An upper bound on the edge number of a simple planar graph

We will use Euler's formula.

Theorem. *Let G be a connected planar graph with v vertices, e edges and a planar drawing with f faces. Then*

$$v + f = e + 2. \tag{1}$$

Corollary. *Let $G = (V, E)$ be a simple planar graph with $v \geqslant 3$ vertices and e edges. Then $e \leqslant 3v - 6$.*

Proof. We may assume that G is connected, because otherwise we can add edges and still have a simple planar graph. Let f be the number of faces in a planar drawing of G, and let $G^* = (V^*, E^*)$ be the corresponding dual graph. For every face of G, there is one vertex in G^*, hence $|V^*| = f$, and we may list them as $V^* = \{v_1^*, v_2^*, \ldots, v_f^*\}$. We first observe

$$2e = 2|E| \overset{(i)}{=} 2|E^*| \overset{(ii)}{=} \sum_{i=1}^{f} \delta(v_i^*), \tag{2}$$

where (i) comes from the fact that by construction the dual graph has the same number of edges as the original graph, and (ii) is the Handshake theorem applied to the dual graph G^*.

For every edge x on the boundary of a face F_i of G, there is an edge x^* in the dual graph which connects the corresponding dual vertex v_i^* with the dual vertex v_j^* where v_j^* is the vertex corresponding to the face F_j such that the edge x is on the common boundary of the faces F_i and F_j. This is illustrated in Figure 1.

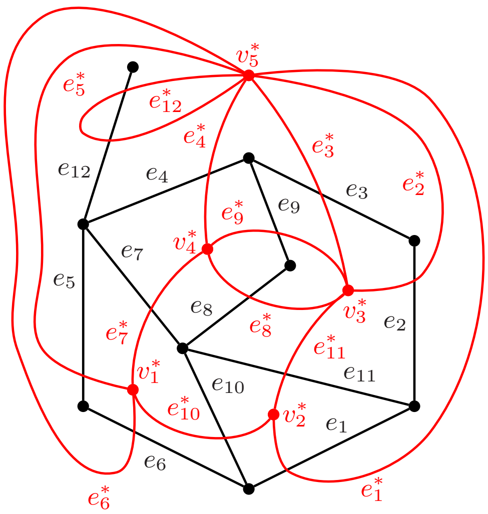

Figure 1: Illustration of the degrees in the dual graphs. For example, the dual vertex v_3^* has degree 5, because the boundary of the corresponding face consists of 5 edges: e_{11}, e_2, e_3, e_9 and e_8. There are two edges between v_3^* and v_4^*, because the corresponding faces have two common edges: e_8 and e_9. Note that on both sides of the edge e_{12} there is the same face, and this corresponds to the loop e_{12}^* at the dual vertex v_5^*. The dual vertex v_5^* corresponds to the infinite face and has degree 8, because the boundary of the infinite face consists of eight edges: e_1, e_2, e_3, e_4, e_5, e_6 and e_{12} counted twice. Note that if e_{12} is drawn inside the face corresponding to the dual vertex v_4^* then we get a loop at v_4^* instead.

Since the the boundary of a face in a planar drawing of a simple graph must contain at least three edges we have $\delta(V_i^*) \geqslant 3$ for all $i \in \{1, 2, \ldots, n\}$. This implies the inequality

$$\sum_{i=1}^{f} \delta(v_i^*) \geqslant 3f, \tag{3}$$

because on the left hand side there is a sum of f terms, and each of these terms is at least 3. Putting together (2) and (3), and using Euler's formula (1), we obtain

$$2e \overset{(2)}{=} \sum_{i=1}^{f} \delta(v_i^*) \overset{(3)}{\geqslant} 3f \overset{(1)}{=} 3(e+2-v).$$

From $3(e+2-v) \leqslant 2e$, it follows that $e \leqslant 3v-6$ (by adding $3v-2e-6$ on both sides). $\square$