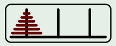
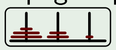
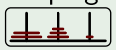
MATH1510 - Discrete Mathematics
Recurrence relations

University of Newcastle

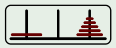
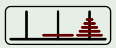
Semester 2

Example (Tower of Hanoi)

Description: The Tower of Hanoi is a puzzle

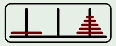
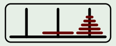
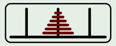
- three pegs with n disks of decreasing size on one peg 
- a legal move takes one disk from peg to peg keeping each stack in decreasing order (small on big)  $\rightarrow$ 
- Goal: move the stack to another peg in minimum moves

To move the stack $\{1, 2, \ldots, n-1, n\}$ from A to B:

- We must move disk n from A to B — can't!
- First move stack $\{1, 2, \ldots, n-1\}$ aside (to C) 
- Then move disk n from A to B 
- Finally move stack $\{1, 2, \ldots, n-1\}$ from C to B. 

Example (Tower of Hanoi)

Description:
Algorithm: Label the disks 1 (tiny) to n (huge) and the pegs A, B and C. To move the stack $\{1, 2, \ldots, n-1, n\}$ from A to B:

- We must move disk n from A to B — can't!
- First move stack $\{1, 2, \ldots, n-1\}$ aside (to C) 
- Then move disk n from A to B 
- Finally move stack $\{1, 2, \ldots, n-1\}$ from C to B. 

Analysis: Let M_n be the number of moves required to move a stack $\{1, 2, \ldots, n\}$ from peg to peg.
The recurrence relation with initial conditions is

$$M_n = M_{n-1} + 1 + M_{n-1} = 2M_{n-1} + 1 \ ; \quad M_1 = 1$$

Example (Merge sort)

Suppose we apply the Merge sort algorithm to a list of n elements where $n = 2^k$. This involves

- Sort the first $n/2 = 2^{k-1}$ elements to get a list A
- Sort the last $n/2 = 2^{k-1}$ elements to get a list B
- Merge lists A and B

Merging two lists of length m requires, in the worst case, $2m - 1$ comparisons. Consequently, if C_k is number of comparisons required (in the worst-case) to merge-sort a list of 2^k elements, then

$$C_k = 2C_{k-1} + 2^k - 1$$

Since $C_1 = 1$, we can find values for $C_2, C_3, C_4, \ldots$

Notation

If we have a recurrence relation (RR) for a sequence $\{a_n\}_{n=0}^{\infty}$,

A solution to the RR is a formula that gives a_n purely in terms of n.

The general solution to the RR is a formula for the sequence that covers all possible choices of initial conditions.

If, in addition to the RR, we have initial conditions given, then

The solution to this problem (RR&IC) is a formula for just one sequence from those described in the general solution, such that it satisfies the initial conditions.

Example

For instance, with the recurrence relation $a_n = 3a_{n-1}$,

- there are many solutions, such as $a_n = 3^n$, $a_n = 0$, $a_n = 7 \times 3^n$.
- The general solution is $a_n = A \times 3^n$.
- With initial conditions $a_0 = 5$, the solution is $a_n = 5 \times 3^n$.

Sums and Products

Not all recurrence relations are easily solved. However, there are certain recurrence relations that we can solve.

The simplest case is where the recurrence relation gives rise to a process that builds up a sum or a product.

Just as with induction, a good strategy is to work out the first few cases, looking for evidence of the process that takes you from one number to the next.

Example (Summation 1)

Recurrence relation $a_n = a_{n-1} + 7$.

- $a_1 = a_0 + 7$
- $a_2 = a_1 + 7 = (a_0 + 7) + 7 = a_0 + 2 \cdot 7$
- $a_3 = a_2 + 7 = (a_0 + 2 \cdot 7) + 7 = a_0 + 3 \cdot 7$
- In general: $a_n = a_0 + 7n$.

Example (Summation 2)

Recurrence relation $s_n = s_{n-1} + 2n$ with initial condition $s_0 = 1$.

