Linear Algebra – Week 3 Homework

30 Sept 2020
Due: 12 Oct 2020

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 2.1.1. With $A = I$ (the identity matrix) draw the planes in the row picture: three sides of a box meet at the solution $\mathbf{x} = (x, y, z) = (2, 3, 4)$.

$$\begin{aligned} 1x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= 3 \\ 0x + 0y + 1z &= 4 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Also, draw the vectors in the column picture: 2 times the first column plus 3 times the second column plus 4 times the third column equal the right side $\mathbf{b}$.

Problem 2.1.4. Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$. Find a third point halfway between.

Problem 2.1.17. Find the matrix P that multiplies (x, y, z) to give (y, z, x). Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z).

Problem 2.1.22. Write the dot product of $(1, 4, 5)$ and (x, y, z) as a matrix multiplication $A\mathbf{x}$. (The matrix A should have one row.) The solutions to $A\mathbf{x} = \mathbf{0}$ lie on a _____ perpendicular to the vector ______. The columns of A are vectors in only ___-dimensional space.

Problem 2.1.26. Draw the row and column pictures for the equations $x - 2y = 0$, $x + y = 6$. (That is, first draw the two lines intersecting in the plane, then draw a picture of $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$ as a linear combination of two column vectors.)

Problem 2.1.29. Start with the vector $\mathbf{u}_0 = (1, 0)$. Multiply again and again by the same "Markov matrix" $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$. The next three vectors are $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$:

$$\mathbf{u}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix}, \qquad \mathbf{u}_2 = A\mathbf{u}_1 = \underline{\qquad}, \qquad \mathbf{u}_3 = A\mathbf{u}_2 = \underline{\qquad}.$$

What property do you notice for all four vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$?

Problem 2.2.6. Choose a coefficient b that makes this system singular (that is, you will get either no solution or infinitely many). Then choose a right side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

Problem 2.2.9. What condition on b_1 and b_2 decides whether these two equations have a solution? How many solutions will they have? Draw the column picture for $\mathbf{b} = (1, 2)$ and $(1, 0)$.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2 \end{aligned}$$

Problem 2.2.13. Apply elimination and back substitution to solve

$$\begin{array}{rcrcrcl} 2x & - & 3y & & & = & 3 \\ 4x & - & 5y & + & z & = & 7 \\ 2x & - & y & - & 3z & = & 5. \end{array}$$

List the three row operations. (Each one has the form: "Subtract ____ times row ____ from row ____.")

Problem 2.2.18. Construct a 3×3 system of linear equations that has 9 different coefficients on the left side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $\mathbf{b} = (1, 10, 100)$ and how many with $\mathbf{b} = (0, 0, 0)$?

Problem 2.2.21. Find the solution for both systems of linear equations:

$$\begin{array}{rcrcrcrcl} 2x & + & y & & & & & = & 0 \\ x & + & 2y & + & z & & & = & 0 \\ & & y & + & 2z & + & t & = & 0 \\ & & & & z & + & 2t & = & 5 \end{array} \qquad \begin{array}{rcrcrcrcl} 2x & - & y & & & & & = & 0 \\ -x & + & 2y & - & z & & & = & 0 \\ & & -y & + & 2z & - & t & = & 0 \\ & & & & -z & + & 2t & = & 5 \end{array}$$

Problem 2.2.27. Elimination in the usual order gives what upper-triangular matrix U and what solution to this "lower triangular" system? We are really solving by "forward substitution":

$$\begin{array}{rcrcrcl} 3x & + & & & & = & 3 \\ 6x & + & 2y & & & = & 8 \\ 9x & - & 2y & + & z & = & 9 \end{array}$$

Graded Problems.

Problem 1. Write the following system of linear equations in two different forms: as a matrix-vector equation involving the coefficient matrix, and as a vector equation expressing $\mathbf{b} = (b_1, b_2)$ as a linear combination of column vectors.

$$2x - 2y = b_1$$
$$-x + 2y = b_2$$

Solve the system for $\mathbf{b} = (1, 0)$ and $\mathbf{b} = (0, 1)$, and show how to write these two vectors as linear combinations of the columns of the coefficient matrix.

Problem 2. Use elimination to try to solve the general 2×2 linear system:

$$ax + by = f$$
$$cx + dy = g$$

What condition on the coefficients a, b, c, d do you need to guarantee that there is a unique solution?