Derivative Security Markets, Market Manipulation, and Option Pricing Theory

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Abstract

This paper studies a new theory for pricing options in a large trader economy. This theory necessitates studying the impact that derivative security markets have on market manipulation. In an economy with a stock, money market account, and a derivative security, it is shown, by example, that the introduction of the derivative security generates market manipulation trading strategies that would otherwise not exist. A sufficient condition is provided on the price process such that no additional market manipulation trading strategies are introduced by a derivative security. Options are priced under this condition, where it is shown that the standard binomial option model still applies but with random volatilities.

I. Introduction

Standard option pricing theory is based on four basic assumptions. The first is that all traders have symmetric information, in that they agree on zero probability events. The second is that markets are complete. The third and fourth assumptions are that markets are frictionless and that all investors act as price takers. Under these four basic assumptions, the absence of arbitrage opportunities gives a unique option price (see Harrison and Pliska (1981) for the formulation). Recent research has endeavored to study option pricing theory under the relaxation of these four assumptions. Investigations into incomplete markets, e.g., Follmer and Schweizer (1991), Hofmann, Platen, and Schweizer (1992), and transaction costs, e.g., Bensaid, Lesne, Pages, and Scheinkman (1992), have been pursued. The purpose of this paper is to investigate option valuation under the relaxation of the price taking assumption. To facilitate understanding and comparison with the existing literature, I maintain the other three assumptions of symmetric information, complete, and frictionless markets.[1]

In relaxing the price taking assumption, I introduce the simplest possible change to the economy, the existence of a monopolist or a large trader. This introduction necessitates investigating the meaning of "no arbitrage opportunities"

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[1]A recent paper by Back (1992) investigates option valuation while simultaneously relaxing all assumptions except frictionless markets.

in this context. This, in fact, relates to the issue of market manipulation, which is a topic of growing concern to financial economists (see Jarrow (1992), Bagnoli and Lipman (1989), Allen and Gale (1992), Gastineau and Jarrow (1991), and Black (1990)). In an earlier paper, Jarrow (1992) studied the existence of market manipulation in an economy consisting of a bond and stock market. The second purpose of this paper is to extend Jarrow's study to investigate what impact, if any, the existence of derivative security markets has on market manipulation. This is a necessary prerequisite towards understanding option valuation. To study this issue, I perform the following "controlled experiment." Using the insights developed in Jarrow (1992), I construct a frictionless economy consisting of a stock and bond market where no market manipulation trading strategies exist for the large trader. Next, I introduce trading in a derivative security on the stock, which by construction, can be synthetically constructed by the large trader. This satisfies the complete markets assumption. I then study whether any additional market manipulation trading strategies exist.

Surprisingly, the answer is yes. Examples are provided to show that without additional structure, market manipulation is now possible when it was not before. One example is through market corners, obtained using derivatives to avoid aggregate stock share holding constraints. A second is through a trading strategy where the manipulator front runs his own trades to take advantage of any leads/lags in price adjustments across the stock and the derivative security market. The intuition underlying this result is that whereas before there was only one market in which to purchase the stock, now there are two (considering the derivative market as the second market). If the two markets are not perfectly aligned, manipulation is possible. The two markets are said to be in *synchrony* if the stock price adjusts instantaneously to reflect the large trader's stock holdings, including those implicit in his holdings of the derivative. Excluding corners, I show that this is a sufficient condition for there to be no market manipulation trading strategies. The satisfaction of this condition, in actual security markets, requires the efficient transmission of information across the two markets.

Given these insights, I then return to the investigation of option valuation when the price taking assumption is relaxed. I do so in an economy with a large trader, but where the stock and option markets are in *synchrony* and there are no market manipulation trading strategies. This is the analog of "no arbitrage opportunities" for the competitive case. For simplicity, I value a European call option using a binomial stock price process. Here, I show the important result that the standard binomial option model still applies, but with a random volatility. Price takers, under a common knowledge assumption, can still apply the standard pricing and hedging procedures to obtain the correct option values. Only the justification for the validity of these techniques is changed. This demonstrates a remarkable robustness of the standard binomial model. Due to the reason for the random volatility, this new theory has the potential to explain some previously puzzling empirical deviations of market prices from the standard Black-Scholes formula.

An outline of this paper is as follows. Section II constructs the economy and summarizes those results needed for the subsequent analysis. Examples of market manipulation trading strategies are provided in Section III. Section IV defines synchronous markets and shows that such markets contain no market manipulation trading strategies. Section V provides the theory of option pricing in a large trader economy, while Section VI concludes the paper.

II. The Model

This section extends the basic model in Jarrow (1992) to include a derivative security. I consider a discrete trading economy with trading dates $\tau = \{0, 1, \ldots, T\}$. The uncertain state of the economy at date T is captured by the state space Ω. The *large trader's* or *speculator's* information flows (about the true state of the economy) over time are represented by the filtration,[2] $\{F_t : t \in \tau\}$. The large trader is endowed with a probability measure, $P: F_T \rightarrow [0, 1]$, representing his subjective beliefs.

Three assets trade. The first is a limited liability risky asset, called a stock, whose price follows a stochastic process,[3] $\{S_t : t \in \tau\}$, adapted to $\{F_t : t \in \tau\}$. This stock pays no dividends and has N shares outstanding for all $t \in \tau$.

The second asset is a money market account. The value of the money market account is represented by the stochastic process, $\{B_t : t \in \tau\}$, which is predictable[4] and initiated with $B_0 \equiv 1$. Predictability means that B_t is known to the speculator at time $t - 1$. I assume that $B_{t+1} \geq B_t$ for all $\omega \in \Omega$ so that spot interest rates are nonnegative. For convenience, I let the money market account serve as the numeraire, and define the relative stock price to be $Z_t \equiv S_t / B_t$. The stochastic process $\{Z_t : t \in \tau\}$ is adapted and nonnegative.

The third asset is a zero net supply, derivative security. For simplicity, I consider only European call options and forward contracts on the stock. The analysis, however, is easily extended to more exotic and path-dependent derivatives. The defining characteristic of these derivative securities is their dollar value at time T, i.e., $[S_T(\omega) - K]$ or $[S_t(\omega) - K]^+$ where $K > 0$ is a strictly positive constant. At this point in the analysis, I make no distinction as to whether the derivative security has cash or physical delivery at expiration. This distinction, however, is relevant when considering market manipulation trading strategies and is discussed in Section III below.

Let $\{D_t : t \in \tau\}$ represent the stochastic process for the derivative security's value, which is adapted to $\{F_t : t \in \tau\}$. By assumption, the derivative security has no cash flows prior to time T. Define its relative price to be $C_t \equiv D_t / B_t$.

The speculator's holdings of the stock, money market account, and derivative security are denoted by a three-dimensional adapted stochastic process $\{(\alpha_t, \beta_t, \gamma_t) : t \in \tau\}$, with α_t equal to the number of shares of the stock, β_t equal to the number of money market account units, and γ_t the number of derivative securities held at time t. This stochastic process is called a trading *strategy*. I endow the speculator with zero initial holdings of the stock, money market account, and the derivative security. This is for convenience.

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[2]A filtration is a nondecreasing family of σ-algebras. Without loss of generality, assume F_t for $t \leq T$ contains all events $A \subset \Omega$ s.t. $A \subset B \in F_t$ and $P(B) = 0$.

[3]A stochastic process $\{S_t : t \in \tau\}$ is a mapping $S: \Omega \times \tau \rightarrow R$. It is adapted if S_t is F_t-measurable for all $t \in \tau$.

[4]A stochastic process $\{B_t : t \in \tau\}$ is predictable if B_t is F_{t-1}-measurable for all $t \in \tau$.

A trading strategy is said to be *self-financing* if

$$\text{(1)} \qquad \beta_{t-1} + \alpha_{t-1} Z_t + \gamma_{t-1} C_t \quad = \quad \beta_t + \alpha_t Z_t + \gamma_t C_t \quad \text{a.s. } P \text{ for all } t \in \tau,$$

where, as a matter of notation, $\alpha_{-1} = \beta_{-1} = \gamma_{-1} = 0$.

By convention, the portfolio position $(\beta_{t-1}, \alpha_{t-1}, \gamma_{t-1})$ is held between time $(t-1)$ and time t. This definition of self-financing parallels the standard one, but contains two subtle differences. First, the relative prices Z_t and C_t are influenced by the trading strategy $(\alpha_t, \beta_t, \gamma_t)$ at time t. This implies the second difference. Self-financing trading strategies, when summed or multiplied by a scalar, need not be self-financing.

