COMP1010 – Week 10
Discrete Mathematics Fundamentals – Part 1

Dr. Raymond Chiong, Dr. Mira Park, Dr. Mark Wallis

COMP1010 – Introduction to Computing

University of Newcastle

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

FACULTY OF ENGINEERING AND BUILT ENVIRONMENT

www.newcastle.edu.au

What is Discrete Mathematics ?

- The study of mathematical structures which are
  - "discrete" rather than "continuous"
    - i.e. Integers (1, 2, 3, 4)
    - NOT reals (i.e. ¾, Pi, infinity)
  - "countable"
    - can you line up the structures and match them to a set of integers – i.e. 1, 2, 3, 4, 5 ?

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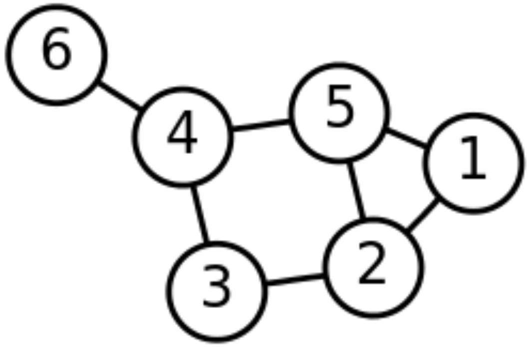
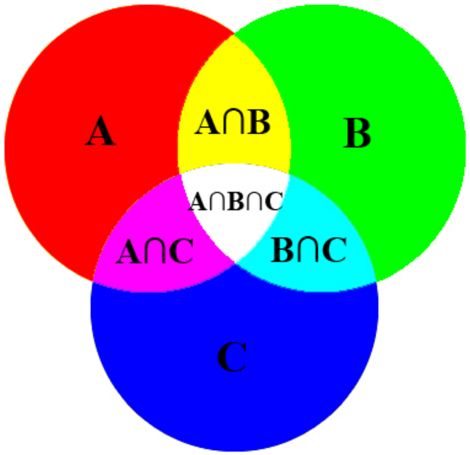
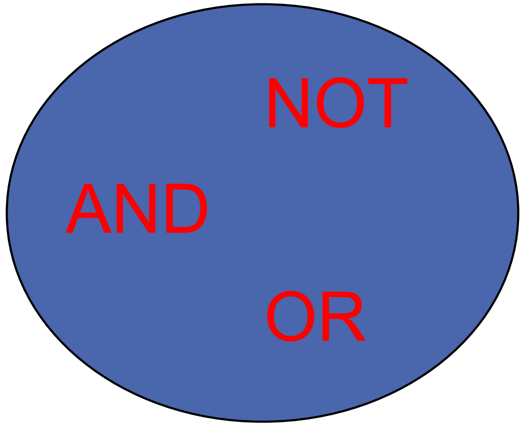


Graph Theory



Set Theory



Logic Statements

and more !

Why are we doing this ?

- Depending on your degree you might have MATH1510 coming up – which is a whole course on Discrete Mathematics

