SCHOOL of ELECTRICAL ENGINEERING & COMPUTING
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Comp3320/6370 Computer Graphics

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Exercises III

Practice exercises for lectures in weeks 6-9

Version 28

This paper provides practice exercises for the lectures given from week 5 to week 6. It extends the material presented on the "Exercises I and II" sheets (that still remain relevant for the whole course and all exams). These questions (some of them with hints or partial solutions) should help for exam preparation. It is recommended to look at the solutions only after you have first tried to solve the exercises yourself. Typically there are several different ways to solve an exercise. Please fill the gaps, try some variations and check if the provided solutions are correct. If you detect errors or have any suggestions for improvement please let us know.

Exercise 15b (Determinants)

(Note: We provide the details of this question for repetition and as a refresher of your pre-requisites for this course. This question will not be asked directly in our exams. However, the knowledge of determinants is of fundamental importance and they may occur implicitly in some tasks. This exercise may therefore help to facilitate general understanding of some of the algebraic aspects associated with computer graphics. Further details can be found in most books on Linear Algebra.)

Question:

Let $A \in M(n \times n, \mathbf{R})$ be a square matrix with real coefficients.

a) What is the general formula to calculate the determinant $\det A$?

b) List some basic properties of determinants.

c) What is the consequence of $\det A = 0$ with respect to invertibitility?

d) If B is a transform in 3D. What are the geometrical consequences of $\det B = 1$?

Answer:

a) The determinant of a square matrix of arbitrary size $(n \times n)$ can be defined by the Leibniz formula or the Laplace formula. The Leibniz formula for the determinant of an $(n \times n)$-matrix **A** is:

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} A_{i,\sigma_i} \tag{1}$$

Here the sum is computed over all permutations σ of a set $\{1, 2, \ldots, n\}$. A permutation is a function that reorders this set of integers. The position of the element i after the reordering σ is denoted σ_i. For example, for $n = 3$, the original sequence $1, 2, 3$ might be reordered to $S = [2, 3, 1]$, with $S_1 = 2,\ S_2 = 3,\ S_3 = 1$. The set of all such permutations (also known as the symmetric group on n elements) is denoted S_n. For each permutation σ, $sgn(\sigma)$ denotes the signature of σ; it is $+1$ for even σ and -1 for odd σ. Evenness or oddness can be defined as follows: the permutation is even (odd) if the new sequence can be obtained by an even number (odd, respectively) of switches of numbers. For example, starting from $[1, 2, 3]$ and switching the positions of 2 and 3 yields $[1, 3, 2]$, switching once more yields $[3, 1, 2]$, and finally, after a total of three (an odd number) switches, $[3, 2, 1]$ results. Therefore $[3, 2, 1]$ is an odd permutation. Similarly, the permutation $[2, 3, 1]$ is even. A permutation cannot be simultaneously even and odd. The term $\prod_{i=1}^{n} A_{i,\sigma_i}$ is notation for the product of the entries at position (i, σ_i), where i ranges from 1 to n:

$$A_{1,\sigma_1} \cdot A_{2,\sigma_2} \cdot \ldots \cdot A_{n,\sigma_n} \tag{2}$$

b) i. If B results from A by interchanging two rows or two columns, then $det(B) = -det(A)$. The determinant is called alternating (as a function of the rows or columns of the matrix).

   ii. If B results from A by multiplying one row or column with a real number c, then $det(B) = c \cdot det(A)$. As a consequence, multiplying the whole matrix by c yields: $det(cA) = c^n det(A)$

   iii. If B results from A by adding a multiple of one row to another row, or a multiple of one column to another column, then $det(B) = det(A)$.

c) The matrix A doesn't have an inverse.

d) The transformation preserves the volume (for example rotation or any rigit body transform but also shearing). And also note that identity map from any finite dimensional space to itself has determinant 1.

