§ A review of matrix ⇔ linear operator   Conjugation ⇔ change of basis

△ We start with a vector space V, a basis $\{e_i\}_{i=1}^n$ for V, and a linear operator $T: V \to V$

Since $\{e_i\}$ is basis, Te_i is a linear combination of $\{e_i\}$.

So there exists column vector A_i s.t.

$$Te_i = (e_1 \cdots e_n) A_i$$

Putting these together, we can write $T(e_1 \cdots e_n) = (e_1 \cdots e_n)(A_1 A_2 \cdots A_n)$

Here $(A_1 \cdots A_n)$ becomes $n \times n$ matrix, and we call it $A_T = (A_1 \cdots A_n)$

So we can write

$$T(e_1 \cdots e_n) = (e_1 \cdots e_n) A_T \quad ①$$

Example: Consider the case $\dim V = 2$ and $\{e_1, e_2\}$ is a basis. Consider T satisfying

$$Te_1 = e_1 + 2e_2, \quad Te_2 = 3e_1 + 4e_2$$

Then we can write $T(e_1\ e_2) = (e_1\ e_2)\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $A_T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

△ Changing basis.

Same notations as above. Choose now a new basis $\{f_i\}_{i=1}^n$ for V.

Since $\{e_i\}$ is a basis, for each f_i, there is a column vector g_i s.t. such that

$$f_i = (e_1 \cdots e_n) g_i$$

Putting together $(f_1 \cdots f_n) = (e_1 \cdots e_n)(g_1 \cdots g_n) \quad ②$

Denote $g = (g_1 \cdots g_n)$ It is a $n \times n$ matrix, which is invertible because $\{f_i\}$ is a basis

multiply by g^{-1} on the right for both sides of ②

$$(f_1 \cdots f_n) g^{-1} = (e_1 \cdots e_n) \quad ③$$

△ Now we can study the same linear operator T in terms of new basis $\{f_i\}$

For the same reason as for ①, there exists a matrix $\tilde{A}_T$, s.t.

$$T(f_1 \cdots f_n) = (f_1 \cdots f_n) \tilde{A}_T \quad ④$$

We want to know the relation between A_T and $\widetilde{A}_T$

To do this, we use ②, ③. So

$$T(f_1 \cdots f_n) \overset{②}{=} T(e_1 \cdots e_n)\, g \overset{①}{=} (e_1 \cdots e_n) A_T g \overset{③}{=} (f_1 \cdots f_n) g^{-1} A_T g$$

Comparing with ④, we conclude that

$$g^{-1} A_T g = \widetilde{A}_T \qquad (\#)$$

We can $g^{-1}A_T g$ the Conjugation of A_T by g. We also say $g^{-1}A_T g$ and A_T are <u>Similar</u>.

Gist: Changing basis $\Longleftrightarrow$ matrix Conjugation

§ Complex numbers

- Motivation: want to solve all polynomial equation

  Ex: $P(x) = x^2 - a \quad a \in \mathbb{R}$.

  Q: Can we find solutions for $P(x) = 0$?

  $$\begin{cases} \text{If } a > 0 \text{ then yes } x = \pm\sqrt{a}. \\ \text{If } a < 0 \text{ e.g. } a = -1, \text{ then } x^2 + 1 = 0 \text{ has no solutions in } \mathbb{R}. \end{cases}$$

- Made up a solution, called "i" s.t. $i^2 + 1 = 0$ or $i^2 = -1$

- Fundamental theorem of algebra: i "produces" solutions for any polynomial equation $P(x) = 0$.

- Make precise "produce"

  Def: The field of complex number $\mathbb{C}$ is a 2-dim vector space over $\mathbb{R}$

  $\mathbb{C} = \mathbb{R}1 \oplus \mathbb{R}i$ together with a multiplication. s.t.

  $$\mathbb{C} \times \mathbb{C} \to \mathbb{C}$$
  $$x, y \mapsto xy$$

  S.t. It is $\begin{cases} \mathbb{R}\text{-linear in both } x \text{ and } y \\ 1 \cdot 1 = 1 \quad 1 \cdot i = i \cdot 1 = i \quad i \cdot i = -1 \\ \cdot \text{ is Commutative distributive and associative} \end{cases}$

- Translate $z \in \mathbb{C}$ $\quad z = a + bi$ $(a + ib)$ $\quad a, b \in \mathbb{R}$. $\quad z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$ (direct sum)

  $z_j = a_j + b_j i \quad j = 1, 2.$

  $cz = ca + cbi$

  $(c_1 z_1)(c_2 z_2) = c_1 c_2\, z_1 \cdot z_2$ $\qquad$ Commutative: $z_1 \cdot z_2 = z_2 \cdot z_1$

distributive ⇒ $z_1 z_2 = (a_1+b_1 i)(a_2+b_2 i) = a_1\cdot a_2 + b_1 a_2 i + a_1 b_2 i + b_1 b_2 i$ ⊛

$= (a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2)$

Δ $\mathbb{C}$ can also be identified with xy-plane $z = x+iy \mapsto (x,y)\in\mathbb{R}^2$

Δ Complex conjugate (not to be confused with matrix conjugation)

Def: For $z = a+ib \in \mathbb{C}$ its conjugate is $\bar z = a-ib$

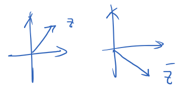
Lem: $\forall z_1, z_2 \in \mathbb{C}$, we have $\overline{z_1\cdot z_2} = \bar z_1 \cdot \bar z_2$

for any

Pf: Suppose $z_j = a_j + ib_j$ for $j=1,2$. Using ⊛, we get $\bar z_1\cdot\bar z_2$ and $\overline{z_1 z_2}$ are both

$(a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1)$ # ⟵ end of proof

Lem: For any $z\in\mathbb{C}$, $z\bar z \in \mathbb{R}_{\ge 0}$ and $z^{-1} = \frac{1}{z\bar z}\bar z$



Pf: Suppose $z = a+ib$ then $\bar z = a-ib$

and $z\cdot\bar z = (a+ib)(a-ib) = (a^2+b^2) + i(-ab+ab) = a^2+b^2 \in \mathbb{R}$ #

Def: $|z| = \sqrt{z\cdot\bar z} = \sqrt{a^2+b^2}$ Consistent with distance in $\mathbb{R}^2$, measures how 'large' is z

Lem:
$$\begin{cases} |z_1+z_2| \le |z_1| + |z_2| \\ |z| = 0 \text{ iff } z = 0 \\ |z_1\cdot z_2| = |z_1|\cdot|z_2| \end{cases}$$

We only check $|z_1\cdot z_2| = |z_1|\cdot|z_2|$

Commutative
associative

Pf: By definition, $|z_1\cdot z_2| = \sqrt{(z_1 z_2)\cdot\overline{(z_1 z_2)}} = \sqrt{(z_1\cdot\bar z_1)\cdot(z_2\cdot\bar z_2)}$

$|z_1|\cdot|z_2| = \sqrt{z_1\cdot\bar z_1}\cdot\sqrt{z_2\cdot\bar z_2} = \sqrt{(z_1\cdot\bar z_1)\cdot(z_2\cdot\bar z_2)}$ #