L13

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Final exam:

1. Time and place

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2. Online+offline;
3. Alternative time slots
4. Can bring notes and books; Expect 7-8 problems, so time will be tight if you do not prepare well
5. Materials of exam: cover whole semester, Homework as a good source

Δ Thm (Perron-Frobenius) Suppose $A > 0$

① $\exists$ an eigenvalue $\lambda_0 \in \mathbb{R}_{>0}$ eigenvector $V_0 > 0$

② $\forall$ other eigenvalue λ, we have $|\lambda| < \lambda_0$

③ V_0 is unique up to scalar

Pf: review pf of part ①. $S = \{ s \mid Ax \geq s x$ for some $\underbrace{x \geq 0}_{x \neq 0} \}$

S is nonempty & bounded from above

$\Rightarrow$ get a maximum λ_0 inside S s.t. $A x_0 \geq \lambda_0 x_0$ for some $x_0 \geq 0$

actually we must have $Ax_0 = \lambda_0 x_0$ by λ_0 maximal

Main ingredient: if $Ax_0 \neq \lambda_0 x_0$, some components $(Ax_0)_j > (\lambda_0 x_0)_j$

then $A(Ax_0) > A(\lambda_0 x_0)$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Pf i-th row of $A\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is $\sum_k \overset{>0}{A_{ik}} \cdot a_k$ → is strictly larger

i-th row of $A\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is $\sum_k A_{ik}\, b_k$ than

for new vector $x = Ax_0$ it satisfies $A \cdot x > \lambda_0 \cdot x$

enlarge λ_0 by a small bit contradicting maximality of λ_0

$\Rightarrow A x_0 = \lambda_0 x_0$ $\begin{pmatrix} a_1 \\ \vdots \\ \boxed{a_n} \end{pmatrix}$ $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$\left\{ \begin{array}{l} x_0 > 0: \quad x_0 \neq 0, \quad x_0 = \frac{1}{\lambda} A x_0 > \frac{1}{\lambda} A \cdot 0 = 0 \\ x_0 \geq 0 \quad \text{same ingredient as above} \end{array} \right.$

③ Suppose in general $\underline{v}$ is an eigenvector with eigenvalue λ_0

then up to a scalar $v \geq 0$

from $Av = \lambda_0 v$ we take absolute value

$$|Av| = |\lambda_0||v| = \lambda_0 |v|$$

$$|\underline{A}v| \leq A|v| \qquad |(Av)_i| = |\sum_j A_{ij} \cdot v_j| \leq \sum A_{ij} |v_j|$$

From above $|v| \geq 0$ then we must have equality $\Rightarrow$ all triangle inequality must be equality

$\Rightarrow$ v_j have to be parallel then after a scalar, we can make them all positive

Suppose we have 2 eigenvectors v_1 v_2 for λ_0, by above we can assume $v_1, v_2 > 0$

then we can form new eigenvectors $\underline{v_1 - a v_2}$

grow a from 0 to large

for some a, one of the components of $v_1 - av_2$ becomes 0

then $v_1 - av_2 = 0$ $\Rightarrow$ $v_1 = av_2$

#

△ Corollary: Suppose $A > 0$, A is diagonalizable, λ_0 is maximal eigenvalue

$$\lim_{n\to\infty} \frac{1}{\lambda_0^n} A^n \cdot x = \underline{C x_0}$$ for some C, x_0 is the eigenvector for λ_0

x_1: $0\sim 19$ x_2: $20\sim 39$ x_3: ≥ 40

Example, $A = \begin{pmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & P_3 \end{pmatrix}$ population group growth model

Corollary $\rightsquigarrow$ Population distributes like eigenvector x_0

$\lambda_0 > 1$	increase
$\lambda_0 = 1$	stabilize
$\lambda_0 < 1$	decay

Issue:

Here $A \geq 0$ some entries $= 0$ Can't apply Thm & Cor directly

$A \cdot v_1 = \lambda_0 \binom{a}{b}$ but here $\underline{A^3} > 0$ apply to A^3

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Side remark: Triangle inequality in $n=2$ case:

$v_1 = \binom{a}{b}$ $a, b \in \mathbb{C}$ $\left| \underbrace{A_{11} a + A_{12} b}_{\overset{\lambda_0 a}{=}} \right| \le A_{11}|a| + A_{12}|b| = \lambda_0 |a|$

equality $\hookrightarrow$ forces a b along same direction

$A \binom{|a|}{|b|} \ge \lambda_0 \binom{|a|}{|b|}$ ① $\Rightarrow$ $A\binom{|a|}{|b|} = \lambda_0 \binom{|a|}{|b|}$

Pf of Corollary : Conclusion is independent of conjugation

In particular we assume $A = \begin{pmatrix} \lambda_0 & & \\ & \lambda & \\ & & \ddots \end{pmatrix}$ $x = (\boxed{v_0} \cdots) \begin{pmatrix} c \\ \vdots \\ \vdots \end{pmatrix}$

then $\lim\limits_{n\to\infty} \frac{1}{\lambda_0^n} A^n x = \lim\limits_{n\to\infty} \frac{1}{\lambda_0^n} \begin{pmatrix} \lambda_0 & & \\ & \ddots & \\ & & \ddots \end{pmatrix}^n \begin{pmatrix} c \\ \vdots \\ \vdots \end{pmatrix}$

$= \lim\limits_{n\to+\infty} \begin{pmatrix} 1 & & \\ & \left(\frac{\lambda}{\lambda_0}\right)^n & \\ & & \ddots \end{pmatrix} \begin{pmatrix} c \\ \vdots \\ \vdots \end{pmatrix}$

part ② ③ of Perron-Frob $\underbrace{\left(\frac{\lambda}{\lambda_0}\right)}_{<1}{}^n \to 0$

