数 学 作 业 纸

（科目：Linear Algebra）

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Problem 3.1.27

Sol. (a) False. Let $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, then $b_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $b_2=\begin{bmatrix}0\\1\end{bmatrix}$ are not in $C(A)$, but

$b_3=b_1+b_2=\begin{bmatrix}1\\1\end{bmatrix}$ is in $C(A)$, so vectors not in $C(A)$ can't form a subspace.

(b) True

(c) True.

(d) False. Let $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, the $A-I=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ and we can see $C(A)$ and $C(A-I)$ is not the same.

Problem 3.1.28

Sol. (i) Let $A=\begin{bmatrix}1&1&0\\1&0&0\\0&1&0\end{bmatrix}$, then $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}$ in $C(A)$ where $\begin{bmatrix}1\\1\\1\end{bmatrix}$ not in $C(A)$.

(ii) $A=\begin{bmatrix}1&0&0\\2&0&0\\3&0&0\end{bmatrix}$ which column space is a line in $\mathbb{R}^3$

Problem 3.2.4

Sol.

$$A=\begin{bmatrix}-1&3&5\\-2&6&10\end{bmatrix}\longrightarrow\begin{bmatrix}-1&3&5\\0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&-3&-5\\0&0&0\end{bmatrix}=R.$$

Special Solution: $x_1=\begin{bmatrix}3\\1\\0\end{bmatrix}$, $x_2=\begin{bmatrix}5\\0\\1\end{bmatrix}$

$$B=\begin{bmatrix}-1&3&5\\-2&6&7\end{bmatrix}\longrightarrow\begin{bmatrix}-1&3&5\\0&0&-3\end{bmatrix}\longrightarrow\begin{bmatrix}1&-3&-5\\0&0&1\end{bmatrix}=R$$

Special Solution: $x=\begin{bmatrix}3\\1\\0\end{bmatrix}$

The number of pivot variables plus the number of free variables is n.

Problem 3.2.12

Sol. $A=\begin{bmatrix}1&-3&-1\end{bmatrix}$, y and z are free variables and special solutions are $\begin{bmatrix}3\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}$

Problem 3.2.20

Sol. $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ has $N(A)=C(A)$.

Problem 3.2.31

Sol.

(a) $A=\begin{bmatrix}4&4&4&4\\4&4&4&4\\4&4&4&4\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1&1\\4&4&4&4\\4&4&4&4\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}=R$, rank $=1$.

(b) $A=\begin{bmatrix}1&2&3&4\\2&3&4&5\\3&4&5&6\end{bmatrix}\rightarrow\begin{bmatrix}1&2&3&4\\0&-1&-2&-3\\0&-2&-4&-6\end{bmatrix}\rightarrow\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{bmatrix}=R$, rank $=2$.

(c) $A=\begin{bmatrix}1&-1&1&-1\\-1&1&-1&1\\1&-1&1&-1\end{bmatrix}\rightarrow\begin{bmatrix}1&-1&1&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}=R$, rank $=1$.

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Problem 3.2.32

Sol. $A^T y = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{cases} -y_1 + y_3 - y_4 = 0 \\ y_1 - y_2 - y_5 = 0 \\ y_2 - y_3 - y_6 = 0 \\ y_1 + y_2 + y_3 = 0 \end{cases}$

$$A^T = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\to \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$. special solutions are $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Problem 3.2.41

Sol. $A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$

Problem 3.2.43

Sol. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R$. which means column 1 and 3 are pivot column,

so $S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ is invertible.

$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = R$, only column 1 (row 1) is pivot column, and $s = \begin{bmatrix} 1 \end{bmatrix}$ is invertible.

$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, column 2 and 3 (row 1 and 3) is pivot column, so $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible.

Problem 3.3.4

Sol. $[A\, b] = \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [R\, d]$

$$x = x_p + x_n = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Problem 3.3.6

Sol.

