COMP1010 – Week 11
Discrete Mathematics Fundamentals – Part 2

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Logical Form and Logical Equivalence

- The central concept of deductive logic is the concept of argument form. An argument is a sequence of statements aimed at demonstrating the truth of an assertion.

- The assertion at the end of the sequence is called the *conclusion*, and the preceding statements are called *premises*.

- To have confidence in the conclusion that you draw from an argument, you must be sure that the premises are acceptable on their own merits or follow from other statements that are known to be true.

Statements

- Most of the definitions of formal logic have been developed so that they agree with the natural or intuitive logic used by people who have been educated to think clearly and use language carefully.

- The differences that exist between formal and intuitive logic are necessary to avoid ambiguity and obtain consistency.

- In any mathematical theory, new terms are defined by using those that have been previously defined. However, this process has to start somewhere. A few initial terms necessarily remain undefined.

Statements

- In logic, the words *sentence*, *true*, and *false* are the initial undefined terms.

• Definition

A **statement** (or **proposition**) is a sentence that is true or false but not both.

Compound Statements

- We now introduce three symbols that are used to build more complicated logical expressions out of simpler ones.

  The symbol ~ denotes *not*, $\wedge$ denotes *and*, and $\vee$ denotes *or*.

- Given a statement *p*, the sentence "~*p*" is read "not *p*" or "It is not the case that *p*" and is called the **negation of *p*.**

Compound Statements

- Given another statement q, the sentence "$p \wedge q$" is read "p and q" and is called the **conjunction of *p* and *q***.
- The sentence "$p \vee q$" is read "p or q" and is called the **disjunction of *p* and *q***.
- In expressions that include the symbol ~ as well as $\wedge$ or $\vee$, the **order of operations** specifies that ~ is performed first.
- For instance, $\sim p \wedge q = (\sim p) \wedge q$.

Compound Statements

- In logical expressions, as in ordinary algebraic expressions, the order of operations can be overridden through the use of parentheses.
- Thus $\sim(p \wedge q)$ represents the negation of the conjunction of p and q.
- In this, as in most treatments of logic, the symbols $\wedge$ and $\vee$ are considered coequal in order of operation, and an expression such as $p \wedge q \vee r$ is considered ambiguous.
- This expression must be written as either $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$ to have meaning.

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Truth Values

- In the previous slides we built compound sentences out of component statements and the terms *not*, *and*, and *or*.

- If such sentences are to be statements, however, they must have well-defined **truth values**—they must be either true or false.

- We now define such compound sentences as statements by specifying their truth values in terms of the statements that compose them.

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Truth Values

- *The negation of a statement is a statement that exactly expresses what it would mean for the statement to be false*.

> **• Definition**
>
> If p is a statement variable, the **negation** of p is "not p" or "It is not the case that p" and is denoted $\sim p$. It has opposite truth value from p: if p is true, $\sim p$ is false; if p is false, $\sim p$ is true.

- The truth values for negation are summarized in a *truth table*.

p	$\sim p$
T	F
F	T

Truth Table for ~*p*

Truth Values

• Definition

If p and q are statement variables, the **conjunction** of p and q is "p and q," denoted $p \wedge q$. It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, $p \wedge q$ is false.

- The truth values for conjunction can also be summarized in a truth table.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table for $p \wedge q$

Truth Values

• Definition

If p and q are statement variables, the **disjunction** of p and q is "p or q," denoted $p \vee q$. It is true when either p is true, or q is true, or both p and q are true; it is false only when both p and q are false.

- Here is the truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth Table for $p \vee q$

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Evaluating the Truth of More General Compound Statements

- Now that truth values have been assigned to $\sim p$, $p \wedge q$, and $p \vee q$, consider the question of assigning truth values to more complicated expressions such as $\sim p \vee q$, $(p \vee q) \wedge \sim(p \wedge q)$, and $(p \wedge q) \vee r$. Such expressions are called *statement forms* (or *propositional forms*).

• Definition

A **statement form** (or **propositional form**) is an expression made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables. The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

Evaluating the Truth of More General Compound Statements

- To compute the truth values for a statement form, follow rules similar to those used to evaluate algebraic expressions.

- For each combination of truth values for the statement variables, first evaluate the expressions within the innermost parentheses, then evaluate the expressions within the next innermost set of parentheses, and so forth until you have the truth values for the complete expression.

Logical Equivalence

The statements

6 is greater than 2 and 2 is less than 6

are two different ways of saying the same thing. Why? Because of the definition of the phrases *greater than* and *less than*. By contrast, although the statements

(1) Dogs bark and cats meow

and

(2) Cats meow and dogs bark

are also two different ways of saying the same thing, the reason has nothing to do with the definition of the words.

Logical Equivalence

- It has to do with the logical form of the statements.

- Any two statements whose logical forms are related in the same way as (1) and (2) would either both be true or both be false.

- You can see this by examining the following truth table, where the statement variables *p* and *q* are substituted for the component statements “Dogs bark” and “Cats meow,” respectively.

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Logical Equivalence

- The table shows that for each combination of truth values for *p* and *q*, $p \wedge q$ is true when, and only when, $q \wedge p$ is true.

- In such a case, the statement forms are called *logically equivalent*, and we say that (1) and (2) are *logically equivalent statements*.

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$p \wedge q$ and $q \wedge p$ always have the same truth values, so they are logically equivalent

Logical Equivalence

- There are two ways to show that statement forms *P* and *Q* are *not* logically equivalent. As indicated previously, one is to use a truth table to find rows for which their truth values differ.

- The other way is to find concrete statements for each of the two forms, one of which is true and the other of which is false.

