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The UNIVERSITY of NEWCASTLE

Comp3320/6370 Computer Graphics

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Semester 2, 2018

LECTURE w06

Transforms II

September 9, 2018

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Motivation

The course so far has repeated/refreshed some basic material about vectors from high school and the introductory maths courses (vectors, basis/coordinates, dot product, cross product, determinants, matrices, matrix inversion, etc.).

We also had a look at computer graphics specific homogeneous coordinates and some fundamental transforms (rotation, translation, shearing and scaling) that are used by OpenGL.

Speed is a major concern in CG. In the present lecture we want to understand how and when we/OpenGL can make some algorithms faster by using a very fast way of matrix inversion.

We also want to understand how we can switch between different coordinate systems.

These two details allow us to obtain a faster way to rotate about an arbitrary axis and to scale in direction of an arbitrary vector.

Basic linear algebra

We know by now:

- What is the difference between the dot product and the cross product?
- How to calculate the norm or length of a vector.
- Projection of one vector onto another.
- What is an *Orthonormal Basis* (ONB)? Given one vector, how can we generate an ONB?
- Matrix product for square matrices and for general rectangular matrices.
- How to calculate the inverse M^{-1} of a matrix M.

Hint: Use the Exercise I and II sheets and also look at your maths books or the material available on the internet.

Transforms I

Recall what are

- Homogeneous coordinates
- Translations: points, vectors
- Rotations
- Euler transform
- Scaling
- Shearing
- Rigit-body transforms

Exercise 13: Additivity of Rotations in 2D

Show that for $\alpha, \beta \in [0, 2\pi]$

$$\mathbf{R}(\alpha)\mathbf{R}(\beta) = \mathbf{R}(\alpha + \beta)$$

Compare with the rule for multiplying two complex numbers $ae^{i\alpha}$ and $be^{i\beta}$ for the special case that $a = b = 1$.

Exercise 14: Rotations in 3D: non-commutativity

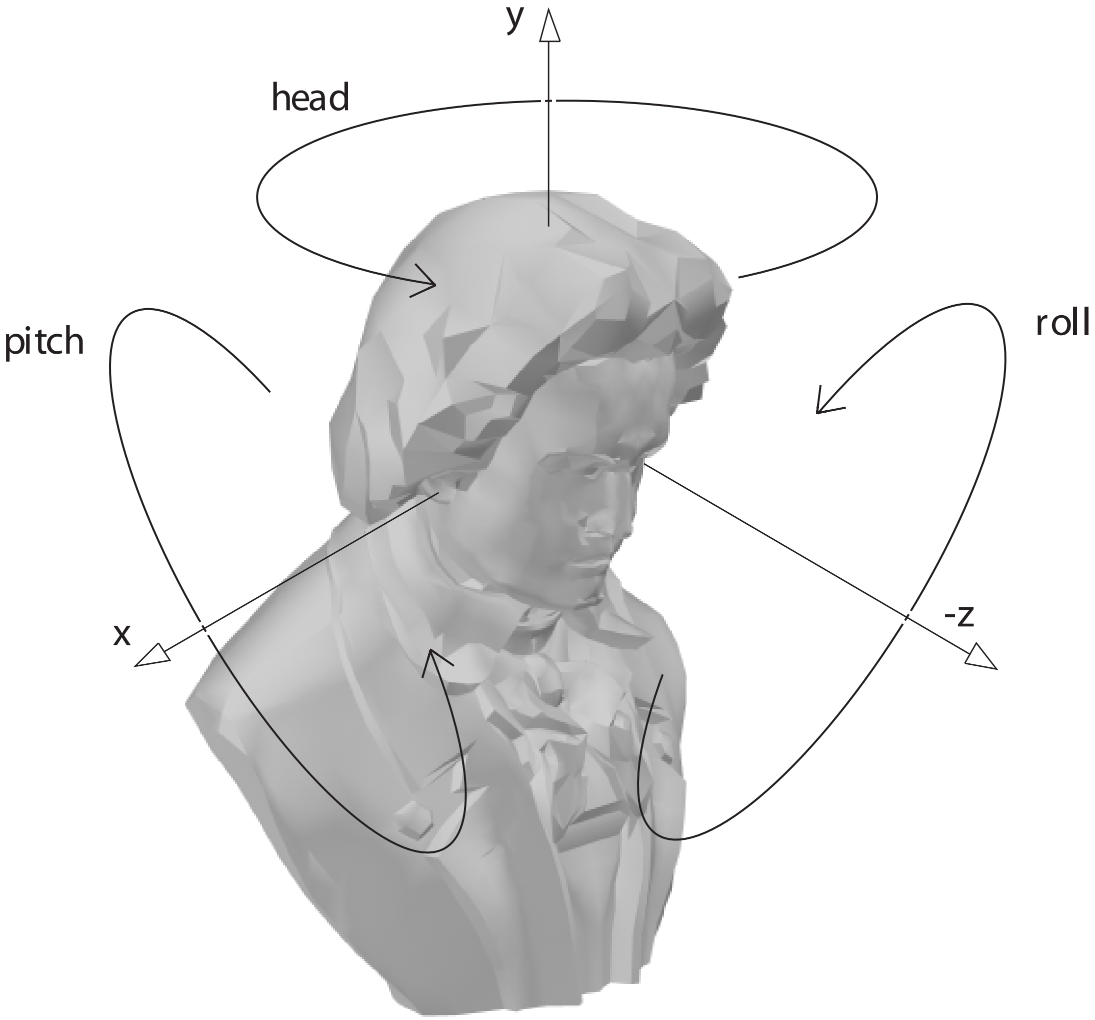
Use rotations in 3D with homogeneous coordinates and calculate the matrix for an x-roll of 30°, followed by an y-roll of 45°, followed by a z-roll of 60°.

