Lecture

Basic Vector Geometry

(Modified from: Hill & Kelley book, chapter 4)

Comp3320/6370 Computer Graphics

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Introduction

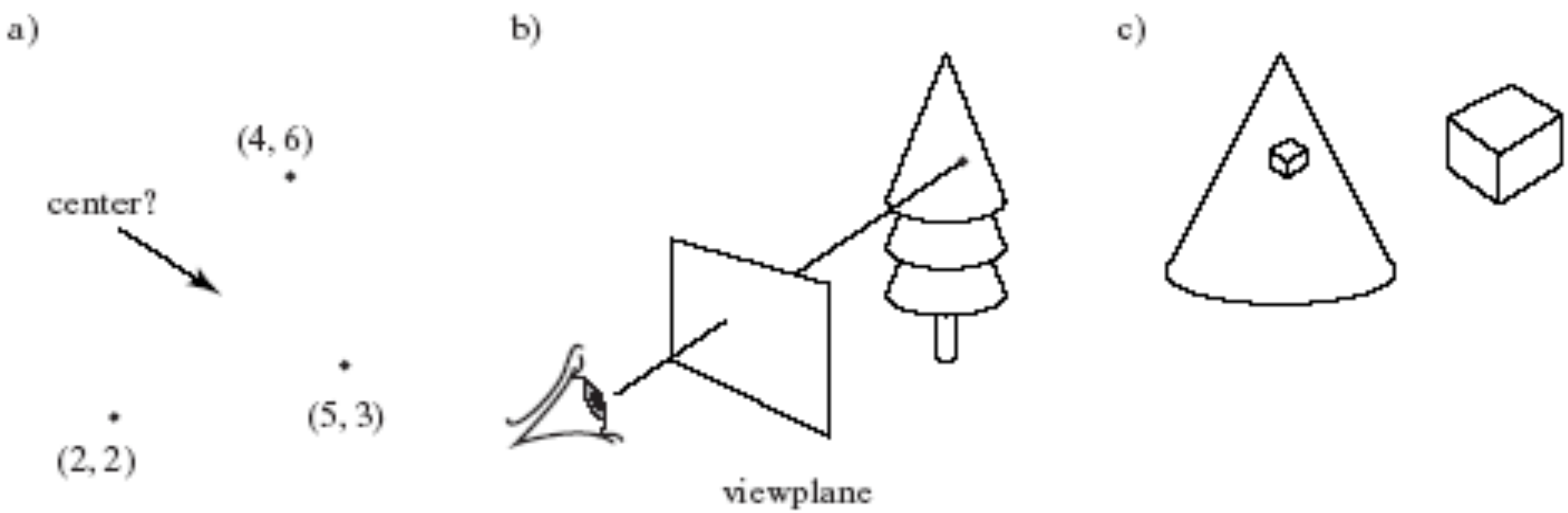
- In computer graphics, we work with objects defined in a three dimensional world (with 2D objects and worlds being just special cases of the 3D situation).
- All objects to be drawn, and the cameras used to draw them, have shape, position, and orientation.
- We must write computer programs that somehow describe these objects, and describe how light bounces around illuminating them, so that the final pixel values on the display can be computed.

Introduction

- The two fundamental tool sets that come to our aid are *vector geometry* and *transformations*. These are supported by corresponding basic facts from *linear algebra*.
- We will develop methods to describe various geometric objects, and we will learn how to convert geometric ideas to numbers.
- This provides a collection of crucial algorithms that we can use in graphics programs.

Easy Problems for Vectors

a) Where is the center of the circle through the 3 points? b) What image shape appears on the viewplane, and where? c) Where does the reflection of the cube appear on the shiny cone, and what is the exact shape of the reflection?



Vectors

- Vectors provide easy ways of solving some tough problems.
- A vector has a length, a direction, and an orientation but not a position (relative to a coordinate system). Its foot is by default at the origin and can be moved anywhere.
- A point has position but not length and direction (relative to a coordinate system).
- A scalar has only size and sign (i.e it is a number).

Vector Magnitude or Norm

The magnitude (length, size or norm) of a *n*-vector w is denoted as $|w|$ or

$$\|w\| = \sqrt{w_1^2 + w_2^2 + \ldots + w_n^2}$$

Example: the magnitude of w = (4, -2) is $\sqrt{20}$
and that of w = (1, -3, 2) is $\sqrt{14}$

Vector Magnitude and Unit Vectors

A unit vector has magnitude $|\mathrm{v}| = 1$.

The unit vector pointing in the same direction as

vector a is $\hat{\mathrm{a}} = \dfrac{\mathrm{a}}{|\mathrm{a}|}$ (if $|\mathrm{a}| \neq 0$).

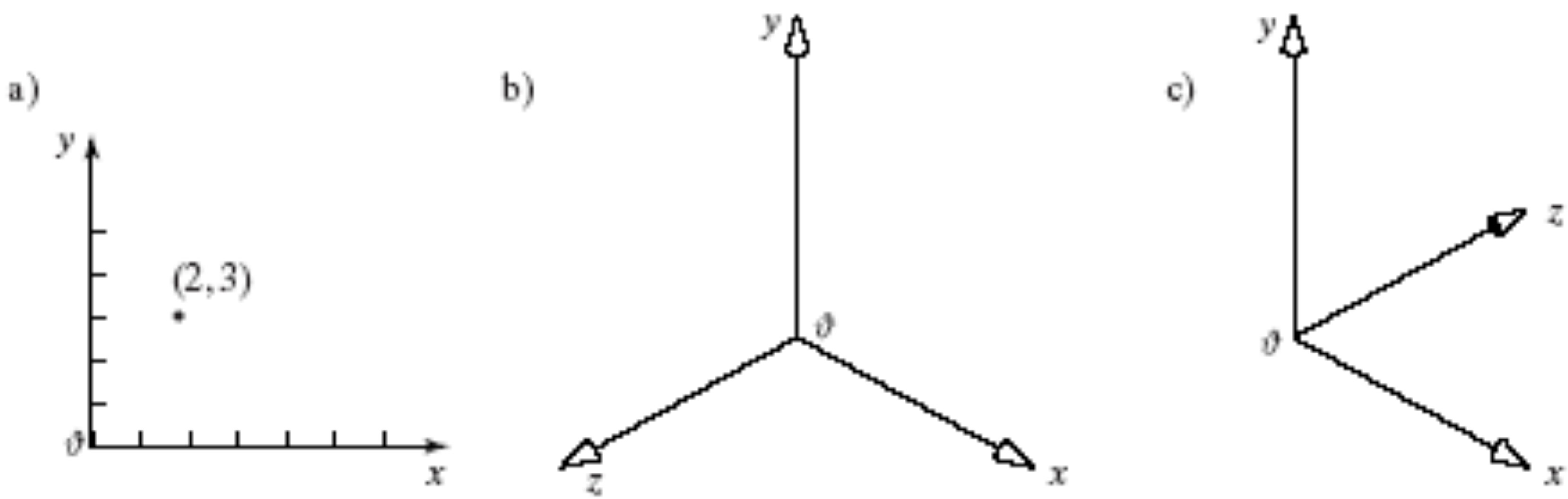
Converting a to $\hat{\mathrm{a}}$ is called *normalizing* vector a.

Vector Magnitude and Unit Vectors (2)

- At times we refer to a unit vector as a **direction**.
- Any vector can be written as its magnitude times its direction: **a** = |a| $\hat{a}$

Basics of Points and Vectors

- All points and vectors are defined relative to some coordinate system. Shown below are a 2D (*x-y)*-coordinate system and a right- and a left-handed 3-D coordinate system.

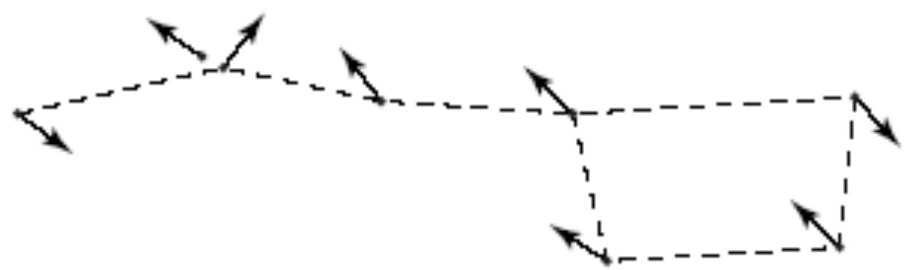


Left and Right Handedness

- In a 3D system, using your right hand, curl your fingers around going from the x-axis to the y-axis. Your thumb is at right angles to your fingers.
  - If your thumb points along the direction of the z-axis, the system is right handed.
  - If your thumb points opposite to the direction of the z-axis, the system is left handed.

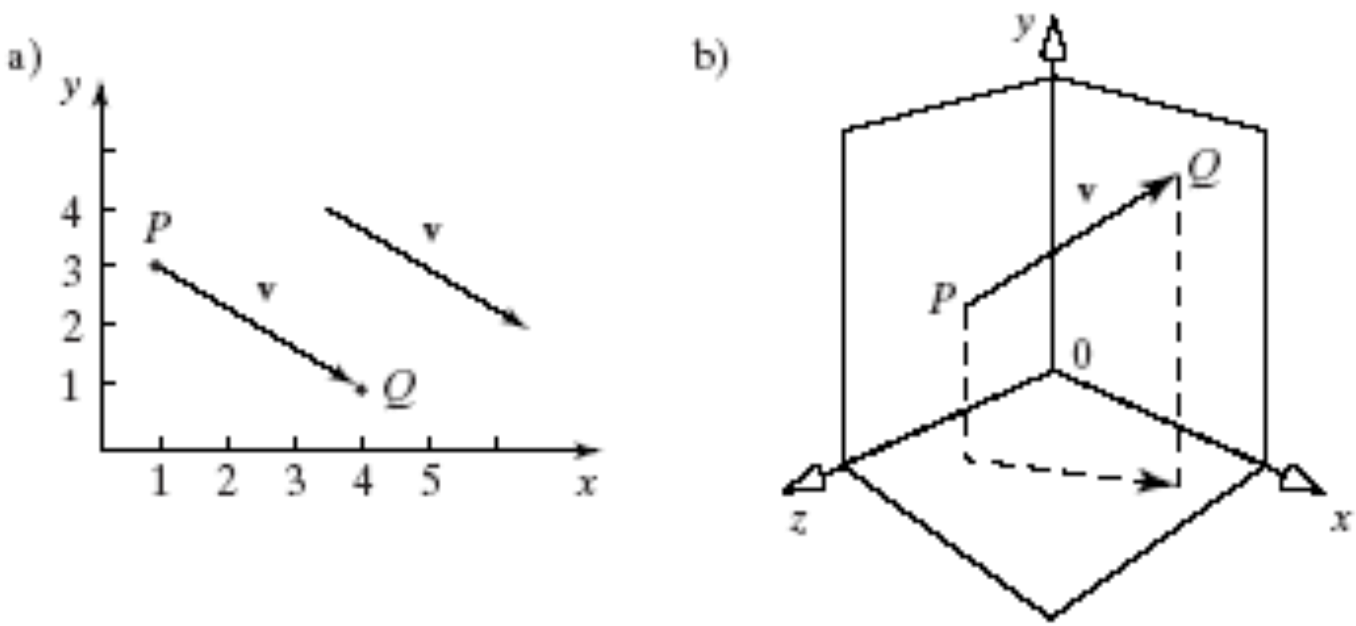
Review of Vectors

- Vectors are drawn as arrows of a certain length pointing in a certain direction.
- A vector can be interpreted as a *displacement* from one point to another. Shown below are displacements of the stars in the Big Dipper over the next 50,000 years.



