△ Review

Given a Taylor expansion

$$f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)\cdot x^2}{2!} + \cdots + \quad \text{Convergent } |x|<R$$

$$\left(= a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots \right)$$

Defined $f(A) = f(0) + f'(0)\cdot A + \frac{f''(0)}{2!} A^2 + \cdots$ △ which is convergent when eigenvalues λ_i for A satisfy $|\lambda_i| < R$

△ Lem:
$$f(J_{\lambda,n}) = \begin{pmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \cdots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \\ & & & & f'(\lambda) \\ 0 & & & & f(\lambda) \end{pmatrix}$$

Pf: Taylor expansion at $x=\lambda$ gives

$$f(x) = f(\lambda) + f'(\lambda)\underset{\underset{J_{\lambda,n}-\lambda I = N}{\downarrow}}{(x-\lambda)} + \frac{f''(\lambda)}{2!}(x-\lambda)^2 + \cdots \frac{f^{(n)}(\lambda)}{n!}(x-\lambda)^n + \cdots$$

Substitute $x = J_{\lambda,n}$. Constant $C \to CI$

$$f(J_{\lambda,n}) = f(\lambda) I + f'(\lambda)\cdot N + \frac{f''(\lambda)}{2!} N^2 + \cdots \frac{f^{(n)}(\lambda)}{n!} N^n + \cdots$$ #

△ $f(x) = e^{\lambda x}$ $\quad \frac{d}{dx} f(x) = \lambda \cdot e^{\lambda x}$

generalize ↪

given A, $f(t) := e^{At}$ is $n\times n$ matrix with each entry functions of t

$\frac{d}{dt} f(t) :=$ taking derivative wrt t for each entry of e^{At}

Lemma Der: $\frac{d}{dt} f(t) := \frac{d}{dt}(e^{At}) = A\cdot e^{At}$

Pf: Step 1: Reduce to Jordan normal form

$$h^{-1}\left(\frac{d}{dt}f(t)\right)h = \frac{d}{dt}\left(\underline{h^{-1}f(t)h}\right)$$

$$\frac{d}{dt}\left(af_1(t)+bf_2(t)\right) = a\frac{d}{dt}f_1(t) + b\frac{d}{dt}f_2(t)$$

Step 2: need to prove for $J_{\lambda,n}$, $\frac{d}{dt}\left(e^{J_{\lambda,n}t}\right) = J_{\lambda,n}e^{J_{\lambda,n}t}$

$$e^{J_{\lambda,n}t} = e^{\begin{pmatrix} \lambda t & t & & 0 \\ & \ddots & \ddots & \\ & & & t \\ 0 & & & \lambda t \end{pmatrix}} = \lambda t I + tN$$

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$$e^x = f(x) = f(\lambda_0) + f'(\lambda_0)(x-\lambda_0) + \cdots \frac{f^{(n)}(\lambda_0)}{n!}(x-\lambda_0)^n + \cdots$$

$x \to J_{\lambda,n}t$ $\qquad f(x) = e^x$

$\lambda_0 \to \lambda t$ $\qquad f^{(n)}(x) = e^x$

$$e^{J_{\lambda,n}t} = e^{\lambda t} + e^{\lambda t}\cdot(\overset{J_{\lambda,n}t-\lambda tI}{tN}) + \cdots \frac{e^{\lambda t}}{(n-1)!}(tN)^{n-1} + 0 \cdots$$

$$LHS = \frac{d}{dt}\left(e^{J_{\lambda,n}t}\right) = \lambda e^{\lambda t} + \boxed{\begin{matrix}\lambda e^{\lambda t}\cdot(tN) \\ + \\ e^{\lambda t}\cdot N\end{matrix}} + \cdots + \begin{matrix}\lambda\frac{e^{\lambda t}}{(n-1)!}(tN)^{n-1} \\ + \\ (n-1)\frac{e^{\lambda t}}{(n-1)!}(tN)^{n-2}\cdot N\end{matrix} + \cdots$$

$$RHS = J_{\lambda,n}\cdot e^{J_{\lambda,n}t} = (\lambda I + N)e^{J_{\lambda,n}t} = \lambda I e^{\lambda t} + \boxed{\begin{matrix}\lambda I\cdot e^{\lambda t}(tN) \\ N\cdot e^{\lambda t}\end{matrix}} + \cdots \begin{matrix}\lambda I\frac{e^{\lambda t}}{(n-1)!}(tN)^{n-1} + \\ N\cdot e^{\lambda t}(tN) + \cdots\end{matrix}$$

#

$\triangle$. System of ODE (ordinary differential equations)

