清華大学 数学作业纸

TSINGHUA UNIVERSITY

班级：CST01　姓名：[name]　编号：2020010869　科目：Linear Algebra　第 1 页

Problem 4.3.1.

Sol. $A=\begin{bmatrix}1&0\\1&1\\1&3\\1&4\end{bmatrix}$　$A^T[A|\vec{b}]=\begin{bmatrix}1&1&1&1\\0&1&3&4\end{bmatrix}\left[\begin{array}{cc|c}1&0&0\\1&1&8\\1&3&8\\1&4&20\end{array}\right]=\left[\begin{array}{cc|c}4&8&112\\8&26&36\end{array}\right]$

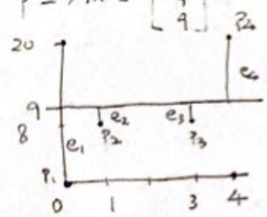
$A^TA\hat{x}=A^T\vec{b}\Rightarrow\begin{bmatrix}4&8\\8&26\end{bmatrix}\cdot\hat{x}=\begin{bmatrix}112\\36\end{bmatrix}\Rightarrow\hat{x}=\begin{bmatrix}1\\4\end{bmatrix}$, $\vec{p}=A\hat{x}=\begin{bmatrix}1\\5\\13\\17\end{bmatrix}$, $\vec{e}=\vec{b}-\vec{p}=\begin{bmatrix}-1\\3\\-5\\3\end{bmatrix}$

$E=\|e\|^2=44$

Problem 4.3.5.

Sol. $A=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$　$A^T[A|\vec{b}]=[1\cdots1]\left[\begin{array}{c|c}1&0\\1&8\\1&8\\1&20\end{array}\right]=[4\,|\,36]$, $\hat{x}=9$, $\vec{p}=A\hat{x}=\begin{bmatrix}9\\9\\9\\9\end{bmatrix}$

best height is $C=9$, $\vec{e}=\vec{b}-\vec{p}=\begin{bmatrix}0\\8\\8\\20\end{bmatrix}-\begin{bmatrix}9\\9\\9\\9\end{bmatrix}=\begin{bmatrix}-9\\-1\\-1\\11\end{bmatrix}$



Problem 4.3.10

Sol. $A=\begin{bmatrix}1&0&0&0\\1&1&1&1\\1&3&9&27\\1&4&16&64\end{bmatrix}$　$[A\vec{b}]=\left[\begin{array}{cccc|c}1&0&0&0&0\\1&1&1&1&8\\1&3&9&27&8\\1&4&16&64&20\end{array}\right]\rightarrow$

$\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&1&8\\0&3&9&27&8\\0&4&16&64&20\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&1&8\\0&0&6&24&-16\\0&0&12&60&-12\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&1&8\\0&0&1&4&-\frac{8}{3}\\0&0&0&12&20\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&0&\frac{19}{3}\\0&0&1&0&-\frac{28}{3}\\0&0&0&1&\frac{5}{3}\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&\frac{47}{3}\\0&0&1&0&-\frac{28}{3}\\0&0&0&1&\frac{5}{3}\end{array}\right]$

So $\begin{bmatrix}C\\D\\E\\F\end{bmatrix}=\begin{bmatrix}0\\\frac{47}{3}\\-\frac{28}{3}\\\frac{5}{3}\end{bmatrix}$, which means $\vec{p}=\vec{b}=\begin{bmatrix}0\\8\\8\\20\end{bmatrix}$　$\vec{e}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

Problem 4.3.12.

Sol. (a) $a^Ta=[1\cdots1]\cdot\begin{bmatrix}1\\\vdots\\1\end{bmatrix}=m$, $a^Tb=[1\cdots1]\begin{bmatrix}b_1\\\vdots\\b_m\end{bmatrix}=\sum_{i=1}^{m}b_i$, $\hat{x}=\frac{a^Tb}{a^Ta}=\frac{\sum_{i=1}^{m}b_i}{m}$ is the mean of b's.