Vectors and Coordinate Systems

- A vector v pointing from point P = (1, 3) to Q = (4, 1), has the components (3, -2). It is calculated by subtracting the coordinates of the two points component wise (Q – P).
- To "go" from P to Q, we move down by 2 and right by 3. Since v has no position, the two arrows labeled v are the same vector. The 3D case is also shown.



Vector Operations

- The difference between 2 points is a vector: $\mathbf{v} = Q - P$.

- The sum of a point and a vector is a point: $P + \mathbf{v} = Q$.

- We represent a 3-dimensional vector by a 3-tuple of its components, e.g. $\mathbf{v} = (v_x, v_y, v_z) = (v_1, v_2, v_3)$. (In graphics we will usually use 2-, 3- or 4-dimensional vectors but most of the concepts can be generalised to n dimensions).

Vector Representations

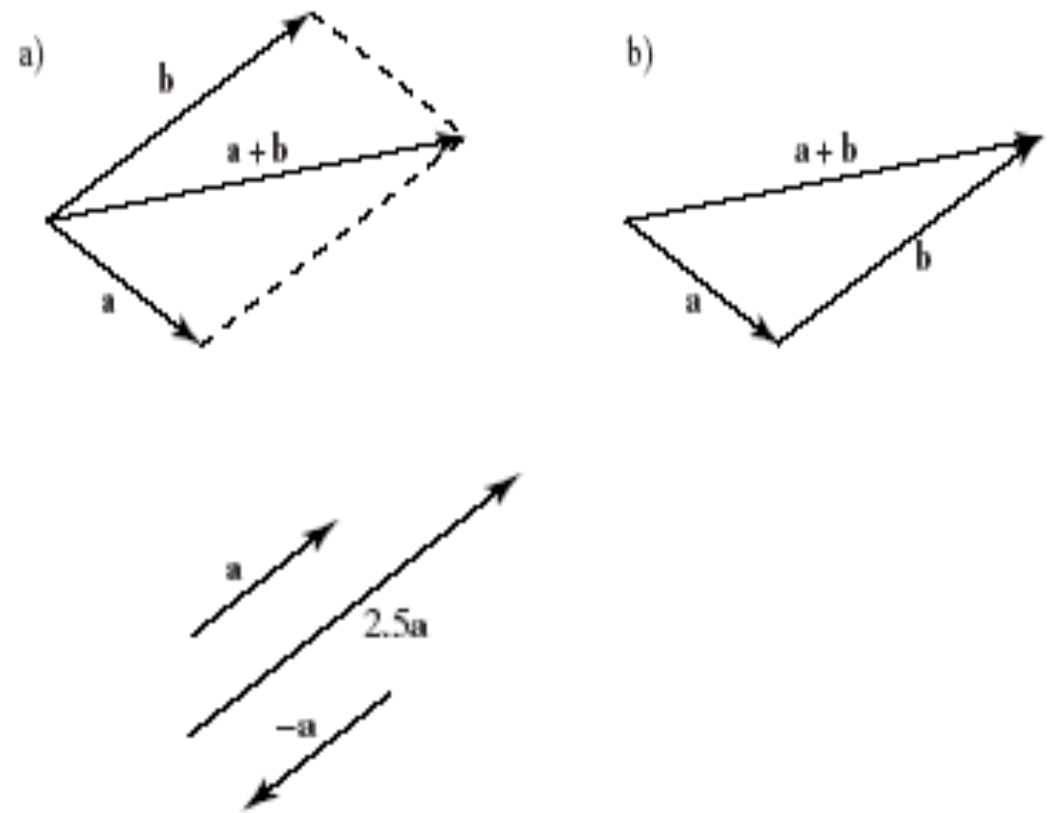
- A vector **v** = $(33, 142.7, 89.1)$ is a row vector.
- A vector **v** = $(33, 142.7, 89.1)^T$ is a column vector (The “T” superscript stands for “Transposed”). It is the same as

$$v = \begin{pmatrix} 33 \\ 142.7 \\ 89.1 \end{pmatrix}$$

- If we talk about a vector we assume by default that it is a column vector. If the context is clear the “T” is often left out.

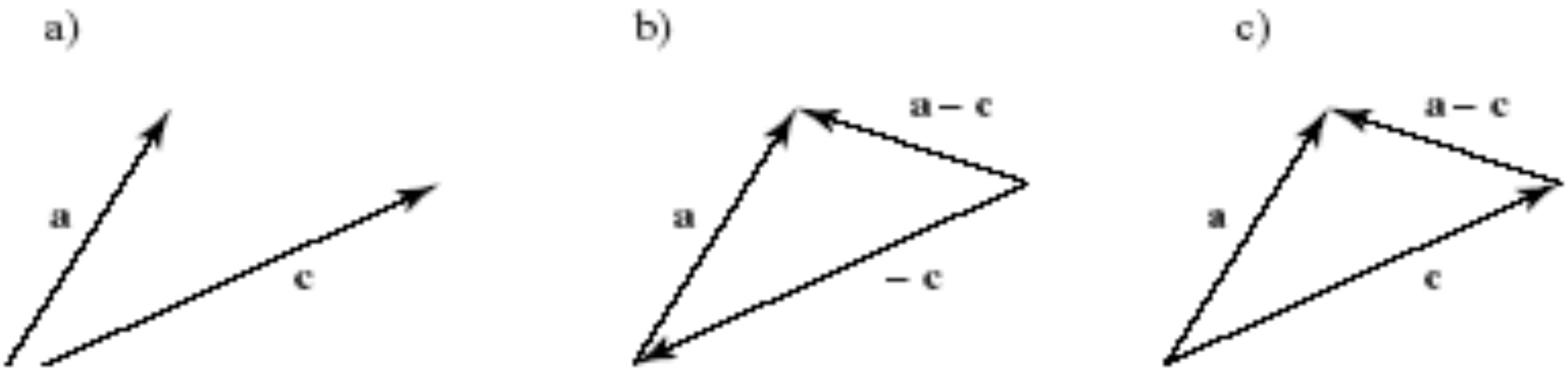
Vector Operations (2)

- Vectors have 2 fundamental operations: addition
  of 2 vectors and multiplication by a scalar.
- If **a** and **b** are vectors, so is **a** + **b**, and so is s**a**, where s is a scalar.



Vector Operations (3)

Subtracting **c** from **a** is equivalent to adding **a** and (**-c**), where **–c** = (-1)**c**.



Vector Space Operations

- $\mathbf{v}_1 \pm \mathbf{v}_2 = (v_{1x} \pm v_{2x}, v_{1y} \pm v_{2y}, v_{1z} \pm v_{2z})$
- $s\mathbf{v} = (sv_x, sv_y, sv_z)$
- Both operations are required to form “linear combinations of vectors”

Linear Combinations of Vectors

A linear combination of the m vectors

$\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$ is $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_m\mathbf{v}_m$.

Example:

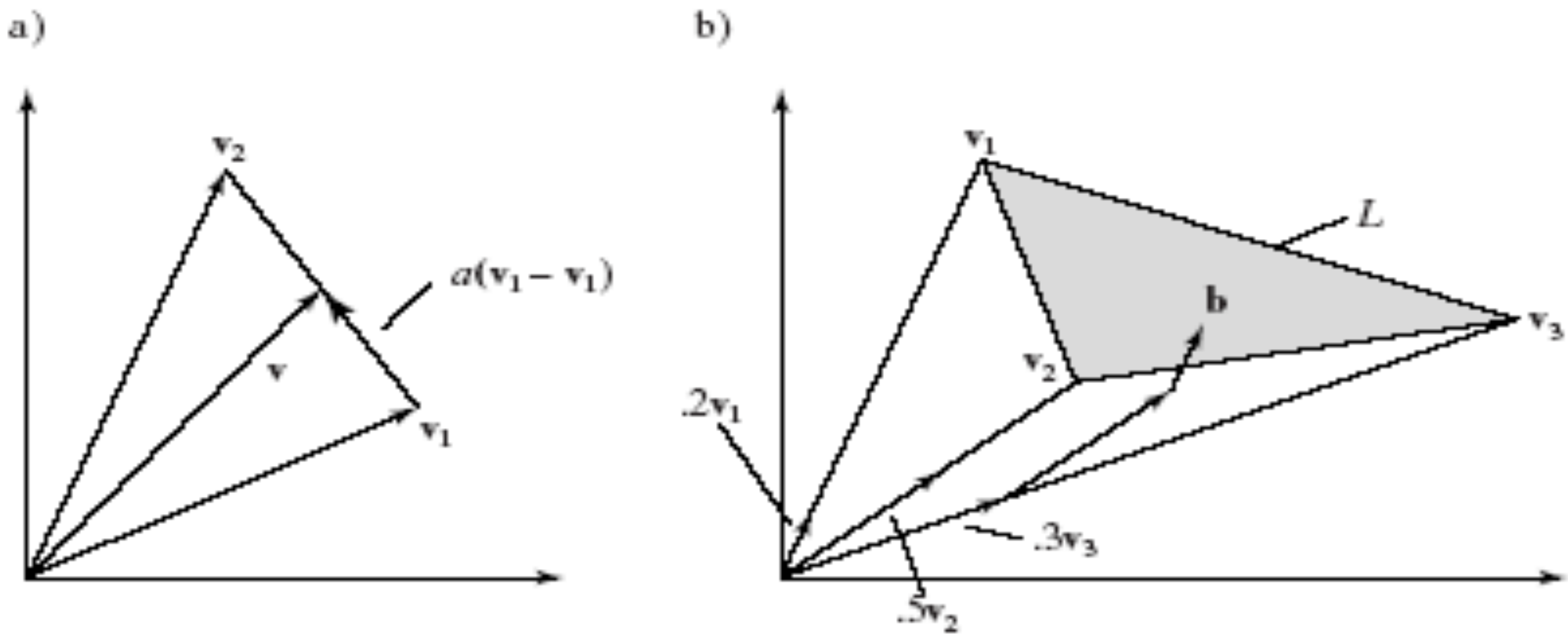
2(3, 4,-1) + 6(-1, 0, 2) forms the vector (0, 8, 10).

