MATH1510 - Discrete Mathematics
Graphs

University of Newcastle

UoN

Definitions

Definition

A graph is a pair $G = (V, E)$, where V is a set of vertices and E is a set of edges, such that each edge $e \in E$ is incident on either 1 or 2 vertices.

Definition

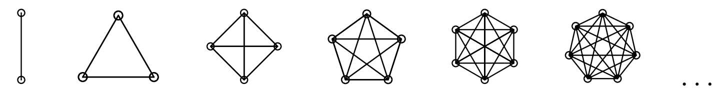
A simple graph is a pair $G = (V, E)$ where V is a set of vertices, and E is a set of edges, each edge being a subset of V of cardinality 2.

Example

$$G = \Big(\{a, b, c, d, e\}\ ,\ \ \{\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{d, e\}\} \Big)$$

Complete graph

The Complete simple graph on n vertices, written K_n, is a simple graph with n vertices and an edge between every possible pair of vertices. We often label the vertices $1, 2, \ldots, n$ and then the edges are $(1, 2)$, $(1, 3)$, ..., $(1, n)$, $(2, 3)$, ..., $(2, n)$, $(3, 4)$, ..., $(n-1, n)$.



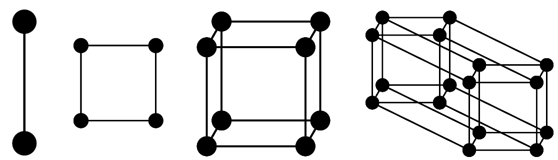
K_2 K_3 K_4 K_5 K_6 K_7 $\cdots$

n-cube

The n-cube is a model for efficient connection of processors in certain parallel computing architectures, as used in the Connection machine.

vertices processors

edges direct connection



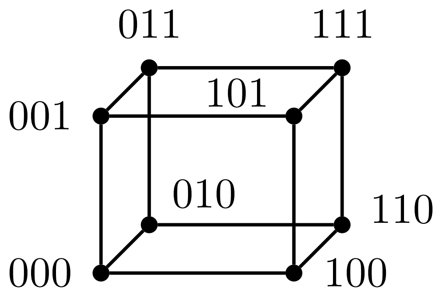
Q_1 Q_2 Q_3 Q_4

n-cubes can be characterised via a geometric algorithm or symbolically via binary strings.

n-cube

The n-cube Q_n is the graph that can be described as follows:

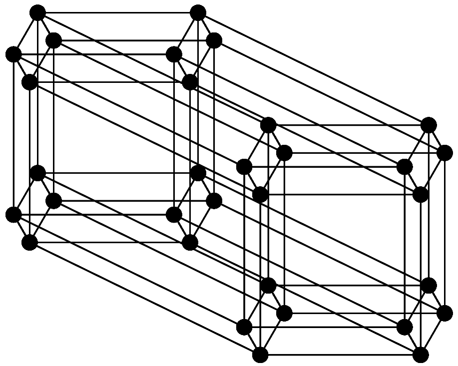
- The vertex set is the set of $(0,1)$-strings of length n.
- Two vertices are connected by an edge if and only if they differ in exactly one position.



What is the distance between the vertices 01101 and 00110 in Q_5?

- A 1
- B 3
- C 4
- D 5

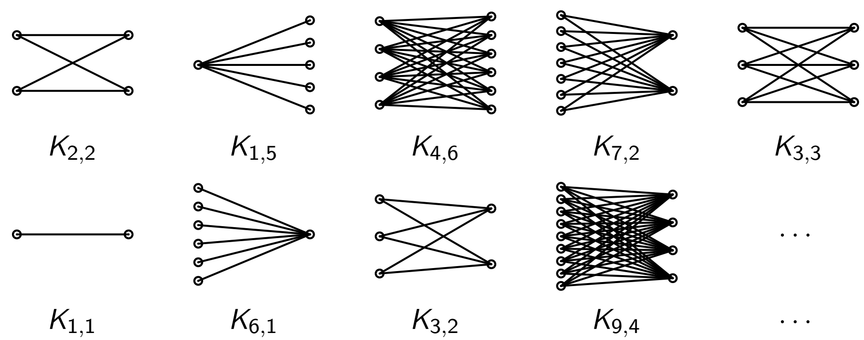
What is the diameter of the 5-cube?



