MATH1510 - Discrete Mathematics
Probability 1

University of Newcastle

UoN

Introduction

In 1654, Antoine Gombaud, a French writer, asked the mathematicians Blaise Pascal and Pierre de Fermat to help him work out what is now known as the problem of points. This led to the formulation of the theory of probability.

Probability is important in many fields, such as statistics, genetics, psychology, actuarial science, economics and computer science.

Within the fields of mathematics and computing, probability has applications in many diverse areas such as artificial intelligence, pattern recognition, (pure) number theory and numerical computations.

Definitions – Experiment

Definition

An experiment is any process that generates a result we can measure or count. This is a fundamental concept in probability.

The result of the experiment is called an outcome.

We often use coins and dice in experiments to illustrate key ideas.

Example

An experiment consists of rolling a die 3 times and recording the results. The outcome is the set of 3 numbers which are recorded. The set of all possible outcomes consists of every combination of the 3 dice rolls.

Definitions – Event

Definition

An event is a set containing some outcomes from an experiment.

The sample space is the event consisting of all possible outcomes.

Examples

- A bag contains 5 red and 2 black marbles. One marble is selected from the bag. An event might be ‘selecting a red marble’.
- Three coins are tossed and we count the number of heads. The possible outcomes are none, one, two or three heads. An event might be ‘obtaining 2 or more heads’.

Definitions – Equally Likely

Definition

Equally likely outcomes are those that have an equal chance of occurring.

Examples

- When rolling a (fair) die, each outcome is equally likely by symmetry.
- When rolling two dice, the chance of rolling double 2 is the same as the chance of rolling double 6, but is not the same as the chance of rolling a 2 and a 6 in any order.

Definitions – Mutually Exclusive

Definition

Mutually exclusive events are events which cannot occur together. In other words, events A and B are mutually exclusive if

$$A \cap B = \varnothing.$$

Example

Suppose 2 dice are rolled. Event A is the set of outcomes in which at least one of the dice shows a 2. Event B is the set of outcomes in which the numbers on the dice add to more than 8.

Then the events A and B are mutually exclusive.

Key Definition

Definition

If the outcomes in S are equally likely, we define the probability of an event E, written $P(E)$, as:

$$P(E) = \frac{|E|}{|S|},$$

where
$|E|$ = number of outcomes making up the event E,
$|S|$ = total number of outcomes
= number of elements in the sample space.

This means many problems in probability are reduced to problems in *enumeration*.

Examples

Example

Suppose three coins are tossed simultaneously. Let A be the event that at least two heads were obtained.

$$|A| = 4, \ |S| = 8, \text{ so } P(A) = \frac{1}{2}.$$

Examples

- An integer in $\{1, 2, 3, \ldots, 25\}$ is selected at random. Find the probability that it is prime.
- Find the probability of winning a first division prize in lotto with one standard game. That is, if you select 6 numbers from 44, what is the chance that they will be the same 6 that are drawn by the lotto machine?

Example

A box has 6 white, 5 black and 3 green balls. Two balls are drawn one after the other, without replacement. What is the probability:

- that the first is green and the second is black?
- that both balls drawn are black?

Example

Three children each order a sandwich from the canteen. Sarah orders salad (s), Peter orders peanut butter (p) and Hannah orders ham (h). If the sandwiches are sealed, placed in plain bags and distributed randomly, one per child, what is the probability that:

- each child has the correct sandwich?
- no child has the correct sandwich?
- at least one child has the correct sandwich?

Important Results

Below are the most important basic results in probability (sometimes they are taken to be the *axioms*).

- $0 \leq |E| \leq |S|$.
  So $0 \leq P(E) \leq 1$.
- $P(\varnothing) = 0$ and $P(S) = 1$.
  The closer that $P(E)$ is to 1, the more likely it is that the event E will occur.
- If $E_1, E_2, \ldots, E_n$ are (pairwise) mutually exclusive outcomes that partition the sample space, then

$$P(E_1) + P(E_2) + \cdots + P(E_n) = 1$$

Complements

Definition

The complement of an event E, written as $\bar{E}$, is the set of all elements in the sample space which do not belong to E (so $\bar{E}$ is the set complement of E).

Example

Suppose two dice are rolled and A is the event that contains all outcomes in which the two numbers rolled are different. The complement $\bar{A}$ is the set of all outcomes in which a double is rolled.

$$P(E) + P(\bar{E}) = 1.$$

Enumerating the Sample Space

We can sometimes find the probability of an event by listing the sample space. Two ways to organise information are given below.

Example

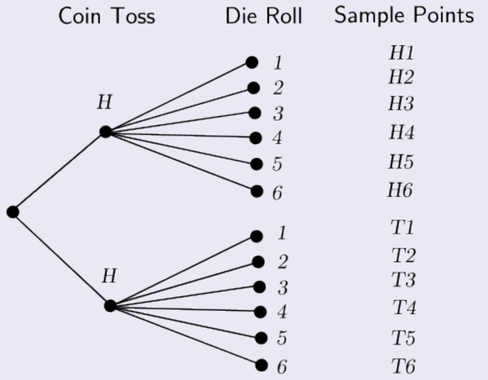
Two dice are rolled. The outcomes in the sample space are:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

Tree Diagrams

Example

A die is rolled and a coin is tossed. What is the probability of obtaining a head and a 4?



Tree Diagrams and Tabular Forms

Tree diagrams (like the one in the last example) are one way of listing the sample space.

