The University of Newcastle
School of Information and Physical Sciences

COMP2230/COMP6230 Algorithms

Tutorial 10 Solutions

7 - 8 October 2021

Tutorial

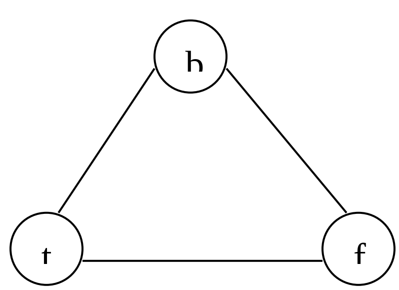
1. Prove that the graph 3-clolourability is NP-complete.

Solution:
To prove that a problem A is NP-complete we need to:
a) prove that it is in NP, and
b) we need to provide a polynomial time reduction from some known NP-complete problem to problem A. That shows that every problem in NP can be reduced to A in polynomial time.

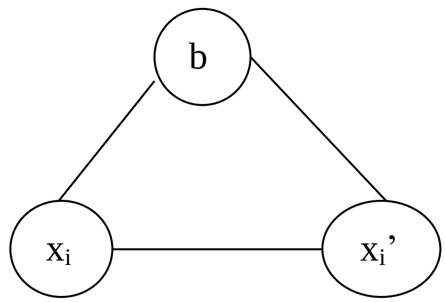
a) 3-colourability problem: Given graph G, is there a proper colouring of G with 3 colours, where a proper colouring assigns a colour to each vertex of the graph such that no two adjacent vertices are coloured with a same colour. Given a 3-colouring of a graph, it is easy to check whether it is a proper colouring by checking the colours of the two end vertices of each edge. This can be done in polynomial time (O(|E|)). Thus 3-colourability is in NP.

b) Reduction from 3SAT. Here we have to provide 4 things:
1. Construct the reduction.
2. Show that the reduction can be performed in polynomial time.
3. Show that if an instance of 3SAT is satisfiable, the graph is 3-colourable (IF).
4. Show that if an instance of 3SAT is not satisfiable, the graph is not 3-colourable (ONLY IF). Alternatively, you can show that if the graph is 3-colourable than the 3SAT instance is satisfiable.

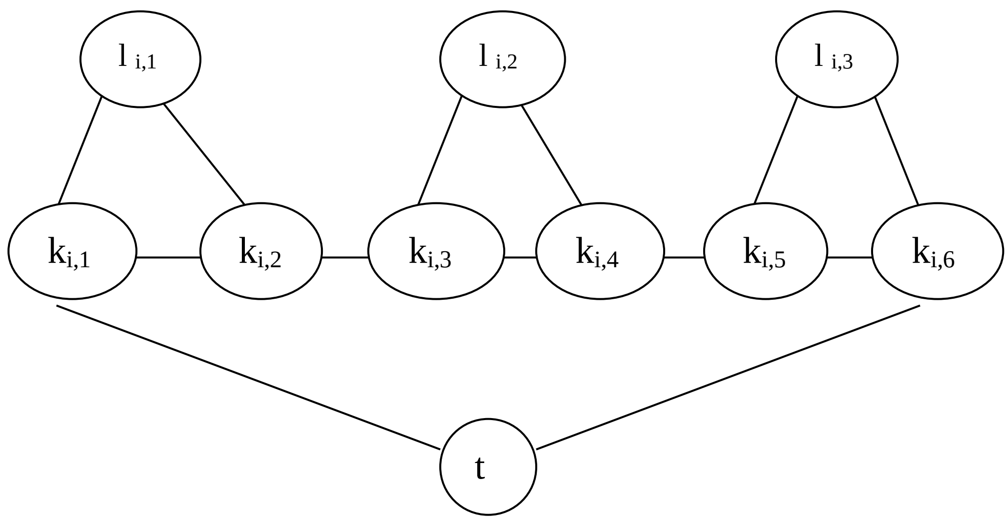
1. Reduction from 3SAT can be classified as component design. Start from an arbitrary instance of 3SAT and construct a graph in a following way.
   i. First construct a triangle with vertices t, f and b. Since this is a triangle we need 3 colours for proper colouring. Without loss of generality we use colours T, F and B to colour vertices t, f and b respectively. Note that colours T and F are chosen to correspond to TRUE and FALSE, respectively.



ii. Then for each variable x_i construct a triangle x_i , x_i' and b. Note that this triangle ensures that each variable is coloured either T or F and that if a variable is T, the negated variable is F, and vice versa.



iii. Then for each clause i with literals $l_{i,1}$, $l_{i,2}$ and $l_{i,3}$ construct the following gadget:



2. We next show that the above reduction can be performed in polynomial time, that is, that the instance of 3-colourability can be constructed in polynomial time, starting from the instance of 3SAT. Indeed that is the case: first we construct 3 vertices (t,b,f) and 3 edges among them; then for each variable we need to add 2 vertices and 3 edges, which is 2n+3 vertices and 3n+3 edges in total; finally, for each clause we add 6 vertices and 13 edges, which brings the total number of vertices to 2n+6m+3 and the total number of edges to 3n+13m+3, where n is the number of variables and m the number of clauses in the instance of 3SAT.

3. We show that the above gadget can be properly coloured with colours T, F and B if and only if the corresponding clause has at list one TRUE literal.
<u>IF:</u> Suppose that the corresponding clause has at least one true literal. There are 5

cases: TFF, FTF, TTF, TFT, TTT. It is easy to show that in each of these cases there is a proper 3-colouring of the gadget (left as an exercise).

4. <u>ONLY IF:</u> Suppose that all literals in the corresponding clause are FALSE. Then either $k_{i,3}$ or $k_{i,4}$ are T. Without loss of generality, suppose that $k_{i,3}$ is T. Then vertex $k_{i,2}$ is B and $k_{i,1}$ is T, which is not a proper colouring since $k_{i,1}$ is connected to vertex t which is coloured T.

More Exercises

2. Show that determining whether a graph G has a vertex cover of size at most k is NP-complete.

Solution idea:
Reduction from Independent set. Start from an arbitrary instance of Independent set, which comprises a graph G and an integer k and show that G has a vertex cover of size at most n-k (where n is the number of vertices in G) if and only if G has an Independent set of size at least k.

3. Show that determining whether a graph G has a clique of size at least k is NP-complete.

Solution idea:
Reduction from Independent set. Start from an arbitrary instance of Independent set, which comprises a graph G and an integer k and show that the complement G' of G has a clique of size at least k if and only if G has an Independent set of size at least k.

4. Show that if any NP-complete problem is not in P, then no NP-complete problem is in P.

Solution:
Suppose that the problem A is NP-complete and is not in P. Consider another NP-complete problem, say B. As B is NP-complete, there is a polynomial time reduction from any problem in NP to problem B. Thus there is a polynomial time reduction from A to B: $A \leq_p B$. Then if A is intractable, so is B. Thus if A is intractable, so is any NP-complete problem.

5. Find a planar graph not containing a triangle that needs at least 3 colours in a colouring.

Solution:
Any graph which is not bipartite requires at least 3 colours, e.g., a 5-cycle.