CSCI-UA.0480-004
Algorithmic Problem Solving

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Lecture 13: Graphs

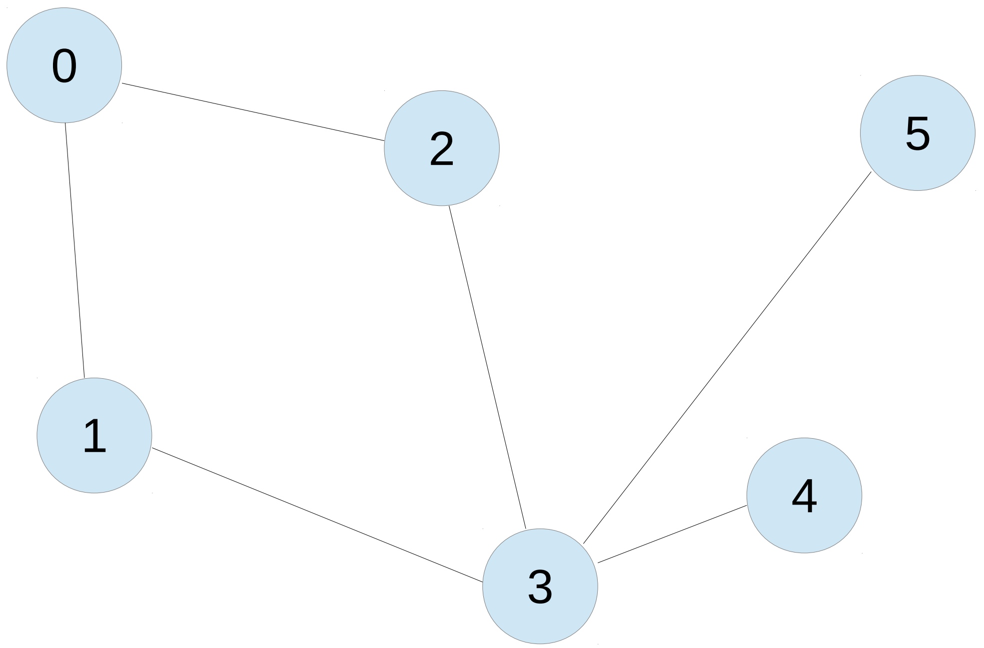
Graph Traversal Algorithms

- Many problems rely on traversing elements in a graph
  - e.g., UVa 469 – Wetlands of Florida
    - You're given a 2D grid, each cell of which can be “water” or “land”
    - Cells adjacent on the major axes or diagonals are adjacent
    - For a given water (x, y) coordinate on the grid, determine the area of the connected water
  - These problems look hard if you're not familiar with graph traversals

Graph Traversal Algorithms

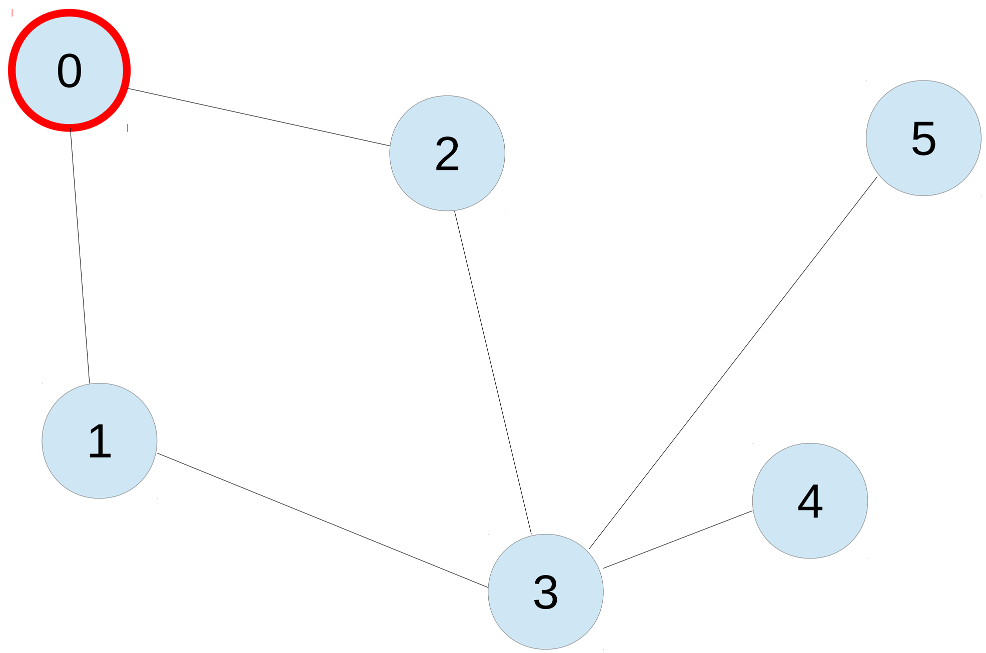
- Depth-first search
  - The first and most natural way to solve this problem is by visiting every node using recursion
  - As the name implies, visit the furthest nodes from the originating node
  - Perform backtracking

Graph Traversal Algorithms



<u>Adjacency list</u>
0: 1, 2
1: 1, 3
2: 0, 3
3: 1, 2, 4, 5
4: 3
5: 3

Graph Traversal Algorithms



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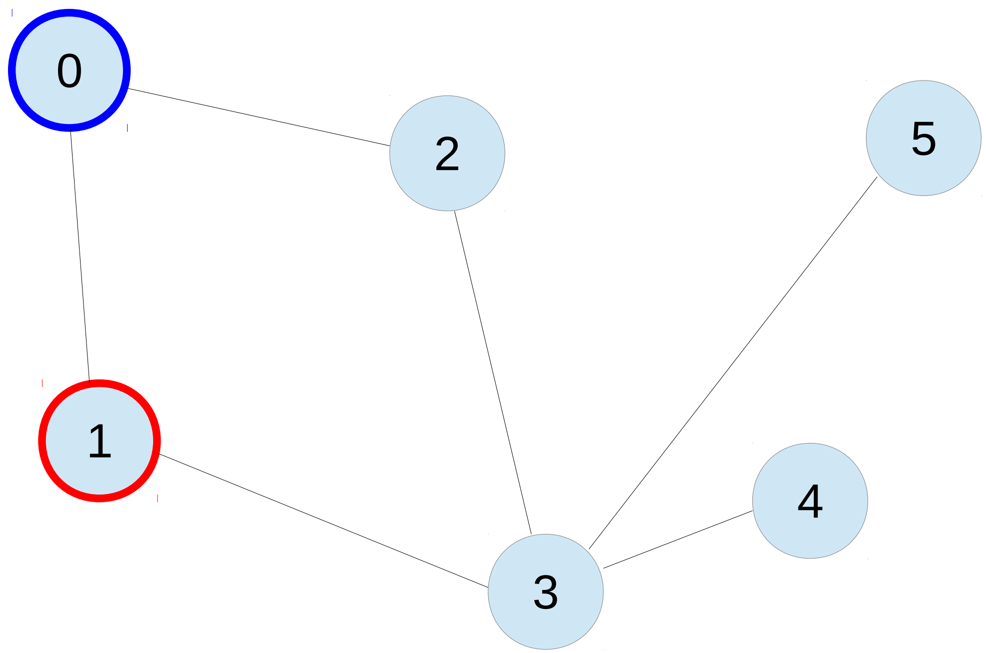
<u>Stack</u>
dfs(0)

Graph Traversal Algorithms



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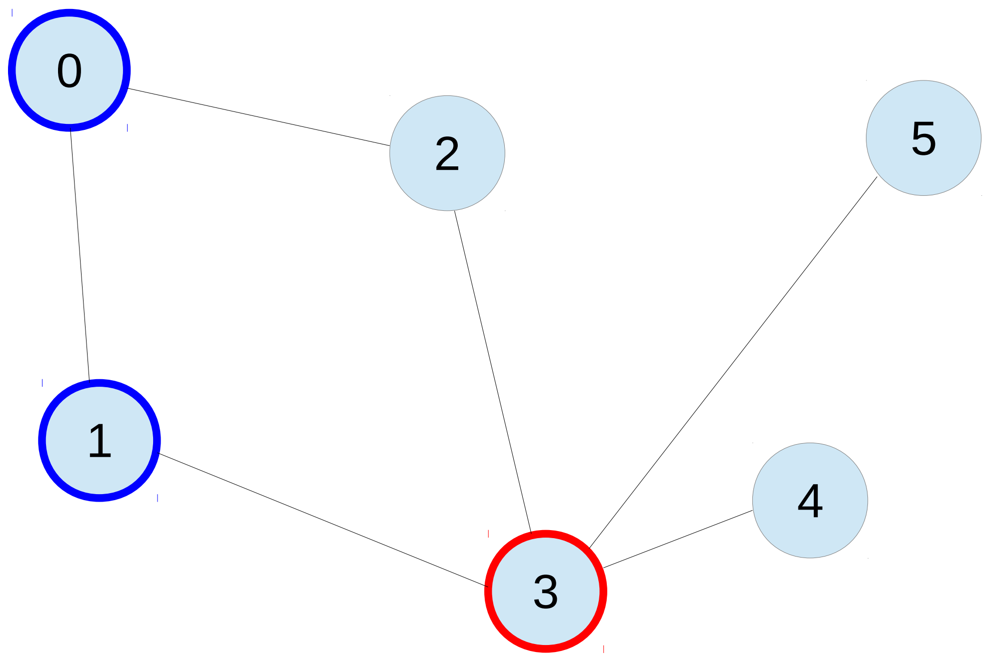
dfs(1)

Graph Traversal Algorithms



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Stack
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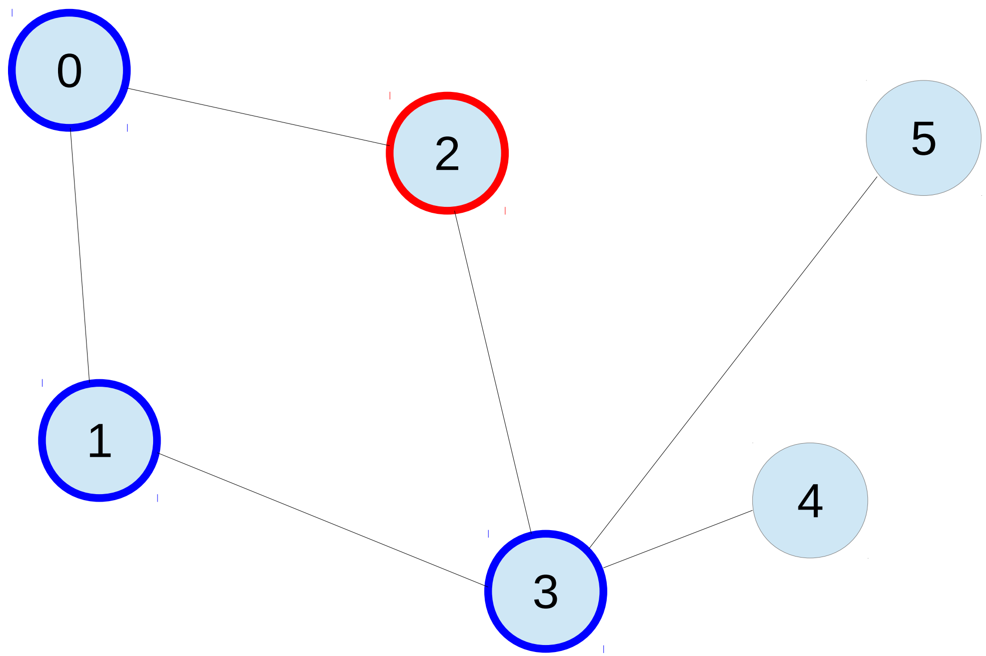
dfs(3)

Graph Traversal Algorithms



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Stack
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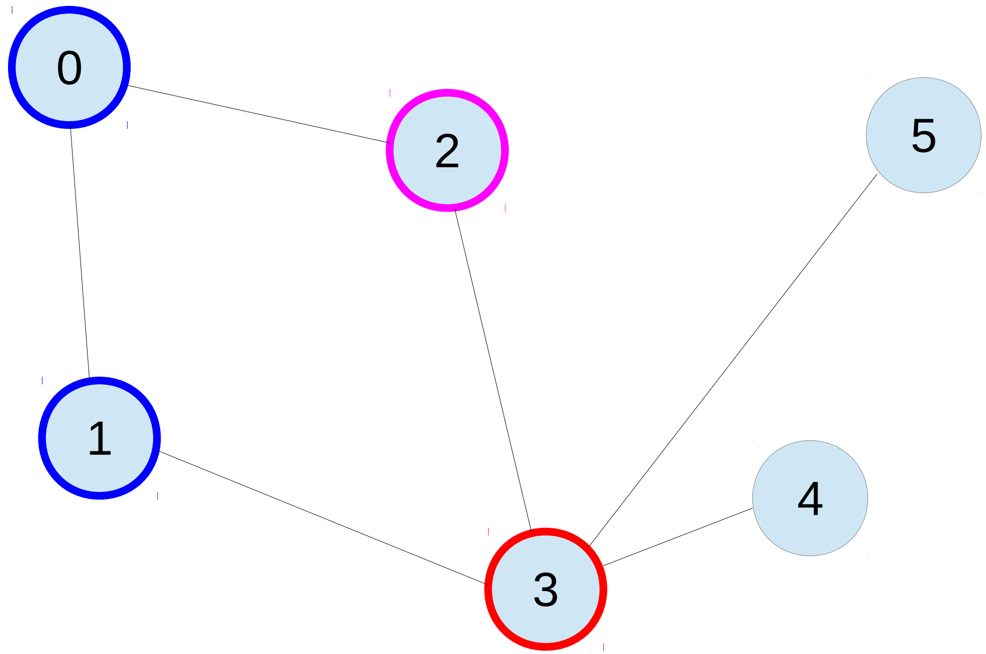
dfs(1)
dfs(3)
dfs(2)

Graph Traversal Algorithms



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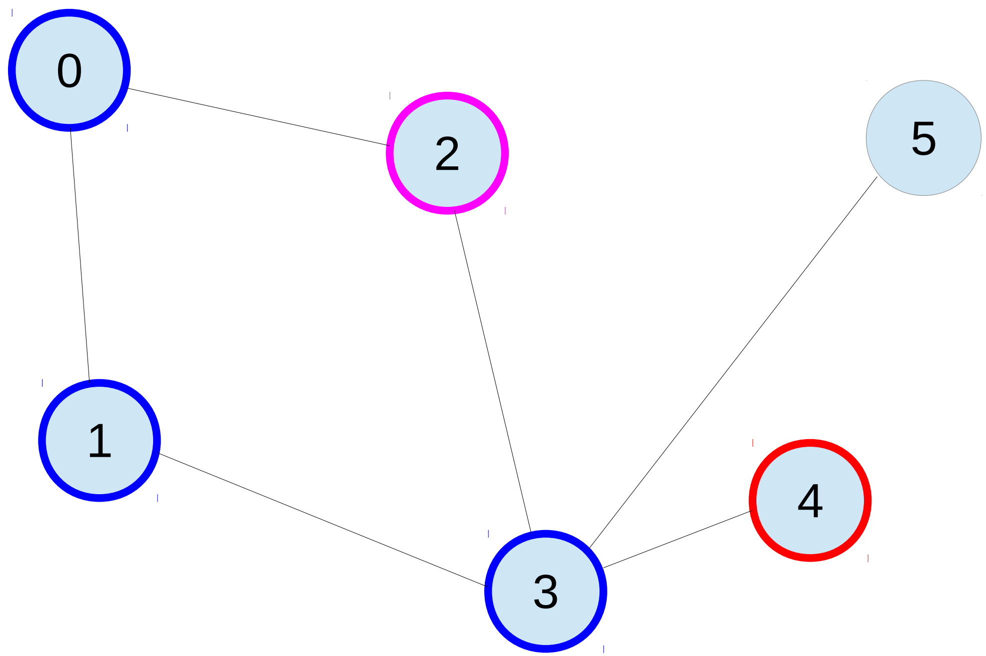
Graph Traversal Algorithms



