Calculus A(1): Homework 6

Each assigned exercise is worth 20 points. The bonus exercise is optional. We may (or may not) decide to grade the bonus and use it to replace one assigned exercise (if it improves your total grade). We refer to Thomas' Calculus book (whose PDF is available on the weblearn) for the exercises given by a paragraph and number. If you are using your own Thomas' Calculus book, make sure that the numbering of exercises is identical with the PDF.

Routine exercises (do not hand-in)

§4.2, Exercises 3, 8, 9, 23, 25, 51, 52
§4.3, Exercises 3, 8, 22, 36, 37, 43
§4.4, Exercises 3, 34, 38, 47, 50, 64, 65, 67, 70, 81, 82
§4.6, Exercises 2, 6, 10, 21, 28, 31. 34
page 324, Exercise 32

Assigned exercises (hand-in)

A1. Find $\lim_{x\to 0} \frac{8x-4\sin(2x)}{x^3}$ (if it exists).

A2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that f is differentiable and for all x we have $f'(x) = x^2(x-1)(x-2)^3$. Find the local extrema of f and specify which are local maxima or minima. For which of these local extrema does the second derivative test apply?

A3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and convex. Prove that if f has a local minimum at $x = c$, then f has a global minimum at $x = c$ and this global minimum is attained only at $x = c$.

A4. Let $f : [1, +\infty) \to \mathbb{R}$ be continuous on $[1, +\infty)$ and differentiable on $(1, +\infty)$. Prove that if $\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to+\infty} f'(x)$ exists, then $\lim_{x\to+\infty} f'(x) = 0$. (Hint: apply the MVT on each segment $[n, n + 1]$ for $n \in \mathbb{N}$.)

A5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that if f is convex (i.e. by definition f' is increasing), then the graph of f is above all its tangents, i.e. for any $a \in \mathbb{R}$, for all $x \neq a$ we have $f(x) > L_a(x)$ where $L_a(x)$ is the tangent line of f at a.

Bonus exercises (optional)

B1. Prove the converse of A5: if the graph of f is above all its tangents, then f is convex (i.e. f' is increasing).

B2. Let $a < b$ and $f : [a, b] \to \mathbb{R}$ such that $f(a) = f(b) = 0$, the derivative f' of f exists, f' is continuous on $[a, b]$ and f' is differentiable on (a, b). In particular, the second derivative f''

exists on (a, b). Show that for any $d \in (a, b)$, there exists $c \in (a, b)$ such that

$$f(d) = \frac{f''(c)}{2} \cdot (d - a)(d - b) \ .$$

(Hint: apply the MVT theorem to a well-chosen function $g(x)$ depending on f.)