Math1510 - Discrete Mathematics
Relations 1

University of Newcastle

UoN

The Cartesian Product of 2 Sets

Definition

Given two sets A and B, the *Cartesian product* of A and B, denoted $A \times B$, is the set of **all ordered pairs** (a, b), where a is in A and b is in B.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

a is called the first coordinate and b is called the second coordinate.

Example: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

Which is false?

A $\{s_1, s_2\} \times \{t_1, t_2\} = \{(s_1, t_1), (s_1, t_2), (s_2, t_1), (s_2, t_2)\}$

B $\{s_1, s_2\} \times \{t_1, t_2\} = \{(s_1, t_1), (s_2, t_2)\}$

C $\{s\} \times \{t_1, t_2\} = \{(s, t_1), (s, t_2)\}$

D $\{\} \times \{t_1, t_2\} = \{\}$

Equality of ordered pairs

Definition

Two ordered pairs (a_1, a_2) and (b_1, b_2) are equal if and only if

$$a_1 = b_1 \text{ and } a_2 = b_2.$$

Is this true?

$$(1, 2) = (2, 1)?$$

A True

B False

Equality of ordered pairs

Definition

Two ordered pairs (a_1, a_2) and (b_1, b_2) are equal if and only if

$$a_1 = b_1 \text{ and } a_2 = b_2.$$

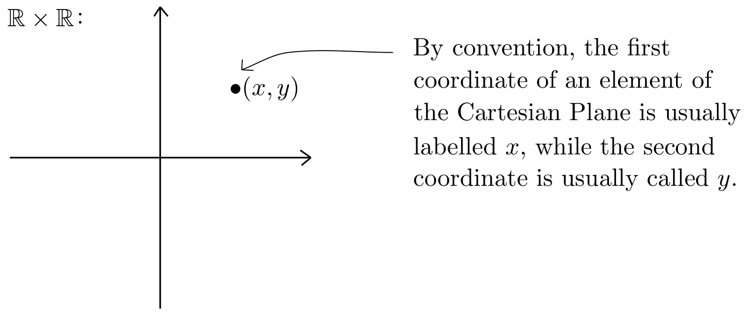
Is this true?

$$(1, 2) = (2, 1)?$$

- A True
- B **False**, because order matters.

The Cartesian Plane

The Cartesian product $\mathbf{R} \times \mathbf{R}$ is often represented by the *Cartesian Plane:*



Two points in the plane are equal (i.e. the *same* point) if and only if both coordinates agree.

Functions

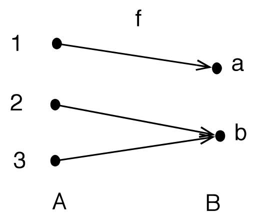
Definition

Given two sets A and B, a function is a subset $f \subseteq A \times B$ of the Cartesian Product which satisfies:

1. For every $a \in A$, there exists some $b \in B$ such that $(a, b) \in f$; and
2. If $(a, b_1) \in f$ and $(a, b_2) \in f$ then $b_1 = b_2$.

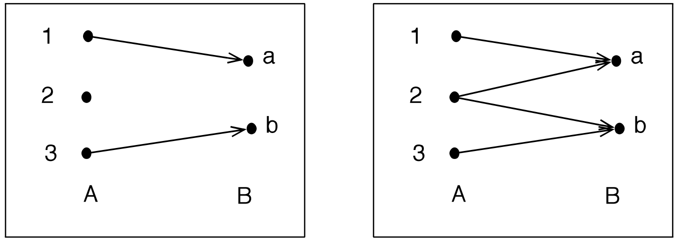
Example function

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then the Cartesian Product $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$ and an example of a function which is a subset of this Cartesian Product is $f = \{(1, a), (2, b), (3, b)\}$. We may draw arrow diagram:



An alternative representation is to use the symbol f in a slightly different way and write $f(1) = a$, $f(2) = b$ and $f(3) = b$. Yet another alternative is the notation $f : A \to B$ by $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto b$.

Some assignments that are **not** functions



Why not?

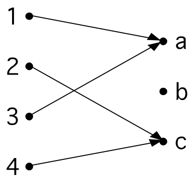
Terminology

Definition

We write $f : A \to B$ and say "f maps A to B" for a function $f \subseteq A \times B$.