Linear Combinations of Vectors

- The linear combination becomes an affine combination if $a_1 + a_2 + \ldots + a_m = 1$.
  - Example: 3**a** + 2**b** - 4**c** is an affine combination of **a**, **b**, and **c**, but 3**a** + **b** - 4**c** is not.
  - (1-*t*) **a** + (*t*) **b** is an affine combination of **a** and **b**.
- The affine combination becomes a convex combination if $a_i \geq 0$ for $1 \leq i \leq m$.
  - Example: 0.3**a**+0.7**b** is a convex combination of **a** and **b**, but 1.8**a** -0.8**b** is not.

The Set of All Convex Combinations of 2 or 3 Vectors

$\mathbf{v} = (1 - a)\mathbf{v}_1 + a\mathbf{v}_2$, as a varies from 0 to 1, gives the set of all convex combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. An example is shown below.



Dot Product (or Scalar Product)

The dot product of n-vectors **v** and **w** is

$$\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + \ldots + v_nw_n$$

- The dot product is commutative: $\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}$
- The dot product is distributive: $(\mathbf{a} \pm \mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} \pm \mathbf{b}\cdot\mathbf{c}$
- The dot product is associative over multiplication by a scalar: $(s\mathbf{a})\cdot\mathbf{b} = s(\mathbf{a}\cdot\mathbf{b})$
- The dot product of a vector with itself is its magnitude squared: $\mathbf{b}\cdot\mathbf{b} = |\mathbf{b}|^2$

Applications: Angle Between 2 Vectors

Let **b** = (|**b**| cos ϕ_b, |**b**| sin ϕ_b),

and **c** = (|**c**| cos ϕ_c, |**c**| sin ϕ_c)

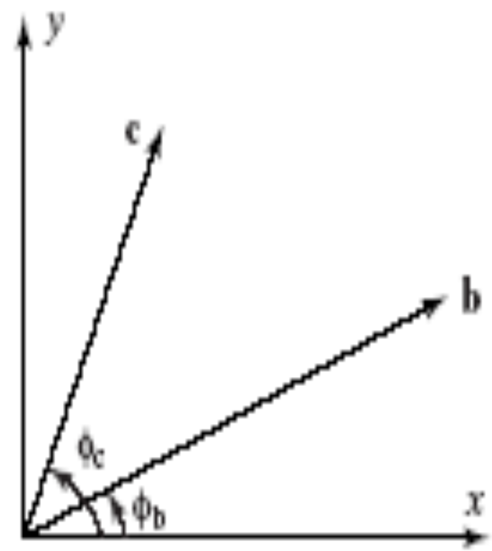
Then

b·**c** = |**b**||**c**| cos ϕ_c cos ϕ_b +

|**b**||**c**| sin ϕ_b sin ϕ_c

= |**b**||**c**| cos (ϕ_c- ϕ_b)

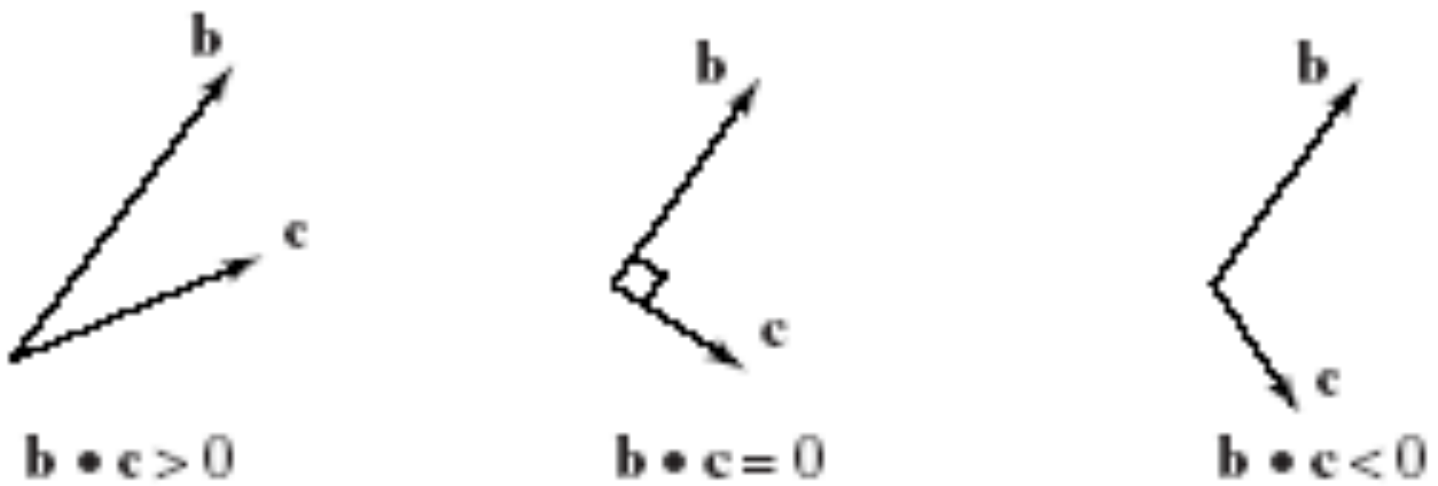
= |**b**||**c**| cos θ, where θ = ϕ_c- ϕ_b is the smaller angle between **b** and **c**.



$$\cos(\theta) = \hat{\mathbf{b}} \cdot \hat{\mathbf{c}}$$

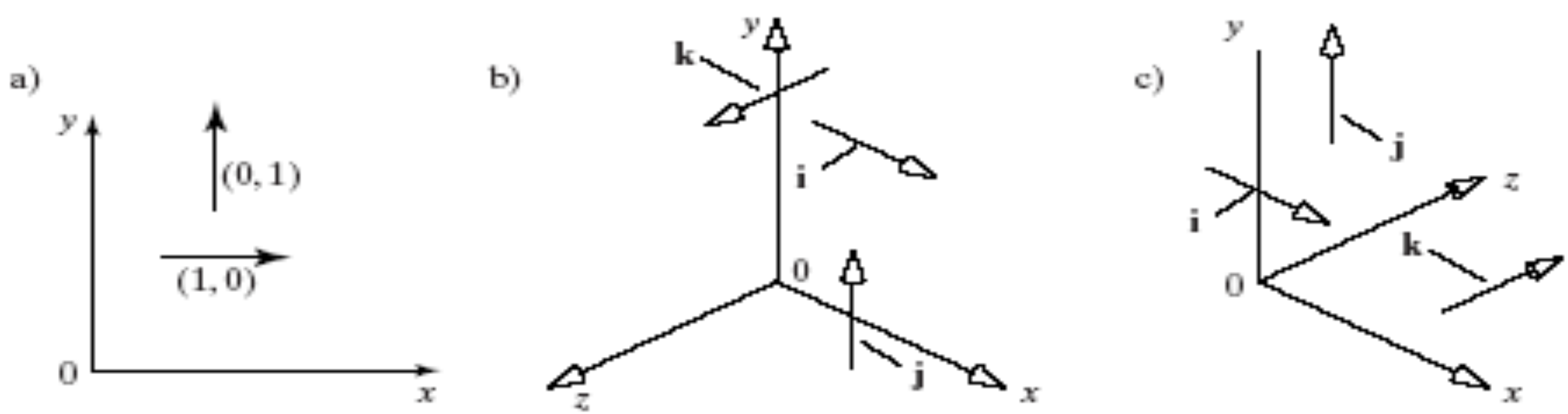
Angle Between 2 Vectors (2)

- The cosine is positive if $|\theta| < 90^o$, zero if $|\theta| = 90^o$, and negative if $\theta > 90^o$.
- Vectors **b** and **c** are perpendicular (orthogonal, normal) if $\mathbf{b} \cdot \mathbf{c} = 0$.



Standard Unit Vectors

- The standard unit vectors in 3D are
  $\mathbf{e_1} = \mathbf{i} = (1,0,0)$,
  $\mathbf{e_2} = \mathbf{j} = (0, 1, 0)$, and
  $\mathbf{e_3} = \mathbf{k} = (0, 0, 1)$.
  k always points in the positive z direction
- In 2D, **i** = (1,0) and **j** = (0, 1).
- The standard unit vectors are orthogonal (check!).