$f_1(x) \cdots f_n(x)$ $\quad$ n-function on x $\qquad a_{ij} \in \mathbb{C}$

$f_1(x) \cdots f_n(x)$ n-function in x $a_{ij} \in \mathbb{C}$

$$(*) \begin{cases} \frac{d}{dx} f_1 = a_{11} f_1 + a_{12} f_2 + \cdots a_{1n} f_n \\ \frac{d}{dx} f_2 = a_{21} f_1 + \qquad\qquad + a_{2n} f_n \\ \vdots \\ \frac{d}{dx} f_n = a_{n1} f_1 + \cdots \quad \cdots a_{nn} f_n \end{cases}$$

$$\vec{f}(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} \qquad (*) \Rightarrow (*) \ \underline{\frac{d}{dx} \vec{f}(x) = A \cdot \vec{f}(x)} \qquad A = (a_{ij})_{ij}$$

Cor: (of Lem Der) $\forall \ \vec{c} = \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix}$, $\vec{f}(x) = \underline{e^{A \cdot x}} \cdot \underline{\vec{c}}$ is a solution to $(*)$

Rmk: any solution to $(*)$ is a linear combination of this shape

Pf: Indeed, $\frac{d}{dx}\left(e^{Ax} \cdot \vec{c}\right) \overset{\frac{d}{dx} \text{ is linear operation}}{=} \underline{\frac{d}{dx}\left(e^{Ax}\right)} \cdot \vec{c}$

$$\overset{\text{Lem Der}}{=} A \cdot e^{Ax} \cdot \vec{c}$$

$$\frac{d}{dx} \vec{f}(x) = A \cdot \vec{f}(x) \qquad \#$$

Δ System of ODEs with initial condition

$$(**) \begin{cases} (*) \ \frac{d}{dx} \vec{f}(x) = A \cdot \vec{f}(x) \\ \vec{f}(0) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \leftarrow \text{initial condition} \end{cases}$$

Cor: $\vec{f}(x) = e^{Ax} \cdot \begin{pmatrix} a_1 \\ \vdots \end{pmatrix}$ is the solution to $(**)$

Cor: $\vec{f}(x) = \underline{e^{Ax} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}$ is the solution to $(**)$

Pf: Just check $\vec{f}(0) \overset{?}{=} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ ✓ but $e^{A \cdot 0} = e^0 = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = I$ #

Δ Eco-System $f_1(t)$ = population of rabbits at time t

$f_2(t)$ = — wolves —

$$(**) \begin{cases} f_1(t) = 10^6 \quad f_2(t) = N \\ (*) \begin{cases} \frac{d}{dt} f_1(t) = 4 \cdot f_1(t) - 100 f_2(t) \\ \frac{d}{dt} f_2(t) = f_2(t) \end{cases} \end{cases}$$

Ask: what should be N if $f_1(t) \to 0$?

Solution: $(*) \Leftrightarrow \frac{d}{dt} \vec{f}(t) = \underbrace{\begin{pmatrix} 4 & -100 \\ 0 & 1 \end{pmatrix}}_{A} \vec{f}(t)$

Cor $\Rightarrow$ Need to Compute $\underline{e^{At}} \cdot \begin{pmatrix} 10^6 \\ N \end{pmatrix}$

If $h^{-1} A h$ is Jordan then $e^{At} = h (e^{h^{-1} A t h}) h^{-1}$

Step 1 find h, s.t. $h^{-1} A h$ is Jordan $A = \begin{pmatrix} 4 & -100 \\ 0 & 1 \end{pmatrix}$

$\lambda_1 = 4 \quad \lambda_2 = 1$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 100 \\ 3 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 100 \\ 0 & 3 \end{pmatrix}$

Compute $h^{-1} = \frac{1}{\det h} \begin{pmatrix} 3 & -100 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{100}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$

Compute $h^{-1} = \frac{1}{\det h}\begin{pmatrix}3 & -100\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -\frac{100}{3}\\ 0 & \frac{1}{3}\end{pmatrix}$

$h^{-1}Ah = \begin{pmatrix}4 & \\ & 1\end{pmatrix}$

Step 2: $e^{h^{-1}Aht} = e^{\begin{pmatrix}4t & \\ & t\end{pmatrix}} = \begin{pmatrix}e^{4t} & \\ & e^{t}\end{pmatrix}$