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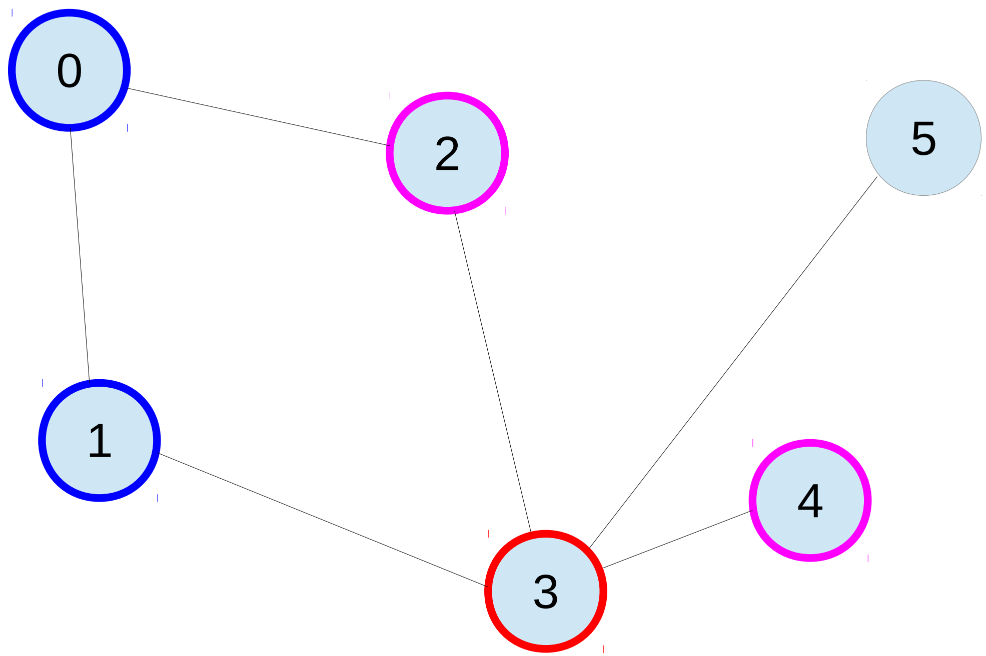
Graph Traversal Algorithms



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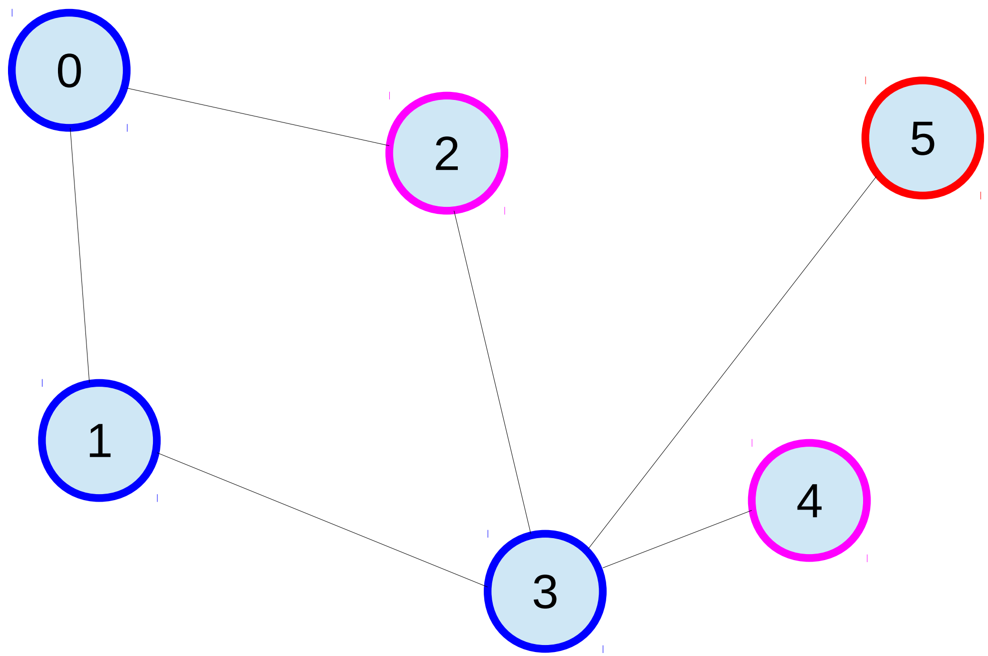
Graph Traversal Algorithms



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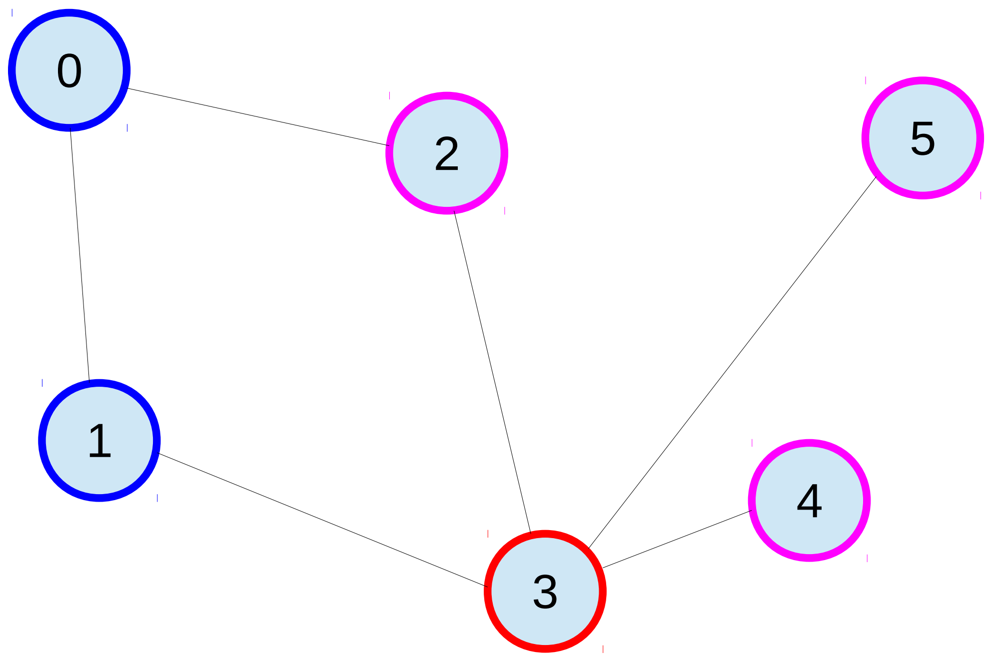
Graph Traversal Algorithms



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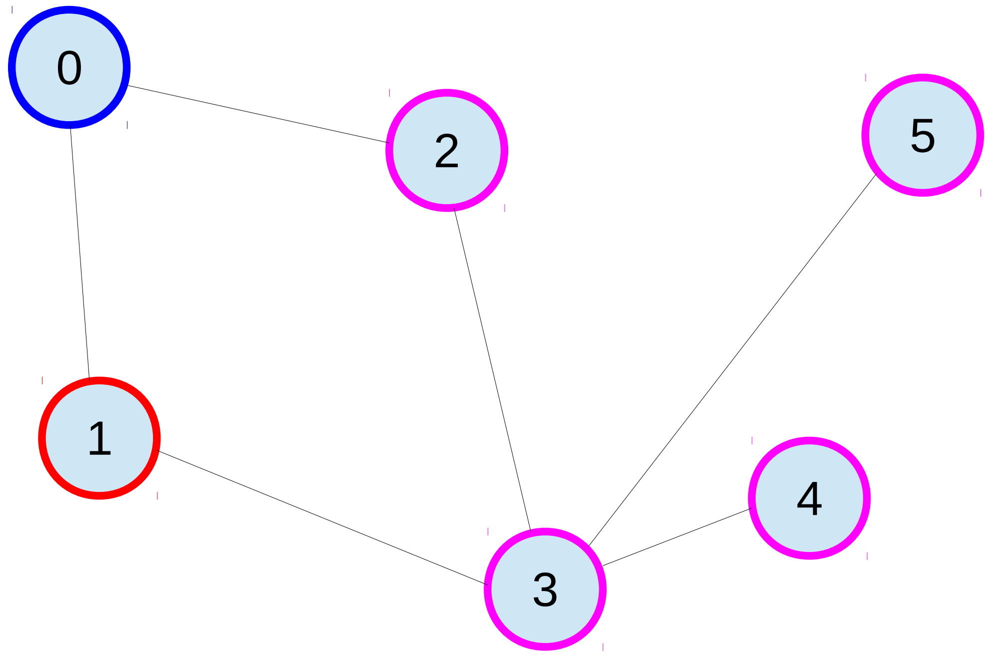
Graph Traversal Algorithms



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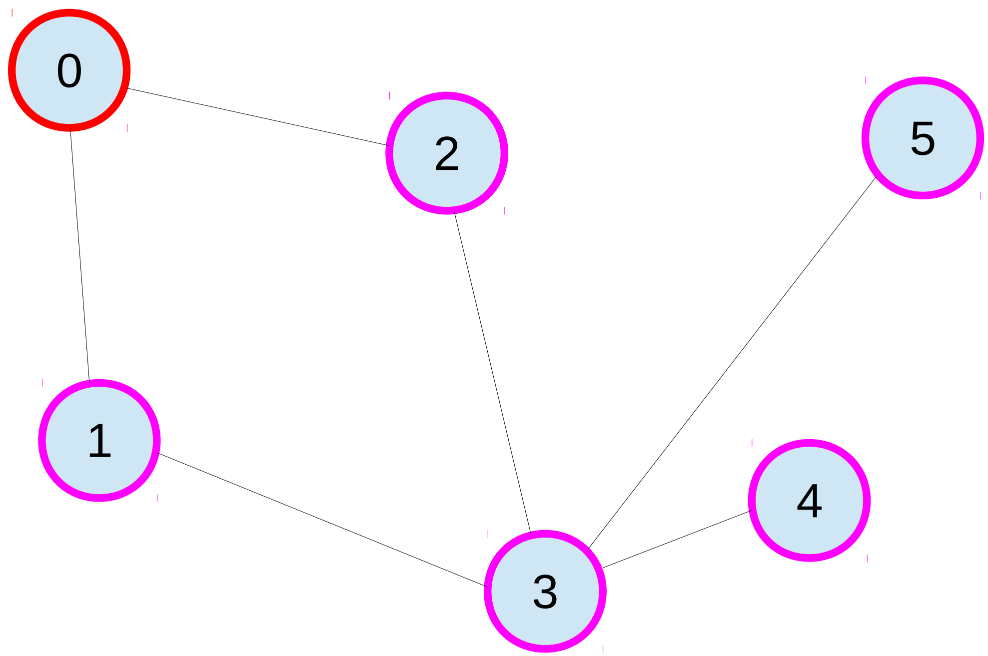
Graph Traversal Algorithms



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Stack
dfs(0)

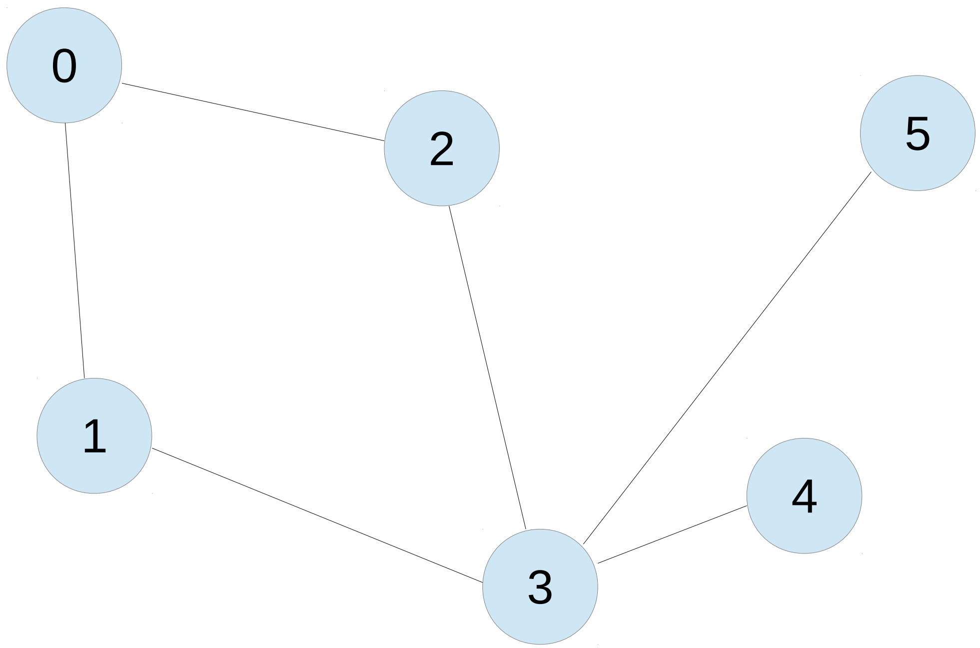
Graph Traversal Algorithms

```
ArrayList<ArrayList<Integer>> adjList; // prefilled with adjacents

int dfs(int node) { // returns # of nodes visited from node idx
    int res = 0;

    visited[node] = true; // mark this node as visited

    for (int i = 0; i < adjList.get(node).size(); i++) {
        int neighbor = adjList.get(node).get(i);
        if (!visited[neighbor]) {
            res += dfs(neighbor); // add number of dfs nodes visited
        }
    }

    return res+1; // the +1 refers to visiting the present node
}

public static void main(String[] args) {
    System.out.println(dfs(0));
}
```

Graph Traversal Algorithms

- Breadth-first search
  - Visit nodes closest to the originating node before diving down into the tree
  - Implemented using Queue

