Practice Midterm Exam

1. Consider the system of linear equations:

$$\begin{array}{rcrcrcrcr} -3x_1 & - & 4x_2 & + & 4x_3 & + & 4x_4 & = & 2 \\ x_1 & + & 2x_2 & - & x_3 & - & 2x_4 & = & -2 \\ -3x_1 & - & 2x_2 & + & 5x_3 & + & 2x_4 & = & \alpha \end{array}$$

where α is any real number.

(a) Find the only value of α for which the system has solutions.

(b) For this value of α, find all solutions to the system of equations.

(c) Identify the reduced row echelon form of the coefficient matrix of this system of equations.

2. (a) A matrix A is skew-symmetric if $A^T = -A$. Use the rules of transposes and the three properties of a subspace to show that the set S of all skew-symmetric $n \times n$ matrices is a subspace of the vector space $\mathbf{M}$ of $n \times n$ matrices.

   (b) Find a spanning set for the subspace of skew-symmetric 2×2 matrices that has only one matrix in it. (That is, show that all skew-symmetric 2×2 matrices are multiples of one particular matrix.)

3. (a) Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

(b) Use A^{-1} to solve the linear system of equations $A\mathbf{x} = (0, 1, 0, 0)$.

4. Suppose $A\mathbf{x} = \mathbf{b}$ is a linear system of n equations in n variables and $\mathbf{x}_1$, $\mathbf{x}_2$ are two solutions with $\mathbf{x}_1 \neq \mathbf{x}_2$.

   (a) Is the matrix A invertible? Explain.

   (b) Show that $\mathbf{x} = \mathbf{x}_1 + \alpha(\mathbf{x}_1 - \mathbf{x}_2)$ is also a solution to $A\mathbf{x} = \mathbf{b}$ for any scalar α. Which value of α gives the solution $\mathbf{x}_2$?

5. Set

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}.$$

(a) Find the LU decomposition of A.

(b) Use the LU decomposition to solve the system of equations $A\mathbf{x} = (1, 2, 3)$ (that is, solve the two triangular systems $L\mathbf{y} = (1, 2, 3)$ and $U\mathbf{x} = \mathbf{y}$).

6. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 5 & 5 & 4 \\ 3 & 6 & 7 & 8 & 5 \end{bmatrix}$$

(a) Find a spanning set (the special solutions) for the null space $\mathbf{N}(A)$.

(b) Find a linear relation on b_1, b_2, b_3 that guarantees that $\mathbf{b} = (b_1, b_2, b_3)$ is a vector in the column space $\mathbf{C}(A)$.

7. Determine whether or not the following sets of vectors are bases for $\mathbb{R}^3$. In case the vectors are *not* linearly independent, find a way to write one vector as a linear combination of the other two.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$

8. Find numbers c that give dependent columns, so that a combination of the columns equals 0. For each value of c that you find, write one column of each matrix as a linear combination of the other two.

   (a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & c \end{bmatrix}$

   (b) $B = \begin{bmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

   (c) $C = \begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$