For simplicity of notation, characters with a superscript represent a history vector beginning at time 0 and ending at time t, i.e., $\alpha^t(\omega) \equiv (\alpha_t(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega))$ for all $\omega \in \Omega$, $t \in \tau$, with $\beta^t(\omega)$ and $\gamma^t(\omega)$ defined similarly. Furthermore, define a vector of zeros by a bold character, i.e., $\mathbf{0} \equiv (0, 0, \ldots, 0)$. The dimension of the zero vector will be clear from the context in which it is used.

Let Φ denote the set of all self-financing trading strategies $\{(\alpha_t, \beta_t, \gamma_t) \colon t \in \tau\}$. Also, let Φ_0 denote the subset of these strategies with identically zero holdings of the derivative security for all times $t \in \tau$, i.e., $\gamma_t \equiv 0$. It represents an active position only in the stock and the money market fund.

I now characterize the economy through Assumptions A.1–A.5.

A.1. Frictionless Markets

There are no transaction costs nor short sale restrictions imposed upon the large trader's holdings $\{\alpha_t, \beta_t, \gamma_t \colon t \in \tau\}$.

A.2. The Relative Stock Price Process

There exists a sequence of functions $\{g_t\}_{t \in \tau}$ with $g_t \colon \Omega \times R^{2(t+1)} \to R$ for all $t \in \tau$ such that for any trading strategy $\{\alpha_t, \beta_t, \gamma_t \colon t \in \tau\}$ of the large trader, the composition mapping $Z \colon \Omega \times \tau \to R$ defined by $Z_t(\omega) = g_t(\omega, \alpha^t(\omega), \gamma^t(\omega))$ for all $\omega \in \Omega$, $t \in \tau$, represents the stochastic process for the relative stock price. It is nonnegative and $\{F_t \colon t \in \tau\}$ adapted. Assume that the large trader knows the sequence of functions $\{g_t\}_{t \in \tau}$.

This assumption is very weak and merely determines the reaction function of the market to the speculator's trades, as reflected in the equilibrium price. The equilibrium price is assumed to be only a function of the large trader's holdings in the stock and the derivative security.

It is important to emphasize that the speculator's trading strategy $\{\alpha_t, \beta_t, \gamma_t \colon t \in \tau\}$ is a function of the state of the world $\omega \in \Omega$. This, however, is a very general specification. For example, this includes as a special case, trading strategies that can be written as functions of the relative stock price process $\{Z_t \colon t \in \tau\}$. To see this, note that Assumption A.2 implies that given $\{\alpha_t, \beta_t, \gamma_t \colon t \in \tau\}$, the relative stock price is a deterministic function of ω. Thus, for a given trading strategy $\{\alpha_t, \beta_t, \gamma_t \colon t \in \tau\}$, the relative stock price process $\{Z_t \colon t \in \tau\}$ generates an information filtration. If the speculator's trading strategy happens to be measurable with respect to the information filtration generated by $\{Z_t \colon t \in \tau\}$, then the trading strategy $\{\alpha_t, \beta_t, \gamma_t \colon t \in \tau\}$ can alternatively be written as a function of the stock price process $\{Z_t \colon t \in \tau\}$. This alternate specification would implicitly incorporate any simultaneity due to the relative stock price depending on the trading strategy $\{\alpha_t, \beta_t, \gamma_t: t \in \tau\}$.

Assume that the large trader knows the sequence of price functions $\{g_t\}_{t \in \tau}$. The exogenous specification of the price process as in Assumption A.2, however, does not make explicit the information and the structure of the economy that is common knowledge to the remaining traders. An explicit determination of what is common knowledge and to whom requires a complete specification of the underlying equilibrium (or perhaps, disequilibrium) economy. Without such a construction, a number of interesting and important issues remain indeterminate.[5] An illustration of this indeterminacy is presented in Section V below. This indeterminacy is a weakness of the partial equilibrium analysis. Paradoxically, it is also its strength, as it allows the analysis to be simultaneously consistent with numerous different common knowledge structures and equilibrium constructs. For additional elaboration of Assumption A.2, see Jarrow (1992).

A.3. Speculator as a "Large Trader"

For all $t \in \tau$, $\{\alpha_t, \beta_t, \gamma_t: t \in \tau\} \in \Phi_0$ and $\omega \in \Omega$,

a) if $\alpha_t(\omega) > \alpha_{t-1}(\omega)$ then $g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega); \mathbf{0}) > g_t(\omega; \alpha_{t-1}(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega); \mathbf{0})$,

b) if $\alpha_t(\omega) < \alpha_{t-1}(\omega)$ and $[\alpha_{t-1}(\omega) \leq N$ or $\alpha_t(\omega) < N]$ then $g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega); \mathbf{0}) < g_t(\omega; \alpha_{t-1}(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega); \mathbf{0})$.

This is Assumption A.3 of Jarrow (1992) with the derivative security holdings set identically equal to zero. This assumption states that as the speculator's stock holdings increase, everything else constant, relative prices increase. Similarly, as his share holdings decrease, everything else remains constant *and* there is no market corner and short squeeze, then the relative price decreases. A *market corner* and *short squeeze* in state $\omega \in \Omega$ at time $t \in \tau$ is defined as a pair of shares holding α_t, α_{t-1} such that $[\alpha_{t-1}(\omega) > \alpha_t(\omega) \geq N]$. A market corner and short squeeze are excluded from the specification of the equilibrium price process because when they occur, the market process breaks down.

A.4. No Arbitrage Opportunities Based on the Speculator's Information

a) Let the remaining set of traders be indexed by the set I and endowed with probability beliefs P_i equivalent to P for all $i \in I$.[6]

b) There exists an equivalent probability measure $\bar{P}$ such that for all $t \in \tau$ and $\{\alpha_t, \beta_t, \gamma_t: t \in \tau\} \in \Phi_0$, if $\alpha_t(\omega) = \alpha_{t+1}(\omega)$ a.s. then $\bar{E}\{g_{t+1}(\omega; \alpha_{t+1}(\omega), \alpha_t(\omega), \ldots, \alpha_0(\omega); \mathbf{0}) | F_t\} = g_t(\omega; \alpha_t(\omega), \alpha_{t-1}(\omega), \ldots, \alpha_0(\omega); \mathbf{0})$ a.s. where $\bar{E}(\cdot)$ denotes expectation with respect to $\bar{P}$.

This assumption imposes some additional structure on the remaining set of traders in the economy. The remaining set of traders is indexed by the set $i \in I$ and endowed with equivalent probability beliefs P_i. Each of these traders possesses

[5]An example is whether the price function $g_t(\omega, \alpha^t(\omega), \gamma^t(\omega))$ is common knowledge to the remaining traders. Even if it is, the large trader's trades $(\alpha^t(\omega), \gamma^t(\omega))$ need not be common knowledge nor even deductible in an equilibrium. This follows because the realization of $g_t(\omega, \alpha^t(\omega), \gamma^t(\omega))$ need not be invertible in $(\alpha^t(\omega), \gamma^t(\omega))$.

[6]Two probability measures are said to be *equivalent* if they agree on zero probability events. For the two equivalent probability measures P and P_i, "P_i-a.s." and "P-a.s." have the same meaning, hence, for brevity, I subsequently write these as "a.s."

an information filtration $\{F_t^i: t \in \tau\}$, which has no prespecified relationship with respect to the large trader's information $\{F_t: t \in \tau\}$, i.e., they can be "smaller," "larger," or neither. The information sets $\{F_t^i: t \in \tau\}$ are assumed to contain the information set (σ-algebra) generated by the price processes $\{S_t, B_t, D_t: t \in \tau\}$. In Example 2 below and in Section V, I view this set of traders as "price takers."

This assumption implies that if the speculator's information were available to a price taker and if the speculator's holdings in the stock over the trading period $[t, t+1]$ were held constant, then no arbitrage opportunities exist. This is the extension of the basic "symmetric information" and no arbitrage opportunity assumption of standard option pricing theory. This assumption is quite mild, and satisfied by most models used in financial economics. Again, for a more detailed elaboration of this assumption, see Jarrow (1992).

A.5. Price Process Independence of Large Trader's Past Holdings

For all $t \in \tau$, $\omega \in \Omega$ and for all $\{\alpha_t, \beta_t, \gamma_t: t \in \tau\}$, $\{\hat{\alpha}_t, \hat{\beta}_t, \hat{\gamma}_t: t \in \tau\} \in \Phi_0$, if $\alpha_t(\omega) = \hat{\alpha}_t(\omega)$, then $g_t(\omega, \alpha^t(\omega), 0) = g_t(\omega, \hat{\alpha}^t(\omega), 0)$.

This assumption is imposed because, subsequently, it is sufficient to guarantee that market manipulation trading strategies do not exist using the stock and money market fund alone. An example of an exogenously specified price process satisfying Assumptions A.1–A.5 is given by the following.

