MATH1510 - Discrete Mathematics
Recurrence relations

University of Newcastle

Semester 2

Linear recurrence relations

- Consider a second-order linear reccurrence relation

$$s_n = as_{n-1} + bs_{n-2}.$$

- Suppose $s_n = r^n$, i.e., the sequence $(r^0, r^1, r^2, \ldots)$, is a solution.
- For $n = 2$, we get $r^2 = ar + b$. This is a necessary condition.
- Conversely, if $r^2 = ar + b$ then

$$r^n = r^2 r^{n-2} = (ar + b)r^{n-2} = ar^{n-1} + br^{n-2},$$

so the condition is also sufficient.

Theorem

The sequence $(1, r, r^2, r^3, \ldots)$ is a solution of the RR $s_n = as_{n-1} + bs_{n-2}$ if and only if r is a solution of the characteristic equation $x^2 - ax - b = 0$.

New solutions from old solutions

- Suppose the sequences $s = (s_0, s_1, s_2, \ldots)$ and $s' = (s'_0, s'_1, s'_2, \ldots)$ are solutions of the RR
$$s_n = as_{n-1} + bs_{n-2}.$$
- Consider the sequence $s'' = (s''_0, s''_1, s''_2 \ldots)$ with $s''_n = s_n + s'_n$.
$$\begin{aligned} s''_n = s_n + s'_n &= (as_{n-1} + bs_{n-2}) + (as'_{n-1} + bs'_{n-2}) \\ &= a(s_{n-1} + s'_{n-1}) + b(s_{n-2} + s'_{n-2}) = as''_{n-2} + bs''_{n-2}. \end{aligned}$$
- Similarly, if $\lambda \in \mathbb{R}$ and $s'' = (s''_0, s''_1, s''_2 \ldots)$ with $s''_n = \lambda s_n$ then
$$s''_n = \lambda s_n = \lambda\left(as_{n-1} + bs_{n-2}\right) = a\lambda s_{n-1} + b\lambda s_{n-2} = as''_{n-1} + bs''_{n-2}$$

Observation

- The sum of two solutions is again a solution.
- Multiplying any solution by a real gives another solution.

Characterizing all solutions (first case)

- Suppose the characteristic equation of the RR $s_n = as_n + bs_{n-2}$ has two real roots $r_1 \neq r_2$, i.e., $x^2 - ax - b = (x - r_1)(x - r_2)$.
- The previous observation tells us that every sequence (s_n) of the form
$$s_n = Ar_1^n + Br_2^n$$
is a solution.
- Now let the sequence $u = (u_0, u_1, u_2, \ldots)$ be any solution.
- Consider the linear system of equations
$$A + B = u_0, \qquad Ar_1 + Br_2 = u_1.$$
which has the solution
$$A^* = \frac{r_2 u_0 - u_1}{r_2 - r_1}, \qquad B^* = \frac{u_1 - r_1 u_0}{r_2 - r_1}.$$

Characterizing all solutions (first case cont'd)

- We obtain the particular solution $v = (v_0, v_1, v_2, \ldots)$ with
$$v_n = A^* r_1^n + B^* r_2^n.$$
- By our previous observation, we know that the sequence $w = (w_0, w_1, w_2, \ldots)$, defined by $w_n = u_n - v_n$ is another solution.
- Now
$$\begin{aligned} w_0 &= u_0 - v_0 = u_0 - \left(A^* r_1^0 + B^* r_2^0\right) = u_0 - A^* - B^* = 0, \\ w_1 &= u_1 - v_1 = u_1 - \left(A^* r_1^1 + B^* r_2^1\right) = u_1 - A^* r_1 - B^* r_2 = 0. \end{aligned}$$
- Since w is a solution, $w_n = aw_{n-1} + bw_{n-2}$ for all $n \geqslant 0$, and by induction $w_n = 0$ for all n.
- This implies that $u_n = v_n$ for all n, so every solution has the form $Ar_1^n + Br_2^n$.

