A graduate course on complex manifolds
Course calendar and homework assignments
2022-2023学年第二学期复流形
课程进度和作业布置

Instructor: Bo Yang

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说明

此文件定期更新中，**每次阅读前参考上方的红色标识的最后更新时间。**

本课程的教材和参考书是[1].

本课程的先修课程：复变函数(本科)，微分流形。要求学生熟悉微分形式的语言。如果自学过微分流形的同学可与任课老师联系确定是否适合修读。

上课时间和地点：每周二5-7节(14:30-15:15, 15:25-16:10, 16:40-17:25)在实验楼106。

教学大纲和教科书等相关资料都已上传到课程QQ群的文件夹，QQ群号: 733850534.

课程最终成绩由平时作业和一次期末作业的成绩确定。我们鼓励大家用Latex写好后打印提交作业，解答可以用英文或中文，欢迎同学之间讨论题目的解法，但须独立书写解答。

The main goal

If time permits, we hope to cover Chapter 1 and 2 of [1], including

1. Complex manifolds, definitions and examples.
2. Basic Hermitian geometry and Kähler geometry.
3. Basics on linear elliptic partial differential equations, in particular Theorem 2.12 in [1]. We also use Chapter 2 in [3] as a reference on Schauder estimates.
4. The proof of the Newlander-Nirenberg theorem, if time permits.

Remark: We do not have to cover any nonlinear theory of elliptic PDEs. For anyone who wish to study further, we refer to Prof. Lei Ni's online course on Monge-Ampère equations.

http://tianyuan.xmu.edu.cn/cn/MiniCourses/3168.html

Week 3: 02/28/2023, This is the first lecture

Summary of lectures: We follow Section 1.1 and 1.2 of [1].

1. An overview of the course, from uniformization theorem on Riemannian 2-manifolds to Kähler-Einstein metrics in higher dimensions.

2. Holomorphic functions in a domain in $\mathbb{C}^n$, we follow parts of [2].

   (a) Definition. Let D is a domain in $\mathbb{C}^n$, and $f(z_1, \cdots, z_n)$ a continuous function. For any $1 \le k \le n$ and fixing $z_1, \cdots, z_{k-1}, z_{k+1}, \cdots, z_n$, if $f(\cdots, z_k, \cdots)$ is holomorphic on z_k where z_k is defined. We call f is holomorphic of n variables $(z_1, \cdots, z_n) \in D$.

   (b) Cauchy integral formula and its consequences (convergent power series).

Week 4: 03/07/2023

Summary of lectures:

1. Holomorphic functions in a domain in $\mathbb{C}^n$ (continued), we follow [2]. Holomorphic mapping. Open mapping, Inverse mapping, and Implicit mapping theorem.

2. Definition of complex manifolds, we follow [1] 1.1.

3. Examples: Riemann sphere, complex projective space $\mathbb{CP}^n$, projective manifolds $V \subset \mathbb{CP}^n$.

4. Almost complex structure, and any complex manifold admits a natural almost complex structure, here we follow [1] 1.2.

5. The statement of Newlander-Nirenberg theorem.

Week 5: 03/14/2023

Summary of lectures:
Definition of Hermitian and Kähler metrics. We cover [1] Sec 1.3, p.6-p.7.

Week 6: 03/21/2023

Summary of lectures:
Curvature of Kähler metrics. We cover [1] Sec 1.3, p.9-p.13.

Homework 1, assigned on 03/21, due on 06/13 in class

Instruction: Choose at least 3 problems among the following 8 problems.

(1) (Definitions of holomorphic functions/maps)

Part 1: Let D is a domain in $\mathbb{C}^n$, we consider a map $f : U \to \mathbb{C}^m$ in the form

$$f(z_1, \cdots, z_n) = (f_1, \cdots, f_m).$$

Definition A: Fix any $1 \le l \le m$, then $f_l(z_1, \cdots, z_n)$ a continuous function. Moreover, for any $1 \le k \le n$ and fixing $z_1, \cdots, z_{k-1}, z_{k+1}, \cdots, z_n$, if $f_l(\cdots, z_k, \cdots)$ is holomorphic on z_k where z_k is defined. Then we call $f : U \to \mathbb{C}^m$ is holomorphic.

