Linear Algebra – Homework 5

21 Oct 2020
Due: 29 Oct 2020

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 2.5.6.

(a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

(b) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two different matrices such that $AB = AC$.

Problem 2.5.7. If A has row $1+$ row $2 =$ row 3, show that A is not invertible:

(a) Explain why $A\mathbf{x} = (0, 0, 1)$ cannot have a solution. Add eqn 1 + eqn 2.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\mathbf{x} = \mathbf{b}$?

(c) In elimination, what happens to equation 3?

Problem 2.5.11.

(a) Find invertible matrices A and B such that $A + B$ is not invertible.

(b) Find non-invertible matrices A and B such that $A + B$ is invertible.

Problem 2.5.21. There are sixteen 2×2 matrices whose entries are 1's and 0's. How many of them are invertible?

Problem 2.5.25. Find A^{-1} and B^{-1} (*if they exist*) by elimination on $[\,A\ I\,]$ and $[\,B\ I\,]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Problem 2.5.28. Exchange rows and continue with elimination to find A^{-1}:

$$[\,A\ I\,] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

Problem 2.5.31. This matrix has a remarkable inverse. Find A^{-1} by elimination on $[\,A\ I\,]$. Extend to a 5×5 "alternating matrix" and guess its inverse; then multiply to confirm.

$$\text{Invert} \quad A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and solve} \quad A\mathbf{x} = (1, 1, 1, 1).$$

Problem 2.5.39. A is a 4×4 matrix with 1's on the diagonal and $-a$, $-b$, $-c$ on the diagonal above. Find A^{-1} for this bidiagonal matrix.

Problem 2.6.6. What elimination matrices E_{21} and E_{32} put A into upper triangular form, $E_{32}E_{21}A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}.$$

Problem 2.6.8. This problem shows how the elimination matrix inverses E_{ij}^{-1} multiply to give L. You see this best when A is already lower triangular with 1's on the diagonal. Then $U = I$.

$$A = L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

The elimination matrices E_{21}, E_{31}, E_{32} contain $-a$ then $-b$ then $-c$.

(a) Multiply $E_{32}E_{31}E_{21}$ to find the single matrix E that produces $EA = I$.

(b) Multiply $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ to bring back L.

The multipliers a, b, c are mixed up in E but perfect in L!

Problem 2.6.13. Compute L and U for the symmetric matrix A:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to guarantee that systems $A\mathbf{x} = \mathbf{b}$ will have unique solutions.

Problem 2.6.15. Solve the triangular system $L\mathbf{c} = \mathbf{b}$ to find $\mathbf{c}$. Then solve $U\mathbf{x} = \mathbf{c}$ to find $\mathbf{x}$:

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}.$$

For safety, multiply LU and solve $A\mathbf{x} = \mathbf{b}$ as usual. Circle $\mathbf{c}$ when you see it.

Problem 2.6.16. Solve $L\mathbf{c} = \mathbf{b}$ to find $\mathbf{c}$. Then solve $U\mathbf{x} = \mathbf{c}$ to find $\mathbf{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Graded Problems.

Problem 1. Find the inverse to the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Use A^{-1} to solve the system of linear equations $A\mathbf{x} = (1, 0, 0, 1)$.

Problem 2. Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

Then solve the system $A\mathbf{x} = (1, 2, 3, 4)$ for $\mathbf{x}$ by solving the two triangular systems $L\mathbf{y} = (1, 2, 3, 4)$ and $U\mathbf{x} = \mathbf{y}$.