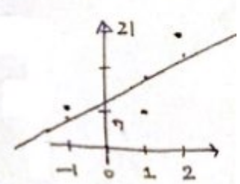
(b) $e=b-a\hat{x}=\begin{bmatrix}b_1-\text{mean}\\\vdots\\b_m-\text{mean}\end{bmatrix}$, $\|e\|^2=(b_1-\text{mean})^2+\cdots+(b_m-\text{mean})^2$　$\|e\|=\sqrt{(b_1-\text{mean})^2+\cdots+(b_m-\text{mean})^2}$

(c) $e=b-p=\begin{bmatrix}-2\\-1\\3\end{bmatrix}$, $p^Te=[3\,3\,3]\begin{bmatrix}-2\\-1\\3\end{bmatrix}=0$　projection matrix $P=\frac{aa^T}{a^Ta}=\frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

Problem 4.3.17

Sol. $\begin{bmatrix}1&-1\\1&1\\1&2\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix}=\begin{bmatrix}7\\7\\21\end{bmatrix}$　$A^T[A|b]=\begin{bmatrix}1&1&1\\-1&1&2\end{bmatrix}\left[\begin{array}{cc|c}1&-1&7\\1&1&7\\1&2&21\end{array}\right]=\left[\begin{array}{cc|c}3&2&35\\2&6&42\end{array}\right]\Rightarrow\begin{bmatrix}3&2\\2&6\end{bmatrix}\hat{x}=\begin{bmatrix}35\\42\end{bmatrix}\Rightarrow\hat{x}=\begin{bmatrix}9\\4\end{bmatrix}$

line $b=9+4t$



Problem 4.3.22

Sol. $A=\begin{bmatrix}1&-2\\1&-1\\1&0\\1&1\\1&2\end{bmatrix}$　$A^T[Ab]=\begin{bmatrix}1&1&1&1&1\\-2&-1&0&1&2\end{bmatrix}\left[\begin{array}{cc|c}1&-2&4\\1&-1&2\\1&0&-1\\1&1&0\\1&2&0\end{array}\right]=\left[\begin{array}{cc|c}5&0&5\\0&10&-10\end{array}\right]\Rightarrow\hat{x}=\begin{bmatrix}1\\-1\end{bmatrix}$, the best line is $b=1-t$.

Problem 4.4.2

Sol. $q_1 = \frac{1}{3}\begin{bmatrix}2\\2\\-1\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{1}{3}\end{bmatrix}$, $q_2 = \frac{1}{3}\begin{bmatrix}-1\\2\\2\end{bmatrix} = \begin{bmatrix}-\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix}$ $Q = \begin{bmatrix}\frac{2}{3} & -\frac{1}{3}\\\frac{2}{3} & \frac{2}{3}\\-\frac{1}{3} & \frac{2}{3}\end{bmatrix}$ $Q^TQ = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ $QQ^T = \frac{1}{9}\begin{bmatrix}5&2&-4\\2&8&2\\-4&2&5\end{bmatrix}$

Problem 4.4.7

Sol. $Q^TQ\hat{x} = Q^Tb$, as Q is orthonormal, so $Q^TQ = I$, then $\hat{x} = Q^Tb$.

Problem 4.4.10.

Sol. (a) $c_1q_1 + c_2q_2 + c_3q_3 = 0 \Rightarrow c_1 q_1\cdot q_1 + c_2q_2\cdot q_1 + c_3\cdot q_3\cdot q_1 = 0 \Rightarrow c_1 + 0 + 0 = 0 \Rightarrow c_1 = 0$

$c_1q_1 + c_2q_2 + c_3q_3 = 0 \Rightarrow c_1q_1\cdot q_2 + c_2q_2 q_2 + c_3q_3 q_2 = 0 \Rightarrow c_2 = 0$

$c_1q_1 + c_2q_2 + c_3q_3 = 0 \Rightarrow c_1q_1\cdot q_3 + c_2q_2 q_3 + c_3q_3q_3 = 0 \Rightarrow c_3 = 0$

so $q's$ are independent.

(b) $Qx = 0 \Rightarrow Q^TQx = 0 \Rightarrow Ix = 0 \Rightarrow x = 0$

Problem 4.4.18.

Sol. $A = \vec{a} = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$ $A^T[A|\vec{b}] = [1\ -1\ 0\ 0]\left[\begin{array}{c|c}1&0\\-1&1\\0&-1\\0&0\end{array}\right] = [2|-1] \Rightarrow \vec{p} = -\frac{1}{2}A$, $B = \vec{b} - \vec{p} = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix} - \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-1\\0\end{bmatrix}$