$= \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix} \begin{pmatrix} \boxed{c} \\ \vdots \\ \vdots \end{pmatrix} = c$ #

Critical typo, should be $\begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Another remark: If you don't like how I wrote $x = (e_1 \cdots e_n) \begin{pmatrix} c \\ \vdots \end{pmatrix}$, just consider A already diagonal under a proper basis. Using computation above, we have $\lim \frac{1}{\lambda_0^n} A^n x = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Also by direct computing eigenvector for λ_0, we see $x_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, then $\lim \frac{1}{\lambda_0^n} A^n x = c \cdot x_0$

Δ Fourier transform and fast Fourier transform

Δ Consider the collection of continuous functions $f: X = [0,1] \to \mathbb{C}$ st

$\triangle$ Consider the collection of $\overset{\text{continuous}}{\vee}$ functions $f: X = [0,1] \to \mathbb{C}$ s.t.

$f(0) = f(1)$ (periodic)

(FT) $n \in \mathbb{Z}$ $\quad a_n = \hat{f}(n) = \int_0^1 f(x)\, e^{2\pi i n x}\, dx$

(FI = Fourier inversion) $\quad f(x) = \sum_{n \in \mathbb{Z}} a_n\, e^{-2\pi i n x}$

Rmks: ① This collection of f $\quad C_p(X) \leftarrow \infty$-dim vector space

② $e^{2\pi i n x}$ is indeed periodic. $\quad e^{2\pi i n \cdot 1} = 1 = e^{2\pi i n \cdot 0}$

③ $\displaystyle \int_0^1 e^{2\pi i n x}\, dx = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

e.g. $n = 0$ $\quad \int_0^1 1\, dx = 1$ $\qquad n \neq 0$ $\quad \int e^{2\pi i n x} = \left.\frac{e^{2\pi i n x}}{2\pi i n}\right|_0^1 = 0$

$e^{i\theta} = \cos\theta + i \sin\theta$

Substitute $f(x) = \sum_n a_n\, e^{-2\pi i n x}$ $\quad$ in $a_m \overset{?}{=} \int_0^1 f(x)\, e^{2\pi i m x}\, dx$

$$\text{RHS} = \int_0^1 \sum_n a_n\, e^{-2\pi i n x} \cdot e^{2\pi i m x}\, dx$$

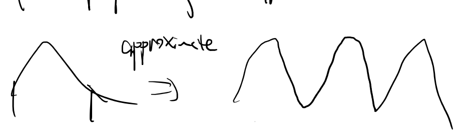
$$= \sum_n a_n \int_0^1 e^{2\pi i (m-n) x}\, dx \quad \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

$$= a_m$$

$\triangle$ Motivation / importance of FT

(1) represents general f by 'easier' function $e^{2\pi i n x}$

(2) Allows greater generalizations $\{$ for proper f on $\mathbb{R}$



(3) $\hat{f}$ has special properties (related to ODE)

(4) Appears naturally in physics

$$f(t) = \sum a_n\, e^{-2\pi i n t}$$

$$\underline{f(t)} = \sum \underset{\text{I}}{a_n} \, e^{\cdots}$$

Sound $= \sum$ strength $\cdot$ note

light $= \sum$ strength $\cdot$ pure color

- In practice we just record $f\left(\frac{j}{N}\right)$ N very large, $0 \le j < N$

  $[0,1]$

(DFT) (discrete)

$$a_n = \hat{f}(n) = \sum f\left(\frac{j}{N}\right) e^{2\pi i \frac{nj}{N}} \qquad x \to \frac{j}{N}$$

$$= \left(e^{2\pi i \frac{n}{N} 0}, e^{2\pi i \frac{n}{N} 1}, \ldots, e^{2\pi i \frac{n(N-1)}{N}}\right) \begin{pmatrix} f(0) \\ f\left(\frac{1}{N}\right) \\ \vdots \\ f\left(\frac{N-1}{N}\right) \end{pmatrix}$$

$$\left(e^0, e^{\frac{2\pi i n}{N}}, e^{\frac{2\pi i n \cdot 2}{N}}, \ldots, e^{\frac{2\pi i n}{N}(N-1)}\right)$$

a_n for $0 \le n \le N-1$ $w = e^{\frac{2\pi i}{N}}$

$$(F)_{ij} = \boxed{w^{(i-1)(j-1)}}$$

Then claim: $\begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \end{pmatrix} = F \begin{pmatrix} f(0) \\ \vdots \\ f\left(\frac{N-1}{N}\right) \end{pmatrix}$ $(*)$

indeed a_n is $n+1$-th row of LHS.

Thus for right side we take $i = n+1$ $(F)_{i=n+1,j} = \left(w^{n \cdot 0}, w^{n \cdot 1}, w^{n \cdot 2}, \ldots, w^{n(N-1)}\right)$

(DFI) $\begin{pmatrix} f(0) \\ \vdots \\ f\left(\frac{N-1}{N}\right) \end{pmatrix} = F^{-1} \begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \end{pmatrix}$

$$\left(\tfrac{1}{\sqrt{N}} F\right)^* \tfrac{1}{\sqrt{N}} F = I \Rightarrow \tfrac{1}{N} F^* \cdot F = I$$

- Lemma. $\frac{1}{\sqrt{N}} F$ is unitary, so in particular F is invertible $F^{-1} = \frac{1}{N} F^*$