- $s_1 = s_0 + 2 \cdot 1$
- $s_2 = s_1 + 2 \cdot 2 = s_0 + 2 \cdot 1 + 2 \cdot 2 = s_0 + 2(1+2)$
- $s_3 = s_2 + 2 \cdot 3 = s_0 + 2(\cdot 1 + 2) + 3 \cdot 2 = s_0 + 2(1+2+3)$
- In general:

$$s_n = s_0 + 2(1 + 2 + \cdots + n) = 1 + 2 \cdot \frac{n(n+1)}{2} = n(n+1) + 1.$$

Example (Product 1)

Recurrence relation $q_n = \frac{1}{10} q_{n-1}$

- $q_1 = \frac{1}{10} q_0$
- $q_2 = \frac{1}{10} q_1 = \frac{1}{10} \cdot \frac{1}{10} q_0 = \left(\frac{1}{10}\right)^2 q_0$
- $q_3 = \frac{1}{10} q_2 = \frac{1}{10} \cdot \left(\frac{1}{10}\right)^2 q_0 = \left(\frac{1}{10}\right)^3 q_0$
- In general, $q_n = \frac{1}{10} q_{n-1} = \frac{1}{10} \cdot \left(\frac{1}{10}\right)^{n-1} q_0 = \left(\frac{1}{10}\right)^n q_0$

Example (Product 2)

Recurrence relation $F_n = nF_{n-1}$ with initial condition $F_0 = 1$.

- $F_1 = 1F_0 = 1$
- $F_2 = 2F_1 = 2 \cdot 1 = 2$
- $F_3 = 3F_2 = 3 \cdot 2 = 6$
- $F_4 = 5F_3 = 4 \cdot 6 = 24$
- In general, $F_n = nF_{n-1} = n(n-1)! = n!$.

Example (Product 3)

Recurrence relation $b_n = \frac{n}{2} \cdot b_{n-1}$ with initial condition $b_0 = 4$.

- $b_1 = \frac{1}{2}b_0$
- $b_2 = \frac{2}{2}b_1 = \frac{2}{2} \cdot \frac{1}{2}b_0 = \frac{2 \cdot 1}{2^2}b_0$
- $b_3 = \frac{3}{2}b_2 = \frac{3}{2} \cdot \frac{2 \cdot 1}{2^2}b_0 = \frac{3 \cdot 2 \cdot 1}{2^3}b_0$
- $b_4 = \frac{4}{2}b_3 = \frac{4}{2} \cdot \frac{3 \cdot 2 \cdot 1}{2^3}b_0 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2^3}b_0$
- In general,

$$b_n = \frac{n}{2}b_{n-1} = \frac{n}{2} \cdot \frac{(n-1)!}{2^{n-1}}b_0 = \frac{n!}{2^n}b_0 = \frac{n!}{2^n} \cdot 4 = \frac{n!}{2^{n-2}}.$$

Example (Hanoi)

Recurrence relation $M_n = 2M_{n-1} + 1$ with initial condition $M_1 = 1$.

- $M_2 = 2M_1 + 1 = 2 + 1$
- $M_3 = 2M_2 + 1 = 2(2 + 1) + 1 = 4 + 2 + 1$
- $M_4 = 2M_3 + 1 = 2(4 + 2 + 1) + 1 = 8 + 4 + 2 + 1$
- $M_5 = 2M_4 + 1 = 2(8 + 4 + 2 + 1) + 1 = 16 + 8 + 4 + 2 + 1$
- In general,

$$\begin{aligned} M_n = 2M_{n-1} + 1 &= 2(2^{n-2} + 2^{n-3} + \cdots + 2 + 1) + 1 \\ &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 = \frac{2^n - 1}{2 - 1} = 2^n - 1. \end{aligned}$$

Example (Investment with fees)

Recurrence relation $a_n = 1.005a_{n-1} - 10$ with initial condition $a_0 = 10000$.

$$\begin{aligned} a_1 &= 1.005a_0 - 10 \\ a_2 &= 1.005a_1 - 10 = 1.005(1.005a_0 - 10) - 10 \\ &= 1.005^2 a_0 - 10(1 + 1.005) \\ a_3 &= 1.005a_2 - 10 = 1.005\left[1.005^2 a_0 - 10(1 + 1.005)\right] - 10 \\ &= 1.005^3 a_0 - 10(1 + 1.005 + 1.005^2) \\ a_4 &= 1.005a_3 - 10 = 1.005\left[1.005^3 a_0 - 10\left(1 + 1.005 + 1.005^2\right)\right] - 10 \\ &= 1.005^3 a_0 - 10\left(1 + 1.005 + 1.005^2 + 1.005^3\right) \end{aligned}$$