Characterizing all solutions (second case)

- Suppose the characteristic equation of the RR $s_n = as_n + bs_{n-2}$ has one real roots $r \neq 0$, i.e., $x^2 - ax - b = (x - r)^2$.
- Note that $x^2 - ax - b = (x - r)^2$ implies $a = 2r$ and $b = -r^2$.
- Using this we obtain for the sequence $s_n = nr^n$,
$$\begin{aligned} as_{n-1} + bs_{n-2} &= a(n-1)r^{n-1} + b(n-2)r^{n-2} \\ &= 2r(n-1)r^{n-1} - r^2(n-2)r^{n-2} = r^n\left[2(n-1) - (n-2)\right] \\ &= nr^n = s_n. \end{aligned}$$

Conclusion

If the characteristic equation has a single real root r, then
$$Ar^n + Bnr^n$$
is a solution for any real numbers A and B.

Characterizing all solutions (second case cont'd)

- Let the sequence $u = (u_0, u_1, u_2, \dots)$ be any solution.
- Consider the system of linear equations

$$Ar^0 + B \cdot 0r^0 = A = u_0, \qquad Ar^1 + B \cdot 1r^1 r = Ar + Br = u_1.$$

which has the solution

$$A^* = u_0, \qquad B^* = \frac{u_1 - u_0 r}{r}.$$

- (v_n) with $v_n = A^* r^n + B^* nr^n$ is a solution, and so is $w_n = u_n - v_n$.

$$w_0 = u_0 - v_0 = u_0 - A^* r^0 = u_0 - A^* = 0,$$
$$w_1 = u_1 - v_1 = u_1 - \left(A^* r^1 + B^* r\right) = u_a - A^* r - B^* r = 0.$$

- Since w is a solution, $w_n = aw_{n-1} + bw_{n-2}$ for all $n \geqslant 0$, and by induction $w_n = 0$ for all n.
- This implies that $u_n = v_n$ for all n, so every solution has the form $Ar^n + Bnr^n$.

Example (Scheduling)

A TV station wants to fill a break between programs with advertisements, which are either half a minute or a full minute. How many ways can this be done? We need to fill n slots (half minutes). Let a_n be the number of ways of filling these n slots.

- There is one way of filling 0 slots (no spots at all): $a_0 = 1$
- There is one way of filling 1 slot (one half-minute spot): $a_1 = 1$
- There are two ways of filling two slots (one full-minute spot or two half-minute spots): $a_2 = 2$
- There are three ways of filling three slots ((full,half), (half,full), (half,half,half)): $a_2 = 3$
- In general: half-minute spot into slot $n \implies a_{n-1}$ possibilities for the remaining $n-1$ slots. full-minute spot in slots $n-1$ and $n \implies a_{n-2}$ possibilities to fill the remaining $n-2$ slots.
- Hence $a_n = a_{n-1} + a_{n-2}$, and we get the Fibonacci numbers again.

Example (Properties of Fibonacci numbers)

Sometimes the applications feed back into the mathematics. We can use the previous example to prove the identity

$$f_m f_n + f_{m-1} f_{n-1} = f_{m+n}.$$

Consider the scheduling of an advertising break of $m+n$ slots. There are two possibilities for where we are at the end of m slots

- The end of m slots is the end of an advert. So we have filled m slots, and n slots.
- The end of m slots is the middle of a long advert. So we have filled $m-1$ slots, and $n-1$ slots.

Example (Bell numbers)

We can use similar principles to count all partitions of a set. The nth Bell number B_n is the number of partitions of $\{1, \ldots, n\}$. For instance $B_1 = 1$, $B_2 = 2$ and $B_3 = 5$.
To find a recurrence relation for this sequence, consider a partition of $\{1, \ldots, n\}$. The symbol n will be in a cell $[n]$ of the partition, say with j elements $(1 \leqslant j \leqslant n)$ and the other $k = n - j$ elements will be partitioned. There are $\binom{n-1}{j-1} = \binom{n-1}{k}$ ways to choose the elements in $[n]$. There are B_k ways to partition the k elements not in $[n]$.
Hence the number of ways to partition $\{1, \ldots, n\}$ so that the cell containing n has $j = n - k$ elements is $\binom{n-1}{k} B_k$ and so in total

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

The initial condition $B_0 = 1$ comes from a degenerate case.

Non-homogenous recurrence relations

- Consider a recurrence relation of the form

$$s_n = as_{n-1} + bs_{n-2} + f(n).$$

- Take two solutions $s = (s_0, s_1, s_2, \ldots)$ and $s' = (s'_0, s'_1, s'_2, \ldots)$.
- Let the sequence $u = (u_0, u_1, \ldots)$ be defined by $u_n = s_n - s'_n$. Then

$$\begin{aligned} u_n = s_n - s'_n &= [(as_{n-1} + bs_{n-2} + f(n)] - \left[(as'_{n-1} + bs'_{n-2} + f(n)\right] \\ &= a(s_{n-1} - s'_{n-1}) + b(s_{n-1} - s'_{n-1}) = au_{n-1} + bu_{n-2}, \end{aligned}$$

so (u_n) is a solution of the (homogeneous) RR $s_n = as_{n-1} + bs_{n-2}$.

