SCHOOL of ELECTRICAL ENGINEERING & COMPUTING
FACULTY of ENGINEERING & BUILT ENVIRONMENT
The UNIVERSITY of NEWCASTLE

Comp3320/6370 Computer Graphics

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Exercises V

This paper provides practice exercises for the lectures given from week 11 to week 12. It extends the material presented on the "Exercises I-IV" sheets (that still remain relevant for the whole course and all exams). These questions (some of them with hints or partial solutions) should help for exam preparation. It is recommended to look at the solutions only after you have first tried to solve the exercises yourself. Typically there are several different ways to solve an exercise. Please fill the gaps, try some variations and check if the provided solutions are correct. If you detect errors or have any suggestions for improvement please let us know.

Exercise 33 (Parameterised Curves)

Question: Describe the graphs of the following parameterised curves:

a) $\gamma : \mathbf{R} \longrightarrow \mathbf{R}^2,\ t \mapsto \gamma(t) = (t, t^2)$.

b) $\gamma : [0, 2\pi] \longrightarrow \mathbf{R}^2,\ t \mapsto \gamma(t) = (\cos(t), \sin(t))$.

c) $\gamma : [0, 2\pi] \longrightarrow \mathbf{R}^2,\ t \mapsto \gamma(t) = (2\cos(t), 3\sin(t))$.

d) $\gamma : [0, 2\pi] \longrightarrow \mathbf{R}^2,\ t \mapsto \gamma(t) = (\cos(t/4), \sin(t/4))$.

e) $\gamma : [0, 2\pi] \longrightarrow \mathbf{R}^3,\ t \mapsto \gamma(t) = (\cos(t), \sin(t), t)$.

f) $\gamma : [0, 2\pi] \longrightarrow \mathbf{R}^3,\ t \mapsto \gamma(t) = (\cos(t), \sin(t), \frac{t}{2\pi})$.

g) $\gamma : [0, 2\pi] \longrightarrow \mathbf{R}^3,\ t \mapsto \gamma(t) = (\cos(4t), \sin(4t), \frac{t}{2\pi})$.

h) $\gamma : [0, 2\pi] \longrightarrow \mathbf{R}^3,\ t \mapsto \gamma(t) = (\cos(t), \sin(t), t^2)$.

i) $\gamma : [0, 2\pi] \longrightarrow \mathbf{R}^2,\ t \mapsto \gamma(t) = (2e^{3t}\cos(t), 2e^{3t}\sin(t))$.

j) $\gamma : [0, 1] \longrightarrow \mathbf{R}^2,\ t \mapsto \gamma(t) = (\cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t))$.

k) $\gamma : [0, 1] \longrightarrow \mathbf{R}^2,\ t \mapsto \gamma(t) = (\sqrt{1-t^2}, t)$.

l)* $\gamma : [0, 1] \longrightarrow \mathbf{R}^2,\ t \mapsto \gamma(t) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$.

What about the speeds of the curves in j), k) and l)?

Answer: These should be easy. Please don't hesitate to ask us if you get stuck. Note: All three of j),k) and l) describe a quater circle.

Exercise 34 (Speed of a Curve)

Question: Let $\gamma : [0,1] \longrightarrow \mathbf{R^2}$, $t \mapsto \gamma(t) = (x(t), y(t))$ be a $\mathbf{C^1} - continuous$ parameterised curve. Then the function $\nu(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$ is called the *speed* of the curve γ. The *arc length* of the curve $\gamma(t)$ is given by $\mathbf{L}(\gamma) = \int_0^1 \nu(s)ds$ and $\mathbf{L}_\gamma(t) = \int_0^t \nu(s)ds$ is called the *arc length function*. The arc length function measures the length of the curve segment from the initial point $(x(0), y(0))$ to the point $(x(t), y(t))$.

a) Calculate the speed of the half circle curve $\gamma : [0,\pi] \longrightarrow \mathbf{R^2}$, $t \mapsto \gamma(t) = (2\cos(t), 2\sin(t))$

b) Calculate the the arc length function, and the length of the half circle curve γ above.

c) Show that a rotation has no effect on the speed of the above curve γ.

d) Let $a, b \in \mathbf{R}$. What is the speed and arc length function of the following curve $\mu(t) = (x(t), y(t)) = (ae^{bt}\text{cos}(t), ae^{bt}\sin(t))$? Can you describe the graph of the curve? Determine the unit normal vector for any point of this curve?

