24. $V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$

$= \int_0^1 \int_0^{1-x} (2-2z)\, dz\, dx$

$= \int_0^1 \left[2z - z^2\right]_0^{1-x} dx$

$= \int_0^1 (1-x^2)\, dx$

$= \left[x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3}$

30. $V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\, dy\, dx$

$= \int_0^2 \int_0^{4-x^2} 4-x^2-y\, dy\, dx$

$= \int_0^2 \left[4y - x^2y - \frac{1}{2}y^2\right]_0^{4-x^2} dx$

$= \int_0^2 8 - 4x^2 + \frac{1}{2}x^4\, dx$

$= \left[8x - \frac{4}{3}x^3 + \frac{1}{10}x^5\right]_0^2 = \frac{128}{15}$

50. $V = \int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{2}} \int_0^a \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$= \int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{2}} \frac{1}{3}a^3 \sin\phi\, d\phi\, d\theta$

$= \int_0^{\frac{\pi}{6}} \left(-\frac{1}{3}a^3 \cos\frac{\pi}{2} + \frac{1}{3}a^3 \cos 0\right) d\theta$

$= \frac{1}{3}a^3 \cdot \frac{\pi}{6} = \frac{\pi a^3}{18}$

54. $V = 4\int_0^{\frac{\pi}{2}} \int_0^1 \int_{r^2}^{r^2+1} dz \cdot r\,dr \cdot d\theta$

$= 4\int_0^{\frac{\pi}{2}} \int_0^1 1\, r\,dr\, d\theta$

$= 4\int_0^{\frac{\pi}{2}} \frac{1}{2}\, d\theta$

$= \pi$

62. first solve $x^2+y^2+z^2=2$ and $z=x^2+y^2$, which gives $z^2+z=2$ or $(z+2)(z-1)=0 \Rightarrow z=-2$ or $z=1$

but $z=x^2+y^2 \geq 0$, so $z=1$, gives $r=\sqrt{x^2+y^2}=1$

then $V = 4\int_0^{\frac{\pi}{2}} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz\, r\,dr\, d\theta$

$= 4\int_0^{\pi/2} \int_0^1 \left(\sqrt{2-r^2} - r^2\right) \cdot r\,dr\, d\theta$

$= 4\int_0^{\pi/2} \left[-\frac{1}{4}r^4 - \frac{1}{3}(2-r^2)^{\frac{3}{2}}\right]_0^1 d\theta$

$= 4\int_0^{\pi/2} \frac{2\sqrt{2}}{3} - \frac{7}{12}\, d\theta$

$= \frac{(8\sqrt{2}-7)\pi}{6}$

66. ∵ average $= \frac{1}{\frac{2}{3}\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \sin\phi \cos\phi\, d\rho\, d\phi\, d\theta$

$= \frac{1}{\frac{2}{3}\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{1}{4} \sin\phi \cos\phi\, d\phi\, d\theta$

$= \frac{3}{16\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin 2\phi\, d\phi\, d\theta$

$= \frac{3}{16\pi} \int_0^{2\pi} \left[-\frac{\cos 2\phi}{2}\right]_0^{\frac{\pi}{2}} d\theta$

$= \frac{3}{16\pi} \int_0^{2\pi} d\theta$

$= \frac{3}{16\pi} \cdot 2\pi$

$= \frac{3}{8}$

14. $x = u + \frac{v}{2}$, $y = v$ $\Rightarrow$ $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1/2 \\ 0 & 1 \end{vmatrix} = 1$

boundary of R	boundary of G	simplified
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y+4}{2}$	$u + \frac{v}{2} + \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

So $\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2} \, dx \, dy$

$= \int_0^2 \int_0^2 v^3 \cdot 2u \cdot e^{4u^2} J(u,v) \, du \, dv$

$= \int_0^2 \int_0^2 v^3 \cdot 2u \cdot e^{4u^2} du \, dv$

$= \int_0^2 \left[\frac{1}{4} v^3 \cdot e^{4u^2} \right]_0^2 dv$

$= \int_0^2 \frac{1}{4} v^3 (e^{16} - 1) \, dv$

$= \frac{1}{16} \left[v^4 (e^{16} - 1) \right]_0^2$

$= e^{16} - 1$

• as $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$

we have $J(\rho, \phi, \theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$

$= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} - (-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$

$= \rho^2 \sin\phi \cos^2\phi + \rho^2 \sin^3\phi$

$= \rho^2 \sin\phi$