In general,

$$a_n = 1.005^n a_0 - 10\left(1 + 1.005 + \cdots + 1.005^{n-1}\right) = 1.005^n a_0 - 10 \cdot \frac{1.005^n - 1}{0.005}.$$

Example (Merge sort)

Recurrence relation $C_k = 2C_{k-1} + 2^k - 1$ with initial condition $C_1 = 1$.

$$\begin{aligned}
C_2 &= 2C_1 + 2^2 - 1 = 2 + 2^2 - 1 \\
C_3 &= 2C_2 + 2^3 - 1 = 2(2 + 2^2 - 1) + 2^3 - 1 \\
&= 4 + 2 \cdot 2^3 - 2 - 1 \\
C_4 &= 2C_3 + 2^4 - 1 = 2(4 + 2 \cdot 2^3 - 2 - 1) + 2^4 - 1 \\
&= 8 + 3 \cdot 2^4 - 4 - 2 - 1 \\
C_5 &= 2C_4 + 2^5 - 1 = 2(8 + 3 \cdot 2^4 - 4 - 2 - 1) + 2^5 - 1 \\
&= 16 + 4 \cdot 2^5 - 8 - 4 - 2 - 1
\end{aligned}$$

$$\begin{aligned}
C_k &= 2^{k-1} + (k-1)2^k - (1 + 2 + \cdots + 2^{k-2}) \\
&= 2^{k-1} + (k-1)2^k - (2^{k-1} - 1) = (k-1)2^k + 1.
\end{aligned}$$

- With $n = 2^k$ this is $n \log_2 n - n + 1 = n\frac{\ln(n)}{\ln(2)} - n + 1$.

A very special case

A constant-coefficient homogeneous linear recurrence relation of the form

$$\begin{aligned}
& s_n = a s_{n-1} && \text{(first order)} \\
\text{OR}\quad & s_n = a s_{n-1} + b s_{n-2} && \text{(second order)} \\
\text{OR}\quad & s_n = a s_{n-1} + b s_{n-2} + c s_{n-3} && \text{(third order)}
\end{aligned}$$

etc.

where $a, b, c, \ldots$ are constants.

We will call these linear recurrence relations.

Solution of first-order case

We have seen the first-order case $s_n = as_{n-1}$ in Examples before.
It gives rise to a product

$$s_n = s_0 \left(\prod_{i=1}^{n} a \right) = s_0 a^n$$

Which means $\{s_n\}_{n=0}^{\infty}$ is a geometric sequence.

It is useful to view

$$1, a, a^2, a^3, a^4, \ldots$$

as a "basic solution" and any other solution is a multiple of this.

Solution of second-order case

A second-order linear RR $s_n = as_{n-1} + bs_{n-2}$ does not lead to a product as occurred in the first-order case.

An example is the Fibonacci Recurrence Relation $f_n = f_{n-1} + f_{n-2}$. Which has solution

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

It is **NOT** the case that there is a constant c such that each term is c times the previous term.

However, the first step towards solving second-order equations is to note that *certain* solutions **are** geometric progressions — these will then feature in the general solution.

Basic solutions for the second-order case

Theorem

Suppose $s_n = as_{n-1} + bs_{n-2}$ is a second-order linear recurrence relation. Then the geometric series $1, r, r^2, r^3, \ldots$ is a solution if, and only if, r is a solution of the quadratic equation $x^2 = ax + b$

Proof.

If $s_n = r^n$ is a solution, then with $n = 2$ we have $r^2 = ar + b$. Conversely, if $r^2 = ar + b$ then the sequence $s_n = r^n$ satisfies

$$s_n = r^n = r^2 r^{n-2} = (ar + b)r^{n-2} = ar^{n-1} + br^{n-2} = as_{n-1} + bs_{n-2} \quad \square$$

The equation $x^2 = ax + b$ is known as the characteristic equation.