Answer:

a) $\nu(t) = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{4\cos^2(t) + 4\sin 2(t)} = \sqrt{4 \cdot 1} = 2.$

b) The arc length function is $\mathbf{L}_\gamma(t) = \int_0^t \nu(s)ds = \int_0^t 2ds = 2t$. The curve has length $\mathbf{L}_\gamma(\pi) - \mathbf{L}_\gamma(0) = 2\pi$.

c) Hint: Use $\gamma_\theta(t) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and the fact that $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

d) Don't forget to apply product and chain rule when calculating the derivatives.

Exercise 35: Let $\gamma(t) = (x(t), y(t), z(t))$ a curve with speed $\nu(t) \neq 0$ and arc length function $\mathbf{L}_\gamma(t)$. Assume that $\phi(s) = (\mathbf{L}_\gamma^{-1})(s)$ is the inverse of the arc length function. What is the speed of the curve $\tilde{\gamma} : [0,1] \longrightarrow \mathbf{R^3}$, $s \mapsto \tilde{\gamma}(s) = (x(\phi(s)), y(\phi(s)), z(\phi(s)))$?

Solution hint:

Let $f = \mathbf{L}_\gamma$ and use the following theorem from calculus:

Inverse Mapping Theorem
Let U be open in E, let $x_0 \in U$, and let $f : U \longrightarrow F$ be a C^1-map. Assume that the derivative $f'(x_0) : E \longrightarrow F$ is invertible. Then f is locally C^1-invertible at x_0. If ϕ is its local inverse, and $y = f(x)$, then

$$\phi'(y) = \frac{1}{f'(x)}$$

Exercise 36 (Frenet Vectors)

Question: Let $\gamma : [0, L] \longrightarrow \mathbf{R^3}$, $s \mapsto \gamma(s) = (x(s), y(s), z(s))$ be a three times continuously differentiable parameterised curve. Assume γ is parameterised by arclength and has length L. Define for each parameter $s \in [0, L]$ three vectors

$t(s) = \gamma'(s)$ *the unit tangent vector*
(Note that $\|t(s)\| = 1$ because γ is parameterised by arclength.)

$n(s) = \frac{t'(s)}{||t'(s)||}$ *the unit normal vector*

$b(s) = t(s) \times n(s)$ *the bi-normal vector*

Explain why for all $s \in [0, L]$ the following dot products are zero:

a) $t(s) \cdot n(s) = 0$

b) $t(s) \cdot b(s) = 0$

c) $n(s) \cdot b(s) = 0$

d) $n(s) \cdot n'(s) = 0$

Answer:

a) $t(s)$ is the unit tangent vector. Therefore $t(s) \cdot t(s) = 1$ (this is just $\|t(s)\|$). If we take the derivative of this equation (using the product rule) we obtain $t(s) \cdot t'(s) = 0$. Consequently $t(s) \cdot \frac{t'(s)}{||t'(s)||} = 0$ and hence $t(s)$ and $n(s)$ are perpendicular. (Note: Now we have another way to find an orthogonal vector to the first (basis) vector! Compare Exercise 20 (Rotation about an arbitrary axis)).

b) Use standard properties of dot and cross product that we addressed in the previous exercise sheet and in the lectures. It follows $t(s) \cdot (t(s) \times n(s)) = 0$ which is true for general vectors.

c) $n(s) \cdot (t(s) \times n(s)) = 0$ and therefore by definition $n(s) \cdot b(s) = 0$.

d) By definition $n(s)$ is the unit normal vector. Therefore $n(s) \cdot n(s) = 1$. If we take the derivative of this equation (using the product rule) we obtain $n(s) \cdot n'(s) = 0$.

Exercise 37 (Parameterised Surfaces)

Question: Give parameterised representations of the following 2-dimensional surfaces in $\mathbf{R^3}$:

a) The torus T^2

b) The 2-sphere $S^2 = \{x \in \mathbf{R^3}; \|x\| = 1\}$.