Graph Traversal Algorithms



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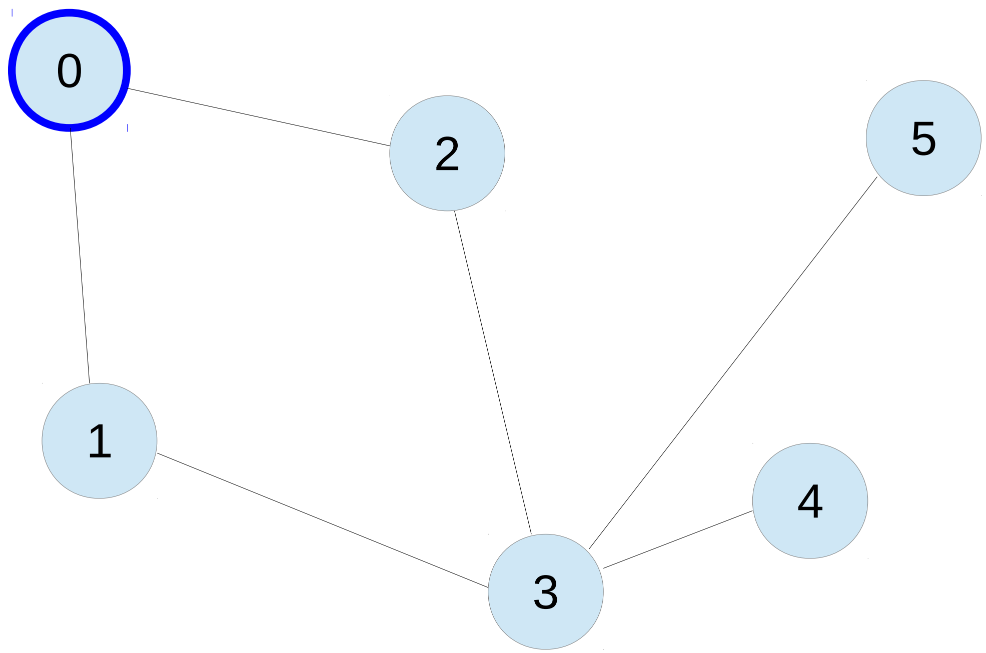

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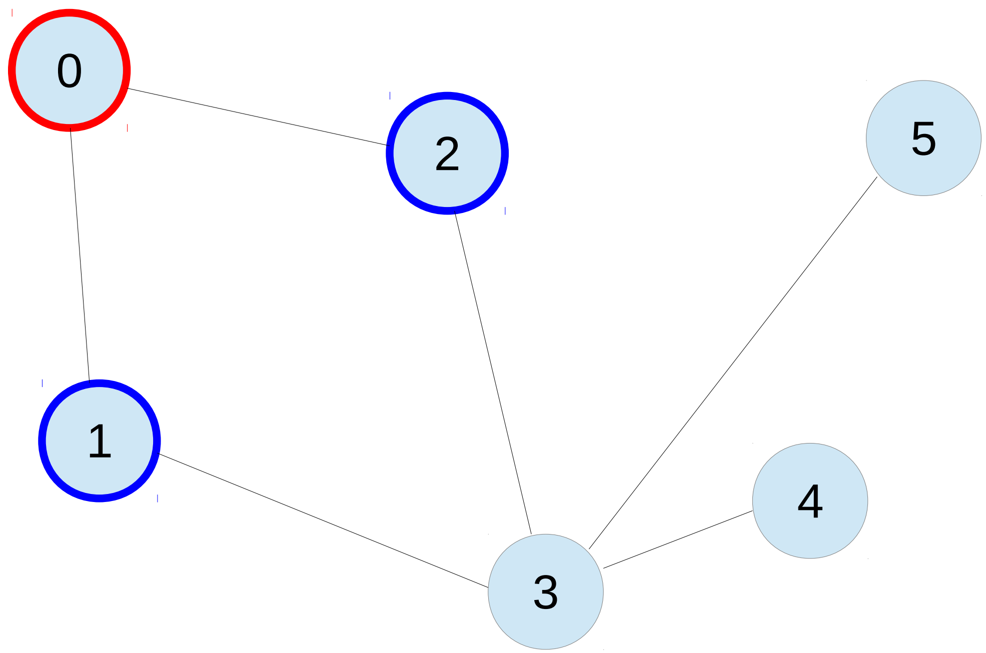
Queue
0

Graph Traversal Algorithms



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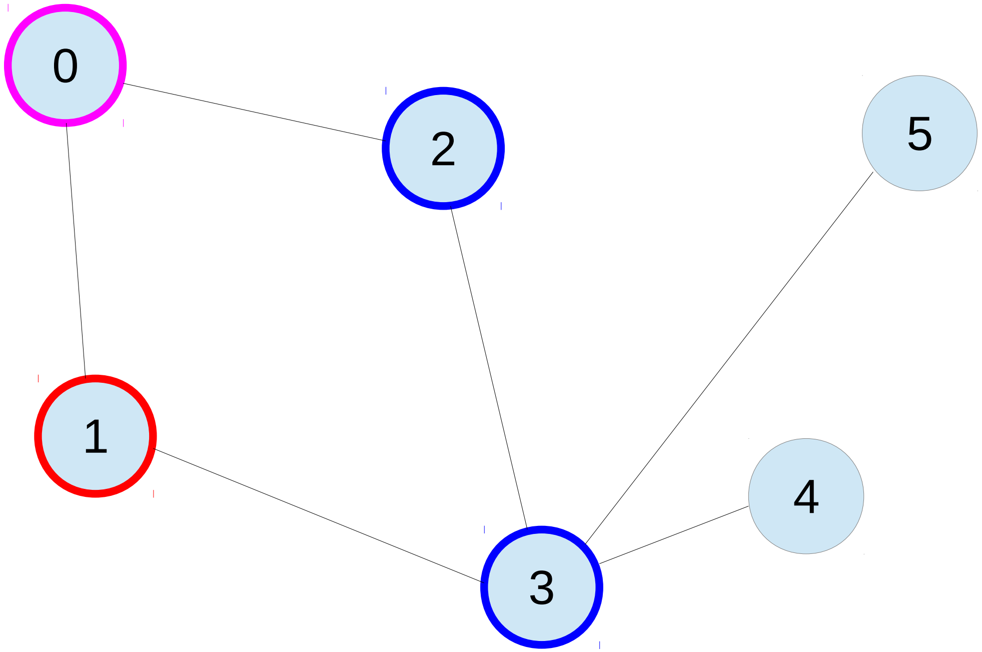
Queue
1
2

Graph Traversal Algorithms



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<u>Queue</u>
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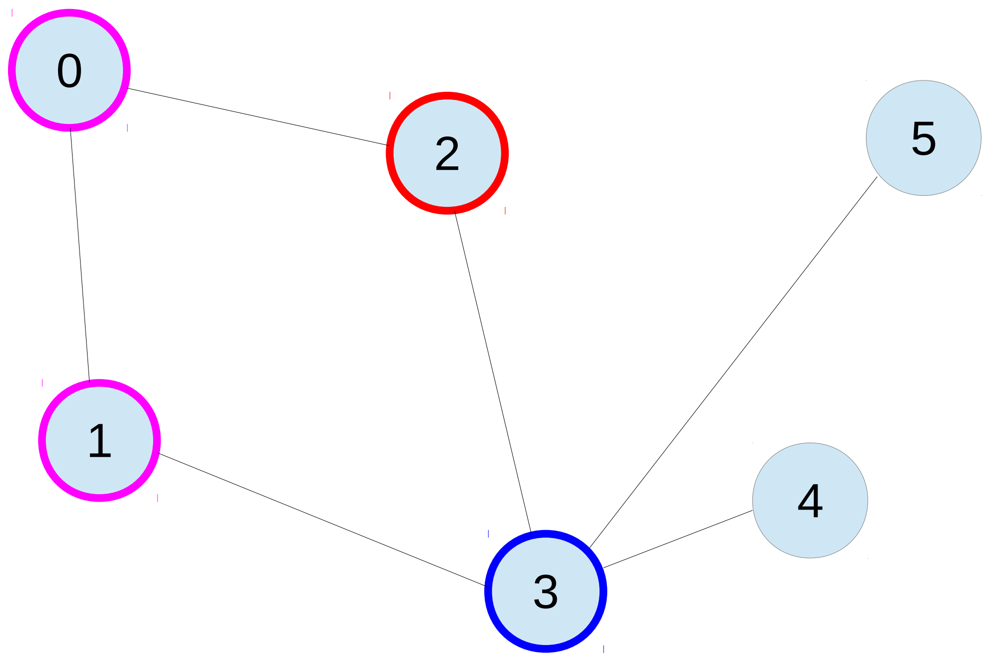
Graph Traversal Algorithms



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<u>Queue</u>
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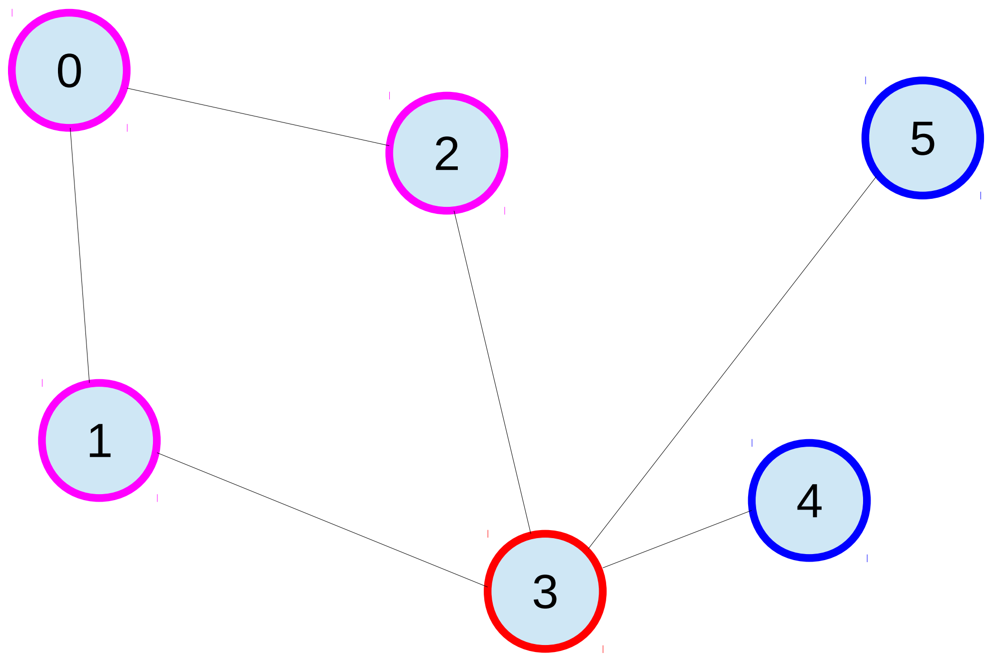
Graph Traversal Algorithms



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Queue
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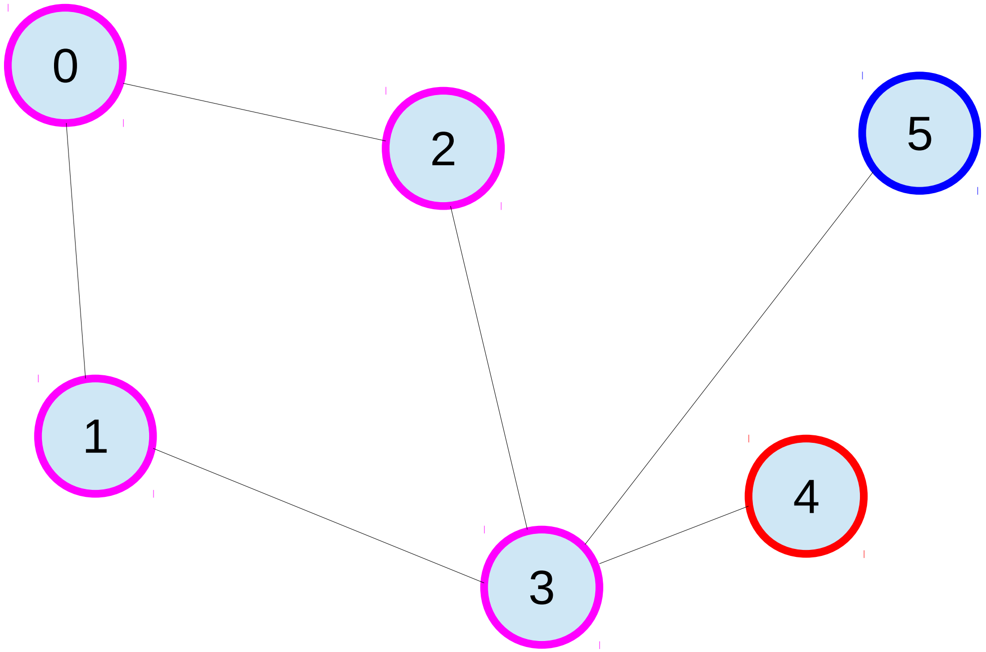
Graph Traversal Algorithms



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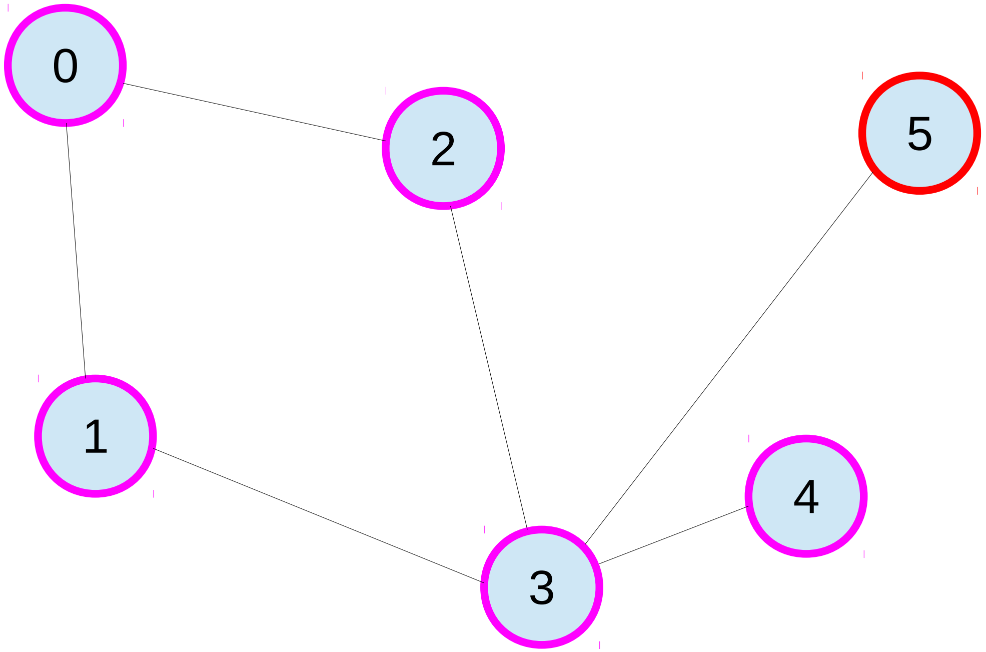
Graph Traversal Algorithms



