清華大學
Tsinghua University

§ 2.6 Exercise 39

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2-1 = 3^2-1 = 8$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2ax = 2a\cdot 3 = 6a$

so $6a = 8 \to a = \frac{4}{3}$

§ 2.6 Exercise 46

suppose $f(x) = \cos x - x$, and $f(\pi) = 1-\pi < 0$, $f(-\pi) = 1+\pi > 0$

According to the IVT, we know that $\exists c \in [\pi, -\pi]$, $f(c)=0$ because $f(x)$ is continue.

so $\cos c - c = 0$, and c is the solution of $\cos x = x$

§ 2.6 Exercise 60

Let $\varepsilon = \left|\frac{f(c)}{2}\right| > 0$, because f is continue on c. so $\exists \delta > 0$, such that $\forall |x-c| < \delta, \Rightarrow |f(x)-f(c)| < \varepsilon$

if $f(c) > 0$, $-\varepsilon < f(x) - f(c) < \varepsilon \Rightarrow -\frac{1}{2}f(c) < f(x) < \frac{1}{2}f(c) \Rightarrow \frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c)$

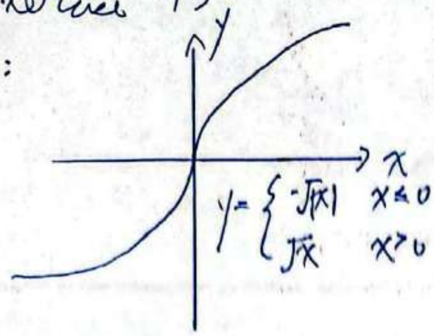
so $\exists \delta$, such that $(c-\delta, c+\delta)$ where f has the same sign as $f(c)$.

if $f(c) < 0$ $-\varepsilon < f(x) - f(c) < \varepsilon \Rightarrow \frac{1}{2}f(c) < f(x) - f(c) < -\frac{1}{2}f(c) \Rightarrow \frac{3}{2}f(c) < f(x) < \frac{1}{2}f(c)$

so $\exists \delta$, such that $(c-\delta, c+\delta)$ where f has the same sign as $f(c)$.

§2.7 Exercise 43

a. graph:



the graph appear to have vertical tangents at $x=0$

b. $\lim\limits_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim\limits_{x\to 0^+} \frac{\sqrt{x}-0}{x} = \lim\limits_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty$

$\lim\limits_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim\limits_{x\to 0^-} \frac{-\sqrt{-x}-0}{x} = \lim\limits_{x\to 0^-} \frac{1}{\sqrt{x}} = +\infty$

so y has a vertical tangent at $x=0$

P. 145 Exercise 19

Let midnight point as zero point, and use longitude to represent all the point at Earth's equator. and suppose the funtion of temperature $T(x) = T(x+2\pi)$

if $T(0) - T(0+\pi) = 0$. there exist a pair of antipodal point where the temperature are same

if $T(0) - T(0+\pi) = b \neq 0$, so $T(\pi) - T(\pi+\pi) = -b \neq 0$

because $0 \in (-|b|, |b|)$ so there exist $c \in (0, \pi)$

$\Rightarrow T(c) - T(c+\pi) = 0$, because $T(x)$ is continuoly

In sum, there always exist a pair of point where the temperature are same

§ 3.1 Exercise 58

(a) $|f(0)| \le 0^2$, so $f(0) = 0$. $\lim\limits_{h\to 0} \frac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{f(h)}{h}$

because $-h^2 \le f(h) \le h^2 \Rightarrow -|h| \le \frac{f(h)}{h} \le |h| \Rightarrow \lim\limits_{h\to 0} \frac{f(h)}{h} = 0$ by sandwich Theorem

so $f'(0) = 0$

(b) if $x \neq 0$, we know that $f(x) = x^2 \cdot \sin(\frac{1}{x}) \le x^2$, so $\lim\limits_{x\to 0} f(x) = 0$ by sandwich Theorem and $f(0) = \lim\limits_{x\to 0} f(x)$.

also $|f(x)| \le x^2$, we know that $f'(0) = 0$ by part (a)

§ 3.2 Exercise 53

(a) $\frac{d(uvw)}{dx} = \frac{du}{dx} \cdot vw + \frac{d(vw)}{dx} \cdot u = \frac{du}{dx} vw + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$

$= u'vw + uv'w + uvw'$

(b) $\frac{d(u_1 u_2 u_3 u_4)}{dx} = \frac{du_1}{dx} \cdot (u_2 u_3 u_4) + \frac{d(u_2 u_3 u_4)}{dx} \cdot u_1 = \frac{du_1}{dx} \cdot (u_2 u_3 u_4) + u_1 \frac{du_2}{dx} u_3 u_4$

$+ u_1 u_2 \frac{du_3}{dx} u_4 + u_1 u_2 u_3 \frac{du_4}{dx} = u_1' u_2 u_3 u_4 + u_1 u_2' u_3 u_4 + u_1 u_2 u_3' u_4 + u_1 u_2 u_3 u_4'$

(c) $\frac{d(u_1 u_2 u_3 \dots u_n)}{dx} = u_1' u_2 u_3 \dots u_n + u_1 u_2' u_3 \dots u_n + u_1 u_2 u_3' \dots + u_n + \dots + u_1 u_2 u_3 \dots u_n'$

§ 3.4 Exercise 48

if we want $f(x)$ is continuous at $x=0$, then: $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x)$

$\Rightarrow \lim_{x \to 0^-} (x+b) = \lim_{x \to 0^+} (\cos x) \Rightarrow b = 1$, so if $b=1$, it will be continuely

and $\frac{d(x+b)}{dx} = 1$, $\frac{d(\cos x)}{dx} = 0$, the left-hand and right-hand derivative are not the same. so, there doesn't exist a value b that makes g differentiable.

P.145 Exercise 17

(a) $\forall \varepsilon > 0$, $\exists \delta = \varepsilon$, such that $\forall |x-0| < \delta \rightarrow |f(x) - 0| < \varepsilon$

Proof: if $x \in (-\delta, \delta)$ is rational, $f(x) = x \rightarrow$ if $|x-0| < \delta$, $|f(x) - 0| < \varepsilon$

if $x \in (-\delta, \delta)$ is irrational, $f(x) = 0 \rightarrow |f(x) - x| = 0 < \varepsilon$

so $\lim_{x \to 0} f(x) = 0$, and $f(0) = 0$. so $f(x)$ is continue at 0

(b) $\forall c \neq 0$, Let $\varepsilon = c$, and we choose a $\delta > 0$

if c is rational number, we know that there exist a irrational number $b \in (c-\delta, c)$, and $f(b) = 0$, so $|f(b) - f(c)| > |0 - c| = \varepsilon$,

if c is irrational number, we know that there exist a rational number b that $|b| \in (|c|, |c| + \delta)$ and $f(b) = b$, so $|f(b) - f(c)| > ||c| - 0| = \varepsilon$. so f is not continuous at any non zero point