The University of Newcastle
School of Electrical Engineering and Computer Science

COMP2230/COMP6230 Algorithms

Tutorial Week 3 Solutions

2-6 August 2021

Tutorial

1. Find Θ for the following functions

i. $6n^3 + 12n + 1$

Solution: $\Theta(n^3)$

ii. $(n + 1)(n + 3) /(n + 2)$

Solution: $\Theta(n)$

2. Find Θ for the number of times the statement x=x+1 is executed.

i.
```
for i=1 to n
    for j=1 to i
        for k=1 to j
            x=x+1
```

Solution:

i	Number of times the statement x=x+1 is executed
1	1
2	1+2
3	1+2+3
…	
i	1+2+…+i = i(i+1)/2
…	
n	1+2+…+n

$$\sum_{p=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2}\left(\sum_{p=1}^{n} i^2 + \sum_{p=1}^{n} i\right) = \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right) = \theta(n^3)$$

ii.

```
i=2
while (i < n){
   i=i*i
   x=x+1
}
```

Solution:

$$2, 2^2, (2^2)^2 = 2^4, 2^8, 2^{16}$$

$$2^{2^0}, 2^{2^1}, 2^{2^2}, \dots, 2^{2^k}$$

$$2^{2^k} = n$$
$$\lg\lg 2^{2^k} = \lg\lg n$$

$$k = \lg\lg n$$

$\Theta(lg\ lg\ n)$

3. Use iteration to solve the following recurrence relations:

i. $a_n = a_{n-1} + 3, n > 1; a_1 = 2$

Solution:
$a_n = a_{n-1} + 3$
$= a_{n-2} + 3 + 3 = a_{n-2} + 2\times3$
$= a_{n-3} + 3 + 2\times3 = a_{n-3} + 3\times3$
….
$= a_{n-i} + 3i$
…
$= a_{n-(n-1)} + 3(n-1)$
$= a_1 + 3(n-1)$
$= 2 + 3(n-1) = 2 + 3n - 3 = 3n - 1$

ii. $a_n = 2a_{n-1}, n > 0; a_0 = 1$

Solution: $a_n = 2^n$

4. True or false?

i. $n^2 = O(n^3)$

Solution: ***T***

ii. $n^2 = \Omega(n^3)$

Solution: ***F***

iii. $n^2 = \Theta(n^3)$

Solution: ***F***

5. Arrange the following functions in ascending order in their growth rate. That is, if a function *g(n)* comes after function *f(n)* then *f(n) = O(g(n))*. Prove your answers.

$$n^2, n^3, 100n^2, n \lg n, 2^n$$

Solution: $n \lg n, n^2, 100n^2, n^3, 2^n$ (note that there is another possible ordering!) To prove that $f(n) = O(g(n))$ we need to show that there are constants $c > 0$ and n_1 such that $f(n) \leq c\, g(n)$ for all $n \geq n_1$. Here for all $n \geq 100$ we have $n \lg n \leq n^2 \leq 100n^2 \leq n^3 \leq 2^n$

6. Prove the following:

iii. $n! = O(n^n)$

Solution: $n! = n(n-1)\ldots 1 \leq n \cdot n \cdot n \cdot \ldots \cdot n = n^n,\ n \geq 1$

iv. $\Sigma_{i=1}^{n}\, i \lg i = \Theta(n^2 \lg n)$

Solution:

$$\sum_{i=1}^{n} i\, lg\, i \leq \sum_{i=1}^{n} n\, lg\, n = n(n\, lg\, n) = n^2 lg\, n, \quad n \geq 1$$

Therefore $\sum_{i=1}^{n} i\, lg\, i = O(n^2 lg\, n)$

$$\sum_{i=1}^{n} i\, lg\, i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i\, lg\, i \geq \left(\frac{n}{2}\right)^2 lg\left(\frac{n}{2}\right) \geq \frac{n^2 lg\, n}{8}$$

for $n \geq 4$, therefore $\sum_{i=1}^{n} i\, lg\, i = \Omega(n^2 lg\, n)$

7. Prove that

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!}$$

Solution: See Theorem 2.1.18 on page 23 in *Algorithms* (Johnsonbaugh and Schaeffer, 2004).