Example 1. Price Process Satisfying A.1–A.5

For $\{\alpha_t, \beta_t, \gamma_t: t \in \tau\} \in \Phi$, $\omega \in \Omega$, and $t \in \tau$, let

$$g_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right) = e^{\eta\left[\alpha_t(\omega)+\delta_t(\omega)\gamma_t(\omega)\right]} y_t(\omega), \tag{2}$$

where $\eta > 0$ is a positive constant, $\{\delta_t: t \in \tau\}$ is a nonnegative stochastic process adapted to $\{F_t: t \in \tau\}$, and $\{y_t: t \in \tau\}$ is a nonnegative stochastic process adapted to $\{F_t: t \in \tau\}$, which is a $\bar{P}$-martingale.

I interpret $\{y_t: t \in \tau\}$ as the fundamental process underlying the relative price, and $\{e^{\eta[\alpha_t(\omega)+\delta_t(\omega)\gamma_t(\omega)]}: t \in \tau\}$ as the speculator-controlled portion of the price. The difference between $g_t(\omega)$ and $y_t(\omega)$ is a bubble. Inspection of Expression (2) reveals that Assumptions A.1–A.5 are satisfied. □

The next assumption characterizes the derivative security in this economy. This is the analog of the complete market assumption in standard option pricing theory.

A.6. Characterization of the Derivative Security

There exists unique, nonnegative, Borel measurable $n_t: \Omega \times R^{2(t+1)} \rightarrow R$, $m_t: \Omega \times R^{2(t+1)} \rightarrow R$ for all $t \in \tau$ such that given $(\alpha_t, \beta_t, \gamma_t) \in \Phi$ where $\alpha_t + n_t\gamma_t \leq N$ for all $t \in \tau$,

$$C_t(\omega) = n_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right) Z_t(\omega) - m_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right), \quad \text{and} \tag{3}$$

$$\begin{aligned} & n_{t-1}\left(\omega, \alpha^{t-1}(\omega), \gamma^{t-1}(\omega)\right) Z_t(\omega) - m_{t-1}\left(\omega, \alpha^{t-1}(\omega), \gamma^{t-1}(\omega)\right) \\ &= n_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right) Z_t(\omega) - m_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right), \end{aligned} \tag{4}$$

for $\omega \in \Omega$ and $t \in \tau$, where

$$C_T(\omega) = \left(Z_T(\omega) - K/B_T(\omega)\right)^+ \quad \text{or} \tag{5a}$$

(5b) $$C_T(\omega) \quad = \quad Z_T(\omega) - K/B_T(\omega),$$

for $\omega \in \Omega, K > 0$ a positive constant.

Assumption A.6 characterizes the market determined price for the derivative security (C_t), relative to the market determined stock price (Z_t). The derivative security corresponds to one of two examples; either a European type call option on the stock or a forward contract on the stock. Expression (3) gives the unique trading strategy in the stock and money market account that duplicates the derivative security: (n_t) shares of the stock and ($-m_t$) units of the money market account. Expression (4) guarantees that this trading strategy is self-financing. For simplicity, I arbitrarily choose the derivative security to be equivalent to a levered position in the stock. This explains the nonnegativity restrictions on n_t and m_t. Generalizations to other derivative securities (exotic or otherwise) follow in a straightforward manner.

Assumption A.6 contains a qualification. The market process operates *only if* the large trader has not cornered the market and is squeezing the shorts. The qualification is obtained via the *limit position* constraint imposed on the total holdings of the stock plus derivative security ($\alpha_t + n_t\gamma_t \leq N$ for all $t \in \tau$). Given physical delivery settlement, this constraint cannot be relaxed. For example, consider a call option on the stock that expires out-of-the-money. Its stock equivalence in value is zero. Yet, the contract could be exercised at a loss, in order to obtain physical delivery of the stock. If these shares could be used to simultaneously create a corner and squeeze the shorts, then the market price for the call would not reflect its private value to the large trader. This private value is equal to the wealth extractable from the squeezed shorts due to the corner (obtained via physical delivery). This argument provides a justification for implementing cash delivery settlement, rather than physical delivery settlement in the specification of an option's contract. In the case of cash settlement, the limit constraint can be relaxed to ($\alpha_t \leq N$ for all $t \in \tau$).

From Expression (3) and Assumption A.2, see that the derivative security's price depends on the large trader's holdings $\{\alpha_t, \beta_t, \gamma_t, t \in \tau\}$ through $Z_t(\omega) = g_t(\omega, \alpha^t(\omega), \gamma^t(\omega))$. To simplify the notation, define a sequence of functions c_t: $\Omega \times R^{2(t+1)} \to R$ for $t \in \tau$ such that

(6) $$\begin{aligned} c_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right) \quad = \quad & n_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right) g_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right) \\ & - m_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right), \end{aligned}$$

$$\text{for all } \omega \in \Omega, \text{ and } \{\alpha_t, \beta_t, \gamma_t : t \in \tau\} \in \Phi.$$

Example 2. A Theory for Pricing Forward Contracts

This example illustrates Assumption A.6 with respect to forward contracts in a one-period economy. The analysis is easily generalized to multiple periods.

Consider a forward contract on the stock with expiration date $T = 1$. For this example, it does not matter whether there is cash or physical delivery.

By definition, the normalized value of the forward contract at time 1 is

(7) $$c_1(\omega;\alpha_1(\omega),\alpha_0;\gamma_1(\omega),\gamma_0) \quad = \quad g_1(\omega;\alpha_1(\omega),\alpha_0;\gamma_1(\omega),\gamma_0) - K/B_1.$$

The contract equals the value of the stock ($g_1(\omega)$), less the forward price, (K/B_1).

Next, value the contract at time 0. To obtain this value, add some additional structure. First, let the remaining set of traders indexed by the set I be price takers. Assume that they also trade without frictions. Finally, assume that there are no arbitrage opportunities for price takers with respect to the price process described by Assumptions A.1–A.5. This additional assumption is distinct from Assumption A.5, which concentrated on the large trader's behavior.

Given these additional assumptions, the standard techniques can now be applied by price takers to construct a synthetic forward. They simply buy the stock, store it, and borrow K/B_1 dollars to finance the purchase. This strategy duplicates the forward contract's time 1 payoff, therefore, no arbitrage implies

$$c_0(\alpha_0;\gamma_0) \quad = \quad g_0(\alpha_0;\gamma_0) - K/B_1. \tag{8}$$

By market convention, the forward price K is set such that the time 0 value of the forward contract is zero, i.e., $g_0(\alpha_0;\gamma_0) - K/B_1 = 0$.This completes the derivation.

Note that using the notation of Assumption A.6, $n_0(\alpha_0,\gamma_0) = n_1(\omega;\alpha_1(\omega),\alpha_0;\gamma_1(\omega),\gamma_0) = 1$ and $m_0(\alpha_0,\gamma_0) = m_1(\omega;\alpha_1(\omega),\alpha_0;\gamma_1(\omega),\gamma_0) = K/B_1$. Hence, Expressions (7) and (8) correspond to Expressions (3) and (4), and Assumption A.6 is satisfied.

This no arbitrage argument was possible because price takers could replicate the forward contract synthetically, independent of the large trader's position. This same argument cannot be extended, however, to price call options. A more elaborate pricing theory needs to be developed, and this is done in Section V below. □

The large trader's *real wealth* at time $t \in \tau$ under state $\omega \in \Omega$ for a trading strategy $\{\alpha_t, \beta_t, \gamma_t: t \in \tau\} \in \Phi$ is defined by

$$\begin{aligned} V_t(\omega) \quad \equiv \quad & \alpha_{t-1}(\omega) g_t\left(\omega; 0, \alpha^{t-1}(\omega); 0, \gamma^{t-1}(\omega)\right) + \beta_{t-1}(\omega) \\ & + \gamma_{t-1}(\omega) c_t\left(\omega; 0, \alpha^{t-1}(\omega); 0, \gamma^{t-1}(\omega)\right). \end{aligned} \tag{9}$$

Real wealth corresponds to the liquidation value of the portfolio at time t in units of the money market account. It is the liquidation value at time t since in Expression (9), I have set both $\alpha_t(\omega) \equiv 0$ and $\gamma_t(\omega) \equiv 0$.

A *market manipulation trading strategy* is defined to be any self-financing trading strategy $\{\alpha_t, \beta_t, \gamma_t: t \in \tau\} \in \Phi$ such that

$$V_0 \quad = \quad 0, \quad V_T \quad \geq \quad 0 \quad \text{a.s. and} \quad P(V_T > 0) \quad > \quad 0. \tag{10}$$

This is an arbitrage opportunity in real wealth.