Observation

The difference of any two solutions of a non-homogeneous RR is a solution of the associated homogeneous RR.

Solution of non-homogeneous recurrence relations

- The general solution of a non-homogeneous RR can be written as the sum of any particular solution and the general solution of the associated homogeneous RR. More precisely:

Theorem

Let $u = (u_0, u_1, \ldots)$ be any solution of the RR

$$s_n = as_{n-1} + bs_{n-2} + f(n) \tag{1}$$

1. *If the characteristic equation $x^2 - ax - b = 0$ has two real solutions $r_1 \neq r_2$, then the general solution of (1) has the form*

$$s_n = u_n + Ar_1^n + Br_2^n.$$

2. *If the characteristic equation $x^2 - ax - b = 0$ has a single real solution r, then the general solution of (1) has the form*

$$s_n = u_n + Ar^n + Bnr_2^n.$$

Solving inhomogeneous RR

$$s_n = as_{n-1} + bs_{n-2} + f(n)$$

1. Solve the corresponding homogeneous problem.
2. Find a particular solution for the non-homogeneous problem.
3. Use the initial values to determine the free parameters.

Note that this doesn't tell us anything of how to do step 2. A method that often works is to assume that the solution looks like $f(n)$ and then determine the free parameters from the recurrence.

Example

Solve $a_n = a_{n-1} + a_{n-2} + 3n + 1$ with $a_0 = 2$ and $a_1 = 3$.

Step 1. Solve the homogeneous problem.

- The characteristic equation is $x^2 - x - 1 = 0$ with solutions $\frac{1 \pm \sqrt{5}}{2}$.
- The general solution of the homogeneous RR $a_n = a_{n-1} + a_{n-2}$ is

$$A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Example

Solve $a_n = a_{n-1} + a_{n-2} + 3n + 1$ with $a_0 = 2$ and $a_1 = 3$.

Step 2. Solve the non-homogeneous problem.

- $f(n) = 3n + 1$, so we guess $u_n = \alpha n + \beta$.
- From $u_n = u_{n-1} + u_{n-2} + 3n + 1$ we obtain

$$\alpha n + \beta = \alpha(n-1) + \beta + \alpha(n-2) + \beta + 3n + 1$$

  or $(3 + \alpha)n + (\beta - 3\alpha + 1) = 0$.
- Since this has to be satisfied for all n, we get $\alpha = -3$ and $\beta = -10$.
- A solution of the RR $a_n = a_{n-1} + a_{n-2} + 3n + 1$ is $u_n = -3n - 10$.

General solution

$$a_n = -3n - 10 + A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Example

Solve $a_n = a_{n-1} + a_{n-2} + 3n + 1$ with $a_0 = 2$ and $a_1 = 3$.

Step 3. Find the constants.

The general solution is $a_n = -3n - 10 + A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$, and the initial conditions say

$$A + B = 12, \qquad A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 16.$$

This gives $A = 6 + 2\sqrt{5}$ and $B = 6 - 2\sqrt{5}$.

Solution

$$a_n = -3n - 10 + (6 + 2\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^n + (6 - 2\sqrt{5})\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Example

Solve $a_n = 2a_{n-1} - a_{n-2} + 2^n$ with $a_0 = 1$ and $a_1 = 2$.

Step 1. Solve the homogeneous problem.

- The characteristic equation is $x^2 - 2x + 1 = 0$ with only one solution: $x = 1$.
- The general solution of the homogeneous RR $a_n = 2a_{n-1} - a_{n-2}$ is

$$a_n = A + Bn.$$

Example

Solve $a_n = 2a_{n-1} - a_{n-2} + 2^n$ with $a_0 = 1$ and $a_1 = 2$.

Step 2. Solve the non-homogeneous problem.

