Solution of HW9.

1. (a). $\overline{\langle f, g \rangle} = \overline{\int_0^1 \bar{f}(x) g(x) dx} = \int_0^1 \overline{(\bar{f}(x) g(x))}\, dx$

$= \int_0^1 f(x) \bar{g}(x) dx$

$= \langle g, f \rangle$

$\forall a, b \in \mathbb{C}$

(b) $\langle af_1 + bf_2, g \rangle = \int_0^1 \overline{(af_1(x) + bf_2(x))}\, g(x) dx$

$= \int_0^1 \bar{a} \bar{f}_1(x) g(x) dx + \int_0^1 \bar{b} \bar{f}_2(x) g(x) dx$

$= \bar{a} \langle f_1, g \rangle + \bar{b} \langle f_2, g \rangle$

(c) $\langle f, f \rangle = \int_0^1 \bar{f}(x) \cdot f(x) dx \geq 0$

If $f = 0$ clearly $\langle f, f \rangle = 0$

If $\langle f, f \rangle = 0$. then $\int_0^1 \bar{f}(x) f(x) dx = 0$

Since $f(x)$ is continuous, we see that $f(x) = 0$.

So $\langle \ , \ \rangle$ is an inner product.

2. $v^* H v$ is a number,

Consider

$\overline{v^* H v}^T = \left(\overline{v^*} \cdot \overline{H} \cdot \overline{v} \right)^T$

$= \overline{v}^T \cdot \overline{H}^T \cdot \left(\overline{v^*} \right)^T$

$= v^* \cdot H \cdot v$

$\Rightarrow \overline{v^* H v} = v^* H v$. So $v^* H v$ is a real number.

3. • If $g_{\mathbb{R}} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ is symmetry.

then $A^T = A$, $B^T = -B$.

so $g_{\mathbb{C}}^* = (A+Bi)^* = (A-Bi)^T = A^T - B^T i = A+Bi = g_{\mathbb{C}}$

$\Rightarrow$ $g_{\mathbb{C}}$ is Hermitian

• If $g_{\mathbb{C}}$ is Hermitian, then $g_{\mathbb{C}}^* = g_{\mathbb{C}}$

$\Rightarrow$ $A^T - B^T i = A + Bi$ $\Rightarrow$ $A = A^T$, $B = -B^T$

$\Rightarrow$ $g_{\mathbb{R}}$ is Symmetry.

4. Cleary g is NOT Hermitian since $e^{\frac{2\pi i}{3}}$ is not real.

Notice that $g^* = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ 1 & e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} \end{pmatrix}$

And $g^* \cdot g = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ so g is unitary.

Solution of HW 10.

1. ① $e_1 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $e_3 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$

$\langle e_1, e_1 \rangle = 2. \Rightarrow \tilde{e}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$

$e_2' = e_2 - \langle \tilde{e}_1, e_2 \rangle \tilde{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{i}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$

$\tilde{e}_2 = e_2' / |e_2'| = \sqrt{\frac{2}{3}} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$

$e_3' = e_3 - \langle \tilde{e}_1, e_3 \rangle \tilde{e}_1 - \langle \tilde{e}_2, e_3 \rangle \tilde{e}_2$

$= \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} - 0 \cdot \tilde{e}_1 - \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{2}{3}} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{2i}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{2}{3} \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}$

$\Rightarrow \tilde{e}_3 = \frac{\sqrt{3}}{3} \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}$

So $\tilde{e}_1 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$ $\tilde{e}_2 = \frac{\sqrt{6}}{3} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$ $\tilde{e}_3 = \frac{\sqrt{3}}{3} \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}$

is an orthonormal basis

② $W^\perp = \langle \tilde{e}_3 \rangle$ since $W = \langle e_1, e_2 \rangle = \langle \tilde{e}_1, \tilde{e}_2 \rangle$

and $\tilde{e}_3, \tilde{e}_1, \tilde{e}_2$ form an orthonormal basis.

2. For any $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.

then $\langle v, v \rangle = v^* H v = \lambda_1 v_1 \cdot \overline{v_1} + \lambda_2 v_2 \cdot \overline{v_2} + \cdots + \lambda_n v_n \cdot \overline{v_n}$

So if $\lambda_i > 0$ for $1 \le i \le n$. then $\langle v, v \rangle \geq 0$.

and if $\langle v, v \rangle = 0$. $\Rightarrow \lambda_1 v_1 \cdot \overline{v_1} + \lambda_2 v_2 \cdot \overline{v_2} + \cdots + \lambda_n v_n \cdot \overline{v_n} = 0$

$\Rightarrow v_i = 0 \Rightarrow v = 0$

Conversely, if we have $\lambda_i \le 0$ for some i.

then consider $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$. then

$$\langle e_i, e_i \rangle = \lambda_i \le 0.$$

Since $e_i \neq 0$. this means $\langle \cdot, \cdot \rangle_H$ is NOT positive-definite.

3. Eigenvalues of H:

$$\begin{vmatrix} \lambda - 4 & 0 & -3 - i\sqrt{3} \\ 0 & \lambda - 3 & 0 \\ -3 + i\sqrt{3} & 0 & \lambda - 3 \end{vmatrix} = (\lambda-4)(\lambda-3)^2 - (\lambda-3)(-3-i\sqrt{3})(-3+i\sqrt{3})$$

$$= (\lambda-3)\big((\lambda-4)(\lambda-3) - (9+3)\big)$$

$$= (\lambda-3)(\lambda-7)\cdot\lambda$$

$\Rightarrow$ eigenvalues are $0, 3, 7$.

