§2.3 Exercise 52

we know that $\forall \varepsilon>0, \exists \delta>0$ s.t $\forall\ 0<|x-c|<\delta$, we have $|f(x)-L|<\varepsilon$

and let $x=h+c$, we have $0<|x-c|<\delta \Leftrightarrow 0<|h-0|<\delta$

so if $\lim_{x\to c} f(x)=L \Leftrightarrow \forall \varepsilon>0, \exists \delta>0$ s.t. $\forall\ 0<|x-c|<\delta$, we have $|f(x)-L|<\varepsilon \Leftrightarrow$

$\forall \varepsilon>0, \exists \delta>0$ s.t. $\forall\ 0<|h-0|<\delta$, we have $|f(h+c)-L|<\varepsilon \Leftrightarrow \lim_{h\to 0} f(h+c)=L$

§2.3 Exercise 54.

Let $f(x)=x$, if we choose $L=1$ and $x_0=0$, $\forall \varepsilon>0$, there exists $|f(1)-L|=0<\varepsilon$

but actually $\lim_{x\to x_0} f(x)=0$, not L.

because we want to find a area $(x_0-\delta, x_0+\delta)$ that in which every $f(x)$ is close enough L. but in this method, we just find "one" value that close enough L. so we can find $f(1)=1$ in this example, but $f(x_0)$ is not "close enough." so it will not become the limit.

§2.4 Exercise 5.

(a) No, $\lim_{x\to 0^+} f(x)$ doesn't exist because:

if $\varepsilon=\frac{1}{2}$, we can't find any δ and L that satisfied $\forall\ 0<x-0<\delta \Rightarrow |f(x)-L|<\varepsilon$

(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$

(C) No, because $\lim_{x\to 0^+}$ doesn't exist

§2.4 Exercise 69

At most 1 horizontal asymptote can the graph have.

because if $\lim_{x\to +\infty} f(x)=L$, we will know that $f(x)=\dfrac{L+\frac{a_2}{x}+\frac{a_3}{x^2}+\dots\frac{a_{n+1}}{x^n}}{\frac{b_2}{x}+\frac{b_3}{x^2}+\dots\frac{b_{n+1}}{x^n}}$

so, if $x\to-\infty$, $f(x)$ also have a limit L,

we know that there a most 1 horizontal asymptotes can the graph have.

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A1. (a) let $\{x\} = x - \lfloor x \rfloor$, we know that $\{x\} < 1$

so $0 \leq x^2 \cdot \{\frac{1}{x}\} \leq x^2 \cdot \frac{1}{x}$, and $\lim\limits_{x \to 0} x^2 \cdot \frac{1}{x} = 0$, so $\lim\limits_{x \to 0} x^2 \cdot \{\frac{1}{x}\} = 0$

and $\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} x^2 \cdot \lfloor \frac{1}{x} \rfloor = \lim\limits_{x \to 0} x^2 \cdot \frac{1}{x} - x^2 \cdot \{\frac{1}{x}\} = \lim\limits_{x \to 0} x - x^2 \{\frac{1}{x}\} = 0$

(b) $-1 \leq \cos\frac{1}{x} \leq 1 \Rightarrow -|x| \leq x\cos\frac{1}{x} \leq |x|$

cause $\lim\limits_{x \to 0} -|x| = \lim\limits_{x \to 0} |x| = 0$

so $\lim\limits_{x \to 0} x\cos\frac{1}{x} = 0$

A2. if $\lim\limits_{x \to +\infty} f(x) = L$, we know that $\forall \varepsilon > 0, \exists M$, s.t $\forall x > M, \Rightarrow |f(x) - L| < \varepsilon$

because $f(x)$ is even function, we know that $\forall x < -M \Rightarrow |f(x) - L| < \varepsilon$

so we can choose $M' = -M \Rightarrow \forall \varepsilon > 0, \exists M'$. s.t. $\forall x < M' \Rightarrow |f(x) - L| < \varepsilon$

then $\lim\limits_{x \to -\infty} f(x) = L$

A3. if $\lim\limits_{x \to +\infty} f(x) = L$, we know that $\lim\limits_{x \to +\infty} f(x+1) = L$ because we can choose $M' = M$ when we use the definition of limit to proof it

Similarly, we know that $\lim\limits_{x \to +\infty} f(x-1) = L$ if we choose $M' = M+1$.

so $\lim\limits_{x \to +\infty} g(x) = \lim\limits_{x \to +\infty} f(x)^2 - f(x+1) \cdot f(x-1) = L^2 - L \cdot L = 0$

A4. $\lim\limits_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim\limits_{x \to 0} \frac{1 - (1 - 2\sin^2\frac{x}{2})}{x^2} = \lim\limits_{x \to 0} \frac{2\sin^2\frac{x}{2}}{4 \cdot (\frac{x}{2})^2} = \lim\limits_{x \to 0} \frac{2}{4} \cdot \frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2} = \lim\limits_{x \to 0} \frac{1}{2} \cdot 1^2$

$= \frac{1}{2}$