Given this economy, the following proposition is true.[7]

[7]To interpret this economy in the context of a full equilibrium model, consider embedding this structure into an extended economy. Let the extended economy contain an additional time period $(T+1)$ with a correspondingly augmented state space and filtration. Let there be no trading at time $T+1$, and let the asset prices $(S_{T+1}(\omega), B_{T+1}(\omega), D_{T+1}(\omega))$ be determined entirely by the state ω, i.e., they are exogenously determined and independent of the traders' actions or the equilibrium. In this setting, real wealth and paper wealth are identical at time $T+1$. This extended economy can be studied using standard techniques.

The structure in the paper and all the examples provided can be considered in the context of this expanded economy. I thank the referee for pointing out this interpretation.

Proposition 1. No Market Manipulation in Stock and Money Market Account

Given Assumptions A.1–A.5, there are no market manipulation trading strategies $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\} \in \Phi_0$ with $\alpha_t \leq N$ a.s. for all $t \in \tau$.

Proof. This is a direct application of the proposition in Jarrow (1992). To see this, identify Φ with Φ_0, $g_t(\omega, \alpha^t(\omega))$ with $g_t(\omega, \alpha^t(\omega), \mathbf{0})$, and Assumptions A.1–A.5 with A.1–A.5 of Jarrow (1992). □

This proposition states that under Assumptions A.1–A.5, if the speculator only considers trading strategies with *zero* shares of the derivative security, then there are no market manipulation trading strategies in the stock and money market alone (as long as corners in the stock holdings are excluded by keeping $\alpha_t \leq N$). But, what happens when trading in a derivative security is considered? The next section demonstrates, by example, that under Assumptions A.1–A.6, market manipulation strategies exist involving the derivative security.

III. Market Manipulation Using the Derivative Security

This section demonstrates, by example, that the introduction of a derivative security market can introduce market manipulation possibilities that otherwise would not exist. Two examples are provided. The first illustrates how market corners can be obtained using the derivative security to circumvent the explicit share holding constraint ($\alpha_t \leq N$ for all $t \in \tau$). The second is a form of "front-running," where the speculator front runs his own trades to his advantage.

Example 3. Market Corners Using Derivative Securities

This possibility can be illustrated with a simple two-period model ($T = 2$). Let the derivative security be a *physical delivery*, forward contract on the stock that matures at time 1. By construction, at date 1, the forward contract is equal to one share of the stock (physical delivery) less the dollar forward price of the contract (denoted K). K is determined in the market so that the time 0 forward price, $c_0(\omega; N, 1) = 0$. See Example 2 for details.

Consider the following zero investment, self-financing trading strategy where $(\beta_2, \beta_1, \beta_0)$ are defined by Expression (1) and $(\alpha_2, \alpha_1, \alpha_0) = (0, N-1, N)$, $(\gamma_2, \gamma_1, \gamma_0) = (0, 1, 1)$. At time 0, the large trader "effectively" owns $(N + 1)$ shares of the stock. At time 1, the speculator "calls in the short" by reducing his holdings. The short position in the forward contract needs to cover his position. He goes to the market and must buy from the large trader (who owns all N shares of the stock). The speculator therefore determines the market price at time 1, $g_1(\omega; N-1, N; 1, 1)$, because he *is* the market. Note, I am using a convention that at time 1, the share in the stock received from the forward contract is accounted for by $\gamma_1 = 1$. The speculator's time 2 real wealth is (the calculations are given in the Appendix)

$$\text{(11)} \qquad V_2(\omega) = N\left[g_2(\omega; 0, N-1, N; 0, 1, 1) - g_0(N; 1)\right] + g_1(\omega; N-1, N; 1, 1) - \left(K/B_1\right).$$

Since the speculator can determine $g_1(\omega; N-1, N; 1, 1)$ arbitrarily and $g_2(\omega; 0, N-1, N; 0, 1, 1) \geq 0$ a.s., if the speculator chooses $g_1(\omega; N-1, N; 1, 1) = M +$

$Ng_0(\omega; N; 1) + K/B_1)$ where $M > 0$ is a large constant, then $V_2(\omega) \geq M > 0$ a.s. By construction, this is a market manipulation trading strategy.

Notice that this strategy always satisfies the aggregate share constraint since $\alpha_0 \leq N, \alpha_1 \leq N$, and $\alpha_2 \leq N$. This is true even including the time 1 delivery of the share from the forward contract. By using the derivative security, the speculator "avoids" the share constraint to corner the market and create a short squeeze.

Because the speculator avoids the market mechanism through such a strategy, exclusion of this manipulation requires government regulation. The approach used in Assumption A.6 is to impose aggregate share restrictions *jointly* on $(a_t + n_t\gamma_t)$ such that manipulation is excluded. A second approach is to require *cash delivery* settlement rather than physical delivery settlement of the stock. Combining this condition with the previous regulatory constraint that ($\alpha_t \leq N$ for all $t \in \tau$) would remove these manipulation strategies as well. □

Example 4. The Speculator Front Running His Own Trades

This example illustrates that Assumptions A.1–A.6 allow market manipulation, even excluding corners. The rough intuition underlying the example is that the speculator, because of a lagged price adjustment across the two markets, can "front run" his own trades.

Consider a single-period economy ($T = 1$) where the derivative security is again a forward contract on the stock. It does not matter whether there is cash or physical delivery. The forward contract matures at time 1, at which time the forward contract yields the stock less the time 0 forward price, denoted K. The forward price is determined such that the time 0 value of the forward contract is zero. I impose the additional assumptions contained in Example 2 so that Expressions (7) and (8) are satisfied, and Assumption A.6 holds.

Suppose the speculator faces the following price process: given $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\} \in \Phi$,

$$g_0(\alpha_0; \gamma_0) = \begin{cases} 1/2 & \text{if } \alpha_0 = 0,\ \gamma_0 = 1 \\ \exp(\alpha_0 + \gamma_0) & \text{otherwise,} \end{cases} \tag{12}$$

$$\text{and} \quad g_1(\omega; \alpha_1(\omega), \alpha_0(\omega); \gamma_1(\omega), \gamma_0(\omega)) = \exp(\alpha_1(\omega) + \gamma_1(\omega))\, y(\omega), \tag{13}$$

where $y(\omega)$ is F_1-measurable and satisfies $1/2 < y(\omega)$ a.s. with $\bar{E}(y(\omega)) = 1$, where $\bar{P}$ is defined as in Assumption A.4.

First, I show that Expressions (12) and (13) satisfy Assumptions A.1–A.5. To see this, note that if $\gamma_0 \equiv \gamma_1 \equiv 0$, then

$$\begin{aligned} g_0(\alpha_0, 0) &= e^{\alpha_0}, \\ g_1(\omega, \alpha_1(\omega), \alpha_0; 0, 0) &= e^{\alpha_1(\omega)} y(\omega). \end{aligned} \tag{14}$$

This is identical to Example 1 given in Section II, which satisfies Assumptions A.1–A.5.

It is easy to see that a market manipulation trading strategy is $(\alpha_1, \alpha_0) = (0, 0)$ and $(\gamma_1, \gamma_0) = (0, 1)$, where β_0 is chosen to satisfy Expression (1). That is, take a long position in the forward contract at time 0 and hold it until time 1, at which time sell the stock. Indeed, a calculation in the Appendix yields the time 1 real wealth to be

$$
\begin{aligned}
(15) \qquad V_1(\omega) &= g_1(\omega; 0, 0; 0, 1) - g_0(0, 1) \\
&= y(\omega) - 1/2 \quad > \quad 0 \quad \text{a.e.} \quad \bar{P}.
\end{aligned}
$$

By using the forward market to take a position in the stock, this trading strategy takes advantage of an asymmetry and price adjustment lag present across the two markets. The spot market lags the forward market. To see this, consider Expression (12). If the speculator took a levered position in the stock at time 0, the spot price for the stock is ($e = \exp(1)$). Yet, if he takes the same position in the forward contract, the spot price for the stock is (1/2). At time 1, however, the two spot prices for the stock are again equal at ($ey(\omega)$) for either position. In essence, by construction, the spot market does not evaluate the position in the forward at time 0 to be equivalent to a time 0 position in the stock. One interpretation could be that the spot market was not aware of this transaction until the maturity date of the forward contract at time 1, when the two markets again coincide. The speculator's trading strategy takes advantage of this time lag and subsequent reversal. □

These two examples motivate the need to identify sufficient conditions on the price process such that derivative security markets admit no market manipulation trading strategies. This is the content of the next section.

IV. Synchronous Markets

This section investigates a sufficient condition on the stock price process such that a derivative security market adds no market manipulation trading strategies. The condition is motivated by Example 4 in the preceding section. In that example, manipulation was possible because the two different markets were not aligned.

