△ Definition $M_{m\times n} = \{ m\times n \text{ matrices} \}$

$M_{m\times n}(\mathbb{C}) = \{ g \ m\times n \text{ matrices with entries} \in \mathbb{C} \}$

$M_n := M_{n\times n}$

$GL_n = \{ g \in M_{n\times n},\ \det(g) \neq 0 \}$

$= \{ g \in M_{n\times n},\ g^{-1} \text{ exists} \}$ $\quad g^{-1}\cdot g = g\cdot g^{-1} = I$

△ $T : V \to V$ a linear operator $\dim V = n$

$T(v_1+v_2) = Tv_1 + Tv_2 \qquad T(cv) = cTv$

Given a basis $B_V = \{e_1 \cdots e_n\}$

It suffices to describe T by describing Te_i

$Te_i = e_1 \cdot g_{1i} + e_2 g_{2i} + \cdots e_n g_{ni} = (e_1 \cdots e_n)\begin{pmatrix} g_{1i} \\ g_{2i} \\ \vdots \\ g_{ni} \end{pmatrix}$

$T(e_1 \cdots e_n) = (e_1 \cdots e_n)\begin{pmatrix} & g_{ij} & \end{pmatrix}$

Definition $g = (g_{ij})$ is the matrix associated to T and basis $\{e_i\}$

Denote $g = g_T^{\{e_i\}}$

△ Changing basis: Pick a new basis $B_V' = \{e_i'\}$

$\exists\ g_T'$ s.t. $T(e_1' \cdots e_n') = (e_1' \cdots e_n')\, g_T'$

Question: How is g_T & g_T' related?

because $\{e_i\}$ is basis $\quad e_j' = \sum_i e_i h_{ij} = (e_1 \cdots e_n)\begin{pmatrix} h_{1j} \\ \vdots \\ h_{nj} \end{pmatrix}$

$$(\text{☆})\quad (e_1' \cdots e_n') = (e_1 \cdots e_n)\, h \qquad h = (h_{ij})$$

$\{e_i'\}$ is basis $\Rightarrow$ $\exists\, h'$ s.t.

$$(\triangle)\quad (e_1 \cdots e_n) = (e_1' \cdots e_n') \cdot h'$$

Substitute $(\triangle)$ into (☆) $\quad (e_1' \cdots e_n') = (e_1' \cdots e_n')\, h' \cdot h$

$$\Leftrightarrow \quad h' \cdot h = I. \qquad \Rightarrow \begin{cases} h \in Gl_n \\ h' = h^{-1} \end{cases}$$

(1) $\quad T(e_1' \cdots e_n') = (e_1' \cdots e_n')\, g_T'$

(2) $\quad T(e_1 \cdots e_n) = (e_1 \cdots e_n)\, g_T$

Sub (☆) & $(\triangle)$ into (1) $\quad \overset{\text{LHS}+(\text{☆})}{T(e_1 \cdots e_n) \cdot h} = \overset{\text{RHS}+(\text{☆})}{(e_1 \cdots e_n)\, h\, g_T'}$

multiply the equality on right-hand side by h^{-1}

$$(T(e_1 \cdots e_n))(h \cdot h^{-1}) = (e_1 \cdots e_n)\, h\, g_T'\, h^{-1}$$

$$T(e_1 \cdots e_n) = (e_1 \cdots e_n)\, h\, g_T'\, h^{-1}$$

Compare with (2) we get $\quad (e_1 \cdots e_n)\, g_T = (e_1 \cdots e_n)\, h\, g_T'\, h^{-1}$

$$\Leftrightarrow \quad g_T = h\, g_T'\, h^{-1}$$

$$\Leftrightarrow \quad h^{-1} g_T h = g_T'$$

$\triangle$ Definition: two matrices g' & g are said to be conjugate to each other if $\exists\, h \in Gl_n$ s.t. $g' = h^{-1} g h$.

We also say g' & g are similar matrices. $\quad g'$ is similar to g

△ Recall : $T: V \to V$ defined an invariant subspace W $(TW \subset W)$

Lem : starting with T g_T is conjugate to blockwise diagonal $\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$ iff $V = W_1 \oplus W_2$