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De Morgan's laws

- The following two logical equivalences are known as **De Morgan's laws** of logic in honor of Augustus De Morgan, who was the first to state them in formal mathematical terms.

De Morgan's Laws

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

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De Morgan's laws

- Symbolically we can represent the two logic equivalences as:

$$\boxed{\sim(p \wedge q) \equiv \sim p \vee \sim q}$$

and

$$\boxed{\sim(p \vee q) \equiv \sim p \wedge \sim q.}$$

Conditional Statements

Let *p* and *q* be statements. A sentence of the form "If *p* then *q*" is denoted symbolically by "$p \to q$"; *p* is called the *hypothesis* and *q* is called the *conclusion*. For instance, consider the following statement:

$$\text{If } \underbrace{4{,}686 \text{ is divisible by } 6}_{\text{hypothesis}}, \text{ then } \underbrace{4{,}686 \text{ is divisible by } 3}_{\text{conclusion}}$$

Such a sentence is called *conditional* because the truth of statement *q* is conditioned on the truth of statement *p*.

Conditional Statements

The notation $p \to q$ indicates that $\to$ is a connective, like $\wedge$ or $\vee$, that can be used to join statements to create new statements.

To define $p \to q$ as a statement, therefore, we must specify the truth values for $p \to q$ as we specified truth values for $p \wedge q$ and for $p \vee q$.

As is the case with the other connectives, the formal definition of truth values for $\to$(if-then) is based on its everyday, intuitive meaning.

Conditional Statements

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for $p \to q$

- **Definition**

If p and q are statement variables, the **conditional** of q by p is "If p then q" or "p implies q" and is denoted $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).

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Example - A Conditional Statement with a False Hypothesis

Consider the statement:

If $0 = 1$ then $1 = 2$.

As strange as it may seem, since the hypothesis of this statement is false, the statement as a whole is true.

Conditional Statements

In expressions that include $\rightarrow$ as well as other logical operators such as $\wedge$, $\vee$, and ~, the **order of operations** is that $\rightarrow$ is performed last.

Thus, according to the specification of order of operations, ~ is performed first, then $\wedge$ and $\vee$, and finally $\rightarrow$.

The Equivalence between $\sim p \vee q$ and $p \rightarrow q$

Rewrite the following statement in if-then form.

Either you get to work on time or you are fired.

Solution:

Let $\sim p$ be

You get to work on time.

and q be

You are fired.

Then the given statement is $\sim p \vee q$. Also p is

You do not get to work on time.

So the equivalent if-then version, $p \rightarrow q$, is

If you do not get to work on time, then you are fired.

The Negation of a Conditional Statement

By definition, $p \to q$ is false if, and only if, its hypothesis, p, is true and its conclusion, q, is false. It follows that

The negation of "if p then q" is logically equivalent to "p and not q."

This can be restated symbolically as follows:

$$\boxed{\sim(p \to q) \equiv p \wedge \sim q}$$

The Contrapositive of a Conditional Statement

One of the most fundamental laws of logic is the equivalence between a conditional statement and its contrapositive.

• Definition

The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p.$$

Symbolically,

$$\text{The contrapositive of } p \to q \text{ is } \sim q \to \sim p.$$

A conditional statement is logically equivalent to its contrapositive.

The Converse and Inverse of a Conditional Statement

The fact that a conditional statement and its contrapositive are logically equivalent is very important and has wide application. Two other variants of a conditional statement are *not* logically equivalent to the statement.

• Definition

Suppose a conditional statement of the form "If p then q" is given.

1. The **converse** is "If q then p."
2. The **inverse** is "If $\sim p$ then $\sim q$."

Symbolically,

$$\text{The converse of } p \to q \text{ is } q \to p,$$

and

$$\text{The inverse of } p \to q \text{ is } \sim p \to \sim q.$$

The Converse and Inverse of a Conditional Statement

1. A conditional statement and its converse are *not* logically equivalent.

2. A conditional statement and its inverse are *not* logically equivalent.

3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Only If and the Biconditional

To say “*p* only if *q*” means that *p* can take place *only* if *q* takes place also. That is, if *q* does not take place, then *p* cannot take place.

Another way to say this is that if *p* occurs, then *q* must also occur (by the logical equivalence between a statement and its contrapositive).

• Definition

It p and q are statements,

p **only if** q means “if not q then not p,”

or, equivalently,

“*if* p then q.”

Only If and the Biconditional

• Definition

Given statement variables p and q, the **biconditional of p and q** is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words *if and only if* are sometimes abbreviated **iff.**

The biconditional has the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table for $p \leftrightarrow q$

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Only If and the Biconditional

In order of operations $\leftrightarrow$ is coequal with $\rightarrow$. As with $\wedge$ and $\vee$, the only way to indicate precedence between them is to use parentheses.

The full hierarchy of operations for the five logical operators is:

Order of Operations for Logical Operators

1.	$\sim$	Evaluate negations first.
2.	$\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3.	$\rightarrow, \leftrightarrow$	Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

Only If and the Biconditional

According to the separate definitions of *if* and *only if*, saying "p if, and only if, q" should mean the same as saying both "p if q" and "p only if q."

The following annotated truth table shows that this is the case:

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$
always have the same truth values,
so they are logically equivalent

Truth Table Showing that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Fallacies

- A **fallacy** is an error in reasoning that results in an invalid argument. Three common fallacies are **using ambiguous premises,** and treating them as if they were unambiguous, **circular reasoning** (assuming what is to be proved without having derived it from the premises), and **jumping to a conclusion** (without adequate grounds).

- In this section we discuss two other fallacies, called *converse error* and *inverse error*, which give rise to arguments that superficially resemble those that are valid by modus ponens and modus tollens but are not, in fact, valid.