△ Review ① $g \sim \underset{h^{-1}gh}{\parallel} \begin{pmatrix} J_{\lambda_1,n_1} & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_k,n_k} \end{pmatrix}$

② $N = \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}}_{n}$ $\qquad N^i = \begin{pmatrix} & \overbrace{\quad}^{i+1\text{-th}} 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & \end{pmatrix}$

$N^n = 0$

△. Given a function $f(x)$: polynomial, Taylor expansion

$f(x) = x^2+1 \qquad f(x) = e^x$

square matrix g

How to compute $f(g)$

△ Example: $g = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ $\qquad P_g(\lambda) = (\lambda-1)(\lambda-3) = \lambda^2 - 4\lambda + 3$

$P_g(g) = (g-I)(g-3I) = g^2 - 4g + 3$

open bracket & $g \cdot I = I \cdot g = g$

$g - I = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$ $\qquad g - 3I = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$

multiply $= 0$

△ Thm (Cayley - Hamilton) In general $P_g(g) = 0$

pf: Step 1. Observe that $P_g(g) = 0$ iff $P_g(h^{-1}gh) = 0 \quad \forall h \in GL_n(\mathbb{C})$

$P_g(\lambda) = \underline{(\lambda-\lambda_1)^{n_1} \cdots (\lambda-\lambda_k)^{n_k}} = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$

$\underline{P_g(h^{-1}gh)} = (h^{-1}gh)^n + a_{n-1}(h^{-1}gh)^{n-1} + \cdots + a_1 h^{-1}gh + a_0$

$(h^{-1}gh)^i$

$= \underbrace{(h^{-1}gh)(h^{-1}gh)\cdots}_{i\text{-copy}}$

$= h^{-1} g^i h$

$= h^{-1} g^n h + a_{n-1} h^{-1} g^{n-1} h + \cdots + a_0$

$= h^{-1}(g^n + a_{n-1} g^{n-1} + \cdots \quad a_0)h$

$= h^{-1} \underline{P_g(g)} h$

$\Rightarrow$ $P_g(g) = 0$ iff $P_g(h^{-1}gh) = 0$

h is invertible $\Rightarrow$ $P_g(g)=0$ iff $P_g(h^{-1}gh)=0$

Step 1 $\Rightarrow$ just need to check $P_g(g)=0$ when g is Jordan

$\Rightarrow$ $P_g(J_{\lambda_i,m_i})=0$ for any Jordan block of g, $m_i \le n_i$

Step 2. plug J_{λ_i,m_i} into $P_g(\lambda)=\underbrace{(\lambda-\lambda_1)^{n_1}\cdots(\lambda-\lambda_k)^{n_k}}$

$$P_g(J_{\lambda_i,m_i}) = \underbrace{(J_{\lambda_i,m_i}-\lambda_1 I)^{n_1}\cdots(\underbrace{J_{\lambda_i,m_i}-\lambda_i I}) ^{n_i}}\cdots(\cdots)=0$$

$J_{\lambda_i,m_i}-\lambda_i I$ is nilpotent $(J_{\lambda_i,m_i}-\lambda_i I)^{m_i}=0$

#

Remark: ① $f(\underbrace{h^{-1}g}h) = h^{-1}f(g)h$

$\Updownarrow$

$f(g) = h\, f(\underbrace{h^{-1}gh})\, h^{-1}$ way to compute $f(g)$ using Jordan

goal Jordan

② Overkill to get product $=0$

Δ Def: $q_g(\lambda) = \prod_i (\lambda-\lambda_i)^{m_{\lambda_i}}$ λ_i eigenvalues for g

m_{λ_i} is maximal size of Jordan blocks associated to λ_i

Call $q_g(\lambda)$ the minimal polynomial of g.

Cor: $q_g(g)=0$ Pf same as above thm.

Lem: $f(x)$ is a polynomial, then $f(g)=0$ iff $f(x)=q_g(x)\cdot f_0(x)$

where $f_0(x)$ is a polynomial

Pf: ① If $f(x)=q_g\cdot f_0$ then $f(g)=0$ as $q_g(g)=0$

$$f(g) = q_g(g)\cdot f_0(g)$$

② Now suppose $f(g)=0$, and $f(x)=q_g(x)\cdot f_0(x)+r(x)$

where $\deg r(x) < \deg q_g(x)$ $r(x)\neq 0$.

Factorize $r(x) = \prod (x-\lambda_i)^{l_i} \prod (x-\mu_j)^{k_j}$

because $\deg r < \deg q_g$, there must be some i s.t. $l_i < m_{\lambda_i}$

$$r(J_{\lambda_i, m_{\lambda_i}}) = \underbrace{(J_{\lambda_i, m_{\lambda_i}} - \lambda_1 I)^{l_1}}_{\text{invertible}} \cdots \underbrace{(J_{\lambda_i, m_{\lambda_i}} - \lambda_i I)^{l_i}}_{\text{nilpotent}} (\cdots) \neq 0$$

$l_i < m_{\lambda_i}$

$(J_{\lambda_i, m_{\lambda_i}} - \lambda_i I)^{l_i} \neq 0$

But by Condition

$\underset{0}{f(g)} = \underset{0}{q_g(g)} f_1(g) + \underset{\neq 0}{r(g)}$ Contradiction

#

$J \sim \begin{pmatrix} & -I \\ I & \end{pmatrix}$ $J \in Gl_n(\mathbb{R})$

$\triangle$ Example: Recall J gives complex structure if $J^2 = -I$

then $q_J(x) = x^2 + 1$

pf: J satisfies $f(x) = x^2 + 1 = 0$ $f(x) = (x+i)(x-i)$

Lem above $\Rightarrow$ $q_J \mid f(x)$ Check it's impossible for $q_J = x+i$ or $x-i$