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Queue

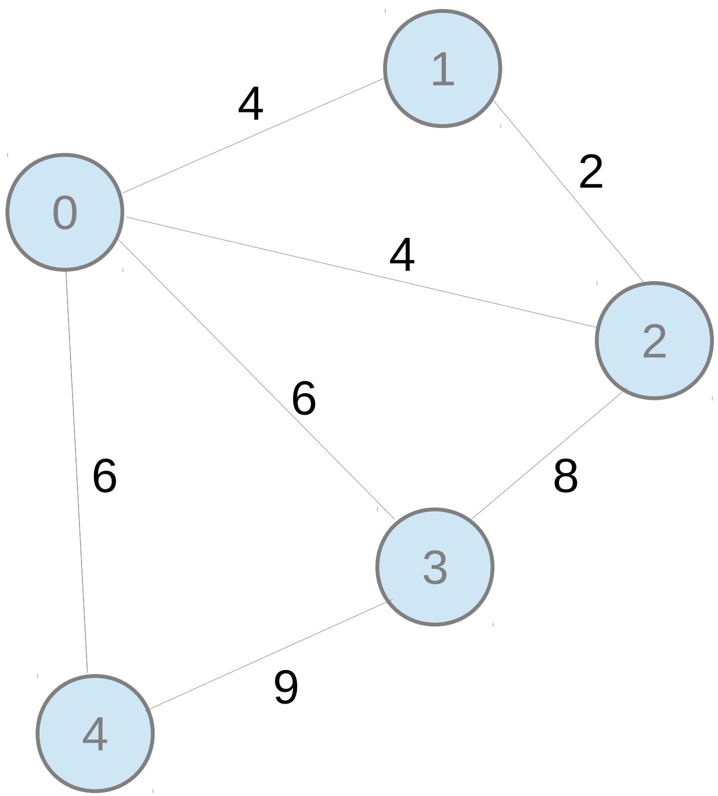
Graph Traversal Algorithms

```
int bfs(ArrayList<ArrayList<Integer>> adjList) {
    Queue<Integer> q = new LinkedList<Integer>();
    boolean visited[] = new boolean[N]; // keep track of visited nodes

    q.add(0); visited[0] = true; // add to traversal queue and mark
    int count = 1; // example: keep count of nodes traversed

    while (!q.isEmpty()) {
        int node = q.poll();

        for (int i = 0; i < adjList.get(node).size(); i++) {
            int neighbor = adjList.get(node).get(i);
            if (!visited[neighbor]) { // do not visit nodes twice
                q.add(neighbor); // add to traversal queue
                visited[neighbor] = true; // mark as visited
                count++; // visited a new node! Keep count
            }
        }
    }

    return count;
}
```

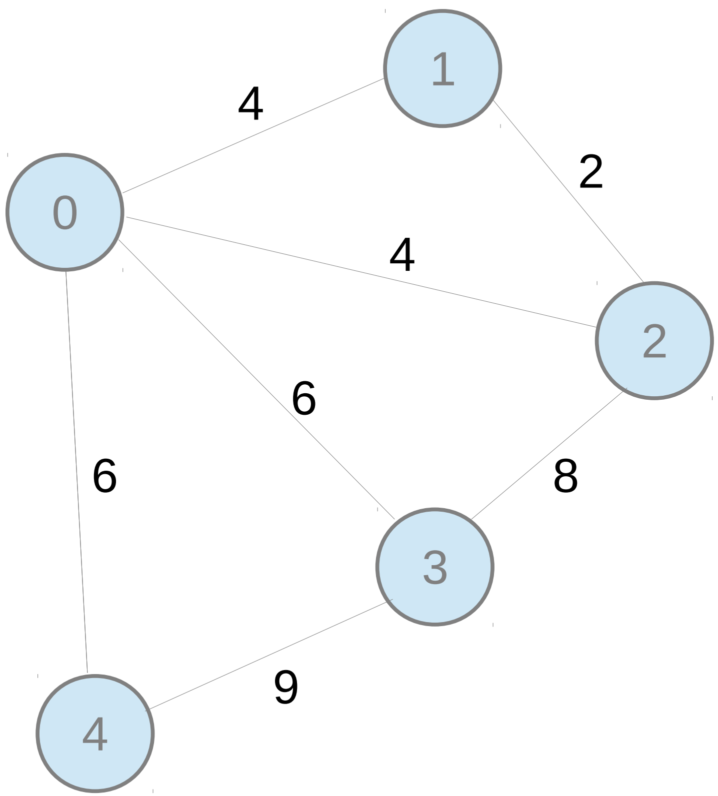
Minimum Spanning Trees

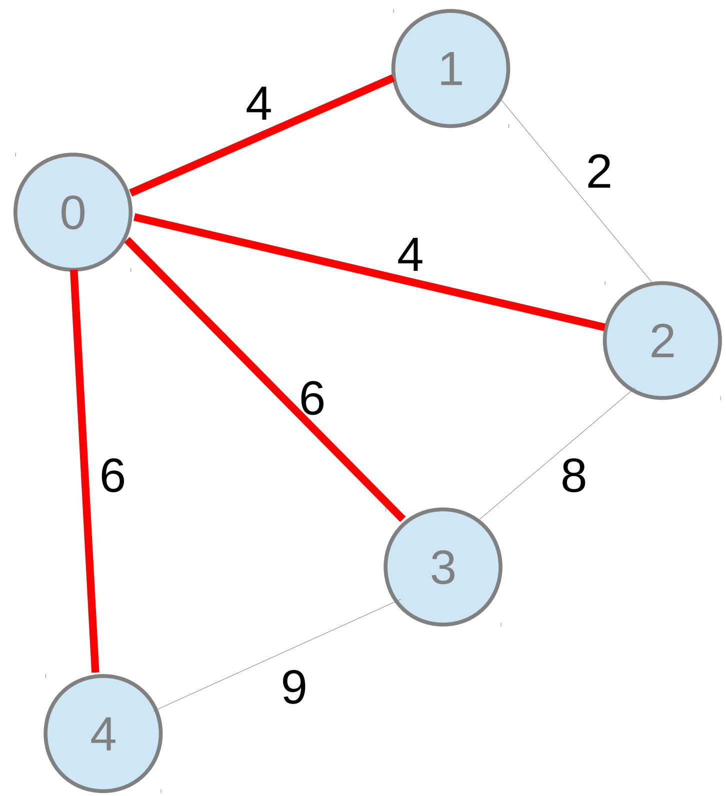
- Spanning tree
    - Given: a connected, undirected graph $G = (V, E)$
        - (V is the set of vertices, E is the set of edges)
    - A spanning tree is a set of edges that is a tree and “covers” all vertices V
        - There can be several trees
    - The spanning tree with the minimum cost (sum of edge weights) is called the **Minimum Spanning Tree**

Minimum Spanning Trees



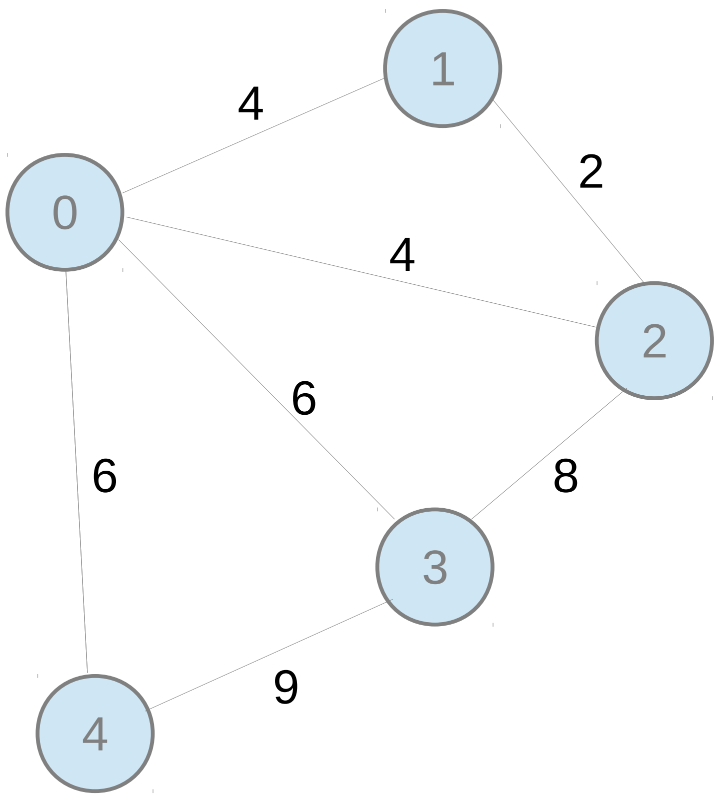
Minimum Spanning Trees

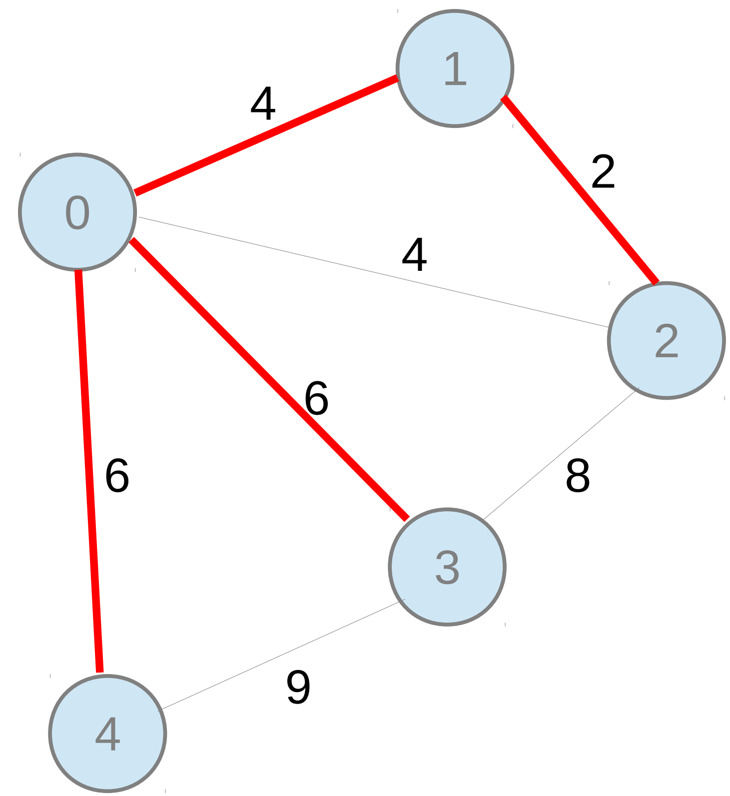




A spanning tree
Cost: 4 + 4 + 6 + 6 = 20

Minimum Spanning Trees





Minimum spanning tree

Cost: 4 + 2 + 6 + 6 = 18

Minimum Spanning Trees

- Kruskal's algorithm for finding the MST
  - Repeatedly finds edges with minimum costs that does not form a cycle
  - Greedy algorithm, provably correct

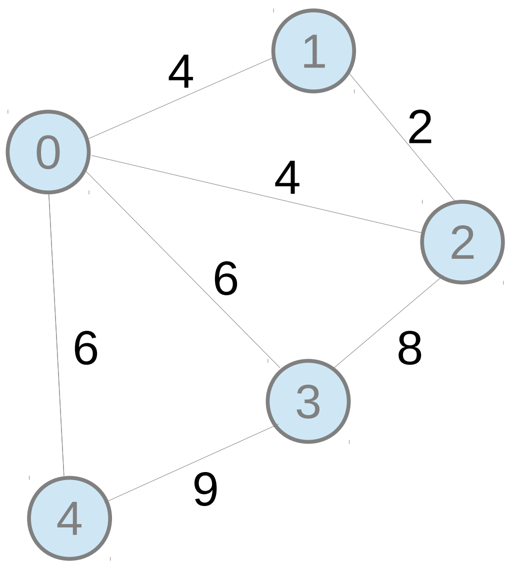
Minimum Spanning Trees

- Kruskal's algorithm pseudocode
  1) Sort edges by increasing weight
  2) While there are unprocessed edges left
     1) Pick an edge *e* **with minimum cost**
     2) If adding *e* to the MST **does not form a cycle,** add *e* to MST

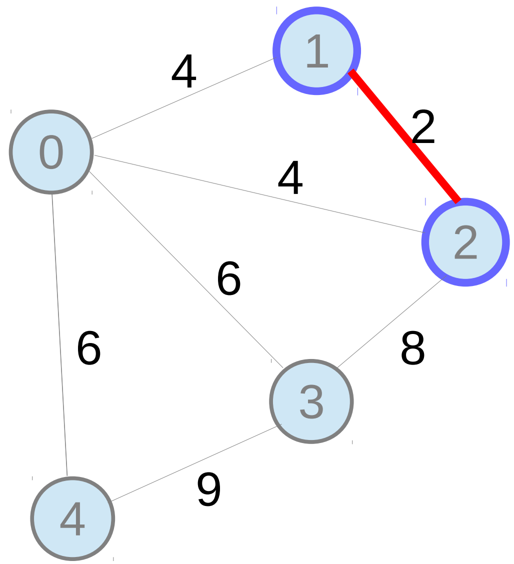
Minimum Spanning Trees

- Kruskal's algorithm pseudocode
  - How to store and sort edges?
    - Using an edge list and Collections.sort
  - How to test for cycles?
    - using disjoint sets and union-find
  - Runtime?
    - Sort: $O(|E| \log |E|)$; Processing: $O(|E|)$
    - Total: $O(|E| \log |E|) = O(|E| \log |V|)$

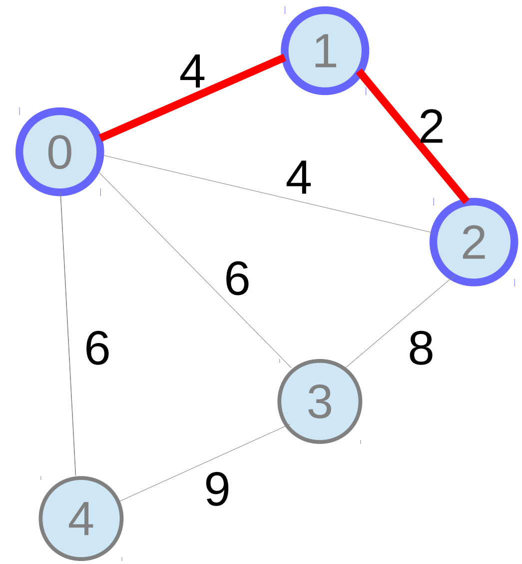
Minimum Spanning Trees



Pick smallest edge



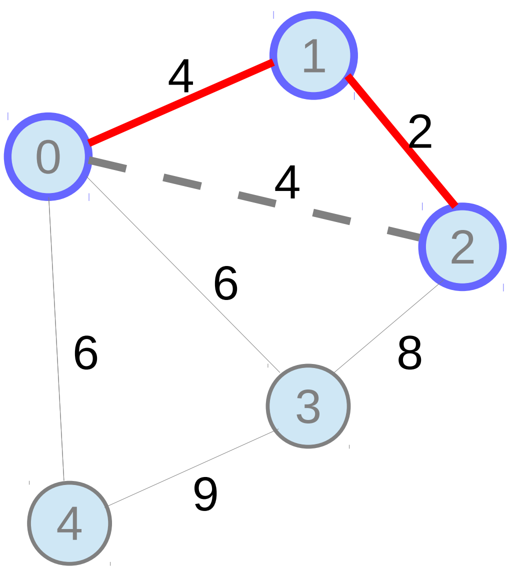
Pick smallest edge
No cycle



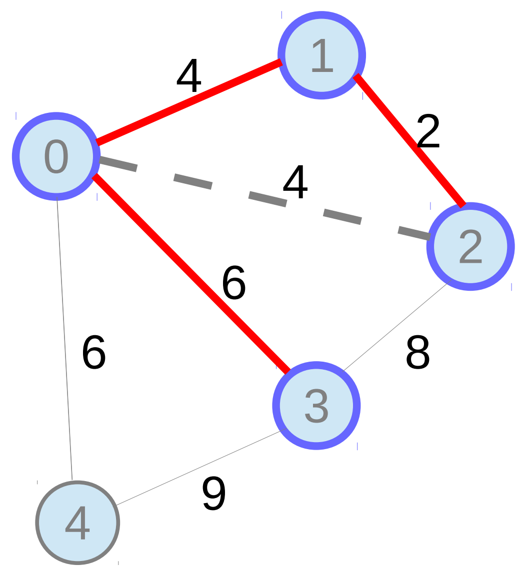
<u>Weighted adjacency list by (index, weight)</u>
0: (1, 4), (2, 4), (3, 6), (4, 6)
1: (0, 4), (2, 2)
2: (0, 4), (1, 2), (3, 8)
3: (0, 6), (2, 8), (4, 9)
4: (0, 6), (3, 9)

