Math1510 - Discrete Mathematics
Relations 2

University of Newcastle

UoN

A given binary relation may turn out to be:

- Reflexive
- Symmetric
- Antisymmetric
- Transitive

We will discuss each property in turn.

Reflexive Relations

Definition

A relation R on a set A is reflexive if

$$\text{for all } a \in A,\ (a,a) \in R.$$

That is, $a\ R\ a$ for all $a \in R$.

- $\leq$, $\geq$, $=$ and $\subseteq$ are all reflexive relations.
- "is the same colour as" is a reflexive relation.
- $<$, $>$, $\neq$ and $\subset$ are not reflexive relations.
- "is the parent of" is not a reflexive relation.

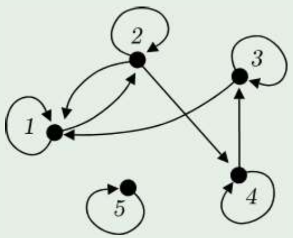
Example of a Reflexive Relation

Example

Consider the set $\{1, 2, 3, 4, 5\}$. The relation on this set given by

$$R = \{(1,1),(1,2),(2,1),(2,2),(2,4),\\ (3,1),(3,3),(4,4),(4,3),(5,5)\}$$

is reflexive with associated digraph as follows.



The adjacency matrix of a reflexive relation

The adjacency matrix of the previous reflexive relation is

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

What property which is true of this matrix must be true of all adjacency matrices of reflexive relations?

Which adjacency matrix is **not** of a **reflexive** relation?

A $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

B $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

C $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

D $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Symmetric Relations

Definition

A relation R on a set A is symmetric if

$$\text{for all } a, b \in A, \text{ if } (a, b) \in R \text{ then } (b, a) \in R.$$

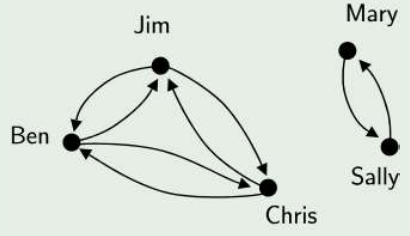
That is, if $a\ R\ b$ then $b\ R\ a$.

- $=$ and $\neq$ are symmetric relations.
- "is the same colour as" is a symmetric relation.
- "is a relative of" (for people) is a symmetric relation.
- $\subseteq$, $\leq$, $\geq$, $<$ and $<$ are not symmetric relations.
- "is the parent of" is not a symmetric relation.

Example of a Symmetric Relation

Example

Consider the set of people $\{\text{Mary}, \text{Jim}, \text{Chris}, \text{Sally}, \text{Ben}\}$. Let R be the relation "is the same sex as" represented in the following digraph:



This relation is symmetric.

The adjacency matrix of a symmetric relation

The adjacency matrix of the previous symmetric relation is

$$\begin{array}{c c} & \begin{array}{ccccc} B & C & J & M & S \end{array} \\ \begin{array}{c} B \\ C \\ J \\ M \\ S \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

What property which is true of this matrix must be true of all adjacency matrices of symmetric relations?

Which adjacency matrix is **not** of a **symmetric** relation?

A $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

B $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

C $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

D $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Antisymmetric Relations

Definition

A relation R on a set A is antisymmetric if for all $a, b \in A$,

$$\text{if } (a, b) \in R \text{ **and** } (b, a) \in R, \text{ then } a = b.$$

In other words, you never have both aRb and bRa unless $a = b$.

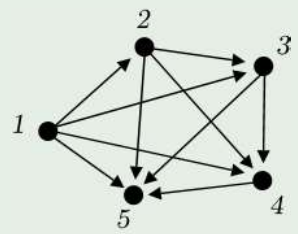
- $\leq, \geq, =, <, >$ and $\subseteq$ are all antisymmetric relations.
- "is the parent of" is an antisymmetric relation.
- $\neq$ is not an antisymmetric relation.
- "is a relative of" is not an antisymmetric relation.

Note: symmetric and antisymmetric are not mutually exclusive, e.g. consider "=".

Example of an Antisymmetric Relation

Example

Consider the relation $<$ on the set $\{1,2,3,4,5\}$.



This relation is antisymmetric.

The adjacency matrix of an anti-symmetric relation

The adjacency matrix of the previous anti-symmetric relation is

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

What property which is true of this matrix must be true of all adjacency matrices of anti-symmetric relations?