First calculate each of the the matrices $R_x(30^\circ)$, $R_y(45^\circ)$, $R_z(60^\circ)$ and the product $R_z(60^\circ) \cdot R_y(45^\circ) \cdot R_x(30^\circ)$.

Is it different from $R_x(30^\circ) \cdot R_y(45^\circ) \cdot R_z(60^\circ)$?

Which is the correct order and why?

Euler Angles



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Exercise 15: The inverse of some transforms

Let $\mathbf{F} \in M(4 \times 4, \mathbf{R})$ have the general form of a 3D transform in homogeneous coordinates. That is, if the submatrix $\mathbf{M} \in M(3 \times 3, \mathbf{R})$ is either a 3D rotation, scaling or shearing matrix and the submatrix $\mathbf{T} \in M(3 \times 1, \mathbf{R})$ is a translation vector then $\mathbf{F}$ can be written as:

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{T} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccc|c} m_{00} & m_{01} & m_{02} & t_x \\ m_{10} & m_{11} & m_{12} & t_y \\ m_{20} & m_{21} & m_{22} & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Let $\mathbf{N} = \mathbf{M}^{-1}$. Show that the inverse of $\mathbf{F}$ is given by:

$$\mathbf{F}^{-1} = \left[\begin{array}{c|c} \mathbf{N} & -\mathbf{NT} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccc|c} n_{00} & n_{01} & n_{02} & -(\mathbf{NT})_x \\ n_{10} & n_{11} & n_{12} & -(\mathbf{NT})_y \\ n_{20} & n_{21} & n_{22} & -(\mathbf{NT})_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

The Orthogonal Group $O(n)$

DEF. $GL_n(\mathbf{R}) = \{A \in M(n \times n, \mathbf{R}); det(A) \neq 0\}$ is called the *General Linear Group*. It is the group of invertible $n \times n$ matrices with real coefficients.
$O(n) := \{A \in GL_n(\mathbf{R});\ A \cdot A^T \ = \ A^T \cdot A = Id\}$ is called the *Orthogonal Group of* $(n \times n)$*–matrices*.

