L8

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△. Ideas in proof of Jordan normal form:

$T: V \to V$ $\quad$ $\underline{T}$ is $\underline{\text{nilpotent}}$. Goal $\underline{g} \sim \begin{pmatrix} J_{0,m_1} & & \\ & \ddots & \\ & & J_{0,m_k} \end{pmatrix}$

$$V^{(j)} = \ker(T^j) = \{\vec{v} \mid \underline{g^j \vec{v} = 0}\}$$

$$V = \underline{V}^{(n)} \supset \underline{V}^{(n-1)} \supset V^{(n-2)} \cdots \supset V^{(1)} = V^{(0)} = \{0\}$$

$$d_j = \dim V^{(j)}$$

$$U^{(j)} = V^{(j)}/V^{(j-1)} \qquad C_j = \dim U^{(j)} = \dim V^{(j)} - \dim V^{(j-1)} = d_j - d_{j-1}$$

△ Idea 1: $\{T^{j-1}v, T^{j-2}v, \cdots, Tv, v\}$ provide good basis

T acts on them like Jordan blocks

Idea 2: Lemma Basis: $W \subset V$ invariant subspace, $U = V/W$

Given basis $\{\bar{e}_i\}$ for U, $\{f_j\}$ for W, pick for any $\bar{e}_i$

Then $\{e_i, f_j\}$ is basis for V $\qquad e_i \qquad \bar{e}_i = e_i + W$

If you can give basis for quotient spaces (subspaces) give basis for V

Generalize: $V^{(n)} \xrightarrow{T} V^{(n-1)} \xrightarrow{T} \cdots \xrightarrow{T} V^{(1)} \supset V^{(0)}$

$$\underbrace{U^{(n)} \xrightarrow{T} U^{(n-1)} \xrightarrow{T}}_{\text{quotient spaces}} \cdots \xrightarrow{T} \underline{U^{(1)} = V^{(1)}/V^{(0)}} \to \text{subspace}$$

Lemma Basis: If given basis for each $\underline{U^{(i)}}$ $\Rightarrow$ get basis for V

Check: $\underline{\sum} \dim U^{(j)} = \dim V$

Pf: $$\sum C_j = \sum_{j=1}^{n} (d_j - d_{j-1}) = (\boxed{d_1} - \underset{0}{d_0}) + (d_2 - \boxed{d_1}) + \cdots + (\underset{n}{d_n} - d_{n-1})$$

Pf: $\sum c_j = \sum_{j=1}^{n} (d_j - d_{j-1}) = (d_1 - d_0) + (d_2 - d_1) + \cdots + (d_n - d_{n-1})$

$= n - 0 = n = \dim V$

∘ Idea 3: Lemma 3: Given a set of L.I vectors $v_1 \cdots v_c \in V^{(j)}$ but not in $V^{(j-1)}$ s.t. $\{\bar{v}_1 \cdots \bar{v}_c\}$ is L.I,

then $\{T^{m-1}v_1, T^{m-2}v_1, \cdots v_1, T^{m-1}v_2, \cdots, Tv_2, v_2 \cdots\}$

$= \{T^j v_k\}_{k=1\cdots c}^{j=0,\cdots m-1}$ is L.I.

Lemma 3', T induces maps $\bar{T}: \underline{U^{(i)}} \to \underline{U^{(i-1)}}$

Furthermore, $\bar{T}$ is injective : $\begin{cases} \text{sends L.I vectors to L.I. vectors} \\ \ker \bar{T} = \{0\} \end{cases}$

Sketch of proof: $T: V^{(i)} \to V^{(i-1)}$

$\bar{T}: \overset{U^{(i)}}{\bar{v} = v + V^{(i-1)}} \mapsto Tv + T(V^{(i-1)}) = Tv + V^{(i-2)}$

$T(V^{(i-1)}) \subseteq V^{(i-2)}$, $U^{(i-1)}$

The proof for $\bar{T}$ sending L.I vectors to L.I vectors

is similar to our proof for Lemma 3.

Corollary for Lemma 3': $\dim U^{(i)} \leq \dim U^{(i-1)}$ $\qquad$ $\underline{c_n \leq c_{n-1} \leq \cdots \leq c_1}$

Pf: $\bar{T}: U^{(i)} \to U^{(i-1)}$

$\dim U^{(i-1)} \geq \dim \mathrm{Im}(\bar{T}) = \dim U^{(i)} - \underline{\dim(\ker(\bar{T}))}$

