The University of Newcastle
School of Information and Physical Sciences

COMP2230/6230 Algorithms

Tutorial Week 11 Solutions

14 - 15 October 2021

Tutorial

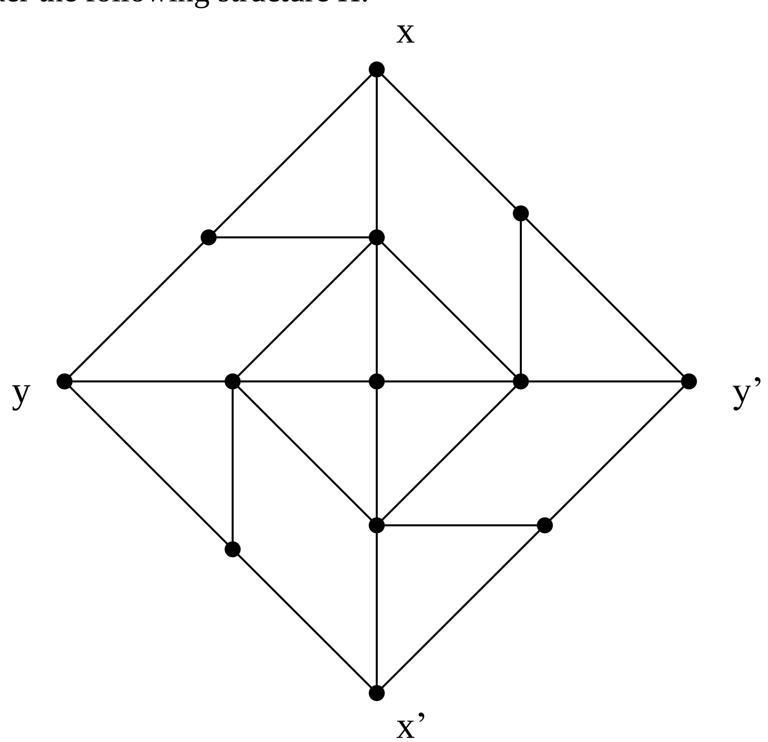
1. Prove that the planar graph 3-colourability is NP-complete.

Solution idea:
We first show that graph 3-clolourability is in NP. Indeed, if we are given 3 colouring of the graph, we can check in polynomial time that the colouring is proper, that is, that no two adjacent vertices are coloured with the same colour. For each vertex, we need to check the colours of all its neighbours, and that can be done in $O(n^2)$.

To show that all problems in NP can be reduced to planar graph 3-colourability in polynomial time we use reduction from 3-clolourability. We start from any instance of 3-colourability, which is an arbitrary graph G. Then we construct a planar graph G' such that G' is 3-colourable if and only if G is 3-colourable.

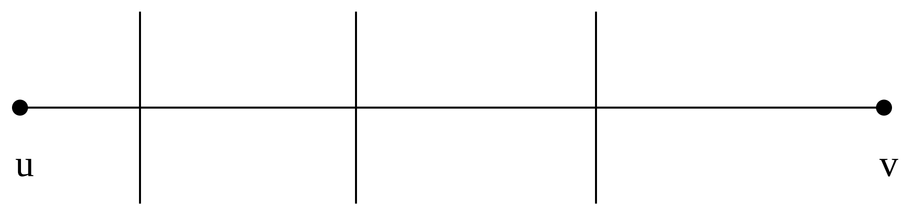
Constructing G' from G:

a. first embed G in the plane (that is, draw G in the plane – sheet of paper); as G is not necessarily planar, some edges will cross, but we do not allow more than two edges to cross in the same point; similarly, we do not allow an edge to go through any vertex (except, of course, the two end vertices on the edge).
b. Consider the following structure H:

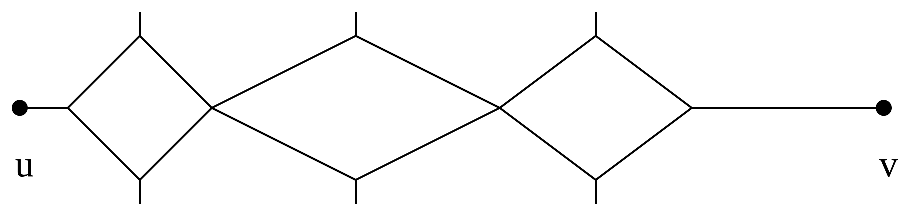


Note that in the above structure H, for any proper 3 colouring, x and x' will always receive the same colour; similarly, y and y' will always receive the same colour; however, x and y can receive either the same or different colours.