Which adjacency matrix is **not** of an **anti-symmetric** relation?

A $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

B $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

C $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

D $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

Transitive Relations

Definition

A relation R on a set A is transitive if for all $a, b, c \in A$

$$\text{if } (a, b) \in R \text{ and } (b, c) \in R \text{ then } (a, c) \in R.$$

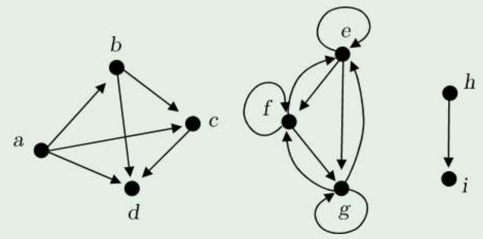
That is, if $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$.

- $\leq, \geq, =, <, >$ and $\subseteq$ are all transitive relations.
- "is the same colour as" is a transitive relation.
- $\neq$ is not a transitive relation.
- "is a parent of" is not transitive.

Example of a Transitive Relation

Example

Consider the set $\{a, b, c, d, e, f, g, h, i\}$ and the relation given by the following digraph.



This relation is transitive.

The adjacency matrix of the previous transitive relation is:

$$
\begin{array}{c|ccccccccc}
 & a & b & c & d & e & f & g & h & i \\
\hline
a & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
f & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
g & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Is it possible to recognise transitivity from an adjacency matrix? Yes you can, but it is less apparent than the other properties. To recognise transitivity we need to be able to do matrix multiplication so we will do a quick detour to learn about matrix multiplication.

Matrix Multiplication

- Let A be a matrix of size $p \times q$. Let B be a matrix of size $q \times s$.

$$
\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix}}_{p \times q}
\times
\underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qs} \end{bmatrix}}_{q \times s}
=
\underbrace{\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1q}b_{q1} & \cdots & a_{p1}b_{11} + a_{p2}b_{21} + \dots + a_{pq}b_{q1} \\ a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1q}b_{q2} & \ddots & a_{p1}b_{12} + a_{p2}b_{22} + \dots + a_{pq}b_{q2} \\ \vdots & & \vdots \\ a_{11}b_{1s} + a_{12}b_{2s} + \dots + a_{1q}b_{qs} & \cdots & a_{p1}b_{1s} + a_{p2}b_{2s} + \dots + a_{pq}b_{qs} \end{bmatrix}}_{p \times s}
$$

Only works when these numbers are equal

- Then the product AB has size $p \times s$, and the $(i,j)^{\text{th}}$ entry of AB is equal to
$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{iq}b_{qj}.$$

Matrix Multiplication

- If you have seen the dot product then notice this product is just a bunch of dot products

$$
\begin{bmatrix} row_1 \\ row_2 \\ \vdots \\ row_p \end{bmatrix}
\begin{bmatrix} col_1 & col_2 & \cdots & col_s \end{bmatrix}
=
\begin{bmatrix} row_1.col_1 & row_1.col_2 & \cdots & row_1.col_s \\ row_2.col_1 & row_2.col_2 & \cdots & row_2.col_s \\ \vdots & \vdots & \ddots & \vdots \\ row_p.col_1 & row_p.col_2 & \cdots & row_p.col_s \end{bmatrix}
$$

Example of Matrix Multiplication

- We multiply a 2×3 matrix by a 3×3 matrix to get a 2×3 matrix:

$$
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 1 \\ 5 & 7 & -2 \\ 0 & -5 & 3 \end{bmatrix}
=
\begin{bmatrix} 1+10+0 & 0+14-15 & 1-4+9 \\ 4+25+0 & 0+35-30 & 4-10+18 \end{bmatrix}
$$
$$
= \begin{bmatrix} 11 & -1 & 6 \\ 29 & 5 & 12 \end{bmatrix}
$$

AB usually does not equal BA

- For example,

$$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&-1\\2&2\\-3&7\end{bmatrix}=\begin{bmatrix}1+4-9&-1+4+21\\4+10-18&-4+10+42\end{bmatrix}$$
$$=\begin{bmatrix}-4&24\\-4&48\end{bmatrix}$$

- but

$$\begin{bmatrix}1&-1\\2&2\\-3&7\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}=\begin{bmatrix}1-4&2-5&3-6\\2+8&4+10&6+12\\-3+28&-6+35&-9+42\end{bmatrix}$$
$$=\begin{bmatrix}-3&-3&-3\\10&14&18\\25&29&33\end{bmatrix}.$$

Sometimes AB exists and BA does not.