Proposition: Let $A \in GL_n(\mathbf{R})$ be an invertible $n \times n$ matrix with real coefficients. The following three statements are equi-valent:

(i) $A^{-1} = A^T$ (i.e. A is an element of $O(n)$)

(ii) $\forall\, x \ \in \mathbf{R}^n \quad \|Ax\| = \|x\|$ (preserves lengths)

(iii) $\forall\, x\ , y \in \mathbf{R}^n \quad Ax \cdot Ay = x \cdot y$ (preserves angles)

Proof. Homework exercise 16.

The Special Orthogonal Group $SO(n)$

DEF. $SO(n) := \{A \in O(n);\ det(A) = 1\}$ is called the *Special Orthogonal Group of* $(n \times n)$*–matrices*.

Exercise 17. Describe in geometrical terms what is the difference between the elements of $O(n)$ and $SO(n)$?

Some properties of $O(n)$

Notes.

1. $A \in O(n) \implies A^{-1} \in O(n)$
2. $A, B \in O(n) \implies A \cdot B \in O(n)$
3. $A \in O(n) \implies |detA| = 1$

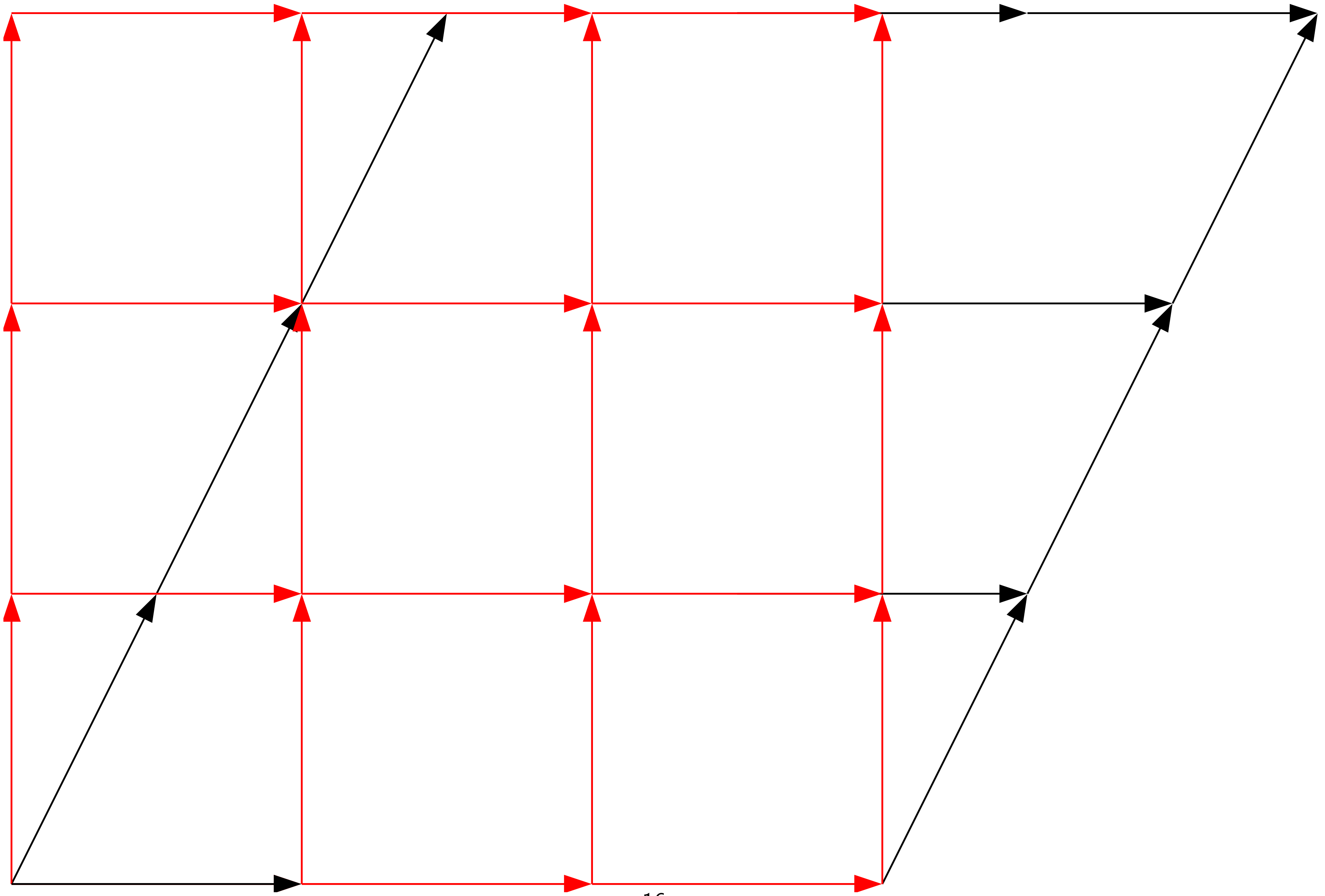
Lemma. If the vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ form an *orthonormal* set (i.e. each of them has length 1 and they are pairwise perpendicular/orthogonal) then the $(n \times n)$-matrix formed by setting the j-th column equal to $\mathbf{v}_j$ for all $1 \leq j \leq n$ is *orthogonal*, i.e. an element of $O(n)$.