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1) \times \ldots \times (n-k+1)}{k!} \leq \frac{n^k}{k!}$$

$$\binom{n}{k} = \frac{n(n-1) \times \ldots \times \big(n-(k-1)\big)}{k(k-1) \times \ldots \times 1} = \frac{n}{k} \times \frac{n-1}{k-1} \times \ldots \times \frac{\big(n-(k-1)\big)}{\big(k-(k-1)\big)} \geq \left(\frac{n}{k}\right)^k,$$

as

$$\frac{n-i}{k-i} \geq \frac{n}{k}, \text{for every } i \ \leq k$$

Indeed, by manipulating the expression above we get
$k(n-i) \geq n(k-i)$

$kn - ki \geq nk - ni$

$-ki \geq -ni$

$-k \geq -n$

$n \geq k,$ which is always satisfied for $\binom{n}{k}$.

8. Prove that n^k is a smooth function

Solution: From example 2.4.13, page 62 of *Algorithms* (Johnsonbaugh and Schaeffer, 2004).

The function $f(n) = n^k$ is smooth because

a. for any positive integer $m \geq 2$, $f(mn) = (mn)^k = m^k n^k = Cf(n)$ for all $n \geq 1$, where $C = m^k$, and
b. $f(n) \leq f(n+1)$ for all $n \geq 1$.

9. Prove that $T(n)$ is well defined for all n by recurrence relation $T(n)=aT(n/b)+cn^k$ when n/b denotes $\lfloor n/b \rfloor$.

Solution: From solutions to exercise 2.4.38, page 660 of *Algorithms* (Johnsonbaugh and Schaeffer, 2004).

We use induction to prove that $T(n)$ is well-defined for all n.

Base Case: $T(0)$ is well-defined since its value is given as an initial condition.

Inductive Hypothesis: Assume that $T(n)$ is well-defined for all $n < l$.

Inductive Step: Prove that $T(n)=aT(n/b)+cn^k$ is well-defined for $n = l$. Since l > 0 and $b \geq 2$, $l/b < l$. Therefore, $\left\lfloor \frac{l}{b} \right\rfloor < l$. By the inductive assumption $T\left(\left\lfloor \frac{l}{b} \right\rfloor\right)$ is well-defined and it follows that $T(l) = aT\left(\left\lfloor \frac{l}{b} \right\rfloor\right) + cl^k$ is well-defined as well.

10. Use the Main (Master) Recurrence Theorem to find Θ for each of the following functions:

i. $T(n) = 2T(n/2) + f(n); f(n) = n^2$

ii. $T(n) = 2T(n/2) + f(n); f(n) = 5$

Solution:

i. $T(n) = \Theta(n^2)$ because $a = b = k = 2$ (i.e. first case, $a < b^k$).

ii. $T(n) = \Theta(n)$ because $a = b = 2, k = 0$ (i.e. third case, $a > b^k$).

Homework

11. Find Θ for the following functions

i. $(6n + 1)^2$

Solution: $\Theta(n^2)$

ii. $3n^2 + 2n \lg n$

Solution: $\Theta(n^2)$

12. Find Θ for the number of times the statement x=x+1 is executed.

i.
```
for i=1 to 2n
    x=x+1
```

Solution: $\Theta(n)$

ii.
```
for i=1 to n
    for j=1 to i
        for k=1 to i
            x=x+1
```

Solution: $\Theta(n^3)$

13. Use iteration to solve the following recurrence relations:

i. $a_n = 2a_{n-1} + 1, n > 1; a_1 = 1$

Solution: $a_n = 2^n - 1$

ii. $a_n = 2^n a_{n-1}, n > 0; a_0 = 1$

Solution: $a_n = 2^{n(n+1)/2}$

14. True or false?

i. $2^n = O(2^{n+1})$

Solution: ***T***

ii. $2^n = \Omega(2^{n+1})$

Solution: ***T***

iii. $2^n = \Theta(2^{n+1})$

Solution: ***T***

15. Arrange the following functions in ascending order in their growth rate. That is, if a function $g(n)$ comes after function $f(n)$ then $f(n) = O(g(n))$. Prove your answers.

