SCHOOL of ELECTRICAL ENGINEERING & COMPUTING
FACULTY of ENGINEERING & BUILT ENVIRONMENT
The UNIVERSITY of NEWCASTLE

Comp3320/6370 Computer Graphics

Semester 2, 2018

Exercises I

Practice exercises for lectures in weeks 1-3

Version 28

This paper provides (partial) solutions to some of the exercises given in the lectures from week 1 to week 3. These partial solutions should help for exam preparation. It is recommended to look at the solutions only after you have first tried to solve the exercises yourself. Typically there are several different ways to solve an exercise. Please fill the gaps, try some variations and check if the provided solutions are correct. If you need help or have questions please let us know.

Exercise 1

Question: Calculate the midpoint of the circle in 2D that contains the following three points:

$$\begin{aligned} A &= (1,1) \\ B &= (3,0) \\ C &= (5,1) \end{aligned}$$

Solution: The three points have to lie on the same circle. Therefore the distances from the centre point $X = (x, y)$ of the circle to all three points A, B, and C must be equal.

$$\begin{aligned} \text{Distance from } X \text{ to } A &= \text{Distance from } X \text{ to } B \\ \text{Distance from } X \text{ to } B &= \text{Distance from } X \text{ to } C \\ \text{Distance from } X \text{ to } C &= \text{Distance from } X \text{ to } A \end{aligned}$$

or using formulas

$$\begin{aligned} \left|\left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|\right| &= \left|\left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right|\right| \\ \left|\left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right|\right| &= \left|\left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix} \right|\right| \\ \left|\left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix} \right|\right| &= \left|\left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|\right| \end{aligned}$$

Using the Euclidean distance formula we obtain the following set of equations:

$$\sqrt{(x-1)^2+(y-1)^2} = \sqrt{(x-3)^2+(y-0)^2} \quad (1)$$

$$\sqrt{(x-3)^2+(y-0)^2} = \sqrt{(x-5)^2+(y-1)^2} \quad (2)$$

$$\sqrt{(x-5)^2+(y-1)^2} = \sqrt{(x-1)^2+(y-1)^2} \quad (3)$$

Squaring each of the above equations, we obtain:

$$(x-1)^2+(y-1)^2 = (x-3)^2+(y-0)^2 \quad (4)$$
$$(x-3)^2+(y-0)^2 = (x-5)^2+(y-1)^2 \quad (5)$$
$$(x-5)^2+(y-1)^2 = (x-1)^2+(y-1)^2 \quad (6)$$

We can now start solving equations (4), (5), (6) for x and y.
Using (4) we obtain:

$$\begin{aligned}
\left(x^2+1-2x\right)+\left(y^2+1-2y\right) &= \left(x^2+9-6x\right)+\left(y^2\right) \\
x^2+y^2-2x-2y+2 &= x^2+y^2-6x+9 \\
6x-2x-2y+2-9 &= 0 \\
4x-2y &= 7 \quad (7)
\end{aligned}$$

Using (5) we obtain:

$$\begin{aligned}
\left(x^2+9-6x\right)+\left(y^2\right) &= \left(x^2+25-10x\right)+\left(y^2+1-2y\right) \\
x^2+y^2-6x+9 &= x^2+y^2-10x-2y+26 \\
10x-6x+2y &= 26-9 \\
4x+2y &= 17 \quad (8)
\end{aligned}$$

Now we have already two linear equations for two variables and using (7) and (8) we can obtain the values for x and y as:

$$\begin{aligned}
x &= 3 \\
y &= \frac{5}{2}
\end{aligned}$$

Therefore centre of the circle is $\left(3, \frac{5}{2}\right)$ and the radius is $||X-B|| = \sqrt{0+(\frac{5}{2})^2} = 2.5$.

Exercise 2

Question: Show that the following two definitions of the dot product are equivalent:

Def 1: $\vec{v}\cdot\vec{w} = v_1w_1+v_2w_2+v_3w_3 \quad (1)$

Def 2: $\vec{v}\cdot\vec{w} = ||\vec{v}||\,||\vec{w}||\cos\theta,$ where θ is the angle between the two vectors. (2)

Solution: We can show the equivalence using the law of cosines applied to vectors $\vec{v}$ and $\vec{w}$:

$$||\vec{v}-\vec{w}||^2 = ||\vec{v}||^2+||\vec{w}||^2-2||\vec{v}||||\vec{w}||\cos\theta$$

i.e.

