Linear Algebra – Homework 11

9 Dec 2020
Due: 17 Dec 2020

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 5.1.2. If a 3×3 matrix has $\det A = -1$, find $\det(\frac{1}{2}A)$, $\det(-A)$, $\det(A^2)$, and $\det(A^{-1})$.

Problem 5.1.7. Find the determinants of rotations and reflections:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 - 2\cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1 - 2\sin^2\theta \end{bmatrix}.$$

Problem 5.1.13. Reduce A to U and find $\det A =$ product of the pivots:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

Problem 5.1.18. Use row operations to show that the 3×3 "Vandermonde determinant" is

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

Problem 5.1.22. From $ad - bc$, find the determinants of A and A^{-1} and $A - \lambda I$:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}.$$

Which two numbers λ lead to $\det(A - \lambda I) = 0$? Write down the matrix $A - \lambda I$ for each of those two numbers λ — it should not be invertible.

Problem 5.1.30. (Calculus question) Show that the partial derivatives of $\ln(\det A)$ give A^{-1}!

$$f(a,b,c,d) = \ln(ad - bc) \qquad \text{leads to} \qquad \begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = A^{-1}.$$

(If you are not familiar with partial derivatives, you can calculate $\partial f/\partial a$ as follows: Treat b, c, d in $f(a,b,c,d)$ as constants and consider f as just a function of one variable a. Then take the derivative of f with respect to a.)

Problem 5.2.1. Compute the determinants of A, B, C from six terms. Are their rows independent?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Problem 5.2.4. Find two ways to choose nonzeros from four different rows and columns:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

(B has the same zeros as A). Is $\det A$ equal to $1+1$ or $1-1$ or $-1-1$? What is $\det B$?

Problem 5.2.15. The tridiagonal 1, 1, 1 matrix of order n has determinant E_n:

$$E_1 = |1| \qquad E_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \qquad E_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \qquad E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

(a) By cofactors show that $E_n = E_{n-1} - E_{n-2}$.

(b) Starting from $E_1 = 1$ and $E_2 = 0$, find $E_3, E_4, \ldots, E_8$.

(c) By noticing how these numbers eventually repeat, find E_{100}.

Problem 5.2.19. The goal of this problem is to find the 4×4 Vandermonde determinant

$$V_4 = \det \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{bmatrix}.$$

(a) Explain why V_4 is a cubic polynomial in the variable x.

(b) Find three possible values r_1, r_2, r_3 for x that make V_4 equal to 0. These are the roots of V_4 as a polynomial in x.

(c) Explain why $V_4 = A(x-r_1)(x-r_2)(x-r_3)$ for some value A, and show that the value of A is the 3×3 Vandermonde determinant from Problem 5.1.18.

(d) Finally, write down a formula for V_4 in terms of a, b, c, x.

Problem 5.2.31. Find the determinant of this cyclic P by cofactors of row 1 and then the "big formula." How many exchanges reorder $4, 1, 2, 3$ into $1, 2, 3, 4$? Is $|P^2| = 1$ or -1?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Problem 5.2.34. This problem shows in two ways that $\det A = 0$ (the x's are any numbers; they don't have to all be the same):

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}.$$

(a) How do you know that the rows are linearly dependent?

(b) Explain why all 120 terms are zero in the big formula for $\det A$.

Graded Problems.

Problem 1. Use row operations to calculate the determinant:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}.$$

Problem 2. Use cofactors to calculate the determinant:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & -2 & 0 & 1 \\ 0 & -2 & 0 & 2 & 0 \end{bmatrix}.$$