

## 1.3 解线性方程组的消元法

**例 3** 求下列线性方程组的解

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 4 \\ 2x_1 + 4x_2 - 6x_3 + 4x_4 = 8 \\ 2x_1 - 3x_2 + x_3 - x_4 = 2 \\ 3x_1 + 6x_2 - 9x_3 + 7x_4 = 9. \end{cases} \quad (\mathbf{L}_1)$$

解

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 4 \\ 2x_1 + 4x_2 - 6x_3 + 4x_4 = 8 \\ 2x_1 - 3x_2 + x_3 - x_4 = 2 \\ 3x_1 + 6x_2 - 9x_3 + 7x_4 = 9 \end{cases}$$

Diagram illustrating row operations:

- Row 1 is multiplied by  $-2$  and added to Row 2 (indicated by a purple arrow and  $\times(-2)$ ).
- Row 1 is multiplied by  $-2$  and added to Row 3 (indicated by a blue arrow and  $\times(-2)$ ).
- Row 1 is multiplied by  $-3$  and added to Row 4 (indicated by a green arrow and  $\times(-3)$ ).

第1个方程乘以 $(-2)$ 分别加到第2,第3个方程,

第1个方程乘以 $(-3)$   
加到第4个方程,得

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 4 \\ 2x_2 - 2x_3 + 2x_4 = 0 \\ -5x_2 + 5x_3 - 3x_4 = -6 \\ 3x_2 - 3x_3 + 4x_4 = -3 \end{cases} \quad (L_2)$$

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 4 \\ 2x_2 - 2x_3 + 2x_4 = 0 \\ -5x_2 + 5x_3 - 3x_4 = -6 \\ 3x_2 - 3x_3 + 4x_4 = -3 \end{cases} \quad (\mathbf{L}_2)$$

第 2 个方程乘以  $\frac{1}{2}$ , 得

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 4 \\ x_2 - x_3 + x_4 = 0 \\ -5x_2 + 5x_3 - 3x_4 = -6 \\ 3x_2 - 3x_3 + 4x_4 = -3 \end{cases} \quad (\mathbf{L}_3)$$

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 4 \\ x_2 - x_3 + x_4 = 0 \\ -5x_2 + 5x_3 - 3x_4 = -6 \\ 3x_2 - 3x_3 + 4x_4 = -3 \end{cases} \quad (L_3)$$

Diagram illustrating row operations on the system (L<sub>3</sub>):

- The second equation is multiplied by 5 (indicated by a purple arrow labeled  $\times 5$ ) and added to the third equation.
- The second equation is multiplied by -3 (indicated by a blue arrow labeled  $\times (-3)$ ) and added to the fourth equation.

第2个方程乘以5 加到第3个方程,

第2个方程乘以(-3)

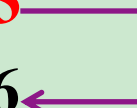
加到第4个方程, 得

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 4 \\ x_2 - x_3 + x_4 = 0 \\ 2x_4 = -6 \\ x_4 = -3 \end{cases} \quad (L_4)$$

$$\left\{ \begin{array}{l} x_1 + x_2 - 2x_3 + x_4 = 4 \\ x_2 - x_3 + x_4 = 0 \\ 2x_4 = -6 \\ x_4 = -3 \end{array} \right. \quad (\mathbf{L}_4)$$

互换第 3, 第 4 两个方程, 得

$$\left\{ \begin{array}{l} x_1 + x_2 - 2x_3 + x_4 = 4 \\ x_2 - x_3 + x_4 = 0 \\ x_4 = -3 \\ 2x_4 = -6 \end{array} \right. \quad (\mathbf{L}_5)$$

$$\left\{ \begin{array}{l} x_1 + x_2 - 2x_3 + x_4 = 4 \\ x_2 - x_3 + x_4 = 0 \\ x_4 = -3 \times (-2) \\ 2x_4 = -6 \end{array} \right. \quad (\mathbf{L}_5)$$


第3个方程乘以(-2)加到第4个方程,  
并去掉方程 $0=0$ 得

$$\left\{ \begin{array}{l} x_1 + x_2 - 2x_3 + x_4 = 4 \\ x_2 - x_3 + x_4 = 0 \\ x_4 = -3 \end{array} \right. \quad (\mathbf{L}_6)$$

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 4 \\ x_2 - x_3 + x_4 = 0 \\ x_4 = -3 \end{cases} \quad (\text{L}_6)$$

