MATH1510 - Discrete Mathematics
Graphs

University of Newcastle

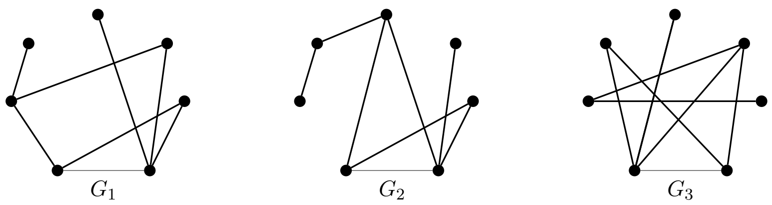
Definition (Isomorphism)

vague Two graphs are isomorphic if they can be drawn so that they look the same.

descriptive Graphs G and H are isomorphic if there is a *bijection* between the vertices of G and the vertices of H in such a way that 2 vertices in G are adjacent if and only if the 2 corresponding vertices in H are adjacent.

precise Graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic if there are bijections $f : V \to W$ and $g : E \to F$ such that for each $v \in V$, $e \in E$, edge e is incident on vertex v if and only if edge $g(e)$ is incident on vertex $f(v)$.

Which graphs are isomorphic?



- A G_1 and G_2
- B G_1 and G_3
- C G_2 and G_3
- D all three

How to test for isomorphism?

To prove whether two graphs are isomorphic, we have to find bijections matching corresponding vertices and edges, as in the definition.

Finding such mappings can be difficult: there are *too many bijections* between any two sets of the same size. This task can be made easier by identifying certain types of common features to match.

These features must be of the type that are *preserved* by the isomorphism, otherwise we may be trying to match vertices or edges that needn't be matched.

For instance, we would only try matching degree-3 vertices of G with degree-3 vertices of H. So if we sort vertices in both graphs according to their degree, then our search for an isomorphism is more efficient. This works, since the isomorphism *must preserve the degree of each vertex*.

In general, properties that are preserved by isomorphisms are useful in determining *how* or *whether* graphs are isomorphic.
To show two graphs are *not* isomorphic, we only need to find some way that they differ where isomorphic graphs should not.
For example, a pair of isomorphic graphs have:

- the same number of vertices and edges
- the same degree sequence
- the same number of components, articulation points, cycles, etc.
- the same diameter

So if two graphs have (for instance) a different number of cycles, they cannot be isomorphic.

Definition

An invariant of a graph G is a property of the graph that is identical in any graph isomorphic to G. (That is, it does not depend on how the graph is drawn.)

Sometimes it is intuitively easy to see why a given property is an invariant (based on the “drawn differently” idea) and yet it may be difficult to formally prove that a given property is an invariant.

On the previous slide, each of the properties “number of vertices”, “number of edges”, “degree sequence”, ... is an invariant.

Which is not a graph invariant?

A minimum number of colours to colour the vertices such that no two adjacent vertices receive the same colour.

B number of pairs of crossing edges in a drawing of G

C number of connected components

D minimal length of a cycle in G

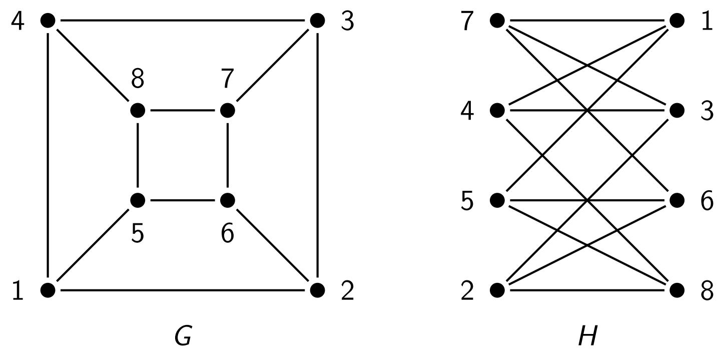
Non-isomorphic graphs

To show that 2 graphs G and H are not isomorphic, find some invariant of G which is different in H.

Sometimes this is simple as G and H having a different number of vertices.

Note however that the converse does not work — just because two graphs agree on some invariant (such as degree sequence) the graphs may not be isomorphic.