Minimum Spanning Trees

Pick smallest edge
Cycle formed, ignore



Pick smallest edge
No cycle



Pick smallest edge
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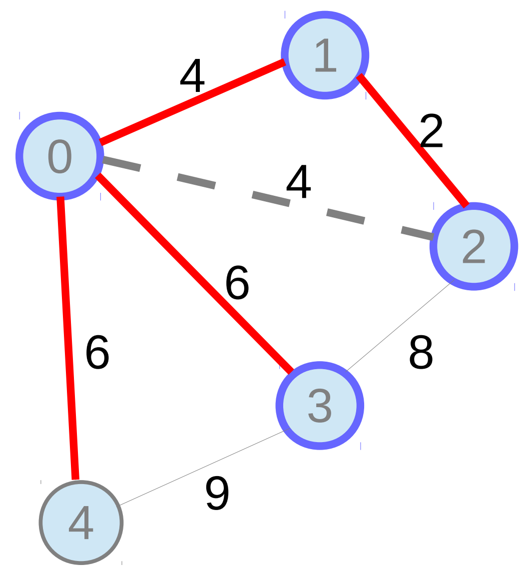

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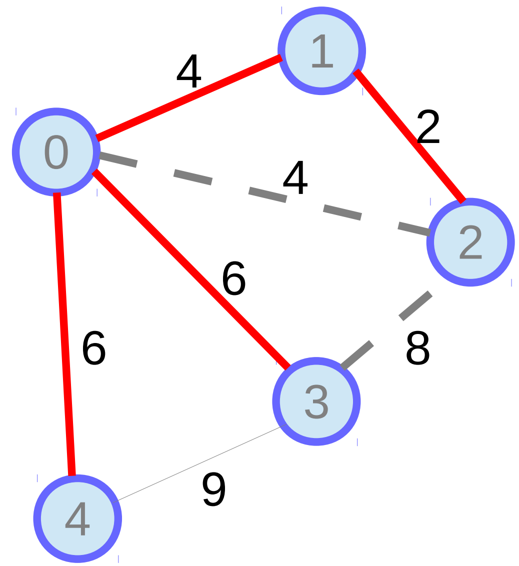
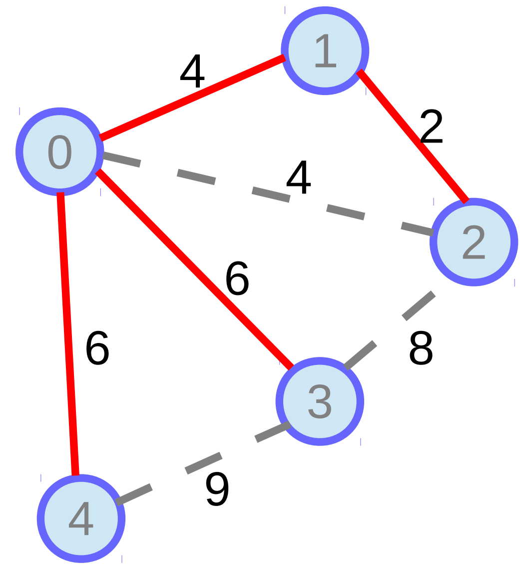
Minimum Spanning Trees

Algorithm not done!
The edge list hasn't
yet been exhausted

Pick smallest edge
Cycle formed, ignore



Pick smallest edge
Cycle formed, ignore



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3: (0, 6), (2, 8), (4, 9)
4: (0, 6), (3, 9)

Minimum Spanning Tree

- Kruskal's code
  - An edge list, a sort, and union-find
    - Follows...

Minimum Spanning Tree

```java
class Edge implements Comparable<Edge> {
    int A, B, w;

    public Edge(int A, int B, int w) {
        this.A = Math.min(A, B);
        this.B = Math.max(A, B);
        this.w = w;
    }

    public int compareTo(Edge e) {
        if (w != e.w) {
            return w < e.w ? -1 : 1;
        } else {
            return 0;
        }
    }
}
```

```java
class UnionFind {
    int uf[];

    public UnionFind(int size) {
        uf = new int[size];
        for (int i = 0; i < size; i++) uf[i] = i;
    }

    public boolean isSameSet(int A, int B) {
        return find(A) == find(B);
    }

    public void union(int A, int B) {
        uf[find(A)] = find(B);
    }

    public int find(int A) {
        int res = uf[A];
        while (uf[res] != res) res = uf[res];
        return uf[A] = res;
    }
}
```

Minimum Spanning Tree

```java
ArrayList<Edge> edgeList = parseEdgeList();
Collections.sort(edgeList);

int mstCost = 0;
UnionFind uf = new UnionFind(nVertices);

for (Edge e : edgeList) { // for each edge
   if (!uf.isSameSet(e.A, e.B)) { // if no cycle
      mstCost += e.w; // add it
      uf.union(e.A, e.B);
   }
}

System.out.println(mstCost);
```

Minimum Spanning Tree

- Prim's algorithm
  - Not covered, see the textbook or Wikipedia for a good overview
  - Also has O(|E| log |V|) runtime

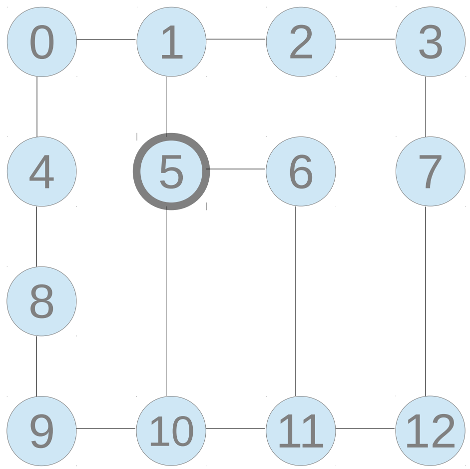
Single source shortest paths

- Classic problem in computer science
  - Given a node on a graph, find the shortest paths to all other nodes

Single source shortest paths

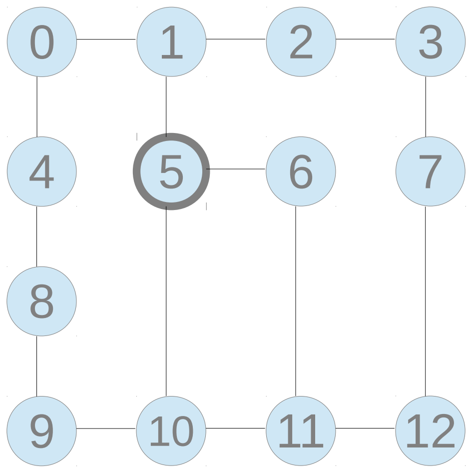
- For <u>undirected</u>, <u>unweighted</u> graphs:
  - Use BFS!
  - E.g., Uva 336 (A Node Too Far)
    - Given an undirected and unweighted graph G = (V, E) and a vertex v in V, find the number of nodes unreachable in n hops

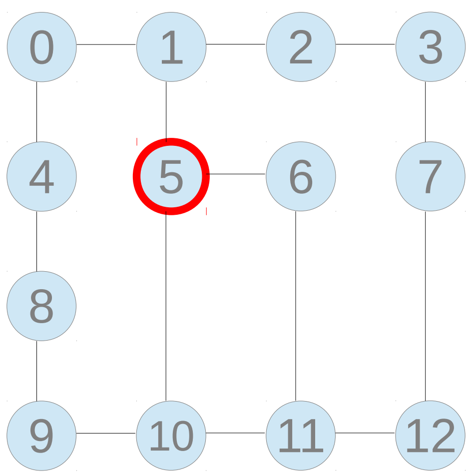
Single source shortest paths



From node 5, find # of nodes > 3 hops away

Single source shortest paths



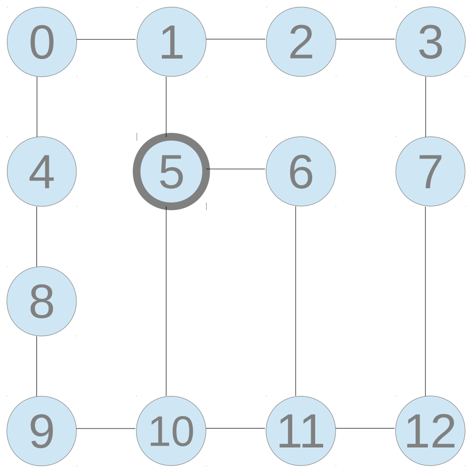


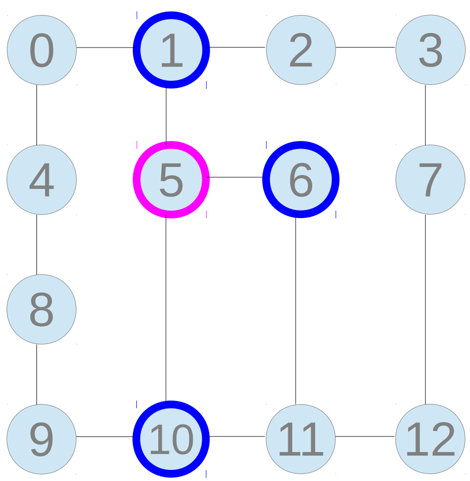
<u>Queue</u>
5

<u>Distances</u>
D[5] = 0

From node 5, find # of nodes > 3 hops away

Single source shortest paths

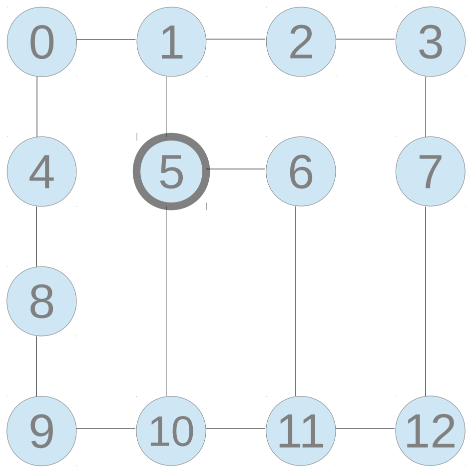


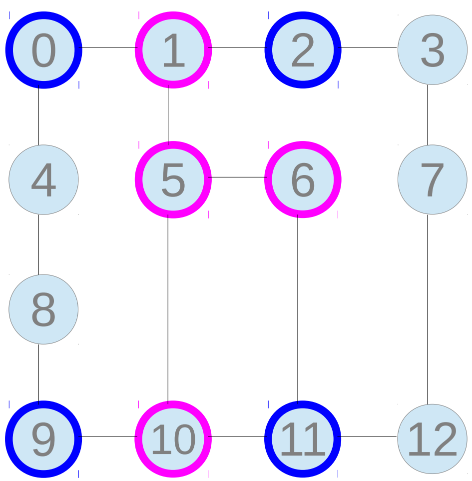


Queue	Distances
1	D[5] = 0
6	D[1] = D[5] + 1 = 1
10	D[6] = D[5] + 1 = 1
	D[10] = D[5] + 1 = 1

From node 5, find # of nodes > 3 hops away

Single source shortest paths



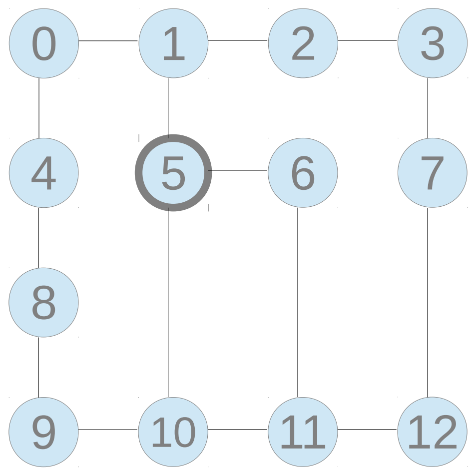


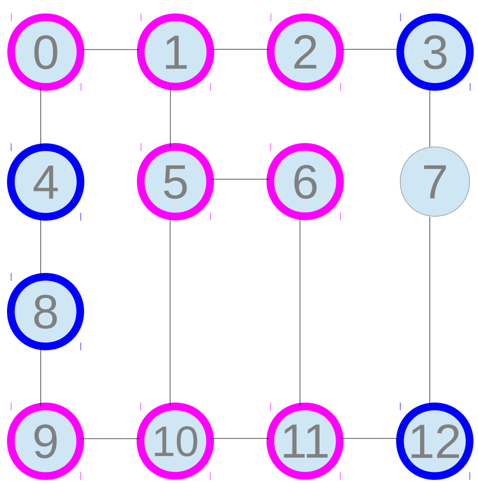
<u>Queue</u>
0
2
9
11

<u>Distances</u>
D[5] = 0
D[1] = 1
D[6] = 1
D[10] = 1
D[0] = D[1] + 1 = 2
D[2] = D[1] + 1 = 2
D[9] = D[10] + 1 = 2
D[11] = D[10] + 1 = 2

From node 5, find # of nodes > 3 hops away

Single source shortest paths



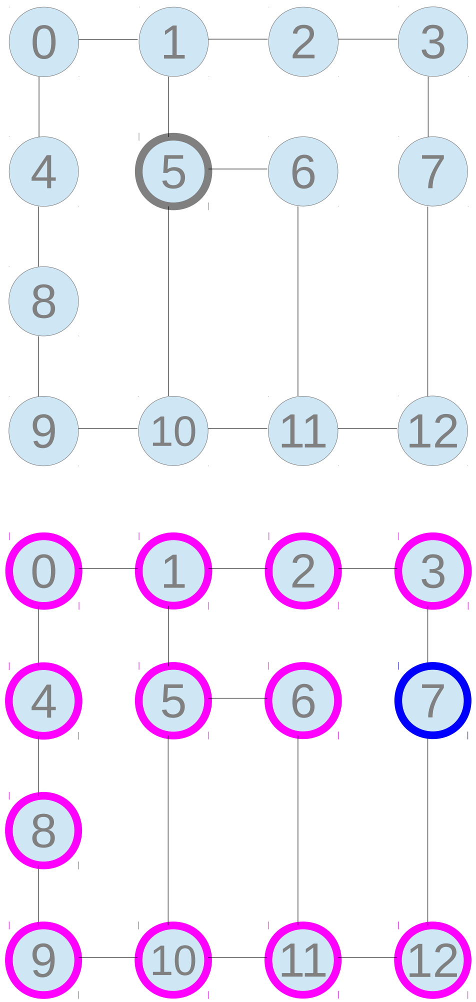


Queue
3
4
8
12

Distances
D[5] = 0
D[1] = 1
D[6] = 1
D[10] = 1
D[0] = 2
D[2] = 2
D[9] = 2
D[11] = 2
D[3] = D[2] + 1 = 3
D[4] = D[0] + 1 = 3
D[8] = D[9] + 1 = 3
D[12] = D[11] + 1 = 3