Definition B: We identify the real vector space $\mathbb{R}^{2p}$ with the complex vector space $\mathbb{C}^p$ by

$$(x_1, \cdots, x_p, y_1, \cdots, y_p) \longleftrightarrow (x_1 + \sqrt{-1}y_1, \cdots, x_n + \sqrt{-1}y_n).$$

We say a linear map $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is **complex linear** if $\phi(\sqrt{-1}v) = \sqrt{-1}\phi(v)$ for $v \in \mathbb{C}^n$ under the above identification. Now we say a map $f : U \to \mathbb{C}^m$ is holomorphic if it is C^1 (continuously differentiable) when viewed as a map $f : U \to \mathbb{R}^{2m}$ and its Jacobian, as a linear map $df : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$, is complex linear.

Question: Prove Definition A and Definition B are equivalent.

Part 2: Recall we define a holomorphic map between two complex manifolds by local holomorphic charts.

Let (M, J_1) and (N, J_2) are two complex manifolds, and $f : M \to N$ be a smooth map so that the tangent map $df : T_pM \to T_{f(p)}N$ commutes with J_1 and J_2, i.e.

$$df \circ J_1 = J_2 \circ df.$$

Then show f is holomorphic.

(2) (The space of complex homogeneous polynomials)

Let $\mathcal{P}_k$ and $\mathcal{P}_{\leq k}$ be homogeneous polynomials on $\mathbb{C}^n$ with degree k and degree at most k respectively. Prove

$$\dim(\mathcal{P}_k) = \binom{n+k-1}{n-1} \quad \text{and} \quad \dim(\mathcal{P}_{\leq k}) = \binom{n+k}{n}.$$

Hint: there is a standard combinatorial proof.

Remark: A remarkable result of Siegel says the transcendence degree of the field of meromorphic function on a complex manifold is at most its complex dimension. The proof relies on the above combinatorial result.

(3) (A smooth hypersurface)

Let $V_f = \{[w_0, \cdots, w_n] \in \mathbb{CP}^n \mid f(w) = 0\}$ where f is a homogeneous degree-d polynomial of $w_0, \cdots, w_n$. Prove V_f is a complex manifold if and only if

$$\bigcap_{i=1}^{n} \{[w_0, \cdots, w_n] \in V \mid \frac{\partial f}{\partial w_i} = 0\} = \{[0, \cdots, 0]\}.$$

Hint: Apply the implicit function theorem.

(4) (The volume form on a Kähler manifold)

Given a Riemannian manifold (M, g) of real dimension n, we may define the Riemannian volume form $dV = \sqrt{\det g_{ij}}dx^1 \wedge \cdots \wedge x^n$. It is a standard fact that the above form is independent of the choice of local coordinates. Now we work on a Kähler manifold (M, J, g) of complex dimension n, with the corresponding Kähler form

$$\omega(X, Y) = g(JX, Y), \quad \text{where } X, Y \in T_pM, \quad p \in M.$$

After we extend T_pM following $T_pM \bigotimes \mathbb{C} = T_p^{1,0}M \oplus T_p^{0,1}M$, we may write

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\overline{z_j}.$$

Now prove the Riemannian volume form the Kähler manifold is given by $\dfrac{\omega^n}{n!}$.

(5) (Curvature tensors of the Fubini-Study metric)

Recall we introduce the Fubini-Study metric on $\mathbb{CP}^n$. Let $[w_0, \cdots, w_n]$ be homogeneous coordinates and define

$$U_0 = \{[w_0, \cdots, w_n] \in \mathbb{CP}^n \mid w_0 \neq 0\}.$$

Then we choose local holomorphic coordinates $z_i = \frac{w_i}{w_0}$ on U_0 and define a Kähler metric by $\omega = \sqrt{-1}\partial\bar{\partial}\ln(1 + |z|^2)$ where $|z|^2 = \sum_{i=1}^n |z_i|^2$. We may define ω similarly on other

$$U_k = \{[w_0, \cdots, w_n] \in \mathbb{CP}^n \mid w_k \neq 0\}.$$

Note that ω is independent of the choice of local coordinates in the intersection $U_i \cap U_j$. This is called the Fubini-Study metric on $\mathbb{CP}^n$.

