L12

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△ Review: Hermitian form $\langle\ ,\ \rangle : V\times V \to \mathbb{C}$ $\left\{ \begin{matrix} \cdots \\ \cdots \\ \cdots \end{matrix}\right.$

$v_1 \perp v_2 \iff \langle v_1, v_2\rangle = 0$

Given a matrix g, $g^* = \overline{g^T}$ $(g_1 g_2)^* = g_2^* \cdot g_1^*$

Hermitian matrix $H^* = H$

Once we fix a standard Hermitian form $V = \mathbb{C}^n$ $\langle \vec{x}, \vec{y}\rangle = \sum_{i=1}^n \overline{x_i}\cdot y_i$

Unitary matrix U, TFAE

$\left\{ \begin{array}{l} U^* \cdot U = U\cdot U^* = I \\ \langle U\cdot v_1, U\cdot v_2\rangle = \langle v_1, v_2\rangle \quad \forall v_1, v_2 \in V \\ U \text{ change ONB to ONB} \end{array}\right.$

△ Thm: Given (another) Hermitian matrix H. $U^* = U^{-1}$

①: $\exists$ Unitary matrix U, s.t. $U^* \cdot H \cdot U$ is diagonal with entries $\in \mathbb{R}$

$(U^* H U)^* = U^* H U$

②: $\exists$ ONB $\{f_i\}$ which is also the set of eigenvectors

Def: $W \subset V$ subspace define $W^\perp = \{v \mid \langle v, w\rangle = 0 \ \forall w \in W\}$

Lem: $W^\perp$ is also a vector subspace

Pf: for $v \in W^\perp$, it suffices to check only for a basis $\{e_i\}$ for W

$w = \sum a_i e_i$

Indeed $\langle v, w\rangle = \sum a_i \langle v, e_i\rangle = 0$ $\langle v, w\rangle = v^* A w$

$\langle v, e_i \rangle = \vec{v}^* \cdot e_i$ as column vectors

$\vec{v}^* \cdot \underline{e_i} = 0$ is a linear equation

Set of equations $(\vec{v}^* \cdot e_i = 0)_i$ is a system of linear equation

Solutions is always a vector space #

Lem: $V = W \oplus W^\perp$

Pf: $\Leftrightarrow$ $\forall v \in V$, $v = w + w^\perp$ uniquely for $w \in W$, $w^\perp \in W^\perp$

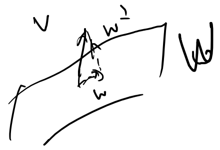
Uniqueness: $v = w_1 + w_1^\perp = w_2 + w_2^\perp$ (need to show $w_1 = w_2$)

$\Rightarrow x = w_1 - w_2 = w_2^\perp - w_1^\perp$ suppose $w_1 \neq w_2$

$\Rightarrow$ find $x \neq 0$, $x \in W$, $x \in W^\perp$

but $\langle \underset{W}{x}, \overset{W^\perp}{x} \rangle = 0$ positive definite $\Rightarrow x = 0$ Contradiction

Existence: Pick ONB for W: $\{e_i\}$



define $w = \sum_i \underline{\langle e_i, v \rangle} \cdot e_i$

$\langle w, e_j \rangle = \langle v, e_j \rangle$

$\langle \sum \langle e_i, v \rangle e_i, e_j \rangle \overset{?}{=} \langle v, e_j \rangle$

$\sum_i \overline{\langle e_i, v \rangle} \delta_{ij}$

$= \overline{\langle e_j, v \rangle} = \langle v, e_j \rangle$

define $w^\perp = v - w$

then indeed $v = w + w^\perp$

Need to check $w^\perp \in W^\perp$, or equivalently $\langle w^\perp, e_i \rangle = 0$

$\langle w^\perp, e_i \rangle = \langle v - w, e_i \rangle = \underline{\langle v, e_i \rangle} - \langle w, e_i \rangle = 0$ #

