Linear Algebra – Homework 12

16 Dec 2020
Due: 24 Dec 2020

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 5.3.1. Solve these linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

$$\text{(a)} \quad \begin{array}{rcrcl} 2x_1 & + & 5x_2 & = & 1 \\ x_1 & + & 4x_2 & = & 2 \end{array} \qquad\qquad \text{(b)} \quad \begin{array}{rcrcrcl} 2x_1 & + & x_2 & & & = & 1 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & + & 2x_3 & = & 0 \end{array}$$

Problem 5.3.5. If the right side $\mathbf{b}$ is the first column of A, solve the 3×3 system $A\mathbf{x} = \mathbf{b}$. How does each determinant in Cramer's Rule lead to this solution $\mathbf{x}$?

Problem 5.3.6. Find A^{-1} from the cofactor formula $C^T / \det A$. Use symmetry in part (b).

$$\text{(a)} \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix} \qquad\qquad \text{(b)} \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Problem 5.3.8. Find the cofactors of A and multiply AC^T to find $\det A$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \quad \text{and} \quad AC^T = \underline{\qquad\qquad}.$$

If you change that 4 to 100, why is $\det A$ unchanged?

Problem 5.3.15. For $n = 5$, the matrix C contains ________ cofactors. Each 4×4 cofactor contains ________ terms and each term needs ________ multiplications. How many total multiplications to compute C? Compare with $5^3 = 125$ total multiplications for the Gauss-Jordan computation of A^{-1} in Section 2.4.

Problem 5.3.17 A box has edges from $(0, 0, 0)$ to $(3, 1, 1)$, to $(1, 3, 1)$, and to $(1, 1, 3)$. Find its volume. (You do not have to answer the second part of this textbook problem, since we did not cover cross products in class.)

Problem 5.3.23. When the edge vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are perpendicular, the volume of the box should be $\|\mathbf{a}\|$ times $\|\mathbf{b}\|$ times $\|\mathbf{c}\|$. Check this formula using determinants: The matrix $A^T A$ is ________. Then find $\det A^T A$ and $|\det A|$.

Problem 6.1.6. Find the eigenvalues of A, B, AB, and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \qquad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

Problem 6.1.12. Find three eigenvectors for this projection matrix P (you may assume that the eigenvalues of P are 1 and 0):

$$P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

Problem 6.1.15. Every permutation matrix leaves $\mathbf{x} = (1, 1, \ldots, 1)$ unchanged, so one eigenvalue is $\lambda = 1$. Find two more λ's (possibly complex) for these permutations, from $\det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Problem 6.1.16. Show that the determinant of A equals the product of eigenvalues $\lambda_1 \lambda_2 \cdots \lambda_n$: Start with the polynomial $\det(A - \lambda I) = 0$ separated into its n factors (always possible as long as you allow the λ's to be complex numbers). Then set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \quad \text{so} \quad \det A = \underline{\qquad\qquad}.$$

Problem 6.1.24. This matrix is singular with rank one. Find all λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

Problem 6.1.27. Find the rank and all eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Problem 6.1.32. Suppose A has eigenvalues 0, 3, 5 with independent eigenvectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $A\mathbf{x} = \mathbf{v} + \mathbf{w}$. Find all solutions.

(c) $A\mathbf{x} = \mathbf{u}$ has no solution: If it did, then ________ would be in the column space.

Graded Problems.

Problem 1.

(a) Find the volume of the box in $\mathbf{R}^4$ determined by the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 8 \\ -8 \end{bmatrix}.$$

(b) If Q is *any* 4×4 orthogonal matrix, what is the volume of the box determined by $Q\mathbf{x}_1$, $Q\mathbf{x}_2$, $Q\mathbf{x}_3$, $Q\mathbf{x}_4$? *Hint:* What does $Q^T Q = I$ tell you about $\det Q$?

Problem 2. Find all eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 2 \\ 2 & -4 & 3 \end{bmatrix}.$$

Does $\mathbf{R}^3$ have a basis consisting of eigenvectors for A?