Last lecture

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△ $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$(-1)^{i+j}$

$= aei - afh - bdi + bgf + cdh - cge$

$\begin{pmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{pmatrix}$

$= aei + bfg + cdh - ceg - afh - bdi$

△ Review: Given V, defined $\hat{V} = \{L : V \to \mathbb{C}\}$

L is determined by its values on basis $(Le_i)_i$

Lem $\dim \hat{V} = \dim V = n$ $\quad\quad \hat{\hat{V}} = V$

Dual basis $\{L_i\}$ for $\{e_i\}$ s.t. $L_j e_i = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

△ Lem: If $Lv = 0$ $\forall L \in \hat{V}$, then $v = 0$

△ Lem: let $\{e_i\}$ be basis for V, $\{L_i\}$ be dual basis for $\hat{V}$,

Let $v = \sum a_i e_i \in V$ (A), then $a_i = L_i v$

Pf: Apply L_i to (A) $\quad L_i v = \sum_j L_i(a_j e_j) = a_i L_i e_i = a_i$

△ last thm: $\begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix} \cdot (e_1 \cdots e_n) = I$

△ Lemma: let $\{e_1' \cdots e_n'\}$ be a new basis $(e_1' \cdots e_n') = (e_1 \cdots e_n) h$

dual basis $\{L_1' \cdots L_n'\}$ for $\{e_i'\}$

$$\underline{(L_1' \cdots L_n') = (L_1 \cdots L_n)\cdot (h^T)^{-1}}$$

Pf: $I = \begin{pmatrix} L_1' \\ \vdots \\ L_n' \end{pmatrix} (e_1' \cdots e_n') = \begin{pmatrix} L_1' \\ \vdots \\ L_n' \end{pmatrix} (e_1 \cdots e_n)\, h$ $\quad\Big\downarrow \cdot h^{-1}$ on right

$$h^{-1} = \begin{pmatrix} L_1' \\ \vdots \\ L_n' \end{pmatrix} (e_1 \cdots e_n) \quad\Big\downarrow h\cdot \text{ on left}$$

$$I = h \begin{pmatrix} L_1' \\ \vdots \\ L_n' \end{pmatrix} (e_1 \cdots e_n)$$

Compare with $I = \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix} (e_1 \cdots e_n)$

we must have $\begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix} = h\cdot \begin{pmatrix} L_1' \\ \vdots \\ L_n' \end{pmatrix}$

Taking transpose $(L_1' \cdots L_n')\cdot h^T = (L_1 \cdots L_n)$ $\quad\Big\downarrow (h^T)^{-1}$ on right

$$(L_1' \cdots L_n') = (L_1 \cdots L_n)\,(h^T)^{-1}$$

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△ Adjoint operator

$$\begin{tikzcd} T: V \arrow[r, "T"] \arrow[dr, dashed, "P\circ T"'] & W \arrow[d, "P"] \\ & \mathbb{C} \end{tikzcd}$$

Adjoint operator $T^*: \hat{V} \leftarrow \hat{W}$

$$T^*: (P: W \to \mathbb{C}) \longmapsto (P\circ T : V \to \mathbb{C})$$

△ Pick basis $\{e_i\}$ for V $\quad \{f_i\}$ for W $\qquad$ $\dim V = n$

dual basis $\{L_i\}$ for $\hat{V}$ $\quad \{P_i\}$ for $\hat{W}$ $\qquad$ $\dim W = m$

Lem: $T(e_1 \cdots e_n) = (f_1 \cdots f_m)\, g$

$$T^*(P_1 \cdots P_m) = (L_1 \cdots L_n)\cdot \underline{g^T}$$

Pf: $(*)\ \big(T^*(P_1 \cdots P_m)\big)^T (e_1 \cdots e_n)$

Pf: $(*)\ \left(T^*(p_1 \cdots p_m)\right)' (e_1 \cdots e_n)$

On one hand $(*) = \left((L_1 \cdots L_n)\, ?\right)^T (e_1 \cdots e_n)$

$$= ?^T \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix} (e_1 \cdots e_n) = ?^T \cdot I$$

On the other hand $(*) = \begin{pmatrix} T^* p_1 \\ \vdots \\ T^* p_n \end{pmatrix} (e_1 \cdots e_n) = \left(T^* p_i \cdot e_j \right)$

$$\begin{array}{ccc} T: V & \longrightarrow & W \\ & {\scriptstyle T^* p_i} \searrow & \downarrow {\scriptstyle p_i} \\ & & \mathbb{C} \end{array}$$

$$T^* p_i\, e_j = p_i \circ T(e_j)$$

$$(*) = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} T(e_1 \cdots e_n) = \underline{\begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} (f_1 \cdots f_m)}\, g = g$$

$$\Rightarrow \quad ?^T = g \qquad ? = g^T$$

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Δ: Let V now be a vector space with <u>a positive def</u> Hermitian form $(\ ,\ ): V \times V \to \mathbb{C}$.