Exercise 16 (The Orthogonal Group $O(n)$)

Question: Consider the following definition:

DEF. $GL_n(\mathbf{R}) = \{A \in M(n \times n, \mathbf{R}); det(A) \neq 0\}$ is called the *General Linear Group*. It is the group of invertible $n \times n$ matrices with real coefficients.

$O(n) := \{A \in GL_n(\mathbf{R});\ A \cdot A^T\ =\ A^T \cdot A = Id\}$ is called the *Orthogonal Group of* $(n \times n)-$ *matrices*.

Give a proof of the following proposition:

Proposition: Let $A \in GL_n(\mathbf{R})$ be an invertible $n \times n$ matrix with real coefficients. The following three statements are equivalent:

(i) $A^{-1} = A^T$ (i.e. A is an element of $O(n)$)

(ii) $\forall x\ \in \mathbf{R}^n \quad \|Ax\| = \|x\|$ (preserves lengths)

(iii) $\forall x\ , y \in \mathbf{R}^n \quad Ax \cdot Ay = x \cdot y$ (preserves angles)

Answer:

Proof of the above proposition is left as an advanced exercise.

Exercise 17 (Orthogonal group)

Question: Describe in geometrical terms what is the difference between the elements of $O(n)$ and $SO(n)$?

Answer. $O(n)$ contains the rotations and reflections. All matrices of $SO(n)$ have a positive determinant, i.e. the corresponding mappings do not allow any change of orientation. Therefore no reflections and only the rotations are contained in $SO(n)$.

Exercise 18 (Some properties of $O(n)$)

Question: Show that

1. $A \in O(n) \implies A^{-1} \in O(n)$
2. $A, B \in O(n) \implies A \cdot B \in O(n)$
3. $A \in O(n) \implies |detA| = 1$

Answer. Left as an advanced exercise.

Exercise 19 (Change of basis)

Question:

Let $E = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be the standard basis and let

$F = \left((1,2,2)^T, (0,1,2)^T, (1,0,2)^T\right)$ and

$G = \left((1,2,3)^T, (1,1,3)^T, (5,2,1)^T\right)$ be other bases of $\mathbf{R}^3$.

Determine:

1. $[Id]_E^F$
2. $[Id]_F^E$
3. $[Id]_F^G$

Answer.

Note: The given bases in this exercise are general bases and not necessarily ONBs. Hence in this situation the calculation of the inverse of the basis transform is usually not accomplished by taking the transpose. Instead we have to apply the standard way of calculating the inverse (see your favourite Linear Algebra book for details or use some software) .

Part 1:

$$\begin{aligned} [Id]_E^F &= \left[\vec{f}_1\ \vec{f}_2\ \vec{f}_3\right] \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix} \end{aligned}$$

Part 2:

$$
\begin{aligned}
[Id]_F^E &= \left([Id]_E^F\right)^{-1} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ -1 & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}
\end{aligned}
$$

Part 3:

$$
\begin{aligned}
[Id]_F^G &= \left([Id]_E^F\right)^{-1} \cdot [\vec{g_1}\ \vec{g_2}\ \vec{g_3}] \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ -1 & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ 2 & 1 & 2 \\ 3 & 3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{13}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{9}{2} \\ \frac{1}{4} & \frac{3}{4} & \frac{7}{4} \end{bmatrix}
\end{aligned}
$$

Exercise 20 (Rotation about an arbitrary axis)

Question: What is the transform that rotates some entity by angle $\alpha = \frac{\pi}{2}$ about the axis defined by vector $\vec{r} = (1, 2, 2)^T$?

Solution:
First we want to find two orthonormal vectors $\vec{s}$, $\vec{t}$ that together with $\vec{r} = (1, 2, 2)^T$ form an orthonormal basis (ONB) (Note: orthogonal is not sufficient (why?)).

