CMSE 820 HW5

This HW is due on Oct 20th at 11:59 pm.

Question 1: Let $X \in \mathbb{R}^{p\times n}$ be a matrix. Show that the matrix $\ell_{2,1}$ norm of X is defined as

$$\|X\|_{2,1} = \sum_{j=1}^{n} \|X_{\cdot,j}\|_2 = \sum_j \sqrt{\sum_{i=1}^{p} X_{ij}}.$$

a. Prove that $\|X\|_{2,1}$ is a convex function of X.

b. Show that the subgradient of $\|X\|_{2,1}$ is given by

$$\left(\partial \|X\|_{2,1}\right)_{ij} = \begin{cases} \frac{X_{ij}}{\|X_{\cdot,j}\|_2}, & \text{if } X_{\cdot,j} \neq 0 \\ W_{ij} : \text{with } \|W_{\cdot,j}\|_2 \leq 1, & \text{if } X_{\cdot,j} = 0 \end{cases}$$

c. Show that the optimal solution of

$$\arg\min_A \frac{1}{2}\|X - A\|_F^2 + \tau\|A\|_{2,1}$$

is given by

$$A = X\mathcal{S}_\tau\big(\text{diag}(x)\big)\text{diag}(x)^{-1},$$

where x is a vector with $x_{[j]} = \|X_{\cdot,j}\|_2$, and $\text{diag}(x)$ is a diagonal matrix with the entries of x along its diagonal. By convention, if $x_{[j]} = 0$ then the jth of $\text{diag}(x)^{-1}$ is also zero.

Question 2:

a. If $f_s(\cdot)$ is convex for any $s \in S$, prove that $f(x) = max_{s\in S} f_s(x)$ is convex. Note that the set S here (number of functions $f_s(\cdot)$) can be infinite.

b. Prove that the dual problem for any general minimization problem is a convex optimization problem.

Question 3: Consider quadratic program:

$$\min_x \frac{1}{2}x^T Qx + c^T x$$
$$\text{subject to } Ax = b, x \geq 0,$$

where $x \in \mathbb{R}^p$ and $Q \in \mathbb{R}^{p\times p}$ is a positive definite matrix. Now, derive its dual problem.

Question 4: Use a small subset of the Yale B data set (provided in HW 3) that contains photos of ten individuals under various illumination conditions. Specifically, you will use only images from the first three individuals under ten different illumination conditions. Remove uniformly at random 0%, 10%, 20%, 30%, and 40% of the entries of all images of individual 1. You will generate X by stacking the image into a long vector. Here, each column of X is an image, and the number of row is the total number of pixel for each image. Apply the low-rank matrix completion (LRMC) algorithm introduced in lecture 18 to X to fill in the missing entries. Plot the completed faces and comment on the quality of completion as a function of the percentage of missing entries by visually comparing the original images (before removing the missing entries) to the completed ones. Plot also the error (Frobenius norm) between the original images and the completed ones as a function of the percentage of missing entries and comment on your results. Repeat for individuals 2 and 3.