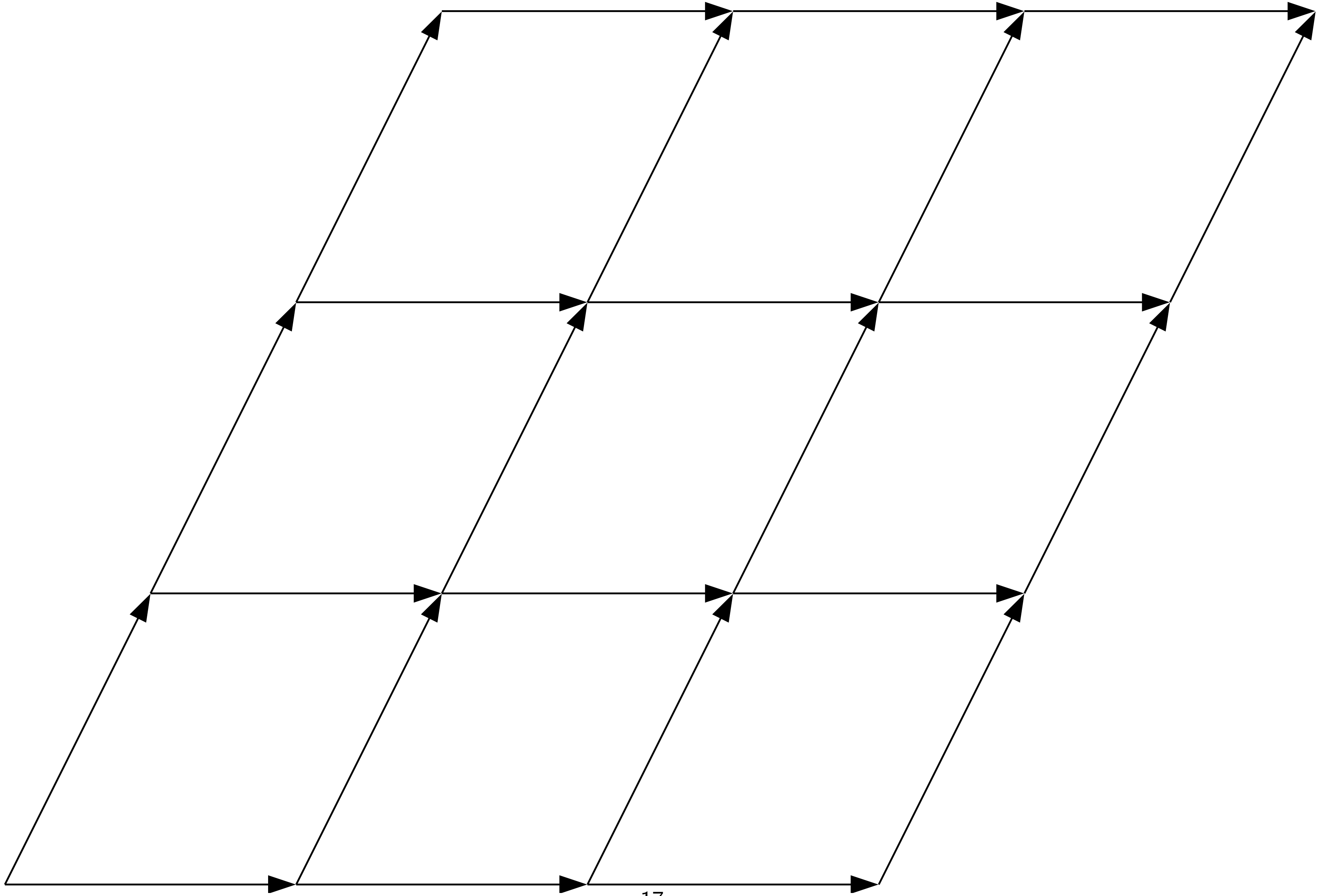
Why is the lemma useful and how can we be sure it is true? (Exercise 18)

