The University of Newcastle
School of Electrical Engineering and Computer Science

COMP2230/6230 Algorithms
Tutorial Week 2

26 - 30 July 2021

Tutorial

1. For each of the following sequences, determine whether it is increasing, decreasing, nonincreasing or nondecreasing.

i. *2, 3, 88, 89, 100*

ii. *2, 3, 3, 88, 89, 100*

iii. *2, 3, 88, 89, 3, 100*

iv. *2*

v. *2, 1*

vi. *2, 2*

2. Which of the following Boolean expressions are in the conjunctive normal form and which are in the disjunctive normal form?

i. $p \wedge \overline{(q \wedge r)} \vee (\overline{p} \wedge s) \vee (q \wedge r)$

ii. $(p \vee q \vee r) \wedge (\overline{p} \vee s) \wedge (q \vee r) \wedge \overline{s}$

iii. $p \vee (\overline{q} \wedge r) \vee \overline{q}$

iv. $\overline{p}$

v. $p \vee q$

3. Prove the two De Morgan's Laws:

i. $\overline{(p \wedge q)} = \overline{p} \vee \overline{q}$

ii. $\overline{(p \vee q)} = \overline{p} \wedge \overline{q}$

4. Simplify expressions (i.e., express without logarithms of products, powers or quotients), where $\log x$ denotes $\log_{10} x$, $\lg x$ denotes $\log_2 x$ and $\ln x$ denotes $\log_e x$.:

i. $\log\sqrt{1000}$

ii. $\log_3 3^{5x}$

iii. $e^{\ln 8}$

iv. $2^{x \lg 3}$

v. $\lg(x^3 2^x)$

vi. $\log(\frac{\sqrt{x}\sin x}{x+4})$

5. Prove that for all n ≥ 1, $\frac{1}{2n} \le \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$

6. The Fibonacci sequence is defined as $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove by mathematical induction that

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1-\sqrt{5}}{2}\right)^{-n}\right)$$

7. Suppose that the only randomness which your algorithm has access to is a fair die (i.e., possible outcomes are 1, 2, 3, 4, 5 and 6, all with equal probabilities). Can you use this function to write an algorithm that implements a random fair coin toss?

8. Show that the following algorithm shuffles a deck of cards (52 cards) in a fair manner, that is, every possible permutation of 1 through 52 is obtained with the same probability.

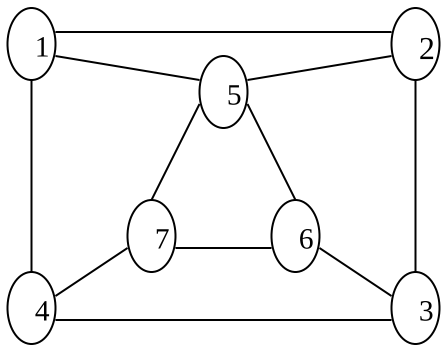
```
shuffle(a) {for i = 1 to 52
        swap(a[i], a[rand(i, 52)])}
```

9. If p(x) and q(x) are polynomials of degrees 10 and 5 respectively, what can you say about the degrees of p(x) + q(x) and p(x) - q(x) ?

10. If A and B are two closed intervals such that $A \cap B = I \neq \varnothing$. Which of the following can interval I be:

i. a closed interval
ii. an open interval
iii. a half-open interval?

11. Given a graph G:



i. Find the degree of each vertex.
ii. Find the neighborhood of each vertex.
iii. Write an adjacency matrix of the graph G.
iv. Is G bipartite? Why?
v. Find all simple paths from vertex 1 to vertex 3.
vi. Does G have a Hamiltonian cycle? If yes, find it.
vii. Does G have Euler cycle? If yes, find it. If not, why not?