Standard Unit Vectors

The standard unit vectors in n dimensions are

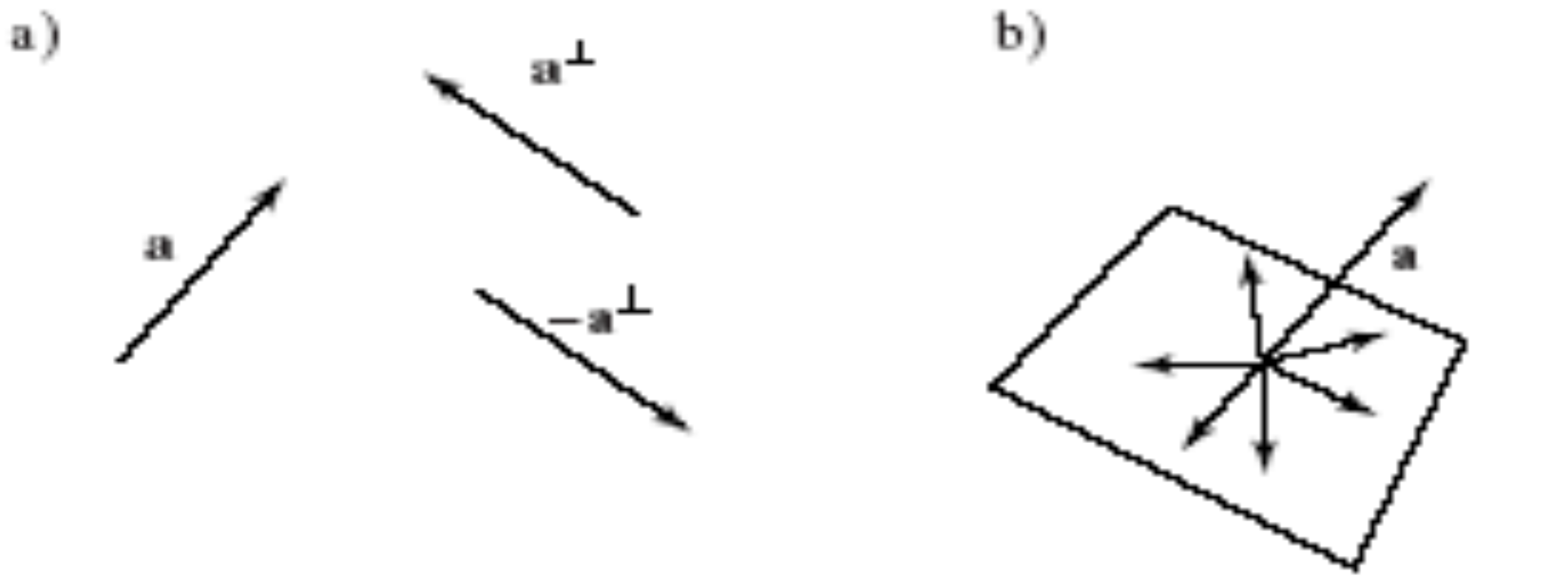
$$\mathbf{e_1} = (1,0, 0, \ldots, 0),$$
$$\mathbf{e_2} = (0, 1, 0, \ldots,0)$$
$$\vdots$$
$$\mathbf{e_n} = (0, 0, \ldots , 0, 1).$$

The standard unit vectors are orthogonal (check!).

DEF. We say they from an **orthonormal basis (ONB)** of the R^n.

Finding a 2D Perpendicular Vector

- If vector **a** = (a_x, a_y), then the vector perpendicular to **a** in the *counterclockwise* sense is $\mathbf{a}^{\perp} = (-a_y, a_x)$, and in the *clockwise* sense it is $-\mathbf{a}^{\perp}$.
- In 3D, any vector in the plane perpendicular to **a** is a perpendicular vector.



Properties of $\perp$ in 2D

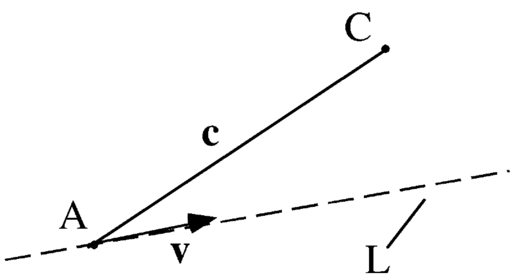
- $(\mathbf{a} \pm \mathbf{b})^{\perp} = \mathbf{a}^{\perp} \pm \mathbf{b}^{\perp}$;
- $(s\mathbf{a})^{\perp} = s(\mathbf{a}^{\perp})$;
- $(\mathbf{a}^{\perp})^{\perp} = -\mathbf{a}$
- $\mathbf{a}^{\perp} \cdot \mathbf{b} = -\mathbf{b}^{\perp} \cdot \mathbf{a} = -a_y b_x + a_x b_y$;
- $\mathbf{a}^{\perp} \cdot \mathbf{a} = a \cdot \mathbf{a}^{\perp} = 0$;
- $|\mathbf{a}^{\perp}| = |\mathbf{a}|$;

Orthogonal Projections and Distance from a Line

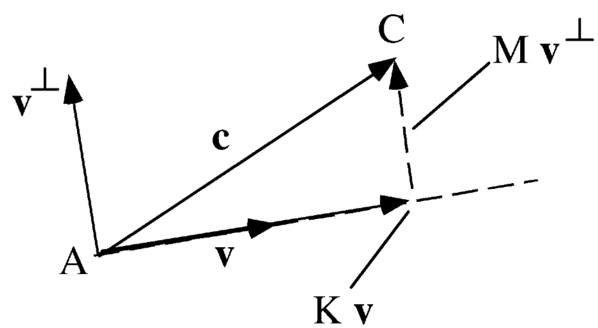
- We are given 2 points A and C and a vector **v**. The following questions arise:
  - How far is C from the line L that passes through A in direction **v**?
  - If we drop a perpendicular line onto L from C, where does it hit L?
  - How do we decompose a vector **c** = C – A into a part along L and a part perpendicular to L?

Illustration of Questions

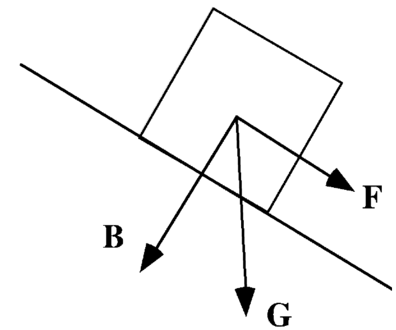
a).



b).



c).



Answering the Questions

We may write $\mathbf{c} = K\mathbf{v} + M\mathbf{v}^{\perp}$. If we take the dot product of each side with $\mathbf{v}$, we obtain $\mathbf{c}\cdot\mathbf{v} = K\mathbf{v}\cdot\mathbf{v} + M\mathbf{v}^{\perp}\cdot v = K|\mathbf{v}|^2$ (why?), or $K = \mathbf{c}\cdot\mathbf{v}/|\mathbf{v}|^2$.

Likewise, taking the dot product with $\mathbf{v}^{\perp}$ gives $M = \mathbf{c}\cdot\mathbf{v}^{\perp}/|\mathbf{v}|^2$. (Why not $|\mathbf{v}^{\perp}|^2$?)

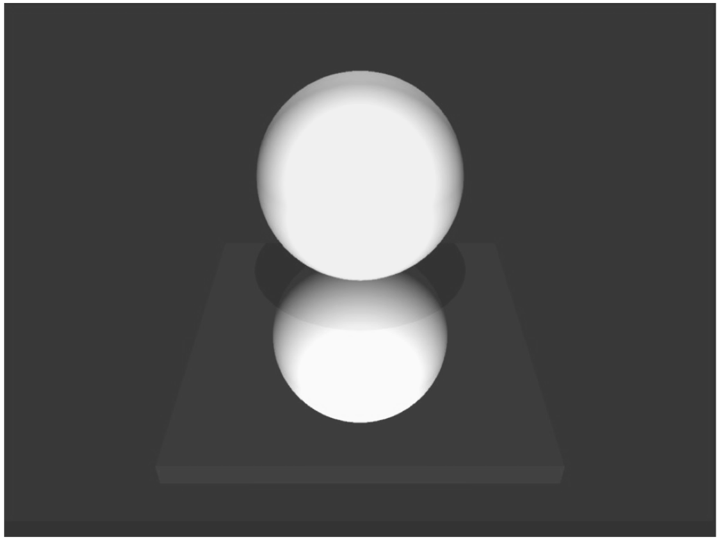
This answers the original questions by using:

$\mathbf{c}=M\mathbf{v}^{\perp}+ K\mathbf{v}$

Distance = $|M\mathbf{v}^{\perp}|$

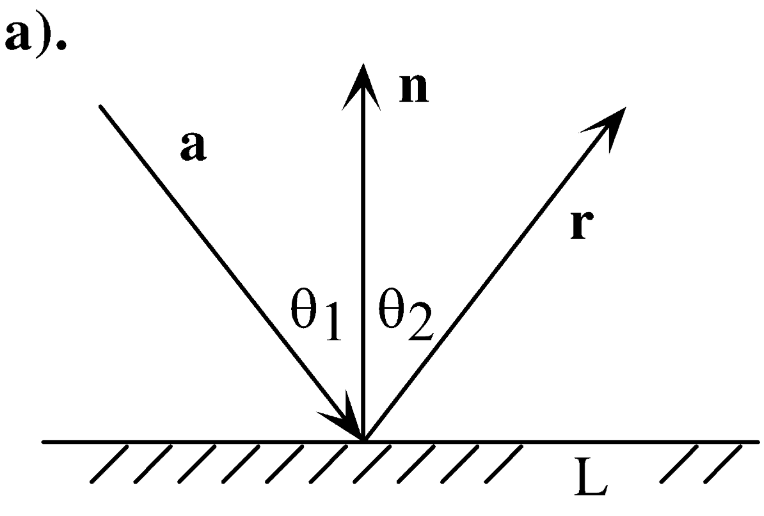
Application of Projection: Reflections

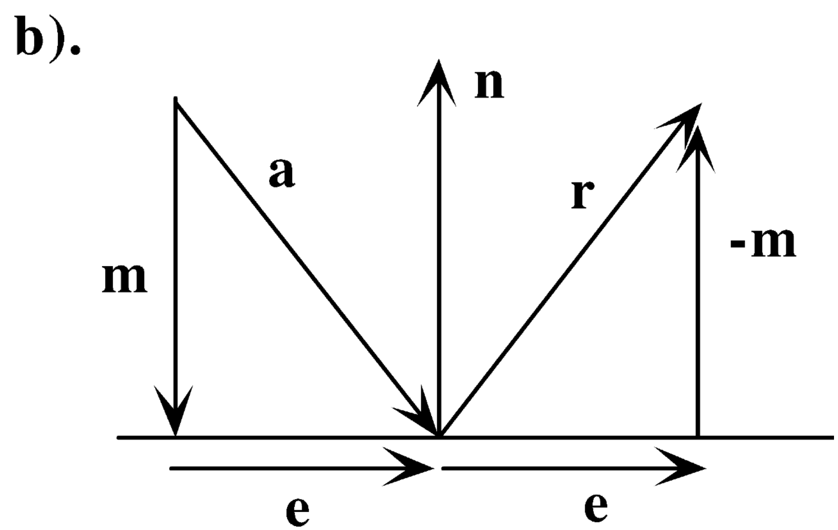
A reflection occurs when light hits a shiny surface (below) or when a billiard ball hits the wall edge of a table.