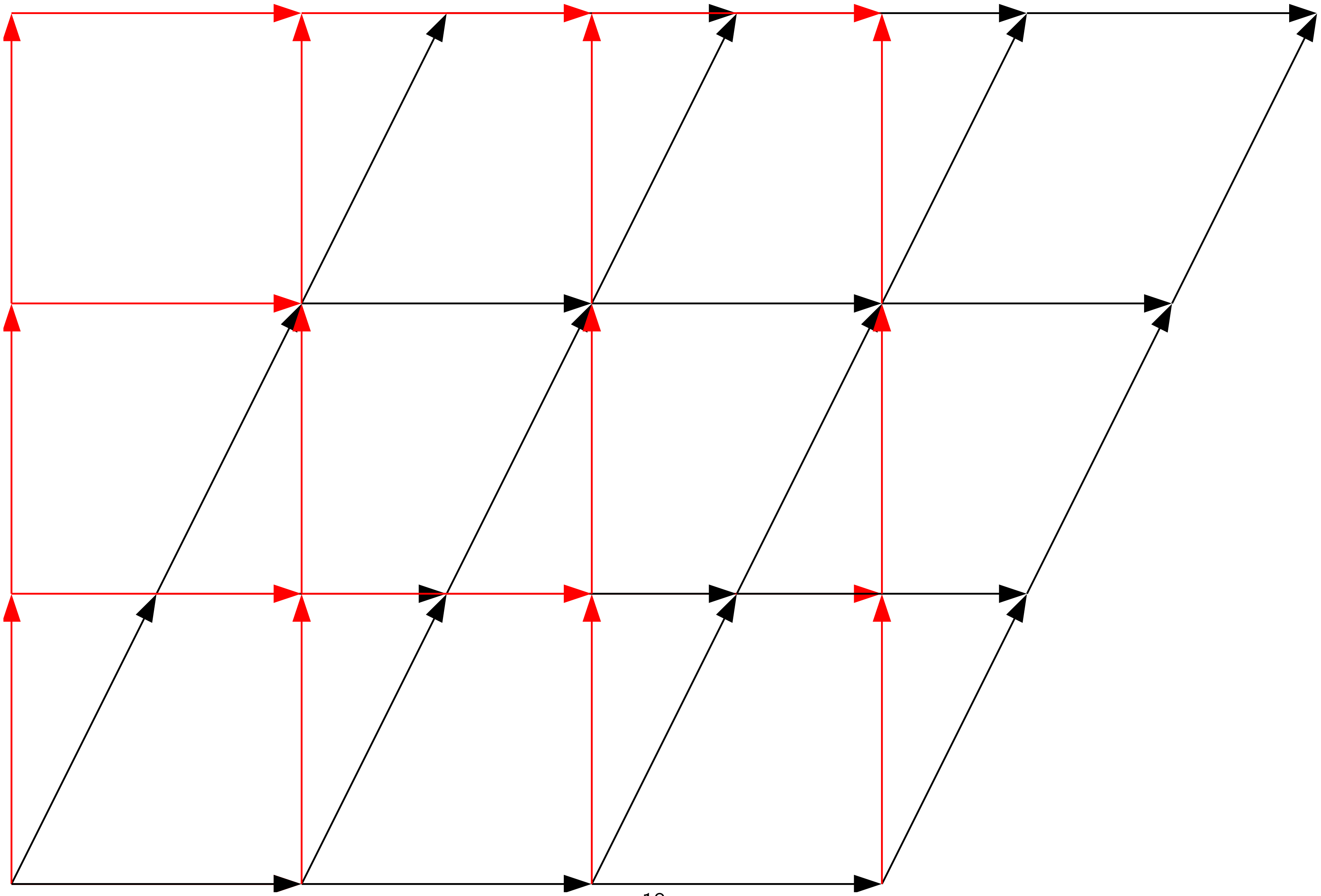
Transforms II

Today look into

- Change of basis
- Rotation about an arbitrary axis







Change of basis

Let $\mathbf{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the standard basis, and $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ be another basis of $\mathbf{R}^3$.

Let $\mathbf{v} \in \mathbf{R}^3$ be a vector. Let $[\mathbf{v}]_{\mathbf{E}}$ be $\mathbf{v}$ expressed in terms of $\mathbf{E}$ (note that $[\mathbf{v}]_{\mathbf{E}} = \mathbf{v}$) and let $[\mathbf{v}]_{\mathbf{F}}$ be $\mathbf{v}$ expressed in terms of $\mathbf{F}$.

Let $[\mathbf{Id}]_{\mathbf{E}}^{\mathbf{F}}$ be the change of basis matrix from $\mathbf{F}$ to $\mathbf{E}$ i.e.

$$[\mathbf{v}]_{\mathbf{E}} = [\mathbf{Id}]_{\mathbf{E}}^{\mathbf{F}} \cdot [\mathbf{v}]_{\mathbf{F}} \tag{1}$$

Note 1. The first basis element of $\mathbf{F}$ is $\mathbf{f}_1$ and therefore $[\mathbf{f}_1]_{\mathbf{F}} = (1\ 0\ 0)^T$. If we take as change of basis matrix the matrix with the basis vectors $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ as columns, i.e.,