Lem A: W is an invariant subspace for action of H

then $W^\perp$ is also an $\underline{\text{invariant}}$ subspace for H

Pf: $H^* = H \Rightarrow \langle v, Hw \rangle = \langle Hv, w \rangle$ ~> standard Hermitian form

This is because LHS $= v^*(H \cdot w) = v^* H^* w$

RHS $= (Hv)^* \cdot w = v^* \cdot H^* \cdot w$

$$RHS = (Hv)^* \cdot w = v^* \cdot H^* \cdot w$$

need to show for $w^\perp \in W^\perp$, that $Hw^\perp \in \underline{W}^\perp$

by definition need $\langle w, Hw^\perp \rangle = 0 \quad \forall w \in W$

$$LHS = \langle Hw, w^\perp \rangle$$

by condition $Hw \in W \Rightarrow \langle Hw, w^\perp \rangle = 0$ #

Pf of thm for version ②

By FTA we can find eigenvalue λ_1, eigenvector $\underline{v_1}$ for H

furthermore, we can assume $\langle v_1, v_1 \rangle = 1$

Lem A: $W = \text{Span}\{v_1\}$, define $W^\perp$

then $W, W^\perp$ are invariant subspaces

Apply <u>same process</u> to $W^\perp$ which is dim $n-1$

(find λ_2, $v_2 \in W^\perp$ $\langle v_2, v_2 \rangle = 1$)

form $W' = \text{Span}\{v_1, v_2\}$ & $W'^\perp$

This is done in finite steps (n steps) dim $V = n$ $\{v_i\}$ are eigenvectors & ONB #

△ Given a Hermitian matrix H

find <u>unitary</u> h, s.t. $h^{-1}Hh$ is diagonal

Similar to find Jordan normal form, but need to make sure eigenvectors form ONB

Example

$$H = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$$

Step 1. find eigenvalues & eigenvectors

$$P_H(\lambda) = \det\begin{pmatrix} \lambda-2 & -1-i \\ -1+i & \lambda-3 \end{pmatrix} = \underbrace{(\lambda-2)(\lambda-3)} - (-1-i)(-1+i)$$

$$= \lambda^2 - 5\lambda + 6 - 2 = \lambda^2 - 5\lambda + 4$$

$\lambda_1 = 1 \qquad \lambda_2 = 4$

For $\lambda_1 = 1$ $\quad (H - \lambda_1 I)v = 0 \qquad H - \lambda_1 I = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$ $\quad a \quad b$

find a solution $v_1 = \begin{pmatrix} -2 \\ 1-i \end{pmatrix}$

For $\lambda_2 = 4$ $\quad \begin{pmatrix} -2 & 1+i \\ 1-i & -1 \end{pmatrix} v = 0 \qquad v_2 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

Step 2. make sure to get ONB

$\left(\langle v_1, v_2 \rangle = \overline{(-2)} \cdot 1 + \overline{(1-i)}(1-i) = -2 + 2 = 0 \right)$

$\langle v_i^0, v_i^0 \rangle = 1 \qquad v_i^0 = \dfrac{v_i}{\langle v_i, v_i \rangle^{\frac{1}{2}}}$

$\langle v_1, v_1 \rangle = |-2|^2 + |1-i|^2 = 4 + 2 = 6 \qquad v_1^0 = \frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1-i \end{pmatrix}$

$\langle v_2, v_2 \rangle = |1|^2 + |1-i|^2 = 1 + 2 = 3 \qquad v_2^0 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

Step 3. $h = (v_1^0, v_2^0) = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1-i}{\sqrt{6}} & \frac{1-i}{\sqrt{3}} \end{pmatrix} \qquad h^{-1} H h = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 4 \end{pmatrix}$

△ Cor : positive definite of $H \iff \lambda_i > 0$

Pf (sketch) $\langle v, v \rangle_H \geq 0$, with $= 0$ iff $v = 0$

By main thm, we can reduce to checking $\langle v, v \rangle_H \geq 0$

for $H = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ $\quad \langle v, v \rangle_H = v^* \cdot H \cdot v = (\bar{v}_1 \cdots \bar{v}_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$$= \sum \lambda_i \bar{v}_i v_i = \sum \underbrace{\lambda_i |v_i|^2}_{\geq 0}$$

