Final practice problems - solutions

29. The direction of fastest change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i}+\mathbf{j} \quad \Leftrightarrow \quad (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = k\,(\mathbf{i}+\mathbf{j}) \quad \Leftrightarrow \quad k = 2x-2$ and $k = 2y-4$. Then $2x-2 = 2y-4 \quad \Rightarrow$ $y = x+1$, so the direction of fastest change is $\mathbf{i}+\mathbf{j}$ at all points on the line $y = x+1$.

67. Let $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$. Then we know that at the given point, $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}}\, f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y\rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\,D_{\mathbf{u}}\, f - b\,D_{\mathbf{v}}\, f}{ad-bc}, \frac{a\,D_{\mathbf{v}}\, f - c\,D_{\mathbf{u}}\, f}{ad-bc} \right\rangle.$$

51. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000\ \text{cm}^3$. Then $f(x,y) = xy + [64{,}000(x+y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x,y) = [(2)(64{,}000)]^2 x^{-3} y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

We are given $Y(N,P) = kNPe^{-N-P}$ where nitrogen level is N and phosphorus level is P in the soil (measured in appropriate units) and k is a constant. We need to find levels of nitrogen and phosphorus which shall result in the best yield. Hence we shall find the partial derivatives and then equate them to 0 to find the critical points.

$Y_N(N,P) = kP(Ne^{-N-P}(-1) + e^{-N-P}) = kPe^{-N-P}(1-N)$
$Y_P(N,P) = kN(Pe^{-N-P}(-1) + e^{-N-P}) = kNe^{-N-P}(1-P)$

When we equate them to 0, we get the critical points (1,1) as we can not have the level of P and N to be 0.

Using second derivatives test, we get:
$Y_{NN} = kP(e^{-N-P}(-1) + (1-N)e^{-N-P}(-1)); Y_{NN}(1,1) = -ke^{-2}$
$Y_{PP} = kN(e^{-N-P}(-1) + (1-P)e^{-N-P}(-1)); Y_{PP}(1,1) = -ke^{-2}$
$Y_{NP} = k(1-N)(1-P)e^{-N-P}; Y_{NP}(1,1) = 0$

We know that $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$

$D(1,1) = k^2e^{-4} > 0; Y_{NN} < 0$
Hence the point (1,1) is a point of maxima.

21. In cylindrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So

$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, 0 \le z \le 2r\}$ and

$$\iiint_E x^2\, dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \left[r^3\cos^2\theta\, z\right]_{z=0}^{z=2r} dr\, d\theta = \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos^2\theta\right]_{r=0}^{r=1} d\theta = \tfrac{2}{5}\int_0^{2\pi}\cos^2\theta\, d\theta = \tfrac{2}{5}\int_0^{2\pi}\tfrac{1}{2}(1+\cos 2\theta)\, d\theta = \tfrac{1}{5}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = \tfrac{2\pi}{5}$$

25. In spherical coordinates, E is represented by $\left\{(\rho, \theta, \phi)\,\middle|\, 0 \le \rho \le 1, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\right\}$. Thus

$$\iiint_E xe^{x^2+y^2+z^2}\, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)e^{\rho^2}\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi/2}\sin^2\phi\, d\phi\int_0^{\pi/2}\cos\theta\, d\theta\int_0^1 \rho^3 e^{\rho^2}\, d\rho$$

$$= \int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\phi)\, d\phi\int_0^{\pi/2}\cos\theta\, d\theta\left(\left[\tfrac{1}{2}\rho^2 e^{\rho^2}\right]_0^1 - \int_0^1 \rho e^{\rho^2}\, d\rho\right)$$

$\left[\text{integrate by parts with } u = \rho^2,\ dv = \rho e^{\rho^2}\, d\rho\right]$

$$= \left[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right]_0^{\pi/2}\left[\sin\theta\right]_0^{\pi/2}\left[\tfrac{1}{2}\rho^2 e^{\rho^2} - \tfrac{1}{2}e^{\rho^2}\right]_0^1 = \left(\tfrac{\pi}{4} - 0\right)(1-0)\left(0 + \tfrac{1}{2}\right) = \tfrac{\pi}{8}$$

39. $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle t - \sin t, 3 - \cos t\rangle\cdot\langle 1 - \cos t, \sin t\rangle\, dt$

$$= \int_0^{2\pi}(t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cos t)\, dt$$

$$= \int_0^{2\pi}(t - t\cos t + 2\sin t)\, dt = \left[\tfrac{1}{2}t^2 - (t\sin t + \cos t) - 2\cos t\right]_0^{2\pi} \qquad \left[\begin{array}{c}\text{integrate by parts}\\ \text{in the second term}\end{array}\right]$$

$$= 2\pi^2$$

28. $\nabla f(x, y) = \cos(x - 2y)\,\mathbf{i} - 2\cos(x - 2y)\,\mathbf{j}$

(a) We use Theorem 2: $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0,0) = \sin 0 = 0$ and $f(\pi,\pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0,0)$ to (π,π); that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j},\ 0 \le t \le 1$.