$$[\mathbf{Id}]_{\mathbf{E}}^{\mathbf{F}} = [\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3] \tag{2}$$

then we obtain

$$\mathbf{f_1} = [\mathbf{f_1}]_{\mathbf{E}} = [\mathbf{Id}]_{\mathbf{E}}^{\mathbf{F}} \cdot [\mathbf{f_1}]_{\mathbf{F}} = [\mathbf{Id}]_{\mathbf{E}}^{\mathbf{F}} \cdot (1\ 0\ 0)^T = [\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3] \cdot (1\ 0\ 0)^T \tag{3}$$

The same works for $\mathbf{f}_2$ and $\mathbf{f}_3$.

$$\begin{tikzcd}
{[\mathbf{v}]_{\mathbf{G}}} \arrow[r, "{[\mathbf{Id}]_{\mathbf{F}}^{\mathbf{G}}}"] \arrow[d, "{[\mathbf{Id}]_{\mathbf{E}}^{\mathbf{G}}}"'] & {[\mathbf{v}]_{\mathbf{F}}} \arrow[d, "{[\mathbf{Id}]_{\mathbf{E}}^{\mathbf{F}}}"] \\
{[\mathbf{v}]_{\mathbf{E}}} \arrow[r, "{[\mathbf{Id}]_{\mathbf{E}}^{\mathbf{E}}}"'] & {[\mathbf{v}]_{\mathbf{E}}}
\end{tikzcd}$$

Change of basis, cont.

Note 2. The change of basis in the opposite direction from $\mathbf{E}$ to $\mathbf{F}$ is given by

$$[\mathbf{e}_i]_{\mathbf{F}} = [\mathbf{Id}]^{\mathbf{E}}_{\mathbf{F}} \cdot [\mathbf{e}_i]_{\mathbf{E}} \tag{4}$$

where $[\mathbf{Id}]^{\mathbf{E}}_{\mathbf{F}} = ([\mathbf{Id}]^{\mathbf{F}}_{\mathbf{E}})^{-1}$ is applied to $\mathbf{e}_i$, $i = 1, 2, 3$.