Recall the Riemann curvature tensor of ω is defined by

$$R(X, Y, Z, W) = g(R(U, V)Z, W) \text{ where } X, Y, Z, W \in T_pM.$$

In class we show after extension to $T_pM \bigotimes \mathbb{C} = T_p^{1,0}M \oplus T_p^{0,1}M$ it satisfies

$$R_{i\bar{j}k\bar{l}} = -\partial_k \partial_{\bar{l}} g_{i\bar{j}} + g^{p\bar{q}} \partial_k g_{i\bar{q}} \partial_{\bar{l}} g_{p\bar{j}}$$

Questions:

(A) Prove the Matrix Determinant Lemma

$$\det(A + uv^T) = (1 + v^T A^{-1} u) \det(A)$$

where A is an invertible matrix, and both u, v are column vectors. and the Sherman-Morrsion Lemma

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}$$

where A is invertible and $1 + v^TA^{-1}u \neq 0$.

Remark: These two elementary lemmas are quite useful in calculations on geometry of hypersurfaces in $\mathbb{R}^n$, and Kähler geometry of $\mathbb{CP}^n$.

(B) Show the Fubini-Study metric satisfies

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}} g_{i\bar{j}} + g_{i\bar{l}} g_{k\bar{j}}.$$

(6) (Comparing Ricci curvatures in Kähler geometry and Riemannian geometry)

Let (M, J, g) be a Kähler manifold, prove the Kähler Ricci curvature is the same with Riemannian Ricci curvature.

Namely if $X \in TM$ is a unit vector, and $u = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX)$. it amounts to check $Ric(u) = Ric(X)$. Precisely, we need to prove

$$\sum_{i=1}^{n} R(u, \overline{u}, v_i, \overline{v_i}) = \sum_{i=1}^{n} \big(R(X, e_i, e_i, X) + R(X, Je_i, Je_i, X)\big).$$

where $\{e_i, Je_i\}$ is an orthonormal basis of TM, and $\{v_i = \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}Je_i)\}$ the corresponding unitary frame of $T^{1,0}M$.

Hint: Riemannian curvature tensor satisfies the first Bianchi identity.

(7) (Examples of holomorphic vector fields)

Prove the following two results.

(1) Any holomorphic vector field v on $\mathbb{CP}^1$ has a form $v = (az^2+bz+c)\frac{\partial}{\partial z}$ where a, b, and c are complex numbers. Here z is a local coordinate defined by $z = \frac{w_1}{w_0}$ where $[w_0, w_1] \in \mathbb{CP}^1$.

(2) Any holomorphic vector field v on $\mathbb{CP}^2$ has a form

$$v = (a_1 z_1^2 + a_2 z_1 z_2 + L_1(z_1, z_2))\frac{\partial}{\partial z_1} + (a_1 z_1 z_2 + a_2 z_2^2 + L_2(z_1, z_2))\frac{\partial}{\partial z_2},$$

where L_1 and L_2 are linear functions in the from of $b_i z_1 + c_i z_2 + d_i$ and all constants are real numbers. Here z_1, z_2 are local coordinates defined by $z_i = \frac{w_i}{w_0}$ where $[w_0, w_1, w_2] \in \mathbb{CP}^2$.

Hint: Use the transition function to understand what happens if z tends to infinity.

Remark: It is a fact that the group of biholomorphic self-maps of $\mathbb{CP}^n$ is $PGL(n+1, \mathbb{C})$, which, as a complex Lie group, has complex dimension $n^2 + 2n$. We see this number coincide with the numbers of free parameters in the above.

