L7

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Reference: Linear algebra done right Chapter 8, with slightly different approach

△ Midterm cover homework from this week.

△ Review

(1) Very weak Jordan $g \sim \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & * & \\ & & \lambda_1 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_2 & \\ & 0 & & & & & \ddots \\ & & & & & & & \lambda_k \\ & & & & & & & & \ddots \\ & & & & & & & & & \lambda_k \end{pmatrix}$ with block sizes $n_1, n_2, \ldots, n_k$

(2) Nilpotent $g - \lambda_1 I \sim \begin{pmatrix} 0 & & * & & * \\ & \ddots & & & \\ 0 & & 0 & & \\ & & & * & \\ 0 & & & & \ddots \end{pmatrix}$ ($\triangle$)

nilpotent

(3) Define generalized eigenspace $V_{\lambda_1} = \ker(g-\lambda_1 I)^{n_1}$

(i) $\dim V_{\lambda_1} = n_1$

(ii) $\ker(g-\lambda_1 I)^{i} = \ker(g-\lambda_1 I)^{n_1}$ $i \geq n_1$

△ Remark: (i) (ii) hold for any V_{λ_i} because for very weak Jordan

Our proof has freedom to choose the first eigenvalue

△ Lemma A: Each V_{λ_i} is an invariant subspace for T

Pf: pick $v \in V_{\lambda_i}$ need $Tv \in V_{\lambda_i}$

$\Downarrow$

$(g-\lambda_i I)^{n_i} v = 0$ To check $g v \overset{?}{\in} V_{\lambda_i}$

need to check

$(g-\lambda_i I)^{n_i}(gv) \overset{?}{=} 0$

Now $(g-\lambda_i I)\cdot g = g\cdot(g-\lambda_i I)$ $(g-\lambda_i I)^{n_i} g = g\cdot(g-\lambda_i I)^{n_i}$

LHS $= g^2 - \lambda_i g =$ RHS

$\downarrow$

$g\cdot(g-\lambda_i I)^{n_i} v = g\cdot 0 = 0$ #

△ $P(g)\cdot g = g\cdot P(g)$ $\forall$ polynomial g

△ Lemma B: $T-\lambda_j$ is invertible on V_{λ_i} $\forall j \neq i$ $(g-\lambda_j)$

Pf: To prove $T-\lambda_j$ is invertible on V_{λ_i}

we need to show $\forall v \in V_{\lambda_i}$ $v \neq 0$ $(T-\lambda_j)v \neq 0$

Prove by contradiction Suppose $(T-\lambda_j)v = 0$. ($\triangle$)

Condition $v \in V_{\lambda_i}$ $\Rightarrow$ $(T-\lambda_i)^j v = 0$ (1)

Suppose j is minimal which means $(T-\lambda_i)^{j-1} v \neq 0$ (2)

Suppose j is minimal which means $(T-\lambda_i)^{j-1} v \neq 0$ (2)

$j \geq 1$ because $(T-\lambda_i)^0 v = I \cdot v = v \neq 0$

Apply $(T-\lambda_i)^{j-1}$ to (*)

$$(T-\lambda_i)^{j-1}(\underbrace{T-\lambda_j}_{=})v = 0$$

$$(T-\lambda_i + \lambda_i - \lambda_j)\, v = 0$$

$$(T-\lambda_i)^j v + (T-\lambda_i)^{j-1}(\lambda_i-\lambda_j)\, v = 0$$

$$\overset{(1)}{(T-\lambda_i)^j v = 0} \Rightarrow (\underbrace{\lambda_i-\lambda_j}_{\neq 0})(T-\lambda_i)^{j-1}v = 0 \Rightarrow (T-\lambda_i)^{j-1}v = 0$$

Contradiction (2) #

$\triangle$ Prop: $V = \bigoplus_{\lambda_i} V_{\lambda_i}$ in particular $g \sim \begin{pmatrix} \begin{matrix}\lambda_1 & * \\ 0 & \lambda_1\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}\lambda_2 & * \\ 0 & \lambda_2\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}\lambda_k & * \\ 0 & \lambda_k\end{matrix} \end{pmatrix}$

