The University of Newcastle
School of Electrical Engineering and Computer Science

COMP2230/6230 Algorithms

Tutorial Week 3

2 - 6 August 2021

Tutorial

1. Find Θ for the following functions

i. $6n^3 + 12n + 1$

ii. $(n + 1)(n + 3) /(n + 2)$

2. Find Θ for the number of times the statement x=x+1 is executed.

i.
```
for i=1 to n
    for j=1 to i
        for k=1 to j
            x=x+1
```

ii.
```
i=2
while (i < n){
   i=i*i
   x=x+1
}
```

3. Use iteration to solve the following recurrence relations:

i. $a_n = a_{n-1} + 3, n > 1; a_1 = 2$

ii. $a_n = 2a_{n-1}, n > 0; a_0 = 1$

4. True or false?

i. $n^2 = O(n^3)$

ii. $n^2 = \Omega(n^3)$

iii. $n^2 = \Theta(n^3)$

5. Arrange the following functions in ascending order in their growth rate. That is, if a function $g(n)$ comes after function $f(n)$ then $f(n) = O(g(n))$. Prove your answers.

$$n^2,\ n^3,\ 100n^2,\ n \lg n,\ 2^n$$

6. Prove the following:

   i. $n! = O(n^n)$

   ii. $\Sigma_{i=1}^{n}\ i \lg i = \Theta(n^2 \lg n)$

7. Prove that

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!}$$

8. Prove that n^k is a smooth function.

9. Prove that $T(n)$ is well defined for all n by recurrence relation $T(n)=aT(n/b)+cn^k$ when n/b denotes $\lfloor n/b \rfloor$.

10. Use the Main (Master) Recurrence Theorem to find Θ for each of the following functions:

   i. $T(n) = 2T(n/2) + f(n);\ f(n) = n^2$

   ii. $T(n) = 2T(n/2) + f(n);\ f(n) = 5$

Homework

11. Find Θ for the following functions

   i. $(6n + 1)^2$

   ii. $3n^2 + 2n \lg n$

12. Find Θ for the number of times the statement x=x+1 is executed.

i.
```
for i=1 to 2n
    x=x+1
```

ii.
```
for i=1 to n
    for j=1 to i
        for k=1 to i
            x=x+1
```

13. Use iteration to solve the following recurrence relations:

i. $a_n = 2a_{n-1} + 1,\ n > 1;\ a_1 = 1$

ii. $a_n = 2^n a_{n-1},\ n > 0;\ a_0 = 1$

14. True or false?

i. $2^n = O(2^{n+1})$

ii. $2^n = \Omega\,(2^{n+1})$

iii. $2^n = \Theta\,(2^{n+1})$

15. Arrange the following functions in ascending order in their growth rate. That is, if a function $g(n)$ comes after function $f(n)$ then $f(n) = O(g(n))$. Prove your answers.

$$10^n,\ n^{1/3},\ n^n,\ \lg n,\ 2\text{^}((\lg n)^{1/2})$$

16. Prove the following:

i. $2^n = O(n!)$

ii. $\lg\,(n^k + c) = \Theta(\lg n)$, for every fixed $k > 0$ and $c > 0$

17. Prove that $n^{\log_b a}$ is a smooth function.

18. Use the Main (Master) Recurrence Theorem to find Θ for each of the following functions:

i. $T(n) = 4T(n/2) + f(n); f(n) = n$

ii. $T(n) = 4T(n/2) + f(n); f(n) = n^2$

19. Solve the following homogeneous recurrence:
$T(n) = 6T(n-1) + 9T(n-2), T(0) = 0, T(1) = 3$

More Exercises

20. Find Θ for the following function: $2 + 4 + 6 + \ldots + 2n$

21. Find Θ for the number of times the statement x=x+1 is executed.

```
j=n
while (j ≥ 1){
  for i=1 to j
    x=x+1
  j=j/3
}
```

22. Use iteration to solve the following recurrence relations:

$$a_n = 2 + \Sigma_{i=1}^{n-1} a_i, \;\; n > 1; \;\; a_1 = 1$$

23. True or false?

i. $n! = O((n+1)!)$

ii. $n! = \Omega((n+1)!)$

iii. $n! = \Theta((n+1)!)$

24. Arrange the following functions in ascending order in their growth rate. That is, if a function $g(n)$ comes after function $f(n)$ then $f(n) = O(g(n))$. Prove your answers.

$$n^{2.5}, (2n)^{1/2}, n+10, 10^n, 100^n, n^2 \lg n$$

25. Prove that $H_n = \Sigma_{i=1}^{n} (1/i) = \Theta (\log n)$.
(Hint: In the previous tutorial you proved that $1/n \leq lg (n+1) - lg\ n < 2/n$.)

26. Prove the following:

$1^k + 2^k + \ldots + n^k = \Theta(n^{k+1})$

27. Prove that $\log (n!) = \Theta (n \log n)$.

28. Consider the following algorithm that computes a^n. Let c_n be the number of multiplications required to compute a^n.

```
exp( a,n) {
     if ( n = = 1)
     return a
     m = ⌊n / 2⌋
     return exp( a, m)* exp( a, n-m)
}
```

i. Find a recurrence relation and initial conditions for the sequence $\{c_n\}$.

ii. Solve the recurrence relation in case n is a power of 2.

iii. Solve the recurrence relation for every positive integer n.

29. Prove that if a and b are numbers such that $0 \leq a < b$ then
$(n+1)a^n < (b^{n+1} - a^{n+1})/(b - a) < (n+1)b^n$.

30. Prove that the sequence $\{(1+1/n)^n\}$ *is* increasing and bounded above by 4.

31. Prove that $1/n \leq lg(n+1) - lg\ n < 2/n$.

32. Prove that $n^k \log_b n$ is a smooth function.

33. Use the Main (Master) Recurrence Theorem to find Θ for the following function:
$T(n) = 2T(n/2) + f(n); f(n) = n^3$