SCHOOL *of* ELECTRICAL ENGINEERING *and* COMPUTING
FACULTY *of* ENGINEERING *and* BUILT ENVIRONMENT
The UNIVERSITY *of* NEWCASTLE

Comp3320/6370 Computer Graphics

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Semester 2, 2018

LECTURE

Curves and Surfaces

October 22, 2018

OVERVIEW

Note: These lecture slides use concepts, statements and figures from chapters 11 and 12 of the book Akenine-Möller and Haines (2002) but also some other general material not available in these books.

Curves and Surfaces

- Curves and surfaces can be described by equations (implicitly or in parametric form). The equations are evaluated, triangles are created and sent to the rendering pipeline. General curves and surfaces allow for more detail and precision than polygons and splines.
- Parametric curves can move the viewer or some object along a predefined path. How can we control the speed? Is there a a natural local coordinate system at each point of the curve?
- Parametric surfaces are an extension of parametric curves. Classification of surfaces allows us to determine what different topology types of surfaces are possible.

Curves in R^2

Give examples for parameterised curves

$$\gamma : [0,1] \longrightarrow \mathbf{R^2},\ t \mapsto \gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

E.g.

- Moving from one point to another.
- The circle curve S^1.
- What variations of the circle curve can you think of?
- ...

Curves in $\mathrm{R}^{\mathbf{3}}$

Give examples for parameterised curves

$$\gamma : [0,1] \longrightarrow \mathbf{R^3},\ t \mapsto \gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

E.g.

- Moving from one point to another.
- The circle curve S^1.
- What variations of the circle curve can you think of?
- ...

Examples Curves

a) $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2,\ t \mapsto \gamma(t) = (t, t^2)$.

b) $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2,\ t \mapsto \gamma(t) = (\cos(t), \sin(t))$.

c) $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2,\ t \mapsto \gamma(t) = (2\cos(t), 3\sin(t))$.

d) $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2,\ t \mapsto \gamma(t) = (\cos(t/4), \sin(t/4))$.

e) $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^3,\ t \mapsto \gamma(t) = (\cos(t), \sin(t), t)$.

f) $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^3,\ t \mapsto \gamma(t) = (\cos(t), \sin(t), \frac{t}{2\pi})$.

g) $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^3,\ t \mapsto \gamma(t) = (\cos(4t), \sin(4t), \frac{t}{2\pi})$.

h) $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^3,\ t \mapsto \gamma(t) = (\cos(t), \sin(t), t^2)$.

i) $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2,\ t \mapsto \gamma(t) = (2e^{3t}\cos(t), 2e^{3t}\sin(t))$.

Quater Circle Curves

j) $\gamma : [0,1] \longrightarrow \mathbb{R}^2,\ t \mapsto \gamma(t) = (\cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t))$.

k)* $\gamma : [0,1] \longrightarrow \mathbb{R}^2,\ t \mapsto \gamma(t) = (\sqrt{1-t^2}, t)$.

l)* $\gamma : [0,1] \longrightarrow \mathbb{R}^2,\ t \mapsto \gamma(t) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$.

Exercise 33

Describe the graphs of the above parameterised curves a) to l).

(*Note: The two quarter circle examples k, and l are given as additional background info and are not examined in the exams.)

The Speed of a Curve

Let $\gamma : [0,1] \longrightarrow \mathbf{R^3}$, $t \mapsto \gamma(t) = (x(t), y(t), z(t))$ be a continuously differentiable curve (i.e. $\mathbf{C^1}$).

The function $\nu(t) = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ is called the *speed* of γ.

The *arc length* of the curve $\gamma(t)$ is given by $\mathbf{L}(\gamma) = \int_0^1 \nu(s)ds$.

$\mathbf{L}_\gamma(t) = \int_0^t \nu(s)ds$ is called the *arc length function*.

The arc length function measures the length of the curve segment from the initial point $(x(0), y(0), z(0))$ to the point $(x(t), y(t), z(t))$.

We have $\mathbf{L}(\gamma) = \mathbf{L}_\gamma(1) - \mathbf{L}_\gamma(0)$.

Exercise 34 (Speed of a Curve)

a) Calculate the speed of the half circle curve $\gamma : [0, \pi] \longrightarrow \mathbf{R^2}$, $t \mapsto \gamma(t) = (2\cos(t), 2\sin(t))$

b) Calculate the the arc length function, and the length of the half circle curve γ above.

c) Show that a rotation has no effect on the speed of the above curve γ.

d) Let $a, b \in \mathbb{R}$. What is the speed and arc length function of the following curve $\mu(t) = (x(t), y(t)) = (ae^{bt}\text{cos}(t), ae^{bt}\sin(t))$? Can you describe the graph of the curve? Determine the unit normal vector for any point of this curve?

