Calculus A(2) Spring 2021 Midterm Exam

Name and Student ID: _____

1. (40 points) For each of the following questions choose **one answer** from A to D.

   (a) A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L. Which of the following statements is true?

   A. There is always n such that $a_n = L$.
   B. There is always n such that $a_n < L$ or $a_n > L$.
   C. There is always n such that $L - 1 < a_n < L + 1$.
   D. None of the above

   (b) Which of the following statements is true about the infinite series $\sum_{n=1}^{\infty}(-1)^n/(n^2+n)$?

   A. The series converges absolutely.
   B. The series converges conditionally.
   C. The series diverges.
   D. The convergence of the series depends on how the summation is performed.

   (c) What is the coefficient of the second (nonvanishing) term in the Taylor series generated by $\sin x$ at $x = \pi/4$?

   A. 1
   B. $1/\sqrt{2}$
   C. $-1/3!$
   D. None of the above

   (d) Halley's Comet has an orbital period of 75.32 years. Which of the following best describes its orbit?

   A. A circle
   B. An ellipse
   C. A parabola
   D. A hyperbola

   (e) A point in a plane has polar coordinates $(r, \theta) = (1, 125\pi/2)$. What is the Cartesian coordinate (x, y) of this point?

   A. $(1, 0)$
   B. $(0, 1)$
   C. $(-1, 0)$
   D. $(0, -1)$

   (f) What is the dot product $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j}$?

   A. 0
   B. 1
   C. $-(2\cos t)\mathbf{i} - (2\sin t)\mathbf{j} + \mathbf{k}$
   D. None of the above

(g) What is the cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 6\mathbf{j} + 9\mathbf{j}$?

A. 0 (scalar)
B. $\mathbf{0}$ (vector)
C. 42
D. None of the above

(h) Which of the following equations describes the line in space that is parallel to the vector $\mathbf{i} + \mathbf{j}$ and goes through point $(x, y, z) = (1, 1, 1)$?

A. $\mathbf{r}(t) = (1 + 2t)\mathbf{i} + (1 + 2t)\mathbf{j} + \mathbf{k}, \ -\infty < t < \infty$
B. $\mathbf{r}(t) = (1 + t)\mathbf{i} + (1 + t)\mathbf{j} + (1 + 2t)\mathbf{k}, \ -\infty < t < \infty$
C. $\mathbf{r}(t) = (1 + t)\mathbf{i} + (1 + t)\mathbf{j} + t\mathbf{k}, \ -\infty < t < \infty$
D. None of the above

(i) A point P_0 lies in a plane in space, and two different vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are both perpendicular to this plane. Which of the following equations describes this plane (as the set of points P satisfying it)?

A. $\mathbf{n}_1 \cdot \overrightarrow{P_0P} = \mathbf{n}_2 \cdot \overrightarrow{P_0P}$.
B. $\mathbf{n}_1 \times \overrightarrow{P_0P} = \mathbf{n}_2 \times \overrightarrow{P_0P}$.
C. $(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \overrightarrow{P_0P} = 0$.
D. None of the above

(j) A particle is moving with constant acceleration along the circle $x^2 + y^2 = 1$ in space. Which of the following statements is true about the position vector $\mathbf{r}$ and the velocity $\mathbf{v}$ of the particle?

A. $\mathbf{r} \cdot \mathbf{v} = 0$.
B. $\mathbf{r} \times \mathbf{v} = 0$.
C. $|\mathbf{v}|$ is constant.
D. None of the above is true.

2. (18 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n} - \sqrt{n-1}}{n} x^n \, .$$

3. (12 points) Calculate

$$\lim_{x \to 0} \frac{1}{7!} \cdot \frac{[\ln(1+x)]^7}{\sin x - x + \frac{x^3}{6} - \frac{x^5}{120}} \, .$$

4. (12 points) Consider the infinite series $\sum_{n=0}^{\infty} 3(-2\cos\theta)^n$.

(a) For what values of θ does the series converge?

(b) For the values of θ found in (a), let $f(\theta)$ be the sum of the series. Sketch the graph of the equation $r = f(\theta)$ in polar coordinates (r, θ).

5. (18 points) Alice is in a spaceship that is moving with velocity $\mathbf{v}(t) = -2\sin(2t)\mathbf{i} + \mathbf{j} + 4\cos(2t)\mathbf{k}$ at time t.

   (a) Alice is at point $(x, y, z) = (1, 1, 1)$ at $t = 0$. Find the acceleration $\mathbf{a}(t)$ and the position $\mathbf{r}(t)$ of Alice at time t.

   (b) Sketch the trajectory of Alice, and sketch its projection to the xz-plane.

   (c) Bob observes Alice from another spaceship moving with velocity $\mathbf{w}(t) = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{k}$. Find the arc length that Alice travels from $t = 0$ to $t = 1$, as measured by Bob (that is, measured in a coordinate system whose origin is Bob's location).