From node 5, find # of nodes > 3 hops away

Single source shortest paths

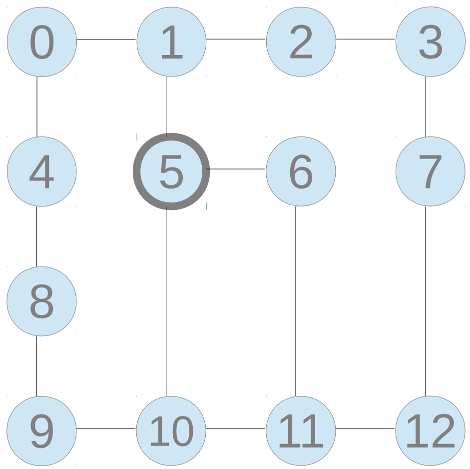


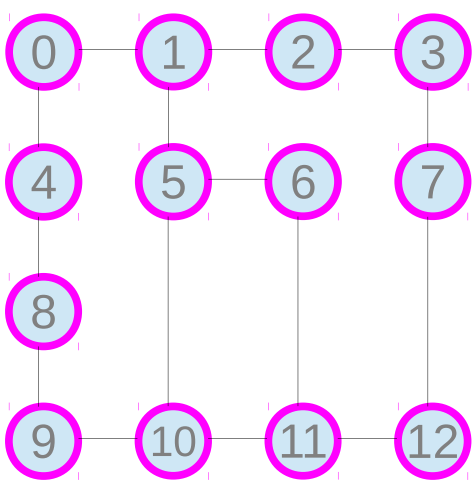
<u>Queue</u>
7

<u>Distances</u>
D[5] = 0
D[1] = 1
D[6] = 1
D[10] = 1
D[0] = 2
D[2] = 2
D[9] = 2
D[11] = 2
D[3] = 3
D[4] = 3
D[8] = 3
D[12] = 3
D[7] = D[3] + 1 = 4

From node 5, find # of nodes > 3 hops away

Single source shortest paths





Queue

Distances

D[5] = 0
D[1] = 1
D[6] = 1
D[10] = 1
D[0] = 2
D[2] = 2
D[9] = 2
D[11] = 2
D[3] = 3
D[4] = 3
D[8] = 3
D[12] = 3
D[7] = 4

Answer: 1

From node 5, find # of nodes > 3 hops away

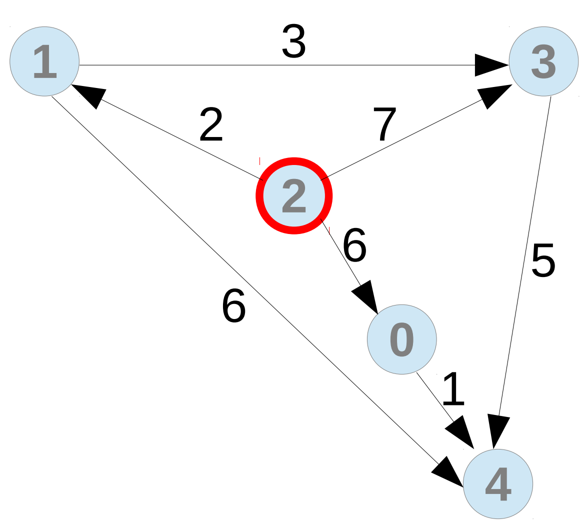
Single source shortest paths

- For undirected, unweighted graphs:
  - Will the same method as above work?
    - Yes

Single source shortest paths

- For <u>directed</u>, <u>weighted</u> graphs:
  - *Without negative weights*
  - Use Dijkstra's! O((|V| + |E|) log |V|)
  - This can be done using a priority queue
    - Works kind of like a greedy, modified BFS
  - Shown by example...

Single source shortest paths



Question: Shortest paths from 2 to all other nodes?

Priority Queue (distance, node index)

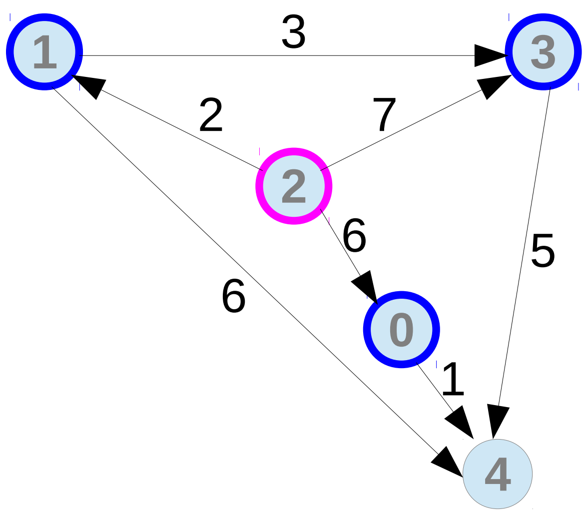
{ (0, 2) }

i	0	1	2	3	4
d[i]	INF	INF	0	INF	INF

Distance table

Start from node 2

Single source shortest paths



Question: Shortest paths from 2 to all other nodes?

Priority Queue (distance, node index)

~~{ (0, 2) }~~

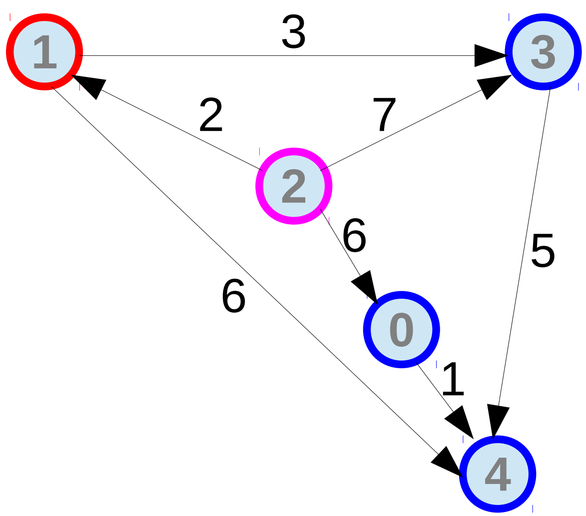
{ (2, 1), (6, 0), (7, 3) }

i	0	1	2	3	4
d[i]	6	2	0	7	INF

Distance table

Add all unvisited nodes from node 2 to the priority queue.
The PQ sorts the distances so the “next closest” node floats to the top.
Right now the closest node is 1, followed by 0, then 3.

Single source shortest paths



Question: Shortest paths from 2 to all other nodes?

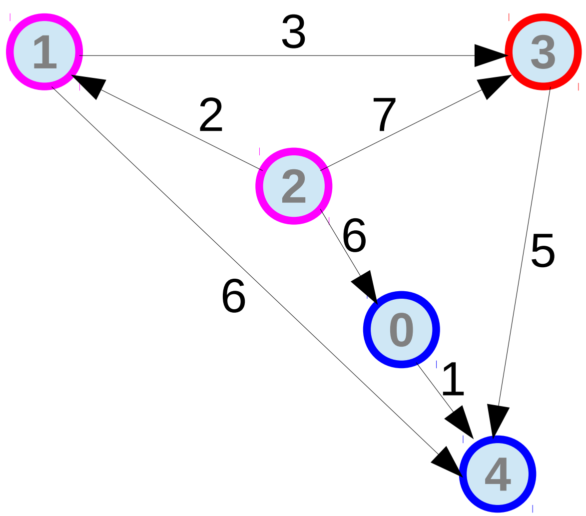
<u>Priority Queue (distance, node index)</u>
~~{ (0, 2) }~~
{ ~~(2, 1)~~, (6, 0), (7, 3) }
{ (5, **3**), (6, 0), (7, **3**), (8, 4) }

i	0	1	2	3	4
d[i]	6	2	0	5	8

Distance table

Poll from the PQ to get node 1.
Add all neighboring nodes to node 1 that haven't been polled yet.
BUT be sure to add all nodes that may already be in the queue with longer distances – there may be a shorter way to reach them

Single source shortest paths



Question: Shortest paths from 2 to all other nodes?

Priority Queue (distance, node index)

~~{ (0, 2) }~~

{ ~~(2, 1)~~, (6, 0), (7, 3) }

{ ~~(5, 3)~~, (6, 0), (7, 3), (8, 4) }

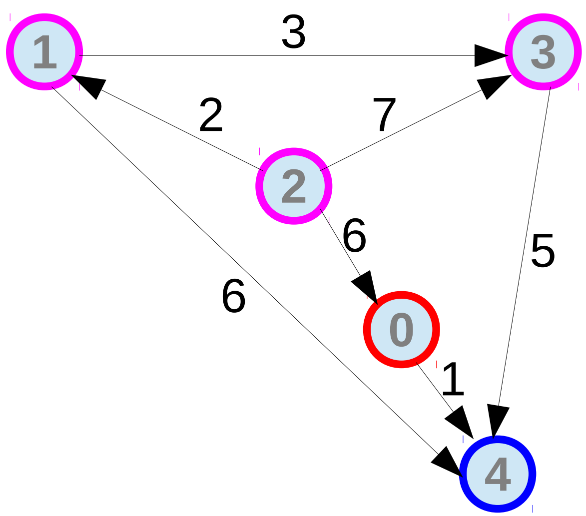
{ (6, 0), (7, 3), (8, 4) }

i	0	1	2	3	4
d[i]	6	2	0	5	8

Distance table

Poll from the PQ to get node 3.
Since we know there is a faster way to get node 4, don't bother adding node 4 to PQ

Single source shortest paths



Question: Shortest paths from 2 to all other nodes?

<u>Priority Queue (distance, node index)</u>

~~{ (0, 2) }~~

{ ~~(2, 1)~~, (6, 0), (7, 3) }

{ ~~(5, 3)~~, (6, 0), (7, 3), (8, 4) }

{ ~~(6, 0)~~, (7, 3), (8, 4) }

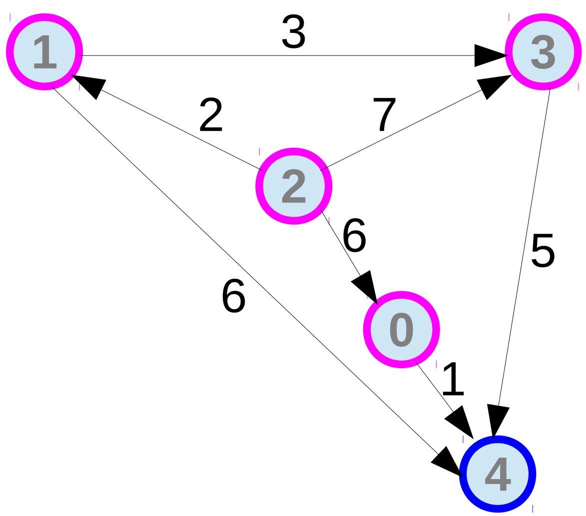
{ (7, 3), (7, 4), (8, 4) }

i	0	1	2	3	4
d[i]	6	2	0	5	7

Distance table

Poll from the PQ to get node 0.
Since we know there is a faster way to get node 4, don't bother adding node 4 to PQ

Single source shortest paths



Question: Shortest paths from 2 to all other nodes?

<u>Priority Queue (distance, node index)</u>

~~{ (0, 2) }~~

{ ~~(2, 1)~~, (6, 0), (7, 3) }

{ ~~(5, 3)~~, (6, 0), (7, 3), (8, 4) }

{ ~~(6, 0)~~, (7, 3), (8, 4) }

{ ~~(7, 3)~~, (7, 4), (8, 4) }

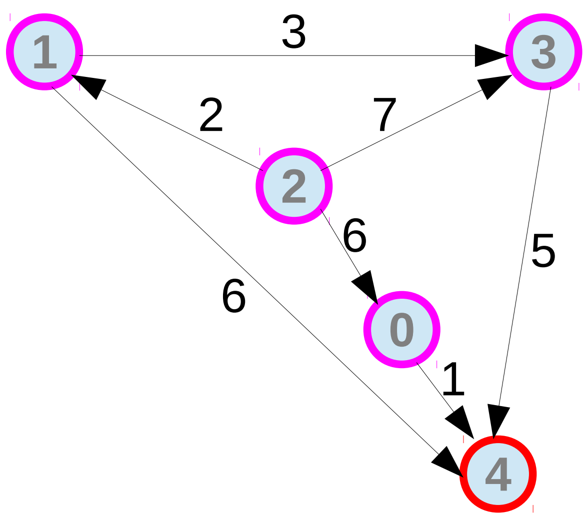
{ (7, 4), (8, 4) }

i	0	1	2	3	4
d[i]	6	2	0	5	7

Distance table

Now the (7, 3) state is ignored because it's been determined that 7 is a longer path than another existing path to node 3

Single source shortest paths



Question: Shortest paths from 2 to all other nodes?

Priority Queue (distance, node index)

~~{ (0, 2) }~~

{ ~~(2, 1)~~, (6, 0), (7, 3) }

{ ~~(5, 3)~~, (6, 0), (7, 3), (8, 4) }

{ ~~(6, 0)~~, (7, 3), (8, 4) }

{ ~~(7, 3)~~, (7, 4), (8, 4) }

{ ~~(7, 4)~~, (8, 4) }

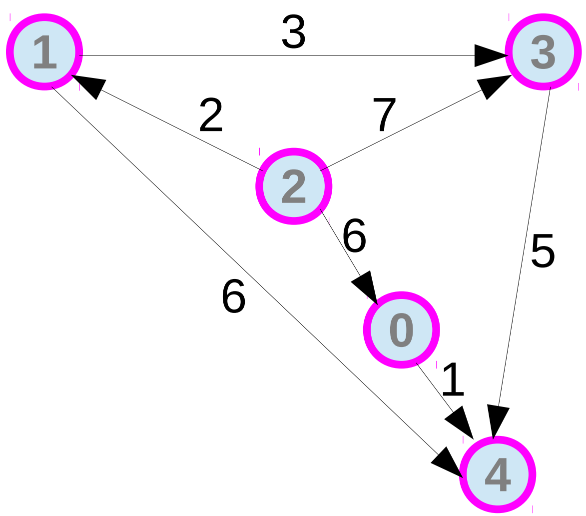
{ (8, 4) }

i	0	1	2	3	4
d[i]	6	2	0	5	7

Distance table

Nowhere to go, so nothing is added to the PQ

Single source shortest paths



Question: Shortest paths from 2 to all other nodes?

