Linear Algebra – Homework 7

4 Nov 2020
Due: 12 Nov 2020

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 3.1.27. True or false (with a counterexample if false):

(a) The vectors $\mathbf{b}$ that are not in the column space $\mathbf{C}(A)$ form a subspace.

(b) If $\mathbf{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A (test this).

Problem 3.1.28. Construct a 3×3 matrix whose column space includes $(1, 1, 0)$ and $(1, 0, 1)$, but not $(1, 1, 1)$. Construct a 3×3 whose column space is only a line.

Problem 3.2.4 For the matrices A and B, find the special solutions to $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$. For any $m \times n$ matrix, the number of pivot variables plus the number of free variables is ________.

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$$

Problem 3.2.12. The equation $x - 3y - z = 0$ determines a plane in $\mathbf{R}^3$. What is the matrix A in this equation? Which variables are free? What are the two special solutions?

Problem 3.2.20. Construct a 2×2 matrix whose nullspace equals its column space. This is possible.

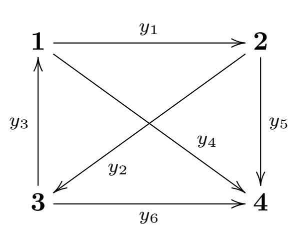
Problem 3.2.31. Find the reduced row echelon forms R and the rank of these matrices:

(a) The 3×4 matrix with all entries equal to 4.

(b) The 3×4 matrix with $a_{ij} = i + j - 1$.

(c) The 3×4 matrix with $a_{ij} = (-1)^j$.

Problem 3.2.32. Kirchhoff's Current Law $A^T\mathbf{y} = \mathbf{0}$ says that *current in* = *current out* at every node. At node 1, this is $y_3 = y_1 + y_4$. Write the four equations for Kirchhoff's Law at the four nodes (arrows show the positive direction of each y). Reduce A^T to R and find three special solutions in the nullspace of A^T (4×6 matrix).



$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Problem 3.2.41. Choose vectors $\mathbf{u}$ and $\mathbf{v}$ so that $A = \mathbf{u}\mathbf{v}^T$ = column times row:

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix}.$$

Problem 3.2.43. If A has rank r, then it has an $r \times r$ submatrix S that is invertible. Remove $m - r$ rows and $n - r$ columns to find an invertible submatrix S inside A, B, and C. You could keep the pivot rows and pivot columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3.3.4. Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Problem 3.3.6. What conditions on b_1, b_2, b_3, b_4 make each system solvable? Find $\mathbf{x}$ in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Problem 3.3.7. Show by elimination that (b_1, b_2, b_3) is in the column space if $b_3 - 2b_2 + 4b_1 = 0$.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}.$$

What combination of the rows of A gives the zero row?

Problem 3.3.13. Explain why these are all false:

(a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$.

(b) A system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution.

(c) The solution $\mathbf{x}_p$ with all free variables zero is the shortest solution (minimum length $\|\mathbf{x}\|$). Find a 2×2 counterexample.

(d) If A is invertible there is no solution $\mathbf{x}_n$ in the nullspace.

Graded Problems.

Problem 1. Find the reduced row echelon form R and a spanning set (the special solution(s)) for the null space $\mathbf{N}(A)$:

$$A = \begin{bmatrix} -3 & -1 & 0 & 1 \\ -4 & -1 & 1 & 1 \\ -1 & 1 & 4 & -1 \end{bmatrix}$$

Problem 2. Find a condition on b_1, b_2, b_3 that guarantees $\mathbf{b} = (b_1, b_2, b_3)$ is in the column space $\mathbf{C}(A)$ of the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Then find the *complete* solution to the system of linear equations

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$