- A 32
- B 10
- C 5
- D 25

Complete bipartite graph

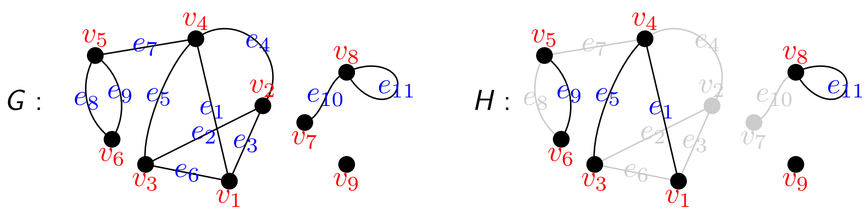
The Complete bipartite graph on m and n vertices, denoted $K_{m,n}$ is a simple graph with $m+n$ vertices and an edge between every pair of vertices (i,j) where i is in the first m vertices and j is in the last n vertices.



Subgraph

Definition

A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ with $W \subseteq V$ and $F \subseteq E$.

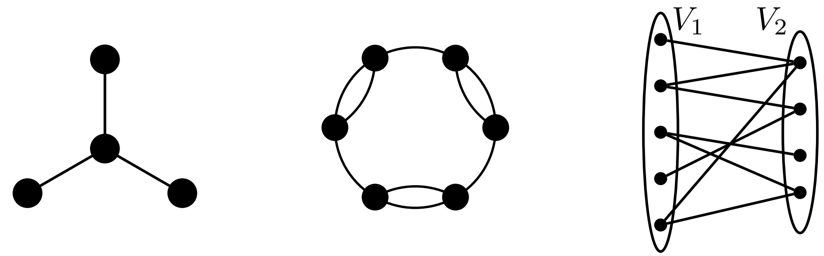


$W = \{v_1, v_3, v_4, v_5, v_6, v_8\}$,
$F = \{e_1, e_5, e_9, e_{11}\}$

Bipartite graphs

Definition

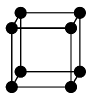
A graph $G = (V, E)$ is bipartite if there is a partition of V into two sets $V = V_1 \cup V_2$ such that every edge $e \in E$ is incident on one element of V_1 and one element of V_2.

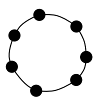


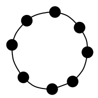
Remark

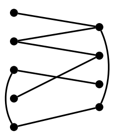
This is equivalent to saying that the vertices can be coloured with two colours such that no two adjacent vertices receive the same colour (proper colouring).

Exactly one of these graphs is not bipartite. Which one?

A 

B 

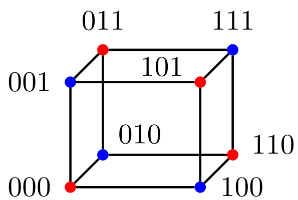
C 

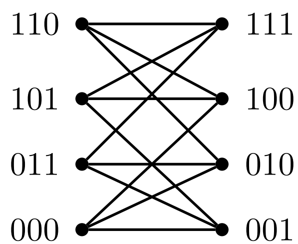
D 

Example

The n-cube is bipartite:

- V_1 = set of $(0, 1)$-strings with an even number of 1-entries
- V_2 = set of $(0, 1)$-strings with an odd number of 1-entries





The handshake theorem

Theorem

Let $G=(V,E)$ be a graph with vertex set $V=\{v_1,v_2,\dots,v_n\}$ and $m=|E|$ edges. Let $\delta(v_i)$ denote the degree of vertex v_i. Then

$$\sum_{i=1}^{n}\delta(v_i)=2m.$$

Proof.

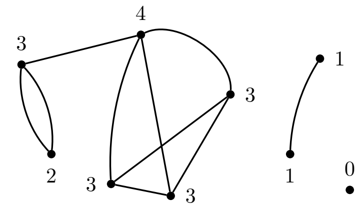
Let us count the pairs (v,e) where v is a vertex and e is an edge incident with v (where the pair (v,e) for a loop e at v is counted twice).