<u>Priority Queue (distance, node index)</u>

~~{ (0, 2) }~~

{ ~~(2, 1)~~, (6, 0), (7, 3) }

{ ~~(5, 3)~~, (6, 0), (7, 3), (8, 4) }

{ ~~(6, 0)~~, (7, 3), (8, 4) }

{ ~~(7, 3)~~, (7, 4), (8, 4) }

{ ~~(7, 4)~~, (8, 4) }

{ ~~(8, 4)~~ }

{ }

i	0	1	2	3	4
d[i]	6	2	0	5	7

Distance table

State (8, 4) is ignored because 8 > 7

Dijkstra's in code

```
public static void main(String[] args) {
    LinkedHashMap<Integer, LinkedHashMap<Integer, Integer>> adj;
    // State is a pair (dist, index)
    PriorityQueue<State> pq = new PriorityQueue<State>();
    int dist[] = new int[V]; Arrays.fill(dist, 1 << 20); // INF
    pq.add(new State(2, 0)); // Initial state

    while (!pq.isEmpty()) {
        State s = pq.poll();
        if (s.dist == dist[s.index]) { // true if has not been updated
            LinkedHashMap<Integer, Integer> nbors = adj.get(s.index);
            for (Map.Entry<Integer, Integer> e : nbors.getEntrySet()) {
                int nbor = e.getKey();
                int nborDist = e.getValue();

                if (nborDist + dist[s.index] < dist[nbor]) {
                    // have found a closer path
                    dist[nbor] = nborDist + dist[s.index];
                    pq.add(new State(nbor, nborDist + dist[s.index]));
                }
            }
        }
    }
}
```

Single source shortest paths

- For <u>undirected</u>, <u>weighted</u> graphs:
  - Will the same method as above work?
    - Yes

Single source shortest paths

- For directed, weighted graphs with negative weights:
  - Dijkstra's algorithm does not work and will get stuck in an infinite loop if there is a cycle, always finding a better path
  - Instead, use Bellman-Ford algorithm:
    - Repeat the “relaxing” part of Dijkstra's |V|-1 times, regardless of how close the nodes are

Single source shortest paths

```
ArrayList<ArrayList<Edge>> adjList = parseAdjList();

int dist[] = new int[N]; // Distance from node 0 to each
Arrays.fill(dist, 1 << 20); // INF
dist[0] = 0;

for (int i = 0; i < N-1; i++) {
    for (int u = 0; u < N; u++) {
        for (int j = 0; j < adjList.get(u).size(); j++) {
            Edge e = adjList.get(u).get(j);
            // min((distance to j), (distance to u) + (distance from u to j)
            dist[e.B] = Math.min(dist[e.B], dist[u] + e.w);
        }
    }
}
```

Single source shortest paths

- Why Bellman-Ford works:
  - Performing relaxing on the graph |V|-1 times guarantees shortest paths are found
    - Proof omitted
  - If relaxing can happen after |V|-1 loops, then a negative cycle exists
  - And it runs in O(|V| * |E|) time with an adjacency list, much greater than Dijkstra's

All pairs shortest paths

- What happens if you want to find the shortest distance between **all** pairs of nodes?
    - On a weighted, connected graph, use Floyd Warshall algorithm
    - Implement in ~4 lines of code
    - O(V^3) instead of N Dijkstra's algorithm, which would be O(V^3 log V)

Floyd Warshall in code

```
// inside int main()
// precondition: m[i][j] contains the weight of edge (i, j)
// or INF if there is no such edge
// (m is an adjacency matrix)

for (int k = 0; k < V; k++)
    for (int i = 0; i < V; i++)
        for (int j = 0; j < V; j++)
           m[i][j] = min(m[i][j], m[i][k] + m[k][j]);

// common error: remember that loop order is k->i->j
```

Graph algorithms

- Now you've seen the bread and butter of graph algorithms
  - There are many more problems associated with graphs
    - Find the width of a graph, find strongly connected components
  - There are special kinds of graphs and smarter algorithms for them
    - Trees, directed acyclic graphs (DAGs), bipartite graphs, eulerian graphs

Map problems tricks

- Demo in Eclipse

Readings from this class

- Readings:
  - Sections 4.1-4.5
    - Mostly what we went over in class, plus more
  - Look at Table 4.4 in the book to have a brief rundown of what we covered today

Forming Quiz Teams

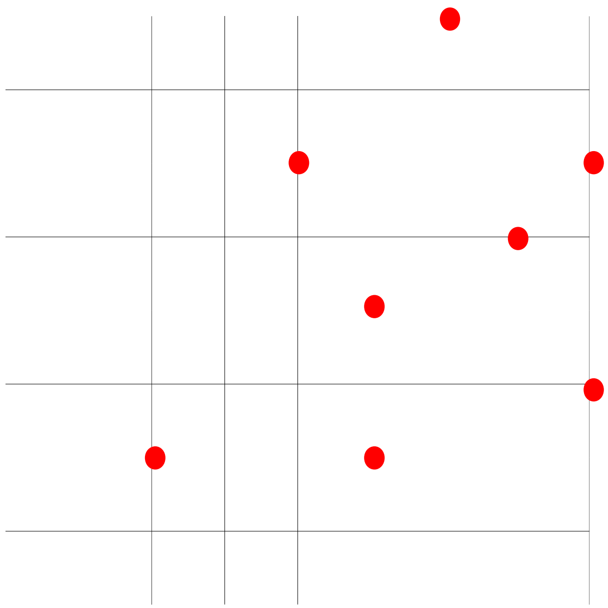
- Given 2*N points on a grid, make N pairs so that the sum of the distances of the paired points is minimized
    - 1 <= N <= 8, so 16 points

Forming Quiz Teams

- Recurrence:
  - dp[used grid points] = the minimum sum of distances between all remaining grid points
  - The answer is dp[all grid points]
  - The recursive step is to find the minimum sum by trying matching each pair of remaining grid points
  - There are a lot of overlapping states, so store the subresults

Forming Quiz Teams

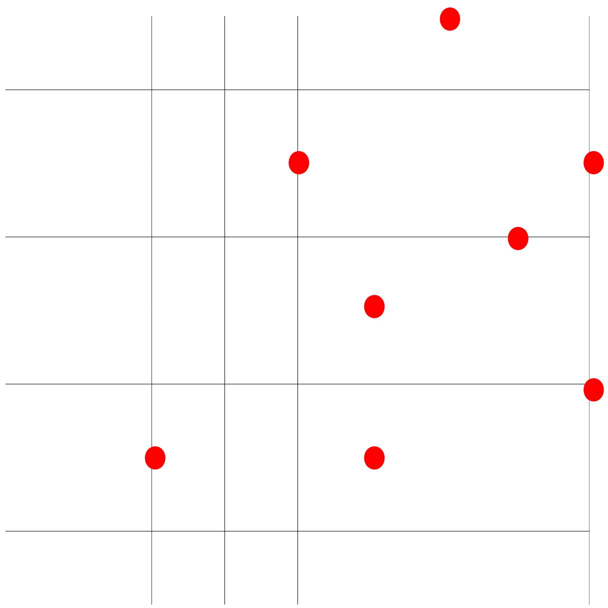
- Example:



(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Forming Quiz Teams

- Example:



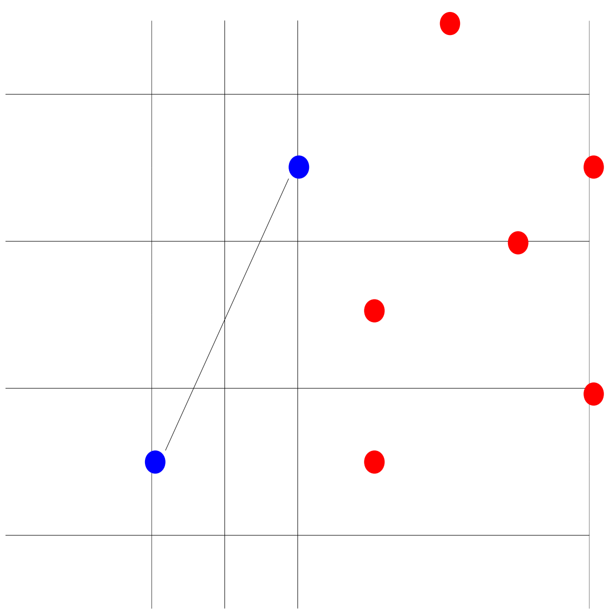
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 0

What is the minimum sum for all the points?

Forming Quiz Teams

- Example:



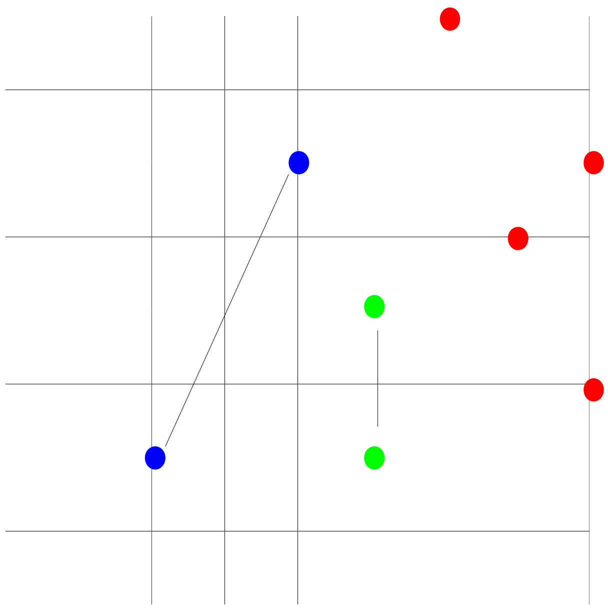
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 1

What is the minimum sum for the 2nd, 3rd, 4th, 5th, 6th, and 7th points?

Forming Quiz Teams

- Example:



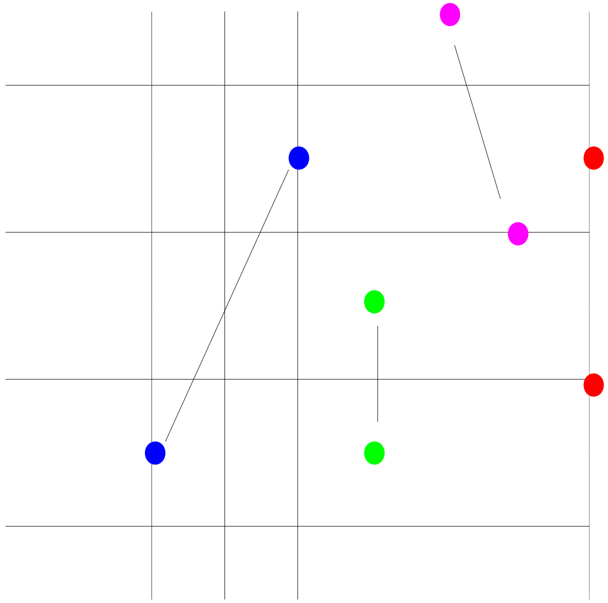
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 2

What is the minimum sum for the 4th, 5th, 6th, and 7th points?

Forming Quiz Teams

- Example:



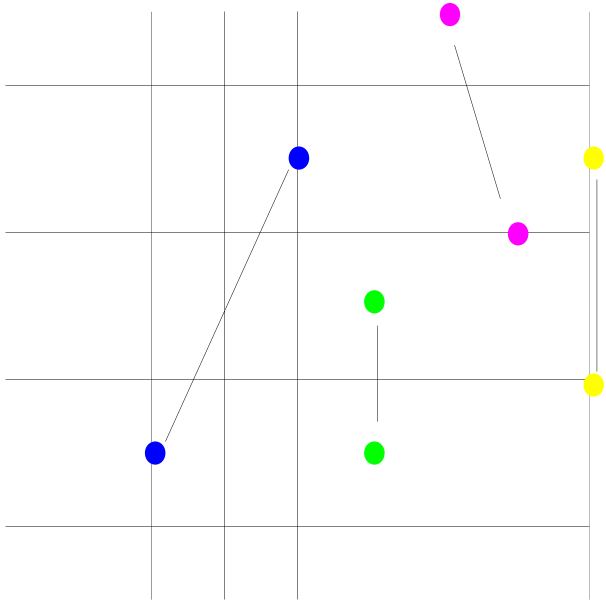
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 3

What is the minimum sum for the 6th and 7th points?

Forming Quiz Teams

- Example:



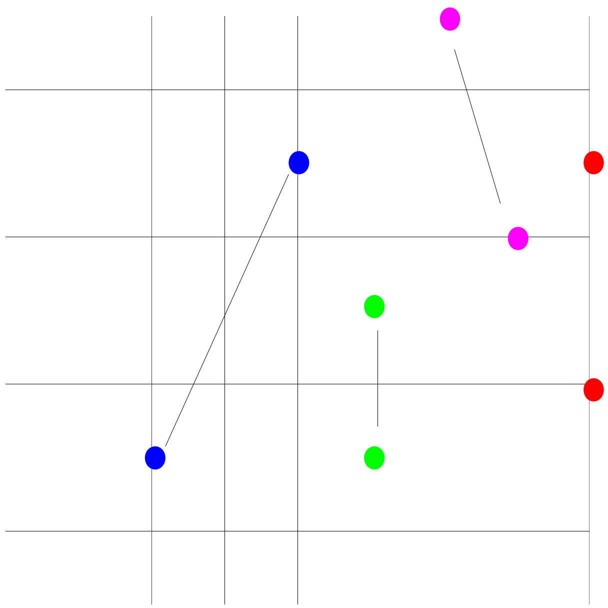
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 4

What is the minimum sum no points?
Answer: Base case, 0

Forming Quiz Teams

- Example:



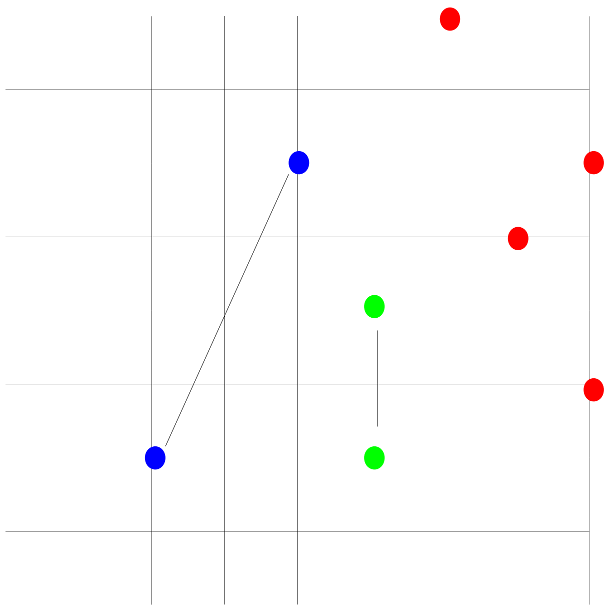
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 3

What is the minimum sum for the 6th and 7th points?
Answer: 0 + dist(P[6], P[7]) = 3

Forming Quiz Teams

- Example:



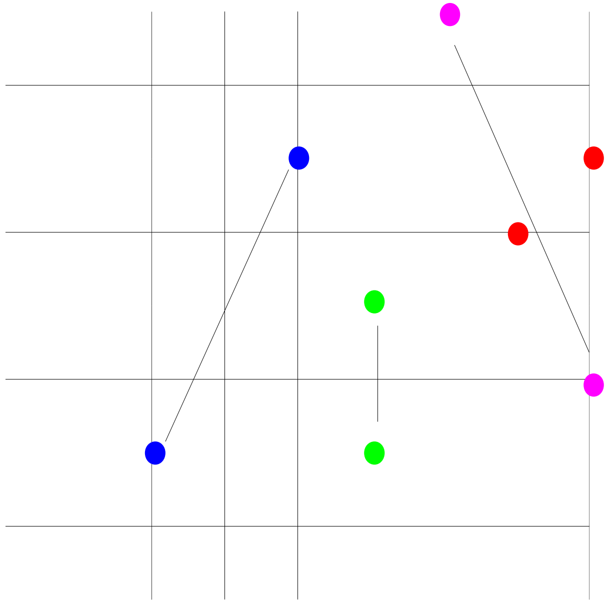
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 2

What is the minimum sum for the 4th, 5th, 6th, and 7th points?
So far: 3 + dist(P[4], P[5]) = 3 + 3.16 = 6.16

Forming Quiz Teams

- Example:



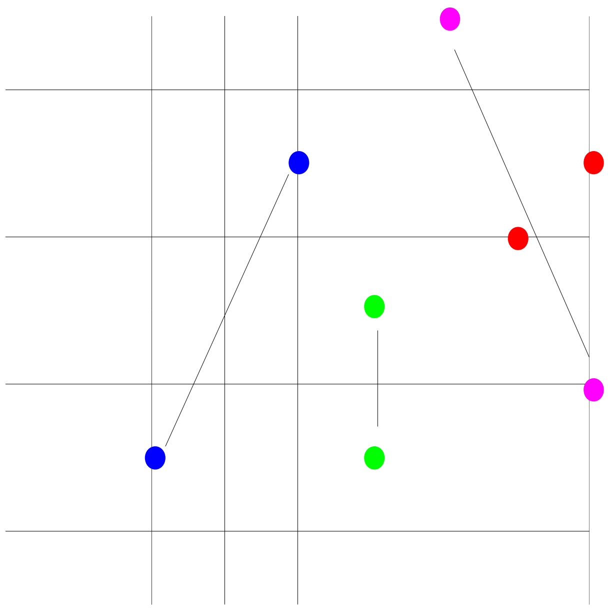
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 3

What is the minimum sum for the 5th and 7th points?

