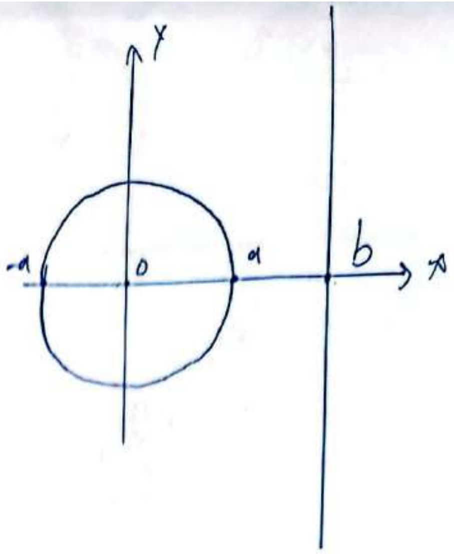
A1

volume $= \pi \int_{-a}^{a} (R(y))^2 - r(y)^2) dy$

$= \pi \int_{-a}^{a} (b + \sqrt{a^2 - y^2})^2 - (b - \sqrt{a^2 - y^2})^2 \, dy$

$= \pi \int_{-a}^{a} (4b\sqrt{a^2 - y^2}) dy$

$= 4b\pi \int_{-a}^{a} \sqrt{a^2 - y^2} \, dy$



let $t = \frac{y}{a}$, $dt = \frac{1}{a} dy$

$4b\pi \int_{-a}^{a} \sqrt{a^2 - y^2} \, dy$

$= 4b\pi \int_{-1}^{1} \sqrt{1 - t^2} \, a^2 dt$

$= 4b\pi \cdot \left(\frac{a^2\pi}{2}\right) = 2a^2 b\pi$

A2. $S = 2\pi \int_0^h f(x)\sqrt{1 + f'(x)^2} \, dx$

$f(x) = \sqrt{R^2 - x^2}$, $(f'(x))^2 = \left(\frac{1}{2} \frac{-2x}{\sqrt{R^2 - x^2}}\right)^2 = \frac{x^2}{R^2 - x}$

$2\pi \int_0^h f(x)\sqrt{1 + f'(x)^2} dx = 2\pi \int_0^h \sqrt{R^2 - x^2} \cdot \sqrt{\frac{R^2}{R^2 - x}} dx = 2\pi R \int_0^h dx = 2\pi R h$

A3. (a) $f'(x) = e^x > 0$, $f''(x) = e^x > 0$, so the graph of $y = e^x$ is always concave

(b) Area of ABCD $< \int_{\ln a}^{\ln b} e^x dx <$ Area of AEFD $\Rightarrow \frac{1}{2}(AB + CD)(\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x dx < \frac{1}{2}(e^{\ln a} + e^{\ln b})(\ln b - \ln a$

because M is the mid point of BC, so $e^{\frac{\ln a + \ln b}{2}} = \frac{1}{2}(AB + CD)$

$\Rightarrow e^{\frac{\ln a + \ln b}{2}}(\ln a + \ln b) < \int_{\ln a}^{\ln b} e^x dx < \frac{1}{2}(e^{\ln a} + e^{\ln b})(\ln b \cdot \ln a)$

清華大學
Tsinghua University

(c) $\int_{\ln a}^{\ln b} e^{x}\,dx = e^{\ln b} - e^{\ln a} = b-a$

so $e^{\frac{\ln a+\ln b}{2}}(\ln b-\ln a) < b-a < \left(\frac{e^{\ln a}+e^{\ln b}}{2}\right)(\ln b-\ln a)$

$\Rightarrow e^{\frac{\ln a+\ln b}{2}} < \frac{b-a}{\ln b-\ln a} < \frac{e^{\ln a}+e^{\ln b}}{2}$

$\Rightarrow \sqrt{e^{\ln a}}\cdot\sqrt{e^{\ln b}} < \frac{b-a}{\ln b-\ln a} < \frac{e^{\ln a}}{2} + \frac{e^{\ln b}}{2}$

$\Rightarrow \sqrt{ab} < \frac{b-a}{\ln b-\ln a} < \frac{a+b}{2}$

A4 let $y=\log x^{x} = x\log x$, $x^{x} = e^{y}$

when $0<x<e$:

$0<\log x<1 \Rightarrow 0 < x\log x < x \Rightarrow \lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} x = 0$ (Sandwich)

so $\lim_{x\to 0^+} x^{x} = \lim_{x\to 0^+} e^{x\log x} = e^{0} = 1$

A5 $\sec^{-1}(x) \in [0,\pi]$, and let $\sec^{-1}(x)=\theta$

we know that $\sec(\pi-\theta) = \frac{1}{\cos(\pi-\theta)} = \frac{1}{-\cos(\theta)} = -\sec\theta = -x$

then $\sec^{-1}(-x) = \pi-\theta = \pi - \sec^{-1}(x)$

B1 length of $C' = \int_c^d \sqrt{\left(\frac{d\, f\circ\varphi}{du}\right)^2 + \left(\frac{d\, g\circ\varphi}{du}\right)^2}\, du$

Let $t = \varphi(u)$

then $\int_c^d \sqrt{\left(\frac{d\, f\circ\varphi}{du}\right)^2 + \left(\frac{d\, g\circ\varphi}{du}\right)^2}\, du$

$= \int_c^d \sqrt{\left(\frac{df}{d\varphi} \times \frac{d\varphi}{du}\right)^2 + \left(\frac{dg}{d\varphi} \times \frac{d\varphi}{du}\right)^2}\, du$

$= \int_c^d \sqrt{\left(\frac{d\varphi}{du}\right)^2 \times \left(\left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)\right)^2}\, du$

$= \int_c^d \sqrt{\left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)^2} \times \frac{d\varphi}{du}\, du$

$= \int_a^b \sqrt{\left(\frac{df}{d\varphi}\right)^2 + \left(\frac{dg}{d\varphi}\right)^2}\, d\varphi$

$= \int_a^b \sqrt{f'(t) + g'(t)}\, dt$ since $t = \varphi(u)$

= length of C

so the curve is independent of the parametrization.