L4

Recall:

$\mathcal{B}_{\mathbb{R}} = \{e_1 \cdots e_n, ie_1 \cdots ie_n\} \longleftarrow \mathcal{B}_{\mathbb{C}} = \{e_1 \cdots e_n\}$

Vector space $/\mathbb{R}$ even dim $2n$ $\longleftrightarrow$ Vectorspace $/\mathbb{C}$ dim n.

$\uparrow$

I_V realizing multiplication by i

$J^2 = -I$

$I_V^2 = -I$

$$I_V = \underbrace{\left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right)}_{n}\Big\}n$$

Lemma: $\forall J^2 = -I$ will be conjugate to I_V. For simplicity $\dim_{\mathbb{R}} V = 4$

Pf: Idea is to say under a proper basis J acts like I_V

To construct a basis Start with any $v \in V$, $v \neq 0$.

get 2nd element $J \cdot v$

Check Jv, v are L.I (linearly indep)

Assume $av + bJv = 0$ $(*)$ $a, b \in \mathbb{R}$

$J \cdot (*)$ $aJv - bv = 0$ $(\triangle)$

$a \times (*) - b \times (\triangle)$, $(a^2 + b^2)v = 0 \Rightarrow a^2 + b^2 = 0 \Rightarrow \begin{cases} a = 0 \\ b = 0 \end{cases}$

3rd: pick any other $u \in V$ s.t. $\{v, Jv, u\}$ are L.I.

4th: Ju

Pf by Contradiction: Suppose (✩) is nontrivial, assume $d \neq 0$

Check: $\{v, Jv, u, Ju\}$ are L.I.

$J^2 = -I$

$av + bJv + cu + dJu = 0$ (✩)

$J \cdot$ (✩): $aJv - bv + cJu - du = 0$ (△)

$c \times$ (✩) $- d \times$ (△): $(\ \ )v + (\ \ )Jv + (c^2 + d^2)u = 0$

as $\{v, Jv, u\}$ L.I. $\Rightarrow$ all coeff $= 0$ $\Rightarrow$ $c^2 + d^2 = 0$ $\Rightarrow$ $c = d = 0$

Now using $\{v, u, Jv, Ju\}$ as basis

$$J(v, u, Jv, Ju) = (v, u, Jv, Ju)\underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{= I_v}$$

LHS $= (Jv, Ju, -v, -u)$

∘ $$T/\mathbb{R} \xrightarrow[\text{if } T \circ I_v = I_v \circ J]{} T/\mathbb{C}$$

In matrix terms $T \cdot I_v = I_v \cdot T$

Def: we say A, B commute if $A \cdot B = B \cdot A$

Ex. $\dim_{\mathbb{R}} V = 2$ $\quad T = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$ $\quad I_v = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

$= 3 \cdot I + 1 \cdot I_v \leftrightarrow T_{\mathbb{C}} = 3 + i$ ↘ 1×1 Complex matrix

Question: $\det(T)$ relation with $\det(T_{\mathbb{C}})$?

Example. $\det(T) = 9 - (-1) = 10 \neq \det(T_{\mathbb{C}}) = 3 + i$

$= (3+i)(3-i) = 9 + 1$

Lem: $\det(T) = \det(T_{\mathbb{C}}) \cdot \overline{\det(T_{\mathbb{C}})}$

way to compute 4×4 determinant for special T

§ Jordan normal form «Linear algebra done right» Chapter 5, 8

· Jordan normal form

Idea: g being $\overset{n\times n}{\text{square matrix}}$, what's simplest form for g when we do conjugations $h \in GL_n(\mathbb{C})$ $\underline{h^{-1} g h}$?

As linear operator what's simplest form after a change of basis

Definition: a Jordan block of eigenvalue λ, size n (≥ 1) is matrix of form

$$J_{\lambda,n} = \underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{n}$$

Example: $n=1$ $J_{\lambda,1} = \lambda$

$n=2$ $J_{\lambda,2} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

$n=3$ $J_{\lambda,3} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

Thm: $\forall \overset{n\times n \text{ matrix}}{g}$ is conjugate to $\begin{pmatrix} J_{\lambda_1,n_1} & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_k,n_k} \end{pmatrix}$ $\sum n_i = n$.

Rmk: allow λ_i to be same

Ex: a special case g is conjugate $\underline{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}$

Δ Observations $\begin{cases} ① \ \underline{\text{Blockwisely diagonal}} \\ ② \ J_{\lambda,n} \text{ are upper triangular} \end{cases}$

Translating ① & ② into linear operator stuff

Δ Definition: Given $T: V \to V$, sub vector space W

we call W an $\underline{\text{invariant}}$ subspace if $TW \subset W$

Lemma: g is conjugate to blockwisely diagonal matrix $\begin{array}{c} \\ n_1\{ \\ n_2\{ \end{array}\begin{pmatrix} \overset{n_1}{A_1} & 0 \\ 0 & \underset{n_2}{A_2} \end{pmatrix}$ $n_1+n_2=n$

if and only if there exists two invariant subspaces W_1, W_2, s.t.