第3个方程乘以 $(-1)$ 分别加到第1,第2个方程,得

$$\begin{cases} x_1 + x_2 - 2x_3 = 7 \\ x_2 - x_3 = 3 \\ x_4 = -3 \end{cases} \quad (\text{L}_7)$$



$$\begin{cases} x_1 + x_2 - 2x_3 = 7 \\ \quad \quad \quad x_2 - x_3 = 3 \\ \quad \quad \quad \quad x_4 = -3 \end{cases} \quad \begin{matrix} \leftarrow \\ \leftarrow \end{matrix} \times(-1) \quad (L_7)$$

第2个方程乘以(-1)加到第1个方程,得

$$\begin{cases} x_1 - x_3 = 4 \\ \quad x_2 - x_3 = 3 \\ \quad \quad \quad x_4 = -3 \end{cases} \quad (L_8)$$

因此,原方程组

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 4 \\ 2x_1 + 4x_2 - 6x_3 + 4x_4 = 8 \\ 2x_1 - 3x_2 + x_3 - x_4 = 2 \\ 3x_1 + 6x_2 - 9x_3 + 7x_4 = 9 \end{cases}$$

与方程组  $\begin{cases} x_1 - x_3 = 4 \\ x_2 - x_3 = 3 \\ x_4 = -3 \end{cases}$  是同解的.

将方程组  $\begin{cases} x_1 - x_3 = 4 \\ x_2 - x_3 = 3 \\ x_4 = -3 \end{cases}$  中每个方程的第1个

未知数项留在等式左边, 其余项都移到等式右边, 得

$$\begin{cases} x_1 = 4 + x_3 \\ x_2 = 3 + x_3 \\ x_4 = -3 \end{cases} \quad (\mathbf{L}_9)$$

在  $\begin{cases} x_1 = 4 + x_3 \\ x_2 = 3 + x_3 \\ x_4 = -3 \end{cases}$  中, 对  $x_3$  的任意一个赋值  $x_3 = c$ ,

可得方程组  $\begin{cases} x_1 - x_3 = 4 \\ x_2 - x_3 = 3 \\ x_4 = -3 \end{cases}$  的一个解:  $\begin{cases} x_1 = 4 + c \\ x_2 = 3 + c \\ x_3 = c \\ x_4 = -3 \end{cases}$

而且,如果  $s_1, s_2, s_3, s_4$  是方程组

$$\begin{cases} x_1 - x_3 = 4 \\ x_2 - x_3 = 3 \\ x_4 = -3 \end{cases}$$

的一个解,那么令  $s_3 = c$ , 则有

$$\begin{cases} s_1 = 4 + c \\ s_2 = 3 + c \\ s_3 = c \\ s_4 = -3 \end{cases} \quad \begin{cases} x_1 = 4 + c \\ x_2 = 3 + c \\ x_3 = c \\ x_4 = -3 \end{cases}$$

因此, 方程组  $(L_1)$  的解的一般形式为

$$\begin{cases} x_1 = 4 + c \\ x_2 = 3 + c \\ x_3 = c \\ x_4 = -3 \end{cases}$$

其中  $c$  为任意常数. 

例题中的方程组有无穷多个解,最后给出的对所有解的描述称为方程组的**通解**.

$$\begin{cases} x_1 = 4 + c \\ x_2 = 3 + c \\ x_3 = c \\ x_4 = -3 \end{cases} \quad \text{其中 } c \text{ 为任意常数.}$$

例题中解方程组的方法称为**消元法**.

从前面的讨论以及解线性方程组的过程可以看出  
以下几个事实:

- (1) 方程作为整体参加运算.
- (2) 在求解方程组的过程中, 运算在相同未知数的系数之间以及常数项之间进行, 未知数不参加运算.
- (3) 线性方程  $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$  与有序数组  $(a_1, a_2, \cdots, a_n, b)$  一一对应.



#### (4) $m \times n$ 线性方程组

[illegible]

与 $m$ 个按顺序排列的有序数组 一一对应:

$$(a_{11}, a_{12}, \dots, a_{1n}, b_1)$$

$$(a_{21}, a_{22}, \dots, a_{2n}, b_2)$$

.....

$$(a_{m1}, a_{m2}, \dots, a_{mn}, b_m)$$