MATH1510 - Discrete Mathematics
Logic 1

University of Newcastle

UoN

Introduction

- Mathematics is an area in which absolute truth can be established. We will learn methods of correct reasoning ('logic') in order to prove mathematical statements.
- Proofs are obtained by deductive reasoning, rather than by observing a number of cases. Proofs make guesses into *facts*.
- We use symbols to simplify complex combinations of statements, and develop an algebra to help us decide valid reasoning.
- We will learn proof by contradiction.

Which not a proposition?

Definition

A proposition is a statement that is either TRUE or FALSE. We refer to TRUE or FALSE as the truth value of the statement.

A $1000 > 5$.

B There are more than 6 billion people on earth.

C There is life in the oceans of Titan.

D Why do the buses always run late?

Propositional Functions

Is the following statement a proposition?

$$x^2 + 3x + 2 = 0.$$

Definitions

Let $P(x)$ be a statement involving a variable x, and let D be some set. If $P(x)$ is a proposition for every $x \in D$, then we call $P(x)$ a predicate, or propositional function.

D is called the domain of discourse or just the domain.

Compound Propositions

Even if you find it hard to decide which, the important thing is that a proposition has a truth value.

We use variables such as p, q and r to represent propositions. To define p to be the proposition "$3 < 5$", we write

$$p : 3 < 5.$$

We will look at how to combine propositions and determine the truth value of the combination.

Definition

A compound proposition is a proposition formed by combining simpler ones.

Binary Operations on Propositions

We deduce the truth value of the compound proposition from the truth values of its components according to some rules.

All compound proposition can be made from simpler ones using 3 operations:

- Conjunction ("and")
- Disjunction ("or")
- Negation ("not")

Conjunction

Definition

The conjunction of p and q is the proposition "p AND q", which is written as

$$p \wedge q.$$

$p \wedge q$ is true whenever both p and q are true. It is false otherwise.

If

$$p : 1 + 3 = 4$$
$$q : 10 < 3$$

then $p \wedge q$ is the proposition "$1 + 3 = 4$ AND $10 < 3$", which is false.

Disjunction

Definition

The disjunction of p and q is the proposition "p OR q", which is written as

$$p \vee q.$$

$p \vee q$ is true whenever p is true or q is true, or both. It is false only if both p and q are false. This is the inclusive sense of the word "or".

If

p : it is Friday

q : it is daytime

then $p \vee q$ is the proposition "it is Friday OR it is daytime".

Negation

Definition

The negation of p is the proposition “NOT p”, which is written as

$$\neg p.$$

$\neg p$ is true whenever p is false and is false whenever p is true.

If

$$p : \text{it is hot}$$

then $\neg p$ is the proposition “it is not hot”.

Inclusive and Exclusive or

In maths “or” is inclusive unless stated otherwise. Occasionally we may use “exclusive or”, where both cannot be true at the same time. We denote this by XOR ($\oplus$).
Express XOR in terms of the other 3 operations.

The similarity between sets and symbolic logic

Note the similarity in notation between $\wedge$ and $\cap$, $\vee$ and $\cup$, and $\neg$ and $\bar{\ }$.
They are similar in more than notation!

Suppose $P(x)$ and $Q(x)$ are propositional functions and D is their domain. Let $P = \{x \in D : P(x) \text{ is true}\}$ and $Q = \{x \in D : Q(x) \text{ is true}\}$. Draw a venn diagram of this situation.

What do union, intersection, complement and symmetric difference correspond to?

OR, AND, NOT, XOR

Determine the truth value of the statement $(-1 < 16) \wedge (3 > 4)$

- A TRUE.
- B FALSE.
- C Not sure.

Determine the truth value of the statement $(-1 < 16) \vee (3 > 4)$

- A TRUE.
- B FALSE.
- C Not sure.

Determine the truth value of the statement $\neg\big((-1 < 16) \vee (3 > 4)\big)$

- A TRUE.
- B FALSE.
- C Not sure.

Let A be the statement "the earth is only 6000 years old" and B be the statement "everything you read is true". Determine:

- $A \vee \neg B$
- $A \wedge \neg B$
- $\neg A \wedge \neg B$

The Truth Table for Conjunction

A truth table lists the truth values of a compound proposition, given the truth values of its components.

The truth table for $p \wedge q$ is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The Truth Table for Disjunction and Negation

The truth table for $p \vee q$ is

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The truth table for $\neg p$ is

p	$\neg p$
T	F
F	T

The Truth Table for XOR

The truth table for p XOR q is

A

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

B

p	q	$p \oplus q$
T	T	T
T	F	T
F	T	T
F	F	F

Evaluating Compound Propositions

- What is the truth value of $(p \vee q) \wedge r$, when p is T, q is T, r is F?
- What is the truth value of $p \vee (q \wedge r)$, when p is T, q is T, r is F?
- What is the truth value of $(p \vee \neg q) \wedge \neg(q \wedge r)$ when p is T, q is F, r is T?