$V = W_1 \oplus W_2$ ...

W_1, W_2, s.t.

$$V = W_1 \oplus W_2 \quad \text{and} \quad T_g|_{W_i} \sim A_i$$

Pf: "$\Rightarrow$" Then $\exists$ basis $\{e_1 \cdots e_{n_1}, e_{n_1+1} \cdots e_n\}$ s.t.

$$(\ast) \quad T_g(e_1 \cdots e_{n_1}, e_{n_1+1} \cdots e_n) = (e_1 \cdots e_{n_1}, e_{n_1+1} \cdots e_n)\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

Let $W_1 = \mathrm{Span}\{e_1 \sim e_{n_1}\}$ $W_2 = \mathrm{Span}\{e_{n_1+1} \sim e_n\}$

$$T_g|_{W_1}(e_1 \cdots e_{n_1}) \overset{\ast}{=} (e_1 \cdots e_{n_1} \cdots e_n)\begin{pmatrix} A_1 \\ 0 \end{pmatrix}$$

$$= (e_1 \cdots e_{n_1}) A_1$$

Similarly $T_g|_{W_2} \rightsquigarrow A_2$

"$\Leftarrow$" Starting with $V = W_1 \oplus W_2$ ($v \in V$ can be uniquely written as $w_1 + w_2$ $w_i \in W_i$)

$\Rightarrow$ if we pick basis $(e_1 \cdots e_{n_1})$ for W_1 $(e_{n_1+1} \cdots e_n)$ for W

then $(e_1 \cdots e_{n_1}, e_{n_1+1} \cdots e_n)$ is a basis for V

$$T_g(e_1 \cdots e_{n_1}) = (e_1 \cdots e_{n_1}) A_1 \quad \text{for some } A_1$$

$$= (e_1 \cdots e_n) \begin{pmatrix} A_1 \\ 0 \end{pmatrix}$$

W_2 invariant $T_g(e_{n_1+1} \cdots e_n) = (e_{n_1+1} \cdots e_n) A_2$

$$= (e_1 \cdots e_n)\begin{pmatrix} 0 \\ A_2 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \#$$

$\triangle$ translate upper triangular (blockwisely upper triangular)

$$g = \begin{pmatrix} \lambda_1 & n \\ 0 & \lambda_2 \end{pmatrix} \quad \underset{(T_g e_1 \ \ T_g e_2)}{T_g(e_1 \ e_2)} = (e_1 \ e_2)\begin{pmatrix} \lambda_1 & n \\ 0 & \lambda_2 \end{pmatrix}$$

$g = \begin{pmatrix} \lambda_1 & n \\ 0 & \lambda_2 \end{pmatrix}$ $\quad T_g(e_1\ e_2) = (e_1\ e_2)\begin{pmatrix} \lambda_1 & n \\ 0 & \lambda_2 \end{pmatrix}$

$(T_g e_1, T_g e_2)$

$T_g e_1 = \lambda_1 e_1$ $\quad W_1 = \mathrm{span}\{e_1\}$

$\Rightarrow W_1$ is invariant

$T_g e_2 = n\cdot e_1 + \lambda_2 e_2$

'Forget' e_1 part $\nearrow$ $\quad T_g e_2 \sim \lambda_2 e_2$

Side remark: Integers, Congruence between two numbers

$3 \div 2 = 1$ remainder 1

$3 \equiv 1 \ (2)$

$3 \equiv 1 \pmod 2$ $\quad 2 \mid 3-1$ $\quad 3-1 \in 2\mathbb{Z}$

△ Make precise 'Forget'

Def. Given $W \subset V$ a subspace, define an equivalence relation $v_1 \sim v_2$ iff $v_1 - v_2 \in W$

Define a quotient vector space $U = V/W = \{v \in V\}/\sim$

elements of U can be identified with $\bar{v} = v + W = \{v' \mid v - v' \in W\}$ $\quad = v' + W$

U have a structure as vector space
$$\begin{cases} \bar{v}_1 + \bar{v}_2 = v_1 + v_2 + W \\ c\cdot\bar{v}_1 = cv_1 + W \end{cases} \qquad \bar{v}_1 = v_1 + W, \quad \bar{v}_2 = v_2 + W$$