The Fibonacci recurrence relation (FRR)

$$f_n = f_{n-1} + f_{n-2}$$

is linear, and we can solve its characteristic equation $x^2 = x + 1$ using the quadratic formula:

$$x^2 - x - 1 = 0 \quad \Rightarrow \quad x = \frac{1 \pm \sqrt{5}}{2} \approx 1.618, -0.618$$

By the theorem, there are two exponential solutions to the FRR:

$$\begin{array}{rlccccc}
\text{Solution 1:} & 1, & \frac{1+\sqrt{5}}{2}, & \left(\frac{1+\sqrt{5}}{2}\right)^2, & \left(\frac{1+\sqrt{5}}{2}\right)^3, & \left(\frac{1+\sqrt{5}}{2}\right)^4, & \ldots \\
\approx & 1, & 1.618, & 2.618, & 4.236, & 6.854, & \ldots \\
\\
\text{Solution 2:} & 1, & \frac{1-\sqrt{5}}{2}, & \left(\frac{1-\sqrt{5}}{2}\right)^2, & \left(\frac{1-\sqrt{5}}{2}\right)^3, & \left(\frac{1-\sqrt{5}}{2}\right)^4, & \ldots \\
\approx & 1, & -0.618, & 0.382, & -0.236, & 0.146, & \ldots
\end{array}$$

Both of these sequences satisfy the FRR, but neither is the Fibonacci sequence. They are not integers!

However, we can get some integer solutions by manipulating the two solutions above.
Adding the two solutions above gives

$$2,\ 1,\ 3,\ 4,\ 7, \dots$$

which is an integer solution to the FRR. However this is not the Fibonacci sequence either.
Subtracting gives approximately

$$0,\ 2.236,\ 2.236,\ 4.472,\ 6.708, \dots \quad \text{or exactly:} \quad 0,\ \sqrt{5},\ \sqrt{5},\ 2\sqrt{5},\ 3\sqrt{5}, \dots$$

which is $\sqrt{5}$ times the Fibonacci sequence. Consequently

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Theorem

Suppose $s_n = as_{n-1} + bs_{n-2}$ is a second-order linear recurrence relation for a sequence $s_0, s_1, s_2, \dots$. If the quadratic equation $x^2 = ax + b$ has two distinct (real) solutions r_1, r_2 then for any constants A, B, the sequence

$$s_n = Ar_1^n + Br_2^n$$

is a solution to the recurrence relation, and every solution is of this form.

Sketch proof.

By the theorem, both $s_n = r_1^n$ and $s_n = r_2^n$ are solutions. Also, the set of solutions is closed under sums and constant multiples, so $s_n = Ar_1^n + Br_2^n$ is a solution for any A, B.

If $u_0, u_1, u_2, \dots$ was a sequence that was a solution, then solve $A + B = u_0$ and $Ar_1 + Br_2 = u_1$ simultaneously to find A, B. Then $t_n = u_n - Ar_1^n - Br_2^n$ will be a solution with $t_0 = t_1 = 0$. Then $t_n = 0$ for all n, so $u_n = Ar_1^n + Br_2^n$. □

Example

Solve the 2nd order linear recurrence relation

$$a_n = 4a_{n-1} - 3a_{n-2}.$$

- The characteristic equation is $x^2 - 4x + 3 = (x-3)(x-1)$.
- general solution: $a_n = A \cdot 3^n + B \cdot 1^n = A3^n + B$.

Example

Solve the 2nd order linear recurrence relation with initial conditions

$$a_n = 4a_{n-1} - 3a_{n-2} \; ; \quad a_0 = 0 \; , \quad a_1 = 1$$

- initial conditions: $0 = a_0 = A \cdot 3^0 + B = A + B$ and $1 = a_1 = A \cdot 3^1 + B$.
- Solving for A and B yields $A = 1/2$, $B = -1/2$.
- Solution: $a_n = \frac{1}{2} \cdot 3^n - \frac{1}{2}$.

In the case where the characteristic equation has only one solution, this method doesn't work. Instead of r_1^n and r_2^n being the basic solutions, we use r^n and nr^n.

Theorem

Suppose $s_n = as_{n-1} + bs_{n-2}$ is a second-order linear recurrence relation for a sequence $s_0, s_1, s_2, \ldots$. If the quadratic equation $x^2 = ax + b$ has only one real solution r then for any constants A, B, the sequence

$$s_n = Ar^n + Bnr^n$$

is a solution to the recurrence relation, and every solution is of this form.

Example

Solve the 2nd order linear recurrence relation with initial conditions

$$a_n = 4a_{n-1} - 4a_{n-2}\ ; \quad a_0 = 0\ , \quad a_1 = 1$$

- Characteristic equation: $x^2 - 4x + 4 = (x-2)^2$.
- basic solutions 2^n and $n2^n$.
- general solution: $a_n = A2^n + Bn2^n = (A + Bn)2^n$.
- initial conditions: $0 = a_0 = (A + B \cdot 0)2^0 = A$ and $1 = a_1 = (A + B \cdot 1)2^1 = 2(A + B)$.
- Solving for A and B yields $A = 0$ and $B = 1/2$.
- Solution: $a_n = \frac{1}{2} \cdot n2^n = n2^{n-1}$.