Lem A: Each V_{λ_i} is invariant subspace

weak Jordan

Previous lemma $V = \bigoplus W_i$ $g \sim \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$

Pf: To prove $V = \bigoplus_{\lambda_i} V_{\lambda_i} \overset{\text{definition}}{\Leftrightarrow} \forall\, v \in V$, $\exists$ unique $v_i \in U_{\lambda_i}$ $v = \sum v_i$

Unique part: $v = \sum v_i = \sum v_i'$ $v_i, v_i' \in V_{\lambda_i}$

then $\sum_i (v_i - v_i') = 0$ $v_i - v_i' \in V_{\lambda_i}$

let $w_i = v_i - v_i'$

Reduce to show: if $\sum w_i = 0$ $(\triangle)$ for $w_i \in U_{\lambda_i}$, then $w_i = 0$?

Prove by contradiction, lets say $w_{i_0} \neq 0$.

By condition $(T-\lambda_i)^{n_i} w_i = 0$ Lemma B $\Rightarrow (T-\lambda_i)^{n_i} w_j \neq 0$.

Apply $\prod_{i \neq i_0} (T-\lambda_i)^{n_i}$ to $(\triangle)$ $\prod(\ )\sum w_i = 0$

$\prod_{i \neq i_0} (T-\lambda_i)^{n_i} w_i = 0$ $\forall i \neq i_0$

$\prod_{i \neq i_0} (T-\lambda_i)^{n_i} w_{i_0} \neq 0$. Contradiction

Up to now $\bigoplus V_{\lambda_i}$ is indeed a direct sum

$\oplus V_{\lambda_i}$ is a linear subspace of $\underline{V}$

Just need to show $\dim \oplus V_{\lambda_i} = \dim V$

$$\dim \oplus V_{\lambda_i} = \sum_i \dim \underline{V_{\lambda_i}} = \sum_i n_i = n = \dim V$$

$$\sum_i \underline{\# \lambda_i} = \# \text{ of solutions to } P(\lambda) = 0$$

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△ This proposition allow us to reduce the proof Jordan form for g to proof of Jordan form for $\underline{g_{\lambda_i} = \begin{pmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix}}$ diagonal block

△ We can further look at $g_{\lambda_i} - \lambda_i I$, which is nilpotent

Lemma: $g_{\lambda_i} - \lambda_i I \sim \begin{pmatrix} J_{0,m_1} & 0 & \cdots & 0 \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & J_{0,m_k} \end{pmatrix} = \overline{\begin{pmatrix} J_{\lambda_i,m_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & \cdots & 0 & J_{\lambda_i,m_k} \end{pmatrix}} - \lambda_i I$ $\quad J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix}$

$\Updownarrow$

$g_{\lambda_i} \sim \begin{pmatrix} J_{\lambda_i,m_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & \cdots & 0 & J_{\lambda_i,m_k} \end{pmatrix}$

$\Downarrow$

Pf: Suppose $\exists h$, $h^{-1} g_{\lambda_i} h =$

$$h^{-1}(g_{\lambda_i} - \lambda_i I) h = h^{-1} g_{\lambda_i} h - h^{-1}(\lambda_i I) h$$
$$= h^{-1} g_{\lambda_i} h - \underline{\lambda_i} I$$
$$= \text{desired shape}$$

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△ We are left to prove the following:

Prop. If g is nilpotent then $g \sim \begin{pmatrix} J_{0,m_1} & & \\ & \ddots & \\ & & J_{0,m_k} \end{pmatrix}$

△ Def: Suppose T / g is nilpotent $T^n = g^n = 0$.

define $V^{(i)} = \ker(T^i) \quad \forall\ 1 \le i \le n$.

