清华大学 TSINGHUA UNIVERSITY 数学作业纸

班级：计01 姓名：容逸朗 编号：2020010869 科目：概统 第 1 页

11. $P(A) = P(A|C)P(C) + P(A|C^c)P(C^c) \geq P(B|C)P(C) + P(B|C^c)P(C^c) = P(B)$

16. (i) $P((A_1 \cup A_2) \cap A_3 \cap (A_4^c \cup A_5^c))$

$= P((A_1^c A_2^c)^c A_3 (A_4 A_5)^c)$

$= P((A_1^c A_2^c)^c A_3) - P((A_1^c A_2^c)^c A_3 A_4 A_5)$

$= P(A_3) - P(A_1^c A_2^c A_3) - P(A_3 A_4 A_5) + P(A_1^c A_2^c A_3 A_4 A_5)$

$= P(A_3) - P(A_1^c)P(A_2^c)P(A_3) - P(A_3)P(A_4)P(A_5) + P(A_1^c)P(A_2^c)P(A_3)P(A_4)P(A_5)$

$= (P(A_3) - P(A_3)P(A_1^c)P(A_2^c))(1 - P(A_4)P(A_5))$

$= P(A_3 \cdot (A_1 \cup A_2)) \cdot P(A_4^c \cup A_5^c)$.

(ii) $P((A_1 \cup A_2) \cap (A_3 \cap A_4))$

$= P((A_1^c A_2^c)^c A_3 A_4)$

$= P(A_3 A_4) - P(A_1^c A_2^c A_3 A_4)$

$= P(A_3)P(A_4) - P(A_1^c)P(A_2^c)P(A_3)P(A_4)$

$= P(A_3 A_4) - P(A_3 A_4)P(A_1^c)P(A_2^c)$

$= P(A_3 A_4) - P(A_3 A_4)P((A_1 \cup A_2)^c)$

$= P(A_3 A_4) \cdot P(A_1 \cup A_2)$

$P((A_1 \cup A_2) \cap A_5^c)$

$= P((A_1^c A_2^c)^c A_5^c)$

$= P(A_5^c) - P(A_1^c A_2^c A_5^c)$

$= P(A_5^c) - P(A_1^c)P(A_2^c)P(A_5^c)$

$= P(A_5^c)(1 - P((A_1 \cup A_2)^c))$

$= P(A_5^c)P(A_1 \cup A_2)$

$P(A_3 \cap A_4 \cap A_5^c)$

$= P(A_3)P(A_4)P(A_5^c)$

$= P(A_3 A_4) \cdot P(A_5^c)$

$P((A_1 \cup A_2) \cap (A_3 \cap A_4) \cap A_5^c)$

$= P((A_1^c A_2^c)^c A_3 A_4 A_5^c)$

$= P(A_3 A_4 A_5^c) - P(A_1^c A_2^c A_3 A_4 A_5^c)$

$= P(A_3)P(A_4)P(A_5^c) - P(A_1^c)P(A_2^c)P(A_3)P(A_4)P(A_5^c)$

$= P(A_5^c)P(A_3 A_4)(1 - P((A_1 \cup A_2)^c))$

$= P(A_5^c)P(A_3 A_4)P(A_1 \cup A_2)$

17. 记：A：4抽2中奖，B：第5抽中奖，U_i：使用第i部老虎机.

$P(B|A) = P(U_1)P((B|A)|U_1) + P(U_2)P((B|A)|U_2) + P(U_3)P((B|A)|U_3)$

$= \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{2}{3} = \frac{1}{2}$

18. 对第一个不合格的考试做讨论，记A为被录取，B_4为全部通过，B_{3i}为只有第i次不通过的情况 $(1 \le i \le 4)$

$P(A) = P(B_4) + P(B_3)$

$= P(B_4) + P(B_{31}) + P(B_{32}) + P(B_{33}) + P(B_{34})$

$= p^4 + (1-p)\frac{p}{2} \cdot p^2 + p(1-p) \cdot \frac{p}{2} \cdot p + p^2(1-p) \cdot \frac{p}{2} + p^3(1-p)$

$= p^3 + \frac{3}{2}p^3(1-p)$.