- The domain of f is A.
- The codomain of f is B.
- The range of f is the set $\{b \in B : (a, b) \in f\}$

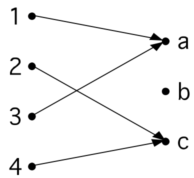
Let $A = \{1, 2, 3, 4\}$. Let $B = \{a, b, c\}$. Define f by the arrow diagram:



Which is the domain of f?

- A $\{1, 2, 3, 4\}$
- B $\{a, b, c\}$
- C $\{a, c\}$

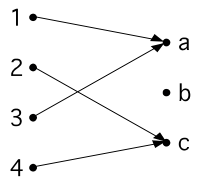
Let $A = \{1, 2, 3, 4\}$. Let $B = \{a, b, c\}$. Define f by the arrow diagram:



Which is the codomain of f?

- A $\{1, 2, 3, 4\}$
- B $\{a, b, c\}$
- C $\{a, c\}$

Let $A = \{1, 2, 3, 4\}$. Let $B = \{a, b, c\}$. Define f by the arrow diagram:



Which is the range of f?

A $\{1, 2, 3, 4\}$

B $\{a, b, c\}$

C $\{a, c\}$

We now explore some properties which functions may have

- Surjectivity
- Injectivity
- Bijectivity

Surjective Functions

Definition

A function f from a set A to a set B is said to be surjective (or onto) if the range of f is B. That is,

$$\text{for each } b \in B \text{ there is an } a \in A \text{ such that } b = f(a).$$

Example

Let f be the function from the set $\{a, b, c, d\}$ to the set $\{1, 2, 3\}$, defined by

$$f(a) = 2, \quad f(b) = 1,$$
$$f(c) = 2, \quad f(d) = 3.$$

Then f is surjective.

Injective Functions

Definition

A function f from A to B is said to be injective (or one-to-one) if for each $b \in \text{range}(f)$, there is at most one $a \in A$ with $f(a) = b$. That is,

$$\text{if } f(a_1) = f(a_2) \text{ then } a_1 = a_2.$$

Example

Let f be the function from the set $\{a, b, c, d\}$ to the set $\{1, 2, 3, 4, 5\}$, defined by

$$f(a) = 2, \quad f(b) = 1,$$
$$f(c) = 5, \quad f(d) = 3.$$

Then f is injective (but not surjective).

Bijective Functions

Definition

A function $f : A \to B$ that is both injective (one-to-one) and surjective (onto) is called bijective.

The function is called a bijection, and provides a *"one-to-one correspondence"* between the elements of A and B.

Example

Let f be the function from the set $\{a, b, c, d\}$ to the set $\{1, 2, 3, 4\}$, defined by

$$f(a) = 1, \ \ f(b) = 2,$$
$$f(c) = 3, \ \ f(d) = 4.$$

Then f is a bijection.

Another example of a bijection

The digits of my left hand are in one-to-one correspondence with the digits of my right hand, via the map:

left little finger $\mapsto$ right little finger
left ring finger $\mapsto$ right ring finger
left middle finger $\mapsto$ right middle finger
left index finger $\mapsto$ right index finger
left thumb $\mapsto$ right thumb

In other words, this map defines a bijection.

Which claim is false?

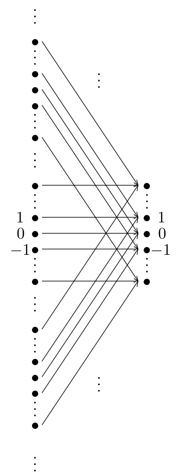
- A $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = -x$ is a bijection
- B $g : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ by $g(x) = |x|$ is surjective
- C $g : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ by $g(x) = |x|$ is injective
- D $h : \mathbb{Z} \to \mathbb{Z}$ by $g(x) = x$ is surjective, injective and bijective

An Example important in Computer Science:

- Let B be the set of *ints* in some computer language, and let the map

  $$f : \mathbb{Z} \to B,$$

  be specified as it is in the computer language specified - think of 'C' or 'Fortran' for instance. Then f is **not** *a bijection* because it fails to be *injective*, as we can see schematically by the partial arrow diagram:



Computers cannot store all of the numbers in $\mathbb{Z}$, so instead most use a limited set which in some way mimics the behaviour of the integers, but not in all aspects. If you were to keep adding '1' to an *int*, instead of always getting bigger and bigger, your cumulative total would eventually wrap back around to some value that it has already been.

Can we 'undo' a function? Not always. Sometimes, yes.