Eg: $i = n$ then $V^{(n)}$ is generalized eigenspace for T and $\lambda = 0$

and it's V itself because $T^n = 0$

Properties (1) $\underline{V = V^{(n)} \supset V^{(n-1)} \supset V^{(n-2)}} \cdots V^{(1)} \supset \{0\}$

Pf: $\underline{V^{(i)}} \overset{?}{\supset} V^{(i-1)}$ by def $v \in V^{(i-1)}$ means

$T^{i-1} \cdots \quad T(T^{i-1} \cdots$

pf: $V^{(i)} \supset V^{(i-1)}$ by def $v \in V^{(i-1)}$ means

$T^{i-1} v = 0$ so $T(T^{i-1}v) = T^i v = 0$

$\Rightarrow v \in V^{(i)}$

(2) Each $V^{(i)}$ is an invariant subspace for T

Furthermore $T V^{(i)} \subset V^{(i-1)} \subset V^{(i)}$

Pf: $\forall v \in V^{(i)}$, $Tv \overset{?}{\in} V^{(i-1)}$

$(T^{i-1}) Tv \overset{?}{=} 0$

LHS $= T^i v = 0$ because $v \in V^{(i)}$ #

△ Def: $d_i \triangleq \dim V^{(i)}$ e.g. $d_n = \dim V^{(n)} = \ker(T^n) = n$

Property (1) $\Rightarrow$ $d_n \geq d_{n-1} \geq \cdots \geq d_1$

$c_i \triangleq d_i - d_{i-1} = \dim \left(V^{(i)} / V^{(i-1)} \right)$ $\quad U^{(i)} = V^{(i)} / V^{(i-1)}$

△ Lemma C: Suppose $\bar{v}_1 \ldots \bar{v}_c$ are linearly independent in $U^{(i)}$,

Let $v_i \in V^{(i)}$ be any preimage of $\bar{v}_i$ $\quad i = 1, \ldots, c$

$V^{(n)}$
$\cup$
$V^{(n-1)}$ $\quad U^{(n)}$
$\vdots$
$V^{(i)}$ $\quad U^{(i)}$ $\quad \bar{v}_1 \ldots \bar{v}_c \rightarrow$ $\quad v_1 \ldots v_c \in V^{(i)}$
$\cup$
$V^{(i-1)}$ $\quad\quad Tv_1 \ldots Tv_c \in V^{(i-1)}$
$\vdots$ $\quad\quad T^2 v_1 \ldots \in V^{(i-2)}$

$\vdots$

$T^{i-1} v_1 \ldots T^{i-1} v_c \in V^{(1)}$

Then $\{T^j v_k\}_{k=1,\ldots,c}^{j=0,\ldots,i-1}$ are linearly independent

△ Motivation: T acts well on $\{T^{i-1}v_1, \ldots, Tv_1, v_1\}$

$$T(T^{i-1}v_1 \cdots v_1) = (\underbrace{T^i v_1}_{=0}, T^{i-1}v_1, \cdots, Tv_1) = (T^{i-1}v_1 \cdots v_1) \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}}_{= J_{0,i}}$$

△ Proof of Lemma C Suppose we have a linear relation

0,2

△ Proof of Lemma C: Suppose we have a linear relation

$$\sum a_{kj} T^j v_k = 0 \qquad (*)$$

Pf by contradiction $a_{kj_0} \neq 0$ for some k and j_0 is minimal

apply $\underline{T^{i-j_0-1}}$ to $(*)$

$\Rightarrow \quad \sum a_{kj} T^{i-j_0-1+j} v_k = 0$

because j_0 is minimal $\quad LHS = \sum_{j \geq j_0} a_{kj} \cdots$

$$= a_{kj_0} T^{i-1} v_k + \sum_{j > j_0} a_{kj} \underbrace{T^{\overset{\geq i}{i-1+j-j_0}} v_k}_{0}$$

$$= \sum_i a_{kj_0} T^{i-1} v_k$$

$$= T^{i-1}\left(\sum a_{kj_0} v_k\right) = 0$$

$V^{(i)}$ ∪

$\Rightarrow \quad \underline{\sum a_{kj_0} v_k} \in \underline{V^{(i-1)}}$ i.e. $\overline{\sum a_{kj_0} v_k} = 0$ in $V^{(i)} / V^{(i-1)}$