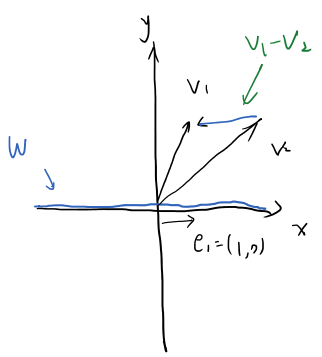
Check they are well-defined: if we pick $v_1' \in v_1 + W$, $v_2' \in v_2 + W$

$$\bar{v}_1 + \bar{v}_2 \overset{?}{=} \bar{v}_1' + \bar{v}_2'$$
$$\| \qquad\qquad \|$$
$$v_1 + v_2 + W \overset{?}{=} v_1' + v_2' + W$$

still same collection of vectors

but we know $v_1 - v_1' \in W$, $v_2 - v_2' \in W$

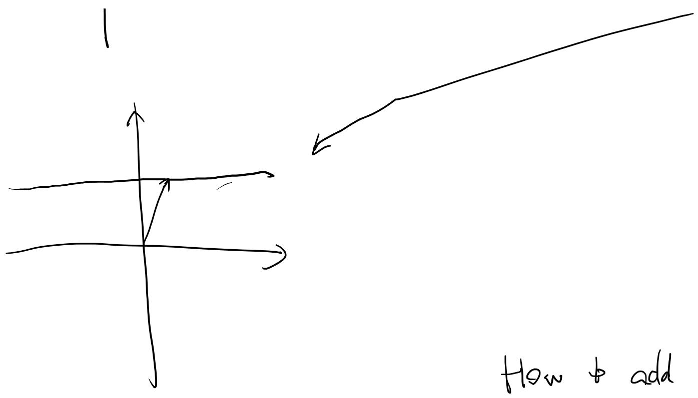
△ Examples. V = xy plane $\quad W$ = x-axis = $\mathrm{span}\{e_1\}$



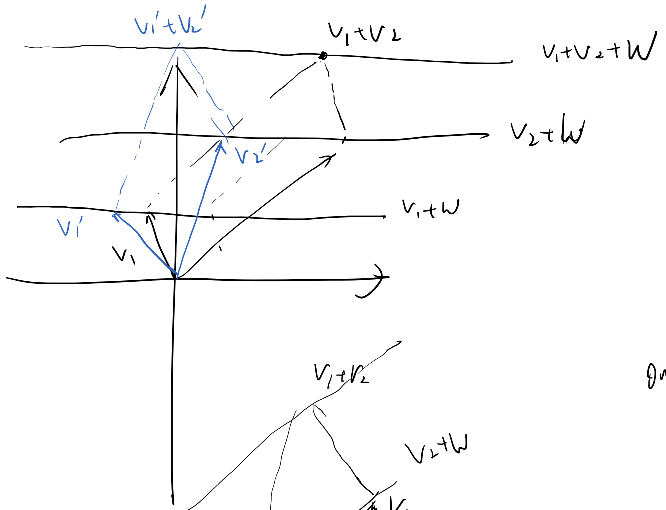
$v_1 \sim v_2$ iff $v_1 - v_2 \in W$ iff $v_1 - v_2$ is parallel to x-axis

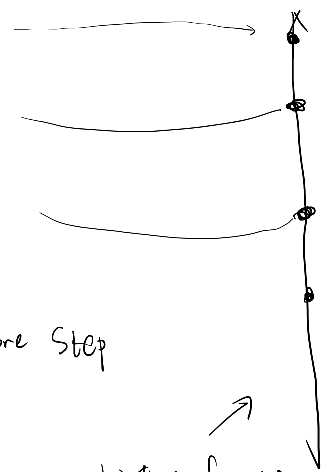
$v + W$ is horizontal line

1



How to add V_1+W V_2+W

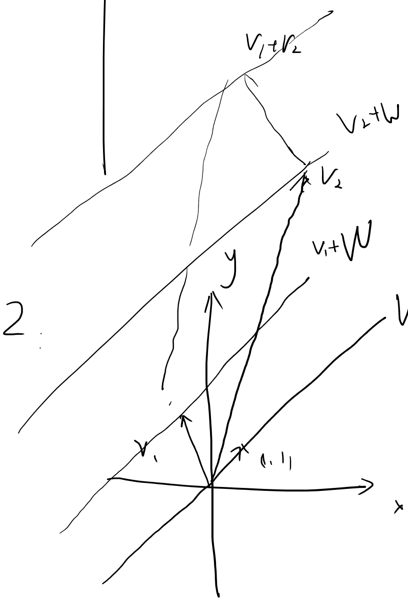




One more Step

picture for V/W

Example 2



$W = \text{Span}(1,1)$

$V+W$