Forming Quiz Teams

- Example:



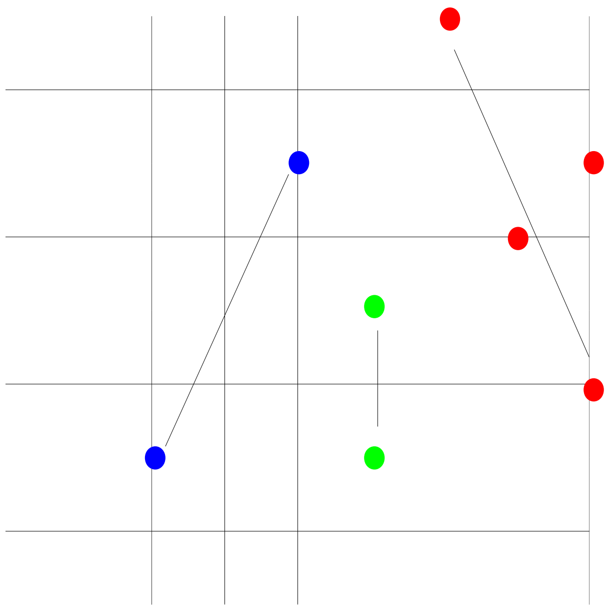
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 3

What is the minimum sum for the 5th and 7th points?
Answer: 0 + dist(P[5], P[7]) = 1.41

Forming Quiz Teams

- Example:



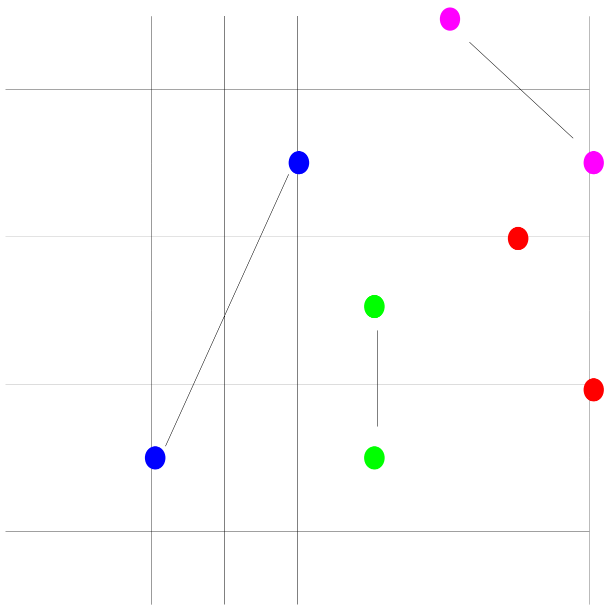
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 2

What is the minimum sum for the 4th, 5th, 6th, and 7th points?
From before: 6.16
Just computed: 1.41 + dist(P[4], P[6]) = 1.41 + 5.39 = 6.80
Still: 6.16

Forming Quiz Teams

- Example:



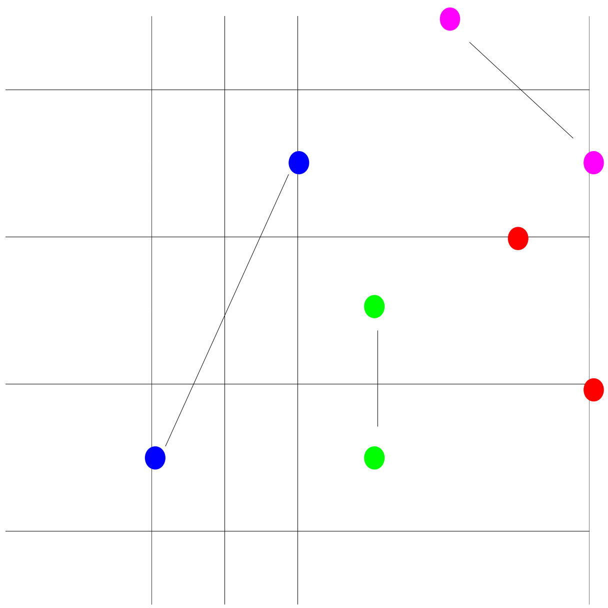
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 3

What is the minimum sum for the 5th and 6th points?

Forming Quiz Teams

- Example:



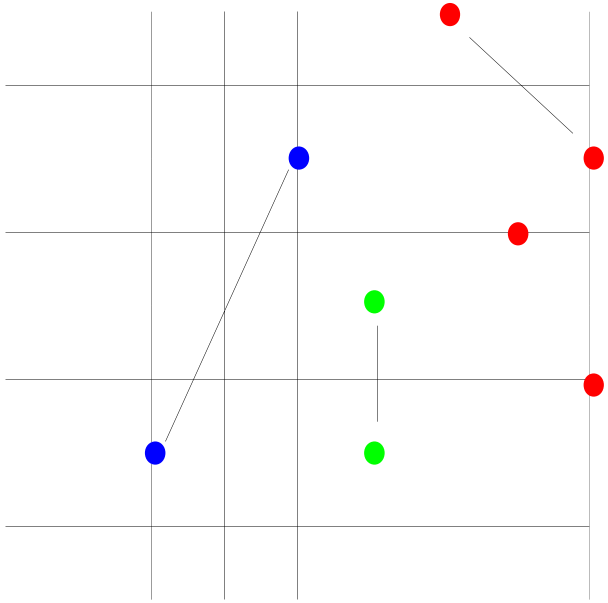
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 3

What is the minimum sum for the 5th and 6th points?
Answer: **2.24**

Forming Quiz Teams

- Example:



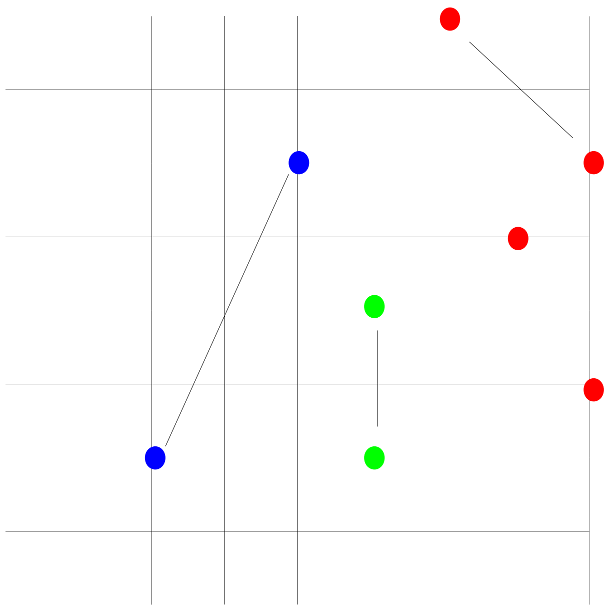
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 2

What is the minimum sum for the 4th, 5th, 6th, and 7th points?
From before: 6.16
Just computed: 2.24 + dist(P[4], P[7]) = 5.07
Now: 5.07

Forming Quiz Teams

- Example:



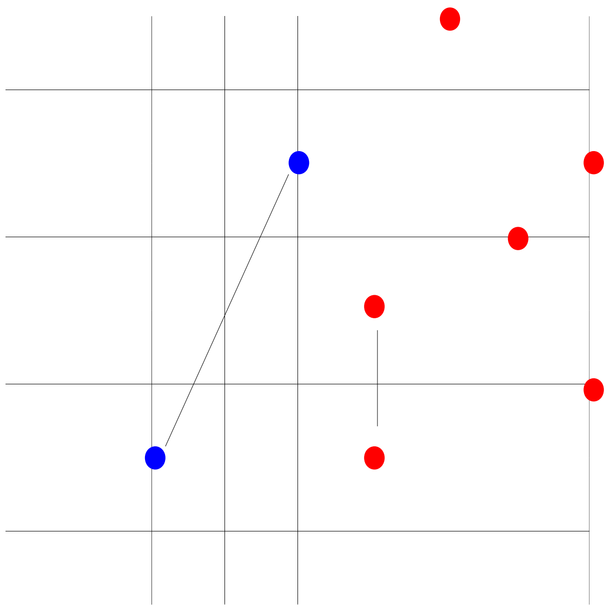
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 2

What is the minimum sum for the 4th, 5th, 6th, and 7th points?
We have tried everything, so we can definitively answer 5.07.

Forming Quiz Teams

- Example:



(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 1

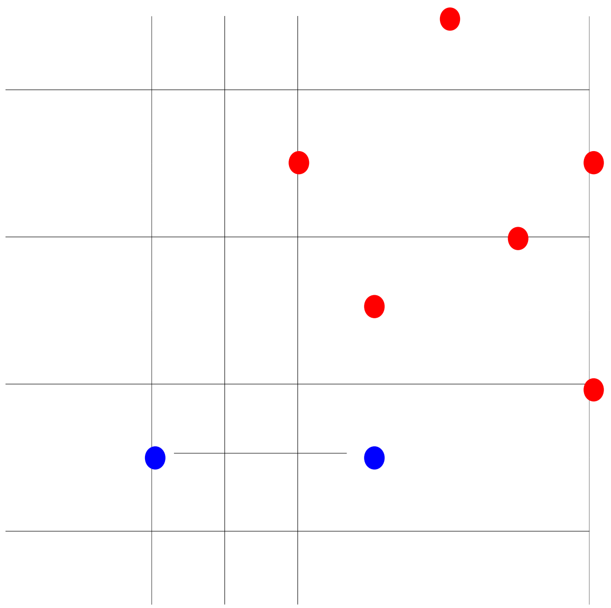
What is the minimum sum for the 2nd, 3rd, 4th, 5th, 6th, and 7th points?
So far: 5.07 + dist(P[2], P[3]) = 7.07

Forming Quiz Teams

- Fast-forward…

Forming Quiz Teams

- Example:



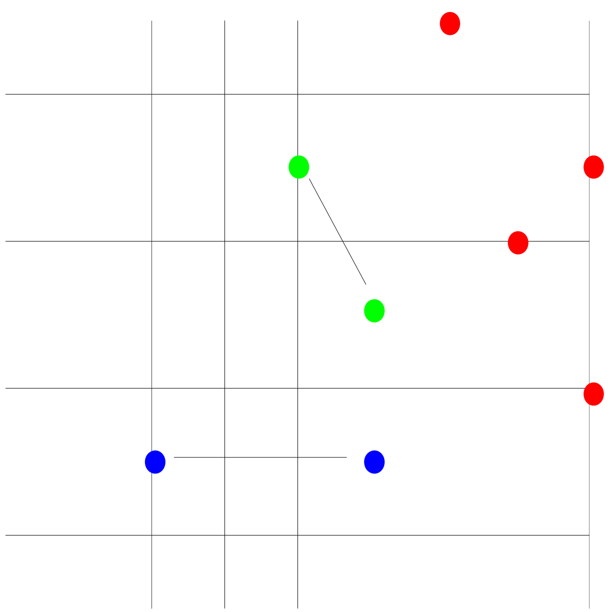
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 1

What is the minimum sum for the 1st, 3rd, 4th, 5th, 6th, and 7th points?

Forming Quiz Teams

- Example:



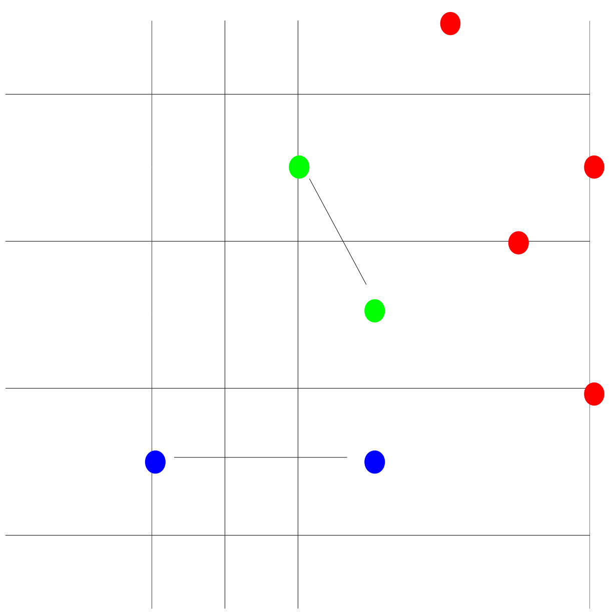
(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

Recursion depth 2

What is the minimum sum for the 4th, 5th, 6th, and 7th points?

Forming Quiz Teams

- Example:



(2, 2)
(4, 6)
(5, 2)
(5, 4)
(6, 8)
(7, 5)
(8, 3)
(8, 6)

What is the minimum sum for the 4th, 5th, 6th, and 7th points?
Answer: 5.07 (from the memoization table)

Forming Quiz Teams

- How to implement:
  - Use bitmasks!
  - Use a memoization table with 2^16 elements
  - Each entry in the table is considered a bitmask representing the set of all grid points chosen
- Another similar DP solution is the O(2^n * n) solution to the Traveling Salesman Problem

Forming Quiz Teams

```
int N;
int x[] = new int[16], y[] = new int[16]; // grid coordinates
double dp[] = new double[1 << 16];        // 2^16 entries

public double solve(int mask) {
    if (dp[mask] >= 0) return dp[mask];   // memoization step
    double res = INFINITY;

    for (int i = 0; i < 2*N; i++) {
        for (int j = i+1; j < 2*N; j++) { // filters out permutations
            if ((((1 << i) | (1 << j)) & mask) == 0) { // unused set elmnts
                double dist = sqrt(pow(x[i] – x[j], 2)
                                 + pow(y[i] – y[j], 2));
                res = min(res, solve(mask | (1 << i) | (1 << j));
            }
        }
    }

    return dp[mask] = res; // store the solution in memo table
}

public void main() {                 // left out the parsing details
    dp[(1 << (N*2)) – 1] = 0.0; // base case: all points used = 0 min dist
    System.out.printf(solve(0)); // 0 = empty bit mask = all points rem
}
```