SCHOOL of ELECTRICAL ENGINEERING & COMPUTING
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The UNIVERSITY of NEWCASTLE

# Comp3320/6370 Computer Graphics

## Semester 2, 2018

### Exercises II

**Practice exercises for lectures in week 3-5**

Version 28

This paper provides (partial) solutions to some of the exercises given in the lectures. These partial solutions should help for exam preparation. It is recommended to look at the solutions only after you have first tried to solve the exercises yourself. Typically there are several different ways to solve an exercise. Please fill the gaps, try some variations and check if the provided solutions are correct.

### Exercise 10 (Additivity of rotations in 2D)

Question: Show that for all $\alpha, \beta \in [0, 2\pi]$, we have $R(\alpha)\, R(\beta) = R(\alpha + \beta)$.

Solution:

$$
\begin{aligned}
R(\alpha) &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\
R(\beta) &= \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}
\end{aligned}
$$

Then,

$$
\begin{aligned}
R(\alpha)\cdot R(\beta) &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} \\
&= R(\alpha+\beta)
\end{aligned}
$$

*Note:* Compare with the rule for multiplying two complex numbers $ae^{i\alpha}$ and $be^{i\beta}$ for the special case that $a = b = 1$. This was discussed in the lectures and the slides.

### Exercise 11 (Rotations in 3D)

Question: Use standard mathematical notation in 3-dimensional space and describe the rotation matrices about the $x$, $y$, $z$ axis for $90°, 180°, 270°, 360°$.

Solution:

For $90^\circ$ :

$$
\begin{aligned}
R_x(90^\circ) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ \\ 0 & \sin 90^\circ & \cos 90^\circ \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
R_y(90^\circ) &= \begin{bmatrix} \cos 90^\circ & 0 & \sin 90^\circ \\ 0 & 1 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
R_z(90^\circ) &= \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

We can calculate rotation matrices for $180^\circ, 270^\circ, 360^\circ$ in the same manner.

<u>Alternative solution using homogeneous notation:</u>

For $90^\circ$ :

$$
\begin{aligned}
R_x(90^\circ) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ & 0 \\ 0 & \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
R_y(90^\circ) &= \begin{bmatrix} \cos 90^\circ & 0 & \sin 90^\circ & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
R_z(90^\circ) &= \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

We can calculate rotation matrices for $180^\circ, 270^\circ, 360^\circ$ in the same manner.

### Exercise 12 (Rotations in 3D: Non-commutativity)

Question: Use rotations in 3D and calculate the matrix for an $x$-roll of $30^\circ$, followed by an $y$-roll of $45^\circ$, followed by a $z$-roll of $60^\circ$.

First calculate each of the the matrices $R_x(30^\circ)$, $R_y(45^\circ)$, $R_z(60^\circ)$ and the product $R_z(60^\circ) \cdot R_y(45^\circ) \cdot R_x(30^\circ)$.

Is it different from $R_x(30^\circ) \cdot R_y(45^\circ) \cdot R_z(60^\circ)$ ?

Which is the correct order and why ?

Answer:

$R_z(60^\circ) \cdot R_y(45^\circ) \cdot R_x(30^\circ)$ is the correct order because in our notation convention (using vectors in column notation) transforms are applied right to left. (Still should do the explicit matrix multiplications here).

**Exercise 13** (Inverse of a matrix)

Question: What is the inverse (if it exists) of:

a) The $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $a, b, c, d \in \mathbf{R}$.

b) The $3 \times 3$ matrix

$$B = \begin{bmatrix} -3 & 5 & -4 \\ 2 & -6 & 12 \\ 1 & -2 & 2 \end{bmatrix}$$

c) The $3 \times 3$ matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Solution (hints):

a)

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

if $\det(A) := |A| := ad - bc \neq 0$.

