Linear Algebra – Homework 10

2 Dec 2020
Due: 10 Dec 2020

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 4.3.1. With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$, set up and solve the normal equations $A^T A\widehat{\mathbf{x}} = A^T\mathbf{b}$. For the best straight line, find the four heights p_i (the y-values for the line at $t = 0, 1, 3, 4$), and the errors e_i (the differences between the y-values of the line and the y-values of the data points). What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

Problem 4.3.5. Find the height C of the best *horizontal* line to fit $\mathbf{b} = (0, 8, 8, 20)$. An exact fit would solve the unsolvable equations $C = 0$, $C = 8$, $C = 8$, $C = 20$. Find the 4×1 matrix A in these equations and solve $A^T A\widehat{x} = A^T\mathbf{b}$. Draw the horizontal line $\widehat{x} = C$ and the four errors in $\mathbf{e}$.

Problem 4.3.10. For the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the same four points in Problem 4.3.1, write down the four equations $A\mathbf{x} = \mathbf{b}$. Solve them by elimination. This cubic now goes exactly through the points. What are $\mathbf{p}$ and $\mathbf{e}$?

Problem 4.3.12. This problem projects $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ onto the line through $\mathbf{a} = (1, 1, \ldots, 1)$. We solve m equations $\mathbf{a}x = \mathbf{b}$ in 1 unknown (by least squares).

(a) Solve $\mathbf{a}^T\mathbf{a}\widehat{x} = \mathbf{a}^T\mathbf{b}$ to show that $\widehat{x}$ is the *mean* (the average) of the b's.

(b) Find $\mathbf{e} = \mathbf{b} - \mathbf{a}\widehat{x}$ and the *variance* $\|\mathbf{e}\|^2$ and the *standard deviation* $\|\mathbf{e}\|$.

(c) The horizontal line $\widehat{x} = 3$ is closest to $\mathbf{b} = (1, 2, 6)$, since 3 is the average of 1, 2, and 6. Check that $\mathbf{p} = (3, 3, 3)$ is perpendicular to $\mathbf{e}$ and find the 3×3 projection matrix P.

Problem 4.3.17. Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution to $\widehat{\mathbf{x}} = (C, D)$ and draw the closest line.

Problem 4.3.22. Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

Problem 4.4.2. The vectors $(2, 2, -1)$ and $(-1, 2, 2)$ are orthogonal. Divide them by their lengths to find orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. Put those into the columns of Q and multiply Q^TQ and QQ^T.

Problem 4.4.7. If Q has orthonormal columns, what is the least squares solution $\widehat{\mathbf{x}}$ to $Q\mathbf{x} = \mathbf{b}$?

Problem 4.4.10. Orthonormal vectors are automatically linearly independent:

(a) Vector proof: When $c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{q}_3 = \mathbf{0}$, what dot product leads to $c_1 = 0$? Similarly, $c_2 = 0$ and $c_3 = 0$. Thus the $\mathbf{q}$'s are independent.

(b) Matrix proof: Show that $Q\mathbf{x} = \mathbf{0}$ leads to $\mathbf{x} = \mathbf{0}$. Since Q might not be square, you can use Q^T but not Q^{-1}.

Problem 4.4.18. Find orthogonal vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ by Gram-Schmidt from $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$$\mathbf{a} = (1, -1, 0, 0), \qquad \mathbf{b} = (0, 1, -1, 0), \qquad \mathbf{c} = (0, 0, 1, -1).$$

$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are both bases for the vectors perpendicular to $\mathbf{d} = (1, 1, 1, 1)$.

Problem 4.4.22. Find orthogonal vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ by Gram-Schmidt from

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

Problem 4.4.31.

(a) Choose c so that Q is an orthogonal matrix:

$$Q = c \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

(b) Project $\mathbf{b} = (1,1,1,1)$ onto the first column. Then project $\mathbf{b}$ onto the plane of the first two columns.

Problem 4.4.32. If $\mathbf{u}$ is a unit vector, then $Q = I - 2\mathbf{u}\mathbf{u}^T$ is a reflection matrix. Find Q_1 from $\mathbf{u} = (0,1)$ and Q_2 from $\mathbf{u} = (0, \sqrt{2}/2, \sqrt{2}/2)$. Draw the reflections when Q_1 and Q_2 multiply the vectors $(1,2)$ and $(1,1,1)$.

Graded Problems.

Problem 1. Physics tells us that near the Earth's surface, the height $h(t)$ of an object dropped at time $t = 0$ from an initial height of h_0 obeys the equation $h(t) = h_0 - \frac{1}{2}gt^2$, where g is the acceleration due to gravity. In an experiment, a ball is dropped from an initial height $h_0 = 50$, and its distances above the ground are measured to be $50, 44, 32, 6$ at times $t = 0, 1, 2, 3$. Use the least squares method to find the parabola $C + Dt + Et^2$ that best fits these data points, and use your value for E to estimate the acceleration g.

Note: You do not need to worry about units in this problem. Also, the numbers in this exercise will get a bit large. You will not have to deal with such large numbers on the final exam.

Problem 2. Use the Gram-Schmidt process as necessary to find orthonormal bases for both the row space and null space of

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

Combine these two orthonormal bases to get an orthonormal basis for $\mathbf{R}^3$.