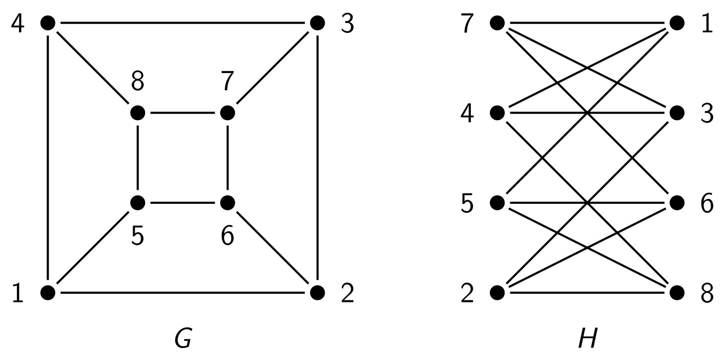
Finding isomorphisms



We need a bijection $f : V(G) \to V(H)$ such that

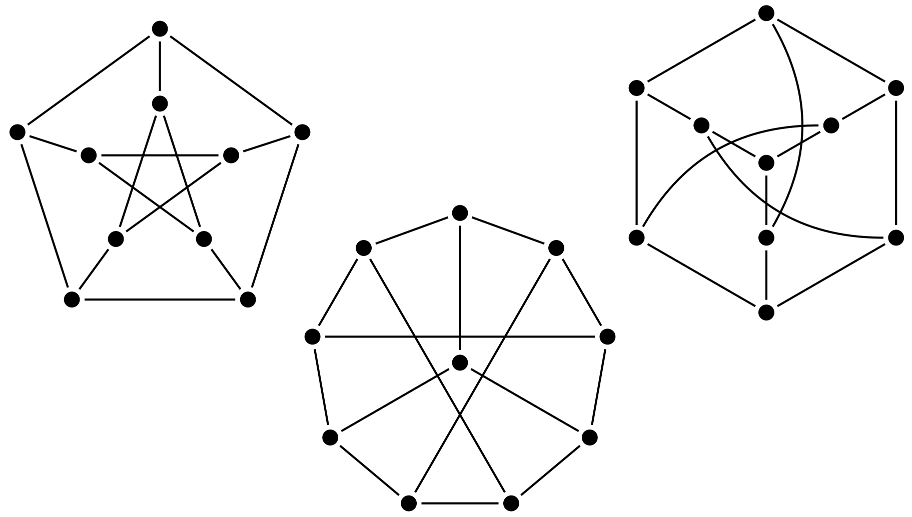
$$\{i,j\} \in E(G) \iff \{f(i), f(j)\} \in E(H).$$

Finding isomorphisms



- Let's try $f(1) = 1$. (That's a choice. There are other options.)
- $f(2)$ is adjacent to $f(1) = 1$ in H, hence $f(2) \in \{4, 5, 7\}$. Let's try $f(2) = 4$.
- $f(3)$ is adjacent to $f(2) = 4$ in H, but $f(3) \neq f(1) = 1$, hence $f(3) \in \{3, 8\}$. Let's try $f(3) = 3$.
- Now everything is forced: $f(4) = 7$, $f(5) = 5$, $f(6) = 8$, $f(7) = 7$, $f(8) = 6$.

The Petersen graph



Examples

1. Find all non-isomorphic graphs with 3 vertices.
2. Find all non-isomorphic graphs with 4 vertices that are simple, connected and have no cycles.
3. Find all non-isomorphic graphs with 5 vertices that are simple, connected and have no cycles.
4. Find all non-isomorphic simple graphs with 4 vertices
5. Find all non-isomorphic simple connected graphs with 5 vertices

Adjacency matrices

If we have two graphs represented in their adjacency matrices, how can we tell whether they are isomorphic? We need to take into account that any graph can have several possible adjacency matrices, depending on how it vertices are ordered.

However, the ordering of the vertices is the only point on which the adjacency matrices of isomorphic graphs can vary. This is encapsulated in the following theorem.

Theorem

Two graphs G_1 and G_2 are isomorphic if, and only if, there is some ordering of the vertices of each such that the adjacency matrices A_1 and A_2 are equal.

How difficult is it to decide if two (large) graphs G_1 and G_2 are isomorphic? Using the theorem above, there is an algorithm:

- Fix an ordering of the vertices of G_1 and find its adjacency matrix A_1
- For each permutation (ordering) of the vertices of G_2:
  - Find the adjacency matrix A_2
  - If $A_1 = A_2$ then halt, G_1 and G_2 are isomorphic
- G_1 and G_2 are not isomorphic

Q: How much work is required by this algorithm?
Q: Is this the best possible algorithm?

It is not known if there is an efficient (i.e. polynomial) algorithm to decide if two given graphs G and H are isomorphic.

Isomorphisms of directed graphs

With a digraph, the direction of an edge is significant, so an isomorphism of digraphs must preserve the direction.

So when we match the vertices, we must be careful to check that the matching edges go in the corresponding direction.

How would we find all non-isomorphic digraphs with 3 vertices?

Planar graphs

Definition

A graph G is planar if a representation of it can be drawn on the plane without any edges crossing. (Such a drawing we call a planar configuration of G.)

Desirable for, e.g.

