Practice Final Exam

1. (a) Find the unique value of c such that the system of equations has at least one solution:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 3 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= c \\ x_1 - x_2 - 3x_3 - 5x_4 &= 1 \end{aligned}$$

   (b) For the value of c you found, find *all* solutions to the system of equations.

2. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues of A.

(b) Find a basis of $\mathbf{R}^3$ consisting of eigenvectors for A.

(c) Find the angles between the eigenvectors in the basis.

3. Determine whether the following sets of vectors are bases for $\mathbb{R}^3$. In case a set of vectors is *not* linearly independent, show how to write one of the vectors as a linear combination of the others:

(a) $\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix} \right\}$

4. Find bases for the null space, column space, row space, and left null space of the matrix:

$$A = \begin{bmatrix} -3 & 1 & 4 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 5 & 2 \end{bmatrix}$$

5. (a) Diagonalize the matrix A: write $A = X\Lambda X^{-1}$ where X is invertible and Λ is diagonal.

$$A = \begin{bmatrix} 10 & 12 \\ -6 & -7 \end{bmatrix}$$

(b) Show that the matrix B is *not* diagonalizable:

$$B = \begin{bmatrix} 8 & 9 \\ -4 & -4 \end{bmatrix}$$

6. Consider the (x, y) data points $(-1, 0)$, $(0, 1)$, $(1, 3)$, and $(2, 9)$.

   (a) Find the best least squares fit by a linear function $y = mx + b$ to the data.

   (b) Plot your linear function from part (a) along the data on a coordinate system.

   (c) Find the error of the least squares approximation.

7. (a) Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 14 \end{bmatrix}$$

(b) Use A^{-1} to solve the linear system of equations $A\mathbf{x} = (1, 2, 1)$.

8. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 2 \end{bmatrix}.$$

(a) Find the LU decomposition of A.

(b) Find the volume of the box in $\mathbf{R}^3$ that is spanned by the columns of A.

9. (a) Find the determinants of the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -3 & 0 & 3 \end{bmatrix}$$

(b) Are the matrices A and B invertible?

10. (a) Find an orthonormal basis for the subspace V of $\mathbf{R}^4$ spanned by $(1, 1, 1, 1)$ and $(3, 2, 2, 1)$.

    (b) Find the projection matrix P for the orthogonal projection onto V, and compute the projection of $(0, 0, 1, 1)$ onto V.