Week 3

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My English is poor, so stop me any time if you have any question during the class

What is this excerce class about?

① Do the homeworks.

② Answer your questions in

2.1 Questions during doing homework

2.2 Questions during the class, but you didn't figure out

2.3 Any question else.

③ Give you some exercises. Do it during the exercise class, not necerssary complete.

Deadline for homework:

Homework 2 and Homework 3 should be signed before 26th March!!

$$Z = e^{i\frac{2\pi}{3}}$$

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1. If a complex number $z \in \mathbb{C}$ satisfies equation
$$z^n = 1$$
for some integer n, z is called an **$n-$th root of unity**.
   (a) Check by direct computations (without using exponential functions) that $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a third root of unity;
   (b) Recall the formula
   $$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$
   Show that $-\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{i\frac{2\pi}{3}}$.
   (c) Can you write down in general all $n-$th root of unity?

e^{Z}

$z^n = 1$

-1 is second root of unity since $(-1)^2 = 1$

Remark: 1 is n-th root of unity for any n!!

(a) we should prove $z^3 = 1$ → what is written in blue is not mathematics but noises in my head

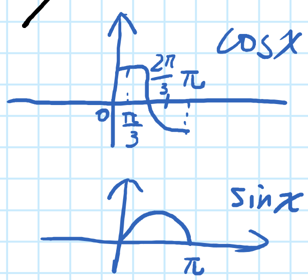
$$Z^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= \left(-\frac{1}{2}\right)^2 + 2\times\left(-\frac{1}{2}\right)\times\left(\frac{\sqrt{3}}{2}i\right) + \left(\frac{\sqrt{3}}{2}i\right)^2$$
$$= \frac{1}{4} + \left(-\frac{\sqrt{3}}{2}i\right) + \frac{3}{4}\cdot i^2$$
$$= \frac{1}{4} + \left(-\frac{\sqrt{3}}{2}i\right) + \frac{3}{4}(-1)$$
$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$Z^3 = Z^2 \cdot Z = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= \left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2$$
$$= \frac{1}{4} - \frac{3}{4}\cdot(i)^2$$
$$= \frac{1}{4} + \frac{3}{4} = 1$$

$\Longrightarrow$ Z is a third root of unity

$$\boxed{\cos x = -\cos(\pi - x)}$$

(b)
$$e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$= -\cos\left(\pi - \frac{2\pi}{3}\right) + i\sin\left(\pi - \frac{2\pi}{3}\right)$$
$$= -\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}$$



(c): $e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1$

$e^{i\frac{2\pi}{3}}$ is a third root of unity.

Notice the $\underline{e^x \cdot e^y = e^{x+y}}$

what's the general n-th root of unity?

The general n-th root of unity should be $\boxed{e^{i\frac{2k\pi}{n}}}$

for $k = 0, 1, 2, \cdots$

since $\left(e^{i\frac{2k\pi}{n}}\right)^n = e^{n\cdot\left(i\frac{2k\pi}{n}\right)} = e^{i2k\pi} = 1$

Question: Are these n-th root of unity distinct?

$e^{i\frac{2k\pi}{n}}$, $k = 0, 1, 2, 3, \cdots, n-1, n, n+1, \cdots$

claim $e^{i\frac{2k_1\pi}{n}} = e^{i\frac{2k_2\pi}{n}}$ $\underline{\text{if and only if}}$ $n \mid k_1 - k_2$

i.e. $\frac{k_1-k_2}{n} \in \mathbb{Z}$

Do it yowself if you want!

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2. Consider the following polynomial equation

$$p(z) = z^3 + (3+i)z^2 + (2+3i)z + 2i = 0.$$

(a) Check that $z = -i$ is a solution.
(b) Use long division to fully factorize $p(z)$.

if $p(z) = (z-a)\cdot P_1(z)$
then a is a root of $P(z)$.
i.e $p(a) = 0$.
the converse is still true.
i.e. if $p(b) = 0$ for some $b \in \mathbb{C}$.
then $p(z) = (z-b)P_2(z)$ for some $P_2(z)$.

(a) Want $p(-i) = 0$.

$$p(-i) = (-i)^3 + (3+i)(-i)^2 + (2+3i)(-i) + 2i$$
$$= -i^3 + (3+i)(-1) + (2+3i)(-i) + 2i$$
$$= i - 3 - i - 2i - 3i^2 + 2i$$
$$= 0$$

so $z = -i$ is a solution of this equation.

(b) By (a). $-i$ is a root of $P(z)$. so $z+i \mid p(z)$
what is $P(z)/(z+i)$?

