Linear Algebra – Week 2 Homework

23 Sept 2020
Due: 1 Oct 2020

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 1.2.7. Find the angle θ (from its cosine) between these pairs of vectors:

(a) $\mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

(c) $\mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$

(d) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Problem 1.2.11. If $\mathbf{v} \cdot \mathbf{w}$ is negative, what does this say about the angle between $\mathbf{v}$ and $\mathbf{w}$? Draw a 3-dimensional vector $\mathbf{v}$ (an arrow), and show where to find all $\mathbf{w}$'s with $\mathbf{v} \cdot \mathbf{w} < 0$.

Problem 1.2.16. How long is the vector $\mathbf{v} = (1, 1, \ldots, 1)$ in 9 dimensions? Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ and a unit vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

Problem 1.2.22. Derive the Schwarz inequality $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$ by algebra instead of trigonometry:

(a) Step 1: Multiply out both sides of the inequality, $|\mathbf{v} \cdot \mathbf{w}|^2 = (v_1 w_1 + v_2 w_2)^2$ and $\|\mathbf{v}\|^2\|\mathbf{w}\|^2 = (v_1^2 + v_2^2)(w_1^2 + w_2^2)$.

(b) Step 2: Show that the difference between those two sides equals $(v_1 w_2 - v_2 w_1)^2$. This difference cannot be negative since it is a square, so the inequality is true.

Problem 1.2.27. Draw a parallelogram with two sides $\mathbf{v}$ and $\mathbf{w}$. Show that the squared diagonal lengths $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2$ add to the sum of four squared side lengths $2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2$.

Problem 1.2.33. Find 4 perpendicular unit vectors of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$: Choose $+$ or $-$.

Problem 1.3.4. Find a combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3$ that gives the zero vector with $x_1 = 1$:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \qquad \mathbf{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Are these vectors dependent or independent? The three vectors lie in a _____.

Problem 1.3.5. Consider the matrix W with columns $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ from Problem 1.3.4. The rows of W are the following vectors (written as columns):

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \qquad \mathbf{r}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Linear algebra says that these vectors must also lie in a plane, so there must be many combinations with $y_1\mathbf{r}_1+y_2\mathbf{r}_2+y_3\mathbf{r}_3 = \mathbf{0}$. Find two sets of y's.

Problem 1.3.10. A forward difference matrix is upper triangular:

$$\Delta\mathbf{z} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 - z_1 \\ z_3 - z_2 \\ 0 - z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{b}.$$

Find z_1, z_2, z_3 from b_1, b_2, b_3. What is the inverse matrix in $\mathbf{z} = \Delta^{-1}\mathbf{b}$?

Problem 1.3.12 The 4×4 matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

has an inverse. Solve $C\mathbf{x} = (b_1, b_2, b_3, b_4)$ to find its inverse in $\mathbf{x} = C^{-1}\mathbf{b}$.

Graded Problems.

Problem 1. Consider the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Find a unit vector pointing in the same direction as $\mathbf{u}$.

(b) Find nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ that are perpendicular to $\mathbf{u}$ and to each other.

Problem 2. Suppose $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 4$.

(a) What are the largest and smallest possible values of $\mathbf{u} \cdot \mathbf{v}$?

(b) What are the largest and smallest possible values of $\|\mathbf{u} + \mathbf{v}\|$?

(c) Draw pictures of vectors $\mathbf{u}$ and $\mathbf{v}$ that illustrate both the largest and possible values of $\|\mathbf{u} + \mathbf{v}\|$.

Problem 3. Consider the 2×2 matrix $A = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$.

(a) Solve the equation $A\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ for $\mathbf{x}$ in terms of b_1, b_2.

(b) What is the inverse matrix A^{-1}?