Note 3. Let $\mathbf{G} = (\mathbf{g_1}, \mathbf{g_2}, \mathbf{g_3})$ be a third basis of $\mathbf{R}^3$. Then the change of basis matrix from $\mathbf{G}$ to $\mathbf{F}$ can be obtained in two steps via transforming into and from the standard basis $\mathbf{E}$ as follows

$$[\mathbf{Id}]^{\mathbf{G}}_{\mathbf{F}} = [\mathbf{Id}]^{\mathbf{E}}_{\mathbf{F}} \cdot [\mathbf{Id}]^{\mathbf{G}}_{\mathbf{E}} = ([\mathbf{Id}]^{\mathbf{F}}_{\mathbf{E}})^{-1} \cdot [\mathbf{Id}]^{\mathbf{G}}_{\mathbf{E}} = [\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3]^{-1} \cdot [\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3] \tag{5}$$

Comment 1. If $[\mathbf{Id}]^{\mathbf{F}}_{\mathbf{E}} \in O(3)$ i.e. it is an orthogonal matrix then $([\mathbf{Id}]^{\mathbf{F}}_{\mathbf{E}})^{-1} = ([\mathbf{Id}]^{\mathbf{F}}_{\mathbf{E}})^T$.

Comment 2 (Some inconsistency in mathematical terminology)**.** If $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ consists of orthogonal (i.e. pairwise perpendicular) vectors then this is not sufficient for the matrix $[\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3]$ to be in $O(3)$. $\mathbf{F}$ must be an ONB (orthonormal basis), i.e. $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ must be pairwise orthogonal vectors of unit length. Then $[\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3] \in O(3)$ and $[\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3]^{-1} = [\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3]^T$.

Exercise 19: Change of basis

Let $\mathbf{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the standard basis and let

$\mathbf{F} = ((1\ 2\ 2)^T, (0\ 1\ 2)^T, (1\ 0\ 2)^T)$ and

$\mathbf{G} = ((1\ 2\ 3)^T, (1\ 1\ 3)^T, (5\ 2\ 1)^T)$ be

other bases of $\mathbf{R}^3$.

Determine:

1. $[\mathbf{Id}]^{\mathbf{F}}_{\mathbf{E}}$
2. $[\mathbf{Id}]^{\mathbf{E}}_{\mathbf{F}}$
3. $[\mathbf{Id}]^{\mathbf{G}}_{\mathbf{F}}$

Rotation about an arbitrary axis

Let a rotation axis be given by a normalised vector $\mathbf{r}$. What is the transform that rotates some entity by an angle α about the axis defined by vector $\mathbf{r}$?

Step 1: Find ONB. The first axis is $\mathbf{r}$ (make sure it is of length 1). Find the second axis $\mathbf{s}$ such that it is orthogonal to $\mathbf{r}$ and of unit length. An orthogonal vector to $\mathbf{r}$ can be found by setting the numerically smallest component of $\mathbf{r}$ to zero, then swapping the remaining two components, and negating the first of these. $\mathbf{s}$ is obtained by normalising the result. Then $\mathbf{t} = \mathbf{r} \times \mathbf{s}$ is the third axis. Now $\mathbf{F} = (\mathbf{r}, \mathbf{s}, \mathbf{t})$ is an ONB.

Rotation about an arbitrary axis, cont.