Example

Consider a half circle curve in 2 dimensions

$$\gamma : [0, \pi] \longrightarrow \mathbf{R^2},\ t \mapsto \gamma(t) = (2\cos(t), 2\sin(t))$$

1. The speed of γ is

$$\nu(t) = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{4\cos^2(t) + 4\sin^2(t)} = \sqrt{4 \cdot 1} = 2.$$

2. The arc length function is

$$\mathbf{L}_\gamma(t) = \int_0^t \nu(s) ds = \int_0^t 2ds = 2t.$$

3. The curve has length

$$\mathbf{L}_\gamma(\pi) - \mathbf{L}_\gamma(0) = 2\pi.$$

4. Using $\gamma_\theta(t) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and the fact that $(\sin\theta)^2 + (\cos\theta)^2 = 1$ we can see that a rotation has no effect on the speed of γ. (Note: Don't forget to apply product and chain rule when calculating the derivatives.)

Parameterisation by Arclength and Exercise 35

Let $\gamma(t) = (x(t), y(t), z(t))$ a curve with $\nu(t) \neq 0$ and arc length function $\mathbf{L}_\gamma(t)$. Parameterisation by arclength allows us to control the speed of a curve. To see this please solve the following exercise.

Exercise 35 (advanced): Assume that $\phi(s) = (\mathbf{L}_\gamma^{-1})(s)$ is the inverse of the arc length function. What is the speed of the curve $\tilde{\gamma} : [0,1] \longrightarrow \mathbb{R}^{\mathbf{3}}$, $s \mapsto \tilde{\gamma}(s) = (x(\phi(s)), y(\phi(s)), z(\phi(s)))$?

Solution hint: Let $f = \mathbf{L}_\gamma$ then the following well-known theorem will help to find a solution:

Inverse Mapping Theorem
Let U be open in E, let $x_0 \in U$, and let $f : U \longrightarrow F$ be a C^1-map. Assume that the derivative $f'(x_0) : E \longrightarrow F$ is invertible. Then f is locally C^1-invertible at x_0. If ϕ is its local inverse, and $y = f(x)$, then

$$\phi'(y) = \frac{1}{f'(x)}$$

Frenet Vectors

Let $\gamma : [0, L] \longrightarrow \mathbf{R^3}$, $s \mapsto \gamma(s) = (x(s), y(s), z(s))$ be a three times continuously differentiable parameterised curve. Assume γ is parameterised by arclength and has length L. Define for each parameter $s \in [0, L]$ three vectors

$t(s) = \gamma'(s)$ *the unit tangent vector*

$n(s) = \frac{t'(s)}{||t'(s)||}$ *the unit normal vector*

$b(s) = t(s) \times n(s)$ *the bi-normal vector*

The triple (t, n, b) is called the *Frenet frame* of the curve. Further define the *curvature* of γ at s as $\kappa(s) = ||\gamma''(s)||$ and the *torsion* τ of γ at s as the amount of $n'(s)$ which is in direction of the bi-normal $b(s)$, that is $\tau(s) = n'(s) \cdot b(s)$ (i.e. dot product).

Note: We can show that $[t(s), n(s), b(s)]$ is an ONB.

Exercise 36 (Frenet Frame)

Part I Explain why for all $s \in [0, L]$ the following dot products are zero:
(a) $t(s) \cdot n(s) = 0$,
(b) $t(s) \cdot b(s) = 0$
(c) $n(s) \cdot b(s) = 0$

Part II (only for COMP6370 students)
Show that $n' = -\kappa t + \tau b$ and $b' = -\tau n$.
Hint: Explain why for all $s \in [0, L]$ the following dot products are zero:
(i) $t(s) \cdot b(s) = 0$ and (ii) $n(s) \cdot n'(s) = 0$

