CMSE 820 HW4

This HW is due on Oct 6th at 11:59 pm.

Question 1: For the statistical view of PCA, finish the proof for the theorem of Principal Components of a Random Vector assuming there is no repeated eigenvalues. Namely, you need to prove u_2 is indeed the second eigenvector of Σ_x and then generalize it to the rest of u_j.

Question 2: Suppose A is a symmetric real matrix $\in \mathbb{R}^{n\times n}$, and $\lambda_i(A)$ is the ith largest eigen value of A. Prove that

a.

$$\lambda_1(A) = \max_{\|v\|_2^2=1} v^T A v.$$

b.

$$\lambda_n(A) = \min_{\|v\|_2^2=1} v^T A v.$$

Question 3: Prove the Weyl's inequality: For symmetric real matrices $A, E \in \mathbb{R}^{n\times n}$. For $1 \leq k \leq n$,

$$\lambda_k(A) + \lambda_n(E) \leq \lambda_k(A+E) \leq \lambda_k(A) + \lambda_1(E),$$

where $\lambda_i(M)$ represents the ith largest eigenvalue of matrix M.
Hint:you can apply the Courant−Fishcer theorem to prove it.

Theorem[Courant-Fishcer] For a symmetric real matrix $A \in \mathbb{R}^{n\times n}$ and $1 \leq k \leq n$, we have

$$\lambda_k(A) = \max_{\dim(V)=k} \min_{v\in V:\|v\|=1} v^T A v; \lambda_k(A) = \min_{\dim(V)=n-k+1} \max_{v\in V:\|v\|=1} v^T A v$$

Question 4: Face recognition using PCA. In this exercise you will use a small subset of the Yale B dataset, which contains photos of ten individuals under various illumination conditions. Specifically, you will use only images from the first three individuals under ten different illumination conditions. Download the file YaleB-Dataset.zip. This file contains the image database along with the MATLAB function loadimage.m. Decompress the file and type help loadimage at the MATLAB prompt to see how to use this function.

a. Write your own code for PCA (you can use any program language).

b. Apply PCA with $d = 2$ to all 10 images from individual

   1. Plot the mean face μ and the first two eigenfaces u_1 and u_2. What do you observe?

   2. Plot $\mu + a_i u_i$ for $a_i = -\sigma_i, -0.8\sigma_i, -0.6\sigma_i, \ldots, 0.6\sigma_i, 0.8\sigma_i, \sigma_i$ with $i \in \{1, 2\}$. Here, σ_i is the standard deviation of $y_{[i]}$ (the ith principal component). What do the first two principal components capture?

. Repeat for individuals 2 and 3.