$\sum a_{kj_0} \bar{v}_k = 0 \qquad a_{kj_0} \neq 0$ for some k

Contrad $\{\bar{v}_k\}$ is L.I. #

△ How to do in general

$V^{(n)} \quad > U^{(n)} = 0$

∪

$V^{(n-1)} \quad > U^{(n-1)} = 0$

⋮

$V^{(i-1)} \quad U^{(m_1)} \neq 0 \quad \bar{v}_1 \cdots \bar{v}_{c_1}$

$V^{(i)}$ ⋮

⋮ $\quad > U^{(1)}$

$v_1 \cdots v_{c_1} \in V^{(i)}$

$Tv_1 \cdots Tv_{c_1} \in V^{(i-1)}$

⋮

$T^{i-1} v_1$

$\dim V^{(i)} = \dim V^{(i-1)} + c_1$

$\dim V^{(i-1)} \geq \dim V^{(i-2)} + c_1$

because $Tv_1 \cdots Tv_c$ L.I. not in $V^{(i-2)}$

Start from $U^{(n)}$

look at first time $U^{(m_1)} \neq 0$

pick a basis $\bar{v}_1 \cdots \bar{v}_{c_1}$

Produce $\{T^j v_k\}^j$

Recall $d_i = \dim V^{(i)}$ $\quad d_n \geq d_{n-1} \geq \cdots \geq d_1$

Lemma: Recall $C_i = d_i - d_{i-1} = \dim \underline{U}^{(i)}$, then $C_n \leq \underset{\substack{\uparrow \\ \text{by Lemma C}}}{C_{n-1}} \leq C_{n-2} \leq \cdots C_1$

May start with $C_n = C_{n-1} \cdots = 0$

Step 1. Take note first non zero C_i call it C_{m_1}

Then we can pick basis $\bar{v}_1 \cdots \bar{v}_{C_{m_1}}$ for $U^{(m_1)}$

and produce $\{T^j v_k\}$

$\{T^j v_k\}_j \leftrightarrow$ Jordan blocks of largest size

Step 2: at $i = m_2$ when $C_i < C_{i-1}$ Second time when C_i jumps

we need to find new Linearly independent vectors

$C_{i-1} - C_i$ of them $\quad C_i = \dim U^{(i)}$ T

$\bar{w}_1 \cdots \bar{w}_{C_{i-1}-C_i}$ $\quad C_{i-1} = \dim \underline{U}^{(i-1)}$ is larger

Form a set $\{T^j w_k\}_{j,k}$

$\{T^j w_k\}_j \leftrightarrow$ smaller Jordan blocks

Continue finitely many steps, we can stop.

Δ Given g can ask (1) what's Jordan normal form

(2) Which h s.t. $h^{-1} g h$ is Jordan

Δ Example: If $\underline{g^n = 0}$, $\underline{g^{n-1}} \neq 0$, then $g \sim J_{0,n}$ $\quad$ $n =$ size of g

Pf: $\dim \ker(g^i)$ $\quad V^{(n)} = V$ $\quad d_n = n$

$\ker(g^i) \neq V$ because $g^{n-1} \neq 0$. $\quad d_{n-1} < d_n$

$C_n = d_n - d_{n-1} > 0.$

$\Rightarrow$ can find $\bar{v} \in U^{(n)} = V^{(n)}/V^{(n-1)} \neq 0$

Can produce $\{T^{n-1}v$, $T^{n-2}v$ $\cdots$ $T \cdots v\}$ $\quad V^{(n)}$ $\quad V^{(1)}$

Can find [illegible]

Can produce $\{\overset{v^{(n)}}{T^{n-1}v}, T^{n-2}v, \dots, Tv, \overset{v^{(1)}}{v}\}$

There are n of them, and L.I. Already a basis

For this basis T behaves as $J_{0,n}$ as before

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