$A^T\vec{c} = [1\ -1\ 0\ 0]\begin{bmatrix}0\\0\\1\\-1\end{bmatrix} = 0 \Rightarrow \vec{p_a} = 0$, $B^T[B|\vec{c}] = [\frac{1}{2}\ \frac{1}{2}\ -1\ 0]\left[\begin{array}{c|c}1/2&0\\1/2&0\\-1&1\\0&-1\end{array}\right] = [\frac{3}{2}|-1] \Rightarrow \vec{p_b} = -\frac{2}{3}\cdot B$

$C = \vec{c} - \vec{p_a} - \vec{p_b} = \begin{bmatrix}1/3\\1/3\\1/3\\-1\end{bmatrix}$

orthogonal vectors : $\vec{q_1} = \frac{A}{\|A\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$, $\vec{q_2} = \frac{B}{\|B\|} = \sqrt{\frac{2}{3}}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-1\\0\end{bmatrix}$, $\vec{q_3} = \frac{C}{\|C\|} = \frac{\sqrt{3}}{2}\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\\-1\end{bmatrix}$

Problem 4.4.22

Sol.

$A = \vec{a} = \begin{bmatrix}1\\1\\2\end{bmatrix}$ $A^T[A|\vec{b}] = [1\ 1\ 2]\left[\begin{array}{c|c}1&1\\1&-1\\2&0\end{array}\right] = [6|0] \Rightarrow \vec{p} = 0$. , $B = \vec{b} - \vec{p} = \vec{b} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$

$\vec{p_a} = \frac{A^Tc}{A^TA}A = \frac{3}{2}A = \begin{bmatrix}3/2\\3/2\\3\end{bmatrix}$, $\vec{p_b} = \frac{B^Tc}{B^TB}B = \frac{1}{2}B = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{bmatrix}$, $C = \vec{c} - \vec{p_a} - \vec{p_b} = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$

so $\vec{q_1} = \frac{A}{\|A\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\2\end{bmatrix}$, $\vec{q_2} = \frac{B}{\|B\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$, $\vec{q_3} = \frac{C}{\|C\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

Problem 4.4.31.

Sol. (a) $c = \frac{1}{2}$ and then every column of the matrix become unit length.

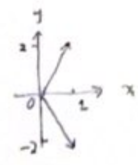
(b) $\vec{p_1} = \frac{a_1^T\vec{b}}{a_1^Ta_1}\cdot a_1 = \frac{-2}{4}a_1 = -\frac{1}{2}\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}$ $\vec{p_2} = \frac{a_2^T\vec{b}}{a_2^Ta_2}a_2 = \frac{-2}{4} = -\frac{1}{2}\begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix}$, so $\vec{p} = \vec{p_1} + \vec{p_2} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$

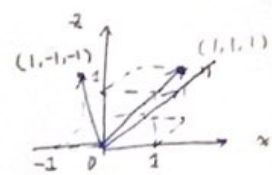
Problem 4.4.32.

Sol. $Q_1 = I - 2uu^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $Q_2 = I - 2uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

(reflect across x-axis) (reflect across plane $y+z=0$)





Problem 1.

Sol. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 50 \\ 44 \\ 32 \\ 6 \end{bmatrix}$, so we have $A^T[A|b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 50 \\ 1 & 1 & 1 & 44 \\ 1 & 2 & 4 & 32 \\ 1 & 3 & 9 & 6 \end{array}\right] = \left[\begin{array}{ccc|c} 4 & 6 & 14 & 132 \\ 6 & 14 & 36 & 126 \\ 14 & 36 & 98 & 226 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3/2 & 7/2 & 33 \\ 0 & 5 & 15 & -72 \\ 0 & 15 & 49 & -236 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3/2 & 7/2 & 33 \\ 0 & 1 & 3 & -72/5 \\ 0 & 0 & 4 & -20 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3/2 & 0 & 101/2 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 248/5 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & -5 \end{array}\right]$

the the parabola is $h(t) = \frac{248}{5} + \frac{3}{5}t - 5t^2$, as $-\frac{1}{2}gt^2 = -5t^2$, so $g = 10$.

Problem 2

Sol. $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $R(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, $N(A) = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$a_1 = a = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $b_1 = b - \frac{a_1^T b}{a_1^T a_1} a_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$q_1 = \frac{a_1}{\|a_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ $q_2 = \frac{b_1}{\|b_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so $\left\{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ is an orthonormal basis of $R(A)$

As $N(A) = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

let $w = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $w_1 = \frac{w}{\|w\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\left\{\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}\right\}$ is an orthonormal dasis of $N(A)$

hence, $\left\{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}\right\}$ is an orthonormal basis for $\mathbb{R}^3$.