$$\begin{aligned}
(v_1-w_1)^2+(v_2-w_2)^2+(v_3-w_3)^2 &= v_1^2+v_2^2+v_3^2+w_1^2+w_2^2+w_3^2-2||\vec{v}||||\vec{w}||\cos\theta \\
-2\left[v_1w_1+v_2w_2+v_3w_3\right] &= -2||\vec{v}||||\vec{w}||\cos\theta \\
\text{Hence, } v_1w_1+v_2w_2+v_3w_3 &= ||\vec{v}||||\vec{w}||\cos\theta
\end{aligned}$$

Exercise 3

Question: Select a plane and a line and calculate the "hitpoint": For the plane you require a normal vector $\vec{n}$ and an anchor point B. For the line you require a direction vector $\vec{c}$ and an anchor point A. Use the example: $\vec{n} = (1,0,1)^T$, $B = (1,2,2)^T$, $\vec{c} = (1,2,0)^T$, $A = (0,1,1)^T$.

Solution: Given:

- A plane with normal vector $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and anchor point $B = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$
- A line with direction vector $\vec{c} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and anchor point $A = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

We want to find the **hit point** P_{hit}.

Suppose the line hits at $t = t_{hit}$, with the equation of line as $A + \vec{c}t = P$.
At $t = t_{hit}$, the line $A + \vec{c}t = P$ and the plane $\vec{n} \cdot (P - B) = \vec{0}$ must have same coordinates. We exploit this fact and substitute the point P in the equation of the plane to calculate the hit point:

$$\begin{aligned}
\vec{n} \cdot (A + \vec{c}t_{hit} - B) &= 0 \\
\Rightarrow t_{hit} &= \frac{\vec{n} \cdot (B - A)}{\vec{n} \cdot \vec{c}} \\
&= \frac{(1,0,1)^T \cdot [1-0, 2-1, 2-1]^T}{(1,0,1)^T \cdot (1,2,0)^T} \\
&= \frac{(1,0,1)^T \cdot (1,1,1)^T}{1+0+0} \\
\text{Hence, } t_{hit} &= 2
\end{aligned}$$

Using t_{hit} the hit point can now be calculated as:

$$\begin{aligned}
P_{hit} &= A + \vec{c}t_{hit} \\
&= (0,1,1)^T + 2\,(1,2,0)^T \\
P_{hit} &= (2,5,1)^T
\end{aligned}$$

Exercise 4

Question: Let $\vec{v}$ be a vector in 3D.

a) How can we obtain a perpendicular vector (i.e. a vector that is orthogonal to $\vec{v}$)?

b) Why does this work or how can we test that the new vector is perpendicular to $\vec{v}$?

c) Is it uniquely determined?

Solution:

a) Hint: Set one coordinate to zero, then swap the other two coordinates and put a minus sign in. A numerically stable solution is described in chapter 3 of the book Real-time Rendering (available online in shortloans).

b) If the dot product of both vectors is zero then they must be perpendicular (cf. Exercise 2 and note that $\cos(90^o) = 0$).

c) It is not uniquely determined. Given $\vec{v}$ there are many perpendicular vectors. They all together form a plane so that $\vec{v}$ is the normal vector.

Exercise 5

Question: Given vectors $\vec{v}$ and $\vec{w}$ in 3D.

How can we obtain the projection of $\vec{v}$ onto $\vec{w}$, i.e. the projection vector $proj_{\vec{w}}(\vec{v})$ and its length $\|proj_{\vec{w}}(\vec{v})\|$.
(Note: We use the notation $proj_{\vec{w}}(\vec{v})$ and not $proj_{\vec{v}}(\vec{w})$.)

Solution: We can use the dot product and then obtain: $||proj_{\vec{w}}(\vec{v})|| = \vec{v} \cdot \frac{\vec{w}}{||\vec{w}||}$. The projection vector is $\frac{\vec{v}\cdot\vec{w}}{||\vec{w}||} \frac{\vec{w}}{||\vec{w}||}$.

Exercise 6

Question: Given vectors $\vec{v} = (2, 2, 1)^T$ and $\vec{w} = (1, -2, 0)^T$ calculate:

a) The dot product $\vec{v} \cdot \vec{w}$

b) The cross product $\vec{v} \times \vec{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$

c) Check that $\vec{v} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{v} \times \vec{w}) = 0$. i.e. the cross product is a vector that is perpendicular to both $\vec{v}$ and $\vec{w}$.

Exercise 7 (Unit ball)

Question: The unit ball in n-dimensional space $\mathbf{R}^n$ is the set of points

$$B^n = \{\vec{v}; \vec{v} \in \mathbf{R}^n \text{ and } \|\vec{v}\| \leq 1\}$$

Describe how the unit ball in $n = 2$ dimensions looks like for:
a) The 1-norm $\|\vec{v}\|_1 = \sum_{i=1}^n |v_i|$
b) The 2-norm $\|\vec{v}\|_2 = (\sum_{i=1}^n |v_i|^2)^{\frac{1}{2}}$, i.e. the standard Euclidean norm.
c) The maximum norm $\|\vec{v}\|_\infty = \max_{i=1,\ldots,n} |v_i|$
(See lecture 1 for the definition of a norm.)

