Solution to Homework 2.

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1. $T = \begin{pmatrix} 3 & 4 \\ -1 & 3 \end{pmatrix} \in M_2(\mathbb{R})$, you need to find $I_v \in M_2(\mathbb{R})$ so that

$$\left.\begin{array}{l} ① \ I_v^2 = -I \\ ② \ T I_v = I_v T \end{array}\right\} \Rightarrow T \text{ is a operator on a complex v.s.}$$

$T = 3I + \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$. need $I_v \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} I_v$.

Let us assume that $I_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a, b, c, d \in \mathbb{R}$.

then $I_v T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & 4a \\ -d & 4c \end{pmatrix}$

$T I_v = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4c & 4d \\ -a & -b \end{pmatrix}$ $\Rightarrow \begin{cases} b = -4c \\ d = a \end{cases}$

and

$$I_v^2 = -I \Rightarrow \begin{pmatrix} a & -4c \\ c & a \end{pmatrix} \begin{pmatrix} a & -4c \\ c & a \end{pmatrix} = \begin{pmatrix} a^2 - 4c^2 & 0 \\ 2ac & -4c^2 + a^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow a = 0. \quad c = \tfrac{1}{2}.$$

$$\Rightarrow I_v = \begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix}$$

Problem itself is a operator problem, how to translate it into a matrix problem? which is computable.

2. "Equivalent" two-sided proof!

pf: "$\Rightarrow$" (Assume that (a) is true, to get (b))

By the lemma in the end of Lecture 3, we see that we have a basis for V such that I_v correspondes to the matrix

$$I_v = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

Now assume that the matrix of T under this basis is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A, B, C, D \in M_2(\mathbb{R})$.

T can be viewed as a linear operator over $\mathbb{C}$, So we have

$T I_V = I_V T$, $\Rightarrow$

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$LHS = \begin{pmatrix} -C & -D \\ A & B \end{pmatrix}. \qquad RHS = \begin{pmatrix} B & -A \\ D & -C \end{pmatrix}$$

$\Rightarrow$ $B = -C$ and $A = D$.

So $T = \begin{pmatrix} A & B \\ -B & D \end{pmatrix}$ under this basis, which means T is conjugate to this matrix

"$\Leftarrow$". We have a basis such that T conjugates to

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

consider $I_V = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. clearly $I_V^2 = -I$.

Now $I_V T = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} -B & -A \\ A & -B \end{pmatrix}$

$T I_V = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -B & -A \\ A & -B \end{pmatrix}$

So $I_V T = T I_V$ Thus T can be viewed as an operator over $\mathbb{C}$.

3. We shall write $T = A + Bi$ but, as an operator over $\mathbb{R}$, T has size 4. as an operator over $\mathbb{C}$, T has size 2. so what happened?

Put $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$.

then $T = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. Let $I_V = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

Assume the that the standard basis of V is $\{e_1, e_2, e_3, e_4\}$

Thus $I_V(e_1) = e_3$, $I_V(e_2) = e_4$.

and $\{e_1, e_2\}$ is a basis for V over $\mathbb{C}$.

Notice that $T(e_1) = (e_1, e_2, e_3, e_4)\begin{pmatrix} 1 & 2 & -5 & -6 \\ 3 & 4 & -7 & -8 \\ 5 & 6 & 1 & 2 \\ 7 & 8 & 3 & 4 \end{pmatrix} = e_1 + 3e_2 + 5e_3 + 7e_4$

$$= e_1 + 3e_4 + 5I_V(e_1) + 7I_V(e_2)$$

Similarly, $T(e_2) = 2e_1 + 4e_2 + 6I_V(e_1) + 8I_V(e_2)$

So $T(e_1, e_2) = (e_1, e_2)\begin{pmatrix} 1+5i & 2+6i \\ 3+7i & 4+8i \end{pmatrix}$

$$= (e_1, e_2)(A + Bi)$$

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From this class, we will NOT talk about homeworks.

So, you should ask me more questions, so I can answer it in exercise class.

① Change of basis and conjugation of matrix

V is a vector space of $\dim = n$ over a field F

$$\underline{T: V \longrightarrow V}$$

is a linear operator i.e $T(v_1+v_2) = T(v_1)+T(v_2)$

$T(\alpha v) = \alpha T(v)$.

