Ex 16

Let $A \in GL_n(\mathbb{R})$ then the following three statements are equivalent:

(a) $A^T = A^{-1}$

(b) $\forall x \in \mathbb{R}^n \quad \|Ax\| = \|x\|$

(c) $\forall x, y \in \mathbb{R}^n \quad Ax \cdot Ay = x \cdot y$

Proof. (a) $\Rightarrow$ (b)

$$\underset{\text{dot}}{u \cdot v} = v^T \underset{\text{matrix}}{\cdot} u$$

$$(*) \quad Au \cdot v = v^T(Au) = (v^TA)u = (A^Tv)^Tu = u \cdot A^Tv$$

$$\|Ax\| = \sqrt{Ax \cdot Ax} \overset{(*)}{=} \sqrt{x \cdot A^TAx} \overset{(a)}{=} \sqrt{x \cdot x} = \|x\|$$

(b) $\Rightarrow$ (c)

$$\|u+v\|^2 = (u+v) \cdot (u+v) = \|u\|^2 + 2(u \cdot v) + \|v\|^2$$
$$\|u-v\|^2 = (u-v) \cdot (u-v) = \|u\|^2 - 2(u \cdot v) + \|v\|^2$$

$$u \cdot v = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) \overset{(b)}{=} \frac{1}{4}\left(\|A(u+v)\|^2 - \|A(u-v)\|^2\right) =$$
$$= \frac{1}{4}\left(\|Au + Av\|^2 - \|Au - Av\|^2\right) = Au \cdot Av$$

(c) $\Rightarrow$ (a) $\quad \forall x, y \in \mathbb{R}^n$

$$x \cdot y = Ax \cdot Ay$$
$$x \cdot y = x \cdot A^TAy$$
$$x \cdot (A^TAy - y) = 0$$
$$x \cdot (A^TA - Id)y = 0$$

Let $x = (A^TA - Id)y$

$$\forall y \in \mathbb{R}^n \quad (A^TA - Id)y \cdot (A^TA - Id)y = 0$$

$$\Rightarrow \forall y \in \mathbb{R}^n \quad (A^TA - Id)y = 0$$

$$\Rightarrow \quad A^TA - Id = 0$$

$$\Rightarrow \quad A^T = A^{-1}$$

q.e.d.