Reflections (2)

- When light reflects from a mirror, the angle of reflection must equal the angle of incidence: $\theta_1 = \theta_2$.
- Vectors and projections allow us to compute the new direction **r**, in either two-dimensions or three dimensions.





Vector Cross Product (3D Vectors Only)

- **a** x **b** = $(a_y b_z - a_z b_y)$**i +**
  $(a_z b_x - a_x b_z)$**j +**
  $(a_x b_y - a_y b_x)$**k.**
- The determinant below also gives the result:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

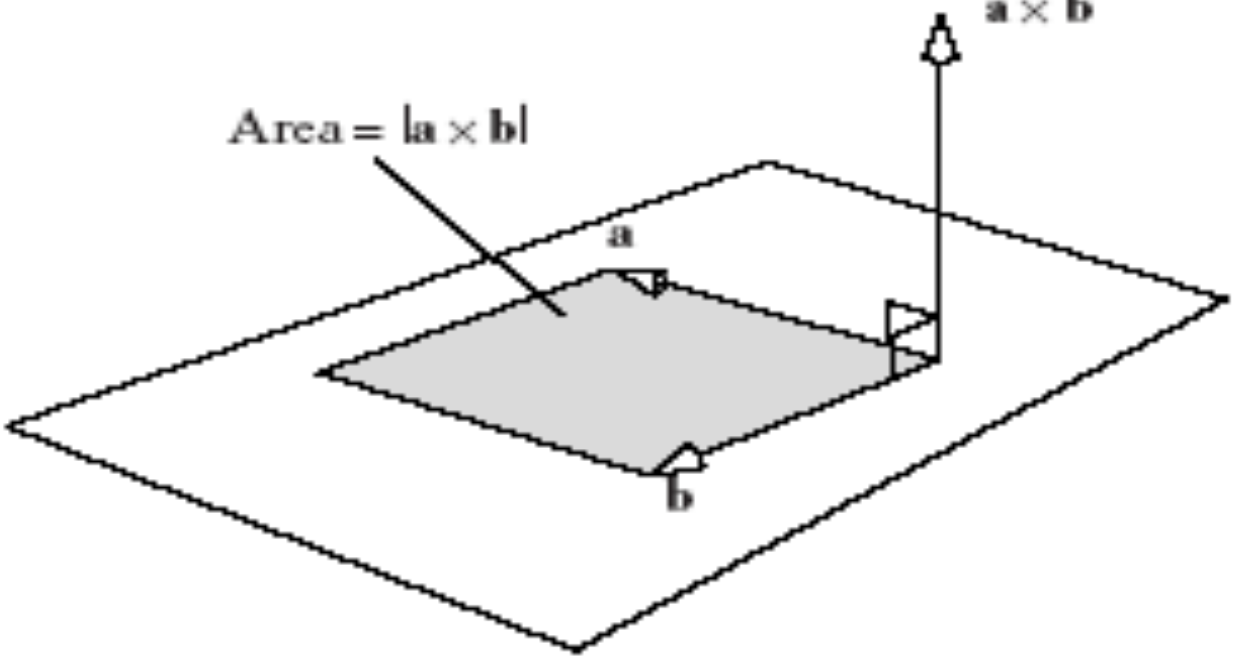
Properties of the Cross-Product

- **i** x **j** = **k**; **j** x **k** = **i**; **k** x **i** = **j**
- **a** x **b** = - **b** x **a**; **a** x (**b** ± **c**) = **a** x **b** ± **a** x **c**; (s**a**) x **b** = s(**a** x **b**)
- **a** x (**b** x **c**) ≠ (**a** x **b**) x **c** – for example, **a** = (a_x, a_y, 0), **b** = (b_x, b_y, 0), **c** = (0, 0, c_z)
- **c** = **a** x **b** is perpendicular to **a** and to **b**. The direction of **c** is given by a right/left hand rule in a right/left-handed coordinate system.

Properties (2)

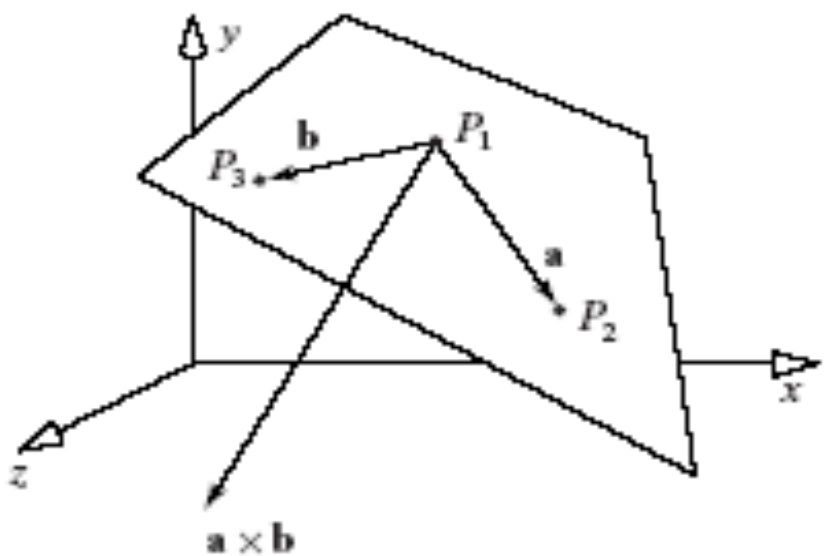
- $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$
- $||\mathbf{a} \times \mathbf{b}|| = |\mathbf{a}||\mathbf{b}| \sin\theta$, where θ is the smaller angle between **a** and **b**.
- $||\mathbf{a} \times \mathbf{b}||$ is also the area of the parallelogram formed by **a** and **b**.
- $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if **a** and **b** point in the same or opposite directions, or if one or both has length 0.

Geometric Interpretation of the Cross Product



Application: Finding the Normal to a Plane

- Given any 3 non-collinear points A, B, and C in a plane, we can find a normal to the plane:
  - **a** = B – A, **b** = C – A, **n** = **a** x **b**. The normal on the other side of the plane is –**n**.



Representations of Key Geometric Objects

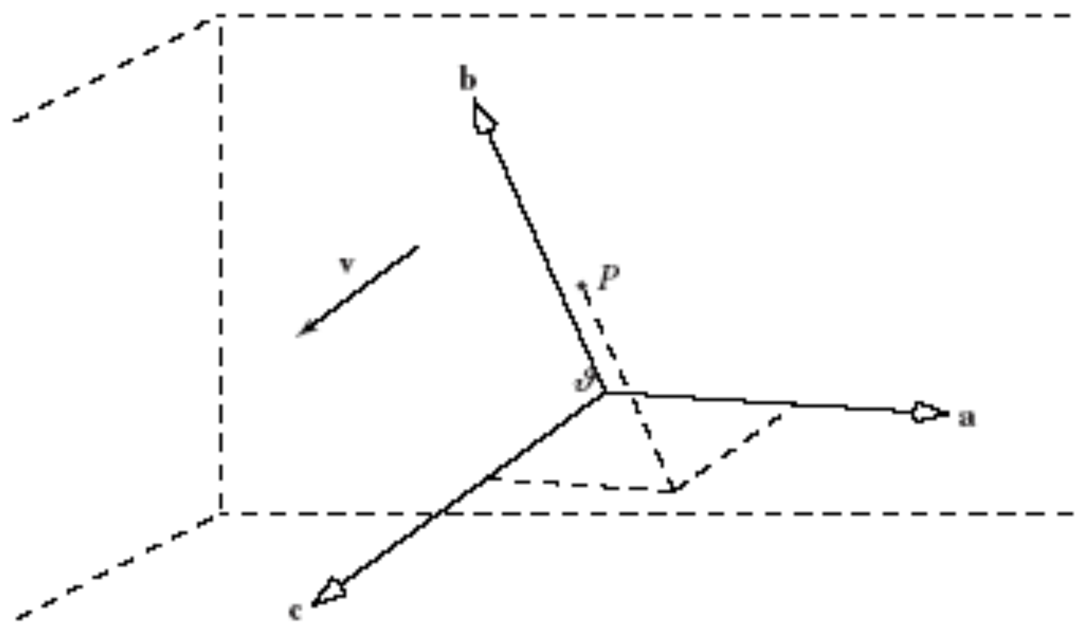
- Lines and planes are essential to graphics, and we must learn how to represent them – i.e., how to find an equation or function that distinguishes points on the line or plane from points off the line or plane.
- It turns out that this representation is easiest if we represent vectors and points using 4 coordinates rather than 3.

Coordinate Systems and Frames

- A vector or point has coordinates in an underlying coordinate system.
- In graphics, we may have multiple coordinate systems, with origins located anywhere in space.
- We define a coordinate frame as a single point (the origin, *O*) with 3 mutually perpendicular <u>unit</u> vectors: **a**, **b**, and **c**.

Coordinate Frames (2)

- A vector $\mathbf{v}$ is represented by (v_1, v_2, v_3) such that $\mathbf{v} = v_1\mathbf{a} + v_2\mathbf{b} + v_3\mathbf{c}$.
- A point $P - O = p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c}$.



Homogeneous Coordinates

- It is useful to represent both points and vectors by the same set of underlying objects, (**a**, **b**, **c**, *O*).
- A vector has no position, so we represent it as (**a**, **b**, **c**, *O*)$(v_1, v_2, v_3,0)^T$.
- A point has an origin (*O*), so we represent it by (**a**, **b**, **c**, *O*)$(v_1, v_2, v_3,1)^T$.

Changing to and from Homogeneous Coordinates

To: if the object is a vector, add a 0 as the 4^{th} coordinate; if it is a point, add a 1.

From: simply remove the 4^{th} coordinate.

OpenGL uses 4D homogeneous coordinates for all its vertices.

- If you send it a 3-tuple in the form (x, y, z), it converts it immediately to $(x, y, z, 1)$.
- If you send it a 2D point (x, y), it first appends a 0 for the z-component and then a 1, to form $(x, y, 0, 1)$.

All computations done within OpenGL are in 4D homogeneous coordinates.