Homework

12. Which sequences are both increasing and decreasing?

13. Which sequences are both nonincreasing and nondecreasing?

14. Show that $p \leftrightarrow q$ and $(\overline{p} \vee q) \wedge (\overline{q} \vee p)$ are equivalent.

15. Simplify (i.e., express without logarithms of products, powers or quotients):

i. $\ln(\sqrt{x}e^{4x})$

ii. $\ln \dfrac{x^2-1}{(x-1)^2}, x>1$

16. Solve each equation:

viii. $100 = 50e^{-x}$

ix. $\dfrac{1}{5} = 5^{3x-2}$

x. $\ln(2x+5) = 0$

xi. $\log_x 6 = \dfrac{1}{3}$

17. Use induction to prove that each equation is true for every positive integer *n*.

i. $\sum_{i=1}^{n}(2i-1) = n^2$

ii. $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

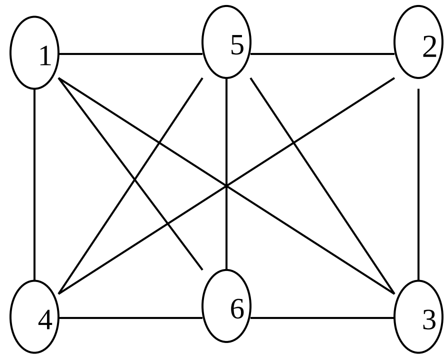
18. Suppose that the only randomness which your algorithm has access to is *rand(1,2)*, that is, a random fair coin toss. Can you use this function to write an algorithm that implements throwing a fair die (i.e., the outcome should be 1, 2, 3, 4, 5 or 6, with equal probability)?

19. If p(x) and q(x) are polynomials of degrees 10 and 5 respectively, what can you say about the degree of p(x) × q(x)?

20. If A and B are two open intervals such that A ∩ B = I ≠ ∅. Which of the following can interval I be:

i. a closed interval
ii. an open interval
iii. a half-open interval?

21. Given a graph G:



i. Find the degree of each vertex.
ii. Find the neighborhood of each vertex.
iii. Write an adjacency matrix of the graph G.
iv. Is G bipartite? Why?
v. Find all simple paths from vertex 1 to vertex 3.
vi. Does G have a Hamiltonian cycle? If yes, find it.
vii. Does G have Euler cycle? If yes, find it. If not, why not?

More Exercises

22. Write an expression equivalent to $p \to q$ involving only $\wedge$ and $\overline{\ }$ operators.

23. Solve each equation:

i. $\ln(0.8x + 0.03) = 0.01$

ii. $0.25 = e^{-0.4x}$

iii. $\lg(\log_3 x) = 4$

iv. $21 \times 2^{x+1} \times 3^{1-x} = 56$

24. Show that if k and n are positive integers satisfying $2^{k-1} < n < 2^k$, then $k - 1 < \lg n < k$, where $\lg n = \log_2 n$.

25. Prove that for real numbers a and $r \neq 1$, $\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$ for all $n \geq 0$.

26. Use induction to prove that each equation is true for every positive integer *n*.

i. $$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

ii. $$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

iii. $$\sum_{i=1}^{n} (-1)^{i+1} i^2 = \frac{(-1)^{n+1} n(n+1)}{2}$$

iv. $$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

27. Mr Con R. Tiste suggests the following modification of the above shuffle algorithm for deck *a* of cards:

```
new_shuffle(a) {
    for i = 1 to 52
        swap(a[i], a[rand(1, 52)])
}
```

Does *new_shuffle(a)* produce every possible permutation of 1 through 52 with the same probability? Prove your claim.

28. If p(x) and q(x) are polynomials, both of degree 5, what can you say about the degrees of p(x) + q(x), p(x) - q(x) and p(x) × q(x)?

29. If A and B are two half-open intervals such that $A \cap B = I \neq \emptyset$. Which of the following can interval I be:

i. a closed interval
ii. an open interval
iii. a half-open interval?

30. Locker doors: There are n lockers in a hallway numbered sequentially from 1 to n. Initially, all the locker doors are closed. You make n passes by the lockers, each time starting with locker #1. On the ith pass, i =1, 2, ..., n, you toggle the door of every i^{th} locker: if the door is closed, you open it, if it is open, you close it. For example, after the first pass every door is open; on the second pass you only toggle the even-numbered lockers (#2, #4, ...) so that after the second pass the even doors are closed and the odd ones are opened; the third time through you close the door of locker #3 (opened from the first pass), open the door of locker #6 (closed from the second pass), and so on. After the last pass, which locker doors are open and which are closed? How many of them are open?