- For example, a 2×3 matrix by a 3×3 matrix gives a 2×3 matrix:

$$\begin{bmatrix}1&2&3\\-1&0&4\end{bmatrix}\begin{bmatrix}1&0&1\\7&-1&3\\-2&-3&1\end{bmatrix}=\begin{bmatrix}1+14-6&0-2-9&1+6+3\\-1+0-8&0+0-12&-1+0+4\end{bmatrix}$$
$$=\begin{bmatrix}9&-11&10\\-9&-12&3\end{bmatrix}$$

- but 3×3 matrix by a 2×3 cannot be multiplied:

$$\begin{bmatrix}1&0&1\\7&-1&3\\-2&-3&1\end{bmatrix}\begin{bmatrix}1&2&3\\-1&0&4\end{bmatrix}$$ - this product is not defined.

What is this product?

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}$$

A $\begin{pmatrix}5&12\\21&32\end{pmatrix}$

B $\begin{pmatrix}19&22\\43&50\end{pmatrix}$

C $\begin{pmatrix}23&34\\31&56\end{pmatrix}$

What is this product?

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\begin{pmatrix}5&6&7&8\end{pmatrix}$$

A $\begin{pmatrix}5&12&21&32\end{pmatrix}$

B $\begin{pmatrix}5\\12\\21\\32\end{pmatrix}$

C $\begin{pmatrix}5&6&7&8\\10&12&14&16\\15&18&21&24\\20&24&28&32\end{pmatrix}$

D (70)

Can transitivity be recognised from the adjacency matrix?

Theorem

Let R be a relation on a set S and let A be its adjacency matrix. R is transitive if and only if for all i and j, the i, jth entry of A is 1 whenever the i, jth entry of A^2 is non-zero.

'Whenever' is another way of saying 'if'. Lets write this down another way to make sure we understand the direction of implications.

Theorem

A relation with adjacency matrix A is transitive iff (if entry (i, j) in A^2 is not zero, then entry (i, j) in A is 1).

Example: The previous relation's matrix is consistent with the theorem.

$$\begin{bmatrix} 0&1&1&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&0&0\\ 0&0&0&0&1&1&1&0&0\\ 0&0&0&0&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}\begin{bmatrix} 0&1&1&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&0&0\\ 0&0&0&0&1&1&1&0&0\\ 0&0&0&0&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&1&2&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&3&3&3&0&0\\ 0&0&0&0&3&3&3&0&0\\ 0&0&0&0&3&3&3&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

We will use the properties of reflexivity, symmetry, antisymmetry and transitivity to build/discover some mathematical constructs such as

- Posets
- Total Orders
- Equivalence Relations

POSETS

Definition

A *POSET*, also called a *Partially Ordered Set*, or *Partial Order*, is a binary relation on a set, which is *reflexive*, *anti-symmetric* and *transitive*.

- $R = \{(a, b) : a \text{ divides } b\}$ is a partial order
- $\leq, \geq, \subseteq, \supseteq, =$ are partial orders
- $<, >, \subset, \supset, \neq$ are **not** partial orders

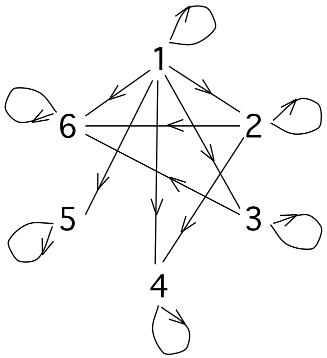
On what sets, for each relation?

Exploring the ‘divides’ relation

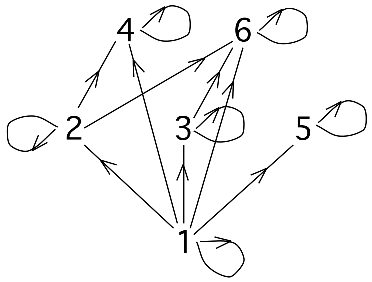
Consider the set of positive integers $S = \{1, 2, 3, 4, 5, 6\}$

Members of the “divides” relation R include: (1,1), (2,4), (3,6). How many elements of R altogether?