Combinations

Linear combinations of vectors and points:

- The difference of 2 points is a vector: the fourth component is $1 - 1 = 0$
- The sum of a point and a vector is a point: the fourth component is $1 + 0 = 1$
- The sum of 2 vectors is a vector: $0 + 0 = 0$
- A vector multiplied by a scalar is still a vector: $a \times 0 = 0$.
- Linear combinations of vectors are vectors.

Point + Vector

- Suppose we add a point A and a vector that has been scaled by a factor t: the result is a point, $P = A + t\mathbf{v}$.
- Now suppose $\mathbf{v} = B - A$, the difference of 2 points: $P = tB + (1-t)A$, i.e. an affine combination of points is a point.

Linear Interpolation of 2 Points

$P = (1-t)A + tB$ is a linear interpolation of 2 points. This is very useful in graphics in many applications,

- $P_x(t)$ provides an x value that is fraction t of the way between A_x and B_x. (Likewise P_y, P_z).

Linear interpolation

- One often wants to compute the point $P(t)$ that is fraction t of the way along the straight line from point A to point B.
- Each component of the resulting point is formed as the linear interpolation of the corresponding components of A and B.

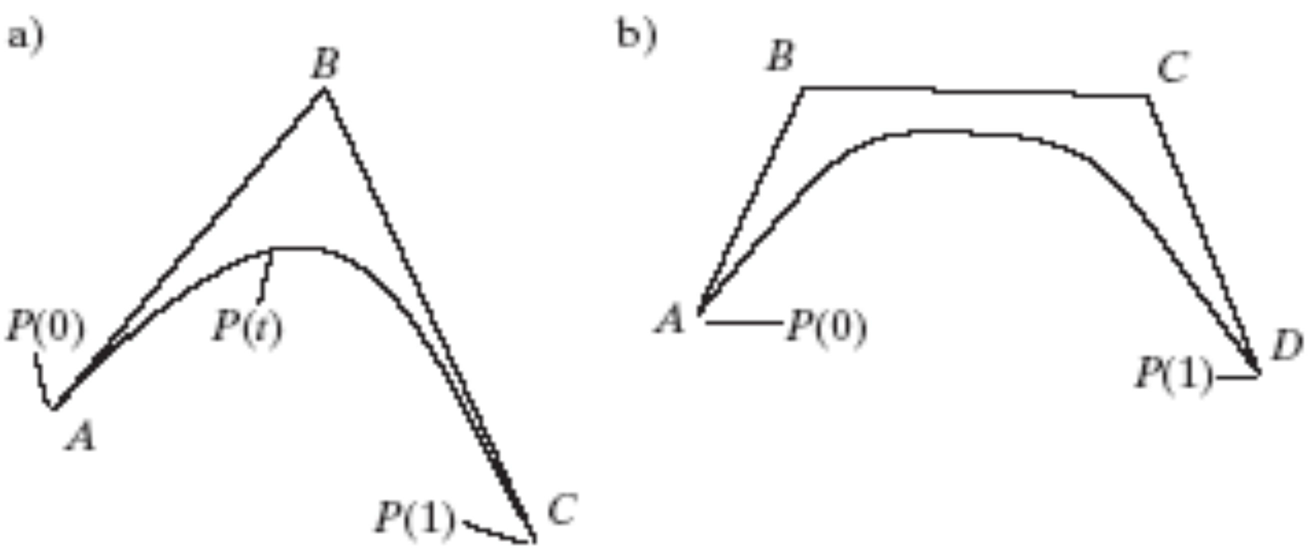
Example

Uses of linear interpolation

- In films, artists draw only the key frames of an animation sequence (usually the first and last).
  - Linear interpolation is used to generate the in-between frames.
- Preview: We want a smooth curve that passes through or near 3 points. We expand $((1-t) + t)^2$ and write $P(t) = (1-t)^2A + 2t(1-t)B + t^2C$

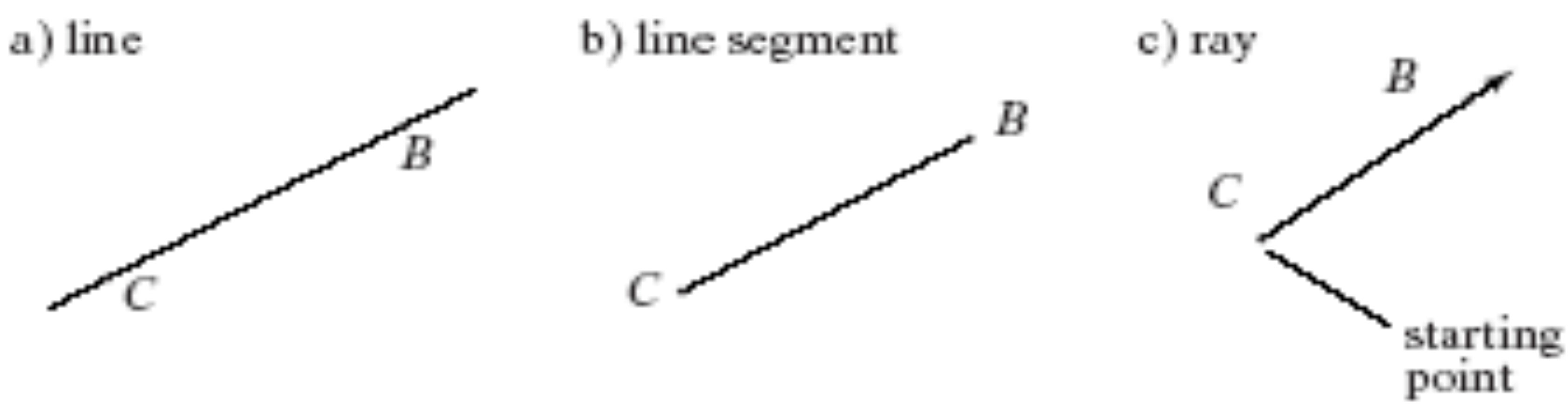
Uses of Tweening (2)

– This is called the Bezier curve for points A, B, and C.
– It can be extended to 4 points by expanding $((1-t) + t)^3$ and using each term as the coefficient of a point.



Representing Lines

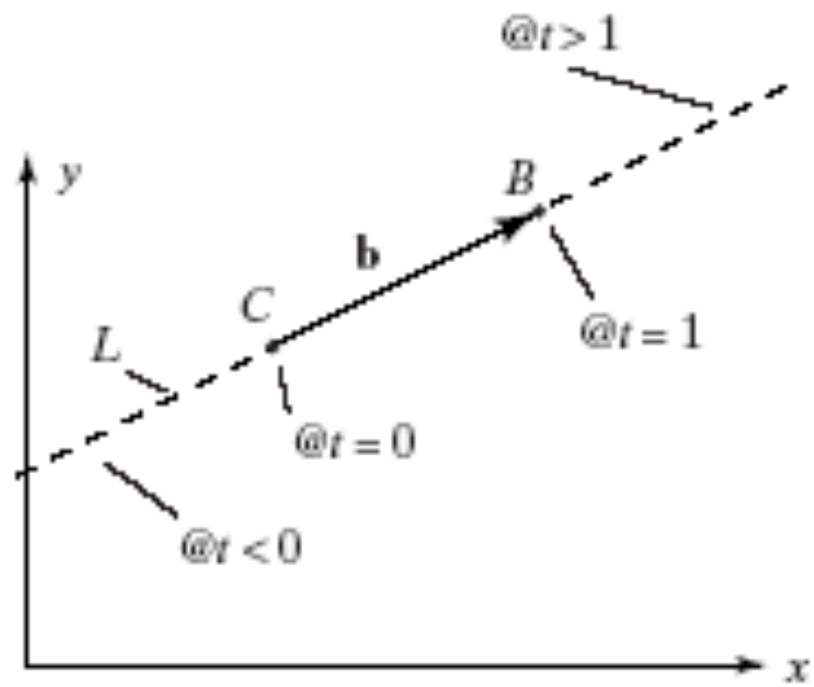
- A line passes through 2 points and is infinitely long.
- A line segment has 2 endpoints.
- A ray has a single endpoint.



Representing Lines (2)

- There are 2 useful line representations:
- Parametric form: we have 2 points, B and C, on the line. P(x, y) is on the line when P = C + **b**t, where **b** = B – C.
  - $0 \le t \le 1$: line segment;

  $-\infty \le t \le \infty$: line; $-\infty \le t \le 0$ or $0 \le t \le \infty$: ray.

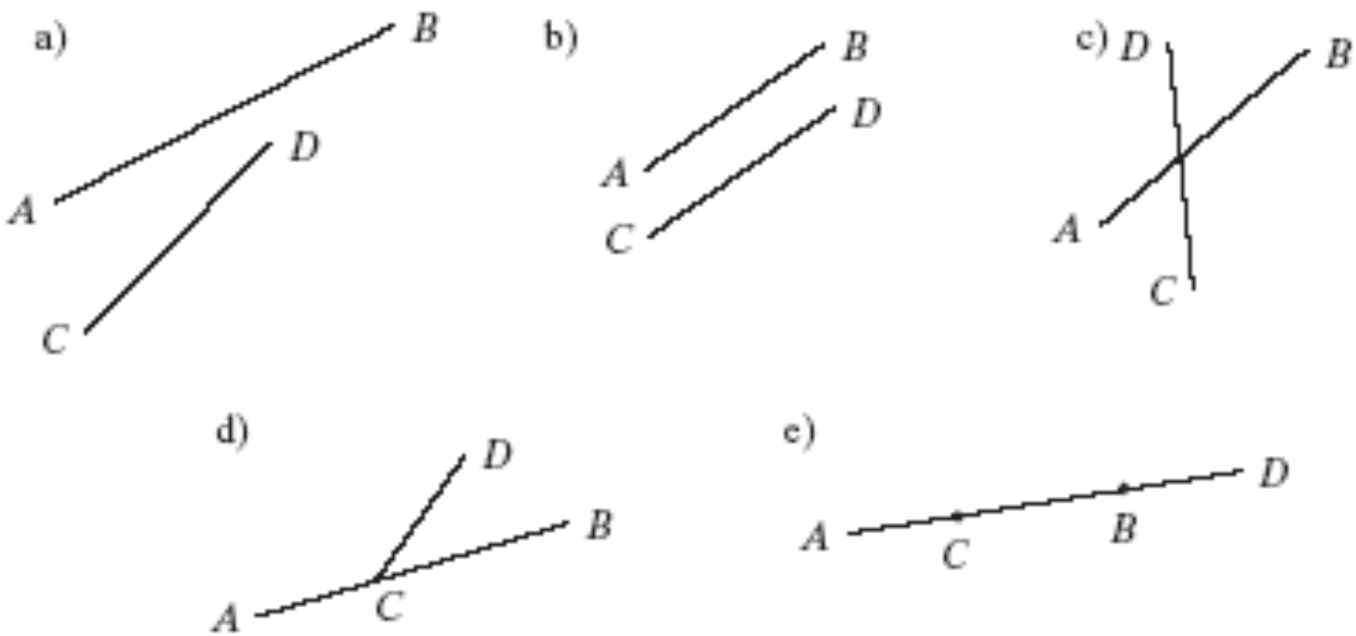


Representing Lines (3)

- $L(t)$ lies fraction t of the way between C and B when t lies between 0 and 1.
- When $t = 1/2$ the point $L(0.5)$ is the **midpoint** between C and B, and when $t = 0.3$ the point $L(0.3)$ is 30% of the way from C to B: $|L(t) – C/ = |\mathbf{b}|\ |t|$ and $|B - C| = |\mathbf{b}|$, so the value of $|t|$ is the ratio of the distances $|L(t) - C|$ to $|B - C|$.