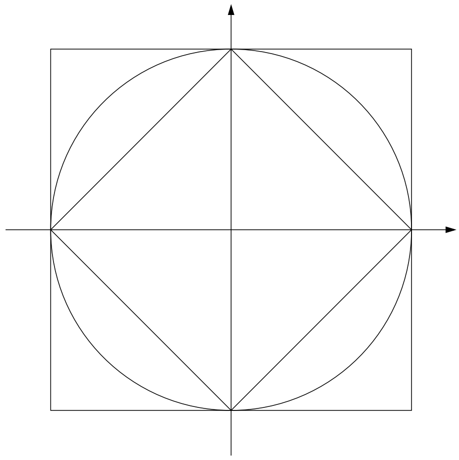
Note:

$$\|\vec{v}\|_\infty \leq \|\vec{v}\|_2 \leq \|\vec{v}\|_1$$

and

$$\|\vec{v}\|_\infty = \lim_{p\to\infty} \|\vec{v}\|_p = \lim_{p\to\infty} \left(\sum_{i=1}^n |v_i|^p \right)^{\frac{1}{p}}$$

Solution hint: Which is which?



Exercise 8 (Matrix product)

Question: Let $M \in M(m \times k, \mathbf{R})$ and $N \in M(k \times n, \mathbf{R})$ be two matrices with real coefficients. Let $P \in M(m \times n, \mathbf{R})$ be the matrix obtained by multiplying M and N:

$$P = (p_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = M \cdot N$$

Give the general formula for p_{ij}, i.e. the coefficients of P.

Solution:

$$\begin{aligned} P &= M \cdot N \\ &= \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1k} \\ m_{21} & m_{22} & \dots & m_{2k} \\ . & . & \dots & . \\ . & . & \dots & . \\ m_{i1} & m_{i2} & \dots & m_{ik} \\ . & . & \dots & . \\ . & . & \dots & . \\ m_{m1} & m_{m2} & \dots & m_{mk} \end{bmatrix} \cdot \begin{bmatrix} n_{11} & n_{12} & \dots & n_{1j} & \dots & n_{1n} \\ n_{21} & n_{22} & \dots & n_{2j} & \dots & n_{2n} \\ . & . & \dots & \dots & \dots & . \\ . & . & \dots & \dots & \dots & . \\ . & . & \dots & \dots & \dots & . \\ n_{k1} & n_{k2} & \dots & n_{kj} & \dots & n_{kn} \end{bmatrix} \\ &= \begin{bmatrix} p_{11} & p_{12} & \dots & \dots & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & \dots & \dots & p_{2n} \\ . & . & \dots & \dots & \dots & . \\ . & . & \dots & \dots & \dots & . \\ . & . & \dots & p_{ij} & \dots & . \\ . & . & \dots & \dots & \dots & . \\ . & . & \dots & \dots & \dots & . \\ p_{m1} & p_{m2} & \dots & \dots & \dots & p_{mn} \end{bmatrix} \end{aligned}$$

where for all $i = 1, \dots, m$ and $j = 1, \dots, n$

$$p_{ij} = \sum_{\nu=1,\dots,k} m_{i\nu} n_{\nu j} = \mathbf{m_{i*}}^T \cdot \mathbf{n_{*j}}$$

is the dot product of the i-th row vector $\mathbf{m_{i*}}^T = [m_{i1}, m_{i2}, \dots, m_{ik}]$ of M

with the j-th column vector $\mathbf{n_{*j}} = \begin{bmatrix} n_{1j} \\ n_{2j} \\ . \\ . \\ . \\ n_{kj} \end{bmatrix}$ of N.

Exercise 9 (Rotations in 2D)

Please visualise by drawing the following two situations:

a) A vector (or point) with coordinates $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ rotates around the origin. Can you see that $\sin^2\alpha + \cos^2\alpha = 1$?

b) Now consider an additional vector $\begin{pmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{pmatrix}$ in the same drawing and show that

$$\begin{pmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

c) If $\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$ describes the rotation by an angle β, what describes the inverse rotation, i.e. the rotation by $-\beta$?

<u>Solution hints:</u>

- Draw every situation carefully in a 2-dimensional coordinate system.
- Remember the following two formulas from highschool

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

- Revise your knowledge about $\sin$ and $\cos$.
- Consider to use complex numbers as an alternative approach.

No guarantee that the solutions are correct yet. Please email any errors that you detect. Check blackboard for updates.