MATH1510 - Discrete Mathematics
Enumeration 2

University of Newcastle

UoN

Combinations with Repetitions

Theorem (Multichoose)

If set X contains t elements (types), then the number of unordered ways to select k element from X, with repetitions allowed, is:

$$C(k+t-1, t-1).$$

Example

In how many ways can ten identical maths books be distributed to 4 tutorial groups?
Here we are dividing 10 books into 4 types, so $t = 4, k = 10$.

Exercise

How many different ways are there to choose a dozen doughnuts from the five varieties at a doughnut shop?

- A $C(17, 5)$
- B $C(16, 4)$
- C $C(12, 5)$
- D $C(17, 4)$

Combinations with Repetitions

Example

Comparing the theorem with our previous example, we have

$$\begin{aligned} X &= \{\mathrm{CS}, \mathrm{Physics}, \mathrm{Statistics}\} \\ k &= 6, \ \ t = 3 \\ k+t-1 &= 8 && \text{the number of places to put a divider} \\ t-1 &= 2 && \text{the number of dividers} \end{aligned}$$

So the number of possibilities is:

$$C(k+t-1, t-1) = C(6+3-1, 3-1) = C(8, 2).$$

Note this won't work if there are < 6 books of any given type.

Binomial Expansion

Recall the expansion of $(a+b)^2$:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+2ab+b^2\\ &\neq a^2+b^2\end{aligned}$$

What about $(a+b)^3$?

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2\\ &= (a+b)(a^2+2ab+b^2)\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

The Binomial Theorem

Where do the 3's come from?

In $(a+b)(a+b)(a+b)$ there are 3 ways of multiplying two lots of a and one b together.

Hence there are $C(3,2)=3$ ways of getting a^2b.

If we think of each term as choosing places for a's and filling the gaps with b's, then we have:

$$C(3,3)a^3+C(3,2)a^2b+C(3,1)ab^2+C(3,0)b^3 = a^3+3a^2b+3ab^2+b^3.$$

Theorem (Binomial Theorem)

For integer $n \geq 0$,

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)a^{n-k}b^k.$$

The Binomial Theorem – Examples

Example

Expand $(x+2)^4$.

Example

- Find the coefficient of x^2y^5 in the expansion of $(2x-3y)^7$.

The ideas in the binomial theorem can be extended to trinomials and so on, such as $(a+b+c)^4$.

Example

Find the coefficient of a^2x^2c in $(a+x+c)^2(a+x+d)^3$.

Exercise

Find the coefficient of s^4t^8 in $(2s-t)^{12}$.

- A $C(12,4)$
- B $2^4C(12,8)$
- C $-2^4C(12,4)$
- D 495

Slightly harder example

- Find the coefficient of $\frac{x^4}{y^6}$ in the expansion of $\left(2x^2 - \frac{3}{xy}\right)^{11}$.

The k-th term will be

$$C(11,k)(2x^2)^{11-k}\left(\frac{-3}{xy}\right)^k = C(11,k)2^{11-k}(-3)^k x^{22-2k}y^{-k}x^{-k}$$
$$= C(11,k)2^{11-k}(-3)^k x^{22-3k}y^{-k}.$$

Therefore we must want the coefficient of the term that corresponds to $k=6$, and hence the coefficient is $C(11,6)2^5(3)^6$.

Pascal's Triangle

The binomial coefficients can be obtained without doing a lot of calculation, using a pattern named after the French mathematician Blaise Pascal (discovered much earlier by the Chinese).

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Pascal's Triangle contains $C(n,r)$.

Pascal's Triangle – Identities

Example

Use Pascal's triangle to write down (no working out) the expansion of $(x+y)^6$.

Example

Find the sum of each row of Pascal's triangle up to row 6. Guess the sum of the nth row and prove it. How does this relate to *power sets*?

We can state Pascal's triangle as an algebraic identity:

Theorem (Pascal's Triangle Identity)

$$C(n+1,k) = C(n,k-1) + C(n,k).$$

Proof of Pascal's Identity.

The number of ways to select sets containing k elements from $n+1$ objects is $C(n+1,k)$. Divide these sets (with k elements) into two groups – those that contain a particular element (say x) and those that do not contain that element.
Those with x: We need to choose $k-1$ more elements from a possible n elements. $C(n,k-1)$ ways to do this.
Those without x: We need to choose all k elements from a possible n elements (all but x). $C(n,k)$ ways to do this.
So $C(n+1,k) = C(n,k-1) + C(n,k)$ as required. □

Pascal's Identity gives us another way to prove $\sum_{k=0}^{n} C(n,k) = 2^n$. Your favourite, induction!