Definition. Synchronous Markets

The stock, money market fund, and derivative security markets under Assumptions A.1–A.6 are said to be in *synchrony* for all $t \in \tau$ if for any self-financing trading strategy of the large trader $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\}$, the following condition is satisfied for all $t \in \tau$ and $\omega \in \Omega$,

$$(16a) \qquad g_t\left(\omega, \bar{\alpha}^t(\omega), \mathbf{0}\right) = g_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right),$$

$$(16b) \qquad \text{where} \quad \bar{\alpha}_t(\omega) \equiv \alpha_t(\omega) + \gamma_t(\omega) n_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right).$$

Expression (16b) gives the speculator's "effective" position $\bar{\alpha}_t$ in the stock at time t, given that he is holding α_t shares of the stock and γ_t shares of the redundant derivative security. This equivalent position is based on Expression (3) where $n_t(\cdot)$ represents the derivative's unique "delta." I point out that $\bar{\alpha}_t$ is F_t-measurable for all $t \in \tau$. Expression (16a) defines the market to be in synchrony if the stock price is the same whether i) the large trader takes his position directly in the stock market $(\bar{\alpha}_t, \ldots, \bar{\alpha}_0)$, or ii) the large trader splits his position in the stock and derivative security market $((\alpha_t, \ldots, \alpha_0), (\gamma_t, \ldots, \gamma_0))$.

When the security markets are synchronous, the market prices adjust *instantaneously* to the true "effective" stock purchases of the large trader, even though he "disguises" some of these purchases in the derivative security market. The following proposition should now come as no surprise.

Proposition 2. No Market Manipulation Strategies Due to Derivative Claims
Given Assumptions A.1–A.6, if markets are in synchrony, then there are no market manipulation trading strategies $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\} \in \Phi$ with $\alpha_t + \gamma_t n_t(\alpha^t, \gamma^t) \leq N$ for all $t \in \tau$.

Proof. In the Appendix.

The proof is delegated to the Appendix. The logic of the proof, however, is straightforward. First, I show that any self-financing trading strategy involving the derivative security can be duplicated by the large trader with a self-financing trading strategy in the stock and money market fund alone. Proposition 1 then gives the result.

This proposition provides sufficient conditions for a derivative security market not to introduce any additional market manipulation strategies for the speculator. First, market corners and short squeezes are excluded by the limit position constraint that $\alpha_t + n_t\gamma_t \leq N$ a.s. This restricts the speculator's total holdings in the stock plus derivative to be less than the total supply outstanding at any date. Second, the markets must be in synchrony. The satisfaction of this condition requires the efficient transmission of information across the two different, but related, markets. Whether or not markets satisfy this later condition is a testable hypothesis. Its satisfaction certainly depends on the ability of arbitrageurs to profit from any deviations.

V. A Theory for Option Pricing

This section develops the theory for option valuation under a relaxation of the price taking assumption. The three alternate basic assumptions of option pricing theory will be maintained, i.e., symmetric information A.4, frictionless markets A.1, and complete markets. The complete markets assumption will be an implication of the stock price process subsequently imposed. The absence of arbitrage is replaced by the *synchronous market* condition of the previous section. Hence, options are priced in a large trader economy such that no market manipulation trading strategies exist. As such, this theory is a generalization of the standard binomial option pricing model (see Jarrow and Rudd (1983), chapter 13). This theory has the important result, under a common knowledge assumption, that to a price taker, the standard binomial model still applies, but with a random volatility. Although the argument generating the option model is more complex, the resulting formula and hedging procedures for the price taker are identical. As a secondary benefit, this section provides an example of an economy satisfying Assumptions A.1–A.6 of the preceding sections showing, in fact, that these assumptions are consistent.

For simplicity, consider a two-period economy ($T = 2$). The multiperiod extension is straightforward. The three traded assets consist of the stock, a money market account, and a European call option on the stock. The European call option matures at date 2 with an exercise price of K. The notation is the same as given in Section II.

The state space consists of four states,

$$(17) \quad \Omega = \{(u,u),(u,d),(d,u),(d,d)\} \quad \text{for} \quad u,d \in R \quad \text{with } u > 0 > d.$$

The quantities u and d will take on an economic interpretation momentarily. The information structure is

$$(18a) \quad F_0 = \{\phi, \Omega\},$$

$$(18b) \quad F_1 = \{\phi, \Omega, \{(u,u),(u,d)\}, \{(d,u),(d,d)\}\},$$

$$(18c) \quad F = F_2 = \text{all subsets of } \Omega.$$

The large trader's probability beliefs P are defined for some $p \in (0,1)$ by

$$(19) \quad \begin{aligned} P((u,u)) &= p^2, \\ P((u,d)) &= P((d,u)) = p(1-p), \quad \text{and} \\ P((d,d)) &= (1-p)^2. \end{aligned}$$

Assume that the other traders in the economy, indexed by $i \in I$, are price takers and have equivalent probability beliefs P^i defined as in Expression (19) with $p^i \in (0,1)$.

The "fundamental" price process $\{y_t : t \in \tau\}$ follows the standard binomial random walk with a constant "volatility," i.e.,

$$(20a) \quad y_0(\omega) = y_0 \quad \text{for all } \omega \in \Omega,$$

$$(20b) \quad y_l(\omega) = \begin{cases} y_0 e^u & \text{if} \quad \omega \in \{(u,u),(u,d)\} \\ y_0 e^d & \text{if} \quad \omega \in \{(d,u),(d,d)\}, \end{cases}$$

$$(20c) \quad y_2(\omega) = \begin{cases} y_0 e^{2u} & \text{if} \quad \omega = (u,u) \\ y_0 e^{u+d} & \text{if} \quad \omega \in \{(u,d),(d,u)\} \\ y_0 e^{2d} & \text{if} \quad \omega = (d,d). \end{cases}$$

The quantity "u" represents the return when the process jumps up, while the quantity "d" represents the return for a jump down. It is easy to see that the probability measure $\bar{P}$, defined as in Expression (19), but with $\bar{p} = [1-e^d]/[e^u - e^d]$, where $1 > \bar{p} > 0$ makes $\{y_t : t \in \tau\}$ a $\bar{P}$-martingale with respect to $\{F_t : t \in \tau\}$.

For any self-financing trading strategy $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\}$, define the price process[8] as

[8] The subsequent analysis does not depend in any significant manner on the functional form of the stock price process given in Expression (21). As the following argument will show, the analysis can be easily generalized to any process satisfying Assumptions A.1–A.5.

$$(21) \qquad g_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right) = \exp\{\eta\left(\alpha_t(\omega) + \delta_t(\omega)\gamma_t(\omega)\right)\}y_t(\omega)$$

$$\text{for all } \omega \in \Omega,\ t \in \tau,$$

where $\eta > 0$ and $\{\delta_t : t \in \tau\}$ is a nonnegative stochastic process adapted to $\{F_t : t \in \tau\}$.

The price process (21) consists of a *controlled* process $\exp\{\eta(\alpha_t + \delta_t\gamma_t)\}$ and the *fundamental* process y_t. The controlled part reflects a sensitivity coefficient $\eta > 0$ to the speculator's share holdings (α_t) plus a sensitivity coefficient $\eta\delta_t$ to the speculator's call option holdings (γ_t). Based on the analysis in Section IV (see Expression (16)), I can interpret δ_t as the *market's perception* of the option's "delta." The self-financing trading strategy $\{\bar{\alpha}_t, \bar{\beta}_t, \bar{\gamma}_t : t \in \tau\}$ defined by Expression (1) with $\bar{\alpha}_t = \alpha_t + \delta_t\gamma_t$ and $\bar{\gamma}_t \equiv 0$, generates the identical equilibrium price process as does $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\}$. Thus, in the market's view, each call long at time t under state ω has the same influence on the stock price as does $\delta_t(\omega)$ shares of the stock. For example, the value of δ_2 at the call's maturity should be 1 if the call is in the money and 0 otherwise.

For convenience, define a sequence of functions $h_t : \Omega \times R \rightarrow R$ by $h_t(\omega;\xi) = e^{\eta\xi}y_t(\omega)$ for all $\omega \in \Omega$, $\xi \in R$. Using Expression (21),

$$(22) \qquad h_t\left(\omega; \alpha_t(\omega) + \delta_t(\omega)\gamma_t(\omega)\right) = g_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right).$$

I will use this reduced notation for the price process.