(8) (Examples of the first Chern class)

Consider the degree-d hypersurface $V_f \subset \mathbb{CP}^n$ defined in Problem (3), Assume the if and only if condition there is satisfied and we get V_f is a smooth manifold. By a slight abuse of notation, let ω_{FS} denote the restriction of the Fubini-Study metric on $\mathbb{CP}^n$ onto V. In class we know ω_{FS} is a Kähler form on V_f, and let $[\omega_{FS}]$ represents the correspond element in $H^2(V_f, \mathbb{R})$. Then prove

$$c_1(V_f) = \frac{n+1-d}{2\pi}[\omega_{FS}]$$

Remark: At this stage we may solve it by a direct differential geometric method. If you need more hints, see Tian's book 'Canonical metrics in Kähler geometry' (Example 2.9 on p.16).

Week 7: 03/29/2023

Summary of lectures:
Curvatures on Kähler manifolds continued. We cover [1] Sec 1.4, p.10-p.13.

Week 8: 04/04/2023

Summary of lectures:
Holomorphic vector bundles and examples. We cover [1] Sec 1.5, p.14-p.15.

Week 9: 04/11/2023

Summary of lectures:

Connections and curvature on line bundles. We cover [1] Sec 1.6, p.16-p.19.

Week 10: 04/18/2023

Summary of lectures:

An example on the Kodaira vanishing theorem and harmonic functions. We cover [1] Sec 1.7 and 2.1, pp.19-21, 23-25.

Week 10: 04/23/2023 (replaces the 05/02 lecture)

Summary of lectures:

Hölder spaces and Schauder estimates on Laplacian equations. We cover [1] Sec 2.3, p.27-p.30, and use the book by Frenández-Real and Ros-Oton [3] for references.

Week 11: 04/25/2023

Summary of lectures:

Schauder estimates on Laplacian and elliptic equations continued. We cover [1] Sec 2.3, p.27-p.30, and [3] Sec 2.2 and 2.3.

Week 13: 05/09/2023

Summary of lectures:

We finish Schauder estimates on Laplacian. We cover [3] pp.41-44.

Homework 2, assigned on 05/15, due on 06/13 in class

Instruction: Choose at least 3 problems among the following 8 problems.

(1) (Poisson Integral formula)

Let $B(0,R)$ be the ball centered at the origin in $\mathbb{R}^n$. For any points $x \in B(0,R)$ and $y \in \partial B(0,R)$, we define

$$K(x,y) = \frac{R^2 - |x|^2}{\omega_n R|x-y|^n}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Given any function $\varphi \in C(\partial B(0,R))$, we introduce

$$u(x) = \begin{cases} \int_{\partial B(0,R)} K(x,y)\varphi(y) dA_y, \ x \in B(0,R) \\ \varphi(x) x \in \partial B(0,R) \end{cases}$$

Question: Prove $u \in C^\infty(B(0,R)) \cap C(\partial B(0,R))$; Moreover, u satisfies $\Delta u = 0$ inside $B(0,R)$ and $u = \varphi$ on $\partial B(0,R)$.

Remark: We use this fact when discussing X.J. Wang's approach to Schauder estimates on Laplacian.

(2) (Counting harmonic polynomials)

Let $\mathcal{P}_k$ be the space of real homogeneous polynomials on $\mathbb{R}^n$ with degree k. By the conclusion of a problem in HW1, we know

$$\dim(\mathcal{P}_k) = \binom{n+k-1}{n-1}.$$

Now let $\mathcal{H}_k$ be homogeneous harmonic polynomials on $\mathbb{R}^n$ with degree k, and $\mathcal{H}_{\leq k}$ be homogeneous harmonic polynomials on $\mathbb{R}^n$ with degree less or equal to k.

Question: follow the steps to study the dimensions of $\mathcal{H}_k$ and $\mathcal{H}_{\leq k}$.