Finding the Intersection of 2 Line Segments

- They can miss each other (a and b), overlap in one point (c and d), or even overlap over some region (e). They may or may not be parallel.



Intersection of 2 Line Segments (2)

- Every line segment has a **parent line**, the infinite line of which it is part. Unless two parent lines are parallel, they will intersect at some point in 2D. We locate this point.
- Using parametric representations for each of the line segments in question, call AB the segment from A to B. Then $AB(t) = A + \mathbf{b}\, t$, where for convenience we define $\mathbf{b} = B - A$.
- As t varies from 0 to 1 each point on the finite line segment is crossed exactly once.

Intersection of 2 Line Segments (3)

- $AB(t) = A + \mathbf{b}t$, $CD(u) = C + \mathbf{d}u$, where $\mathbf{b} = B - A$ and $\mathbf{d} = C - D$.
- Obtain the intersection formula:

  $A + \mathbf{b}t = C + \mathbf{d}u$
- With $\mathbf{c} = C\text{-}A$ we obtain: $\mathbf{b}t = \mathbf{c} + \mathbf{d}u$

  In 2D this gives two linear equations in two unknowns that can be solved in the usual manner.

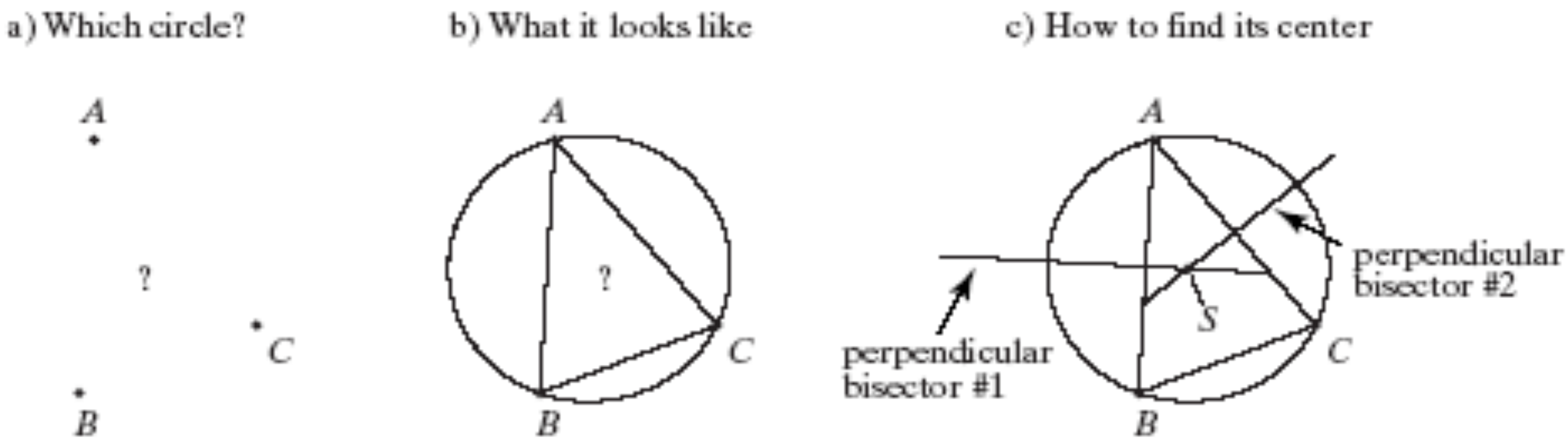
Intersection of 2 Line Segments (4)

- Case1: $\mathbf{b}\cdot\mathbf{d}^{\perp} = 0$ means $\mathbf{d}\cdot\mathbf{b}^{\perp} = 0$ and the lines are either the same line or parallel lines. There is no intersection.
- Case 2: $\mathbf{b}\cdot\mathbf{d}^{\perp} \neq 0$ gives $t = \mathbf{c}\cdot\mathbf{d}^{\perp} / \mathbf{b}\cdot\mathbf{d}^{\perp}$ and $u = -\mathbf{c}\cdot\mathbf{b}^{\perp} / \mathbf{d}\cdot\mathbf{b}^{\perp}$, where $\mathbf{c} = C - A$.
- In this case, the line segments intersect if and only if $0 \leq t \leq 1$ and $0 \leq u \leq 1$, at $P = A + \mathbf{b}(\mathbf{c}\cdot\mathbf{d}^{\perp} / \mathbf{b}\cdot\mathbf{d}^{\perp})$.

Finding A Circle through 3 Points

We want to find the center and the radius of a circle that is defined by three points.

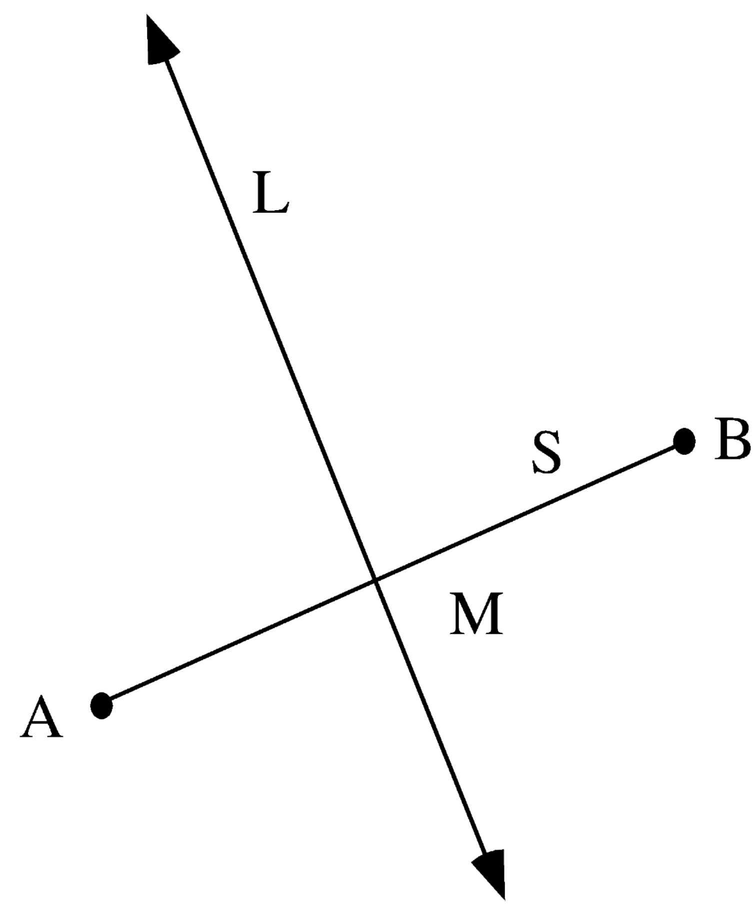
- The 3 points make a triangle, and the center S is where the perpendicular bisectors of two of the sides of the triangle meet.
- The radius is r = |A – S|.



Circle through 3 Points (2)

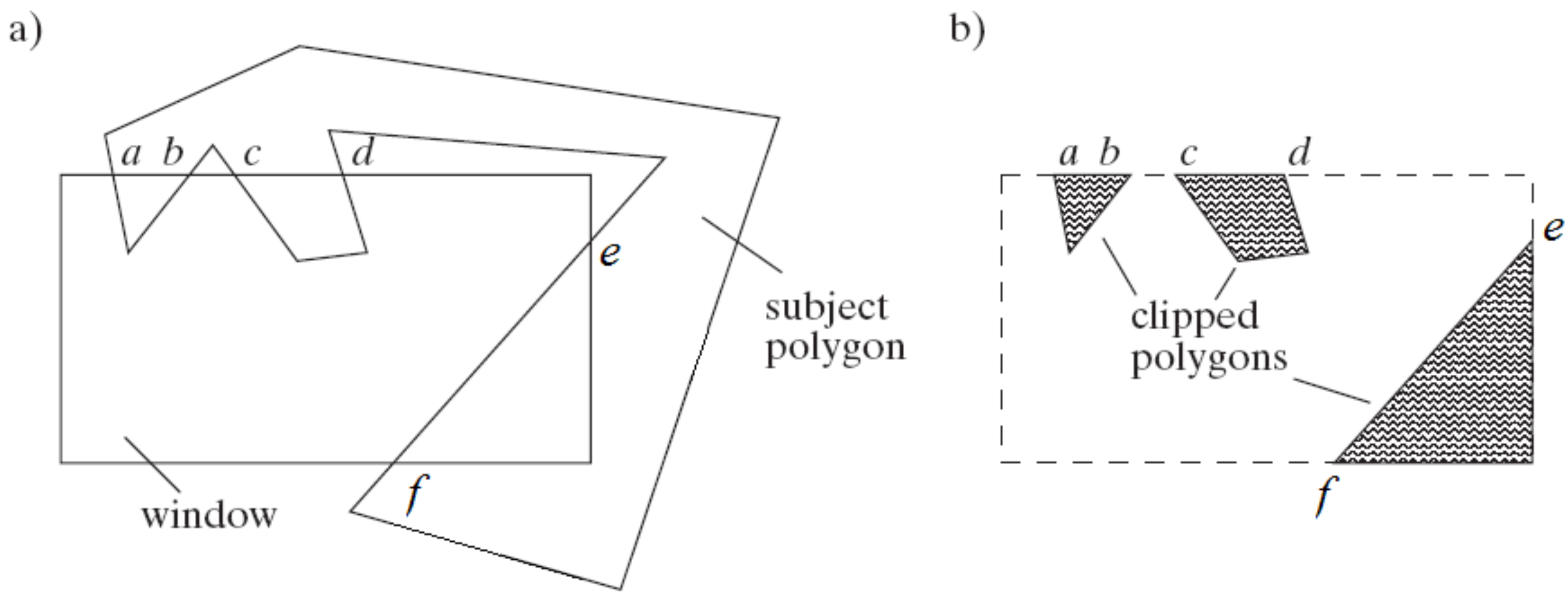
- The perpendicular bisector passes through the midpoint M = ½ (A + B) of the line AB, in the direction $(B - A)^{\perp}$.
- Let **a** = B – A, **b** = C – B, and **c** = A - C.
- Exercise: A = (1,1), B= (3,0), C = (5,1)