$$10^n,\ n^{1/3},\ n^n,\ \lg n,\ 2^\wedge((\lg n)^{1/2})$$

Solution: $\lg n,\ 2^\wedge((\lg n)^{1/2}),\ n^{1/3},\ 10^n,\ n^n$

Hint: *Look at logarithms of functions*

16. Prove the following:

i. $2^n = O(n!)$

Solution: $2^n = 2\cdot 2\cdot 2\cdot \ldots\ 2 \leq 2(2\cdot 3\cdot \ldots \cdot n) = 2n!,\ n \geq 1$

ii. $\lg(n^k + c) = \Theta(\lg n)$, for every fixed $k > 0$ and $c > 0$

Solution:
$\lg(n^k + c) \leq \lg(n^k + c\,n^k) = \lg(c+1) + k\lg n \leq (c+1) + k\lg n \leq (c+1)\lg n + k\lg n = (c+1+k)\lg n$ for $n \geq 2$, as $(c+1) \leq (c+1)\lg n$ for $n \geq 2$.
Thus $\lg(n^k + c) = \boldsymbol{O(\lg n)}$

$\lg(n^k + c) \geq \lg n^k = k\lg n,\ n \geq 1$
Thus $\lg(n^k + c) = \boldsymbol{\Omega(\lg n)}$

It follows that $\lg(n^k + c) = \boldsymbol{\Theta(\lg n)}$, for every fixed $k > 0$ and $c > 0$

17. Prove that $n^{\log_b a}$ is a smooth function.

18. Use the Main (Master) Recurrence Theorem to find Θ for each of the following functions:

i. $T(n) = 4T(n/2) + f(n);\ f(n) = n$

ii. $T(n) = 4T(n/2) + f(n);\ f(n) = n^2$

Solution:
i. $T(n) = \Theta(n^2)$ because $a = 4,\ b = 2,\ k = 1$ (i.e. third case, $a > b^k$).
ii. $T(n) = \Theta(n^2 \log n)$ because $a = 4,\ b = 2,\ k = 2$ (i.e. second case, $a = b^k$).

19. Solve the following homogeneous recurrence:
T(n) = 6T(n-1) + 9T(n-2), T(0) = 0, T(1) = 3

Solution:

Homogeneous recurrence: $T(n) - 6T(n-1) - 9T(n-2) = 0$ for $n > 1; T(0) = 0, T(1) = 3$

Characteristic equation: $x^2 - 6x - 9 = 0$

Roots: $x_1 = 3(1+\sqrt{2}),\ x_2 = 3(1-\sqrt{2})$

General form: $T(n) = C_1 3^n(1+\sqrt{2})^n + C_2 3^n(1-\sqrt{2})^n$

Constants:

$T(0) = C_1 3^0(1+\sqrt{2})^0 + C_2 3^0(1-\sqrt{2})^0 = C_1 + C_2 = 0;$ Thus $C_2 = -C_1$

$T(1) = C_1 3(1+\sqrt{2}) - C_1 3(1-\sqrt{2}) = 3C_1(1+\sqrt{2}-1+\sqrt{2}) = 6C_1\sqrt{2} = 3$; Thus $C_1 = \frac{1}{2\sqrt{2}}$

Finally, we have $T(n) = \frac{3^n}{2\sqrt{2}}(1+\sqrt{2})^n - \frac{3^n}{2\sqrt{2}}(1-\sqrt{2})^n$

More Exercises

20. Find Θ for the following function: $2 + 4 + 6 + \ldots + 2n$

Solution: $\Theta(n^2)$

21. Find Θ for the number of times the statement x=x+1 is executed.

```
j=n
while (j ≥ 1){
  for i=1 to j
    x=x+1
  j=j/3
}
```