- $f(n) = 2^n$, so we guess $u_n = \alpha 2^n + \beta$.
- From $u_n = 2u_{n-1} - u_{n-2} + 2^n$ we obtain

$$\alpha 2^n + \beta = 2\left(\alpha 2^{n-1} + \beta\right) - \left(\alpha 2^{n-2} + \beta\right) + 2^n$$

  or $(4 - \alpha)2^{n-2} = 0$.
- Since this has to be satisfied for all n, we get $\alpha = 4$.
- A solution of the RR $a_n = 2a_{n-1} - a_{n-2} + 2^n$ is $u_n = 4 \cdot 2^n = 2^{n+2}$.

General solution

$$a_n = 2^{n+2} + A + Bn.$$

Example

Solve $a_n = 2a_{n-1} - a_{n-2} + 2^n$ with $a_0 = 1$ and $a_1 = 2$.

Step 3. Find the constants.

The general solution is $a_n = 2^{n+2} + A + Bn$, and the initial conditions say

$$4 + A = 1, \qquad 8 + A + B = 2.$$

This gives $A = B = -3$.

Solution

$$a_n = 2^{n+2} - 3 - 3n = 2^{n+2} - 3(n+1).$$

Example

Solve $a_n = 4a_{n-1} - 3a_{n-2} + 3^n$ with $a_0 = 1$ and $a_1 = 3$.

Step 1. Solve the homogeneous problem.

- The characteristic equation is $x^2 - 4x + 3 = 0$ with solutions 1 and 3.
- The general solution of the homogeneous RR $a_n = 4a_{n-1} - 3a_{n-2}$ is

$$A + B3^n.$$

Example

Solve $a_n = 4a_{n-1} - 3a_{n-2} + 3^n$ with $a_0 = 1$ and $a_1 = 3$.

Step 2. Solve the non-homogeneous problem.

- $f(n) = 3^n$, so we guess $u_n = \alpha 3^n + \beta$.
- From $u_n = 4u_{n-1} - 3u_{n-2} + 3^n$ we obtain

$$\alpha 3^n + \beta = 4\left(\alpha 3^{n-1} + \beta\right) - 3\left(\alpha 3^{n-2} + \beta\right) + 3^n$$

  or $3^n = 0$.

It does not work this way. The 3^n in our guess is not good enough because 3 is a root of the characteristic equation. In this case we need a more sophisticated guess.

Example

Solve $a_n = 4a_{n-1} - 3a_{n-2} + 3^n$ with $a_0 = 1$ and $a_1 = 3$.

Step 2. Solve the non-homogeneous problem.

- $f(n) = 3^n$ and 3 is a root, so we guess $u_n = \alpha n 3^n + \beta$.
- From $u_n = 4u_{n-1} - 3u_{n-2} + 3^n$ we obtain

$$\alpha n 3^n + \beta = 4\left(\alpha(n-1)3^{n-1} + \beta\right) - 3\left(\alpha(n-2)3^{n-2} + \beta\right) + 3^n$$

  or $0 = (9 - 3\alpha(n-2) + 12\alpha(n-1) - 9\alpha n)3^{n-2} = (9 - 6\alpha)3^{n-2}$.
- Since this has to be satisfied for all n, we get $\alpha = 3/2$.
- A solution of the RR $a_n = 4a_{n-1} - 3a_{n-2} + 3^n$ is $a_n = \frac{3}{2} \cdot n3^n = \frac{n3^{n+1}}{2}$.

General solution

$$a_n = \frac{n3^{n+1}}{2} + A + B3^n.$$

Example

Solve $a_n = 4a_{n-1} - 3a_{n-2} + 3^n$ with $a_0 = 1$ and $a_1 = 3$.

Step 3. Find the constants.

The general solution is $a_n = \frac{n3^{n+1}}{2} + A + B3^n$, and the initial conditions say

$$A + B = 1, \qquad \frac{9}{2} + A + 3B = 3.$$

This gives $A = 9/4$ and $B = -5/4$.

Solution

$$a_n = \frac{n3^{n+1}}{2} + \frac{9}{4} - \frac{5 \cdot 3^n}{4} = \frac{(6n-5)3^n + 9}{4}.$$

Summary

Recurrence relations with or without initial conditions

Solution A solution vs the general solution vs the solution to an RR with IC.

Basic solutions Sums and products.

Linear RRs (homogeneous) Characteristic equation to find exponential solutions

Linear RRs (nonhomogeneous) general solution as a combination of the solution for the corresponding homogeneous problem and a special solution for the nonhomogeneous problem