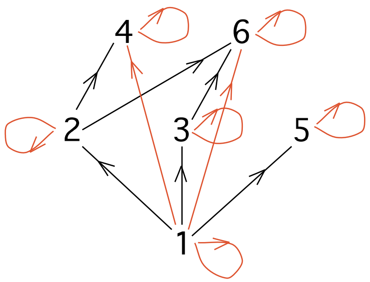
The digraph of R may be drawn:



Neater to redraw with all (non-loop) arrows pointing up:



Neater to redraw with all (non-loop) arrows pointing up:

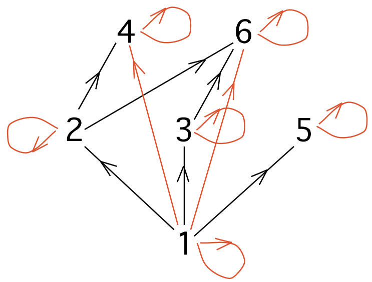


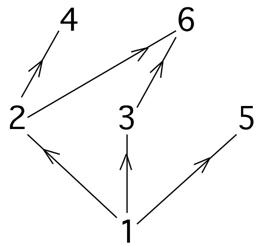
Because we know its a POSET

- reflexivity, i.e. loops
- and transitivity edges

are redundant.

Neater to redraw with all (non-loop) arrows pointing up:





Because we know its a POSET

- reflexivity, i.e. loops
- and transitivity edges

are redundant. Deleting these gives a: *Hasse Diagram*

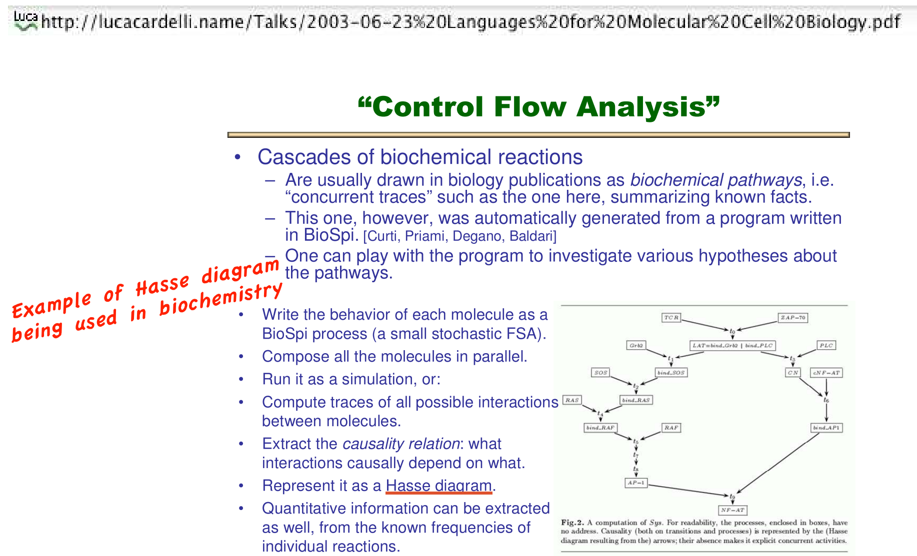
Hasse Diagrams

Definition

A Hasse Diagram is a diagram of a POSET on a finite set constructed as follows:

- Draw the digraph of the POSET, with the elements of the set located so that the non-loop arrows of the relation all point up the page, or all point down the page.
- Delete the loops
- Delete the redundant edges implied by transitivity
- Optionally, delete the arrows on the remaining edges.

Hasse Diagrams are often used in biology, chemistry and computer science: here's an example I found on the web



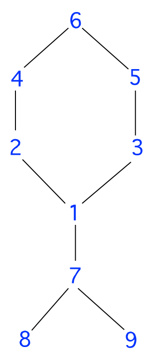
POSETS and *precedence*

If certain events must **precede** certain other events, we may construct relation

$$iRj \qquad \text{if } i = j \textbf{ or } \text{task } i \text{ precedes task } j$$

Example: an overseas trip?

1. pack suitcase
2. book airline ticket
3. book hotel
4. check flight availability
5. check hotel availability
6. decide on destination
7. fly to destination
8. check in to hotel
9. explore destination



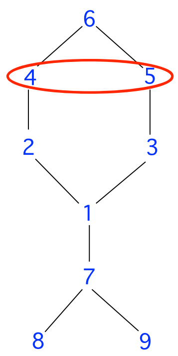
Comparability

Definition

Let R be a binary relation which is a POSET. If xRy or yRx then we say that the elements x and y are comparable. If not, they are incomparable.