s.t. W_i are invariant subspaces

$T|_{W_i}$ is given A_i

△ Recall $W \subset V$ subspace,

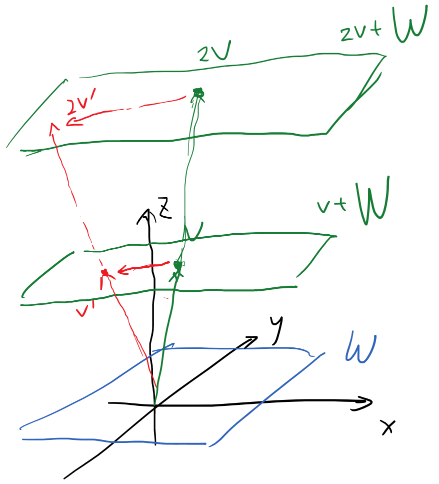
Defined a quotient space is $U = V/W = \{ \bar{v} = v + W \}$

with $\begin{cases} \bar{v}_1 + \bar{v}_2 = v_1 + v_2 + W =: \overline{v_1+v_2} \\ c\bar{v} = cv + W =: \overline{cv} \end{cases}$

Example : $V = \mathbb{R}^3$ space, $W = xy$-plane

$\bar{v} = v + W$ is a vector in $U = V/W$

$2\bar{v} = 2v + W$



△ Define a linear map, called quotient map $\tau : V \to U$

$v \mapsto \bar{v} = v + W$

Check $\begin{cases} \tau \text{ is actually a linear operator} & \tau(v_1+v_2) \overset{\checkmark}{=} \tau v_1 + \tau v_2 \\ & \quad \| \qquad\qquad\qquad \| \\ & v_1+v_2+W \qquad v_1+W + v_2+W \\ & \qquad\qquad\qquad\qquad \| \\ & \qquad\qquad\qquad v_1+v_2+W \\ \tau(cv) = c\tau(v) \end{cases}$

Recall $\ker(\tau) = \{ v \in V, \ \tau(v) = \bar{0} \} = \{ v \in V, \ v + W = 0 + W \}$

$= W$ $\quad v + w_1 = w_2$, $v = w_2 - w_1 \in W$

$$\mathrm{Im}(\tau) = \{u \in U, \text{ s.t. } \exists v \in V, \tau(v) = u\}$$

$$= U \quad (\cancel{\tau \text{ is surjective}})$$

△ Lemma: $\dim V = \dim W + \dim(U/W)$

Pf: $\mathrm{RHS} = \dim(\ker(\tau)) + \dim(\mathrm{Im}(\tau))$

In general $\dim V = \dim(\ker T) + \dim(\mathrm{Im}\, T)$ #

△ Lem: τ map gives 1-1 correspondence

$$\left\{ \begin{matrix} \text{vector subspaces } X \\ W \subset X \subset V \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{vector subspaces } Y \\ \text{of } U \end{matrix} \right\}$$

Sketch of proof: "$\longrightarrow$" $X \mapsto \tau(X)$

"$\longleftarrow$" $\tau^{-1}(Y) = \{v \in V, \text{ s.t. } \tau(v) \in Y\} \longleftarrow Y$

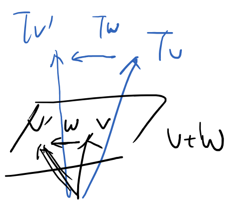
Check: $\tau(X)$, $\tau^{-1}(Y)$ are vector spaces "$\longrightarrow$" "$\longleftarrow$" are inverse to each other #

△ $W \subset V$ invariant subspace, $TW \subset W$

Definition T gives rise to a linear operator $\bar{T}$ on U

$$\bar{T}: U \to U$$

$$\bar{v} = v + W \mapsto \bar{T}\bar{v} = Tv + W$$



Question: Is it well defined? $v+W = v'+W$ if $v-v' \in W$

$$Tv + W \overset{?}{=} Tv' + W$$

This equality holds because $v - v' = w$ for $w \in W$

$$RHS = T(v-w) + W = Tv - \underline{Tw} + W$$

By condition W is invariant $\Rightarrow Tw \in W$ $\quad \overset{\parallel}{Tv + W}$

Δ Lemma: $T: V \to V$, g_T is matrix associated to T with basis $\{e_i\}$

g_T is conjugate to blockwise upper triangular matrix $\begin{array}{c c} & \overbrace{\hphantom{A_1}}^{n_1}\ \overbrace{\hphantom{A_2}}^{n_2} \\ \begin{matrix} n_1 \\ n_2 \end{matrix} & \left(\begin{array}{c|c} A_1 & N \\ 0 & A_2 \end{array}\right) \end{array}$ $\quad n = n_1 + n_2$

$\Leftrightarrow$ $\exists\, W$ invariant subspace, s.t. $T|_W$ is associated to A_1

