Prop. $A \in O(n) \Rightarrow A^{-1} \in O(n)$

Proof. $\| A^T x \| = \sqrt{A^T x \cdot A^T x} = \sqrt{x \cdot A^{TT} A^T x} = \sqrt{x \cdot \underbrace{AA^T}_{A \in O(n)} x} = \sqrt{x \cdot x} = \|x\|$

$\Rightarrow A^T \in O(n) \Rightarrow A^{-1} = A^T \in O(n)$ □

Prop. $A, B \in O(n) \Rightarrow A \cdot B \in O(n)$

Proof. $\|ABx\| = \sqrt{x \cdot (AB)^T (AB) x} = \sqrt{x \; B^T \underbrace{A^T A}_{= Id} B x} = \sqrt{x \; \underbrace{B^T B}_{= Id} x} = \|x\|$ □

Prop. $A \in O(n) \Rightarrow |\det A| = 1$

Proof. $A \cdot A^T = Id$ (because $A \in O(n)$)

$\Rightarrow \det(A \cdot A^T) = 1$

$\Rightarrow \det A \cdot \det A^T = 1$

$\Rightarrow \det A \cdot \det A = 1$

$\Rightarrow |\det A| = 1$ □

Prop. $A, B \in GL_n(\mathbb{R}) \Rightarrow (AB)^{-1} = B^{-1} A^{-1}$

Proof. $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = A \, Id \, A^{-1} = AA^{-1} = Id$

$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1} \, Id \, B = B^{-1}B = Id$ □

Prop. $(A^T)^{-1} = (A^{-1})^T$

Proof. $A^T (A^{-1})^T = (A^{-1}A)^T = Id^T = Id$

$(A^{-1})^T A^T = (AA^{-1})^T = Id^T = Id$ □