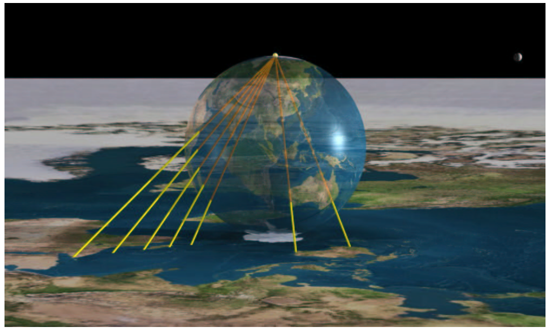
- printed circuit (conducting wires between components shouldn't cross)
- layout of machinery on a shop floor (conveyor belts, access alleys, etc. shouldn't cross)

Definition

When a planar graph is drawn with no edge crossings, its edges divide the plane into regions called faces. The area *outside* the graph is also a face.

Stereographic projection

Any graph that has a planar representation can be projected onto the surface of a sphere, and vice-versa, using the stereographic projection.



We saw this previously with the *edge graphs of polyhedra*.
So a graph G is planar if and only if it can be drawn on the surface of a sphere with no edges crossing.

The dual graph (informally)

If a graph G is in a planar configuration, the dual graph G^* has

- a vertex of G^* for each face of G,
- an edge between two vertices of G^* for each edge separating two faces of G
- a face of G^* for each vertex of G.

Informally, replace vertices with faces, faces with vertices, and an edge separating two faces with an edge joining two vertices.

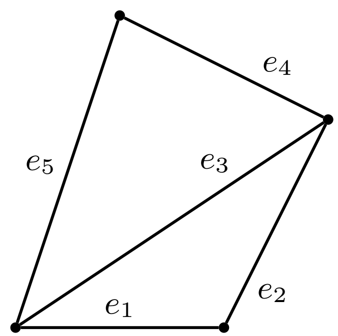
Formal definition of dual graph

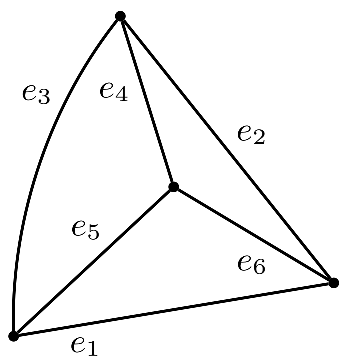
Formally, suppose G is a planar graph with vertices $V = \{v_1, \dots, v_n\}$, edges $E = \{e_1, \dots, e_m\}$ and faces $F = \{f_1, \dots, f_r\}$.

Then the dual graph of $G = (V, E)$ is a planar graph $G^* = (V^*, E^*)$ that has

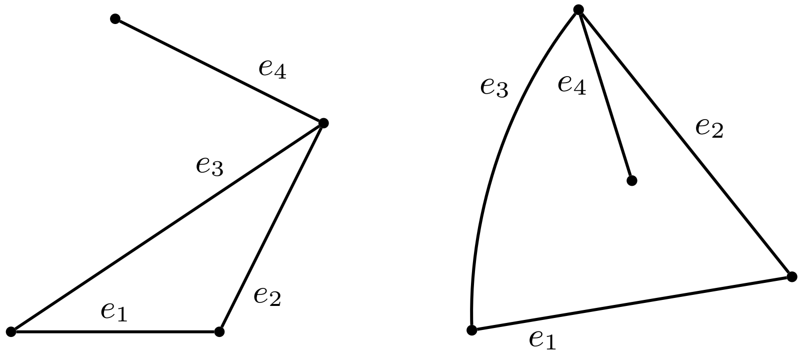
- vertices $V^* = \{v_1^*, \dots, v_r^*\}$ (corresp. to F),
- edges $E^* = \{e_1^*, \dots, e_m^*\}$ (corresp. to E),
- faces $F^* = \{f_1^*, \dots, f_n\}$ (corresp. to V),
- in G^*, e_k^* is incident on vertices v_i^* and v_j^* when the corresponding edge e_k separates faces f_i and f_j.

What are the duals of the following graphs?





And what about these?



Euler's formula

Theorem (Euler, 1750)

If G is a connected planar graph with v vertices, e edges and f faces, then $f + v = e + 2$.

Proof

We proceed by induction on e, the number of edges.
Basis Step: For $e = 1$, there are two cases to check:

- a loop at a single vertex: $v = 1$, $e = 1$, $f = 2$
- and and edge connecting two distinct vertices: $v = 2$, $e = 1$, $f = 1$

Proof (continued)

Inductive Step: Suppose the formula is true for all connected planar graphs with k edges. Let G be any connected planar graph with $k+1$ edges. We will remove an edge giving a new graph G' with only k edges.