To restrict the speculator's market power, assume that the sensitivity coefficient η is determined and $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\}$ is restricted such that

$$(23a) \quad h_2((u,u);\alpha_2+\delta_2\gamma_2) > h_2((u,d);\alpha_2+\delta_2\gamma_2) > h_2((d,d);\alpha_2+\delta_2\gamma_2) \text{ and}$$

$$(23b) \qquad h_1(u;\alpha_1+\delta_1\gamma_1) > h_1(d;\alpha_1+\delta_1\gamma_1),$$

$$\text{where} \quad u \in \{(u,d),(u,u)\} \quad \text{and} \quad d \in \{(d,u),(d,d)\}.$$

This implies that the speculator cannot change the ordinal rankings of the price jumps due to $\{y_t : t \in \tau\}$ and only has "local" price adjustment power. This restriction is imposed for simplicity and it is not necessary for the subsequent analysis.

Let $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\} \in \Phi$ be the speculator's optimal self-financing trading strategy. By *optimal*, I mean that $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\}$ maximizes the speculator's preferences across all possible self-financing trading strategies in Φ. His preferences are left unspecified with the exception that more real wealth at time T is preferred to less.

Given that the speculator knows his optimal self-financing trading strategy $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\}$ in replicating the call, he faces only the fundamental price risk $\{y_t : t \in \tau\}$. Hence, he can replicate the call option using a dynamic trading strategy. I price the call such that i) there are no market manipulation trading strategies (by imposing synchronous markets), and such that ii) the large trader does not want to change his optimal holdings. To make this argument, consider a deviation of $\{n_t, m_t, -1: t \in \{0,1\}\}$ from the speculator's optimal position. With this deviation, the speculator's total holdings are

$$\{\alpha_t + n_t, \beta_t + m_t, \gamma_t - 1: t \in \{0,1\}\} \quad \text{and} \quad \{\alpha_2, \beta_2, \gamma_2\}. \tag{24}$$

The shares $\{n_t, m_t: t \in \{0,1\}\}$ are selected to replicate a long position in the call. Combined with the written call (the -1 in the third argument), *if* the call is properly priced, *then* the speculator will be indifferent between $\{\alpha_t, \beta_t, \gamma_t: t \in \tau\} \in \Phi$ and the deviation strategy given in Expression (26). I now turn to the implementation of this argument.

The holdings $\{n_1, m_1\}$ are functions of $u \in \{(u,d),(u,u)\}$ and $d \in \{(d,d), (d,u)\}$. I determine $(n_1(u), m_1(u))$ first. To duplicate the call, by the standard arguments, I choose $(n_1(u), m_1(u))$ such that

$$\begin{aligned}
(25) \qquad & n_1(u)h_2\left((u,u);\alpha_2+\delta_2\gamma_2\right)+m_1(u) \\
&= \left[h_2\left((u,u);\alpha_2+\delta_2\gamma_2\right)-K/B_2(u)\right]^+ \equiv c_2(u,u), \\
& n_1(u)h_2\left((u,d);\alpha_2+\delta_2\gamma_2\right)+m_1(u) \\
&= \left[h_2\left((u,d);\alpha_2+\delta_2\gamma_2\right)-K/B_2(u)\right]^+ \equiv c_2(u,d).
\end{aligned}$$

The effective holdings of the speculator at time 2 are unchanged from $(\alpha_2 + \delta_2\gamma_2)$, since the speculator's synthetic long call and actual written call negate each other. Due to Expression (23a), there exists a unique solution to this system,

$$(26a) \qquad n_1(u) = \frac{c_2(u,u)-c_2(u,d)}{h_2\left((u,u);\alpha_2+\delta_2\gamma_2\right)-h_2\left((u,d);\alpha_2+\delta_2\gamma_2\right)},$$

$$(26b) \qquad m_1(u) = \frac{c_2(u,d)h_2\left((u,u);\alpha_2+\delta_2\gamma_2\right)-c_2(u,u)h_2\left((u,d);\alpha_2+\delta_2\gamma_2\right)}{h_2\left((u,u);\alpha_2+\delta\gamma_2\right)-h_2\left((u,d);\alpha_2+\delta_2\gamma_2\right)}.$$

Because of Expression (23a), $0 \le n_1(u) \le 1$ and $m_1(u) \le 0$. A similar calculation generates $(n_1(d), m_1(d))$ as identical to $(n_1(u), m_1(u))$, but replacing the first "u" argument in all of Expression (26) with "d."

Now, both strategies $(\alpha_1 + n_1, \beta_1 + m_1, \gamma_1 - 1)$ and $(\alpha_1, \beta_1, \gamma_1)$ generate the *identical* holdings at time 2, i.e., $(\alpha_2, \beta_2, \gamma_2)$. Unless the time 1 value of the positions is equal, the initial portfolio $(\alpha_1, \beta_1, \gamma_1)$ could *not* be an optimum. Indeed, if $(\alpha_1 + n_1, \beta_1 + m_1, \gamma_1 - 1)$ were lower in value, it would be preferred. Conversely, if it were higher in value, then $(\alpha_1 - n_1, \beta_1 - m, \gamma_1 + 1)$ would be preferred. Hence, the optimality condition for $\{\alpha_t, \beta_t, \gamma_t: t \in \tau\}$ implies

$$\begin{aligned}
(27) \qquad & [\alpha_1(\omega)+n_1(\omega)]\, h_1\left(\omega;\alpha_1+n_1+\delta_1(\gamma_1-1)\right) \\
&+ (\beta_1(\omega)+m_1(\omega)) + [\gamma_1(\omega)-1]\, c_1\left(\omega;\alpha_1+n_1+\delta_1(\gamma_1-1)\right) \\
&= \alpha_1(\omega)h_1\left(\omega;\alpha_1+\delta_1\gamma_1\right)+\beta_1(\omega)+\gamma_1(\omega)c_1\left(\omega;\alpha_1+\delta_1\gamma_1\right).
\end{aligned}$$

This condition by itself does not yield a unique determination of the call price. The reason is that the left side of Expression (27) involves $c_1(\omega; \alpha_1 + n_1 + \delta_1(\gamma_1 - 1))$, while the right side involves $c_1(\omega; \alpha_1 + \delta_1\gamma_1)$. These are in general two different quantities. To close the model we assume a synchronous market condition,

$$\text{(28)} \qquad g_t\left(\omega, \bar{\alpha}^t(\omega), \mathbf{0}\right) = g_t\left(\omega, \alpha^t(\omega), \gamma^t(\omega)\right)$$
$$\text{where} \quad \bar{\alpha}_t(\omega) = \alpha_t(\omega) + \gamma_t(\omega)n_t(\omega).$$

Substitution of Expression (21) into (28), yields that

$$\text{(29)} \qquad n_t(\omega) = \delta_t(\omega) \quad \text{for all} \quad t \in \tau \quad \text{and} \quad \omega \in \Omega.$$

This assumption implies that the market's perception of the share equivalent of 1 call is identical to the speculator's actual equivalence. This assumption is sufficient to exclude market manipulation. I will show later under an additional common knowledge assumption that this synchronous market condition also implies that the option can be synthetically constructed by the remaining price takers in the economy, i.e., the complete markets condition holds. Under Expression (28), Expression (27) simplifies,

$$\text{(30)} \quad c_1\left(\omega; \alpha_1(\omega) + n_1(\omega)\gamma_1(\omega)\right) = n_1(\omega)h_1\left(\omega; \alpha_1(\omega) + n_1(\omega)\gamma_1(\omega)\right) + m_1(\omega)$$
$$\text{for all} \quad \omega \in \Omega.$$

This is an exact pricing model for the call. Note that it satisfies Assumption A.6. Substitution of $n_1(\omega)$ and $m_1(\omega)$ into Expression (30) and simplification generates

$$\text{(31)} \quad c_1\left(u; \alpha_1 + n_1\gamma_1\right) = \lambda(u)\left[e^{\eta(\alpha_2(u,u)+n_2(u,u)\gamma_2(u,u))}y_0e^{2u} - K/B_2(u)\right]^+$$
$$+ (1-\lambda(u))\left[e^{\eta(\alpha_2(u,d)+n_2(u,d)\gamma_2(u,d))}y_0e^{u+d} - K/B_2(u)\right]^+,$$
$$\text{where} \quad \lambda(u) = \frac{e^{\eta[\alpha_1(u)+n_1(u)\gamma_1(u)]} - e^{\eta[\alpha_2(u,d)+n_2(u,d)\gamma_2(u,d)]+d}}{e^{\eta[\alpha_2(u,u)+n_2(u,u)\gamma_2(u,u)]+u} - e^{\eta[\alpha_2(u,d)+n_2(u,d)\gamma_2(u,d)]+d}},$$

$$\text{(32)} \quad \text{and} \quad n_2(u,u) = \begin{cases} 1 & \text{if} \quad e^{\eta\{\alpha_2(u,u)+\gamma_2(u,u)\}}y_0e^{2u} - K/B_2(u) \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$
$$n_2(u,d) = \begin{cases} 1 & \text{if} \quad e^{\eta[\alpha_2(u,d)+\gamma_2(u,d)]}y_0e^{u+d} - K/B_2(u) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A similar expression holds for $c_1(d; \alpha_1 + n_1\gamma_1)$ with the first argument "u" in all of (31) and (32) replaced by a "d."