Examples

Using tree diagrams:

- List the three-digit numbers greater than 600 that can be formed from the digits 2, 3, 6, 7 (no repetition of digits).
- List the even three-digit numbers greater than 600 that can be formed from the digits 2, 3, 6, 7 (no repetition of digits).

Sometimes it is convenient to use a tabular form to enumerate the sample space.

Tabular Forms

Example

Two dice are rolled, the table of probabilities for their sum is:

Sum	2	3	4	5	6	7	8	9	10	11	12
P(Sum)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

- What is the probability that the sum is 3?
- What is the probability that the sum is ≥ 7?
- What is the probability that the sum is ≤ 10?

The Addition Rule

Recall that A and B represent mutually exclusive events if they are disjoint. In that case, $|A \cup B| = |A| + |B|$.

Example

Two dice are rolled. Find the probability of obtaining a sum of 3 or 10.

For more than two events:

Definition

Events $E_1, E_2, \ldots, E_n$ are mutually exclusive if $E_i \cap E_j = \varnothing$ for all i, j with $i \neq j$. That is, they are pairwise disjoint.

Mutually Exclusive Events

Theorem

If $E_1, E_2, \ldots, E_n$ are pairwise disjoint then

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = \sum_{i=1}^{n} P(E_i).$$

What if two events are not necessarily disjoint?

Example

A digit is selected from the set

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

What is the probability that it is even or less than 4?

If E_1 and E_2 are events in a sample space S, consisting of equally likely outcomes, then

$$\begin{aligned} P(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} \\ &= \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} \\ &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \end{aligned}$$

Applications of Probability (1)

Monty Hall problem

You face 3 doors, 2 of which conceal a goat and the other one conceals a car. You choose a door at random with the hope of winning the prize behind it, and the host opens one door you didn't choose to reveal a goat. You are offered the chance to switch your choice. Should you switch?

List the sample space. If you don't switch, you have $\frac{1}{3}$ chance of winning the car. If you switch, your chance increases to $\frac{2}{3}$.

Door 1	Door 2	Door 3	Stay with door 1	Swap from door 1
Car	Goat	Goat	Car	Goat
Goat	Car	Goat	Goat	Car
Goat	Goat	Car	Goat	Car

Applications of Probability (2)

In a room of just N people there's a 50-50 chance of two people having the same birthday. What number do you think N is?

The birthday problem

Calculate the probability that among n people, at least 2 people share the same birthday.

We calculate the probability that *no two* people share the same birthday. The 1st person can be born on any day of the year, the 2nd person can be born on any of the remaining 364 days, the 3rd person can be born on any of the remaining 363 days, etc. This probability is given by

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{366-n}{365}.$$

So the probability we seek is $1 - \binom{365}{n} \frac{n!}{365^n}$.

Conditional Probability

Sometimes we wish to find the probability of event B *given* that event A occurs. That is, we want the probability of B while restricting the sample space to A.

This probability is given by

$$\frac{|B \cap A|}{|A|} = \frac{|B \cap A|}{|S|} \frac{|S|}{|A|} = \frac{P(B \cap A)}{P(A)}.$$

If $P(A) \neq 0$, we call this the conditional probability of B given A and denote it by

$$P(B|A) = \frac{P(B \cap A)}{P(A)}.$$

It also implies

$$P(B \cap A) = P(A)P(B|A).$$

Conditional Probability – Example

Example

There are 100 applicants for a teaching position, of whom some have at least 3 years of teaching experience and some have not, some are married and some are not, with the breakdown being:

	Married	Not
$\geq$ 3yrs teaching	12	24
$<$ 3yrs teaching	18	46

- What is the probability that an applicant is married and has at least 3 years of experience?
- Given that an applicant is married, what is the probability that they have at least three years teaching experience?

Harder Example

Example

A person draws two cards from four kings and four aces. Suppose what this person says is true.

1. If the person announces 'I have the ace of hearts', then what is the probability that he has two aces?
2. If the same person says 'I have at least one ace', then what is the probability that he has two aces?
3. If he points to one of the cards and says 'this card is an ace', then what is the probability that he has two aces?

Answers: either use the formula, or list the sample space: $\frac{3}{7}$, $\frac{3}{11}$, $\frac{3}{7}$.

Working for the third scenario.

$$\begin{aligned} P(2\text{ aces}|\text{an ace}) &= \frac{P(2\text{ aces and an ace})}{P(\text{an ace})} \\ &= \frac{P(2\text{ aces})}{P(\text{an ace})} \\ &= \frac{C(4,2)/C(8,2)}{C(4,1)/C(8,1)} \\ &= \frac{3/14}{4/8} \\ &= \frac{3}{7} \end{aligned}$$

Working for the second scenario.

$$\begin{aligned} P(2\text{ aces}|\geq 1\text{ ace}) &= \frac{P(2\text{ aces and at least 1 ace})}{P(\geq 1\text{ ace})} \\ &= \frac{P(2\text{ aces})}{P(\geq 1\text{ ace})} \\ &= \frac{C(4,2)/C(8,2)}{1 - C(4,2)/C(8,2)} \\ &= \frac{3/14}{11/14} \\ &= \frac{3}{11} \end{aligned}$$

Textbook exercises

Exercises Section 6.5:

- 1, 3, 5, 6, 12-14, 16, 17, 19, 20, 22, 25, 27-33, 34, 36, 38, 39, 41, 43, 49-51, 53-56

Exercises Section 6.6:

- 1-3, 4, 8, 12, 14-18, 19, 22-27, 31, 33, 35, 37, 39, 40-43