CMSE 820: Homework #7

Due on Nov 3, 2019 at 11:59pm

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Problem 1

Solution

Let $k_1(\cdot,\cdot)$ and $k_2(\cdot,\cdot)$ be two reproducing kernels associated with an arbitrary RKHS $\mathcal{H}$ over $\mathcal{X}$. Note that $\forall x \in \mathcal{X}, k_1(\cdot,x) \in \mathcal{H}$ and $k_2(\cdot,x) \in \mathcal{H}$. Moreover, by the reproducing property, we have

$$\forall x, y \in \mathcal{X}, \quad k_1(x,y) = k_1(y,x) = \langle k_1(\cdot,x), k_2(\cdot,y)\rangle_{\mathcal{H}} = \langle k_2(\cdot,y), k_1(\cdot,x)\rangle_{\mathcal{H}} = k_2(x,y).$$

So the reproducing kernel is unique.

Problem 2

Solution

$\forall N \in \mathbb{N}^+, \forall x_1, x_2, \cdots, x_N \in \mathcal{X}, \forall c_1, c_2, \cdots, c_N \in \mathbb{R}$

$$\sum_{i,j} c_i c_j f(x_i) k(x_i,x_j) f(x_j) = \sum_{i,j} [c_i f(x_i)][c_j f(x_j)] k(x_i,x_j) \geq 0,$$

which implies immediately the positive semi-definiteness of $\tilde{k}(x,y)$. (The last inequality follows from the positive semi-definiteness of $k(x,y)$.

Problem 3

Solution

Rewrite the kernel function as

$$k(x,y) = \min\{x,y\} = \int_0^1 \mathbf{1}_{[0,x]}(z)\mathbf{1}_{[0,y]}(z)dz.$$

$\forall N \in \mathbb{N}^+, \forall x_1, x_2, \cdots, x_N \in [0,1], \forall c_1, c_2, \cdots, c_N \in \mathbb{R}$,

$$\sum_{i,j} c_i c_j \min\{x_i,x_j\} = \sum_{i,j} \int_0^1 [c_i \mathbf{1}_{[0,x_i]}(z)][c_j \mathbf{1}_{[0,x_j]}(z)]dz = \int_0^1 [\sum_i c_i \mathbf{1}_{[0,x_i]}(z)]^2 dz \geq 0.$$

So k is positive semi-definite.

Problem 4

Proof (1)

$\forall x \in [0,1]$, we can define a function $g_x(z) = \min\{x,z\} = \begin{cases} z, & z \leq x \\ x, & z > x \end{cases}$. It is obvious that $g_x'(z) = \begin{cases} 1, & z < x \\ 0, & z > x \end{cases}$ is bounded and continuous almost everywhere (except at $z = x$), and hence integrable on [0, 1], which shows the absolute continuity of g_x. Moreover, $g_x(0) = 0$ so $g_x \in \mathcal{H}^1$.
Now consider the inner product $\langle f, g_x\rangle_{\mathcal{H}^1}$ for any arbitrary $x \in [0,1]$ and any $f \in \mathcal{H}^1$.

$$\langle f, g_x\rangle_{\mathcal{H}^1} = \int_0^x f'(z)\cdot 1 dz + \int_x^1 f'(z)\cdot 0 dz = \int_0^x f'(z)dz = f(x)$$

Proof (2)

The equation above allows us to rewrite the evaluational functional as $\delta_x(f) = f(x) = \langle f, g_x\rangle_{\mathcal{H}^1}$. We'll show that δ_x is bounded.

$$\forall f \in \mathcal{H}^1, |\delta_x(f)| = |\langle(f, g_x\rangle_{\mathcal{H}^1}| \leq \|g_x\|_{\mathcal{H}^1}\|f\|_{\mathcal{H}^1}$$

The operator norm of δ_x

$$\|\delta_x\| \leq \|g_x\|_{\mathcal{H}^1} = (\langle g_x, g_x\rangle_{\mathcal{H}^1})^{\frac{1}{2}} = \sqrt{x} \leq 1.$$

Thus, δ_x is bounded and $mathcalH^1$ is an RKHS.