Linear Algebra – Fall 2020
Final Exam

NAME:

STUDENT ID:

Instructions:

- Show answers and arguments in the space provided. You may use the back of the pages also, but indicate clearly any such material that you want graded. Answers given without supporting work may receive zero credit.
- This is a closed book exam: no calculators, notes, or formula sheets.

QUESTION	POINTS	SCORE
1	15	
2	12	
3	16	
4	12	
5	15	
6	16	
7	12	
8	15	
9	20	
10	17	
TOTAL	150	

1. (15 points) Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 3x_2 + 4x_3 &= b_2 \\ 3x_1 + 4x_2 + 5x_3 &= b_3 \end{aligned}$$

(a) Find a linear relation involving b_1, b_2, and b_3 that guarantees the system has at least one solution.

(b) For $(b_1, b_2, b_3) = (1, 1, 1)$, find *all* solutions of the system of equations.

2. (12 points) Find the determinants of A and B:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}, \qquad B = \begin{bmatrix} a & 2 & b \\ 2 & 0 & 2 \\ b & 2 & a \end{bmatrix}.$$

What condition on a, b guarantees that B is *not* invertible?

3. (16 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 2 \\ 7 & -5 & 2 & 5 \\ -1 & -7 & -8 & 7 \end{bmatrix}.$$

(a) Find the reduced row echelon form R of A.

(b) Find bases for the null space, row space, column space, and left null space of A.

4. (a) (7 points) If $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent vectors, show that the sums

$$\mathbf{v}_1 = \mathbf{w}_2 + \mathbf{w}_3, \quad \mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_3, \quad \mathbf{v}_3 = \mathbf{w}_1 + \mathbf{w}_2$$

are also independent. (*Hint:* Write $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ in terms of the $\mathbf{w}$'s. Find and solve equations for the c's, to show they are zero.)

(b) (5 points) If $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent vectors, show that the differences

$$\mathbf{v}_1 = \mathbf{w}_2 - \mathbf{w}_3, \quad \mathbf{v}_2 = \mathbf{w}_1 - \mathbf{w}_3, \quad \mathbf{v}_3 = \mathbf{w}_1 - \mathbf{w}_2$$

are dependent. (Find a linear combination of the $\mathbf{v}$'s that gives zero.)

5. (a) (12 points) Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) (3 points) Use A^{-1} to solve the linear system of equations $A\mathbf{x} = (3, 6, 2)$.

6. (a) (6 points) Find all eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(b) (10 points) Find an *orthonormal* basis for $\mathbf{R}^3$ consisting of eigenvectors for A.

7. (a) (4 points) What is the area of the parallelogram in $\mathbf{R}^2$ spanned by $\mathbf{x} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} b \\ d \end{bmatrix}$?

   (b) (8 points) Verify that for any 2×2 reflection matrix $Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, the area of the parallelogram spanned by $Q\mathbf{x}$ and $Q\mathbf{y}$ is the same as your answer for part (a).

8. (a) (10 points) Find the best least squares line $C + Dt$ to fit the data points $(-2, 4)$, $(-1, 2)$, $(0, -1)$, $(1, 0)$, and $(2, 0)$.

   (b) (2 points) Sketch a graph of the data points and your least squares line.

   (c) (3 points) Find the least squares error $\|\mathbf{e}\|$ of the best fit line.

9. Consider 2×2 matrices

$$A = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ 6 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

(a) (8 points) Find an invertible matrix X and a diagonal matrix Λ such that $A = X\Lambda X^{-1}$.

(b) (6 points) Use part (a) to calculate the matrix A^N for any positive integer N. What limit matrix does A^N approach as $N \to \infty$?

(c) (6 points) Show that B is *not* diagonalizable.

10. (a) (6 points) Find a basis for the subspace S of $\mathbf{R}^4$ spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

(b) (5 points) Find a basis for the orthogonal complement $S^\perp$.

(c) (6 points) Find $\mathbf{b}_1$ in S and $\mathbf{b}_2$ in $S^\perp$ so that $\mathbf{b}_1 + \mathbf{b}_2 = (1, 1, 1, 1)$.

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