# MATH1510 - Discrete Mathematics
# Enumeration 1

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## Introduction

Many areas of mathematics involve problems of enumeration, or *counting* cleverly. For example:

- The number of ways something can happen (probability theory, which finds applications in statistics, physics, economics, etc).
- The number of operations required in executing an algorithm (complexity theory).



## The Addition Principle

**Example**

In how many ways can you type one character on a keyboard if the keyboard has 26 letter keys, 10 digit keys and no others?

If one type of object can be selected in $r$ ways and another type can be selected in $s$ ways, then the number of ways of selecting *one* object from these two types is:

$$r + s.$$

In our example it is important that we know that there are no letters which are also digits.



## Disjoint Sets

For two disjoint sets we have $|A \cup B| = |A| + |B|$. This generalises easily to more than two sets.

**Example**

In how many ways can you type one character on a keyboard if the keyboard has 26 letter keys, 10 digit keys and 10 punctuation keys?

- A 10
- B 26
- C 36
- D 46

## Intersecting Sets

What if the sets $A$ and $B$ are not disjoint?

$$|A \cup B| = |A| + |B| - |A \cap B| \quad (\text{inclusion} - \text{exclusion}).$$

### Example

In a class there are 9 students with blond hair and 6 students with blue eyes. Of these there are 4 students who have blond hair and blue eyes. How many students have either blond hair or blue eyes?

- A 15
- B 19
- C 11
- D 7



## The Multiplication Principle

### Example

Between towns A and B there are 3 connecting roads and between towns B and C there are 4 connecting roads. How many different routes are possible in traveling from town A to town C?

- A 7
- B 12
- C 81
- D 64



If an activity can be constructed in $t$ successive steps, where step 1 can be done in $n_1$ ways, step 2 in $n_2$ ways, ... and step $t$ in $n_t$ ways, then the number of different possible activities is

$$n_1 \times n_2 \times \cdots \times n_t.$$

### Exercise

In Morse code, alpha-numeric characters are represented by a string of dots (•) and dashes (—). If strings of length at most $n$ are to be used, what is the smallest value of $n$ needed to encode

(i) the English alphabet?

(ii) the alphabet and the 10 digits?



### Exercise

In Victoria, car number plates usually consists of 3 letter followed by 3 numbers. In NSW, number plates often have 2 letters, followed by 2 numbers, then followed by another 2 letters.

(i) By these rules, how many different number plates can there be in Vic and in NSW? Which state can have more, and why?

(ii) How many NSW number plates can there be, if the 2 (consecutive) numbers can appear anywhere?

## The Multiplication Principle – Examples

**Example**

A six-person committee composed of Alice, Ben, Connie, David, Ellie and Finlay is to select a chairperson, secretary, and treasurer.

- In how many ways can this be done?
- In how many ways can this be done if either Alice or Ben must be the chairperson?
- In how many ways can this be done if Ellie must hold one of the offices?
- In how many ways can this be done if both David and Finlay must hold office?

**Answers:**

$6 \times 5 \times 4 = 120$; $2 \times 5 \times 4 = 40$; $3 \times 5 \times 4 = 60$; $3 \times 2 \times 4 = 24$.



## The Multiplication Principle – Examples

**Example**

In how many ways can you draw a heart and then a jack from a deck of cards if

- the first card is replaced before drawing the second,
- the first card is not replaced?

It may be helpful to think of the addition and multiplication principles in terms of how many ways there are of getting *dressed*.

Suppose you can choose from $t_1$ T-shirts, $t_2$ shirts, $t_3$ singlets, $b_1$ pairs of pants, $b_2$ pairs of shorts, $s_1$ pairs of runners, $s_2$ pairs of sandals, or no shoes. Then the number of ways to get dressed is

$$(t_1 + t_2 + t_3) \times (b_1 + b_2) \times (s_1 + s_2 + 1).$$



## Permutations

**Definition**

An ordering (ordered arrangement) of $n$ distinct objects is called a permutation of the objects.

**Example**

There are 6 permutations of the three letters $a$, $b$, $c$, namely:

$$a\,b\,c \qquad a\,c\,b \qquad b\,a\,c \qquad b\,c\,a \qquad c\,a\,b \qquad c\,b\,a$$

The first position can be any of the three possible letters. The second position must be chosen from the two remaining and the last has to be the only letter left.

