow (A-\lambda_2 I)v = 0 \Rightarrow \begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow h = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}. \quad h^{-1} = -\frac{1}{3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} e^t & e^{4t} \\ e^t & -2e^{4t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

and $h^{-1}Ah = \begin{pmatrix} 1 & \\ & 4 \end{pmatrix}$

$$\Rightarrow e^{At} = e^{h \cdot \begin{pmatrix} 1 & \\ & 4 \end{pmatrix} h^{-1} t} = h \cdot e^{\begin{pmatrix} t & \\ & 4t \end{pmatrix}} \cdot h^{-1} = h \cdot \begin{pmatrix} e & \\ & e^4 \end{pmatrix} h^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} e^t & \\ & e^{4t} \end{pmatrix} \cdot \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \left(-\frac{1}{3}\right)$$

$$= \frac{1}{3}\begin{pmatrix} 2e^t + e^{4t} & e^t - e^{4t} \\ 2e^t - 2e^{4t} & e^t + 2e^{4t} \end{pmatrix}$$

So $F(t) = \frac{1}{3}\begin{pmatrix} 2e^{t}+e^{4t} & e^{t}-e^{4t} \\ 2e^{t}-2e^{4t} & e^{t}+2e^{4t} \end{pmatrix}\begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$

(2). if $f_1(0) = f_2(0) \Rightarrow F_0 = F_1$

$\Rightarrow f_1(t) = \frac{1}{3}\left(2e^{t}F_0 + e^{4t}\cdot F_0 + e^{t}F_0 - e^{4t}F_0\right) = e^{t}F_0$

$f_2(t) = \frac{1}{3}\left(2e^{t}F_0 - 2e^{4t}F_0 + e^{t}F_0 + 2e^{4t}F_0\right) = e^{t}F_0$

$\Rightarrow f_1(t) = f_2(t)$ and $\lim_{t\to\infty} \frac{f_1(t)}{f_2(t)} = 1$

2. Assume $\lambda_1 \dots \lambda_r$ are distinct eigenvalues of A

Since A is diagonable. So we see that V can be decompose as:

$$V = V(\lambda_1) \oplus \cdots \oplus V(\lambda_r)$$

where $V(\lambda_i)$ are eigenspaces of A.

For any $v_i \in V(\lambda_i)$ we see that

$$\begin{aligned} A(Bv_i) &= B(AV_i) \\ &= B(\lambda_i v_i) \\ &= \lambda_i(Bv_i) \end{aligned}$$

$\Rightarrow Bv_i \in V(\lambda_i)$. i.e. $V(\lambda_i)$ is a invariant subspace of B.

So we may take basis in $V(\lambda_i)$ so that $B|_{V(\lambda_i)}$ is of Jordan normal form. We also see that, under same basis, A is of diagonal form since all basis are take from eigenspace. So We see that we have h so that $h^{-1}Ah$ is diagonal and $h^{-1}Bh$ is Jordan normal form

3. (a). $A(-A) = (-A)A = -A^2$.

so $e^{A} \cdot e^{-A} = e^{A+(-A)} = e^{0} = I_n$

Thus $e^{-A} = (e^{A})^{-1}$

(b) For $n=1$. $e^{A} = e^{A} = (e^{A})^{1}$. we are done.

Assume that $e^{kA} = (e^{A})^{k}$.

then $e^{(k+1)A} = e^{kA+A}$ clearly kA commutes with A

so $e^{kA+A} = e^{kA} \cdot e^{A}$

$= (e^{A})^{k} \cdot e^{A} = (e^{A})^{k+1}$

So by induction. we have $e^{nA} = (e^{A})^{n}$ for all $n > 0$.