Step 1: Given any $P, Q \in \mathcal{P}_k$, we introduce

$$< P, Q > \doteq P(\partial)(Q) = \sum_{|\alpha|=k} \alpha! P_\alpha Q_\alpha$$

Here we use $\alpha = (\alpha_1, \cdots, \alpha_n)$ as a multi-index, $\alpha! = \prod_{l=1}^n (\alpha_l!)$, and $P(\partial)$ represents a mixed partial derivative, for example if $P(x) = x_1^2 + x_1x_2$ then $P(\partial) = \partial_1^2 + \partial_1\partial_2$.

Show $< P, Q >$ defines an innner product on $\mathcal{P}_k$, and $\mathcal{H}_k$ is the orthogonal complement of

$$S_k = \left\{ P \in \mathcal{P}_k \;\middle|\; P(x) = |x|^2 P_1(x),\ P_1 \in \mathcal{P}_{k-2} \right\}.$$

In other words, prove the orthogonal (with respect to $< P, Q >$) direct sum decomposition.

$$\mathcal{P}_k = \mathcal{H}_k \oplus |x|^2 \mathcal{P}_{k-2}.$$

Step 2: Prove

$$\begin{aligned} \dim(\mathcal{H}_k) &= \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1} \\ &= \binom{n+k-2}{n-2} + \binom{n+k-3}{n-2}. \end{aligned}$$

Here we set $\binom{n+k-3}{n-1} = 0$ when $k \leq 1$.

Step 3: Prove

$$\dim(\mathcal{H}_{\leq d}) = \binom{n+d-1}{n-1} + \binom{n+d-2}{n-1}.$$

and $\dim(\mathcal{H}_{\leq d}) \sim \frac{2}{(n-1)!} d^{n-1}$ as $d \to \infty$.

Hint for Step 1: Consider the linear map $\varphi : \mathcal{P}_k \to \mathcal{P}_{k-2}$ defined by $\varphi(P) = \Delta P$, show φ is onto, and also note the adjoint map of φ is $\varphi^*(Q) = |x|^2 Q(x)$.

Note: If you need more help, just search harmonic polynomials online, there should be a lot of references coming up. A specific one I have in mind is 苗长兴的书" 调和分析及其在偏微分方程中的应用(第二版)科学出版社" Sec 3.2.

(3) (Gradient estimates for harmonic functions with explicit constants)

On p.24 in [1], Corollary 2.3 leads to gradient (higher order) estimates for harmonic functions. It states:

Proposition A: Let $B(0,R)$ denote the ball centered at the origin with radius R in $\mathbb{R}^n$. For any given $k \in \mathbb{N}$, there exists constants C which only depends n and k so that for any harmonic function $u : B(0,2R) \to \mathbb{R}$,

$$\sup_{B(0,1)} |D^k u| \le \frac{C}{R^k} \sup_{B(0,2R)} |u|.$$

Here we $D^k f$ represents multi-index partial derivatives of u.

It is possible to find an explicit expression of C, thought it might not be sharp. For example in Han-Lin's book ('Elliptic partial differential equations', Prop 1.13 on p.5) it is proved $C = k^n e^{n-1} n!$.

Question: Prove Han-Lin's result mentioned above. Namely Proposition A holds with $C = k^n e^{n-1} n!$.

Note: If you follow the method in Han-Lin, make sure you understand the detailed arguments and write solutions by your own.

(4) (Gradient estimates for (positive) harmonic functions)

Proposition: (Gradient estimates (also called the Harnack inequality)) There exists some constant $C > 0$ which only depends on n so that for any strictly positive harmonic function in an open set Ω containing $B(0,2R)$ in $\mathbb{R}^n$.

$$\sup_{B(0,R)} \frac{|Du|}{u} \le \frac{C}{r}$$

Question: Follow the steps in below to prove the Proposition.