$q_J = x+i$ means $J = -iI \notin Gl_n(\mathbb{R})$

Similar for $q_J = x-i$ impossible

$\triangle$ In general Suppose $f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots$ is Taylor expansion

Suppose it is convergent for $|x| < R$ ← radius of convergence

Given a matrix g, s.t. all its eigenvalues $|\lambda_i| < R$ $= \sqrt{\lambda_i \cdot \bar{\lambda}_i}$

then define $f(g) = a_0 + a_1 g + a_2 g^2 + \cdots$

which is convergent

$\triangle$ Actually compute $f(g)$:

Step 1. find h, h^{-1}, s.t. $h^{-1} g h$ is Jordan

Step 2: Compute $f(h^{-1}gh)$ by computing $f(J_{\lambda_i, m_i})$

Step 3: $f(g) = h \cdot f(h^{-1}gh)\, h^{-1}$

Δ How to compute $f(J_{\lambda,n})$

Lem: If $|\lambda| < R$,

$$f(J_{\lambda,n}) = \begin{pmatrix} f(\lambda) & f'(\lambda) & \frac{f^{(2)}(\lambda)}{2} & \cdots & \frac{f^{(i)}(\lambda)}{i!} & \cdots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ & \ddots & \ddots & \ddots & & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & & \frac{f^{(2)}(\lambda)}{2} \\ & 0 & & & \ddots & \ddots & f'(\lambda) \\ & & & & & & f(\lambda) \end{pmatrix}$$

$i! = i(i-1)\cdots 1$

Pf: $f(x) \overset{\text{Taylor expansion at } x=\lambda}{=} f(\lambda) + f'(\lambda)(x-\lambda) + \frac{f^{(2)}(\lambda)}{2}(x-\lambda)^2 + \cdots + \frac{f^{(i)}(\lambda)}{i!}(x-\lambda)^i + \cdots$

$$f(J_{\lambda,n}) \overset{\downarrow}{=} f(\lambda)I + f'(\lambda)\underbrace{(J_{\lambda,m} - \lambda)}_{\text{is nilpotent}\ \|\ N} + \frac{f^{(2)}(\lambda)}{2}\underbrace{(J_{\lambda,m}-\lambda)^2}_{\| \ N^2} + \cdots + \frac{f^{(i)}(\lambda)}{i!}\underbrace{(J_{\lambda,m}-\lambda)^i}_{\|\ N^i} + \cdots$$

Once $i \geq n$ $\quad \frac{f^{(i)}(\lambda)}{i!}(J_{\lambda,m} - \lambda)^i = 0$

#

Corollary: If $\lambda_1 \cdots \lambda_n$ are eigenvalues for g, then $f(\lambda_1) \cdots f(\lambda_n)$ are eigenvalues for $f(g)$

Δ Example: $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^i + \cdots$

is convergent only for $|x| < 1$ $\quad (R = 1)$

Indeed put $x = 1$ $\quad 1 + 1 + \cdots + 1 + \cdots$ is divergent

Given g s.t. $\boxed{|\lambda_i| < 1}$

we can define $f(g) = 1 + g + g^2 + \cdots g \cdots$ is convergent

$f(g) = (\underline{I - g})^{-1}$ $\quad$ that is $f(g)$ is the inverse matrix to $I - g$

Remark: gives a way to compute inverse matrix under some conditions

Example: $f(x) = e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^i}{i!} + \cdots$

is convergent for any x $\quad (R = +\infty)$

For any matrix g we can define

$$f(g) = e^g = 1 + g + \frac{g^2}{2} + \cdots + \frac{g^i}{i!} + \cdots$$

always convergent!

Example. $g = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ $\qquad e^g = ?$

Solution. Step 1: $P(\lambda) = (\lambda - 1)(\lambda - 3)$

$\lambda_1 = 1$ $\qquad \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \vec{v} = 0$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 3$ $\qquad \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \vec{v} = 0$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad h^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $\quad h^{-1} g h = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$

Step 2: $f(h^{-1} g h) = e^{\begin{pmatrix} 1 & \\ & 3 \end{pmatrix}} = \begin{pmatrix} e^1 & \\ & e^3 \end{pmatrix}$

Step 3: $f(g) = h\, f(h^{-1} g h)\, h^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^1 & \\ & e^3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^1 & e^3 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^1 & \boxed{e^3 - e^1} \\ 0 & e^3 \end{pmatrix} \qquad \#$$

$\triangle$ We know $e^x \cdot e^y = e^{x+y}$

$$\left(1 + x + \frac{x^2}{2} + \cdots\right) \left(1 + y + \frac{y^2}{2} + \cdots\right)$$

Question: $e^A \cdot e^B \overset{?}{=} e^{A+B}$

In general, no.

Lemma: If $A \cdot B = B \cdot A$, then $e^A \cdot e^B = e^{A+B}$

Pf. Same as for $e^x \cdot e^y = e^{x+y}$ with condition $A \cdot B = B \cdot A$

pf: Same as for $e^x \cdot e^y = e^{x+y}$ with condition $A \cdot B = B \cdot A$

$ABA^{-1} = B$

Definition We say A, B commutes with each other if $AB = B \cdot A$

Δ Differential equation, find $f(x)$,

DE: s.t. $f'(x) = a \cdot f(x)$ initial condition $f(0) = C$

This problem has solution $f(x) = C \cdot e^{ax}$

Check: $f'(x) = aC \cdot e^{ax} = a \cdot f(x)$ $\quad ((e^x)' = e^x)$

$f(0) = C \cdot e^0 = C \cdot I = C$

matrix, number

Δ $e^{gx} = 1 + gx + \frac{(gx)^2}{2} + \cdots$

$\frac{d}{dx}\left(e^{gx}\right)$