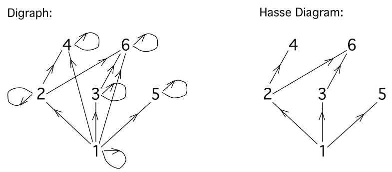
In the overseas trip example,

1. pack suitcase
2. book airline ticket
3. book hotel
4. check flight availability
5. check hotel availability
6. decide on destination
7. fly to destination
8. check in to hotel
9. explore destination



4 and 5 are incomparable because we can't say which must precede which.

For the “divides” relation on $\{1, 2, 3, 4, 5, 6\}$, which claim is false?



- A 2 and 4 are comparable
- B 1 and 4 are comparable
- C 5 and 6 are comparable
- D 2 and 3 are incomparable

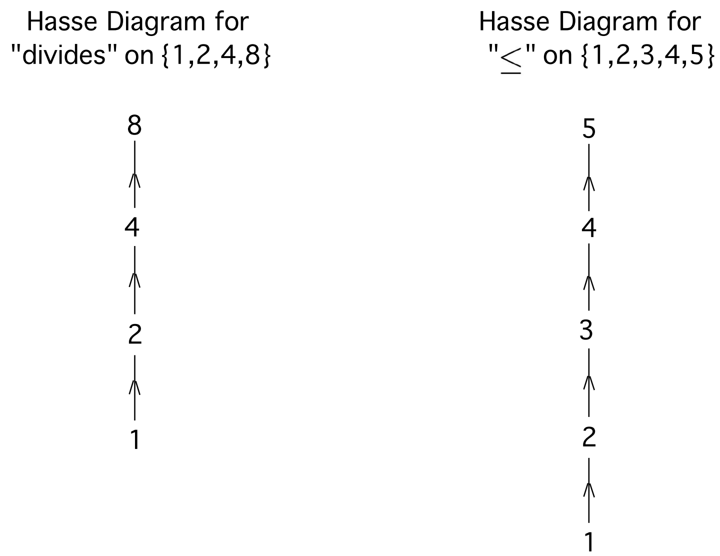
Total Order

Definition

A total order is a partial order with the additional property that any two elements are comparable.

- The “divides” relation is a total order **on some sets**, e.g. $\{1, 2, 4, 8\}$
- $\leq$ and $\geq$ are total orders.

Hasse Diagrams for Total Orders



Which relation is **not** a total order?

- A “$\subseteq$” on the set $\{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$
- B “$\supseteq$” on the set $\{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$
- C “$\subseteq$” on the set $\{\{1\}, \{2\}, \{1, 2\}\}$
- D “$=$” on the set $\{1\}$

Equivalence Relations

Definition

A relation R on a set A is an equivalence relation if it is

- Reflexive
- Symmetric
- Transitive

We often write $a \equiv b$ instead of aRb for an equivalence relation.

Examples:

- having the same "size"
- having the same "colour"
- being "=" (when applied to numbers)
- being "congruent" (when applied to geometric figures)

Equivalence Relations Example

Example

Consider the relation "=" on the set of (non-reduced) fractions given by $\frac{a}{b} = \frac{c}{d}$ if, as integers, $ad = bc$.

This relation is

- Reflexive: $\frac{a}{b} = \frac{a}{b}$.
- Symmetric: if $\frac{a}{b} = \frac{c}{d}$ then $\frac{c}{d} = \frac{a}{b}$.
- Transitive: if $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$ then $\frac{a}{b} = \frac{e}{f}$.

Hence this relation is an equivalence relation.

Which of the following is not an equivalence relation?