Theorem

If a function $f \subseteq A \times B$ is injective (i.e. one-to-one), then the collection of pairs $\{(b, a) : (a, b) \in f\}$ is a function from the range of f to A.

Challenge: Prove this theorem.

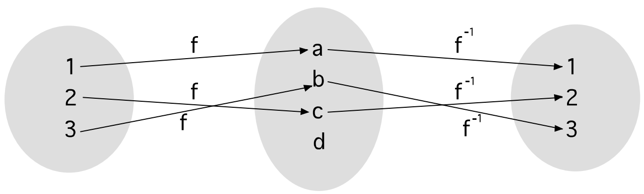
Definition

The new function described in the theorem is denoted f^{-1}, and is called the inverse of f. If $f(a) = b$, then $f^{-1}(b) = a$.

Example

The function $f : \{1, 2, 3\} \to \{a, b, c, d\}$ defined by $f(1) = a, f(2) = c, f(3) = b$ is injective (i.e. one-to one).

The inverse function $f^{-1} : \{a, b, c\} \to \{1, 2, 3\}$ is defined by $f^{-1}(a) = 1, f^{-1}(b) = 3, f^{-1}(c) = 2$.



The process of *composing* one function with another tells us the result of first acting with one function, and then with a second function.

Composition of functions

$$\begin{tikzcd} \bullet x \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \bullet f(x) \arrow[r, "g"] & \bullet g(f(x)) \end{tikzcd}$$

- If the range of function f is a subset of the domain of function g, then we may form the composite function

$$g \circ f$$

  defined such that $(g \circ f)(x) = g(f(x))$.
- The new composite function has domain (abbreviated 'dom')

$$\mathrm{dom}(g \circ f) = \{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\}$$

Observation

The composition of a function with its inverse is the identity function, i.e.

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$$

Example

Let X be the set of odd integers and let f be the function from $\mathbb{Z}$ to X given by $f(n) = 2n + 5$. That is

$$f : \mathbb{Z} \to X, \quad n \mapsto 2n + 5.$$

The function f is one-to-one and onto, hence f is a bijection. (Check this for yourself!) The inverse function is

$$f^{-1} : X \to \mathbb{Z}, \quad n \mapsto \frac{n-5}{2}.$$

Hence $(f^{-1} \circ f)(n) = f^{-1}(2n + 5) = \frac{(2n+5)-5}{2} = n$.

Composition of injective functions

Theorem

If f and g are both injective functions, then $g \circ f$ (if defined) is also an injective function and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof:
Notice $(g \circ f) \circ (f^{-1} \circ g^{-1})(x) = x$.

Definition

A sequence is a function whose domain is a set of consecutive integers.

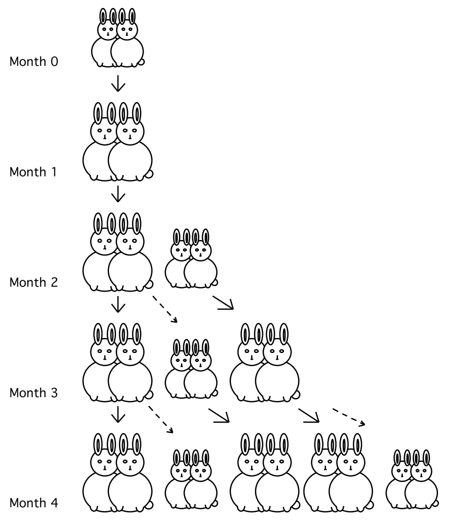
Which one of the following sets could *not* be the domain of a sequence?

- A $\{6, 7, 8, 9, 10, 11, 12\}$
- B $\mathbb{N}$
- C $\{1, 3, 5, 7, 9\}$
- D $\{1, 2, 3, 4, 5\}$
- $\mathbb{Z}$ Note: our textbook's definition allows $\mathbb{Z}$, some others do not.