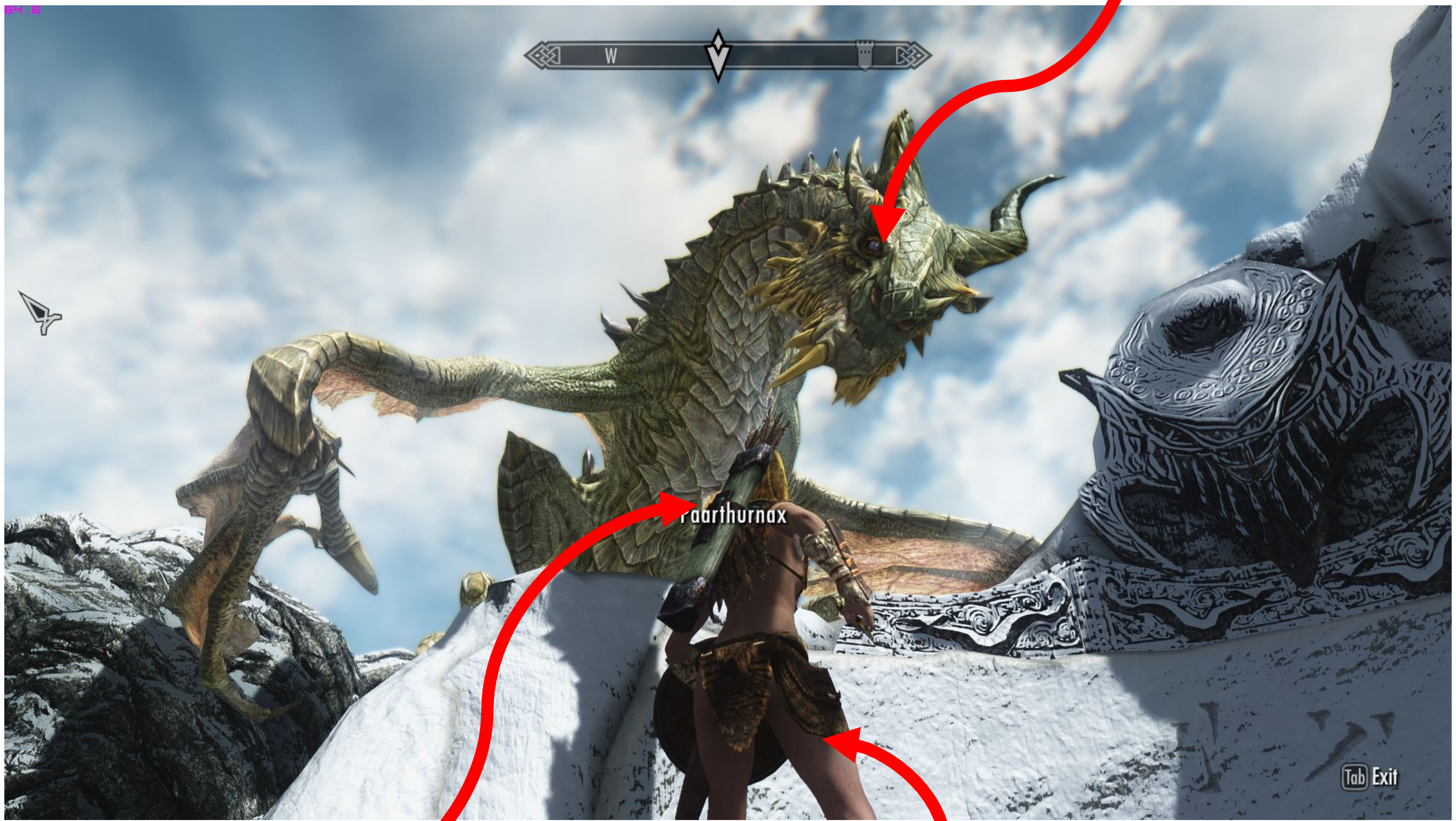
- These two weeks are meant as a light introduction, primarily to ensure you have the prepared for MATH1500

- These two weeks do not replace MATH1510 content, but rather introduce you to the concepts

WHAT RELEVANCE DOES DISCRETE MATHEMATICS HAVE TO ME AS A COMPUTER SCIENTIST / SOFTWARE ENGINEER / INFORMATION TECHNOLOGIST ?

In gaming ….

NPC movement (graph theory)



Damage Calculations
(logic statements)

Inventory (set theory)

In finance …

Trade management (set theory)

Predictive analysis (graph theory)

Trade automational (logic statements)

On the web …

Friend management (graph theory)

Attributes (set theory)

Responsive design (logic statements)

Let’s get started

- This material is based on the following two chapters of your textbook
  - Speaking Mathematically
  - The Logic of Compound Statements

- You will get lots of assistance during the Workshops working through examples of questions from these topics as well.

- The following slides are based on slides provided with the textbook.

Variables

There are two uses of a variable. To illustrate the first use, consider asking

Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

In this sentence you can introduce a variable to replace the potentially ambiguous word "it":

Is there a number x with the property that $2x + 3 = x^2$?

Variables

The advantage of using a variable is that it allows you to give a temporary name to what you are seeking so that you can perform concrete computations with it to help discover its possible values.

To illustrate the second use of variables, consider the statement:

No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

Variables

In this case introducing a variable to give a temporary name to the (arbitrary) number you might choose enables you to maintain the generality of the statement, and replacing all instances of the word "it" by the name of the variable ensures that possible ambiguity is avoided:

No matter what number n might be chosen, if n is greater than 2, then n^2 is greater than 4.

Variables

Here is another example - use variables to rewrite the following sentences more formally.

Q: Are there numbers with the property that the sum of their squares equals the square of their sum?