- Vertex v_i contributes $\delta(v_i)$ pairs $\implies$ $\sum_{i=1}^{n}\delta(v_i)$ pairs in total
- Every edge e contributes 2 pairs $\implies$ $2m$ pairs in total

We have counted the same objects in two different ways, hence the results must be equal. $\square$

The handshake theorem in action

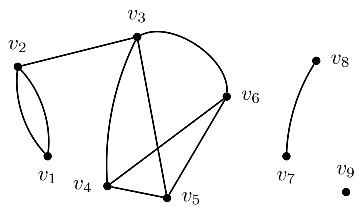
How many edges in this graph?



The graph has

$$\frac{2+3+4+3+3+3+1+1+0}{2}=\frac{20}{2}=10\text{ edges}.$$

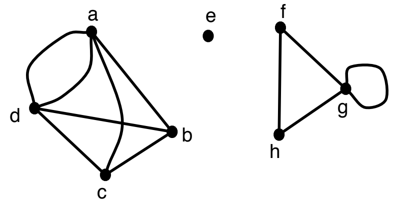
The handshake theorem and adjacency matrices



$$\begin{pmatrix}
0&2&0&0&0&0&0&0&0\\
2&0&1&0&0&0&0&0&0\\
0&1&0&1&1&1&0&0&0\\
0&0&1&0&1&1&0&0&0\\
0&0&1&1&0&1&0&0&0\\
0&0&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0
\end{pmatrix}\qquad
\begin{matrix}2\\3\\4\\3\\3\\3\\1\\1\\0\end{matrix}$$

- $\delta(v_i)$ is the sum of the i-th row.
- Handshake theorem: The sum of the entries of the adjancy matrix equals twice the number of edges.

Incidence matrices (variant 2)

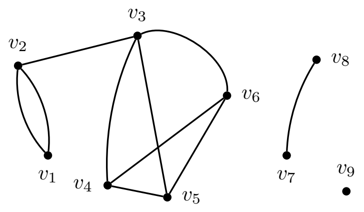


has **incidence matrix:** $\begin{pmatrix}
1&1&1&1&0&0&0&0&0&0\\
0&0&1&0&1&1&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0\\
1&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&1&0&1&2\\
0&0&0&0&0&0&0&1&1&0
\end{pmatrix}$

The $(i,j)^{\text{th}}$ entry is the number of incidences between the i^{th} vertex and the j^{th} edge, i.e.

- it is **1** if the j^{th} edge connects the i^{th} vertex with some other vertex,
- it is **2** if the j^{th} edge is a loop at the i^{th} vertex, and
- it is **0** otherwise.

The handshake theorem and incidence matrices



$$\begin{array}{c|c}
\begin{pmatrix}
1&1&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0\\
0&0&1&1&1&1&0&0&0&0\\
0&0&0&1&0&0&1&1&0&0\\
0&0&0&0&1&0&1&0&1&0\\
0&0&0&0&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0
\end{pmatrix} & \begin{matrix}2\\3\\4\\3\\3\\3\\1\\1\\0\end{matrix}\\
\hline
\begin{matrix}2&2&2&2&2&2&2&2&2&2\end{matrix} & 20
\end{array}$$

- The sum of the i-th row is $\delta(v_i) \implies$ sum of all entries is $\sum_{i=1}^{n} \delta(v_i)$.
- The sum of the j-th column is 2 $\implies$ sum of all entries is $2m$.
- Handshake theorem: The sum of the entries of the incidence matrix equals twice the number of edges.

Two corollaries to the handshake theorem

Corollary

The sum of all degrees in a graph is even.

Corollary

Every graph has an even number of vertices of odd degree.

Why/how does the corollary follow from the handshake theorem?

Which of the following could NOT be the degree sequence of a graph?

A $1, 2, 4$

B $2, 4, 6$

Another question: how many edges in K_4?

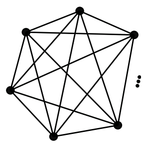
A 4

B 6

C 8

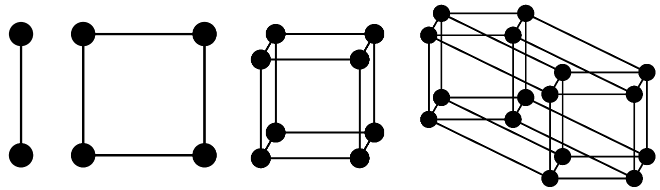
D 9

Generalizing: how many edges in K_n?