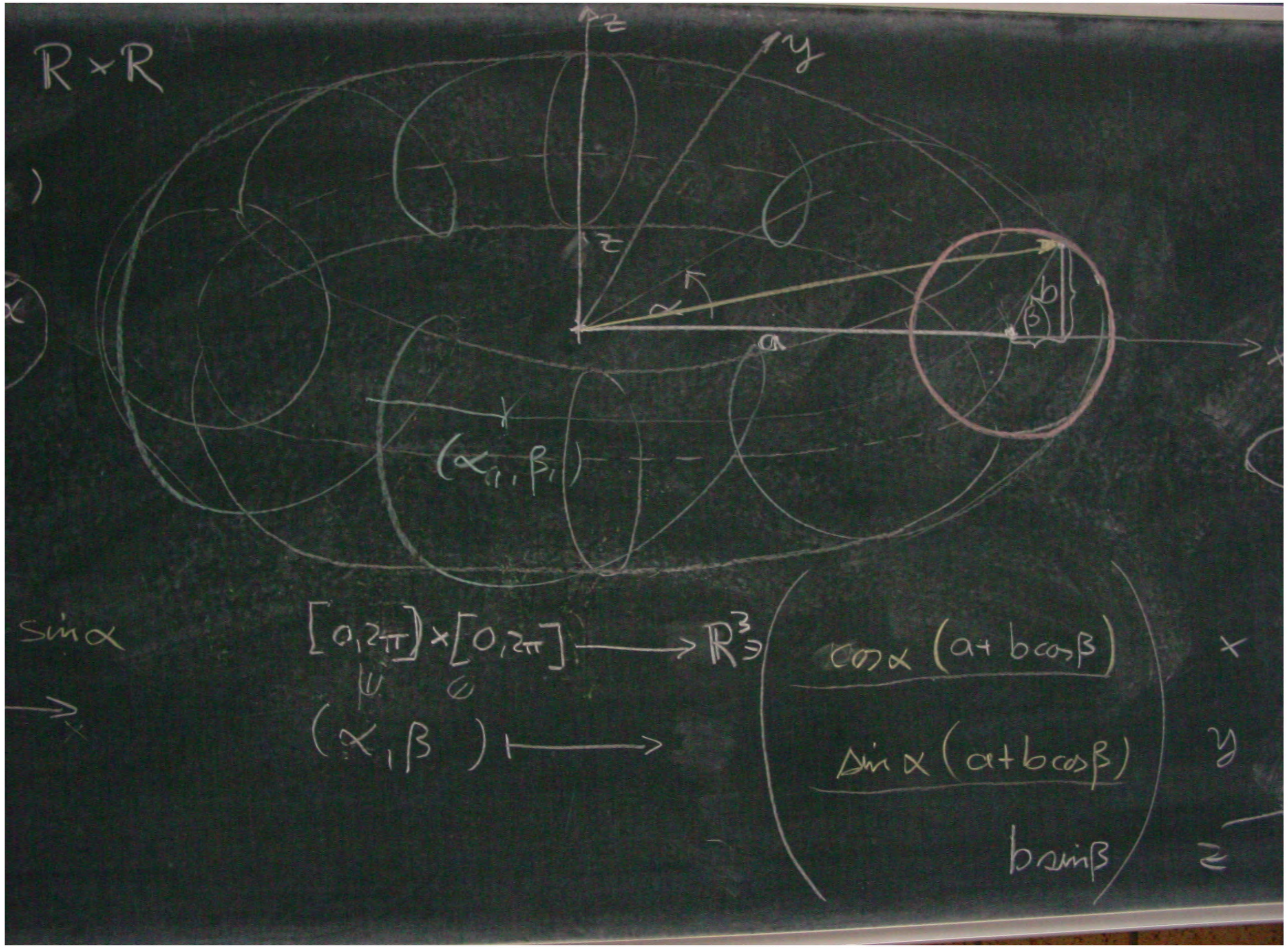
Parameterisation of T^2 (Exercise 37)

Let $a, b \in \mathbf{R}$ two parameters (each is radius of a circle) that represent the width and height of the doughnut/torus, respectively. Then we can describe the 2-dimensional torus T^2 in $\mathbf{R^3}$ by a set of points in 3D that is parameterised by two parameters $\alpha \in [0, 2\pi)$ and $\beta \in [0, 2\pi)$.

$$
\begin{aligned}
[0, 2\pi) \times [0, 2\pi) \;&\longrightarrow\; \mathbf{R^3} \\
(\alpha, \beta) \;&\mapsto\; \begin{pmatrix} (a + b\cos\beta)\cos\alpha \\ (a + b\cos\beta)\sin\alpha \\ b\sin\beta \end{pmatrix}
\end{aligned}
$$

The angle α controls rotation in the $(x - y)$-plane and β controls rotation in the plane spanned by the z-axis and the vector $(\cos\alpha, \sin\alpha)$.