Example (Fibonacci's Population dynamics problem)

A pair of baby rabbits will produce a new pair of rabbits every year from the age 2. How many pairs in n years?

- Let p_n be the total number of pairs in year n.
- $p_1 = p_2 = 1$.
- In year 3, the first pair produces a new pair: $p_3 = 2$.
- In year 4, the first pair produces another pair: $p_4 = 3$.
- In year 5, the first pair and the second pair produce a new pair each: $p_5 = 5$.
- In general, in year n, all the p_{n-2} pairs that were present in year $n-2$ produce new pairs:
$$p_n = p_{n-1} + p_{n-2}.$$
- So the solution is the Fibonacci sequence.

Example (Population dynamics)

A population of animals with life span of 2 years has per-capity fecundity of 1 offspring per 1 year-old parent and 2 offspring per year per 2 year-old parent. In years 0 and 1 there were 60 births each. How many births in subsequent years?

- Let b_n be the number of births in year n.
- In year n, every animal born in year $n-1$ contributes 1 birth, and every animal born in year $n-2$ contributes 2 births.
- This implies $b_n = b_{n-1} + 2b_{n-1}$.
- The characteristic equation is $x^2 - x - 2 = (x+1)(x-2)$, hence the general solution $b_n = A\,2^n + B(-1)^n$.
- The initial conditions $60 = b_0 = A + B$ and $60 = b_1 = 2A - B$ yield $A = 40$ and $B = 60$.
- The number of births in year n is

$$b_n = 40 \cdot 2^n + (-1)^n \cdot 20.$$

Example (Scheduling)

A TV station wants to fill a break between programs with advertisements, which are either half a minute or a full minute. How many ways can this be done? We need to fill n slots (half minutes). Let a_n be the number of ways of filling these n slots.

- There is one way of filling 0 slots (no spots at all): $a_0 = 1$
- There is one way of filling 1 slot (one half-minute spot): $a_1 = 1$
- There are two ways of filling two slots (one full-minute spot or two half-minute spots): $a_2 = 2$
- There are three ways of filling three slots ((full,half), (half,full), (half,half,half)): $a_2 = 3$
- In general: half-minute spot into slot $n \implies a_{n-1}$ possibilities for the remaining $n-1$ slots. full-minute spot in slots $n-1$ and $n \implies a_{n-2}$ possibilities to fill the remaining $n-2$ slots.
- Hence $a_n = a_{n-1} + a_{n-2}$, and we get the Fibonacci numbers again.

Example (Properties of Fibonacci numbers)

Sometimes the applications feed back into the mathematics. We can use the previous example to prove the identity

$$f_m f_n + f_{m-1} f_{n-1} = f_{m+n-1}.$$

Consider the scheduling of an advertising break of $m + n$ slots. There are two possibilities for where we are at the end of m slots

- The end of m slots is the end of an advert. So we have filled m slots, and n slots.
- The end of m slots is the middle of a long advert. So we have filled $m - 1$ slots, and $n - 1$ slots.

Example (Bell numbers)

We can use similar principles to count all partitions of a set. The nth Bell number B_n is the number of partitions of $\{1, \ldots, n\}$.
To find a recurrence relation for this sequence, consider a partition of $\{1, \ldots, n\}$. The symbol n will be in a cell $[n]$ of the partition, say with j elements ($1 \leqslant j \leqslant n$) and the other $k = n - j$ elements will be partitioned. There are $\binom{n-1}{j-1} = \binom{n-1}{k}$ ways to choose the elements in $[n]$. There are B_k ways to partition the k elements not in $[n]$.
Hence the number of ways to partition $\{1, \ldots, n\}$ so that the cell containing n has $j = n - k$ elements is $\binom{n-1}{k} B_k$ and so in total

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

The initial condition $B_0 = 1$ comes from a degenerate case.

Summary

Recurrence relations with or without initial conditions

Solution A solution vs the general solution vs the solution to an RR with IC.

Basic solutions Sums and products.

Linear RRs Characteristic equation to find exponential solutions

Linear RRs Use initial conditions to find values of the free parameters