Step 1: Let $v = \ln u, Q = |\nabla v|^2$, it is straightforward to check

$$\Delta Q + 2 < \nabla v, \nabla Q >= 2|\operatorname{Hess} v|^2 = \sum_{k,l} |v_{kl}|^2.$$

Fix a point $x_0 \in B(0,2R)$, we further assume the gradient of v at x_0 has a specific form $\nabla v(x_0) = (|\nabla v(x_0)|, 0, \cdots, 0)$. Then at x_0, We use Cauchy-Schwarz to get

$$\left|\nabla(|\nabla v|^2)\right|^2 \le 4|\nabla v|^2 \sum_{j=1}^n (v_{1j})^2.$$

and

$$\begin{aligned}\sum_{k,l} |v_{kl}|^2 &\ge v_{11}^2 + 2\sum_{\alpha=2}^n v_{1\alpha}^2 + \sum_{\alpha=2}^n v_{\alpha\alpha}^2 \\ &\ge v_{11}^2 + 2\sum_{\alpha=2}^n v_{1\alpha}^2 + + \frac{(\Delta v - v_{11})^2}{n-1} \\ &\ge \frac{n}{n-1}\sum_{j=1}^n v_{1j}^2 + \frac{Q^2}{n-1} + \frac{2v_{11}Q}{n-1}.\end{aligned}$$

Moreover,

$$2v_1v_{11} = \frac{< \nabla(|\nabla v|^2), \nabla v >}{|\nabla v|}$$

which leads to

$$v_{11} = \frac{< \nabla Q, \nabla v >}{2Q}.$$

We assemble the above all together to get

$$\Delta Q + \frac{2(n-2)}{n-1} < \nabla v, \nabla Q > \geq \frac{2}{n-1}Q^2 + \frac{2n}{n-1}\frac{|\nabla Q|^2}{4Q}.$$

Step 2: Pick a smooth function ϕ defined on $[0, \infty)$ so that $\phi = 1$ on $[0, R)$, $0 < \phi < 1$ on $(R, 2R)$, and $\phi = 0$ on $[R, +\infty)$. Moreover there exists a constant C so that

$$-\frac{C}{R} \leq \frac{\phi'}{(\phi)^{\frac{1}{2}}} \leq 0, \quad x \in (R, 2R)$$

and

$$|\phi''| \leq \frac{C}{R^2} \quad x \in (R, 2R)$$

Now we define the function $G = \phi(|x|)Q$ where $|x| = \sqrt{\sum_i x_i^2}$. Assume x_0 is a point inside $B(0, 2R)$ where G attains its positive maximum inside $B(0, 2R)$.

Explain that we may choose a unitary change of coordinates so that

$$\nabla v(x_0) = (|\nabla v(x_0)|, 0, \cdots, 0).$$

Now we have $\Delta G(x_0) \leq 0$ and $\nabla G(x_0) = 0$.

Step 3: Now we solve $\Delta G(x_0)$. Note that we have

$$\nabla Q = \frac{\nabla G - Q\nabla\phi}{\phi}$$

since $\nabla G(x_0) = 0$. Now we begin with

$$\begin{aligned}
\Delta G(x_0) &= (\Delta\phi)Q + 2 < \nabla\phi, \nabla Q > +\phi\Delta Q \\
&\geq \frac{\Delta\phi}{\phi}G + \frac{2 < \nabla\phi, \nabla G >}{\phi} - \frac{2|\nabla\phi|^2 G}{\phi^2} \\
&\quad + \phi\Big[\frac{2}{n-1}Q^2 + \frac{n}{2(n-1)}\frac{|\nabla Q|^2}{Q} - \frac{2(n-2)}{n-1} < \nabla v, \nabla Q > \Big]
\end{aligned}$$

It can be further simplified as

$$< \nabla v, \nabla Q > = \frac{< \nabla v, \nabla G >}{\phi} - \frac{< \nabla v, \nabla\phi >}{\phi}Q.$$

and

$$|\nabla Q|^2 = \frac{|\nabla G|^2}{\phi^2} + \frac{|\nabla\phi|^2 G^2}{\phi^4} - \frac{2 < \nabla G, \nabla\phi > G}{\phi^3}.$$

