Linear Algebra – Homework 6

28 Oct 2020
Due: 5 Nov 2020

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 2.7.5.

(a) The row vector $\mathbf{x}^T$ times A times the column $\mathbf{y}$ produces what number?

$$\mathbf{x}^T A\mathbf{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \_\_\_\_\_\_$$

(b) This is the row $\mathbf{x}^T A =$ ______ times the column $\mathbf{y} = (0, 1, 0)$.

(c) This is the row $\mathbf{x}^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$ times the column $A\mathbf{y} =$ ______.

Problem 2.7.11. Which permutation makes PA upper triangular? Which permutations make $P_1 A P_2$ lower triangular? Multiplying A on the *right* by P_2 exchanges the _______ of A.

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

Problem 2.7.16. If $A = A^T$ and $B = B^T$, which of these matrices are certainly symmetric?

(a) $A^2 - B^2$

(b) $(A + B)(A - B)$

(c) ABA

(d) $ABAB$

Problem 2.7.22. Find the $PA = LU$ factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Problem 2.7.32. The permutation matrix P that multiplies (x, y, z) to give (z, x, y) is also a rotation matrix. Find P and P^3. The rotation axis $\mathbf{a} = (1, 1, 1)$ doesn't move, it equals $P\mathbf{a}$. What is the angle of rotation around the axis $\mathbf{a}$ as you go from $\mathbf{v} = (2, 3, -5)$ to $P\mathbf{v} = (-5, 2, 3)$?

Problem 2.7.39. Suppose Q^T equals Q^{-1} (transpose equals inverse, so $Q^T Q = I$).

(a) Show that the columns $\mathbf{q}_1, \ldots, \mathbf{q}_n$ of Q are unit vectors: $\|\mathbf{q}_i\|^2 = 1$.

(b) Show that every two different columns of Q are perpendicular: $\mathbf{q}_1^T \mathbf{q}_2 = 0$.

(c) Find a 2×2 example with first entry $q_{11} = \cos\theta$ for some angle θ.

Problem 3.1.4. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2×2 matrices. Write down the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

Problem 3.1.10. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\mathbf{v} = (1, 4, 0)$ and $\mathbf{w} = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \leq b_2 \leq b_3$.

Problem 3.1.14. The subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0, 0, 0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe all subspaces of $\mathbf{D}$, the space of 2×2 diagonal matrices.

Problem 3.1.15.

(a) The intersection of two planes through $(0, 0, 0)$ is probably a _____ in $\mathbf{R}^3$ but it could be a _____. It can't be $\mathbf{Z}$!

(b) The intersection of a plane through $(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a _____ but it could be a _____.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, show that their intersection $\mathbf{S} \cap \mathbf{T}$ is a subspace of $\mathbf{R}^5$. Here $\mathbf{S} \cap \mathbf{T}$ consists of all vectors that lie in *both* subspaces. (Check that $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$ are in $\mathbf{S} \cap \mathbf{T}$ if $\mathbf{x}$ and $\mathbf{y}$ are in both spaces.)

Problem 3.1.19. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Problem 3.1.20. For which right sides (find condition(s) on b_1, b_2, b_3) are these systems solvable?

(a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Problem 3.1.25. Suppose $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b}^*$ are both solvable. Then $A\mathbf{z} = \mathbf{b} + \mathbf{b}^*$ is solvable. What is $\mathbf{z}$? This translates into: If $\mathbf{b}$ and $\mathbf{b}^*$ are in the column space $\mathbf{C}(A)$, then $\mathbf{b} + \mathbf{b}^*$ is in $\mathbf{C}(A)$.

Graded Problems. In these problems, you should check the three properties of a subspace:

1. Does the subset contain the zero vector $\mathbf{0}$?

2. If $\mathbf{v}$ and $\mathbf{w}$ are in the subset, what about $\mathbf{v}+\mathbf{w}$?
3. If $\mathbf{v}$ is in the subset and c is any scalar, what about $c\mathbf{v}$?

Problem 1.

(a) Show that the subset of symmetric matrices is a subspace of the vector space $\mathbf{M}$ of $n \times n$ matrices.

(b) Is the subset of symmetric matrices closed under *matrix* multiplication? That is, if A and B are both symmetric, does AB have to be symmetric?

Problem 2. Show that the subset $\mathbf{P}_n$ of polynomials with degree $\leq n$ is a subspace of the vector space $\mathbf{F}$ of all real-valued functions. (Functions in $\mathbf{P}_n$ look like $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, with $a_0, a_1, \ldots, a_n$ allowed to be any scalars, including 0.)