(b) From (a), $\int_{C_2}\mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0,0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}, 0\right) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2}t\,\mathbf{i}$, $0 \le t \le 1$, the straight line from $(0,0)$ to $\left(\frac{\pi}{2}, 0\right)$.

44. A parametric representation for the hemisphere S is $\mathbf{r}(\phi,\theta) = 3\sin\phi\cos\theta\,\mathbf{i} + 3\sin\phi\sin\theta\,\mathbf{j} + 3\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi = 3\cos\phi\cos\theta\,\mathbf{i} + 3\cos\phi\sin\theta\,\mathbf{j} - 3\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -3\sin\phi\sin\theta\,\mathbf{i} + 3\sin\phi\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi\times\mathbf{r}_\theta = 9\sin^2\phi\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\cos\phi\,\mathbf{k}$. The rate of flow through S is

$$\iint_S \rho\mathbf{v}\cdot d\mathbf{S} = \rho\int_0^{\pi/2}\int_0^{2\pi}(3\sin\phi\sin\theta\,\mathbf{i} + 3\sin\phi\cos\theta\,\mathbf{j})\cdot\left(9\sin^2\phi\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\cos\phi\,\mathbf{k}\right) d\theta\, d\phi$$

$$= 27\rho\int_0^{\pi/2}\int_0^{2\pi}\left(\sin^3\phi\sin\theta\cos\theta + \sin^3\phi\sin\theta\cos\theta\right) d\theta\, d\phi = 54\rho\int_0^{\pi/2}\sin^3\phi\, d\phi\int_0^{2\pi}\sin\theta\cos\theta\, d\theta$$

$$= 54\rho\left[-\tfrac{1}{3}(2 + \sin^2\phi)\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0 \text{ kg/s}$$

5. C is the square in the plane $z = -1$. Rather than evaluating a line integral around C we can use Equation 3:

$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl} \mathbf{F} = x^2 z\, \mathbf{i} + (xy - 2xyz)\, \mathbf{j} + (y - xz)\, \mathbf{k}$. For S_2, we choose $\mathbf{n} = \mathbf{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = y - xz = x + y$ on S_2, where $z = -1$. Thus $\iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} (x + y)\, dx\, dy = 0$ so $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

16. Let S be the surface in the plane $x + y + z = 1$ with upward orientation enclosed by C. Then an upward unit normal vector for S is $\mathbf{n} = \frac{1}{\sqrt{3}} (\mathbf{i} + \mathbf{j} + \mathbf{k})$. Orient C in the counterclockwise direction, as viewed from above. $\int_C z\, dx - 2x\, dy + 3y\, dz$ is equivalent to $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}(x, y, z) = z\, \mathbf{i} - 2x\, \mathbf{j} + 3y\, \mathbf{k}$, and the components of $\mathbf{F}$ are polynomials, which have continuous partial derivatives throughout $\mathbb{R}^3$. We have $\operatorname{curl} \mathbf{F} = 3\, \mathbf{i} + \mathbf{j} - 2\, \mathbf{k}$, so by Stokes' Theorem,

$$\begin{aligned} \int_C z\, dx - 2x\, dy + 3y\, dz &= \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (3\, \mathbf{i} + \mathbf{j} - 2\, \mathbf{k}) \cdot \tfrac{1}{\sqrt{3}} (\mathbf{i} + \mathbf{j} + \mathbf{k})\ dS \\ &= \tfrac{2}{\sqrt{3}} \iint_S dS = \tfrac{2}{\sqrt{3}}\, (\text{surface area of } S) \end{aligned}$$

Thus the value of $\int_C z\, dx - 2x\, dy + 3y\, dz$ is always $\frac{2}{\sqrt{3}}$ times the area of the region enclosed by C, regardless of its shape or location. [Notice that because $\mathbf{n}$ is normal to a plane, it is constant. But $\operatorname{curl} \mathbf{F}$ is also constant, so the dot product $\operatorname{curl} \mathbf{F} \cdot \mathbf{n}$ is constant and we could have simply argued that $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$ is a constant multple of $\iint_S dS$, the surface area of S.]

7. $\operatorname{div} \mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E (3y^2 + 3z^2)\, dV = \int_0^{2\pi} \int_0^1 \int_{-1}^{2} (3r^2 \cos^2\theta + 3r^2 \sin^2\theta)\, r\, dx\, dr\, d\theta \\ &= 3 \int_0^{2\pi} d\theta \int_0^1 r^3\, dr \int_{-1}^{2} dx = 3(2\pi)\left(\tfrac{1}{4}\right)(3) = \tfrac{9\pi}{2} \end{aligned}$$

24. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (2x + 2y + z^2)\, dS$, so $\mathbf{F} \cdot \mathbf{n} = 2x + 2y + z^2$. But for S,

$\mathbf{n} = \dfrac{x\, \mathbf{i} + y\, \mathbf{j} + z\, \mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = x\, \mathbf{i} + y\, \mathbf{j} + z\, \mathbf{k}$. Thus $\mathbf{F} = 2\, \mathbf{i} + 2\, \mathbf{j} + z\, \mathbf{k}$ and $\operatorname{div} \mathbf{F} = 1$.

If $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$, then $\iint_S (2x + 2y + z^2)\, dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$.