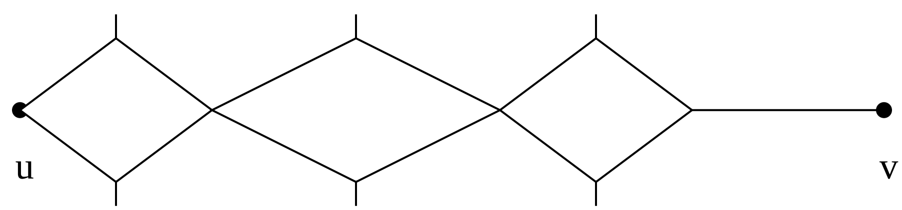
c. Consider the planar embedding of G, and consider an edge uv with all its crossings:



Replace each crossing of two edges by the structure H:



d. Select one end vertex of the edge uv – say vertex u; identify the u vertex with the nearest vertex of the H structure:



An important fact to note here is that as in any proper 3 colouring y and y' in the H structure are of the same colour, and of the same colour as u and thus u and v are of different colours.

e. Repeat c. and d. for all edges with crossings.

It is left to show that G' can be constructed in polynomial time, and that G' is 3-colourable if and only if G is 3-colourable.

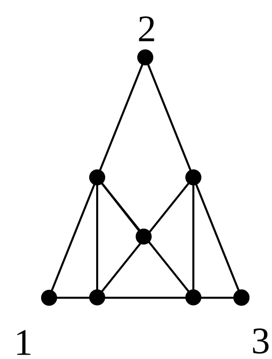
Graph G' has n vertices that correspond to the vertices of the original graph G. Additionally, there are up to 4 extra vertices for each edge crossing, and there are no more than m(m-1)/2 edge crossings; therefore, there are no more than n+2m(m-1) vertices in G', and thus the construction can be done in polynomial time.

Given a 3 colouring of graph G, we construct the colouring of G' as follows: assign to each vertex v' in G' that corresponds to a vertex v in G the colour that vertex v receives in G. The rest of the colouring of G' follows.

2. Prove that the 3-clolourability of a graph with no vertex degree exceeding 4 is NP-complete.

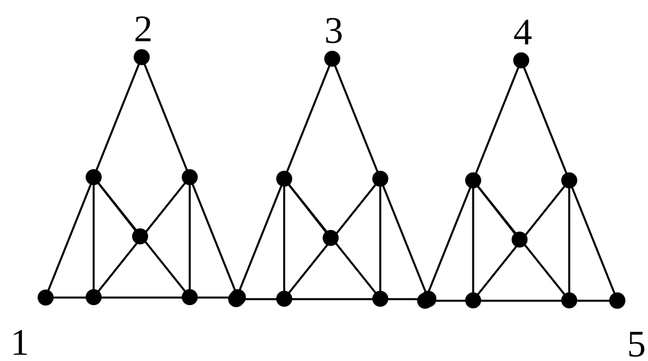
Solution idea: Reduction from 3-clolourability. Start from any instance of 3-colourability, which is an arbitrary graph G. Then construct a graph G' with maximum degree 4 such that G ' is 3-colourable if and only if G is 3-colourable. Constructing G' from G:

a. Consider the following structure H:



Note that for any proper colouring of H vertices 1, 2 and 3 always receive the same colour. Also note that vertices 1, 2 and 3 have degree 2 while the inner vertices of the structure H have degree 4. We call vertices 1, 2 and 3 "outlet" vertices.

b. Replace each vertex of degree d > 4 by d-2 structures H. For example, a vertex with degree 5 will be replaced by 3 structures H as follows:



Here there are 5 outlet vertices with degree 2 and there are 5 vertices adjacent with the original vertex with degree 5. We connect each one of these 5 vertices to one of the outlets of the above structure; thus no vertex will have degree more than 4.

It remains to show that G' can be constructed in polynomial time, and that G' is 3-colourable if and only if G is 3-colourable.

More Exercises

3. Design a polynomial time algorithm for 2SAT, or prove that 2SAT is NP-complete.

4. Show that deciding whether a graph is k-colourable is NP-complete for any fixed $k \geq 3$ by giving a reduction from 3-colourability.

5. Show that testing whether a graph G is a subgraph of graph H is NP-complete.

Solution idea:
Reduction from Hamiltonian cycle (restriction).