Observation: $\forall\, w \in V$, it give rise to a linear functional

$$(w, \cdot) : V \to \mathbb{C}$$
$$v \mapsto \underset{\mathbb{C}\text{-linear}}{\underset{\uparrow}{(w, v)}}$$

Thm (Riez Representation thm): Map sending w to $(w, \cdot)$ is a bijection.

In other words, $\forall\, L: V \to \mathbb{C}$, $\exists\, v_L \in V$, s.t. $L\, v = (v_L, v)$ for all $v \in V$

$\triangleright$ Def: Given two vector spaces V_1, V_2

a multilinear functional $\mathcal{L} : V_1 \times V_2 \to \mathbb{C}$

$(v_1, v_2) \mapsto \mathcal{L}(v_1, v_2)$

which is $\mathbb{C}$-linear in both v_1, v_2

Def: Given V_1, V_2, then tensor product $V_1 \otimes V_2$ is defined to be the space of all multilinear fun on $\hat{V}_1 \times \hat{V}_2$

Given basis $\{e_i\}$ for V_1 $\{f_j\}$ for V_2

Lem: $V_1 \otimes V_2$ has a basis of form $e_i \otimes f_j$, in particular $\dim V_1 \otimes V_2 = (\dim V_1) \times (\dim V_2)$

pf: pick dual basis $\{L_i\}$ for $\hat{V}_1$ $\{P_j\}$ for $\hat{V}_2$

a multilinear functional $\mathcal{L}$ on $\hat{V}_1 \times \hat{V}_2$ is determined by its values on $\mathcal{L}(L_i, P_j)$

pick a basis for $\{\mathcal{L}\}$ as $\mathcal{L}_{ij}$ $\begin{cases} (L_i, P_j) \mapsto 1 \\ \text{other pairs} \mapsto 0 \end{cases}$

but $\mathcal{L}_{ij}$ can be identified with $e_i \otimes f_j$

$(e_i \otimes f_j) : (L_k, P_\ell) \mapsto L_k(e_i) \cdot P_\ell(f_j) = \begin{cases} 1 & k=i,\ \ell=j \\ 0 & \text{otherwise} \end{cases}$

matches definition for $\mathcal{L}_{ij}$

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Reformulate a matrix in terms of tensor product

$T : V \to W$ can be identified with an element in $\hat{V} \otimes W$

basis $\{e_i\}$ for V $\{f_i\}$ for W pick dual basis $\{L_i\}$ for $\hat{V}$

$T(e_1 \cdots e_n) = (f_1 \cdots f_m)\, g$

$g = (g_{ij})$ $\quad T e_i = f_1 g_{1i} + f_2 g_{2i} + \cdots$

associate $\tau = \sum_{ij} g_{ji} L_i \otimes f_j$

τ indeed defines a map $\tau : V \to W$

$v \mapsto \sum g_{..} L_i(v) f_j$

τ indeed defines a map $L: V \to W$

$$v \mapsto \sum g_{ji} L_i(v) f_j$$

it sends $e_i \mapsto \sum_j g_{ji} f_j$

$\triangle$ $T_1 : V \to W$, $T_2 : W \to Z$

$\downarrow$ $\qquad\qquad\qquad \downarrow$

$\tau_1 \in \hat{V} \otimes W \qquad \tau_2 \in \hat{W} \otimes Z$

Matrix multiplication $= T_2 \circ T_1 : V \to Z$

$(\tau_1, \tau_2) \mapsto \hat{V} \otimes Z$ by applying elements in $\hat{W}$ on W

$\triangle$ Pick basis $\{e_i\}$ for V $\{f_j\}$ for W, $\{g_k\}$ for Z $\{L_i\}$ for $\hat{V}$ $\{P_j\}$ for $\hat{W}$

$$\tau_1 = \sum a_{ji} L_i \otimes f_j \qquad \tau_2 = \sum b_{kj} P_j \otimes g_k$$

$$(\tau_1, \tau_2) \longmapsto \sum_{i,j,j',k} b_{kj} a_{j'i} L_i \otimes g_k \left(P_j(f_{j'})\right)$$

$$= \sum_{i,k} L_i \otimes g_k \left(\sum_j a_{ji} b_{kj}\right)$$

$\triangle$ Generalization of matrices and multiplications

matrices $\to V_1 \otimes \hat{V}_2 \otimes \cdots \otimes V_n$

multiplication $\to$ Collapsing vectors space with dual vector space

Example: taken a single vector space V

"3-dim matrix": $V \otimes V \otimes \hat{V}$ $\qquad V \otimes V \otimes \hat{W}$

"multiplications": $(V \otimes V \otimes \hat{V}) \times (V \otimes \hat{V}) \to V \otimes V \otimes \hat{V}$

$$(V \otimes V \otimes \hat{W}) \times (W \otimes Z) \longrightarrow V \otimes V \otimes Z$$