(If you don't remember please look up the definition of a determinant in one of the linear algebra books. It is a basic construct that is not difficult but a bit lengthy to explain. It is usually covered as part of basic maths in the context of solving systems of linear equations.)

b) Check if the following matrix is correct (could be an error in there)

$$B^{-1} = \begin{bmatrix} -3 & 0.5 & -9 \\ -2 & 0.5 & -7 \\ -0.5 & 0.25 & -2 \end{bmatrix}$$

c) In general if $M \in M(n \times n; \mathbf{R})$ is a square matrix, then the inverse is given by

$$M^{-1} = \frac{1}{\det(M)} \operatorname{adj}(M),$$

where the adjoint matrix of $M$ is given by

$$\operatorname{adj}(M) = (b_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$$

with

$$b_{ij} = (-1)^{(i+j)} \det(M(\{j\}', \{i\}'))$$

where $M(\{j\}', \{i\}')$ is the matrix $M$ where the $j$-th row and the $i$-th column have been deleted (note the order of $j$ and $i$).

Accordingly if the following $3 \times 3$ matrix is invertible

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

we obtain the following $3 \times 3$ matrix as its inverse (please check if correct):

$$M^{-1} = \frac{1}{|M|} \begin{bmatrix} \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{12} \\ m_{33} & m_{32} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} \\ \\ \begin{vmatrix} m_{23} & m_{21} \\ m_{33} & m_{31} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{11} \\ m_{23} & m_{21} \end{vmatrix} \\ \\ \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{11} \\ m_{32} & m_{31} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} \end{bmatrix}$$

For larger matrices numerical methods such as LU decomposition can be applied.

**Exercise 14** (Homogeneous and non-homogeneous coordinates):

Question: Below several points (or vectors ?) are given in homogeneous coordinates. Determine for each of them the corresponding non-homogeneous 3D Cartesian coordinates.
a) (3, 6, 5, 1)
b) (2, 4, 6, 4)
c) (0, 0, 2, 0.25)
d) (0, 0, 0, 1)
e) (1, 0, 0, 0)

Solution:
a) (3, 6, 5)
b) (0.5, 1, 1.5)
c) (0, 0, 8)
d) (0, 0, 0)
e) Vector (1, 0, 0).

**Exercise 15 (The inverse of some transforms)**

Question: Let $\mathbf{F} \in M(4 \times 4, \mathbf{R})$ have the general form of a 3D transform in homogeneous coordinates. That is, if the submatrix $\mathbf{M} \in M(3 \times 3, \mathbf{R})$ is either a 3D rotation, scaling or shearing matrix and the submatrix $\mathbf{T} \in M(3 \times 1, \mathbf{R})$ is a translation vector then $\mathbf{F}$ can be written as:

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{T} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccc|c} m_{00} & m_{01} & m_{02} & t_x \\ m_{10} & m_{11} & m_{12} & t_y \\ m_{20} & m_{21} & m_{22} & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Let $\mathbf{N} = \mathbf{M}^{-1}$ be the inverse of $\mathbf{M}$. Show that the inverse of $\mathbf{F}$ is given by:

$$\mathbf{F}^{-1} = \left[\begin{array}{c|c} \mathbf{N} & -\mathbf{NT} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccc|c} n_{00} & n_{01} & n_{02} & -(\mathbf{NT})_x \\ n_{10} & n_{11} & n_{12} & -(\mathbf{NT})_y \\ n_{20} & n_{21} & n_{22} & -(\mathbf{NT})_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Answer:

We have to show that $FF^{-1} = F^{-1}F = I$.

$$\begin{aligned}
FF^{-1} &= \begin{bmatrix} M & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} N & -NT \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} MN & M(-NT) + T \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} I_3 & -T + T \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} I_3 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I_4
\end{aligned}$$

$$\begin{aligned}
F^{-1}F &= \begin{bmatrix} N & -NT \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M & T \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} NM & NT - NT \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} I_3 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I_4
\end{aligned}$$

No guarantee that the solutions are correct yet. Please email any errors that you detect. Check blackboard for updates.