$v, v_1, v_2 \in V$, $\alpha \in F$

Assume that $\{e_1, e_2, \cdots, e_n\}$ is a basis of V

$$\begin{cases} T(e_i) = a_{1i}e_1 + a_{2i}e_2 + \cdots + a_{ni}e_n \\ \Updownarrow \\ = (e_1, e_2, \cdots, e_n)\begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix} \end{cases} \quad 1 \le i \le n$$

i-th column

$\rightsquigarrow T(e_1, e_2, \cdots, e_n) \overset{\text{define}}{:=} (T(e_1), T(e_2), \cdots, T(e_n))$

$$= (e_1, e_2, \cdots, e_n)\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A_T}$$

$$= (e_1, \cdots, e_n) A_T$$

Operator T
$+$ a basis $\{e_1, e_2, \cdots, e_n\}$
$\rightsquigarrow$ a matrix A_T

A matrix A_T
$+$ a basis $\{e_1, \cdots, e_n\}$
$\rightsquigarrow$ operator T

i.e. When a basis of a vector space is given, then linear operator is equivalent to matrix

Q: what happened when the basis change???

what we have now:

$$T(e_1, \cdots, e_n) = (e_1, \cdots, e_n) A_T$$

When I chose another basis, say $\{f_1, f_2 \cdots f_n\}$.

what is the matrix of T under this basis

Notice that $V = \langle e_1, e_2, \cdots, e_n \rangle = \langle f_1, f_2, \cdots, f_n \rangle$

$$\Longrightarrow \begin{cases} e_1 = \alpha_{11} f_1 + \alpha_{21} f_2 + \cdots + \alpha_{n1} f_n \\ \quad \vdots \\ e_i = \alpha_{1i} f_1 + \alpha_{2i} f_2 + \cdots + \alpha_{ni} f_n \\ \quad \vdots \\ e_n = \cdots \end{cases}$$

$$\Rightarrow (e_1, e_2, \cdots e_n) = (f_1, f_2, \cdots, f_n) \underbrace{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}}_{P}$$

i.e. $\boxed{(e_1, e_2, \cdots, e_n) = (f_1, f_2, \cdots f_n) P, \quad (f_1, f_2, \cdots f_n) = (e_1, e_2, \cdots e_n) P^{-1}}$

// P is the connection between $\{e_1, \cdots e_n\}$ and $\{f_1, \cdots f_n\}$

Question: What's the linear operator correspondes to P under the basis $\{e_1, \cdots, e_n\}$?

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$T(e_1, e_2, \cdots, e_n) = (e_1, e_2, \cdots, e_n) A_T$ Known

want: $T(f_1, f_2, \cdots, f_n) = (f_1, f_2, \cdots, f_n) ?$ ☆

$= T(e_1, e_2, \cdots, e_n) P^{-1}$

$= (e_1, e_2, \cdots, e_n) A_T P^{-1}$

$= (f_1, f_2, \cdots, f_n) P A_T P^{-1}$ → conjugation of A_T by P

Answer: If the basis is changed, then the matrix is differ by a conjugation!

Exercise 1. V is a 3-dim v.s with basis $\{e_1, e_2, e_3\}$

$T: V \to V$ is a linear operator s.t.

$$T(e_1) = e_1 + e_2$$

$$T(e_2) = e_2 - 2e_3$$

$$T(e_3) = -e_1 + e_2 + 2e_3$$

① Write down the matrix correspondes to T under the basis $\{e_1, e_2, e_3\}$

② If I have another basis $\{f_1, f_2, f_3\}$ with

$f_1 = e_1$, $f_2 = e_1 + e_2$, $f_3 = e_1 + e_2 + e_3$.

write down the matrix of T under $\{f_1, f_2, f_3\}$

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Answer. ① $T(e_1,e_2,e_3)=(e_1,e_2,e_3)\underbrace{\begin{pmatrix}1&1&-1\\1&0&1\\0&-2&2\end{pmatrix}}_{A_T}$

② $(f_1,f_2,f_3)=(e_1,e_2,e_3)\underbrace{\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}}_{P}$

$(e_1,e_2,e_3)=(f_1,f_2,f_3)\underbrace{\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix}}_{P^{-1}}$

$T(f_1,f_2,f_3)=T(e_1,e_2,e_3)P$

$=(e_1,e_2,e_3)A_TP$

$=(f_1,f_2,f_3)P^{-1}A_TP$

$P^{-1}A_TP=\begin{pmatrix} ? \end{pmatrix}$ ✓

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② Vector space over $\mathbb{C}$ and vector space over $\mathbb{R}$

V is a vector space over $\mathbb{C}$, with dim n

then V can be viewed as a vector space over $\mathbb{R}$, with dimension $2n$ in the following way:

If $\{e_1, e_2, \cdots, e_n\}$ is a basis of V over $\mathbb{C}$, then $\{e_1, e_2, \cdots, e_n, ie_1, ie_2, \cdots, ie_n\}$ is a basis of V over $\mathbb{R}$.

$$\{\text{vector space over } \mathbb{C}\} \longrightarrow \{\text{vector space over } \mathbb{R}\}$$

????