Step 2: Basis transform. The matrix

$$[\mathbf{Id}]_{\mathbf{E}}^{\mathbf{F}} = \left[\, \mathbf{r} \;\; \mathbf{s} \;\; \mathbf{t} \,\right] \tag{6}$$

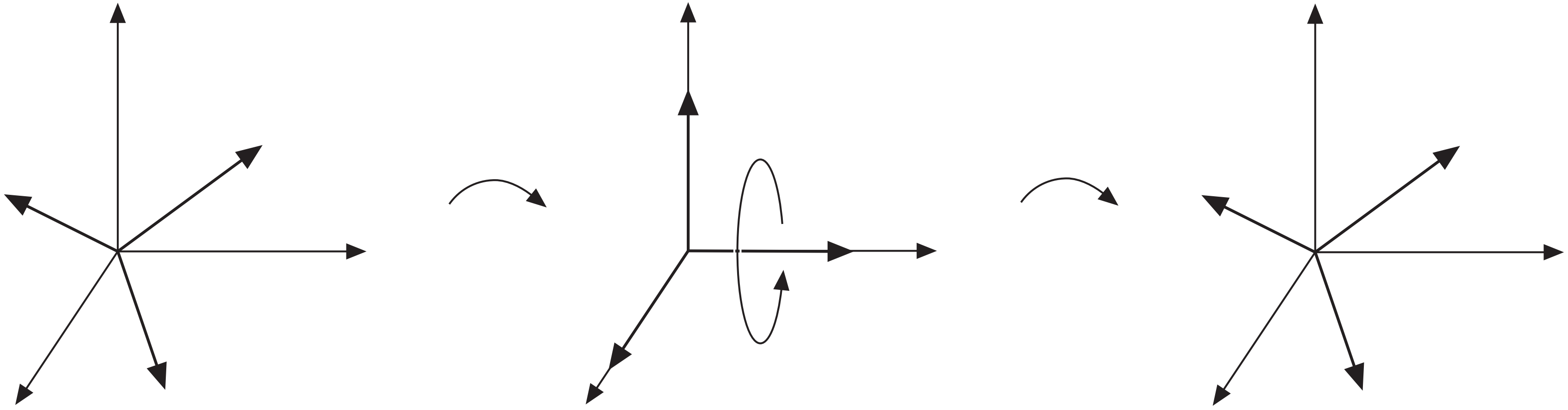
is an element of $O(3)$ and is the transform from $\mathbf{F}$ into the standard basis $\mathbf{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Hence the inverse is $M = ([\mathbf{Id}]_{\mathbf{E}}^{\mathbf{F}})^T$ and it transforms $\mathbf{r}$ into $\mathbf{e}_1$, $\mathbf{s}$ into $\mathbf{e}_2$, and $\mathbf{t}$ into $\mathbf{e}_3$.

Rotation about an arbitrary axis, cont.

Step 3: Concatenation of transforms. The complete transform is given by

$$X = M^T \cdot \mathbf{R}_x(\alpha) \cdot M = \begin{bmatrix} \mathbf{r} & \mathbf{s} & \mathbf{t} \end{bmatrix} \cdot \mathbf{R}_x(\alpha) \cdot \begin{bmatrix} \mathbf{r} & \mathbf{s} & \mathbf{t} \end{bmatrix}^T \tag{7}$$

It transforms the bases such that $\mathbf{r}$ maps onto the x-axis, then it rotates by α about the x-axis, and finally it reverses the initial basis transform.



Exercise 20: Rotation about an arbitrary axis

What is the transform that rotates some entity by angle $\alpha = \frac{\pi}{2}$ about the axis defined by vector $\mathbf{r} = (1\ 2\ 2)^T$?

Exercise 21: Scaling in direction of an arbitrary vector

We would like to scale an entity by factor 2 in direction of a given vector $\vec{r}$. What is the transform ?

Answer.

Step 1: Create an ONB $(f_x = \frac{\vec{r}}{||\vec{r}||}, f_y = \frac{f_x^{\perp}}{||f_x^{\perp}||}, f_z = f_x \times f_y)$.

Step 2: $X = [f_x\ f_y\ f_z] \cdot S(2,1,1) \cdot [f_x\ f_y\ f_z]^T$

$$= [f_x\ f_y\ f_z] \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot [f_x\ f_y\ f_z]^T$$

Suggestion:

Understand this exercise because it sums it all up in a simple way.

Then note the similarities between the approaches in exercises 20 and 21 and check all the details in the previous slides.

LITERATURE

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