清華大學
Tsinghua University

A1. $\frac{d(x^2)}{dx} + \frac{d(xy)}{dx} + \frac{d(y^2)}{dx} = 0 \Rightarrow 2x + y + \frac{dy}{dx}\cdot x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{2x+y}{x+2y}$

so $\begin{cases} x^2 + xy + y^2 = 12 \\ -\frac{2x+y}{x+2y} = 0 \end{cases} \Rightarrow \begin{cases} x = -2 \\ y = 4 \end{cases}$ or $\begin{cases} x = 2 \\ y = -4 \end{cases}$

on $(-2,4)$ and $(2,-4)$ is horizontal tangent.

$\begin{cases} x^2 + xy + y^2 = 12 \\ -\frac{x+2y}{2x+y} = 0 \end{cases} \Rightarrow \begin{cases} x = 4 \\ y = -2 \end{cases}$ or $\begin{cases} x = -4 \\ y = 2 \end{cases}$

on $(4,-2)$ and $(-4,2)$ is vertical tangent

A2. (1) $L(a) = f'(a)(a-a) + f(a) \Rightarrow L(a) = f(a)$

(2) $L'(x) = (f'(a)(x-a) + f(a))' = (f'(a)x)' - (f'(a)a)' + (f(a))'$

$= f'(a) - 0 + 0 = f'(a)$, so $L'(a) = f'(a)$

A3. $P(a) = C = f(a)$

$P'(x) = 2A(x-a) + B$, $P'(a) = B = f'(a)$

$P''(x) = 2A$, $P''(a) = 2A = f''(a)$

so there exist a unique $P(x) = \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)$ of degree 2 is satisfying these conditions.

A4. $-x^3 - 9x = x(x-3)(x+3)$

for $x \in (-\infty, -3]$: $f'(x) = (-x^3 + 9x)' = -3x^2 + 9 = -3(x-\sqrt{3})(x+\sqrt{3})$

$f(-3) = 0$, and doesn't exist x such that $f'(x) = 0$

for $x \in [-3, 0]$: $f'(x) = (x^3 - 9x)' = 3x^2 - 9 = 3(x-\sqrt{3})(x+\sqrt{3})$

$f(0) = 0$, and $f'(-\sqrt{3}) = 6\sqrt{3}$

for $x \in (0, 3]$, $f'(x) = -3(x-\sqrt{3})(x+\sqrt{3})$

$f(3) = 0$. and $f'(\sqrt{3}) = 6\sqrt{3}$

for $x \in (3, +\infty)$, $f'(x) = 3(x-\sqrt{3})(x+\sqrt{3})$

There doesn't exist x such that $f'(x) = 0$

in sum: the global minimum is 0, and doesn't have the global maximum.

and the point $(-3, 0)$, $(0, 0)$, $(3, 0)$ is the extrema are attained by f.

A5. let the length of rectangle is x cm, and the height is $\sqrt{1-x^2}$ cm

the area: $C(x) = x \cdot \sqrt{1-x^2}$. $(0 < x < 1)$ (cm)

$c'(x) = \sqrt{1-x^2} + x \cdot \left(-2x \cdot \frac{1}{2\sqrt{1-x^2}}\right) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$, if $x = \frac{\sqrt{2}}{2}$, $c'(x) = 0$

so the largest area is $C\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$

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Bonus exercises

B1 (a) if $f(x)=\begin{cases} x\sin(\frac{1}{x}) & 0<x\leq 1 \\ 0 & x=0 \end{cases}$, then $f(x)$ is continue

but there doesn't exist δ s.t $\forall x\in(0,\delta) \Rightarrow f(x)\geq 0$.
or $\forall x\in(0,\delta) \Rightarrow f(x)\leq 0$

(b) assume that $f(x)=\begin{cases} x^2\sin(\frac{1}{x}) & 0<x\leq 1 \\ 0 & x=0 \end{cases}$

$f'(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\frac{x^2\sin(\frac{1}{x})}{x}=0$, so $f(x)$ is differentiable.

but there doesn't exist δ s.t $\forall x\in(0,\delta)\Rightarrow f(x)\geq 0$
or $\forall x(0,\delta)\Rightarrow f(x)\leq 0$