Calculus A(2) Spring 2021 Final Exam

Name and Student ID: ______________________________

1. (40 points) For each of the following questions choose **one answer** from A to D.

   (a) Let $f(x,y) = \sin(xy)/(x^2+y^2)$ be a function defined on the plane $\mathbb{R}^2$ except the origin. Which of the following statements is true about $\lim_{(x,y)\to(0,0)} f(x,y)$?

   A. The limit is 0 because $f(0,y) = 0$ for all $y \neq 0$.
   B. The limit does not exist because f is not defined at $(0,0)$.
   C. The limit does not exist because f approaches different values as (x,y) approaches $(0,0)$ from different directions.
   D. None of the above is true.

   (b) Which of the following statements is true about the region in the plane $\mathbb{R}^2$ defined as the union of the set $\{(x,y) \mid x^2+y^2 < 2\}$ and the set $\{(x,y) \mid (x-2)^2+y^2 \leq 1\}$?

   A. The region is open but not closed.
   B. The region is closed but not open.
   C. The region is open and closed.
   D. The region is neither open nor closed.

   (c) Which of the following statements is true about the function $f(x,y) = \cos(xy)$?

   A. $f_{xy}(1,\pi) = 0$.
   B. $f_{xy}(1,\pi) = \pi$.
   C. $f_{xy}(1,\pi) = -\pi$.
   D. $f_{xy}(1,\pi)$ is undefined.

   (d) What is the union of all contour curves of the function $f(x,y) = \sqrt{1-x^2-y^2}$ defined in the region where $x^2+y^2 \leq 1$?

   A. A disk
   B. A sphere
   C. A hemisphere
   D. None of the above

   (e) What is the spherical coordinates (ρ,ϕ,θ) of the point with Cartesian coordinates $(x,y,z) = (-1,-1,-\sqrt{2})$?

   A. $(2,\pi/4,\pi/4)$
   B. $(2,3\pi/4,\pi/4)$
   C. $(2,\pi/4,5\pi/4)$
   D. None of the above

(f) Coordinates (u, v, w) are related to the Cartesian coordinates (x, y, z) by $x = u\cos(v)$, $y = u\sin(v)$, $z = w$. What is the Jacobian $\partial(x, y, z)/\partial(u, v, w)$?

A. u
B. v
C. w
D. None of the above

(g) A two-dimensional object has the shape of a disk of radius 1. Its density function δ is unknown, but we know that the double integral of δ over the object is π. What is the mass of the object?

A. 1
B. π
C. π^2
D. None of the above

(h) Which of the following statements is not true about the standard linear approximation of a function $f(x, y)$ at (a, b)?

A. It is equivalent to considering Taylor's formula for $f(x, y)$ about (a, b) to first order.
B. It is equivalent to considering the tangent plane to the graph $z = f(x, y)$ through the point $(a, b, f(a, b))$.
C. For some choices of f, this approximation gives an exact (that is, not approximate) answer.
D. All of the above are true.

(i) What is the double integral of $x^2 \sin(x + y)$ over the region defined by $-1 \leq x \leq 1$, $-1 \leq y \leq 1$?

A. 0
B. 1
C. -1
D. None of the above

(j) A conservative vector field $\mathbf{F}$ in space has a potential function $f(x, y, z) = x^2 - y^2 - xz + z^2$. What is the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the helix C given by $\mathbf{r}(t) = \cos(2\pi t)\mathbf{i} + \sin(2\pi t)\mathbf{j} + t\mathbf{k}$, where the parameter t varies from 0 to 1?

A. 0
B. $\pi/2$
C. $\sqrt{3}\pi$
D. None of the above

2. (18 points) Consider the function $f(x, y, z) = \sin(\pi xyz)$ and the path C: $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$, $-\infty \leq t \leq \infty$, both defined in space $\mathbb{R}^3$.

   (a) Express the value of f along C as a function of t, and calculate its derivative at $t = 1$.

   (b) Find the direction in which the value of f increases most rapidly at $(x, y, z) = (1, 2, 1)$.

   (c) Calculate the derivative of f in the direction of the path C at $(x, y, z) = (1, 2, 1)$.

3. (12 points) Consider the function $f(x, y) = e^x(x^2 - 2x - y^2)$ defined on the plane $\mathbb{R}^2$.

   (a) Find all critical points of f.

   (b) At each of the critical points found above, determine whether f has a local maximum, local minimum or saddle point.

4. (16 points) Let D be the region in space $\mathbb{R}^3$ such that $\sqrt{3(x^2+y^2)} \leq z \leq \sqrt{1-x^2-y^2}$.

   (a) Sketch the region D. Make sure to include the x-, y- and z-axes.

   (b) What values can each of the spherical coordinates ρ, ϕ, θ take in D? Write down the inequalities the coordinates must satisfy.

   (c) Find the volume of D.

5. (14 points) Let $\mathbf{F}(x, y, z) = x^2 \sin(y)\mathbf{i} + 2x\cos(y)\mathbf{j} + 2\mathbf{k}$ be a vector field in space $\mathbb{R}^3$.

(a) Calculate the divergence $\nabla \cdot \mathbf{F}$.

(b) If S_1 and S_2 are two oriented surfaces that share the same oriented boundary C, then $\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}\sigma = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}\sigma$. Explain why.

(c) Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}\sigma$, where S is the part of the unit sphere defined in the spherical coordinates by $\rho = 1,\ \ 0 \le \phi \le \pi/3$, and $\mathbf{n}$ is the outward normal vector field.