Calculus A(2) Spring 2021 Final Exam – Sample Problems

1. (a) The first and second derivatives of a function $f(x, y)$ are continuous in a disk, and $f_x(a, b) = f_y(a, b) = f_{xy}(a, b) = 0$ at an interior point (a, b) in the disk. Which of the following statements must be true?

   A. f has a local maximum at (a, b).
   B. f has a local minimum at (a, b).
   C. f has a saddle point at (a, b).
   D. None of the above

   (b) Let $\mathbf{v}$ be the gradient vector $\nabla f(a, b, c)$ of a function $f(x, y, z)$ at point (a, b, c). Which of the following statements is true?

   A. $\mathbf{v}$ is normal to the plane tangent to the level surface of f through (a, b, c).
   B. The directional derivative $D_{\mathbf{v}} f$ vanishes at (a, b, c)
   C. The magnitude of $\mathbf{v}$ is equal to $|f_{xx} f_{yy} - f_{xy}^2|$ evaluated at (a, b).
   D. None of the above is true.

2. Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

3. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two non-parallel directions given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find ∇f at this point? If so, how would you do it?

4. A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

5. A model for the yield Y of an agricultural crop as a function of the nitrogen level N and phosphorus level P in the soil (measured in appropriate units) is $Y(N, P) = kNPe^{-N-P}$ where k is a positive constant. What levels of nitrogen and phosphorus result in the best yield?

6. Evaluate $\iiint_D x^2 \mathrm{d}V$, where D is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

7. Evaluate $\iiint_D x e^{x^2+y^2+z^2} \mathrm{d}V$, where D is the portion of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

8. Find the work done by the force field $\mathbf{F}(x, y) = x\mathbf{i} + (y + 2)\mathbf{j}$ in moving an object along an arch of the cycloid $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 + \cos t)\mathbf{j},\ \ 0 \leq t \leq 2\pi$.

9. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ and $\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 1$.

10. Seawater has density $1025\,\mathrm{kg} \cdot \mathrm{m}^3$ and flows in a velocity field $\mathbf{v} = y\mathbf{i} + x\mathbf{j}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9,\ \ z \geq 0$.

11. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, \mathrm{d}\sigma$, where $\mathbf{F}(x, y, z) = xyz\mathbf{i} + xy\mathbf{j} + x^2yz\mathbf{k}$ and S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.

12. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral $\int_C (z\,\mathrm{d}x + 2x\,\mathrm{d}y + 3y\,\mathrm{d}z)$ depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

13. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}\sigma$, where $\mathbf{F}(x, y, z) = 3xy^2\mathbf{i} + xe^z\mathbf{j} + z^3\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

14. Use the Divergence Theorem to evaluate $\iint_S (2x + 2y + z^2)\mathrm{d}\sigma$, where S is the sphere $x^2 + y^2 + z^2 = 1$.