Pigeon Hole Principle

We recall Dirichlet's pigeon hole principle (from the second lecture on logic and proof), which is extremely useful:

If m objects (pigeons) are placed into n boxes (pigeon holes) and $m > n$, then at least one box contains more than one object. We can formalise this idea:

Theorem

(i) *If a set, S, of s elements is partitioned into n subsets, where $s > kn$, then at least one subset has more than k elements.*

(ii) *Let $a_1, a_2, \ldots, a_n$ be real numbers so that $\sum_{i=1}^{n} a_i \geq b$. Then $a_i \geq \frac{b}{n}$ for some i.*

(i) Let $S_1, S_2, \ldots, S_n$ be the partition of S into n subsets ($s > kn$). Now suppose no set S_i contains more than k elements. Then, since the S_i are disjoint

$$\begin{aligned} |S| &= \left|\bigcup S_i\right| \\ &= \sum_{i=1}^{n} |S_i| \\ &\leq kn \\ &< s, \end{aligned}$$

contradicting that $|S| = s$.

(ii) Suppose that $a_i < \frac{b}{n}$ for all i. Then

$$\begin{aligned} \sum_{i=1}^{n} a_i &< n\frac{b}{n} \\ &= b. \end{aligned}$$

contradicting that $\sum_{i=1}^{n} a_i \geq b$.

Pigeon Hole Principle – Examples

Examples

- In any group of 367 people there must be at least one pair with the same birthday.
- Three different pairs of gloves are scrambled in a drawer. How many must one select to guarantee finding a matching pair?
- If there are $n > 1$ people who shake hands with one another, then there is always a pair who will shake hands with the same number of people.
- At least one of your ancestors had parents that were related by blood. An article on this http://edge.org/response-detail/11067
- Among 79 people, at least four must have the same last initial.

Applications of the Pigeon Hole Principle

Most applications involve finding what to use as 'pigeons' and what to use as 'holes'.
Some problems require repeated applications of the pigeon hole principle.

Example

Let A be a set of 7 distinct positive integers, none of which exceeds 24. Show that if you find the sums of all possible sets of no more than 4 numbers chosen from A, the sums cannot all be different.

Solution: we use the total number of such sums as the pigeons, and the range of the sums as the holes.

Applications of the Pigeon Hole Principle

Example

Let S be a unit square. Choose any five points in the interior of S. Show that there are two points which are less than $1/\sqrt{2}$ apart.

Example

In September, October and November, a student solves at least 1 problem per day but no more than 13 problems per week. Show that she solves exactly 12 problems on some consecutive days.

Look at $P(i)$, the cumulative number of problems solved. If $P(i)$, $P(i)+12$ are all different, then we have $91 \times 2 = 182$ distinct positive integers, the largest being at least 182, so $P(91) \geq 182 - 12 = 170$. But there are 13 weeks, so can't solve more than 169 problems.

Example

If $n+1$ numbers are selected from the set $\{1, 2, \ldots, 2n\}$, then one will divide another evenly.

Proof: Let us write each of these $n+1$ selected numbers as a power of two times an odd part, for example,

$$12 = 2^2 \cdot 3, 18 = 2^1 \cdot 9, 23 = 2^0 \cdot 23$$

There are only n different odd parts possible in the numbers from 1 to $2n$. Since we have selected $(n+1)$ numbers, at least two of them have the same odd parts. The smaller power of two will divide the larger evenly, and hence these two numbers satisfy the theorem. This theorem is a "best possible" result because if we relax the conditions of the hypothesis even slightly, the theorem is no longer true. For instance, if we took only n numbers from 1 to $2n$, we might not have a pair that divide evenly.

The Principle of Inclusion-Exclusion

We have seen a simple version of this principle:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example

How many integers in the range 1 to 600 are divisible by 3 or 5?

- A 200
- B 320
- C 120
- D 280

For three sets, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Can you find a similar expression for 4 sets?

The Principle of Inclusion-Exclusion

Example

Among 300 people, 160 like Coke, 150 like Pepsi, 120 like Sprite, 80 like Coke & Pepsi, 66 like Coke & Sprite, 45 like Pepsi & Sprite, and 12 like all three. How many people like none?

In general,

Theorem (Inclusion-Exclusion)

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| = &\sum |A_i| - \sum |A_i \cap A_j| \\ &+ \sum |A_i \cap A_j \cap A_k| - \cdots \\ &- (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|. \end{aligned}$$

Derangements

Example

At a restaurant, 3 men check their hats. In how many ways can the hats be returned so that no man gets his own hat?

- A 1
- B 2
- C 3
- D 4

Derangements

What about for 7 men?

Definition

A derangement is a permutation in which no object appears in its original position.

Example

List all the derangements of $[1, 2, 3, 4]$.

Derangements

Theorem

The number of derangements of n objects is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

Derangements – Applications

A recursive formula for calculating derangements is

$$D_n = (n-1)[D_{n-1} + D_{n-2}].$$

Returning to our hats example. The total number of ways in which none of the 7 men receive his own hat is:

$$D_7 = 7!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^7 \frac{1}{7!}\right] = 1854.$$

Example

For 7 men, in how many ways can their hats be returned so that

- at least one man receives his own hat?

Exercise

For 7 men, in how many ways can their hats be returned so that at least two men receive their own hats?

- A $7!$
- B $7! - 7D_6 - D_7$
- C $7! - 7D_6$
- D $7! - D_7$

Textbook exercises

Exercises Section 6.3:

- 36-45

Exercises Section 6.7:

- 1-4, 13, 15, 16, 17, 22, 28

Exercises Section 6.1:

- 92-99