Complex Numbers

1. What is the multiplicative inverse of the complex number $a + bi \neq 0$?
2. What is the formula for the nth roots of a given complex number $a + bi$?
3. If $a, b \in \mathbb{Z}$ (i.e., integers) what is $(a + bi)/(1 - i)$?
4. Given a complex number $a + bi$ with $a, b \in \mathbb{Z}$, determine the values $c_i \in \{0, 1\}$ such that
$$a + bi = c_0 + c_1(i - 1) + c_2(i - 1)^2 + c_3(i - 1)^3 + \cdots .$$

Solutions

1. $(a - bi)/(a^2 + b^2)$.
2. $a + bi = re^{i\theta} \implies (a + bi)^{1/n} = r^{1/n}e^{i(\theta+2\pi k)/n}$, $k = 0, \ldots, n - 1$.
3. $(a + bi)/(1 - i) = (a + bi)(1 + i)/2 = \frac{a-b}{2} + i\frac{a+b}{2}$.
4. Note that $(a + bi - c_0)/(i - 1)$ must have the form $p + qi$ for $p, q \in \mathbb{Z}$. Thus $c_0 = 1$ if and only if $a + b$ is odd.

Permutations

1. Suppose we have a card shuffling machine which takes the 52 cards 1,2,...,52 and puts them in the order $p(1), p(2), \ldots, p(52)$, for some permutation p.
   (a) After repeatedly applying the shuffler $k > 0$ times, you see the cards are back in the same order they began. Show how, given the permutation p (a list of $p(1), \ldots, p(52)$), to determine the smallest k such that the above occurs.
   (b) Give an algorithm for choosing a permutation of $1, \ldots, 52$ at random such that every permutation is equally likely.
   (c) Suppose you repeatedly shuffle randomly (each time choosing a new permutation using the previous question). What is the expected number of turns before your return to the original order?
   (d) ($\star$) Give an algorithm to determine the final deck after applying the shuffler n times, where $1 \leq n \leq 2^{63} - 1$ and you are given the initial ordering.
   (e) ($\star$) If you choose a shuffler (i.e., a permutation) at random, what is the expected number of cards that the shuffler will leave fixed? That is, their positions are unchanged each shuffle.
   (f) ($\star$) Give an algorithm to compute a permutation that must be applied the maximum number of times before the cards return to their original order.

Solutions

1. Decompose the permutation into cycles. Take LCM of cycle lengths. Assuming everything fits in a long, runtime is linear in the number of cards.

2. For $i = 1, \ldots, n$ randomly swap $a[i]$ with $a[j]$ for some $j = i, \ldots, n$. Runtime is linear in the number of cards.

3. $n!$ times: $X = 1 + (n! - 1)X/n! \implies X = n!$.

4. Decompose the permutation into cycles. Raise each cycle to the nth power. Total runtime is linear (in the number of cards).

5. Let $X_i = 1$ if $p(i) = i$ and 0 otherwise. $E(X_i) = 1/52$ so total expectation is 1.

6. Note that if you choose $p^j q^k$ as the length of one of your cycles, where p, q are prime, then it is always better to choose two different cycles of lengths p^j, q^k. Thus build a list of all primes up to 52, and run a DP algorithm to compute the maximum $p_1^{e_1} \cdots p_m^{e_m})$ where $p_1^{e_1} + \cdots + p_m^{e_m} \leq 100$. Answer is 180180.

Other Topics

1. You are given a list of points in the plane with distinct x coordinates. You start at the leftmost point, travel rightward visiting some of the points. Once you reach the rightmost point, you must pass through all missed points on the way back traveling leftward and ending at the starting point. Determine the shortest path.

2. ($\star\star$) You are given integers p, q denoting $a + b$ and ab respectively. You are also given an integer $n \geq 0$. Determine $a^n + b^n$. The result is guaranteed to fit in a 64-bit integer.

3. ($\star\star\star$) You are controlling a circular disk with center starting at $(0, 0)$, and you are given the width w of the enclosing tunnel (top wall at y-coordinate $w/2$, bottom at $-w/2$). You are also given a list of polygonal obstacles, each given as a list of vertices. The polygons may overlap each other, but will not contain $(0, 0)$. Determine the largest disk radius that still allows you to move past the obstacles (i.e., so you can get to a large x-coordinate). You may move in any direction but cannot touch an obstacle or the tunnel wall. The tunnel extends infinitely to the left and right.

Solutions

1. Think of two trips rightward instead. Solve using DP where your first path begins at p and your second begins at q. Runtime is $O(n^2)$.

2. Here a, b could be complex numbers, but that wont effect us. Note that $a^n + b^n = (a + b)(a^{n-1} + b^{n-1}) - (ab)(a^{n-2} + b^{n-2})$. Thus let $f(n) = a^n + b^n$ giving $f(n) = pf(n-1) - qf(n-2)$. Solve using matrix exponentiation.

3. Binary search on the radius of the disk. To test whether a given radius works, store a matrix of the minimum distance between each polygon (that is, the min over the pairwise distances between their edges). Consider two polygons to be connected if their distance is less than the radius. See if you can find a path in the graph from the bottom wall to the top. If so, there is an obstruction. If not, a path exists.

P vs NP

1. Boolean problems
2. NFA vs DFA
3. TMs and NTMs
4. TMs and Register Machines
5. Solving subset sum wth a Nondeterministic computer
   (a) Draw recursion tree.
   (b) Discuss runtime for a non-deterministic TM/Machine
   (c) Equivalent to first, non-deterministically selecting a binary string, and then using the binary string to make decisions (i.e., put all the non-determinism at the beginning).
6. Define P,NP (maybe mention recursive, non-recursive)
7. Show that NP equivalent to Certificate + Checker
   (a) NP $\implies$ Cert + checker : Certificate is the sequence of choices
   (b) Cert + checker $\implies$ NP : Non-det guess cert; run checker
8. Define NP-Complete
9. Cook's Theorem: SAT is NP-Complete
10. SAT polynomially reduces to 3SAT (3-CNF: conjunction of disjunctions)
11. Can reduce 3SAT to Subset Sum by encoding the truth variables into the bits of numbers