Examples of Sequences

- The function $f : \mathbb{Z}_{\geq 0} \mapsto \mathbb{N}$ defined by $f(n) = n!$ is the sequence which begins
$$1, 1, 2, 6, 24, 120, \dots$$
- The function $f : \mathbb{N} \mapsto \mathbb{Q}$ defined by $f(n) = \frac{1}{n}$ is the sequence which begins
$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$$

A famous sequence



Suppose pairs of rabbits:

* Take one month to mature

* After maturity, produce a new pair of baby rabbits each month

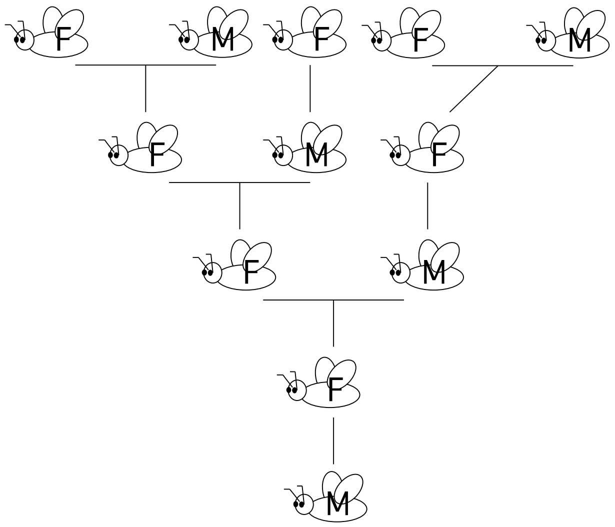
* Never die

How many rabbit pairs after a year? (Posed 1202, Fibonacci)

A moral about mathematical models: even though Fibonacci is a bad model for rabbits, it is good for bees:

- Male honeybees are born from an unfertilized egg, so they only have one parent, a female.
- Female honeybees are born from a fertilized egg, so they have two parents, a female and a male.
- So how many ancestors at each 'level' (parents, grand-parents, great-grandparents etc) does a male honeybee have?

Look familiar?



Cartesian Product of n Sets

We will define relations in terms of a Cartesian Product of n sets, which generalises the definition for 2 sets which we considered previously.

Definition

Given a collection of n sets $\{A_1, A_2, ..., A_n\}$, where n is any positive integer, the *Cartesian product* '$A_1 \times A_2 \times ... \times A_n$' is the set of **all ordered n-tuples** $(a_1, a_2, ..., a_n)$, where a_i is in A_i for each $i = 1, 2, ..., n$.

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) : a_i \in A_i \text{ for each } i\}.$$

Example:

Let $Y = \{2012\}$, $C = \{AUS, USA\}$, $E = \{run, swim\}$, $M = \{gold, silver, bronze\}$

Then the Cartesian Product $Y \times C \times E \times M =$

$$\begin{aligned}\{&(2012, AUS, run, gold), (2012, AUS, run, silver), (2012, AUS, run, bronze),\\ &(2012, AUS, swim, gold), (2012, AUS, swim, silver), (2012, AUS, swim, bronze),\\ &(2012, USA, run, gold), (2012, USA, run, silver), (2012, USA, run, bronze),\\ &(2012, USA, swim, gold), (2012, USA, swim, silver), (2012, USA, swim, bronze)\}\end{aligned}$$

Equality of n-tuples

Definition

The i^{th} entry of a given n-tuple in a Cartesian Product is called the *i^{th} coordinate*.

Definition

Two n-tuples are *equal* if and only if there are **equal in each coordinate**. More precisely, let $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_n)$ both be elements of $A_1 \times A_2 \times ... \times A_n$. Then

$$\mathbf{a} = \mathbf{b} \quad \text{if and only if} \quad a_i = b_i$$

for each $i = 1, 2, ..., n$.

Relations

Definition

Any **subset** of a Cartesian Product is called a Relation.

Example: The relation

$$R = \{(2012, AUS, swim, silver), (2012, USA, run, gold), (2012, AUS, run, bronze)\}$$

is a subset of the Cartesian product

$$\{2012\} \times \{AUS, USA\} \times \{run, swim\} \times \{gold, silver, bronze\}$$

$$\begin{aligned}= \{&(2012, AUS, run, gold), (2012, AUS, run, silver), (2012, AUS, run, bronze),\\ &(2012, AUS, swim, gold), (2012, AUS, swim, silver), (2012, AUS, swim, bronze),\\ &(2012, USA, run, gold), (2012, USA, run, silver), (2012, USA, run, bronze),\\ &(2012, USA, swim, gold), (2012, USA, swim, silver), (2012, USA, swim, bronze)\}\end{aligned}$$

Relations are often expressed as tables. Each row in the table is an element of the relation.