By (a). $z+i$ is a factor of $P(z)$ since $p(-i) = 0$.

$$\begin{array}{r|l} & z^2 + \boxed{3z} + \boxed{2} \\ z+i & z^3 + (3+i)z^2 + (2+3i)z + 2i \ (= p(z)) \\ & z^3 + iz^2 \\ \hline & 3z^2 + (2+3i)z + 2i \\ & 3z^2 + 3iz \\ \hline & 2z + 2i \\ & 2z + 2i \\ \hline & 0 \end{array}$$

$$\Rightarrow P(z)/_{z+i} = z^2 + 3z + 2$$

$$\Rightarrow p(z) = (z+i)(z^2+3z+2)$$
$$= (z+i)(z+1)(z+2)$$

root finding formula.

$$x_i = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

for $ax^2 + bx + c = 0$.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc.$$ $$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$$

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3. Compute the determinant of the following matrix:

$$A := \left(\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right).$$

Block matrix.

3. Let $A_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $A_2 = \begin{pmatrix} 5 & 6 \\ 9 & 10 \end{pmatrix}$ $A_3 = \begin{pmatrix} 7 & 8 \\ 11 & 12 \end{pmatrix}$

So $$A = \underbrace{\begin{pmatrix} A_1 & O \\ A_2 & A_3 \end{pmatrix}}_{\text{block lower trianglar matrix}}$$

$$\underbrace{\det\begin{pmatrix} A_{11} & & * \\ & A_{22} & \\ 0 & \ddots & A_{rr} \end{pmatrix}}_{\text{block upper trianglar matrix}} = \det A_{11} \cdots \cdot \det A_{rr}$$

Thus $\det A = \det A^T = \det A_1^T \cdot \det A_3^T$

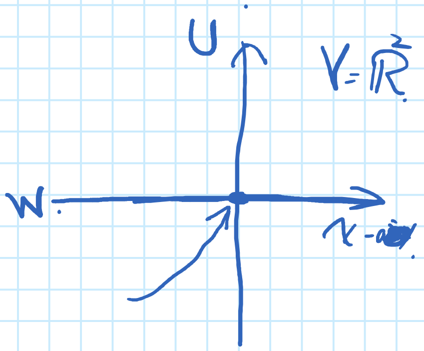
$$A^T = \begin{pmatrix} A_1^T & A_2^T \\ O & A_3^T \end{pmatrix}$$

$$= \det A_1 \cdot \det A_3$$
$$= (1\times 4 - 2\times 3)(7\times 12 - 8\times 10)$$
$$= (-2)(84-80)$$
$$= -8.$$

example of direct sum: $V = \mathbb{R}^2$.

$U = \{(0,y)\}$

$W = \{(x,0)\}$



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4. In the class we mentioned briefly what is a direct sum in general. Here we discuss the direct sum of subspaces (two notions are actually consistent, but do not worry about that here):

Definition. Let U, W be two vector **subspaces** of V. We say V is a direct sum of U, W, denoted by $V = U \oplus W$, if the following two conditions holds:
(1) Any $v \in V$ can be written as $v = u + w$ for some $u \in U, w \in W$. (We can write $V = U + W$ for this property alone.)
(2) $U \cap W = 0$.

Using this definition, prove directly the following lemma:

Lemma 0.1. *Let U, W be two vector subspaces of V. Then $V = U \oplus W$ if and only if for any $v \in V$, there exists* ***unique*** *$u \in U, w \in W$ such that $v = u + w$.*

claim: $V = U \oplus W$

$\forall (x,y) \in V$.

(1) $(x,y) = (0,y) + (x,0)$

claim: For any two 1-dim'e vector subspace U, W of $V = \mathbb{R}^2$. If $U \neq W$. then $V = U \oplus W$.

Do it yourself if you are interested.

Proof of the Lemma:

"if" part (if for any $v \in V$, there exist unique $u \in U$ $w \in W$ s.t. $v = u + w$

($\Longleftarrow$) then $V = U \oplus W$

It is clearly that $V = U + W$.

Now we want to prove $U \cap W = 0$

For any $x \in U \cap W$.

then we see that $\underset{V}{0} = \underset{U}{0} + \underset{W}{0} = \underset{U}{x} + \underset{W}{(-x)}$

By the uniqueness of decomposition. we have $x = 0$

So $U \cap W = 0$. and hence $V = U \oplus W$.

"only if" part. ("$\Longrightarrow$")

By definition. for any $v \in V$. we have $u \in U$ $w \in W$. so that $v = u + w$.

we ~~need to check uniqueness of u and w.~~

If we have $u_1 \in U$, $w_1 \in W$. So that $v = u + w = u_1 + w_1$

we need to check uniqueness of u and w.

If we have $u_1 \in U$, $w_1 \in W$. so that $v = \underline{u+w} = \underline{u_1+w_1}$

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only if part ?

(countinued).

$$v = u + w = u_1 + w_1$$

$$\Rightarrow \underbrace{u - u_1}_{\in U} = \underbrace{w_1 - w}_{\in W}$$

$$V = U \oplus W$$

$u - u_1 = w_1 - w \in U \cap W$

~~By~~ by definition $U \cap W = 0$.

So $u - u_1 = w_1 - w = 0 \Rightarrow u = u_1$ and $w = w_1$

This proves uniqueness of u and w. #

If you have any problems that you have learnt before, but you forget, and we will use in class, you can ask, I will try to remind you in exercise class (If time permit?)