Problem 5.3.1.

Sol.

(a) $|A| = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3 \quad |B_1| = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = -6 \quad |B_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

$x_1 = \frac{|B_1|}{|A|} = -2, \quad x_2 = \frac{|B_2|}{|A|} = 1.$

(b) $|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4 \quad |B_1| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 3 \quad |B_2| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -2 \quad |B_3| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$

so, $x_1 = \frac{|B_1|}{|A|} = \frac{3}{4}, \quad x_2 = \frac{|B_2|}{|A|} = -\frac{1}{2}, \quad x_3 = \frac{|B_3|}{|A|} = \frac{1}{4}$

Problem 5.3.5.

Sol. As the first column of A equal to b, $\det A = \det B_1$, for matrix B_2 and B_3, there exist two same column so the matrix is singular and this make $\det B_2$ and $\det B_3$ equal to zero, therefore $x_1 = \frac{|B_1|}{|A|} = 1$, and $x_2, x_3 = 0$.

Problem 5.3.6.

Sol. (a) $C^T = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{bmatrix} \quad A^{-1} = \frac{C^T}{\det A} = \frac{C^T}{3} = \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -7/3 & 1 \end{bmatrix}$

(b) $C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad A^{-1} = \frac{C^T}{\det A} = \frac{C^T}{4} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$

Problem 5.3.8.

Sol. $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}, \quad AC^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I = \det A \cdot I.$ so $\det A = 3$.

As the cofactor of a_{13} is 0, no matter what a_{13} is, $\det A$ still no change.

Problem 5.3.15.

Sol. For $n=5$, the matrix C contains 25 cofactors, each 4×4 cofactor contains 24 terms and each term needs 3 multiplications, total 1800 multiplication.

Problem 5.3.17.

Sol. $\text{Volume} = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$

Problem 5.3.23.

Sol. $A = [a \ b \ c] \cdot A^TA = \begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix} [a \ b \ c] = \begin{bmatrix} a^Ta & 0 & 0 \\ 0 & b^Tb & 0 \\ 0 & 0 & c^Tc \end{bmatrix} = \begin{bmatrix} \|a\| & 0 & 0 \\ 0 & \|b\| & 0 \\ 0 & 0 & \|c\| \end{bmatrix}$. so $\det A^TA = (\|a\|\cdot\|b\|\cdot\|c\|)^2$

thus, $|\det A| = \|a\|\cdot\|b\|\cdot\|c\|$

Problem 6.1.6

Sol. Let $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$, $\lambda_1 = \lambda_2 = 1$

Let $|B-\lambda I| = 0$, which means $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$, and $\lambda_1 = \lambda_2 = 1$.

Let $|AB-\lambda I| = 0$, then $\begin{vmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 2 = 0$, so $\lambda_1 = 2+\sqrt{3}$ and $\lambda_2 = 2-\sqrt{3}$

hence, we can say the eigenvalues of AB is not equal to eigenvalues of A times eigenvalues of B.

Let $|BA-\lambda I| = 0$, then $\begin{vmatrix} 3-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) - 2 = 0$, so $\lambda_1 = 2+\sqrt{3}$ and $\lambda_2 = 2-\sqrt{3}$.

thus, the eigenvalues of AB is equal to the eigenvalues of BA.

Problem 6.1.12.

Sol. Let $|P-\lambda I| = 0$, which is $\begin{vmatrix} 0.2-\lambda & 0.4 & 0 \\ 0.4 & 0.8-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = [(0.2-\lambda)(0.8-\lambda) - 0.4^2]\cdot(1-\lambda) = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$

and their eigenvector are $\left\{ \begin{array}{l} (P-0I)x = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot x = 0 \text{ gives eigenvector } \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ (P-1I)x = \begin{bmatrix} -0.8 & 0.4 & 0 \\ 0.4 & -0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot x = 0 \text{ gives eigenvector } \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array} \right.$

they share same eigenvalue, which means $\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ is also a eigenvector)

Problem. 6.1.15

Sol. By $\det(A-\lambda I) = 0$, which means $\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0$, so $\lambda_1 = 1$, $\lambda_2 = \frac{-1+\sqrt{3}i}{2}$, $\lambda_3 = \frac{-1-\sqrt{3}i}{2}$

For $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, we can get $\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - (1-\lambda) = 0$, so $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 1$

Problem 6.1.16.

Sol. $\det A = \det(A - 0\cdot I) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

Problem 6.1.24.

Sol. By $\det(A-\lambda I) = 0$, we have $\begin{vmatrix} 2-\lambda & 1 & 2 \\ 4 & 2-\lambda & 4 \\ 2 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2$, so $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 6$.

Let $(P-0I)\cdot x = 0$, then $x = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, which are eigenvectors of the matrix.

when $\lambda = 6$, let $(P-6I)x = 0$ gives eigenvector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

So $\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are the eigenvectors of the matrix A.

Problem 6.1.27.

Sol. As rank(A) = 1 and trace(A) = 4, so the eigenvalues of A are 0, 0, 0, 4.

For C, C is a rank 2 matrix, so there exists two eigenvalues are 0.

As $[1\ 1\ 1\ 1]^T$ is a eigenvector of $\lambda = 2$, with trace(C) = 4, we can see another eigenvector is 2, so 0, 0, 2, 2 is the eigenvalues of C.

Problem 6.1.32.

Sol. (a) As $Au = 0$, $Av = 3v$ and $Aw = 5w$, we can say u is a basis for the nullspace, v and w is a basis for the column space.

(b) From $Av = 3v$ and $Aw = 5w$, we can see $A(\frac{v}{3} + \frac{w}{5}) = v + w$, so $x_p = \frac{v}{3} + \frac{w}{5}$,

$x = x_p + x_n = \frac{v}{3} + \frac{w}{5} + cu$, $c \in \mathbb{R}$.

(c) If it did, then u would be in the column space.

Problem 1.

Sol. (a) Put 4 vectors in matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \end{bmatrix}$

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -3 & 3 & -9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & -6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -12 \end{vmatrix} = 1 \times (-2) \times 3 \times (-12) = 72.$$

(b)

$$|\det(QA)| = |\det(Q)| \cdot |\det(A)| = 1 \cdot 72 = 72.$$

Problem 2

Sol. Let $\det(A - \lambda I) = 0$, so
$$\begin{vmatrix} 1-\lambda & -2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & -4 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -2 & 0 \\ 2 & -3-\lambda & -1-\lambda \\ 2 & -4 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -2 & 0 \\ 0 & 1-\lambda & 0 \\ 2 & -4 & -1-\lambda \end{vmatrix} = (-1-\lambda) \cdot (1-\lambda)^2$$

So the eigenvalues of A are 1, 1, −1.

For $\lambda = 1$, let $(A - I)x = 0$, then $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = -1$, let $(A + I)x = 0$, then $x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

hence, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are the eigenvectors of A.

We all know that 2 vectors can't be a basis of $\mathbb{R}^3$, so $\mathbb{R}^3$ does not have a basis consisting of eigenvectors for A.