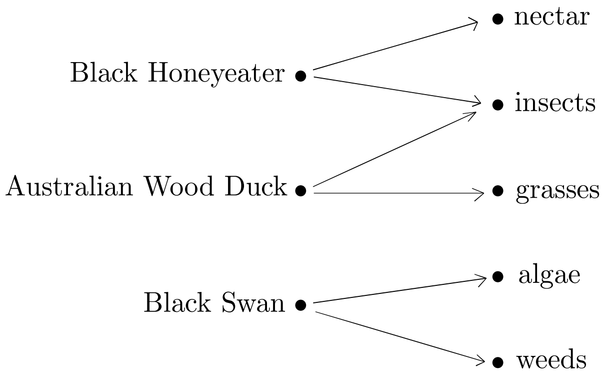
Example: Let B be the set of Australian native birds, H be a set of habitats and F be a set of foods. Then a relation in $B \times H \times F$ is:

Bird	Habitat	Food
Black Honeyeater	open woodlands	insects
Black Honeyeater	open woodlands	nectar
Australian Wood Duck	open woodlands	grasses
Australian Wood Duck	open woodlands	insects
Australian Wood Duck	grasslands	grasses
Australian Wood Duck	grasslands	insects
Australian Wood Duck	wetlands	grasses
Australian Wood Duck	wetlands	insects
Black Swans	wetlands	algae
Black Swans	wetlands	weeds

(Information extracted from http://www.birdsinbackyards.net/finder/)

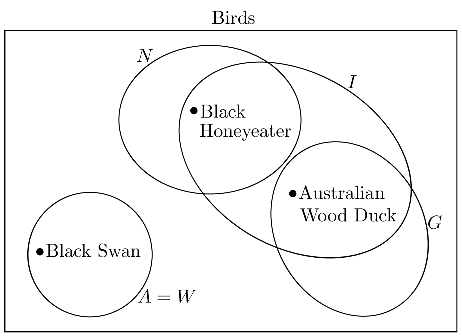
Arrow Diagram

Relations may also be expressed using *arrow diagrams*. A binary relation 'E' which one might call 'eats', from birds to food sources, taken from the same set of data as used on the previous page can be expressed:



The same 'eats' information in the arrow diagram can be expressed using a Venn Diagram. Let

$$N := \{\text{birds that eat Nectar}\}, I := \{\text{birds that eat Insects}\},$$
$$G := \{\text{birds that eat Grasses}\}, A := \{\text{birds that eat Algae}\},$$
$$W := \{\text{birds that eat Weeds}\}.$$



Venn Diagrams often become impractical for large data sets.

Redundancy can be good

Bird	Habitat	Food
Black Honeyeater	open woodlands	insects
Black Honeyeater	open woodlands	nectar
Australian Wood Duck	open woodlands	grasses
Australian Wood Duck	open woodlands	insects
Australian Wood Duck	grasslands	grasses
Australian Wood Duck	grasslands	insects
Australian Wood Duck	wetlands	grasses
Australian Wood Duck	wetlands	insects
Black Swans	wetlands	algae
Black Swans	wetlands	weeds

- The table has a lot of repetition, but it is convenient to look up - this is the basis of the idea of a *relational database*

- The study of ecology, biology, aspects of finance, combinatorics and other disciplines have recently included the collection of some enormous data sets.
- Storing this data in a form such that we can retrieve the information we want from it in a reasonable amount of time is a challenge which is met in a number of different ways.
  - Sometimes a spread sheet is used to create tables like the one of 'Birds, Habitats and Food'; and tools like the Excel 'pivot' command are used to perform set theoretic operations on the data. **Advantage:** readily accessible technology, easy to use. **Disadvantage:** data can be easily accidentally lost.
  - Quite often a purpose-built database is used, and a query language such as 'SQL' (Structured Query Language). A common structure for databases is that of a *relational database*, which is built upon the mathematical concept of a *relation*.

A special case of an *n*-ary relation is a binary relation

- An *n*-ary relation, R, is a subset of an n-ary Cartesian Product, i.e.

$$R \subseteq A_1 \times A_2 \times \ldots \times A_n$$

- A binary relation, R, is a subset of a Cartesian Product of 2 sets, i.e.

$$R \subseteq A \times B$$

We sometimes say that R is the relation *from A to B*.

