COMP2270/6270 – Theory of Computation
Second week

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Exercise 1) Let $\Sigma = \{a, b, c\}$. How many elements are there in Σ^*?

> Countably infinitely many.

Exercise 2) (Chapter 2, Exercise 1 of Ref. [1])
Consider the language $L = \{1^n2^n : n > 0\}$. Is the string 122 in L? Why?

> No. Every string in L must have the same number of 1's as 2's.

Exercise 3) (Chapter 2, Exercise 2 of Ref. [1])
Let $L_1 = \{a^nb^n : n > 0\}$. Let $L_2 = \{c^n : n > 0\}$. For each of the following strings, state whether or not it is an element of L_1L_2:

a) ε.
b) aabbcc.
c) abbcc.
d) aabbcccc.

Justify your decision in each case.

> a) No.
> b) Yes.
> c) No.
> d) Yes.

Exercise 4) Give a formal definition of the ***Kleene star*** **?**

> $L^* = \{ w : w = \varepsilon \lor \exists k > 0 \ (\exists w_1, w_2, ..., w_k \in L \ (w = w_1w_2...w_k))\}$
>
> OR
>
> $L^* = \bigcup_{i\in \mathbb{N}_0} L^i = L^0 \cup L^1 \cup L^2 \ldots$
>
> where, $L^0 = \{\epsilon\}$ and $L^n = LL^{n-1}$ for $n > 0$

Exercise 5) (Chapter 2, Exercise 3 of Ref. [1])
Let L_1 = {peach, apple, cherry} and L_2 = {pie, cobbler, ε}.
List the elements of L_1L_2 in lexicographic order.

`apple`, `peach`, `cherry`, `applepie`, `peachpie`, `cherrypie`, `applecobbler`, `peachcobbler`, `cherrycobbler`
(We list the items shortest first. Within a given length, we list the items alphabetically.)

Exercise 6) With L_1 and L_2 as defined above in Exercise 5, give the Cartesian product of these two languages ?

$L_1 \times L_2$ = { (peach, pie), (peach, cobbler), (peach, ε), (apple, pie), (apple, cobbler), (apple, ε), (cherry, pie), (cherry, cobbler), (cherry, ε)}

Exercise 7) Again, referring to L_1 and L_2 in Exercise 5, give the *power set* of both languages ? Give the formal definition of a *power set* of a language. Is the intersection of both power sets empty or not ? (Justify your answer)

P(L_1) = {∅, {peach}, {apple}, {cherry}, {peach, apple}, {peach, cherry}, {apple, cherry}, {peach, apple, cherry}}
P(L_1) = {∅, { pie}, {cobbler}, {ε}, { pie, cobbler}, { pie, ε}, { cobbler, ε}, { pie, cobbler, ε}}

$P(L) = \{w : w \subseteq L\}$

P(L_1) ∩ P(L_2) = {∅} – it is not an empty set, it is a set containing one element

Exercise 8) (Chapter 2, Exercise 5 of Ref. [1])
Consider the language L of all strings drawn from the alphabet {a, b} with at least two different substrings of length 2.
a. Describe L by writing a sentence of the form $L = \{w \in \Sigma^* : P(w)\}$, where Σ is a set of symbols and P is a first-order logic formula. You may use the function $|s|$ to return the length of s. You may use all the standard relational symbols (e.g., =, ≠, <, etc.), plus the predicate *Substr*(s, t), which is *True* iff s is a substring of t.

$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* : \exists x, y\ (x \neq y \wedge |x| = 2 \wedge |y| = 2 \wedge Substr(x, w) \wedge Substr(y, w))\}$.

b. List the first six elements of a lexicographic enumeration of L.

`aab`, `aba`, `abb`, `baa`, `bab`, `bba`

Exercise 9) (Chapter 2, Exercise 6 of Ref. [1])

For each of the following languages L, give a simple English description. Show two strings that are in L and two that are not (unless there are fewer than two strings in L or two not in L, in which case show as many as possible).

a) $L = \{w \in \{a, b\}^* : \text{exactly one prefix of } w \text{ ends in a}\}$.

L is the set of strings composed of zero or more `b`'s and a single `a`. `a`, `bba` and `bbab` are in L. `bbb` and `aaa` are not.

b) $L = \{w \in \{a, b\}^* : \text{all prefixes of } w \text{ end in a}\}$.

$L = \varnothing$, since ε is a prefix of every string and it doesn't end in `a`. So all strings are not in L, including `a` and `aa`.

c) $L = \{w \in \{a, b\}^* : \exists x \in \{a, b\}^+ (w = axa)\}$.

L is the set of strings over the alphabet {`a`, `b`} whose length is at least 3 and that start and end with `a`. `aba`, and `aaa` are in L. ε, `a`, `ab` and `aa` are not.

Exercise 10) (Chapter 2, Exercise 7 of Ref. [1])

Are the following sets closed under the following operations? If not, what are their respective closures?

a) The language {a, b} under concatenation.

Not closed. $\{w \in \{a, b\}^* : |w| > 0\}$

b) The odd length strings over the alphabet {a, b} under Kleene star.

Not closed because, if two odd length strings are concatenated, the result is of even length. The closure is the set of all strings drawn from the alphabet {`a`, `b`}.

c) $L = \{w \in \{a, b\}^*\}$ under reverse.

Closed. L includes all strings of `a`'s and `b`'s, so, since reverse must also generate strings of `a`'s and `b`'s, any resulting string must have been in the original set.

d) $L = \{w \in \{a, b\}^* : w \text{ starts with a}\}$ under reverse.