能不能??? $\dim_{\mathbb{R}} V = 2n$ ✓

怎么做??? 去找矩阵 I_V 使得 $I_V^2 = -I$ ✓

I_V 有多少???

$$T_{\mathbb{C}}: V_{\mathbb{C}} \to V_{\mathbb{C}} \rightsquigarrow T_{\mathbb{R}}: V_{\mathbb{R}} \to V_{\mathbb{R}}$$

?

????

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Exercise 2. V is a vector space over $\mathbb{C}$, $T: V \to V$ is a linear operator, $\{e_1, e_2\}$ is a basis of V, and

$$T(e_1, e_2) = (e_1, e_2)\begin{pmatrix} 2 & e^{\frac{2\pi}{3}i} \\ 9 & 3 \end{pmatrix} = (e_1, e_2) T_{\mathbb{C}}$$

Since we can view V as a vector space over $\mathbb{R}$.

(a) write down a basis of V as a vector space over $\mathbb{R}$ and write down the matrix of T under the basis you just found, which is $T_{\mathbb{R}}$

(b) Calculate $\det T_{\mathbb{C}}$ and $\det T_{\mathbb{R}}$

Answer. basis $\{e_1, e_2, ie_1, ie_2\}$

$T(e_1) = 2e_1 + 9e_2$

$T(e_2) = e^{\frac{2\pi}{3}i} e_1 + 3e_2$

$= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) e_1 + 3e_2$

$= -\frac{1}{2}e_1 + 3e_2 + \frac{\sqrt{3}}{2}(ie_1)$

$T(ie_1) = iT(e_1)$

$= 2(ie_1) + 9(ie_2)$

$T(ie_2) = -\frac{\sqrt{3}}{2}e_1 - \frac{1}{2}(ie_1) + 3(ie_2)$

$$\Rightarrow T(e_1, e_2, ie_1, ie_2) = (e_1, e_2, ie_1, ie_2)\left(\begin{array}{cc|cc} 2 & -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 9 & 3 & 0 & 0 \\ \hline 0 & \frac{\sqrt{3}}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 9 & 3 \end{array}\right)$$

② $\det T_{\mathbb{C}} = 6 - 9e^{\frac{2\pi}{3}i}$

$\det T_{\mathbb{R}} = ???$

Observation:

If I write $T_{\mathbb{C}} = A + Bi$

then $T_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$

warning: choice of basis !!!

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Question (Continued as in exercise 2)

T as above, assume that V over has another basis $\{f_1, f_2\}$, with

$$e_1 = (1+2i)f_1 + e^{\frac{2\pi i}{3}} f_2$$

$$e_2 = (3+4i)f_1 + 10000 f_2$$

i.e. $(e_1, e_2) = (f_1, f_2)\underbrace{\begin{pmatrix} 1+2i & 3+4i \\ e^{\frac{2\pi i}{3}} & 10000 \end{pmatrix}}_{P}$

So $T(f_1, f_2) = (f_1, f_2) P T_{\mathbb{C}} P^{-1}$

what happened for $T_{\mathbb{R}}$???

basis over $\mathbb{R}$ $\{e_1, e_2, ie_1, ie_2\} \rightsquigarrow \underbrace{\{f_1, f_2, if_1, if_2\}}$

matrix of T under this basis???

Example of $J \in GL_2(\mathbb{R})$ s.t.

① $J^2 = -I$ $\quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

② J is not completely off diagonal

pf. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad ad - bc \neq 0$.

$J = A \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A^{-1}$

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \dfrac{1}{ad-bc}$

$= \cdots\cdots = \begin{pmatrix} \dfrac{bd+ac}{ad-bc} & \dfrac{-b^2-a^2}{ad-bc} \\ \dfrac{d^2+c^2}{ad-bc} & \dfrac{-bd-ac}{ad-bc} \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$

$a = 2$,
$b = c = d = 1$