Answer: a) Let $a, b \in \mathbf{R}$ two parameters (each is radius of a circle) that control the width and height of the doughnut/torus, resp.. See sketch/photo in the lecture slides illustrating the explanation that was given on the blackboard in the lectures.

$$
\begin{array}{rcl}
[0, 2\pi) \times [0, 2\pi) & \longrightarrow & \mathbf{R^3} \\[1em]
(\alpha, \beta) & \mapsto & \begin{pmatrix} (a + b\cos\beta)\cos\alpha \\ (a + b\cos\beta)\sin\alpha \\ b\sin\beta \end{pmatrix}
\end{array}
$$

b) We obtain a 2d sphere of radius r by stacking circles (each parameterised by β) on top of each other along the z-axis in the middle. The radius of each circle in the stack is $r \sin \alpha$. More precisely, we start with a circle of radius 0 (at the North pole $z = r$) and stack circles of increasing radius below it until we reach a big circle of radius r at the equator and then we continue stacking circles with shrinking radius down to 0 (at the South pole $z = -r$).

$$\begin{array}{rcl} [0, \pi) \times [0, 2\pi) & \longrightarrow & \mathbf{R^3} \\ (\alpha, \beta) & \mapsto & \begin{pmatrix} r \sin \alpha \cdot \cos \beta \\ r \sin \alpha \cdot \sin \beta \\ r \cos \alpha \end{pmatrix} \end{array}$$

Exercise 38 (Basic Calculations with Quaternions)

Question:

Let $\hat{\mathbf{q}} = i + 2j + 3k + 4$ and $\hat{\mathbf{r}} = 5i + 6j + 7k + 8$ be two quaternions.

Calculate:

a) The sum $\hat{\mathbf{q}} + \hat{\mathbf{r}}$

b) The conjugate $\hat{\mathbf{q}}^*$

c) The norm $n(\hat{\mathbf{q}}) = ||\hat{\mathbf{q}}||^2 = \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}^* = q_x^2 + q_y^2 + q_z^2 + q_w^2$
(Note: Usually $||\hat{\mathbf{q}}||$ and not $||\hat{\mathbf{q}}||^2$ would be called the "norm". The book RTR [1] calls $||\hat{\mathbf{q}}||^2 = \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}^*$ the norm of $\hat{\mathbf{q}}$ and we decided to adopt the book's convention at this point.)

d) The inverse $\hat{\mathbf{q}}^{-1}$

e) The product $\hat{\mathbf{q}}\hat{\mathbf{r}}$

Solution hint:

Look up the relevant definitions in the lecture slides or in chapter 3 of [1] and put in the above example values. A simple calculation should lead to the desired result.

Exercise 39 (Rotation About an Arbitrary Axis Using Quaternions)

Question:

Let $P = (1, 2, 0, 1)$ be a point.

Which quaternion can rotate P about the axis $u_q = (1, 2, 2, 0)$ by an angle $\alpha = \frac{\pi}{2}$?

Determine the quaternion and the result of the rotation.

Solution hint:
Use method described in the slides or in chapter 3 of [1]. You can verify the result using the solution of exercise 20 from the Exercises III sheet.

Exercise 40 (Some Properties of Quaternions)

Question:

Let $\hat{\mathbf{a}}, \hat{\mathbf{b}} \in \mathbb{H}$ be quaternions.

a) Show that for the imaginary parts $\text{Im}(\hat{\mathbf{a}}) = \mathbf{a}_v$ and $\text{Im}(\hat{\mathbf{b}}) = \mathbf{b}_v$ of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ the quaternion product consists of the cross and the dot product:

$$\mathbf{a}_v \mathbf{b}_v = (\mathbf{a}_v \times \mathbf{b}_v, - < \mathbf{a}_v, \mathbf{b}_v >_{\mathbf{R}})$$

where $\mathbf{a}_v \mathbf{b}_v$ is the quaternion product, $\mathbf{a}_v \times \mathbf{b}_v$ is the cross product in $\mathbf{R}^3$, and $< \mathbf{a}_v, \mathbf{b}_v >_{\mathbf{R}} = \mathbf{a}_v \cdot \mathbf{b}_v$ is the usual dot product in $\mathbf{R}^3$.

b) Describe the real part of the quaternion product.

Answer: The solutions can be obtained by looking up quaternions in the lecture slides or in chapter 3 of [1] and by using the relevant definitions and properties of quaternions.

Final note: No guarantee that the solutions are correct yet. Please email any errors that you detect. Check blackboard for updates regularly.

Bibliography

[1] T. Akenine-Möller and E. Haines. *Real-Time Rendering*. A K Peters, second edition, 2002.

[2] Tomas Akenine-Möller, Eric Haines, Naty Hoffman, Angelo Pesce, Sebastien Hillaire, and Michal Iwanicki. *Real-Time Rendering*. A K Peters/CRC Press, fourth edition, 2018.

[3] John Vince. *Mathematics for Computer Graphics*. Springer-Verlag, 2010.