"Real life" Binary Relations we may wish to model:

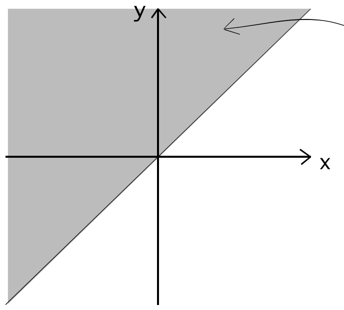
- parent-child e.g. Cameron "is parent of" Sarah
  Sarah "is child of" Cameron
- teacher-student
- student-course
- customer-account type
- data-storage location

Other Binary Relations include:

- is less than
- is less than or equal to
- is a subset of
- is logically equivalent to

We can draw Binary Relations in $\mathbb{R} \times \mathbb{R}$ in the plane

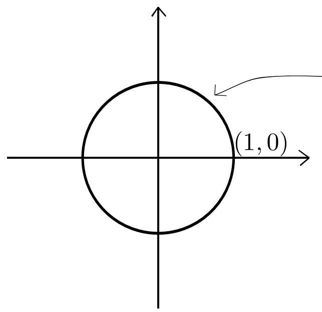
Example:



The $\leq$ relation 'R' contained in the Cartesian Product $\mathbb{R} \times \mathbb{R}$ (illustrated in the Cartesian Plane) is the set of points $(x, y) \in \mathbb{R} \times \mathbb{R}$ that satisfy the rule $x \leq y$

We can draw Binary Relations in $\mathbb{R} \times \mathbb{R}$ in the plane

Example:



The circle relation 'R' contained in the Cartesian Product $\mathbf{R} \times \mathbf{R}$ (illustrated as the Cartesian Plane) is the set of points $(x, y) \in \mathbf{R} \times \mathbf{R}$ that satisfy the rule $x^2 + y^2 = 1$.

- Recall that a *function* is (in this new language) a binary relation satisfying two extra properties (*what are they?*), so notice that all functions are binary relations, but not vice-versa.

Which claim is false?

A $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ is a binary relation

B $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ is a function

C $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \sqrt{1 - x^2}\}$ is a binary relation

D $\{(x, y) \in [-1, 1] \times \mathbb{R} : y = \sqrt{1 - x^2}\}$ is a function

Another notation for binary relations

Suppose R is some binary relation and let the pair (a, b) belong to R, i.e.

$$(a, b) \in R.$$

Then another notation we may use is

$$a\, R\, b,$$

which means that *a is related to b by the relation R*.

Binary Relations you already know

- We may use a different symbol instead of R in the expression '$a\,R\,b$'.
- You already know some examples:
  - The *'less than' relation* contained in the Cartesian Product $\mathbb{R} \times \mathbb{R}$ is the set of all pairs (a, b) such that a is less than b. For a member of this relation, we use the symbol '$<$' and write

$$a < b.$$

Example: $(3.00, 3.14)$ is an element of the "less than" relation and we write:

$$3.00 < 3.14$$

The 'less than' relation

Which of the following is the 'less than' relation on the Cartesian Product $A \times A$, for $A = \{0, 1, 2\}$?

A $R = \{(0, 0), (1, 1), (2, 2)\}$

B $R = \{(0, 1), (1, 2)\}$

C $R = \{(0, 1), (0, 2), (1, 2)\}$

D $R = \{\}$

More Binary Relations you already know...

- The *'proper subset' relation* contained in the Cartesian Product '$\mathcal{P}(U) \times \mathcal{P}(U)$', where $\mathcal{P}(U)$ is the power set of some Universal Set U, is the set of all pairs (X, Y), of sets X and Y whose elements belong to U, such that X is a proper subset of Y. For a member of this relation, we use the symbol '$\subset$' and write

$$X \subset Y.$$

Example: Let U be the set of colours of the rainbow. Then

$$(\{red, yellow\}, \{red, yellow, blue, indigo\})$$

is an element of the proper subset relation and we write:

$$\{red, yellow\} \subset \{red, yellow, blue, indigo\}.$$

More Binary Relations you already know...