This is the standard binomial option pricing model (see Jarrow and Rudd (1983)) with the fundamental stock price process $(y_t(\omega))$ of Expression (20) replaced by the controlled process $(h_t(\omega; \alpha_t(\omega) + n_t(\omega)\gamma_t(\omega)))$ of Expression (22). Otherwise, it is identical. This difference is significant, however, as it implies a random "volatility" for the stock price process as viewed by price takers. This is due to the fact that the large trader's holdings are state dependent.

I now continue the above process to determine the call's price at time 0. As the argument is identical, this analysis proceeds with little discussion. At time 0, I choose (n_0, m_0) such that

$$\begin{aligned} n_0 h_1(u; \alpha_1 + n_1\gamma_1) + m_0 &= c_1(u; \alpha_1 + n_1\gamma_1), \\ n_0 h_1(d; \alpha_1 + n_1\gamma_1) + m_0 &= c_1(d; \alpha_1 + n_1\gamma_1). \end{aligned} \tag{33}$$

Using Expression (23b), there exists a unique solution to (33),

$$n_0 = \frac{c_1(u; \alpha_1 + n_1\gamma_1) - c_1(d; \alpha_1 + n_1\delta_1)}{h_1(u; \alpha_1 + n_1\gamma_1) - h_1(d; \alpha_1 + n_1\gamma_1)}, \tag{34a}$$

$$m_0 = \frac{c_1(d; \alpha_1 + n_1\gamma_1)\, h_1(u; \alpha_1 + n_1\gamma_1) - c_1(u; \alpha_1 + n_1\gamma_1)\, h_1(d; \alpha_1 + n_1\gamma_1)}{h_1(u; \alpha_1 + n_1\gamma_1) - h_1(d; \alpha_1 + n_1\gamma_1)}, \tag{34b}$$

where $0 \le n_0 \le 1$ and $m_0 \le 0$.

Now, both strategies $(\alpha_0 + n_0, \beta_0 + m_0, \gamma_0 - 1)$ and $(\alpha_0, \beta_0, \gamma_0)$ generate the identical holdings at time 1, i.e., $(\alpha_1, \beta_1, \gamma_1)$. Unless the time 0 value of the positions are equal, the initial portfolio $(\alpha_0, \beta_0, \gamma_0)$ could not be an optimum. Hence, the optimality requirement implies

$$\begin{aligned} &\alpha_0 h_0(\omega; \alpha_0 + \delta_0\gamma_0) + \beta_0 + \gamma_0 c_0(\omega; \alpha_0 + \delta_0\gamma_0) \\ &\quad = [\alpha_0 + n_0]\, h_1(\omega; \alpha_0 + n_0 + [\gamma_0 - 1]\,\delta_0) \\ &\qquad + [\beta_0 + m_0] + (\gamma_0 - 1)\, c_0(\omega; \alpha_0 + n_0 + [\gamma_0 - 1]\,\delta_0) \\ &\qquad \text{for all} \quad \omega \in \Omega. \end{aligned} \tag{35}$$

Using the synchronous market condition,

$$n_0(\omega) = \delta_0(\omega). \tag{36}$$

Substitution into Expression (35) and algebra yield

$$c_0(\omega; \alpha_0 + \gamma_0 n_0) = n_0 h_0(\omega; \alpha_0 + \gamma_0 n_0) + m_0. \tag{37}$$

$(\alpha_t + n_t, \beta_t + m_t, \gamma_t - 1)$ is self-financing since at time 1 it equates to $(\alpha_1, \beta_1, \gamma_1)$ and at time 2 it equates to $(\alpha_2, \beta_2, \gamma_2)$; and this later trading strategy is self-financing. Secondly, Assumption A.6, Expression (3) is satisfied by Expressions (30) and (37). Also, by Expressions (30) and (33), $n_0 h_1(\omega; \alpha_1 + n_1\gamma_1) + m_0 = c_1(\omega; \alpha_1 + n_1\gamma_1) = n_1(\omega) h_1(\omega; \alpha_1 + n_1\gamma_1) + m_1(\omega)$. So, Assumption A.6, Expression (4) is satisfied as well.

Substitution of n_0, m_0 into (37) and algebra yields the final result,

$$c_0(\omega; \alpha_0 + n_0\gamma_0) = \lambda_0 \lambda(u) \left[e^{\eta[\alpha_2(u,u) + n_2(u,u)\gamma_2(u,u)]} y_0 e^{2u} - K/B_2(u) \right]^+ \tag{38}$$

$$
\begin{aligned}
&+ \lambda_0((1-\lambda)(u))\left[e^{\eta[\alpha_2(u,d)+n_2(u,d)\gamma_2(u,d)]}y_0e^{u+d} - K/B_2(u)\right]^+ \\
&+ ((1-\lambda_0)(\lambda(d))\left[e^{\eta[\alpha_2(d,u)+n_2(d,u)\gamma_2(d,u)]}y_0e^{d+u} - K/B_2(d)\right]^+ \\
&+ (1-\lambda_0)((1-\lambda)(d))\left[e^{\eta[\alpha_2(d,d)+n_2(d,d)\gamma_2(d,d)]}y_0e^{2d} - K/B_2(d)\right]^+
\end{aligned}
$$

$$
\text{where} \quad \lambda_0 = \frac{e^{\eta(\alpha_0+n_0\gamma_0)} - e^{\eta(\alpha_1(d)+n_1(d)\gamma_1(d))+d}}{e^{\eta[\alpha_1(u)+n_1(u)\gamma_1(u)]+u} - e^{\eta[\alpha_1(d)+n_1(d)\gamma_1(d)]+d}}.
$$

Expression (38) exhibits the call's value at time 0. It depends on the parameters of the fundamental price process (u, d, y_0, η) and the speculator's optimal trading strategy $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\}$. Again, this is the standard binomial option pricing model with the only change being that the controlled price process replaces the fundamental price in the standard formula (see Jarrow and Rudd (1983), chapter 13). This change implies a random "volatility" for the stock.

I now discuss the synthetic construction of the call option by price takers. Let these price takers also trade in a frictionless market. As discussed earlier, a characterization of this situation is indeterminate without the introduction of additional assumptions concerning what is common knowledge in this economy. Two contrasting common knowledge structures will be discussed. The first is where the form of the price function in Expression (22) is not common knowledge, and it is *not* known by the price takers. The second is where it is.

Consider first the scenario where the price takers do not know the price function in Expression (22). In this case, it is still conceivable that the synchronous market condition holds, but because the price takers do not know the price function in Expression (22), they do not know the hedge ratio $(n_t(\omega))$ of the large trader's optimum strategy. Therefore, they cannot synthetically construct the call options. This scenario leads to the large trader being able to synthetically construct the call option, but not the price takers. It is an open question whether this scenario can be supported by some equilibrium model (perhaps with some irrationalities).

The second scenario is perhaps more plausible. In this scenario, it is assumed that the structure of the economy, including the form of the price function in Expression (22), is common knowledge (to the large trader and price takers). Price takers being rational, given the synchronous market condition, therefore, know the call option's delta $(n_t(\omega)$ for $t = 0, 1, 2)$. This knowledge allows them to synthetically construct the call option via Expressions (30) and (37). They simply buy $n_t(\omega)$ shares of the stock, and sell $m_t(\omega) = c_t(\omega) - n_t(\omega)h_t(\omega)$ units of the money market account. They can execute this strategy as the realizations of $(c_t(\omega), h_t(\omega))$ are common knowledge.

It is important to clarify that the optimal trading strategy of the large trader is not assumed to be common knowledge and is not deducible by the price takers in this second scenario. This is true even given the synchronous market condition. The synchronous market condition only allows the price takers to deduce the hedge ratio $n_t(\omega)$. They cannot infer the large trader's position. Indeed, numerous trading strategy selections $(\alpha_t(\omega), b_t(\omega))$ are consistent with a particular hedge ratio. This is most easily seen by looking at $n_2(u, u)$ in Expression (32). Here, any pair $(\alpha_2(u, u), \gamma_2(u, u))$ guaranteeing that the stock expires in the money will yield the same hedge ratio. A similar argument applies to the hedge ratios n_1 in Expression (26) and n_0 in Expression (34).

This insight is significant. It implies that the argument used to price forward contracts in Example 2 cannot be used to price call options. In Example 2, forward contracts could be synthetically constructed by price takers, independent of the large trader's actions. Therefore, the standard no arbitrage opportunity arguments were applicable. In contrast, for call options, the standard no arbitrage argument cannot be applied. The above discussion implies that price takers cannot construct the synthetic call without knowledge of the synchronous market condition. The large trader's optimal position is essential for this calculation, and this is not known by the price takers, nor can it be inferred from the option's price. In summary, under this stronger common knowledge assumption, price takers can price and hedge options, only the justification for this procedure is now different.