Step 3: $e^{At} = h\cdot\left(e^{h^{-1}Aht}\right)h^{-1} = h\begin{pmatrix}e^{4t} & \\ & e^{t}\end{pmatrix}h^{-1}$

$$= \begin{pmatrix}1 & 100\\ & 3\end{pmatrix}\begin{pmatrix}e^{4t} & \\ & e^{t}\end{pmatrix}\begin{pmatrix}1 & -\frac{100}{3}\\ & \frac{1}{3}\end{pmatrix}$$

$$= \begin{pmatrix}e^{4t} & 100e^{t}\\ & 3e^{t}\end{pmatrix}\begin{pmatrix}1 & -\frac{100}{3}\\ & \frac{1}{3}\end{pmatrix}$$

$$= \begin{pmatrix}e^{4t} & \frac{100}{3}(e^{t}-e^{4t})\\ & e^{t}\end{pmatrix}$$

Solution to (**) is $e^{At}\cdot\begin{pmatrix}10^6\\ N\end{pmatrix} = \begin{pmatrix}e^{4t} & \frac{100}{3}(e^{t}-e^{4t})\\ & e^{t}\end{pmatrix}\begin{pmatrix}10^6\\ N\end{pmatrix}$

$$\begin{pmatrix}f_1(t)\\ f_2(t)\end{pmatrix} = \begin{pmatrix}\underline{10^6e^{4t} + N\frac{100}{3}(e^{t}-e^{4t})}\\ Ne^{t}\end{pmatrix}$$

$$f_1(t) = 10^6\underbrace{e^{4t}}_{X^4} + \underbrace{N\frac{100}{3}\underbrace{e^{t}}_{X}}_{\text{'error term'}} - N\frac{100}{3}\underbrace{e^{4t}}_{X^4}$$

$e^{t} = X \to +\infty$

$$= \underline{\left(10^6 - N\frac{100}{3}\right)}\,e^{4t}$$

To control $f_1(t)$ need $10^6 - N\frac{100}{3} < 0$

$$N > \frac{3\cdot 10^6}{100}$$

#

Δ. Recall: A, B say A commutes with B if $\underline{AB = BA}$

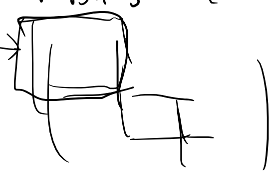
$$\Rightarrow \quad e^{A+B} = e^A \cdot e^B$$

When $AB = BA$ happens?

Δ Lem: If $A\cdot B = B\cdot A$, and V_λ is a $\underline{\text{generalized}}$ eigenspace for A

then V_λ is also an invariant subspace for B

$\Rightarrow$ If $\exists h$, s.t. $h^{-1}Ah = \begin{pmatrix} J_{\lambda_1,n_1} & & \\ & \ddots & \\ & & J_{\lambda_k,n_k} \end{pmatrix}$ then $h^{-1}Bh$ is blockwise diagonal



Pf: Let T_A / T_B be linear operators associated to A / B

$\forall\, v \in V_\lambda$ for T_A means $\quad \underline{(T_A - \lambda I)^{n_1} v = 0} \quad (\Delta)$

to prove Lemma, it suffices to show $T_B \cdot v \in V_\lambda$

it suffices to show $\quad (T_A - \lambda I)^{n_1} \cdot T_B v = 0 \quad (\#)$

Condition $AB = BA \iff T_A \cdot T_B = T_B \cdot T_A$

$$(T_A - \lambda I)^{n_1} T_B = T_B (T_A - \lambda I)^{n_1}$$

$$\text{LHS of } (\Delta) = T_B \boxed{(T_A - \lambda I)^{n_1} \cdot v} \overset{(\Delta)}{=} 0$$

#

Δ: Cor: Suppose A has n-distinct eigenvalues, then $AB = BA$ iff A, B can be diagnonalized simultaneously $\left(\exists h, \begin{matrix} h^{-1}Ah \\ h^{-1}Bh \end{matrix} \right\} \text{diagonal} \Big)$

Pf: '$AB = BA$ $\overset{\text{Lem above}}{\Longrightarrow}$ V_λ for A is also invariant for B

$\overset{\text{n-distinct eigenvalue}}{\Longrightarrow}$ $\dim V_\lambda = 1$ each V_λ is also eigenvector space for B

$\Rightarrow$ can pick common eigenvector

Choosing these common eigenvectors as basis, A, B both become diagonal

If $\exists h$ s.t. $h^{-1}Ah$, $h^{-1}Bh$ are diagonal

then commutes $\Rightarrow$ A, B commutes

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Δ: Definition: Lie bracket for A, B is

$$[A, B] = AB - BA$$

"it measures how commutative are A & B" $[A, B] = 0$ iff $AB = BA$

properties:

not commutative

① $[A, B] = -[B, A]$ $\quad$ ($\overset{\shortparallel}{BA - AB}$)

② not associative $(AB)C = A(BC)$

Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

identity $\quad [A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$

$$[A,[B,C]] = [A, BC-CB] = A(BC-CB)-(BC-CB)A$$

$A \to B \to C \to A$

$$[B,[C,A]] = B(CA-AC)-(CA-AC)B$$

$$[C,[A,B]] = C(AB-BA)-(AB-BA)C$$