Parameterisation of T^2



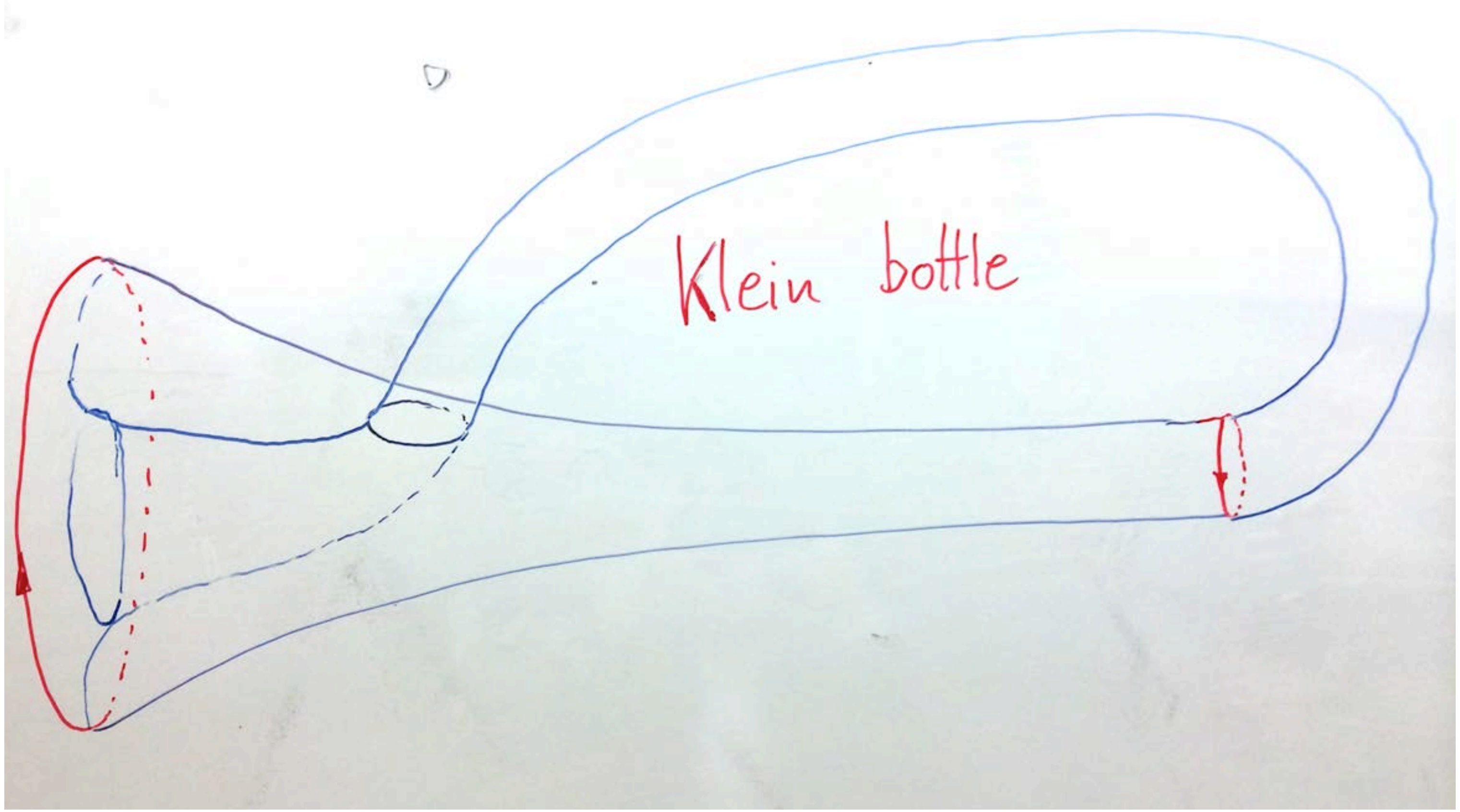
Parameterisation of S^2

We can obtain a sphere of radius r by stacking circles (each parameterised by β) on top of each other along the z-axis in the middle. The radius of each circle in the stack is $r \sin \alpha$. Hence the parameterisation is as follows:

$$\begin{aligned} [0, \pi) \times [0, 2\pi) &\longrightarrow \mathbf{R^3} \\ (\alpha, \beta) &\mapsto \begin{pmatrix} r \sin \alpha \cdot \cos \beta \\ r \sin \alpha \cdot \sin \beta \\ r \cos \alpha \end{pmatrix} \end{aligned}$$

To see this we start with a circle of radius 0 (at the North pole $z = r$) and stack circles of increasing radius below it until we reach a big circle of radius r at the equator and then we continue stacking circles with shrinking radius down to 0 (at the South pole $z = -r$).

Have you heard about K^2 ?



LITERATURE

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