A "Has the same birthday as" on the set of all people

B "Is enrolled in exactly the same subjects as" on the set of all University of Newcastle students

C "Is the sibling of" on the set of all people

D "Evaluates to the same value under a given function" on the domain of a given function

Equivalence Classes

Definition

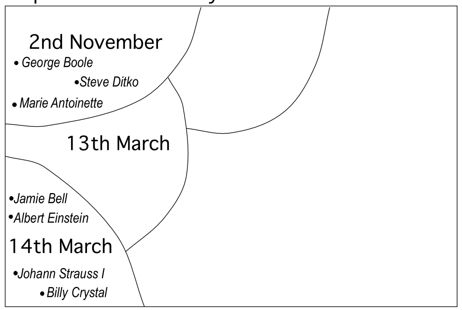
Given an equivalence relation R on a set A, for each element $a \in A$ we can form the equivalence class of a:

$$[a] = \{x \in A : x \; R \; a\}.$$

Example (An Equivalence Class on people)

- March 14th Birthday equivalence class
  = {Albert Einstein, Johann Strauss I, Billy Crystal, Jamie Bell, ... }
- November 2nd Birthday equivalence class
  = {George Boole, Marie Antoinette, Steve Ditko, ... }

All possible dates in a year:



Equivalence Classes define Partitions*

Theorem

If R is an equivalence relation on a set X, then the set of equivalence classes of R is a partition of X.

Example

The set of all people is partitioned into 366 disjoint sets based on their birthdays.

- If you were born on the 14th of March, then you are in the same equivalence class (under the birthday relation) as Albert Einstein.

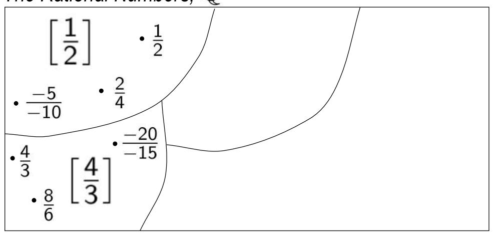
Example (A second Equivalence Classes Example)

- $\left[\frac{1}{2}\right] = \left\{\frac{1}{2}, \frac{2}{4}, \frac{-5}{-10}, \frac{50}{100}, \ldots\right\}$
- $\left[\frac{2}{4}\right] = \left[\frac{1}{2}\right]$
- $\left[\frac{4}{3}\right] = \left\{\frac{4}{3}, \frac{8}{6}, \frac{-20}{-15}, \frac{100}{75}, \ldots\right\}$

Each of these equivalence classes is the set of all fractions representing the same given rational number. We write $\frac{a}{b} = \frac{c}{d}$ iff $ad = bc$ for $b, d \neq 0$.

This relation is a partition on all pairs of integers $\left[\frac{a}{b}\right]$, $b \neq 0$

The Rational Numbers, $\mathbb{Q}$



This relation can be used to *define* $\mathbb{Q}$ in terms of pairs of elements of $\mathbb{Z}$.

Proof of the Partition Theorem*

Theorem

If R is an equivalence relation on a set X, then the set of equivalence classes of R is a partition of X.

Proof outline (see p 161 of textbook for details)

- Define notation. Let X be the set and R be the relation, so that the equivalence class of $a \in X$ is:

$$[a] := \{x \in X : xRa\}$$

and the collection of equivalence classes is

$$S := \{[a] : a \in X\}$$

We need to show:

- for each $a \in X$, the equivalence class $[a]$ is non-empty
- the union of all the classes is the whole set, i.e. $\bigcup S = X$
- pairwise disjointness, i.e. if $x \in X$ and $x \in [a] \cap [b]$ then $[a] = [b]$

The converse is also true

Theorem

Given any partition S on a set X, each of the members of S is an equivalence class, under equivalence relation R given by

$$aRb \iff a \text{ and } b \text{ are in the same member of } S$$

Textbook exercises 1

Appendix A on matrices:

- 1-16 (basic matrix recognition/computations)
- 17, 21 (proofs)
- 18-20 (Optional extra on Proofs and inverses)

Exercises section 3.3

- 22-34 (reflexive, symmetric, antisymmetric, transitive)
- 37-41 (reflexive, symmetric, antisymmetric, transitive)
- 42, 46, 50, 54 (proofs)
- 55 (optional extra, helping understanding of counting and what a relation is)

Textbook exercises 2

Exercises section 3.5

- 11-12, 15 (reflexive, symmetric, antisymmetric, transitive)
- 13-14 (inverses)
- 24-27 (understanding relationship between relations generally and functions specifically)

Exercises Section 3.4

- 1-42 (recognising and defining equivalence classes, with a few proofs)
- 49-66 (as above, with an emphasis on proving. Starred questions 54-56 and 66 are harder, and less relevant)

- Review Exercises section 3.3: 1-6
- Review Exercises section 3.4: 1-4
- Review Exercises section 3.6: 1