$\bar{T}|_{V/W}$ is associated to A_2

Pf: "$\Rightarrow$" If g_T conjugate to $\begin{pmatrix} A_1 & N \\ 0 & A_2 \end{pmatrix}$ then $\exists \{e_1 \cdots\cdots e_n\}$

s.t. $T(e_1 \cdots e_n) = (e_1 - e_n) \left(\begin{array}{c|c} A_1 & N \\ 0 & A_2 \end{array}\right)$

Define $W = \text{Span}\{e_1 \cdots e_{n_1}\}$ $\quad T(\underline{e_1 \cdots e_{n_1}}) = (e_1\, e_{n_1}\, \boxed{e_{n_1+1}\, e_n}) \begin{pmatrix} A_1 \\ \boxed{0} \end{pmatrix}$

$$= (\underline{e_1 \cdots e_{n_1}})\, A_1$$

Indeed W is invariant with T action given by A_1

Now for Quotient space $U = V/W$ pick a basis $\{\bar{e}_{n_1+1}, \cdots \overset{e_n + W}{\bar{e}_n}\}$

$T(e_{n_1+1} \cdots e_n) = (e_1 \cdots e_{n_1}\, e_{n_1+1} \cdots e_n) \begin{pmatrix} N \\ \end{pmatrix}$

$$T(e_{n_1+1} \cdots e_n) = (\cancel{e_1} \cdots e_{n_1}\ e_{n_1+1} \cdots e_n)\begin{pmatrix} N \\ A_2 \end{pmatrix}$$

$$\bar{T}(\bar{e}_{n_1+1} \cdots \bar{e}_n) = (\cancel{\bar{e}_1} \cdots \bar{e}_{n_1}\ \bar{e}_{n_1+1} \cdots \bar{e}_n)\begin{pmatrix} N \\ A_2 \end{pmatrix}$$

$e_1 \cdots e_{n_1} \in W \qquad \bar{e}_1 = \cdots \bar{e}_{n_1} = \bar{0} \in U$

$$\bar{T}(\bar{e}_{n_1+1} \cdots \bar{e}_n) = (\bar{e}_{n_1+1} \cdots \bar{e}_n)(A_2)$$

$\bar{T}$ on U is associated to A_2

"⇐" Given W invariant $\quad T|_W$ given by A_1

$\bar{T}$ on U given A_2

We need to find a basis for V s.t. g_T looks like $\begin{pmatrix} A_1 & N \\ 0 & A_2 \end{pmatrix}$

How? Step 1. find a basis $B_W = \{f_1 \cdots f_{n_1}\}$ for W st. $T \sim A_1$

Step 2 find a basis $B_U\{\bar{f}_{n_1+1} \cdots \bar{f}_n\}$ for U $\bar{T} \sim A_2$

$\forall$ any $\bar{f}_j \in B_U$ find any f_j s.t. $\pi(f_j) = \bar{f}_j$

Claim: $B_V = \{f_1 \cdots f_{n_1}\ f_{n_1+1} \cdots f_n\}$ is a basis for V

Check L.I. for B_V using L.I for B_U, B_W

△ Definition a filtration of vector subspaces is collection of V_i $i = 0 \cdots k$

s.t. $0 = V_0 \subset V_1 \subset V_2 \cdots \subset V_k = V$

Given T, a filtration of invariant vector spaces is as above but all V_i are invariant

△ Lemma: Given T. g_T as before, g_T is conjugate to $\begin{pmatrix} A_1 & * & * \\ 0 & A_2 & \\ 0 & 0 & \ddots \\ & & & A_k \end{pmatrix}$

$\Leftrightarrow$ $\exists$ a filtration of invariant subspaces $0 = V_0 \subset \cdots \subset V_k = V$ s.t.

$$\bar{T}\big|_{V_i/V_{i-1}} \text{ is associated to } A_i$$

Example $k=2$. We need for RHS $0 = V_0 \subset V_1 \subset V_2 = V$

$$\bar{T}\big|_{V/W} = \bar{T}\big|_{V_2/V_1} \text{ is given by } A_2 \qquad (V_1 = W)$$

$$T\big|_W = \bar{T}\big|_{V_1/V_0} \text{ is given by } A_1$$

Example of filtration of invariant subspaces.

Suppose $\exists$ n eigenvectors v_i for T.

then $\underline{\text{span}\{v_i\}}$ is invariant subspace for T $\quad (Tv_i = \lambda_i v_i)$

Construct $V_0 = 0$ $\quad V_1 = \text{span}\{v_1\}$ $\quad V_2 = \text{span}\{v_1, v_2\}$ $\cdots$

$V_k = \text{span}\{v_1 \cdots v_n\} = V$

Another way: $V = \underline{\bigoplus \text{span}\{v_i\}}$ $\Leftrightarrow$ g is conjugate to $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$