MATH1510 - Discrete Mathematics
Summary and Revision

University of Newcastle

Semester 2

Graphs

- Q: What is a graph?

  A: a pair (V, E) of sets V (vertex set) and E (edge set), where every edge is associated with two (not necessarily distinct) vertices, its endpoints.

- What do the terms incident and adjacent mean?

  A: An edge e and a vertex v are incident, if v is an endpoint of e. Two vertices v and w are adjacent, if there is an edge with endpoints v and w.

- What are loops? What are parallel edges?

  A: A loop is an egde whose endpoints coincide. Two edges e and e' are called parallel if they have the same endpoint pairs.

- When do we call a graph simple?

  A graph is simple if it does not have loops or parallel edges.

- What is the degree of a vertex?

  The degree of a vertex v is the number of edges incident with v.

Representing graphs

- Draw the graph with adjacency listing
  - a: b,c,d,d
  - b: a,c,d
  - c: a,b,d
  - d: a,a,b,c
  - e:
  - f: g,h
  - g: f,g,h
  - h: f,g
- What is its adjacency matrix?
- And what is its incidence matrix?

Paths and cycles

- Q: What are paths and cycles in a graph?
  A: A path is a sequence $v_0, e_1, v_1, \ldots, e_k, v_k$ alternating between vertices and edges such that e_i is incident with v_{i-1} and v_i for every i. A cycle is a path with $v_0 = v_k$.
- What is the distance of two vertices in a graph?
  A: The minimal length of a path between the two vertices.
- What is the diameter of a graph?
  A: The maximum distance between any two vertices.
- What is the maximum number of vertices in a graph of diameter 3 in which every vertex has degree at most 2?
  A: 5
- Can you state a sufficient condition for the existence of cycles?
  A: all degrees at least 2

Special graphs

- What is the complete graph K_n? How many edges has this graph?
  A: a simple graph with n vertices and all possible edges. $\binom{n}{2} = \frac{n(n-1)}{2}$.
- What is the complete bipartite graph $K_{m,n}$? How many edges has this graph?
  A: A graph whose vertex set can be partitioned into two sets of A and B with $|A| = m$ and $|B| = n$, such that every edge has one endpoint in A and one endpoint in B, and all these pairs occur.
- What is the n-cube Q_n? How many vertices and edges has Q_n?
  A: The graph on the vertex set $\{0,1\}^n$, where two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are adjacent if and only if they differ in exactly one coordinate.
- Is the n-cube a bipartite graph?
  A: Yes, the partition is given by the parity of the coordinate sums.

The handshake theorem

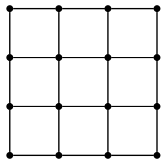
- State the handshake theorem!
  A: The sum of all degrees is twice the number of edges.
- What is its interpretation in terms of adjacency and incidence matrices?
  A: In both cases, the sum of a row equals the degree of the corresponding vertex.
- Is there a graph with degree sequence $3, 4, 4, 5, 5, 6$?
  A: No. The number of odd degree vertices needs to be even.
- Is there a bipartite graph with degree sequence

$$3, 3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6, 6, 6\ ?$$

  A: No, because 58 is not divisible by 3.

Euler cycles

- What is the Königsberg bridge problem?
  A: The origin of graph theory. Is there a walk across the seven bridges of Königsberg using every bridge exactly once.
- What is an Euler cycle in a graph G?
  A: A cycle using every edge of G exactly once.
- How can we decide if a graph has an Euler cycle?
  A: An Euler cycle exists if and only if G is connected and all degrees are even.
- How long is the shortest cycle in this graph that uses all edges?



A: 28

Hamilton cycles

- What is a Hamilton cycle?
  A: A cycle using every vertex exactly once.
- What about the existence of Hamilton cycles?
  A: That's hard. Necessary conditions: connected, no vertices of degree 1, no articulation points. Sufficient condition: all degrees $\geqslant n/2$ (Dirac's theorem)
- What is a Gray code?
  A: A Hamilton cycle in the n-cube which can be constructed by recursion over n.
- What is the Travelling Salesperson Problem (TSP)?
  A: The problem to find a cheapest Hamilton cycle in a weighted directed graph.

Isomorphic graphs

- When are two graphs $G = (V, E)$ and $H = (W, F)$ isomorphic?
  A: If there is a bijection $f : V \to W$ such that
  $\{x, y\} \in E \iff \{f(x), f(y)\} \in F$.
- What are graph invariants?
  A: Properties that are the same for isomorphic graphs.
- Examples?
  A: diameter, degree sequence, number of connected components, minimal length of a cycle
- How do we check if two graphs are isomorphic?
  A: If we can find a separating invariant, we are done. Otherwise it's hard: basically have to check all possible bijections between the vertex sets.