$\overset{\text{injective}}{=} \dim U^{(i)}$

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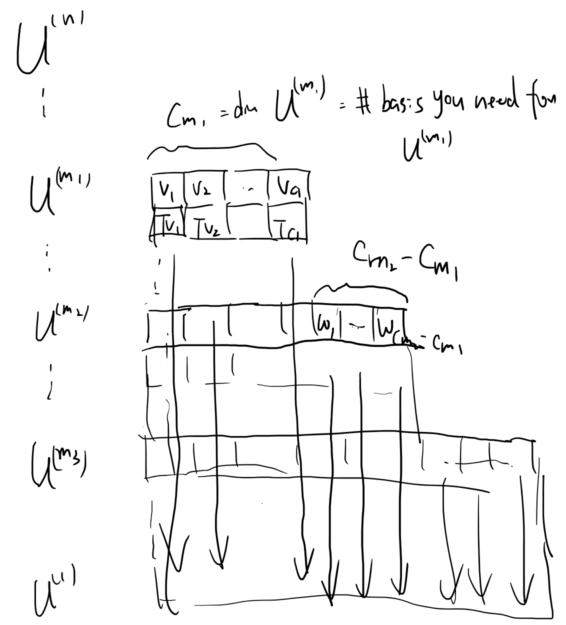
idea: basis for $U^{(i)}$ give L.I vectors for $U^{(i-1)} U^{(i-2)} \cdots U^{(1)}$

by applying T repeatedly

$\triangle$ Combine ideas together

Diagram

$C_{m_1} = \dim U^{(m_1)}$ = # basis you need for $U^{(m_1)}$



by T^j

m_1 is first jump in

$0 \le C_n \le C_{n-1} \cdots \le C_1$

$0 = C_{m_1+1} < C_{m_1}$

m_2 is next jump

$C_{m_1} = \cdots = C_{m_2+1} < C_{m_2}$

pick $v_1 \cdots v_{C_{m_1}}$ for $U^{(m_1)}$

pick additional $w_1 \cdots w_{C_{m_2}-C_{m_1}}$ for $U^{(m_2)}$

Finitely many steps

$\triangle$ General guide for Jordan form

$\begin{cases} \text{given } g\text{, find Jordan} \\ \text{find } h\text{, s.t. } h^{-1}gh \text{ is Jordan} \end{cases}$

Step 1: Find char poly $P_g(x)$, all eigenvalues λ_i with multiplicities n_i

Step 2: Focus on one eigenvalue λ_1, n_1

$g_{\lambda_1}^j := (g - \lambda_1 I)^j \quad j \le n_1$ ($g - \lambda_1 I$ is nilpotent on V_{λ_1})

Compute $d_{\lambda_1, j} = \dim \ker (g - \lambda_1 I)^j$ $\begin{cases} d_{\lambda_1,j} = n - \operatorname{rank}(g - \lambda_1 I)^j \\ \text{Find } \{\vec{v} \mid g_{\lambda_1}^j \vec{v} = 0\} \end{cases}$

Check: $d_{\lambda_1, n_1} \ge d_{\lambda_1, n_1-1} \ge \cdots \ge d_{\lambda_1, 1}$

Check: $d_{\lambda_1,n_1} \geq d_{\lambda_1,n_1-1} \geq \cdots \geq d_{\lambda_1,1}$

Step 3: Compute $C_{\lambda_1,j} = d_{\lambda_1,j} - d_{\lambda_1,j-1}$ get $C_{\lambda_1,n_1} \leq \cdots \leq C_{\lambda_1,1}$

record all jumps in $C_{\lambda_1,n_1} \cdots C_{\lambda_1,1}$

For example first jump previous ones $0 < C_{\lambda_1,m_1}$

record where it jumps $\boxed{m_1}$ Size of Jordan blocks

& size of jump $\underbrace{\boxed{C_{\lambda_1,m_1}}}_{\text{\# of blocks of same size}}$ } C_{λ_1,m_1} number of Jordan block J_{λ_1,m_1}

2nd jump record previous ones $< C_{\lambda_1,m_2}$

where it jumps m_2

size of jump $C_{m_2} - C_{m_1}$ } $(C_{m_2} - C_{m_1})$ number of J_{λ_1,m_2}

Repeat step 2-3 for each λ_i get Jordan form for g

Step 4: Find vectors $v_1 \cdots v_{C_{m_1}}$, $w_1 \cdots w_{C_{m_2}-C_{m_1}}$, apply g_{λ_1} repeatedly to them

$g_{\lambda_1} v_1$

$g^2_{\lambda_1} v_1$

$\vdots$

$g^{m_1-1}_{\lambda_1} v_1$

$v_1 \cdots v_{C_{m_1}} \in V^{(m_1)}$, but not in $V^{(m_1-1)}$

as they should give basis in $V^{(m_1)}$

$g^{m_1}_{\lambda_1} v_1 = 0$

Repeat Step 2 ~ 4 for all λ_i

Step 5: Form matrix h by putting $\{g^j_{\lambda_i} v_k \text{ or } w_k\}$ as column vectors

h is square matrix $\{ \underbrace{g^{j-1}_{\lambda_i} v_k, \cdots g_{\lambda_i} v_k, v_k}\}$

$\lambda_i, v_k, \quad \lambda_i, v_k, v_k)$

are put together

(Step 6 optional if you have plenty of time: check $h^{-1} g h$ is Jordan)

Example 1 $g = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution: Step 1 $P_\lambda(g) = \begin{vmatrix} \lambda-2 & -1 & -1 \\ 1 & \lambda & 1 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-2 & -1 \\ 1 & \lambda \end{vmatrix}$

$= (\lambda-1)^3$

Single eigenvalue $\lambda = 1$ multiplicity 3

Step 2: $g_\lambda = g - \lambda I = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $rk = 1$. $V^{(1)} = \{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mid v_1+v_2+v_3=0\}$

$g_\lambda^2 = 0 \quad V^{(2)} = V \qquad g_\lambda^3 = 0 \quad V^{(3)} = V$

$d_3 = \dim \ker(g_\lambda^3) = 3 \qquad d_2 = 3 \qquad d_1 = 2$

Step 3: $C_3 = 0 < C_2 = 1 < C_1 = d_1 - d_0 = 2$.