Solution:

A: Are there numbers a and b with the property that $a^2 + b^2 = (a + b)^2$?

Or: Are there numbers a and b such that $a^2 + b^2 = (a + b)^2$?

Or: Do there exist any numbers a and b such that $a^2 + b^2 = (a + b)^2$?

Mathematical Statements

Three of the most important kinds of sentences in mathematics are universal statements, conditional statements, and existential statements:

A **universal statement** says that a certain property is true for all elements in a set. (For example: *All positive numbers are greater than zero*.)

A **conditional statement** says that if one thing is true then some other thing also has to be true. (For example: *If 378 is divisible by 18, then 378 is divisible by 6*.)

Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true. (For example: *There is a prime number that is even*.)

Mathematical Statements

Universal Condition Statements

Universal statements contain some variation of the words “for all” and conditional statements contain versions of the words “if-then.”

A ***universal conditional statement*** is a statement that is both universal and conditional. Here is an example:

For all animals *a*, if *a* is a dog, then *a* is a mammal.

Mathematical Statements

Universal Existential Statements

A ***universal existential statement*** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something. For example:

Every real number has an additive inverse.

In this statement the property “has an additive inverse” applies universally to all real numbers.

Mathematical Statements

“Has an additive inverse” asserts the existence of something—an additive inverse—for each real number.

However, the nature of the additive inverse depends on the real number; different real numbers have different additive inverses.

Mathematical Statements

Existential Universal Statements

An ***existential universal statement*** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind.

Mathematical Statements

For example:

There is a positive integer that is less than or equal to every positive integer:

This statement is true because the number one is a positive integer, and it satisfies the property of being less than or equal to every positive integer.

Rewriting:

Some positive integer is less than or equal to every positive integer

Or: there is a positive integer m with property that for all positive integers n, $m \leq n$.

The Language of Sets

Use of the word *set* as a formal mathematical term was introduced in 1879 by Georg Cantor (1845–1918). For most mathematical purposes we can think of a set intuitively, as Cantor did, simply as a collection of elements.

For instance, if C is the set of all countries that are currently in the United Nations, then the United States is an element of C, and if I is the set of all integers from 1 to 100, then the number 57 is an element of I.

The Language of Sets

- **Notation**

If S is a set, the notation $\boldsymbol{x \in S}$ means that x is an element of S. The notation $\boldsymbol{x \notin S}$ means that x is not an element of S. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, **{1, 2, 3}** denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write **{1, 2, 3, . . . , 100}** to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write **{1, 2, 3, . . .}** to refer to the set of all positive integers. (The symbol **. . .** is called an **ellipsis** and is read "and so forth.")

$x \in S$, $x \notin S$

set-roster notation

- {1, 2, 3}
- {1, 2, 3, …,100}
- {1, 2, 3, …}

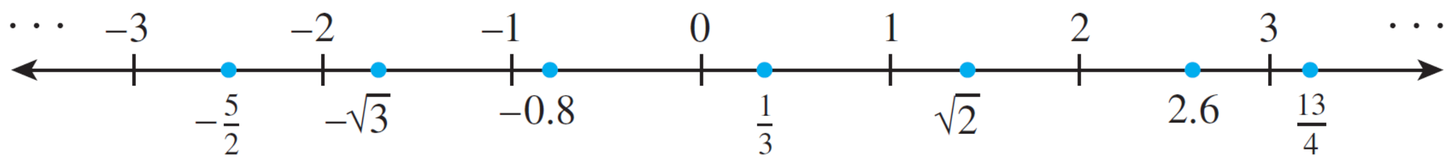
The Language of Sets

Certain sets of numbers are so frequently referred to that they are given special symbolic names. These are summarized in the following table:

Symbol	Set
R	set of all real numbers
Z	set of all integers
Q	set of all rational numbers, or quotients of integers

The Language of Sets

The set of real numbers is usually pictured as the set of all points on a line, as shown below.



The number 0 corresponds to a middle point, called the *origin*.

A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin.