Planar graphs

- What is a planar graph?
  A: A graph that can be drawn in the plane (or on the sphere) without crossing edges.
- What are faces of a planar configuration?
  A: the regions into which the plane is partitioned by the edges
- What is the dual of a planar configuration?
  A: A graph which has vertices corresponding to the faces of the original graph, where two such dual vertices are adjacent if there is an edge of the original graph separating the two faces.
- What is Euler's formula? A: $f + v = e + 2$
- What are bounds following from Euler's formula?
  A: $e \leqslant 3v - 6$ for connected, simple planar graphs, $e \leqslant 2v - 4$ for connected simple, planar graphs without triangular faces

Kuratowski's theorem

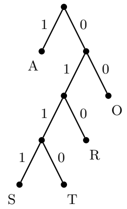
- What are your two favourite non-planar graphs?
  A: K_5 and $K_{3,3}$
- What does Kuratowski's theorem say?
  A: A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$?
- What is a series-reduction?
  A: Take a vertex of degree 2, delete it, and connect its two neighbours by an edge.
- What does it mean for two graphs G and G' to be homeomorphic?
  A: That there is a graph H such that both G and G' can be reduced to H by a sequence of series reductions.

Trees

- Give four definitions of a tree!
  A: (1) acyclic and connected, (2) connected and $|V| - 1$ edges, (3) unique path between any pair of vertices, (4) acyclic and $|V| - 1$ edges
- How many trees are there with 6 vertices?
  A: 6
- What is a rooted tree?
  A: a tree with a special vertex, the root
- What is the level of a vertex of a rooted tree? What is its height?
  A: level = distance from he root, height = maximum level
- What are parent and children vertices of a node v?
  A: The parent of v is the vertex immediatly preceding v on the path between the root and v. A root w is a child of v if v is the parent of v.

Ordered trees

- What is an ordered tree?
  A: A rooted tree together with an ordering for the children of any vertex.
- What is a binary tree?
  A: A rooted tree in which every vertex has at most two children.
- What is the Huffman encoding for the word "STAR"



A: 011101101010

- How do you construct an optimal Huffman code for a given text?
  A: Using a frequency table

Binary search trees

- What is a binary search tree (BST)?
  A: A binary tree in with ordered data associated with the vertices such that for every vertex v the entries in the left subtree of v iare smaller than the entry in v, and the entries in the right subtree are larger than the entry in v.
- How do you search for an entry in a BST?
  A: going down from the root, and asking myself at every vertex if I should turn left or right
- How do you insert a new element in a BST?
  A: I pretend I'm searching, and when I find the empty spot where my element should be, I put it there.

Examples

- Place the following words in a binary search tree of depth less than or equal to four:

  postorder, preorder, inorder, binary, search, tree, algorithm, find

- Write the list of words down in the order that you find them using *inorder* traversal.
- The following is an expression in polish notation (prefix):

$$- \times + ABC / DE$$

  What is the fully parenthesized form of this expression in infix notation?

- As briefly as you can, describe the algorithm that you used to translate between the different forms of expression.

Efficiency

- When is a BST balanced?
  A: When for every vertex the heights of the left and the right subtree differ by at most one.
- Why do we care?
  A: Because balanced BSTs are more efficient for searching and inserting
- How do we balance a BST?
  A: using rotations
- What is the height of a balanced BST with n vertices?
  A: about $\log n$
- Name two more tree structures for storing data?
  A: binary trees, B-trees

Spanning Trees

- What is a spanning tree of a graph G?
  A: A spanning subgraph (i.e. containing all vertices) that is a tree.
- What are breadth first search (BFS) and depth first search (DFS)?
  A: ways to explore all vertices of a connected graph; lead to particular spanning trees; BFS: stay close to the root as long as possible, DFS: get away from the root as quickly as possible
- What is the minimum spanning tree problem?
  A: to find a spanning tree of smallest weight in a weighted graph
- How does Prim's algorithm construct an MST?
  A: starting at any vertex and growing the tree greedily
- How does Kruskal's algorithm construct an MST?
  A: growing a forest greedily and joining trees, until a single tree remains

Algorithms

- Name some sorting algorithms.
  A: selection sort, insertion sort, bubble sort, merge sort
- Which one is most efficient (in the worst case)?
  A: merge sort, run time $O(n \log n)$
- What is a recursive algorithm?
  A: an algorithm that calls itself on smaller instances
- Examples?
  A: Euclidean Algorithm, Merge Sort

Recurrence relation

- What is a recurrence relation (RR)?
  A: a way of describing certain sequences $(s_0, s_1, s_2, \ldots)$ by specifying a relation between s_n and previous terms s_{n-1}, s_{n-2} etc. which holds for all n

- What is a second-order homogeneous, linear RR with constant coefficients?
  A: a RR of the form $s_n = as_{n-1} + bs_{n-2}$ with constants $a, b \in \mathbb{R}$

- How do you solve a RR $s_n = as_{n-1} + bs_{n-2}$ with initial conditions $s_0 = P$ and $s_1 = Q$?
  1. Solve the characteristic equation $x^2 - ax - b = 0$.
  2. If there are two solutions $r_1 \neq r_2$, then find A and B by solving the system $A + B = P$, $r_1 A + r_2 B = Q$. Solution: $s_n = Ar_1^n + Br_2^n$
  3. If there is only one solution r, then find A and B by solving the system $A = P$, $rA + rB = Q$. Solution: $s_n = Ar^n + Bnr^n$