2 jumps

First jump happens at $j=2$, jump size 1 $\Big\} \Rightarrow$ 1 Jordan block $J_{1,2}$

2nd jump at $j=1$, size 1 $\Big\}$ 1 Jordan block $J_{1,1}$

$\Rightarrow g \sim \begin{pmatrix} \boxed{J_{1,2}} & \\ & J_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Step 4: 1st jump find $v \in V^{(2)}$ but $v \notin V^{(1)}$

(need $\bar{v}$ in $V^{(2)}/V^{(1)}$ is basis; but it's 1-dim)

just pick $v = \begin{pmatrix}1\\0\\0\end{pmatrix}$ indeed in $V^{(2)}$ not in $V^{(1)}$

$$g_\lambda v = \begin{pmatrix}1&1&1\\-1&-1&-1\\0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix} \in V^{(1)}$$

$(g_\lambda^2 v = 0)$

For 2nd jump we need find $w \in V^{(1)}$ l.I. from $v, g_\lambda v$

just pick $w = \begin{pmatrix}0\\1\\-1\end{pmatrix}$ (other choices $\underbrace{v_1+v_2+v_3=0}$ $\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $\begin{pmatrix}2\\-1\\-1\end{pmatrix}$, ...)

Step 5 Putting together: $h = (g_\lambda v, v, w) = \begin{pmatrix}1&1&0\\-1&0&1\\0&0&-1\end{pmatrix}$

Example 2: $g = \begin{pmatrix}2&0&0&0\\1&2&0&0\\0&1&2&0\\1&0&1&2\end{pmatrix}$

Step 1. $\lambda = 2$ $n = 4$.

Step 2: $g_\lambda = g - \lambda I = \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\1&0&1&0\end{pmatrix}$ $g_\lambda^2 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix}$

$g_\lambda^3 = \begin{pmatrix}&&&\\&&&\\&&&\\1&&&\end{pmatrix}$ $g_\lambda^4 = 0$.

$V^{(4)} = V$ $d_4 = 4$ $V^{(3)} = \left\{ \begin{pmatrix}1&&&\end{pmatrix}\begin{pmatrix}v_1\\ \vdots\\ v_4\end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix}0\\v_2\\v_3\\v_4\end{pmatrix} \right\}$ $d_3 = 3$

$V^{(2)} = \left\{ \vec{v} \ \begin{pmatrix}1&&&\\&1&&\end{pmatrix}\begin{pmatrix}v_1\\ \vdots\\ v_4\end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix}0\\0\\v_3\\v_4\end{pmatrix} \right\}$ $d_2 = 2$

$V^{(1)} = \left\{ \vec{v} \ \begin{pmatrix}1&&&\\&1&&\\1&&1&\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix}0\\0\\0\\v_4\end{pmatrix} \right\}$ $d_1 = 1$

Step 3: $c_4 = c_3 = c_2 = c_1 = 1$

Step 3: $0 < C_4 = C_3 = C_2 = C_1 = 1$

Only 1 jump at $j = 4$ $\Rightarrow$ 1 Jordan block $J_{2,4}$

$$g \sim \left(J_{2,4} \right) = \begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}$$

Step 4: Just need to find $v \in V^{(4)}$ not in $V^{(3)}$

just pick $v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$g_\lambda v = \begin{pmatrix} & & & \\ 1 & & & \\ & 1 & & \\ 1 & & 1 & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$g_\lambda^2 v = \begin{pmatrix} & & & \\ & & & \\ 1 & & & \\ & 1 & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$g_\lambda^3 v = \begin{pmatrix} & & & \\ & & & \\ & & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$g_\lambda \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

Step 5: $$h = \left(g_\lambda^3 v,\ g_\lambda^2 v,\ g_\lambda v,\ v \right) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

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$$V^{(1)} = \left\{ g_\lambda \cdot \vec{v} = 0 \right\} = \left\{ \begin{pmatrix} & & & \\ 1 & & & \\ & 1 & & \\ 1 & & 1 & \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0 \right\}$$

$$\Rightarrow \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ v_1 + v_3 \end{pmatrix} = 0$$

$\Rightarrow$ $v_1 = 0$ $v_2 = 0$ $v_1 + v_3 = 0$ $\Rightarrow$ $v_3 = 0$

$$V^{(1)} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_4 \end{pmatrix} \right\}$$