For $\lambda_1 = 0$. we have $v_1 = \begin{pmatrix} -3 \\ 0 \\ 3 - i\sqrt{3} \end{pmatrix}$

$\lambda_2 = 3$ we have $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = 7$ we have $v_3 = \begin{pmatrix} 3 + i\sqrt{3} \\ 0 \\ 3 \end{pmatrix}$

Now we need to normalize v_1, v_2, v_3

$$v_1/|v_1| = \frac{\sqrt{21}}{21}\begin{pmatrix}3\\0\\3-i\sqrt{3}\end{pmatrix}$$

$$v_3/|v_3| = \frac{\sqrt{21}}{21}\begin{pmatrix}3+i\sqrt{3}\\0\\3\end{pmatrix}$$

$$v_2/|v_2| = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\Rightarrow \quad h = \frac{\sqrt{21}}{21}\begin{pmatrix}3 & 0 & 3+i\sqrt{3}\\0 & \sqrt{21} & 0\\3-i\sqrt{3} & 0 & 3\end{pmatrix}$$

and $h^* H h = \begin{pmatrix}0 & & \\ & 3 & \\ & & 7\end{pmatrix}$

2 H is NOT positive since it has a zero eigenvalue.

4. $A=(A_{ij})>0$. then $A_{ij}>0$.

$X=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \geq 0$, then $x_i \geq 0$. $X \neq 0 \Rightarrow \exists\, i$ s.t. $x_i \neq 0$

then $AX=\begin{pmatrix} \sum_{j=1}^{n} A_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} A_{nj}x_j \end{pmatrix}$

Since $A_{ij}>0$. $x_j \geq 0 \Rightarrow \sum_{j=1}^{n} A_{ij}x_j \geq 0$.

$\exists\, x_i \neq 0 \Rightarrow \sum_{j=1}^{n} A_{ij}x_j \neq 0$.

$\Rightarrow AX>0$

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HW 11

1. (Solution by 戴振宇)

Consider A^T. we see that $A^T > 0$ and $\lambda_{A^T} = \lambda_A$ be the largest eigenvalue of A^T. v_A be the eigenvector, then $v_A > 0$

$$\Rightarrow \quad A^T v_A = \lambda_A v_A$$

$$\Rightarrow \quad v_A^T A = \lambda_A v_A^T$$

Let v_B be eigenvector of B so that $B v_B = \lambda_B v_B$, then $v_B > 0$

$$\Rightarrow \quad \lambda_A v_A^T v_B = v_A^T A v_B > v_A^T B v_B = \lambda_B v_A^T v_B.$$

Notice that $v_A^T v_B > 0$. $\Rightarrow$ $\lambda_A > \lambda_B$

2. ① Eigenvalues of $A = \frac{1}{3}\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$

$$(\lambda - 3)\lambda - 4 = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4).$$

So eigenvalues of A are $\lambda_0 = \frac{4}{3}$ and $\lambda_1 = -\frac{1}{3}$

Thus whole population would grow since $\frac{4}{3} > 1$

② $$\lim_{n\to\infty} \frac{x_1(n)}{x_2(n)} = \lim_{n\to\infty} \frac{x_1(n)/\lambda_0^n}{x_2(n)/\lambda_0^n}.$$

Let us consider $\lim_{n\to\infty} \frac{1}{\lambda_0^n}\begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} = \lim_{n\to\infty} \frac{1}{\lambda_0^n} A^n x = c x_0$ for some c

Since $x_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $x = \begin{pmatrix} 1000 \\ 0 \end{pmatrix} + k x_1$ $\Rightarrow$ $c \neq 0$

$$\Rightarrow \lim_{n\to\infty} \frac{x_1(n)}{x_2(n)} = 2.$$

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HW12.

① Eigenvalues of $\frac{1}{3}\begin{pmatrix}3 & 2\\ 2 & 0\end{pmatrix}$ are $\lambda_0=\frac{4}{3}$, $\lambda_1=-\frac{1}{3}$

eigenvectors: For λ_0: $\frac{1}{3}\begin{pmatrix}-1 & 2\\ 2 & -4\end{pmatrix}x_0=0 \Rightarrow x_0=\begin{pmatrix}2\\ 1\end{pmatrix}$

λ_1: $\frac{1}{3}\begin{pmatrix}4 & 2\\ 2 & 1\end{pmatrix}x_1=0 \Rightarrow x_1=\begin{pmatrix}1\\ -2\end{pmatrix}$

② $$x=\begin{pmatrix}5\\ 0\end{pmatrix}=2\begin{pmatrix}2\\ 1\end{pmatrix}+\begin{pmatrix}1\\ -2\end{pmatrix}=2x_0+x_1$$

③ $$\lim_{n\to\infty}\frac{1}{\lambda_0^n}A^n x=\lim_{n\to\infty}\frac{1}{\lambda_0^n}A^n(2x_0+x_1)$$
$$=\lim_{n\to\infty}\frac{1}{\lambda_0^n}\left(2\cdot\lambda_0^n\cdot x_0+\lambda_1^n x_1\right)$$
$$=\lim_{n\to\infty}\left(2x_0+\left(\frac{\lambda_1}{\lambda_0}\right)^n x_1\right)$$
$$=2x_0=\begin{pmatrix}4\\ 2\end{pmatrix}$$