CMSE 820 HW6

This HW is due on Oct 27th at 11:59 pm.

Question 1: Show that the kernel function associated with any reproducing kernel Hilbert space must be unique. Namely, given a RKHS $\mathcal{H}$, if there are two kernels, k_1 and k_2, associate with $\mathcal{H}$ (with the reproducing property), then $k_1(x, y) = k_2(x, y)$ for all $x, y \in \mathbb{X}$.

Question 2: Let $k : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be a positive semidefinite kernel, and let $f : \mathbb{X} \to [0, \infty)$ be an arbitrary function. Show that $\widetilde{k}(x, y) = f(x)k(x, y)f(y)$ is also a positive semidefinite kernel.

Question 3: Show that the kernel $k : [0, 1] \times [0, 1] \to \mathbb{R}$ given by $k(x, z) = min\{x, z\}$ is positive semi-definite.

Question 4: A function f over $[0, 1]$ is said to be absolutely continuous if its derivative f' satisfies $\int_0^1 f'(x)dx \leq \infty$, and we have $f(x) = f(0) + \int_0^x f'(z)dz$ for all $x \in [0, 1]$. Now consider the set of functions

$$\mathcal{H}^1[0, 1] = \{f : [0, 1] \to \mathbb{R} | f(0) = 0, \text{ and } f \text{ is absolutely continuous with } f' \in L_2([0, 1])\}.$$

We define an inner product on this space via $\langle f, g \rangle_{\mathcal{H}}^1 = \int_0^1 f'(z)g'(z)dz$, and claim that the resulting Hilbert space is an RKHS. You can prove it by the following steps:

- Prove the representer of evaluation at $x \in [0, 1]$ for $\mathcal{H}$ is $g_x(z) = \min\{x, z\}$. Namely, you need to prove $g_x \in \mathcal{H}$ and $\langle g_x, f \rangle_{\mathcal{H}} = f(x)$ for any $f \in \mathcal{H}$

- Define the evaluation functional δ_x as $\delta_x(f) = \langle f, g_x \rangle_{\mathcal{H}}$ and prove that it is continuos or bounded.