So there are $3 \times 2 \times 1 = 6$ possible permutations.



## Permutations

Suppose now we have $n$ objects to arrange.

- The first object can be any of the $n$ objects.
- The second object can be any of the remaining $(n-1)$ objects.
- The third object can be any of the remaining $(n-2)$ objects.
- $\vdots$
- The last ($n$th) object can only be selected one way.

So the number of permutations of $n$ objects is

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

This expression is called "$n$ factorial" and is written as $n!$.

## Permutations

**Theorem**

*There are $n!$ permutations of $n$ objects.*

Note: for consistency, we define $0! = 1$.

**Exercise**

How many different ways are there of indicating your preferences in the lower house for the seat of Newcastle in a federal election if there are 9 candidates (and you must vote for all 9)?



## Circular Permutations

**Example**

In how many ways can six people be seated around a circular table?

Note: if a seating arrangement is obtained from another by rotation, then they are considered identical.

For each permutation there are 5 others which we consider to be identical. So we divide the total number of permutations by 6 to get

$$\frac{6!}{6} = 5! = 120 \text{ permutations.}$$



## Circular Permutations

**Example**

How many ways can we make a necklace of $n$ beads, if they are all different colours?

There is no distinction between clockwise and anticlockwise arrangements (because we can flip the necklace), so the total number of arrangements is halved, and we get

$$\frac{(n-1)!}{2} \text{ permutations.}$$



## $r$-Permutations

Suppose now we only wish to order $r$ of the $n$ objects, where $r \leq n$.

**Example**

Recall the six person committee example: select a chairperson, secretary and treasurer from $\{A, B, C, D, E, F\}$.

$$6 \times 5 \times 4 = 120 \text{ ways.}$$

In this example we are finding all the permutations of three objects taken from the set of six. We call this number $P(6, 3)$ and the orderings are called 3-permutations.

## $r$-Permutations

### Theorem

*The number of $r$-permutations of $n$ objects is*

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

### Example

How many 2-permutations are there of the set $\{a,b,c\}$?

$$P(3,2) = 3 \times 2 = 6.$$

$$ab,\ ac,\ ba,\ bc,\ ca,\ cb$$



## Combinations

### Example

Suppose that a 6-person committee needs to select a 3-person subcommittee. In how many ways can this be done?

Previously we cared about the order in which the 3 people were selected. Now we don't care about which of the 3! permutations of each selection we are choosing. Consequently we should divide our answer for the number of 3-permutations by 3!:

$$\frac{120}{3!} = 20.$$



## Combinations

### Definition

A selection of objects without regard to order is called a combination. We call a combination in which $r$ objects are selected an $r$-combination.

### Theorem

*The number of $r$-combinations from a set of $n$ objects is*

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}.$$

### Exercise

Which do you think is bigger, $C(15,4)$ or $C(15,11)$?
How big do you think $C(52,13)$ is?



## Alternative Notations

There are other notations you may see:

$$P(n,r) = {}^nP_r,$$
$$C(n,r) = {}^nC_r = \binom{n}{r}.$$

Here are some important properties of $C(n,r)$:

$$C(n,0) = 1,\ C(n,1) = n,\ C(n,2) = \frac{n(n-1)}{2},\ C(n,r) = C(n,n-r).$$

**Exercise**

In how many ways can 12 horses in a race finish as first, second and third?

A $P(12, 3)$

B $C(12, 3)$

C 220

D $12 \times 11 \times 10$



**Exercise**

In how many ways can three of the 12 horses be withdrawn?

A $P(12, 3)$

B $C(12, 3)$

C 220

D $12 \times 11 \times 10$



## Listing Permutations and Combinations

When generating all permutations or combinations of a given type, it is helpful to list them in a structured way. We can use lexicographic order, which generalises alphabetical order.

Given two words, how do we decide which one comes first in the dictionary?
We compare letters. There are 2 possibilities.

(1) Each letter in the shorter word is the same as the corresponding letter in the longer word.

(2) At some position, the letters in the words differ.

In situation (1), we put the shorter word first.

In situation (2), we order the words according to the first letter where the words differ.



## Lexicographic Order

**Example**

In alphabetical order

- “man” precedes “manual”.
- “strain” precedes “stretch”.

These rules also apply to lexicographic ordering, the difference is that instead of letters we may have any ordered sequence.