(i) $[A\ b] = \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 3 & b_4-3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 3 & b_4-3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 0 & b_4+3b_1-3b_3 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 5b_1-2b_3 \\ 0 & 1 & b_3-2b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 0 & b_4+3b_1-3b_3 \end{bmatrix}$ If system solvable, it must have $\begin{cases} b_2-2b_1=0 \\ b_4+3b_1-3b_3=0 \end{cases}$, $x = x_p = \begin{bmatrix} 5b_1-2b_3 \\ b_3-2b_1 \end{bmatrix}$

(ii) $[A\ b] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 3 & 3 & b_4-3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 0 & 0 & b_4+3b_1-3b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 5b_1-2b_3 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_4+3b_1-3b_3 \end{bmatrix}$

Solvable if $\begin{cases} b_2-2b_1=0 \\ b_4+3b_1-3b_3=0 \end{cases}$, then $x = \begin{bmatrix} 5b_1-2b_3 \\ b_3-2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Problem 3.3.7

Sol.

$[A\ b] = \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2-3b_1 \\ 0 & -2 & -2 & b_3-2b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & 1 & 1 & 3b_1-b_2 \\ 0 & 0 & 0 & 4b_1-2b_2+b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -8b_1+3b_2 \\ 0 & 1 & 1 & 3b_1-b_2 \\ 0 & 0 & 0 & 4b_1-2b_2+b_3 \end{bmatrix}$

System solvable if $4b_1-2b_2+b_3=0$, which means $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in $C(A)$

Also, 4 (row 1) − 2 (row 2) + (row 3) gives zero row.

Problem 3.3.13

Sol. (a) The scalar of x_p should always be 1.

(b) Any solution can be x_p.

(c) Let $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, $x_p = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ but there exists $x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

where $\|(4,2)\| = \sqrt{4^2+2^2} = 2\sqrt{5} < 8 = \|(8,0)\|$.

(d) $x_n = 0$ is always in the nullspace.

数 学 作 业 纸

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Problem 1

Sol.

$$A=\begin{bmatrix}-3&-1&0&1\\-4&-1&1&1\\-1&1&4&-1\end{bmatrix}\rightarrow\begin{bmatrix}1&\frac{1}{3}&0&-\frac{1}{3}\\0&\frac{1}{3}&1&-\frac{1}{3}\\0&\frac{4}{3}&4&-\frac{4}{3}\end{bmatrix}\rightarrow\begin{bmatrix}1&0&-1&0\\0&\frac{1}{3}&1&-\frac{1}{3}\\0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&-1&0\\0&1&3&-1\\0&0&0&0\end{bmatrix}=R.$$

the special solutions are $\begin{bmatrix}1\\-3\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$

Problem 2

Sol.

$$[A\ b]=\begin{bmatrix}2&-1&-1&b_1\\-1&2&-1&b_2\\-1&-1&2&b_3\end{bmatrix}\rightarrow\begin{bmatrix}1&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}b_1\\0&\frac{3}{2}&-\frac{3}{2}&\frac{1}{2}b_1+b_2\\0&-\frac{3}{2}&\frac{3}{2}&\frac{1}{2}b_1+b_3\end{bmatrix}\rightarrow\begin{bmatrix}1&0&-1&\frac{2}{3}b_1+\frac{1}{3}b_2\\0&1&-1&\frac{1}{3}b_1+\frac{2}{3}b_2\\0&0&0&b_1+b_2+b_3\end{bmatrix}=[R\ d]$$

If b in $C(A)$, $b_1+b_2+b_3$ should be zero.

when $b=[1\ \ 1\ \ -2]^T$, then:

$$[R\ d]=\begin{bmatrix}1&0&-1&1\\0&1&-1&1\\0&0&0&0\end{bmatrix}$$

$$x=x_p+x_n=\begin{bmatrix}1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}1\\1\\1\end{bmatrix}.$$