It is standard practice in existing option markets to use the binomial option model, reparameterized as a function of the stock's instantaneous volatility, to "price" exchange traded calls. In fact, the usual procedure is to invert this process, and to use market prices to obtain an *implicit volatility*. If valid, the pricing model in Expression (38) suggests that these implicit volatilities will reflect the speculator's "anticipated" trades in the future. These implicit volatilities, being random, would vary stochastically over time and would change dramatically when the large trader's position changes abruptly. This could possibly explain the large increase in market volatilities surrounding the October 19, 1987, stock market crash. Furthermore, this may also explain the dramatic shift in implicit volatilities observed by Rubinstein (1985). To adequately address these and other issues, the stock price process examined in this section needs to be generalized and the methodology refined. This analysis, however, awaits subsequent research.

VI. Conclusion

This paper investigates the impact derivative security markets have on market manipulation. In an economy consisting of a stock, money market account, and a derivative security, I show that a large trader can manipulate the market where without the derivative security, he could not. Two strategies exist: first, the large trader can use the derivative security to corner the market; second, the large trader can exploit leads/lags across the markets to "front run" his own trades.

Sufficient conditions are discovered that exclude manipulation by the large trader. First, quantity limits need to be imposed consistently across the two markets to exclude corners. Secondly, the markets need to be closely linked, a condition I call synchrony. Fast and accurate information concerning trades in the underlying and derivative security need to be available in the corresponding markets.

Finally, this paper develops a theory for option pricing in an economy with a large trader. The absence of market manipulation and the optimality conditions for the large trader's position replace the absence of arbitrage as the pricing argument. It is shown that the standard option pricing model holds, but with a random "volatility" price process. Future research is needed to refine and to extend this model.

Appendix

Example 3. Calculations

time 0) $0 = Ng_0(N;1) + \beta_0 + 1 \cdot c_0(N,1)$ by Expression (1).
But, $c_0(N,1) = 0$ by construction, so
$\beta_0 = -Ng_0(N;1)$.

time 1) By Expression (1),
$Ng_1(N-1,N;1,1) + \beta_0 + c_1(N-1,N;1,1)$
$= (N-1)g_1(N-1,N;1,1) + \beta_1 + c_1(N-1,N;1,1)$. So,
$\beta_1 = \beta_0 + g_1(N-1,N;1,1)$.
But at date 1, $c_1(N-1,N;1,1) = g_1(N-1,N;1,1) - K/B_1$ because the short forward contract delivers the share to the speculator. Recall that each unit of the money market account is worth B_1 dollars at time 1.

time 2) $V_2 = (N-1)g_2(0,N-1,N;0,1,1) + \beta_1 + 1(g_2(0,N-1,N;0,1,1) - K/B_1)$
$= Ng_2(0,N-1,N;0,1,1) + \beta_0 + g_1(N-1,N;1,1) - (K/B_1)$
$= N[g_2(0,N-1,N;0,1,1) - g_0(N;1)] + g_1(N-1,N;1,1) - (K/B_1)$.

This is Expression (11) in the text.

Example 4. Calculations

At time 0, by Expression (1), $0 = \beta_0 + c_0(0,1)$. By definition, $c_0(0,1) = 0$ so $\beta_0 = 0$. The time 1 real wealth is $V_1(\omega) = +\beta_0 + c_1(\omega;0,0;0,1)$. Using Expression (7), and the value of β_0 yields $V_1(\omega) = g_1(\omega;0,0;0,1) - K/B_1$. But by Expression (8), $g_0(0;1) - K/B_1 = 0$. Using this yields $V_1(\omega) = g_1(\omega;0,0;0,1) - g_0(0;1)$.This is Expression (15) in the text.

Proof of Proposition 2. Under Assumption A.5, define a sequence of functions $h_t\colon \Omega \times R \to R$ by $h_t(\omega, \bar{\alpha}_t(\omega)) \equiv g_t(\omega, \bar{\alpha}^t(\omega), \mathbf{0})$ since the right side does not change with $\bar{\alpha}^{t-1}(\omega)$. Using this definition, rewrite the synchronous market expression (16) as $Z_t(\omega) = h_t(\omega; \bar{\alpha}_t(\omega))$ where $\bar{\alpha}_t(\omega) = \alpha_t(\omega) + \gamma_t(\omega)n_t(\omega, \alpha^t(\omega), \gamma^t(\omega))$.

The proof is divided into two steps.

(Step 1) Claim: For every self-financing trading strategy $\{\alpha_t, \beta_t, \gamma_t : t \in \tau\} \in \Phi$, there exists a self-financing trading strategy $\{\bar{\alpha}_t, \bar{\beta}_t, \bar{\gamma}_t : t \in \tau\} \in \Phi_0$ that duplicates its value (with probability one), in fact $\bar{\alpha}_t = \alpha_t + \gamma_t n_t(\alpha^t, \gamma^t), \bar{\beta}_t = \beta_t - \gamma_t m_t(\alpha^t, \gamma^t), \bar{\gamma}_t \equiv 0$.

To prove this claim, first consider the self-financing condition at time 0.

$$\begin{aligned} 0 &= \alpha_0 Z_0 + \gamma_0 C_0 + \beta_0 \\ &= \alpha_0 h_0(\bar{\alpha}_0) + \gamma_0 \left[n_0(\alpha_0, \gamma_0) h_0(\bar{\alpha}_0) - m_0(\alpha_0, \gamma_0)\right] + \beta_0 \text{ by Assumption A.6} \\ &= \bar{\alpha}_0 h_0(\bar{\alpha}_0) + \bar{\beta}_0 \text{ by definition of } \bar{\alpha}_0, \bar{\beta}_0. \end{aligned}$$

Both portfolios have zero value at time 0.

Next, there is a random change from time 0 to time 1 and a portfolio rebalancing. The self-financing condition yields

$$\begin{aligned} \alpha_0 Z_1 + \gamma_0 C_1 + \beta_0 &= \alpha_1 Z_1 + \gamma_1 C_1 + \beta_1 \\ &= \alpha_1 h_1(\bar{\alpha}_1) + \gamma_1 \left[n_1\left(\alpha^1, \gamma^1\right) h_1(\bar{\alpha}_1) - m_1\left(\alpha^1, \gamma^1\right)\right] + \beta_1 \text{ by Assumption A.6} \\ &= \bar{\alpha}_1 h_1(\bar{\alpha}_1) + \bar{\beta}_1 \text{ by definition of } \bar{\alpha}_1, \bar{\beta}_1. \end{aligned}$$

So both portfolios have the the same time 1 value.

To show the trading strategy $(\bar{\alpha}_t, \bar{\beta}_t, \bar{\gamma}_t)$ is self-financing, note that

$$\begin{aligned}\alpha_0 Z_1 + \gamma_0 C_1 + \beta_0 &= \alpha_0 h_1(\bar{\alpha}_1) + \gamma_0\left[n_1\left(\alpha^1, \gamma^1\right) h_1(\bar{\alpha}_1) - m_1\left(\alpha^1, \gamma^1\right)\right] + \beta_0 \\ &= \alpha_0 h_1(\bar{\alpha}_1) + \gamma_0\left[n_0(\alpha_0, \gamma_0) h_0(\bar{\alpha}_1) - m_0(\alpha_0, \gamma_0)\right] + \beta_0 \text{ by Assumption A.6} \\ &= \bar{\alpha}_0 h_1(\bar{\alpha}_1) + \bar{\beta}_0 \text{ by definition of } \bar{\alpha}_0, \bar{\beta}_0.\end{aligned}$$

Continuing this process for all $t \in \tau$ proves the claim.

(Step 2). By Proposition 1, there are no market manipulation strategies for $\{\bar{\alpha}_t, \bar{\beta}_t, \bar{\gamma}_t : t \in \tau\} \in \Phi_0$ such that $\bar{\alpha}_t \leq N$ a.s. P for all $t \in \tau$. The requirement that $\alpha_t + n_t\gamma_t \leq N$ a.e. P for all $t \in \tau$ implies $\bar{\alpha}_t \leq N$ a.s. P for all $t \in \tau$.

By Step 1, each $\{\alpha_t, \beta_t, \gamma_t : t\tau \in \tau\} \in \Phi$ can be duplicated by a $\{\bar{\alpha}_t, \bar{\beta}_t, \bar{\gamma}_t : t \in \tau\} \in \Phi_0$. Hence, there are no market manipulation strategies in Φ, otherwise, they would also appear in Φ_0. □

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