The Language of Sets

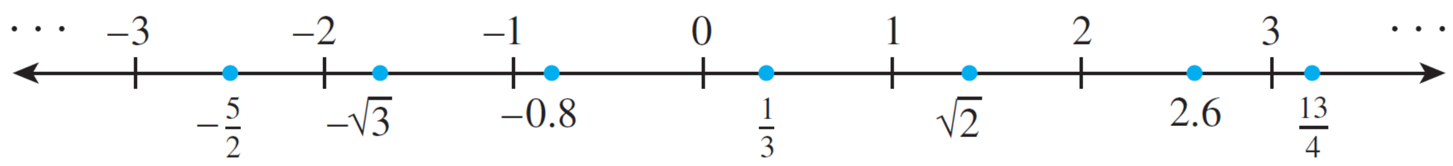
Each point to the left of the origin corresponds to a negative real number, which is denoted by computing its distance from the origin and putting a minus sign in front of the resulting number.

The set of real numbers is therefore divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0.

Note that 0 is neither positive nor negative.

The Language of Sets

Labels are given for a few real numbers corresponding to points on the line shown below.



The real number line is called ***continuous*** because it is imagined to have no holes.

The set of integers corresponds to a collection of points located at fixed intervals along the real number line.

The Language of Sets

Thus every integer is a real number, and because the integers are all separated from each other, the set of integers is called ***discrete***. The name *discrete mathematics* comes from the distinction between continuous and discrete mathematical objects.

Another way to specify a set uses what is called the *set-builder notation*.

• Set-Builder Notation

Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements *x* in *S* such that *P(x)* is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all — such that

Subsets

A basic relation between sets is that of subset.

- **Definition**

If A and B are sets, then A is called a **subset** of B, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of A is also an element of B.

Symbolically:

$$A \subseteq B \quad \text{means that} \quad \text{For all elements } x, \text{ if } x \in A \text{ then } x \in B.$$

The phrases A *is contained in* B and B *contains* A are alternative ways of saying that A is a subset of B.

Subsets

It follows from the definition of subset that for a set *A* not to be a subset of a set *B* means that there is at least one element of *A* that is not an element of *B*.

Symbolically:

$A \not\subseteq B$ means that There is at least one element x such that $x \in A$ and $x \notin B$.

• Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

Distinction between $\in$ and $\subseteq$

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$ **b.** $\{2\} \in \{1, 2, 3\}$ **c.** $2 \subseteq \{1, 2, 3\}$
d. $\{2\} \subseteq \{1, 2, 3\}$ **e.** $\{2\} \subseteq \{\{1\}, \{2\}\}$ **f.** $\{2\} \in \{\{1\}, \{2\}\}$

Solution:

Only (**a**), (**d**), and (**f**) are true.

For (**b**) to be true, the set $\{1, 2, 3\}$ would have to contain the element $\{2\}$. But the only elements of $\{1, 2, 3\}$ are 1, 2, and 3, and 2 is not equal to $\{2\}$. Hence (b) is false.

Distinction between $\in$ and $\subseteq$

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$ **b.** $\{2\} \in \{1, 2, 3\}$ **c.** $2 \subseteq \{1, 2, 3\}$
d. $\{2\} \subseteq \{1, 2, 3\}$ **e.** $\{2\} \subseteq \{\{1\}, \{2\}\}$ **f.** $\{2\} \in \{\{1\}, \{2\}\}$

For (**c**) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of $\{1, 2, 3\}$. This is not the case, so (**c**) is false.

For (**e**) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are $\{1\}$ and $\{2\}$. But 2 is not equal to either $\{1\}$ or $\{2\}$, and so (**e**) is false.

Ordered Pairs

- **Notation**

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Ordered Pairs

a. Is (1, 2) = (2, 1)?

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$

Solution:

a. No. By definition of equality of ordered pairs,

(1, 2) = (2,1) if, and only if, 1 = 2 and 2 = 1.

But 1 ≠ 2, and so the ordered pairs are not equal.

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Ordered Pairs

b. Yes. By definition the ordered pairs are equal

if, and only if, $3 = \sqrt{9}$ and $\frac{5}{10} = \frac{1}{2}$.

Because these equations are both true, the ordered pairs are equal.

Cartesian Products

- **Definition**

Given sets A and B, the **Cartesian product of $\boldsymbol{A}$ and $\boldsymbol{B}$**, denoted $\boldsymbol{A} \times \boldsymbol{B}$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$\boldsymbol{A} \times \boldsymbol{B} = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Cartesian Products

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.

a. Find $A \times B$

b. Find $B \times A$

c. Find $B \times B$

d. How many elements are in $A \times B$, $B \times A$, and $B \times B$?

