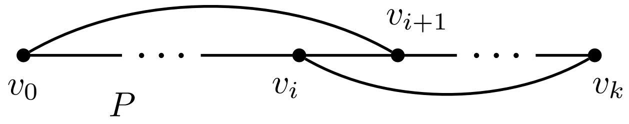
A proof of Dirac's theorem

Theorem (Dirac 1952)**.** *Let $G = (V, E)$ be a simple graph with $|V| = n \geqslant 3$ and every vertex with degree $\geqslant n/2$. Then G contains a Hamiltonian cycle.*

Proof. First note that G is necessarily connected, because otherwise any vertex in a smallest component of G has degree at most $n/2 - 1$. Let $P = (v_0, v_1, \ldots, v_k)$ be a longest simple path in G. By the maximality of P, all the vertices that are adjacent to v_0 or v_k are on P. Hence at least $n/2$ of the vertices $v_0, \ldots, v_{k-1}$ are adjacent to v_k, and at least $n/2$ of the vertices $v_1, \ldots, v_k$ are adjacent to v_0. The last statement is equivalent to saying that at least $n/2$ vertices v_i among $v_0, \ldots, v_{k-1}$ have the property that v_{i+1} is adjacent to v_0. To summarize: Among the at most $n - 1$ vertices $v_0, \ldots, v_{k-1}$, there are at least $n/2$ which are adjacent to v_k, and there are at least $n/2$ whose successor on P is adjacent to v_0.



Consequently, there must be some v_i that has both properties, i.e., v_i is adjacent to v_k and v_{i+1} is adjacent to v_0. Now we claim that the cycle

$$C: \; v_0, v_1, \ldots, v_i, v_k, v_{k-1}, \ldots, v_{i+1}, v_0$$

is a Hamiltonian cycle. Suppose C is not a Hamiltonian cycle. Then, using the connectedness of G, there is a vertex w outside C which is adjacent to v_j for some j. Now we distinguish two cases.

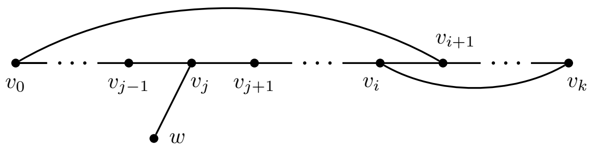


Figure 1: Case 1.

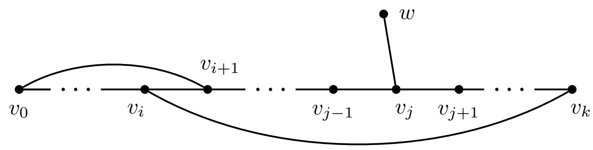


Figure 2: Case 2.

Case 1. If $j \leqslant i$, then $w, v_j, v_{j+1}, \ldots, v_i, v_k, v_{k-1}, \ldots, v_{i+1}, v_0, v_1, \ldots, v_{j-1}$ is a simple path of length $k+1$, contradicting the maximality of P.

Case 2. If $j > i$, then $w, v_j, v_{j+1}, \ldots, v_k, v_i, v_{i-1}, \ldots, v_0, v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ is a simple path of length $k+1$, contradicting the maximality of P. $\square$

Remark

This proof uses two very important principles.

The extremal principle. For a given *finite* set of elements we can always find one which is maximal (or minimal) with respect to some measure. In this case, among the finitely many simple paths in G we picked one with maximal length.

The pigeon hole principle. If $t + 1$ balls are put into t boxes, then there must be at least one box containing at least two balls. More generally, if $rt + 1$ balls are put into t boxes, then there must be at least one box containing at least $r + 1$ balls. In our case, the boxes are labeled with the numbers $0, 1, \ldots, k - 1$, and we distribute the balls in two rounds. In the first round, we put a ball into box i whenever v_i is adjacent to v_k, and in the second round we put a ball in box i whenever v_{i+1} is adjacent to v_0. By the argument in the proof, we have used at least $n/2 + n/2 = n$ balls. The number of boxes is at most $n-1$, so there is a box with two balls, and the label i of such a box gives the required vertex v_i with the property that v_i is adjacent to v_k, and v_{i+1} is adjacent to v_0.