  [Solution?: Center = (3, 2.5)]



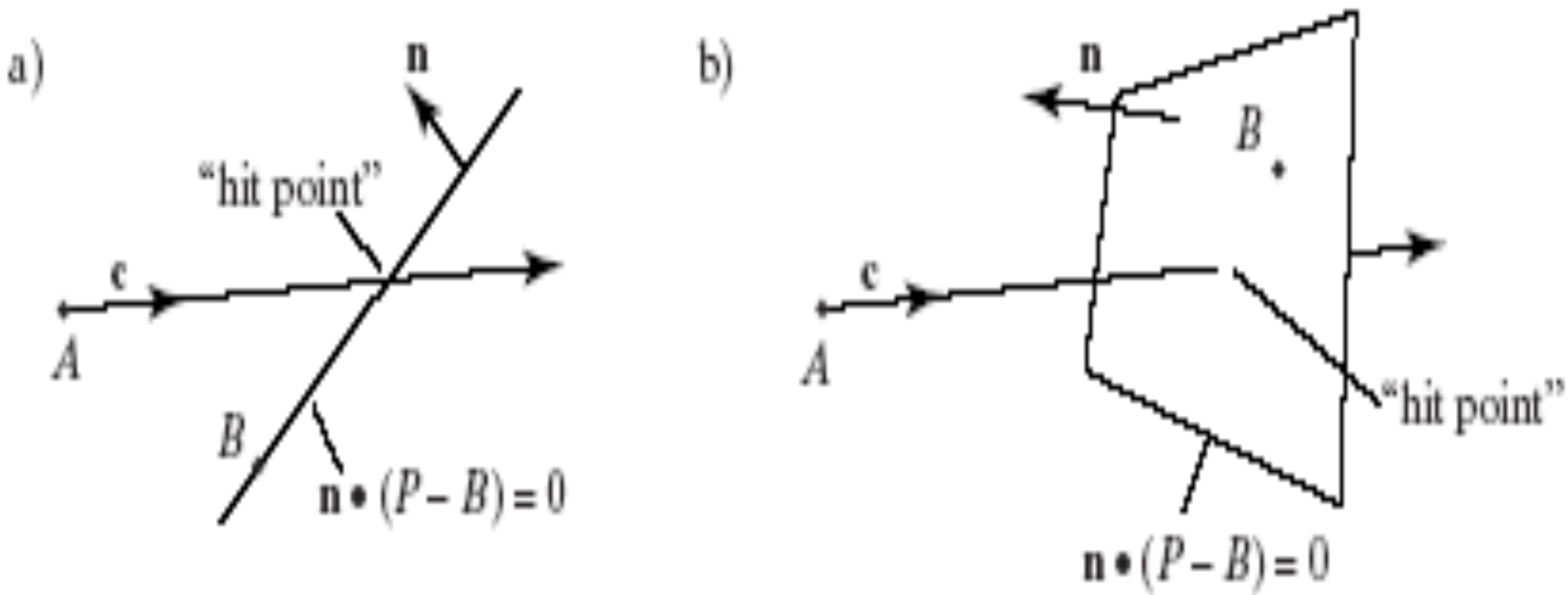
Clipping a Polygon

"To clip a polygon against a window" means to find which part of the polygon is inside the window and thus will be drawn.



Intersections of Lines and Planes

Intersections of a line and a line or plane are used in ray-tracing and 3D clipping: we want to find the “hit point”.



Intersections of Lines and Planes (2)

- Suppose the ray hits at $t = t_{hit}$, the **hit time**.
- At this value of t the ray and line or plane must have the same coordinates, so $A + \mathbf{c}\, t_{hit}$ must satisfy the equation of the point normal form for the line or plane, $\mathbf{n} \cdot (P - B) = 0$.
- When the ray intersects (hits) the line or plane, $A + \mathbf{c}t_{hit} = P$, giving $\mathbf{n} \cdot (A + \mathbf{c}t_{hit} - B) = 0$.

Intersections of Lines and Planes (3)

- Expanding and solving for t_{hit} gives

$$t_{hit} = \mathbf{n} \cdot (B - A) / \mathbf{n} \cdot \mathbf{c}, \text{ if } \mathbf{n} \cdot \mathbf{c} \neq 0.$$

  - If $\mathbf{n} \cdot \mathbf{c} = 0$, the line is parallel to the plane and there is no intersection.

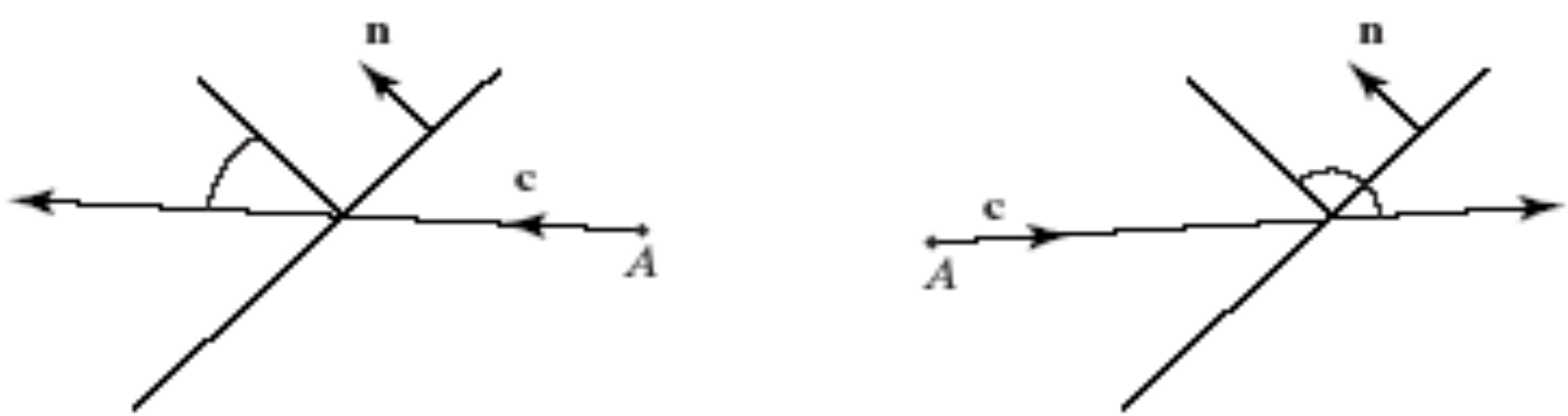
- To find the hit point P_{hit}, substitute t_{hit} into the representation of the ray:

$$P_{hit} = A + \mathbf{c}t_{hit} = A + \mathbf{c}(\mathbf{n} \cdot (B - A) / \mathbf{n} \cdot \mathbf{c}).$$

Direction of Ray

- If **n·c** = 0, the ray is parallel to the line.
- If **n·c** > 0, **c** and **n** make an angle of less than 90º with each other.
- If **n·c** < 0, **c** and **n** make an angle of more than 90º with each other.

a) ray is aimed "along with" **n** b) ray is aimed "counter to" **n**



Exercise 1

Calculate the midpoint of the circle in 2D that contains the following three points:

A = (1,1),
B = (3,0),
C = (5,1)

Comment: We will provide one possible solution later. But please try out different ways to solve this (check with your fellow students how they do it as well.)

Exercise 2

Given the following two definitions of the dot product

DEF1: $\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$

DEF2: $\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos \theta$

Discuss:

a) DEF 1 implies DEF 2
b) DEF 2 implies DEF 1

Comment: The dot product is the most important operation for us.
It is important that you also understand the geometry behind it.
We will tell you more about it in different parts of the course.
In addition it is necessary that you read in your text books about it as much as you can.

Exercise 3

Select a plane and a line and calculate the “hitpoint”:

For the plane you require a normal vector **n** and an anchor point B.

For the line you require a direction **c** and an anchor point A.

Example: n = (1,0,1), B = (1,2,2), c = (1,2,0), A = (0,1,1).

Comment: Solution hints are in the slides and the solution sheet.
Practice variations of this task with different points and vectors.
There will be a very similar exam question.
Please let us help you early if you get stuck while practicing.

Exercise 4

Given a vector v in 3D, how can we obtain a perpendicular vector?

Exercise 5

Given vectors v and w in 3D. What is the projection of v to w?

Comment:

1.) As we have no clear definition of “projection” yet there are two possible basic answers.
a) We can calculate the projection vector, i.e. the vector in direction of w resulting from projecting v to w.
b) We could consider the length of this vector only.
To answer both parts it is best to start with part b.

2.) Next you could think about how to construct a matrix or a transform that given w projects any other vector v onto w. This transform could also be called “projection onto w”.

Exercise 6

Given vectors v = (2,2,1) and w = (1,-2,0).

Calculate:

a) The dot product $v\ w$
b) The cross product $v \times w$
c) The projection of w onto v i.e. $\mathrm{proj}_v\,(w)$

Exercise 7

The unit circle in 2D is the set of points

$$\{v;\ ||v|| = 1\}$$

Describe how the unit circle looks like for:

a) The 1-norm
b) The 2-norm
c) The maximum norm

(See lecture 1 for the definitions.)