23. (a) $P(Y_n \geq 0,\ 1 \leq n \leq 4) = P(Y_1 \geq 0)P(Y_2 \geq 0 | Y_1 \geq 0)P(Y_3 \geq 0 | Y_1, Y_2 \geq 0)P(Y_4 \geq 0 | Y_1, Y_2, Y_3 \geq 0)$

$$= \frac{1}{2} \times 1 \times \frac{3}{4} \times 1 = \frac{3}{8}$$

(b) $P(|Y_n| \leq 2,\ 1 \leq n \leq 4) = P(|Y_1| \leq 2) \cdot P(|Y_2| \leq 2 \mid |Y_1| \leq 2)P(|Y_3| \leq 2 \mid |Y_1|, |Y_2| \leq 2) \cdot P(|Y_4| \leq 2 \mid |Y_1|, |Y_2|, |Y_3| \leq 2)$

$$= 1 \times 1 \times \frac{6}{8} \times 1 = \frac{3}{4}$$

(c) $P(Y_n \geq 0,\ 1 \leq n \leq 4,\ Y_4 = 0) = P(Y_1 \geq 0)P(Y_2 \geq 0 | Y_1 \geq 0)P(Y_3 \geq 0 | Y_1, Y_2 \geq 0)P(Y_4 = 0 | Y_1, Y_2, Y_3 \geq 0)$

$$= \frac{1}{2} \times 1 \times \frac{3}{4} \times \frac{2}{6} = \frac{1}{8}$$

$$P(Y_n \geq 0,\ 1 \leq n \leq 4 | Y_4 = 0) = \frac{P(Y_n \geq 0,\ 1 \leq n \leq 4,\ Y_4 = 0)}{P(Y_4 = 0)} = \frac{\frac{1}{8}}{\frac{6}{16}} = \frac{1}{3}$$

25. (a) 记 A 为出现 k 个正面，B_i 表示色子的点数为 i，抛了 i 个硬币

$P(A) = \sum_{i=1}^{6} P(B_i)P(A|B_i) = \frac{1}{6}\sum_{i=1}^{6}\binom{i}{k}\cdot\frac{1}{2^i}$，$i < k$ 时 $\binom{i}{k} = 0$.

(b) 记 A 为出现 3 个正面，B_i 为色子掷出 i.

$P(B_1|A) = P(B_2|A) = 0$，$P(A) = \frac{1}{6}\sum_{i=3}^{6}\binom{i}{3}\cdot\frac{1}{2^i} = \frac{1}{6}$

$$P(B_n|A) = \frac{P(B_n)P(A|B_n)}{P(A)} = \frac{\frac{1}{6}\times\binom{n}{3}\frac{1}{2^n}}{\frac{1}{6}} = \binom{n}{3}\cdot\frac{1}{2^n}$$

28. $P(X \leq Y) = \sum_{n=0}^{\infty} P_n \cdot \left(\sum_{k=0}^{n} P_k\right)$，$P(X = Y) = \sum_{n=0}^{\infty} P_n^2$

41. 先解柯西函数方程，令 $s = 0$，有 $\varphi(t) = \varphi(0)\varphi(t) \Rightarrow \varphi(0) = 1$

令 $t = s, 2s, \cdots (n-1)s$，有 $\varphi(ns) = \varphi(s)\cdot\varphi((n-1)s) = \varphi^2(s)\cdot\varphi((n-2)s) = \varphi^n(s)$ ①

令①中 $s = \frac{m}{n}$，有 $\varphi(m) = \varphi^n(\frac{m}{n}) \Rightarrow \varphi(\frac{m}{n}) = \sqrt[n]{\varphi(m)} = \varphi(1)^{\frac{m}{n}} = \alpha^{\frac{m}{n}}$ (记 $\alpha = \varphi(1)$)

由于 φ 非上升，故 $\alpha \leq 1$，因此可记 $\alpha = e^{-\lambda}$ ($\lambda \geq 0$).

对于无理数 r，总能找到足够大的 $m, n \in \mathbb{Z}$，使得 $\frac{m}{n} \leq r \leq \frac{m+1}{n}$，此时 $\alpha^{\frac{m+1}{n}} \leq \varphi(r) \leq \alpha^{\frac{m}{n}}$.

令 $n \to \infty$，有 $\varphi(r) = \alpha^r$

由此，对任何实数 t，有 $\varphi(t) = e^{-\lambda t}$. ($t \geq 0$)

下证 $P(T > s+t | T > s) = P(T > t) \Leftrightarrow$ 密度函数指数分布.

由 $F_T(x) = P(T \leq x)$ 知 $P(T > s+t | T > s) = \frac{P(T > s+t)}{P(T > s)} = P(T > t)$

$$\Leftrightarrow P(T > s+t) = P(T > s)P(T > t)$$

$$\Leftrightarrow 1 - F_T(s+t) = (1 - F_T(s))(1 - F_T(t))$$

记 $A(x) = 1 - F_T(x)$，有 $A(s+t) = A(s)A(t)$，故 $A(x) = e^{-\lambda x} = 1 - F_T(x) \Rightarrow F_T(x) = 1 - e^{-\lambda x}$

$f(x) = F_T'(x) = \lambda e^{-\lambda x}$. 是指数分布.