Cartesian Products

a. $A \times B = \{(1, u), (2, u), (3, u), (1, v), (2, v), (3, v)\}$

b. $B \times A = \{(u, 1), (u, 2), (u, 3), (v, 1), (v, 2), (v, 3)\}$

c. $B \times B = \{(u, u), (u, v), (v, u), (v, v)\}$

d. $A \times B$ has six elements. Note that this is the number of elements in A times the number of elements in B.

$B \times A$ has six elements, the number of elements in B times the number of elements in A. $B \times B$ has four elements, the number of elements in B times the number of elements in B.

The Language of Relations and Functions

The objects of mathematics may be related in various ways.

A set A may be said to be related to a set B if A is a subset of B, or if A is not a subset of B, or if A and B have at least one element in common.

A number x may be said to be related to a number y if
$x < y$, or if x is a factor of y, or if $x^2 + y^2 = 1$.

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$ and let us say that an element x in A is related to an element y in B if, and only if, x is less than y.

The Language of Relations and Functions

Let us use the notation *x R y* as a shorthand for the sentence "*x* is related to *y*." Then

$$\begin{array}{lll} 0\ R\ 1 & \text{since} & 0 < 1, \\ 0\ R\ 2 & \text{since} & 0 < 2, \\ 0\ R\ 3 & \text{since} & 0 < 3, \\ 1\ R\ 2 & \text{since} & 1 < 2, \\ 1\ R\ 3 & \text{since} & 1 < 3, \quad \text{and} \\ 2\ R\ 3 & \text{since} & 2 < 3. \end{array}$$

On the other hand, if the notation $x \not R y$ represents the sentence "*x* is not related to *y*," then

$$\begin{array}{lll} 1 \not R 1 & \text{since} & 1 \not< 1, \\ 2 \not R 1 & \text{since} & 2 \not< 1, \qquad \text{and} \\ 2 \not R 2 & \text{since} & 2 \not< 2. \end{array}$$

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Functions

- **Definition**

A **function F from a set A to a set B** is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

$$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

Functions

Properties (1) and (2) can be stated less formally as follows: A relation *F* from *A* to *B* is a function if, and only if:

1. Every element of *A* is the first element of an ordered pair of *F*.

2. No two distinct ordered pairs in *F* have the same first element.

• Notation

If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $\boldsymbol{F(x)}$, which is read "$\boldsymbol{F}$ **of** $\boldsymbol{x}$."

Function Machines

Another useful way to think of a function is as a machine. Suppose *f* is a function from *X* to *Y* and an input *x* of *X* is given.

Imagine *f* to be a machine that processes *x* in a certain way to produce the output *f*(*x*). This is illustrated in Figure 1.3.1

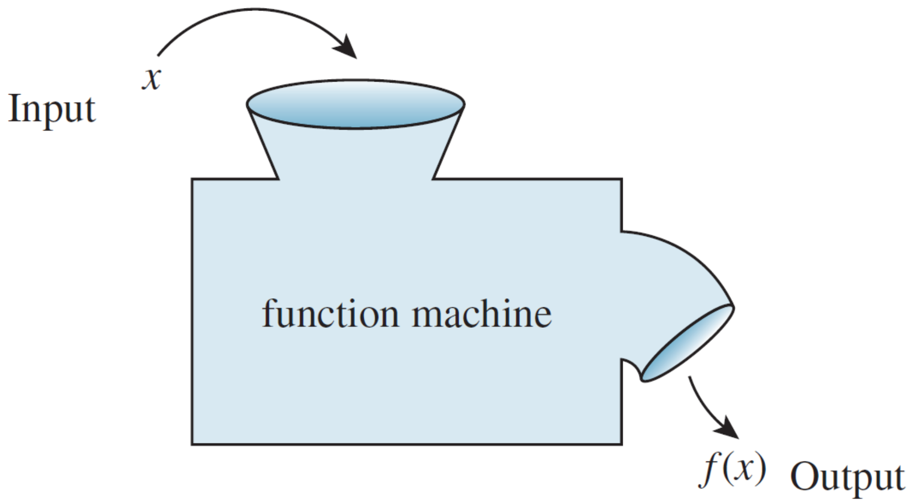

Figure 1.3.1

Function Machines

A function is an entity in its own right. It can be thought of as a certain relationship between sets or as an input/output machine that operates according to a certain rule.

This is the reason why a function is generally denoted by a single symbol or string of symbols, such as *f*, *G*, of log, or sin.

A relation is a subset of a Cartesian product and a function is a special kind of relation.

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