10. $\vec{r}(t) = t\vec{i} + (1-t)\vec{j} + \vec{k}$

$|\vec{v}(t)| = \left|\frac{d\vec{r}(t)}{dt}\right| = |\vec{i} - \vec{j}| = \sqrt{2}$

$f(t) = x - y + z - 2$

$= t - (1-t) + 1 - 2 = 2t - 2$

$\int_C (x-y+z-2)\,ds = \int_0^1 (2t-2)\sqrt{2}\,dt$

$= \sqrt{2}t^2 - 2\sqrt{2}t \Big|_0^1$

$= -\sqrt{2}$

12. $|\vec{v}(t)| = \left|\frac{d\vec{r}(t)}{dt}\right| = \left|-4\sin t\,\vec{i} + 4\cos t\,\vec{j} + 3\vec{k}\right|$

$= \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2} = 5$

$f(t) = \sqrt{x^2+y^2} = \sqrt{(4\cos t)^2 + (4\sin t)^2} = 4$

$\int_C \sqrt{x^2+y^2}\,ds = \int_{-2\pi}^{2\pi} 4 \times 5\,dt$

$= 20t \Big|_{-2\pi}^{2\pi}$

$= 80\pi$

26. let $x = \cos t$, $y = \sin t$, $0 \le t \le \frac{\pi}{2}$

then $\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j}$

and $\vec{F} = \sin t\,\vec{i} - \cos t\,\vec{j}$

$\int_C \vec{F}\,d\vec{r} = \int_0^{\frac{\pi}{2}} \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt$

$= \int_0^{\frac{\pi}{2}} (\sin t\,\vec{i} - \cos t\,\vec{j}) \cdot (-\sin t\,\vec{i} + \cos t\,\vec{j})\,dt$

$= \int_0^{\frac{\pi}{2}} -\sin^2 t - \cos^2 t\,dt$

$= \int_0^{\frac{\pi}{2}} -1\,dt$

$= -\frac{\pi}{2}$

28. pick $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$

gives $\vec{r} = 2\cos t\,\vec{i} + 2\sin t\,\vec{j}$

$\vec{F} = \nabla f = 2(x+y)\vec{i} + 2(x+y)\vec{j}$

$= 4(\cos t + \sin t)\vec{i} + 4(\cos t + \sin t)\vec{j}$.

$\frac{d\vec{r}}{dt} = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$

$W = \int_C \vec{F} \cdot \frac{d\vec{r}}{dt} \cdot dt$

$= \int_0^{2\pi} 8(\cos^2 t - \sin^2 t)\,dt$

$= \int_0^{2\pi} 4 \cdot \cos 2t\,d2t$

$= 4\sin 2t \Big|_0^{2\pi} = 0$

38. (a) as G, m, M are const., let $-GmM = C$, then $F = C \cdot \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$

check $\frac{\partial M}{\partial y} = \frac{-3xyC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z} = \frac{-3xzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial P}{\partial x}$, $\frac{\partial N}{\partial z} = \frac{-3yzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial P}{\partial y} \Rightarrow \vec{F} = \nabla f$

$\frac{\partial f}{dx} = M = \frac{Cx}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f(x,y,z) = -C(x^2+y^2+z^2)^{-1/2} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{Cy}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y} (= N) = \frac{Cy}{(x^2+y^2+z^2)^{3/2}}$

$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{Cz}{(x^2+y^2+z^2)^{3/2}} + h'(z)(= P) = \frac{Cz}{(x^2+y^2+z^2)^{3/2}} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C_1$

so $f(x,y,z) = -C(x^2+y^2+z^2)^{-1/2} + C_1$, let $C_1 = 0$, then:

$f(x,y,z) = \frac{GmM}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ is a potential function for $\vec{F}$.

38 (b). as $s = \sqrt{x^2+y^2+z^2}$

we have $W = \int_{P_1}^{P_2} \frac{GMm}{\sqrt{x^2+y^2+z^2}} = \frac{GMm}{s_2} - \frac{GMm}{s_1} = GMm\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$

18. let $x = r\cos\theta$ and $y = r\sin\theta$, so $z = -x = -r\cos\theta$, $0 \le \theta \le 2\pi$, $0 \le r \le 2$

which gives $\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} - r\cos\theta\cdot\vec{k}$

hence $\vec{r}_r = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} - \cos\theta\,\vec{k}$ and $\vec{r}_\theta = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j} + r\sin\theta\,\vec{k}$

$$|\vec{r}_r \times \vec{r}_\theta| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix} = |r\vec{i} + r\vec{k}| = \sqrt{2}r$$

$$A = \int_0^{2\pi}\int_0^2 \sqrt{2}r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\sqrt{2}}{2}r^2\right]_0^2 d\theta = \int_0^{2\pi} 2\sqrt{2}\,d\theta = 4\sqrt{2}\pi$$

48. As $2x^{3/2} + 2y^{3/2} - 3z = 0 \Rightarrow z = \frac{2}{3}(x^{3/2} + y^{3/2})$

$f_x = \frac{\partial z}{\partial x} = x^{\frac{1}{2}}$ and $f_y = \frac{\partial z}{\partial y} = y^{\frac{1}{2}}$

so $A = \int_0^1\int_0^1 \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy = \int_0^1\int_0^1 \sqrt{x+y+1}\,dx\,dy = \int_0^1 \left[\frac{2}{3}(x+y+1)^{\frac{3}{2}}\right]_0^1 dy$

$= \frac{2}{3}\int_0^1 (y+2)^{\frac{3}{2}} - (y+1)^{\frac{3}{2}}\,dy = \frac{4}{15}\left[(y+2)^{\frac{5}{2}} - (y+1)^{\frac{5}{2}}\right]_0^1 = \frac{4}{15}\cdot 3^{\frac{5}{2}} - \frac{8}{15}\cdot 2^{\frac{5}{2}} + \frac{4}{15}$

$= \frac{12}{5}\sqrt{3} - \frac{32}{15}\sqrt{2} + \frac{4}{15}$

4. let $\vec{r}(\phi,\theta) = (a\sin\phi\cos\theta)\vec{i} + (a\sin\phi\sin\theta)\vec{j} + a\cos\phi\,\vec{k}$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$

then $\vec{r}_\phi = a\cos\phi\cos\theta\,\vec{i} + a\cos\phi\sin\theta\,\vec{j} - a\sin\phi\,\vec{k}$

$\vec{r}_\theta = -a\sin\phi\sin\theta\,\vec{i} + a\sin\phi\cos\theta\,\vec{j}$

and $|\vec{r}_\phi \times \vec{r}_\theta| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} = |a^2\sin^2\phi\cos\theta\,\vec{i} + a^2\sin^2\phi\sin\theta\,\vec{j} + a^2\sin\phi\cos\phi\,\vec{k}|$

$= a^2\sin\phi$

$G(x,y,z) = z^2 = a^2\cos^2\phi$

so $\iint_S G(x,y,z)\,d\sigma = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} a^2\cos^2\phi\cdot a^2\sin\phi\,d\phi\,d\theta = a^4\int_0^{2\pi}\int_0^{\frac{\pi}{2}} -\cos^2\phi\,d\cos\phi\,d\theta$

$= a^4\int_0^{2\pi}\left[-\frac{1}{3}\cos^3\phi\right]_0^{\frac{\pi}{2}} d\theta = a^4\int_0^{2\pi}\frac{1}{3}\,d\theta = \frac{2}{3}\pi a^4$