**Example**

Order the following strings taken from the sequence $1, 2, 3, 4, 5, 6$.

$$15 \qquad 633 \qquad 411 \qquad 42 \qquad 4$$

## Lexicographic Order

### Example
List all the 3-permutations of $\{1, 2, 3, 4\}$ in lexicographic order.

While order doesn't matter for combinations, we will represent combinations by the ordering of least lexicographic value.

### Example
The 5-combinations 57123, 27351, 31752, etc of $\{1, 2, \ldots, 7\}$ all represent the same combination, so we will represent it by 12357.

### Exercise
List all of the 4-combinations of $\{1, 2, \ldots, 6\}$. Represent each one by an increasing string and write your list in lexicographic order.



## Permutations with Repetitions

### Example
How many different arrangements can be formed from the letters of the word DAD ?

We have 3 letters to permute, but some letters are repeated.

Suppose that we identify the repeated letters with subscripts so that we can tell them apart. That is:

$$D_1AD_2$$



## Permutations with Repetitions

### Example continued
With the $D$'s different there are now 3! arrangements. These are:

$$\begin{array}{ccc} D_1AD_2 & & DAD \\ D_1D_2A & & DDA \\ AD_1D_2 & \text{dropping} & ADD \\ AD_2D_1 & \text{subscripts :} & ADD \\ D_2AD_1 & & DAD \\ D_2D_1A & & DDA \end{array}$$

We see that there are only 3 different arrangements.

There are 2! ways of ordering the repeated $D$. So the number of permutations of the letters of the word $DAD$ is $\frac{3!}{2!}$.



## Permutations with Repetitions

### Theorem
*Suppose that a sequence $S$ of $n$ items has $n_1$ identical objects of type 1, $n_2$ identical objects of type 2, ..., and $n_t$ identical objects of type $t$. Then the number of permutations of $S$ is*

$$\frac{n!}{n_1!n_2!\cdots n_t!}.$$

### Example
How many arrangements are there for the sequence AAABBCCC ?

### Example

How many permutations exist of the letters of the word COMMITTEE ?

- A $\frac{9!}{2!2!2!}$
- B $\frac{9!}{6!}$
- C $3!$
- D $6!$



### Example

How many permutations are there of the letters of the word MOLLYCODDLE ?

- A $11!$
- B $\frac{11!}{3!3!}$
- C $5!$
- D $\frac{11!}{2!2!3!}$



Note: This general result can be explained in a different way.

Consider again AAABBCCC

There are $C(8,3)$ ways to choose positions for the three A's. Once the positions for the A's have been determined there are $C(5,2)$ ways to choose positions for the B's. This leaves $C(3,3)$ positions for the three C's.

Thus the number of ways of ordering the letters is

$$C(8,3) \times C(5,2) \times C(3,3)$$



## Combinations with Repetitions

We can also have repetitions in combinations. Remember that in combinations the order doesn't matter.

### Example

A chess set contains 4 rooks, 4 knights, 4 bishops, 2 queens, 2 kings, and 16 pawns. Ignoring colour differences, how many selections of two pieces are possible?

### Example

Consider the set $\{a, b, c\}$. How many selections of size two are possible – allowing repetitions?

Is there a general method?

## Combinations with Repetitions

### Example

How many ways can we select six books from 6 identical Computer Science books, 6 identical Physics books and 6 identical Statistics books?

Choosing one of these possibilities can be thought of as choosing 2 *dividers* (representing the 3 types) to separate 6 crosses (representing the 6 chosen books).

$$\times | \times \times \times | \times \times$$

There are 8 places to choose from (6 for the books and 2 for the dividers). Then the number of possibilities is $C(8, 2) = 28$.



## Textbook exercises

Exercises Section 6.1:

- 1, 3, 4, 6, 10, 15, 16, 25, 29, 37, 38, 42, 46, 49, 52, 55, 58, 64, 65, 67, 70, 71, 74, 77, 78, 79, 80, 81, 82, 83, 86, 89, 91,

Exercises Section 6.2:

- 1, 4, 9, 11, 15, 20, 21, 25, 28, 31, 34, 37, 40, 41, 42, 43, 49, 52, 57, 60, 62, 66,

Exercises Section 6.3:

- 1, 3, 4, 5, 7, 8, 11, 12, 14, 15, 16, 18, 34, 36, 38, 39, 42, 45