If we also note $| < \nabla v, \nabla\phi > | \leq |\nabla\phi|\frac{G^{\frac{1}{2}}}{\phi^{\frac{1}{2}}}$, then finally we can solve

$$\begin{aligned}
\Delta G(x_0) \geq{}& \frac{\Delta\phi}{\phi}G + \frac{n-2}{n-1}\frac{< \nabla\phi, \nabla G >}{\phi} - \frac{3n-4}{2(n-1)}\frac{|\nabla\phi|^2}{\phi^2}G \\
&+ \frac{n}{2(n-1)}\frac{|\nabla G|^2}{G} - \frac{2(n-2)}{n-1} < \nabla v, \nabla G > \\
&- \frac{2(n-2)}{n-1}|\nabla\phi|\frac{G^{\frac{3}{2}}}{\phi^{\frac{3}{2}}} + \frac{2}{n-1}\frac{G^2}{\phi}.
\end{aligned}$$

Recall $\Delta G(x_0) \le 0$ and $\nabla G(x_0) = 0$, we can simplify

$$0 \ge \Delta G(x_0) \ge (\Delta\phi)G - \frac{3n-4}{2(n-1)}\frac{|\nabla\phi|^2}{\phi}G - \frac{2(n-2)}{n-1}|\nabla\phi|\frac{G^{\frac{3}{2}}}{\phi^{\frac{1}{2}}} + \frac{2}{n-1}G^2.$$

Note that $\nabla\phi(x_0) = \phi'\frac{x}{|x|}$ and $\Delta\phi = \phi'' + \phi'\frac{(n-1)}{r}$. Therefore we get $|\Delta\phi| \le \frac{C}{R^2}$.

From this, we get

$$0 \ge -\frac{C}{R^2}G - \frac{C}{R}G^{\frac{3}{2}} + \frac{2}{n-1}G^2.$$

From this we get $G(x_0) \le \frac{C}{R^2}$ for some constant C which only depends on n.

In the end we have

$$\sup_{B(0,R)} |\nabla \ln u| \le \sqrt{G(x_0)} \le \frac{C}{R}.$$

Note: This type of gradient estimate is proved (Lemma 1.32 on p.16) in Han-Lin's book. The steps we outline here is more involved, which is motivated from Peter Li's book 'Geometric analysis' pp.58-60. We choose to follow this way as it could be more helpful if we work on gradient estimates on Riemannian manifolds.

(5) (Absolutely monotonic functions)

We say a smooth function u on $[0,T)$ is **absolutely monotonic** in $[0,T)$ if u and all its derivatives are nonnegative. By smooth at the end point 0, we mean u is continuous on $[0,T)$, then we set $u'(0)$ as the right derivative $\lim_{t\to 0+}\frac{u(t)-u(0)}{t}$ and require $u'(x)$ is continuous on $[0,T)$, moreover we require the same holds for all higher order derivatives.

Question A: Any absolutely monotonic function u on $[0,T)$ is real analytic in $(0,T)$. Moreover, it extends to a real analytic function in $(-T,T)$.

Question B: Let u be absolutely monotonic on $[0,T)$, then $\ln u(e^r)$ is a convex function of r. In other words, $\ln u(x)$ is a convex function of $\ln x$ for any $x \in (0,T)$

Recall a function f defined on (a,b) is real analytic if for any given $x_0 \in (a,b)$ there exists some $\delta > 0$ so that its Taylor series at converges to $f(x)$ on $(x_0-\delta, x_0+\delta)$.

Note: If you need some hints, check Polya-Szego's book 'Problems and theorems in analysis, I' Problem 123 on p.80.

(6) (A version of L^2 three circle theorem for harmonic functions)

Let $u : B(0,R) \to \mathbb{R}$ be harmonic, and we define

$$q_u(r) \doteq \frac{1}{|S^{n-1}(r)|}\int_{S^{n-1}(r)} u^2 dA_r.$$

Question A: Show $q_u(r)$ is absolutely monotonic on $[0,R)$.

Question B: Combined with the conclusion of Problem (5), we get that $\ln q_u(r)$ is a convex function of $\ln r$ for any $r \in (0,R)$.