Δ Perron-Frobenius Strang §10.3

Motivation: In practice many matrices have real entries

even better ≥ 0

> 0

$\not\to$

Example: Divide population into 3 groups ages $0 \sim 19$ x_1

$20 \sim 39$ x_2

≥ 40 x_3

After 20 years, $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & P_3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

F_i newborn rates from each groups (F_2 large F_1, F_3 small)

P_i survival rates (P_2 large P_3 small)

A has all entries ≥ 0

Def in this case we write $A \geq 0$ similarly $A > 0$

Δ Thm (Perron-Frobenius) Suppose $A > 0$

① $\exists$ an eigenvalue $\lambda_0 \in \mathbb{R}_{>0}$ eigenvector $v_0 > 0$

② $\forall$ other eigenvalue λ, we have $|\lambda| < \lambda_0$

③ v_0 is unique up to scalar

Pf: Define a set $S = \left\{ s \in \mathbb{R}_{>0} \;\middle|\; \begin{array}{l} \text{there exist vector } x \geq 0 \\ Ax \geqslant s \cdot x \end{array} \right\}$

S is nonempty: because pick any $x > 0$, then $Ax > 0$

there must be small enough s s.t. $Ax > sx$

S is also bounded from above: $N = \sum_{j,i} A_{ij}$

then $\forall x \geq 0$, $Ax \ngeq N \cdot x$ indeed suppose $x_i = \max\{x_j\}$

$\left(\sum_j A_{ij} x_j \right)_i$ $\quad (Ax)_i = \sum_j A_{ij} x_j$

$< \sum A_{ij} \max\{x_j\}$

$< N \cdot x_i$

(topology)

$\Rightarrow \exists$ maximal value $\lambda_0 \in S$

Claim: λ_0 is actually an eigenvalue > 0

Indeed by $\lambda_0 \in S$ $\Rightarrow \exists x$, s.t. $Ax \geq \lambda_0 x$ Suppose $Ax \neq \lambda_0 x$

Apply A again $A(Ax) > A(\lambda_0 x)$

by enlarging λ_0 a tiny bit, we can guarantee $A(Ax) > (\lambda_0 + \varepsilon) Ax$

this contradicts that λ_0 is maximal

For this x $Ax = \lambda_0 x$ we have $x \geq 0$ $\lambda_0 x = Ax > 0$

This finishes ① of thm.

② Suppose λ is some other eigenvalue $\neq \lambda_0$ and v is eigenvector

$$Ax = \lambda x$$

take $|\cdot|$ entry-wise $\rightarrow$

$$|\lambda x| = |Ax| = \left(\left| \sum A_{ij} x_j \right| \right)$$

$$= |\lambda| \cdot |x|$$

$$\leq \left(\sum |A_{ij}| \, |x_j| \right)$$

$$\underset{\substack{\parallel \\ A_{ij}}}{\uparrow}$$

triangle-ineq

$$|x_1 + x_2| \leq |x_1| + |x_2|$$

"=" hold iff x_1 x_2 parallel

$$= \underline{A} \cdot \underline{|x|}$$

$\Rightarrow |\lambda| \in S$ $\Rightarrow \lambda_0 \geq |\lambda|$ equality holds if all previous $\leq$ are $=$

$\Rightarrow x_i$ are parallel in $\mathbb{C}$ $x = c \cdot x_0$ s.t. $x_0 \geq 0$

$|x| = |c| \, x_0$ which is the eigenvector in ①

$\left.\begin{array}{l} x \text{ as eigenvector for } \lambda \\ x_0 \text{ as eigenvector for } \lambda_0 \end{array}\right\} \Rightarrow \lambda = \lambda_0$ this finishes ②