Solution: $\Theta(n)$

22. Use iteration to solve the following recurrence relations:

$a_n = 2 + \Sigma_{i=1}^{n-1} a_i, n > 1; a_1 = 1$

Solution: $a_n = 3 \cdot 2^{n-2} = (1+2)\ 2^{n-2} = 2^{n-2} + 2^{n-1}$

23. True or false?

i. $n! = O((n+1)!)$

Solution: ***T***

ii. $n! = \Omega((n+1)!)$

Solution: ***F***

iii. $n! = \Theta((n+1)!)$

Solution: ***F***

24. Arrange the following functions in ascending order in their growth rate. That is, if a function $g(n)$ comes after function $f(n)$ then $f(n) = O(g(n))$. Prove your answers.

$$n^{2.5}, (2n)^{1/2}, n+10, 10^n, 100^n, n^2 \lg n$$

Solution: $(2n)^{1/2}, n+10, \ n^2 \lg n, \ n^{2.5}, 10^n, 100^n$

25. Prove that $H_n = \Sigma_{i=1}^{n} (1/i) = \Theta(\log n)$.
(Hint: Use $1/n \le \lg (n+1) - \lg n < 2/n$.)

Solution: *See Theorem 2.3.9 on page 47-48 in Algorithms* (Johnsonbaugh and Schaeffer, 2004).

26. Prove the following:

$1^k + 2^k + \ldots + n^k = \Theta(n^{k+1})$

Solution: See solution to ex. 46, Sec 2.3 on page 658 in *Algorithms* (Johnsonbaugh and Schaeffer, 2004).

27. Prove that $\log (n!) = \Theta (n \log n)$.

Solution:

O: $\log n! = \log n + \log (n-1) + \ldots + 1 + 0 \leq \log n + \log n + \ldots + \log n = n \log n,\ n \geq 1$

Thus $\log n! = O(n \log n)$.

Ω: $\log n! \geq \log n + \log (n-1) + \ldots + \log \lceil n/2 \rceil = \lceil (n+1)/2 \rceil \log \lceil n/2 \rceil \geq \lceil n/2 \rceil \log \lceil n/2 \rceil \geq n/2 \log (n/2) = n/2(\log n - \log 2) \geq n/4 \log n,\ n \geq 4$.

Thus $\log n! = \Omega(n \log n)$

28. Consider the following algorithm that computes a^n. Let c_n be the number of multiplications required to compute a^n.

```
exp( a,n) {
if ( n = = 1)
return a
m = ⌊n / 2⌋
return exp( a, m)* exp( a, n-m)
}
```

i. Find a recurrence relation and initial conditions for the sequence $\{c_n\}$.

ii. Solve the recurrence relation in case n is a power of 2.

iii. Solve the recurrence relation for every positive integer n.

Solution:

i. $c_n = 1 + c_{\lfloor n/2 \rfloor} + c_{\lceil n/2 \rceil}$

ii. $c_n = n - 1$

iii. Show by induction that $c_n = n - 1$

29. Prove that if a and b are numbers such that $0 \leq a < b$ then $(n+1)a^n < (b^{n+1} - a^{n+1})/(b - a) < (n+1)b^n$.

Solution: See Theorem 2.1.28 on page 26 in *Algorithms* (Johnsonbaugh and Schaeffer, 2004).

30. Prove that the sequence $\{(1+1/n)^n\}$ *is* increasing and bounded above by 4.

Solution: See Theorem 2.1.29 on page 26 in *Algorithms* (Johnsonbaugh and Schaeffer, 2004).

31. Prove that $1/n \le lg(n+1) - lg\ n < 2/n$.

Solution: See Theorem 2.1.30 on page 27 in *Algorithms* (Johnsonbaugh and Schaeffer, 2004).

32. Prove that $n^k \log_b n$ is a smooth function.

33. Use the Main (Master) Recurrence Theorem to find Θ for the following function:
$T(n) = 2T(n/2) + f(n); f(n) = n^3$