Not closed. L includes strings that end in `b`. When such strings are reversed, they start with `b`, so they are not in L. But, when any string in L is reversed, it ends in `a`. So the closure is $\{w \in \{a, b\}^* : w \text{ starts with a}\} \cup \{w \in \{a, b\}^* : w \text{ ends with a}\}$.

e) $L = \{w \in \{a, b\}^* : w$ ends in a$\}$ under concatenation.

Closed.

Exercise 11) (Chapter 2, Exercise 8 of Ref. [1])
For each of the following statements, state whether it is *True* or *False*. Prove your answer.

a) $\forall L_1, L_2$ ($L_1 = L_2$ iff $L_1^* = L_2^*$).

False. Counterexample: $L_1 = \{\texttt{a}\}$. $L_2 = \{\texttt{a}\}^*$. $L_1 \neq L_2$ but $L_1^* = L_2^* = \{\texttt{a}\}^* \neq \{\texttt{a}\}$.

b) $(\varnothing \cup \varnothing^*) \cap (\neg\varnothing - (\varnothing\varnothing^*)) = \varnothing$ (where $\neg\varnothing$ is the complement of $\varnothing$).

False. The left hand side equals $\{\varepsilon\}$, which is not equal to $\varnothing$.

c) Every infinite language is the complement of a finite language.

False. Counterexample: Given some nonempty alphabet Σ, the set of all even length strings is an infinite language. Its complement is the set of all odd length strings, which is also infinite.

d) $\forall L$ ($(L^R)^R = L$).

True.

e) $\forall L_1, L_2$ ($(L_1 L_2)^* = L_1^* L_2^*$).

False. Counterexample: $L_1 = \{\texttt{a}\}$. $L_2 = \{\texttt{b}\}$. $(L_1 L_2)^* = (\texttt{ab})^*$. $L_1^* L_2^* = \texttt{a}^*\texttt{b}^*$.

f) $\forall L_1, L_2$ ($(L_1^*L_2^*L_1^*)^* = (L_2 \cup L_1)^*$).

True.

g) $\forall L_1, L_2$ ($(L_1 \cup L_2)^* = L_1^* \cup L_2^*$).

False. Counterexample: $L_1 = \{\texttt{a}\}$. $L_2 = \{\texttt{b}\}$. $(L_1 \cup L_2)^* = (\texttt{a} \cup \texttt{b})^*$. $L_1^* \cup L_2^* = \texttt{a}^* \cup \texttt{b}^*$.

h) $\forall L_1, L_2, L_3$ ($(L_1 \cup L_2)\, L_3 = (L_1 L_3) \cup (L_2 L_3)$).

True.

i) $\forall L_1, L_2, L_3\ ((L_1 L_2) \cup L_3 = (L_1 \cup L_3)\ (L_2 \cup L_3))$.

False. Counterexample: $L_1 = \{\texttt{a}\}$. $L_2 = \{\texttt{b}\}$. $L_3 = \{\texttt{c}\}$.

$(L_1 L_2) \cup L_3 = \{\texttt{ab}, \texttt{c}\}$.

$(L_1 \cup L_3)\ (L_2 \cup L_3) = (\texttt{a} \cup \texttt{c})(\texttt{b} \cup \texttt{c})$

$= \{\texttt{ab}, \texttt{ac}, \texttt{cb}, \texttt{cc}\}$

j) $\forall L\ ((L^+)^* = L^*)$.

True.

k) $\forall L\ (\varnothing L^* = \{\varepsilon\})$.

False. For any L, and thus for any L^*, $\varnothing L = \varnothing$.

l) $\forall L\ (\varnothing \cup L^+ = L^*)$.

False. $\varnothing \cup L^+ = L^+$, but it is not true that $L^+ = L^*$ unless L includes ε.

m) $\forall L_1, L_2\ ((L_1 \cup L_2)^* = (L_2 \cup L_1)^*)$.

True.

Exercise 12) Using mathematical induction:

(a) Prove that $n! \leq n^n$ for any integer $n \geq 1$.

(b) Prove that $3^{2n} - 1$ is divisible by 8 for any integer $n \geq 0$.

(a)

STEP 1 (base case): For n=1 $n! \leq n^n$ is true, since $1! = 1^1$.

STEP 2 (Induction Hypothesis): Suppose $n! \leq n^n$ is true for some $n = k \geq 1$, that is $k! \leq k^k$.

STEP 3 (Induction Step): Now we will show that $n! \leq n^n$ is true for $n = k + 1$, that is $(k + 1)! \leq (k + 1)^{k+1}$. We have $(k + 1)! = k! \cdot (k + 1) \leq k^k \cdot (k + 1) < (k + 1)^k \cdot (k + 1) = (k + 1)^{k+1}$.

(b)

STEP 1 (base case) : For n=0 $3^{2n} - 1$ is divisible by 8, since $3^{2*0} - 1=0$.

STEP 2 (Induction Hypothesis): Suppose $3^{2n} - 1$ is divisible by 8 is true for some $n = k \geq 0$, that is $3^{2k} - 1$ is divisible by 8.

STEP 3 (Induction Step): Now we show that "$3^{2n} - 1$ is divisible by 8" is true for n = k + 1. We have $3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \cdot 9 - 1 = 3^{2k}(8 + 1) - 1 = 3^{2k} \cdot 8 + 3^{2k} - 1$. Both terms is divisible by 8 so $3^{2k+2} - 1$ is by 8 .

REFERENCES

[1] Elaine Rich, Automata Computatibility and Complexity: Theory and Applications, Pearson, Prentice Hall, 2008.