Case 1. If G has a vertex of degree 1, remove it and the incident edge to get a graph G': $v' = v - 1$, $e' = e - 1$, $f' = f$.

Case 2. If G has no vertices of degree 1, then G contains a cycle. Let x be an edge in this cycle, then x separates a face inside the cycle from a face outside the cycle. Remove x to get a graph G': $v' = v$, $e' = e - 1$, $f' = f - 1$.

In both cases the inductive hypothesis $v' + f' = e' + 2$ implies $v + f = e + 2$, and so the inductive step is proven. □

Upper bounds on edges

Using Euler's theorem on a planar simple graph, upper bounds on the number of edges in the graph can be obtained.

Corollary (Upper bound on edges)

If G is a connected, simple planar graph with at least 3 vertices, then $e \leqslant 3v - 6$

Proof.

Let G^* be the dual graph of G then since G is simple $\delta(v_i^*) \geqslant 3$ for each vertex v_i^* if G^*.
Consequently, the sum of $\delta(v_i^*)$ over all dual vertices v_i^* is at least $3|V^*| = 3f$.
But $f = 2 + e - v$, so $2e \geqslant 3(2 + e - v)$ giving $e \leqslant 3v - 6$. □

Proving non-planarity (the first example)

Upper bound on edges

G connected, simple, planar, $v \geqslant 3 \implies e \leqslant 3v - 6$

Corollary

K_5 is not planar.

Proof.

$$e = 10 > 9 = 3 \cdot 5 - 6 = 3v - 6.$$ □

Q: Can we ever get $e = 3v - 6$?

A: Yes, if and only if all vertices v^* in the dual graph G^* have $\delta(v^*) = 3$, meaning all faces in G are triangles — G is a triangulation of the plane.

Proving non-planarity (the second example)

Corollary (Upper bound on edges for bipartite graph)

If a connected, simple planar graph contains no triangular faces, then $e \leqslant 2v - 4$.

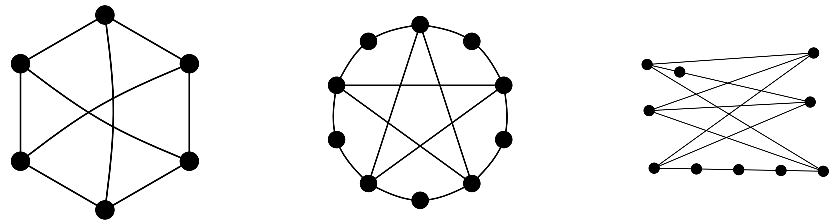
Note: *Bipartite graphs* have no triangular faces, so they must have $e \leqslant 2v - 4$ if they are to be planar.

Corollary

$K_{3,3}$ is not planar.

Graphs containing K_5 and $K_{3,3}$

Q: How do we tell if a graph is *planar* without actually finding a way of drawing it without crossings?
Certainly, it can't contain a subgraph which is a K_5 or a $K_{3,3}$. Nor can it contain subgraphs that are trivial modifications of these:



A: The surprising fact is that this is as bad as it gets: If there is no obstruction in form of a K_5 or $K_{3,3}$ inside our graph G, then G can be drawn in the plane without crossings.

Series reduction

Removing a vertex of degree 2 and joining the edges is called a series reduction. If a graph G_2 is obtained from a graph G_2 by a sequence of series reductions, we say G_1 and G_2 are homeomorphic to each other. More precisely, we have the following

Definition

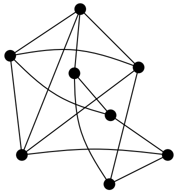
Two graphs G_1 and G_2 are homeomorhic if there is a graph H which can be obtained from both G_1 and G_2 by a sequence of series reductions.

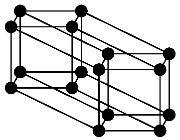
Kuratowski's Theorem

Theorem (Kuratowski's Theorem, 1935)

A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

"Only if" is obvious. "If" is difficult, beyond MATH1510.





Algorithmic complexity

Planarity testing can be done in time that grows linear in the number of vertices.

Summary

Isomorphism. incidence preserving pair of bijections ($V_1 \to V_2$, $E_1 \to E_2$) between two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

Invariants. properties that are preserved by isomorphisms

Planar graphs. can be drawn without crossings in the plane (or on the sphere)

Dual graphs. vertices $\longleftrightarrow$ faces

Euler's formula. $f + v = e + 2$

Series reduction. remove vertex of degree 2 and connect its neighbours by a new edge

Homeomorphic. two graphs that have a common series reduction

Kuratowski's theorem. planar $\iff$ no subgraph homeomorphic to K_5 or $K_{3,3}$