- A $|E(K_n)| = n(n-1)$
- B $|E(K_n)| = n^2$
- C $|E(K_n)| = \frac{n(n-1)}{2}$
- D $|E(K_n)| = \frac{n^2}{2}$

The number of edges of the n-cube



Q_1 Q_2 Q_3 Q_4

$$1,\ 4,\ 12,\ 32,\ \ldots \quad ???$$

- The n-cube has 2^n vertices.
- Every vertex has degree n (one edge for each position of its label).
- This implies $\sum_{i=1}^{2^n} \delta(v_i) = 2^n \cdot n$, hence
$$|E(Q_n)| = \frac{1}{2} \cdot 2^n \cdot n = n \cdot 2^{n-1}.$$

Mathematical Induction

Mathematical Induction is a proof technique which works as follows:

- Set-up We set up $P(n)$ to be a statement we want to prove for all n.
- Base case Prove the statement for the first value of n – usually $P(1)$.
- Inductive step
  - Assume $P(k)$ is true for some **fixed** integer $k \geqslant 1$.
  - Use this assumption to prove that $P(k+1)$ is true.
- Conclusion $P(n)$ is true for all $n \in \mathbb{N}$

Example 1: Mathematical Induction

We will use Mathematical Induction to re-prove $|E(K_n)| = \frac{n(n-1)}{2}$.

- Set-up For every $n \in \mathbb{N}$, $P(n)$ is the claim "$|E(K_n)| = \frac{n(n-1)}{2}$".
- Base case $P(1)$ states that "$|E(K_1)| = \frac{1 \cdot 0}{2} = 0$" which is true since the complete graph on one vertex has no edges.
- Inductive step
  - Assume $P(k)$ is true for some **fixed** integer $k \geqslant 1$.
  - Is P_{k+1} true? Consider the complete graph on $k+1$ vertices. We can build K_{k+1} by taking K_k, adding one new vertex, and connecting the new vertex to the other k vertices with k new edges.
$$|E(K_{k+1})| = |E(K_k)| + k = \frac{k(k-1)}{2} + k = \frac{k^2 - k + 2k}{2}$$
$$= \frac{(k+1)((k+1)-1)}{2} \text{ i.e. claim } P(k+1) \text{ is true.}$$
- Conclusion $|E(K_n)| = \frac{n(n-1)}{2}$ $\forall n \in \mathbb{N}$, by induction. $\square$

What induction isn't

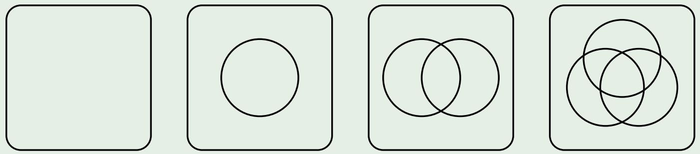
A common error is to observe a pattern from a few instances, and claim that this is an inductive proof. Here you have the initial case(s) without the inductive step.

It is also common to neglect the initial case, and just give the inductive step. This is also incorrect.

Example

A finite number of circles will divide the plane into a finite number of regions. What is the maximum number of regions produced by n circles?

$n = 0$ $n = 1$ $n = 2$ $n = 3$



1 region 2 regions 4 regions 8 regions

Mathematical Induction (Strong form)

Set-up We set up $P(n)$ to be a statement we want to prove for all n.

Base step Prove the statement for the first value of n – usually $P(1)$.

Inductive step Assume the statement is true for all integers up to k, i.e., assume $P(1)$, $P(2)$, ..., $P(k-1)$, $P(k)$ are all true. Use this assumption to prove that $P(k+1)$ is true.

Conclusion $P(n)$ is true for all n.

Example

Every positive integer can be written as a product of one or more prime numbers.

Summary

Subgraphs. Formal (set based) way of saying that one graph is contained in another.

Bipartite graphs. Graphs that can be properly coloured with 2 colours.

Handshake theorem. The sum of the degrees is twice the number of edges.

Induction. A proof method to prove statements for all $n \in \mathbb{N}$.