Using Truth Tables

Construct the truth table for $\neg p \vee q$

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Construct the truth tables for

- $(p \vee w) \wedge (p \wedge q)$
- $(p \vee q) \wedge r$

Order of Operations

In the absence of brackets, the logical operations should be evaluated in the order $\neg$, $\wedge$, $\vee$.

$$p \vee \neg q \wedge r$$

is the same as

$$p \vee ((\neg q) \wedge r).$$

This is analogous to the order of operations in arithmetic where $\times$ is evaluated before $+$.

Logical Equivalence

Definition

Two propositions p and q are logically equivalent, denoted $p \equiv q$, if they have the same truth tables.

- double negative: $\neg(\neg p) \equiv p$
- de Morgan's Law: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Check that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Conditional Propositions

Definition

A conditional proposition is a proposition of the form "if p then q". We write this as

$$p \to q$$

To find the truth value of $p \to q$, we need the truth value of the statement "whenever p is true, then q is also true".
The statement "if it is raining, then it is cloudy" is a conditional proposition.

If it is not raining then this proposition is automatically true, regardless of whether it is cloudy or not. In this case we say the proposition is *vacuously true*. The only time this proposition is false is if it is raining but not cloudy.

The Truth Table for Conditional Propositions

$p \to q$ is defined by its truth table.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Let p : an integer $n > 1$ is a perfect square, and q : an integer n is not a prime. Regardless of what n you start will $p \to q$ is true.
Is $p \to q$ logically equivalent to $p \vee \neg q$? What about to $\neg p \to \neg q$?

Conditional Propositions and Causation

It is important to note that the conditional proposition does not imply causation.

- if $2 < 3$ then the sky is blue (true)
- if $4 < 3$ then the sky is blue (true)
- if $4 < 3$ then the sky is brown (true)
- if $2 < 3$ then the sky is brown (false)

In terms of sets, $p \to q$ can be represented by $\overline{p - q}$.

Equivalence For Conditional Propositions

Equivalent ways to express the conditional proposition: $p \to q$

- p implies q
- q, if p
- p, only if q
- q is necessary for p
- p is sufficient for q

If p is true, q is false and r is true, what is the truth value of $(p \vee q) \to (q \wedge r)$?

A TRUE

B FALSE

Contrapositive and Converse

Definition

The contrapositive of the statement $p \to q$ is the statement $\neg q \to \neg p$.

Definition

The converse of the statement $p \to q$ is the statement $q \to p$.

Note: the converse of a true statement is not always true, though the contrapositive always is.

Show that $p \to q$ is logically equivalent to $\neg q \to \neg p$.

Find the truth value of 'If pigs can fly then $2 + 2 = 4$'.

A TRUE

B FALSE

Find the truth value of 'If pigs can fly then $2 + 2 = 5$'.

A TRUE

B FALSE

Find the truth value of 'If pigs can't fly then $2 + 2 = 4$'.

A TRUE

B FALSE

Find the truth value of 'If pigs can't fly then $2 + 2 = 5$'.

A TRUE

B FALSE

Write the contrapositive and converse of the proposition "all swans are black".
Using truth tables, show that $p \to q$, $\neg(p \to q)$, $p \to \neg q$, $\neg p \to q$ are all different.

Biconditional Proposition

Definition

The biconditional proposition, denoted $p \leftrightarrow q$, is defined as

$$(p \to q) \wedge (q \to p)$$

Important: this is usually read as "p if and only if q", "p is equivalent to q", or "p is necessary and sufficient for q".

For p: the triangle is equilateral,
and q: the triangle has 3 equal angles,
$p \leftrightarrow q$.

The Truth Table for Biconditional Propositions

$p \leftrightarrow q$ can be defined by its truth table.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note that $p \leftrightarrow q$ is true only when p and q have the same truth value (that is, when $p \equiv q$).

Represent $p \leftrightarrow q$ using a Venn diagram, and show that it is the complement of $p \oplus q$.

Tautologies and Contradictions

Definition

A tautology is a compound proposition that is true in all cases.

- $p \vee \neg p$ is a tautology (law of the excluded middle).
- $(p \wedge q) \to p$ is a tautology.

Definition

A contradiction is a compound proposition that is false in all cases.

- $p \wedge \neg p$ is a contradiction.
- $(p \wedge q) \wedge \neg(p \vee q)$ is a contradiction.

Textbook exercises

Exercises Section 1.2:

- 1, 2, 7, 9, 10, 12, 13, 19, 23, 30, 32, 36, 45

Exercises Section 1.3:

- 10, 16, 17, 21, 24, 27, 41, 52, 67, 77