hw 8 - bonus.

班级：计01　　姓名：宋逸朗　　编号：2020010869　　科目：概统　　第 1 页

1. $\int_{-\infty}^{\infty} e^{-(i\xi+x)^2}dx = \int_{-\infty}^{\infty} e^{-x^2-2i\xi x+\xi^2}dx = \int_{-\infty}^{\infty} e^{-x^2}\cdot e^{\xi^2}\cdot[\cos(-2\xi x)+i\sin(-2\xi x)]dx$

由于 e^{-x^2} 为偶函数 e^{ξ^2} 为常数，$\sin(-2\xi x)$ 为奇函数，故 $\int_{-\infty}^{\infty} e^{-x^2}e^{\xi^2}\cdot i\sin(-2\xi x)\,dx = 0$

故只需求：$\int_{-\infty}^{\infty} e^{-x^2}e^{\xi^2}\cos(-2\xi x)dx = e^{\xi^2}\cdot\int_{-\infty}^{\infty} e^{-x^2}\cos(-2\xi x)dx$ ①

取 $I(a) = \int_{-\infty}^{\infty} e^{-x^2}\cos(ax)dx$

则有 $I(a) = -\int_{-\infty}^{\infty} xe^{-x^2}\sin(ax)dx = \frac{1}{2}e^{-x^2}\cdot\sin(ax)\Big|_{-\infty}^{\infty} - \frac{a}{2}\int_{-\infty}^{\infty}\cos(ax)e^{-x^2}dx = 0 - \frac{a}{2}I(a)$

故 $I(a) = c\cdot e^{-\frac{a^2}{4}}$，又 $I(0) = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$，得 $I(a) = \sqrt{\pi}\cdot e^{-\frac{a^2}{4}}$

所以 $\int_{-\infty}^{\infty} e^{-(i\xi+x)^2}dx \overset{①}{=} e^{\xi^2}\int_{-\infty}^{\infty} e^{-x^2}\cos(-2\xi x)dx = e^{\xi^2}\cdot I(-2\xi) = e^{\xi^2}\cdot\sqrt{\pi}\cdot e^{-\xi^2} = \sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}dx$.

2. 由 $X\sim B(n,p)$，$Y\sim B(m,p)$ 知 X的母函数为 $g(z) = \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}z^k = (1-p+pz)^n$

同理可得 Y的母函数 $h(z) = (1-p+pz)^m$

因此，$X+Y$ 的母函数为 $h(z)\cdot g(z) = (1-p+pz)^{m+n} = \sum_{k=0}^{m+n}\binom{m+n}{k}p^k(1-p)^{m+n-k}z^k = \sum_{k=0}^{m+n}B_k(m+n;p)\cdot z^k$

故 $X+Y\sim B(m+n,p)$

3. 下证：$A(x) = \sum_{k=N-x}^{N}\frac{k\cdot(N)_k}{N^{k+1}} = \frac{N!}{x!\,N^{N-x}}$，$0\le x\le N$.

基础：$A(0) = \sum_{k=N}^{N}\frac{k(N)_k}{N^{k+1}} = \frac{N\cdot(N)_N}{N^{N+1}} = \frac{N!}{N^N} = \frac{N!}{0!\,N^{N-0}}$，

归纳：若 $A(n) = \frac{N!}{n!\,N^{N-n}}$ 成立，则 $A(n+1) = \sum_{k=N-n-1}^{N}\frac{k(N)_k}{N^{k+1}} = A(n) + \frac{(N-n-1)(N)_{N-n-1}}{N^{N-n}}$

$$= \frac{N!}{n!\,N^{N-n}} + \frac{(N-n-1)}{N^{N-n}}\frac{N!}{(n+1)!}$$

$$= \frac{(n+1)N! + (N-n-1)N!}{(n+1)!\,N^{N-n}}$$

$$= \frac{N\cdot N!}{(n+1)!\,N^{N-n}}$$

$$= \frac{N!}{(n+1)!\,N^{N-n-1}}$$，满足条件.

故. $\sum_{k=1}^{N}\frac{k\cdot(N)_k}{N^{k+1}} = A(N-1) = \frac{N!}{(N-1+1)!\,N^{N-N+1-1}} = 1$.