- The *'equality' relation* contained in the Cartesian Product '$S \times S$', where S is any specified set, is the set of all pairs (x, y) in $S \times S$ such that x is equal to y. For a member of this relation, we use the symbol '$=$' and write

$$x = y.$$

Example: Let $S = \mathbb{Q}$ be the set of rational numbers. Then

$$\left(\frac{1}{2}, \frac{3}{6}\right)$$

is an element of the equality relation and we write:

$$\frac{1}{2} = \frac{3}{6}.$$

The Inverse Relation

Definition

Let $R \subseteq A \times B$ be a binary relation. The inverse relation R^{-1} is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

Example Let $A = \{1, 2, 3\}$ and let R be the relation "$\leq$" on A, that is, $R = \{(a, b) \in A \times A : a \leq b\}$. Explicitly we may write

$$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\},$$

Then

$$R^{-1} = \{(1, 1), (2, 1), (3, 1), (2, 2), (3, 2), (3, 3)\}.$$

We see that the inverse of $\leq$ is $\geq$.

The adjacency matrix

Definition

Let $R \subseteq A \times B$ be a binary relation, with A and B both countable sets. Let $M = (m_{ij})$ be a matrix whose rows are indexed by elements of the set A (in some order) and whose columns are indexed by the set B (in some order). Then M is the **adjacency matrix** for R provided

$$m_{ij} = \begin{cases} 1 & (i, j) \in R \\ 0 & \text{otherwise} \end{cases}$$

Adjacency matrix example

The relation

$$\{(1, b), (1, c), (2, b), (3, b), (3, c), (4, a)\}$$

from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c\}$ has adjacency matrix

$$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

What's the relation for this adjacency matrix?

$$\begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

A $\{(1, w), (1, x), (1, y), (1, z), (2, w), (2, x), (2, y), (2, z)\}$

B $\{(1, w), (1, z), (2, w), (2, y)\}$

C $\{(x, 1), (y, 1), (x, 2), (z, 2)\}$

D $\{(1, x), (1, y), (2, x), (2, z)\}$

What about its inverse?

If M is the adjacency matrix for binary relation R, what is the adjacency matrix for the inverse relation R^{-1}?

- Eg. if R has adjacency matrix

$$M = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

then R^{-1} has adjacency matrix ...

$$M^T = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

What's the general rule?

We can also use arrow diagrams as we have seen before:

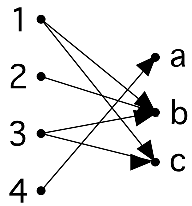
The relation

$$\{(1, b), (1, c), (2, b), (3, b), (3, c), (4, a)\}$$

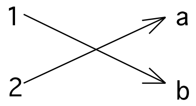
from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c\}$ with adjacency matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

has arrow diagram:



What's the adjacency matrix of this arrow diagram?



A $\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$

B $\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$

C $\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$

D $\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$

A relation *on* a set

Definition

Let $R \subseteq A \times A$ be a binary relation. Then we say R is a relation *on* A.

When R is a relation on A, what shape is its adjacency matrix?

Example: Let R be the 'less-than' relation on $\{1, 2, 3\}$. Then its adjacency matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

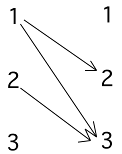
Arrow diagrams and digraphs

- When R is a relation on A, we have a choice how to draw it.
- We may draw our usual arrow diagram, with two copies of A
- Or we may draw a more concise version, with one copy of A

Example: The ‘less-than’ relation on $\{1, 2, 3\}$ with adjacency matrix:

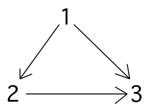
$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

has usual arrow diagram



or

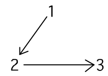
alternative representation, called a `digraph'.



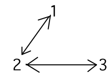
Which digraph represents the relation described by the following adjacency matrix?

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

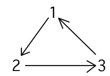
A



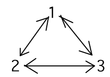
C



B

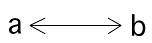


D

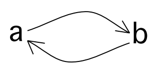


A digraph drawing convention

Note that when (a, b) and (b, a) are both in a relation,



means the same as



We will study digraphs in more generality later on in the course.

Textbook exercises

Exercises Section 1.1:

- 65-72

Exercises Section 3.1:

- 1-5 (Function definition)
- 10-11, 13-31, 48-49 (Injective (one-to-one), Surjective (onto), Bijective)
- 32-47, 50, 69-70 (Composition and Inverses)
- 62-68 (Composition, Inverses, injectivity, surjectivity, bijectivity)
- 74, 89, 97 (Proofs, injectivity, surjectivity, etc.)

Exercises section 3.3

- 1-16 (translating between representations of relations)
- 17-21 (translating between representations, and inverses)
- 18-20 (optional extra on proofs and inverses)

Exercises section 3.5

- 1-10 (Translating between representations)