First we have to normalise $\vec{r} = (1, 2, 2)^T$ to obtain the unit vector $\hat{r}$. The norm of $\vec{r}$ is calculated as follows $||\vec{r}|| = \sqrt{1 + 2^2 + 2^2} = 3$. Hence we obtain

$$
\begin{aligned}
\hat{r} &= \frac{\vec{r}}{||\vec{r}||} \\
&= \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)^T
\end{aligned}
$$

As described in the lecture slides we can calculate $\vec{s}$, by setting the numerically smallest element of $\vec{r}$ to zero, swapping the other two coordinates and negating the first of the non-zero coordinates; We obtain $\vec{s} = \left(0, -\frac{2}{3}, \frac{2}{3}\right)^T$.
Is $\vec{s}$ normalised?—No, not yet because: $||\vec{s}|| = \sqrt{0 + (\frac{2}{3})^2 + (\frac{2}{3})^2} = \sqrt{2}\frac{2}{3}$.
Hence

$$
\begin{aligned}
\hat{s} &= \vec{s}/||\vec{s}|| \\
&= \frac{\left(0, -\frac{2}{3}, \frac{2}{3}\right)^T}{\sqrt{2}\frac{2}{3}} \\
&= \frac{1}{\sqrt{2}}(0, -1, 1)^T.
\end{aligned}
$$

The third vector $\hat{t}$ is obtained by calculating the cross-product

$$\begin{aligned}\hat{t} &= \hat{r} \times \hat{s} \\ &= \frac{1}{3\sqrt{2}}(4,-1,-1)^T\end{aligned}$$

According to the definition of the cross-product $\hat{t}$ is already normalised because both, $\hat{r}$ and $\hat{s}$, have length one (check?).
We can check the pairwise orthogonality of the basis vectors of our new ONB $F = (\hat{r}, \hat{s}, \hat{t})$ by using the dot product, and we must obtain: $\hat{r} \cdot \hat{s} = 0$, $\hat{s} \cdot \hat{t} = 0$, and $\hat{t} \cdot \hat{s} = 0$.

The change of basis transform from F to the standard basis $E = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is the matrix with the three new vectors as columns.

$$[Id]_E^F = \begin{bmatrix} \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{bmatrix}$$

According to our construction $[Id]_E^F$ is an orthogonal matrix (Note: You can check that it is an element in O(3), e.g., by testing the property $A^TA = AA^T = Id$).
Hence $\left([Id]_E^F\right)^{-1} = \left([Id]_E^F\right)^T$.
Let $M = \left([Id]_E^F\right)^T$, then our rotation matrix is given as:

$$\begin{aligned}X &= M^T \cdot R_x(\alpha) \cdot M \\ &= \left[\hat{r}\ \hat{s}\ \hat{t}\right] \cdot R_x(\alpha) \cdot \left[\hat{r}\ \hat{s}\ \hat{t}\right]^T \\ &= \begin{bmatrix} \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ 0 & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix} 1.5 & -4 & 8 \\ 9 & 4 & 1 \\ -3 & 7 & 4 \end{bmatrix}\end{aligned}$$

Note: These calculations were done in standard maths notation but they work the same using homogeneous notation.

Exercise 21 (Scaling in direction of an arbitrary axis)

Question: We would like to scale an entity by factor 2 in direction of vector $\vec{r} = (1, 2, 2)^T$. What is the transform ?

Answer.

Step 1: Create an ONB ($f_x = \frac{\vec{r}}{||\vec{r}||}, f_y = \frac{f_x^{\perp}}{||f_x^{\perp}||}, f_z = f_x \times f_y$).

Step 2: $X = [f_x\ f_y\ f_z] \cdot S(2, 1, 1) \cdot [f_x\ f_y\ f_z]^T$

$= [f_x\ f_y\ f_z] \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot [f_x\ f_y\ f_z]^T$

Discuss similarities between the approaches in exercise 20 and 21.

Final note: No guarantee that the solutions are correct yet. Please email any errors that you detect. Check blackboard for updates.

Bibliography

[1] T. Akenine-Möller and E. Haines. *Real-Time Rendering*. A K Peters, second edition, 2002.

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[5] Gilbert Strang. *Introduction to Linear Algebra*. Wellesley–Cambridge Press, Wellesley, MA, third edition, 2003.