The conclusion in Question B is called a three circle theorem. These results are stated in Lippner-Mangoubi's paper 'Harmonic functions on the lattice: absolute monotonicity and propagation of smallness.' Duke Math. J. 164 (2015), no. 13, 2577-2595.

(7) (Interpolation inequalities for Hölder norms)

Given a bounded domain $\Omega \subset \mathbb{R}^n$, recall we define Hölder spaces $C^{k,\alpha}(\Omega)$ in lecture, following p.27 in [1] or p.6 in [3].

Question: Let $\Omega = B(0,R)$ the ball centered at the origin in $\mathbb{R}^n$. Prove for any $u \in C^{2,\alpha}(\Omega)$ and $\epsilon > 0$, there exists some C_ϵ (independent of u) so that

$$\|u\|_{C^{1,1}(\Omega)} \leq C_\epsilon \|u\|_{L^\infty(\Omega)} + \epsilon [D^2 u]_{C^{0,\alpha}(\Omega)}$$

Note: We use this inequality to reduce Schauder estimates for $\Delta u = f$ to the estimate of the semi-norm $[D^2 u]_{C^{0,\alpha}(\Omega)}$.

Hint: You may adapt the proof of Lemma 6.35 on p.135 in Gilbarg-Trudinger's book.

(8) (Schauder estimates fail when the right hand side is merely functions)

Part A: (From Sec 2.2 in [3])

Consider

$$u(x,y) = (x^2 - y^2) \ln \left| \ln(x^2 + y^2) \right|$$

defined on some neighborhood of the origin in $\mathbb{R}^2$.

Question A: Show we can find some $\delta > 0$ so that u_{xx} and u_{yy} exists, and Δu is continuous on $D(0,\delta)$. However, $u \notin C^{1,1}$ in any open neighborhood of the origin.

Part B: (From p.71 in Gilbarg-Trudinger's book)

Let $D(0,2)$ denote the open disk centered at the origin with radius 2 in $\mathbb{R}^2$. Consider a multi index α with $|\alpha| = 2$, and $P(x)$ a homogeneous harmonic polynomial of degree 2 with $D^\alpha P$ not identically zero. For example, $P = x_1 x_2$ with $\alpha = (1,2)$, then $D^\alpha P = 1$. Choose $\eta \in C_0^\infty(D(0,2))$ with $\eta = 1$ on $D(0,1)$. Define

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k} \Delta(\eta P)(2^k x).$$

Question B: Show f is continuous but any function u which satisfies $\Delta u = f$ is not C^2 in any neighborhood of the origin in $\mathbb{R}^2$.

Week 14: 05/16/2023

Summary of lectures:

We discuss some preliminary results on Schauder estimates on uniformly elliptic equations $\sum_{i,j} a_{ij} u_{ij}$. The idea is to freeze coefficients and use the Schauder estimates for Laplacian. We cover [3] pp.46-51.

Week 15: 05/23/2023

Summary of lectures:

Schauder estimates for uniformly elliptic equations $\sum_{i,j} a_{ij} u_{ij}$, continued

Week 16: 05/30/2023

Summary of lectures:

We discuss another approach to Schauder estimates on uniformly elliptic equations $\sum_{i,j} a_{ij}u_{ij}$. This is the blow up method due to Leon Simon. We cover [3] pp.51-55.

Week 17: 06/06/2023

Summary of lectures:

Poisson equation on Riemannian manifolds.

Week 18: 06/13/2023

Summary of lectures:

Poisson equation on Riemannian manifolds.

References

[1] Szekelyhidi, G.. *An introduction to extremal Kähler metrics.* Graduate Studies in Mathematics, Vol.152, American Mathematical Society, 2014.

[2] Chapter 1 of Kodaira's book '*Complex manifolds and deformation of complex structures*', it contains a consise and nice introduction to basic facts on holomorphic functions of several variables.

[3] Frenández-Real, X. and Ros-Oton, X.. *Regularity theory for elliptic PDE.* Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2022.