MATH1510 - Discrete Mathematics
Probability 2

University of Newcastle

UoN

Independence

Definition

If $P(E_2|E_1) = P(E_2)$, then we say that the event E_2 is independent of event E_1.

E_1 and E_2 are independent if and only if

$$P(E_1 \cap E_2) = P(E_1)P(E_2).$$

Example

Different coin flips are independent events, as are drawing multiple balls from a bag with replacement.

Note: events that are 'independent' in real life can be treated as independent events in probability, however the reverse is not true. The negation of 'independent' is 'not independent'.

Example

You shuffle 9 cards numbered from 1 to 9. A card is selected, noted, replaced and then another card is selected.

- What is the probability that the two digit numeral the cards form is both greater than 70 and divisible by 5?
- What is the probability that the numeral begins with an odd digit and is even?

Example

Box 1 contains 2 black balls and 2 white balls. Box 2 contains 2 black balls and 4 white balls. A ball is selected at random from box 1 and then a ball is selected at random from box 2.

- What is the probability of getting at least one black ball?
- What is the probability of getting a ball of each colour?

Total Probability Formula

If events $A_1, A_2, \ldots, A_k$ partition the sample space S, then for any event B we have

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_k)P(B|A_k).$$

Example

A company purchases a certain part from 3 different sources, A, B and C. Some of the parts are defective.

Source	A	B	C
% purchased	50	20	30
% defective	1	4	2

What is the probability that a randomly selected part is defective?

Solution

Let D is the probability that a part is defective. We are interested in $P(D)$. Without using a formula we can think of it this way:
If we have 1000 parts then 500 will come from source A, 200 from source B and 300 from source C. We would expect 5 of those from A to be defective, 8 of those from source B and 6 of those from source C. So the probability of a part being defective is

$$\begin{aligned} P(D) &= \frac{5+8+6}{1000} \\ &= \frac{19}{1000}. \end{aligned}$$

Equivalently we can calculate $P(D)$ using the formula

$$\begin{aligned} P(D) &= P(A \cap D) + P(B \cap D) + P(C \cap D) \\ &= P(A) \cdot P(D|A) + P(B) \cdot P(D|B) + P(C) \cdot P(D|C) \\ &= \frac{50}{100} \times \frac{1}{100} + \frac{20}{100} \times \frac{4}{100} + \frac{30}{100} \times \frac{2}{100} \\ &= \frac{19}{1000} \end{aligned}$$

Example

Suppose that a part in the previous example is found to be defective, what is the probability that it came from source A? What about from source B? (Use conditional probability.)

Solution

$$\begin{aligned} P(A|D) &= \frac{P(A \cap D)}{P(D)} \\ &= \frac{P(A) \cdot P(D|A)}{P(D)} \\ &= \frac{(50/100) \times (1/100)}{19/1000} \\ &= \frac{5}{19}. \end{aligned}$$

Example

One box has 2 red balls. A second box of identical appearance contains 1 red and 1 white ball. If a box is chosen at random and a ball drawn from it, what is the probability that the first box was selected, if the drawn ball turns out to be red?

Solution

If we label the boxes A and B then the four possible balls can be represented by AR, AR, BR and BW. Clearly $P(A|R)$ is $\frac{2}{3}$. Relating this to our formulas

$$\begin{aligned} P(A|R) &= \frac{P(A \cap R)}{P(R)} \\ &= \frac{1/2}{P(A) \cdot P(R|A) + P(B) \cdot P(R|B)} \\ &= \frac{1/2}{\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2}} = \frac{2}{3}. \end{aligned}$$

Example

A box contains 4 white and 5 black marbles, and 2 marbles are picked out randomly without replacement. Let A be the event that the first marble is white, and B be the event that the second marble is black. Show that A and B are not independent events.

Solution

$P(A) = 4/9, P(B) = 5/9$ but $P(A \cap B) = 5/18$.

Applications of Probability (3)

Medicine

Suppose a disease is rare and affects only 0.5% of the population. A test for the disease exists and is 98% accurate (if you have the disease, it shows that you do with 98% probability, and if you don't have the disease, it shows that you do not with 98% probability). If a person gets tested positive, what is the probability that they actually have the disease?

We wish to find $P(D|P)$ from $P(P|D)$.

$$P(D|P) = \frac{P(D \cap P)}{P(P)} = \frac{P(P|D)P(D)}{P(P)}.$$

$P(P)$ can be calculated from the total probability formula:
$0.005 \times 0.98 + 0.995 \times 0.02$.
The final answer is about only 20%!

Random Variables

Suppose we flip two coins, and are interested in the number of heads obtained. We can represent this measurement by X, so X can take the values of 0, 1 or 2.

X is an example of a random variable, that is, a *numerical* representation of an event. The specific numerical outcomes are usually denoted by x.

We can think of each event E in the sample space S as a (numerical) value and the probability of E as its *function* value.

Example

Let X be the total obtained when rolling two dice.

Sum	2	3	4	5	6	7	8	9	10	11	12
P(Sum)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Probability Function

The resulting function is called a probability function $P(X = x)$ (or just $P(x)$), which satisfies the following conditions:

- $P(x) \geq 0$ for every x,
- $\sum_{\text{all } x} P(x) = 1$.

Example

From a box containing 4 black balls and 2 green balls, 3 balls are drawn with replacement. Find the probability function for the number of green balls.

Solution

$$\begin{aligned}
P(0) &= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \\
&= \frac{8}{27} \\
P(1) &= \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \\
&= \frac{12}{27} \\
P(2) &= \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \\
&= \frac{6}{27} \\
P(3) &= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \\
&= \frac{1}{27}.
\end{aligned}$$

Exercise

Two dice are rolled and the random variable X is defined as the number of even faces shown. Write down $P(X)$.

Example

A die is 'loaded' so that 2, 4, and 6 are equally likely to appear, and 1, 3, and 5 are equally likely to appear, but 1 is twice as likely to appear as 2. What is the probability function?

Solution for loaded die

We know that $P(1) = 2P(2)$, $P(3) = 2P(2)$, $P(4) = P(2)$, $P(5) = 2P(2)$ and $P(6) = P(2)$. We must also have

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

so

$$9P(2) = 1,$$

which implies that $P(2) = \frac{1}{9}$. The other probabilities can now be obtained

$$P(1) = P(3) = P(5) = \frac{2}{9}$$
$$P(2) = P(4) = P(6) = \frac{1}{9}.$$

Example

A coin is flipped repeatedly. Let X be the number of the flip at which a head first appears. Find the probability function for X.

Solution

$P(n) = \frac{1}{2^n}$ and we note that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Expected Values

Suppose events $E_1, E_2, \ldots, E_n$ partition the sample space S, and let X be the random variable which takes the value of x_i on E_i. We wish to find the *average*, or the expected value, of X:

$$\frac{x_1|E_1| + x_2|E_2| + \cdots + x_n|E_n|}{|S|} = x_1P(x_1) + x_2P(x_2) + \cdots + x_nP(x_n).$$

(Sometimes we can let n go to infinity.)

Definition

The expected value of X is

$$E(X) = \sum_{i=1}^{n} x_iP(x_i).$$

Expected Values – Examples

Example

Roll a die and let X represent the value obtained. The expected value is $\frac{1}{6}(1+2+3+4+5+6) = 3.5$, which we can directly observe by performing the experiment many times.

Example

You flip 3 coins. What is the expected number of tails? Does this agree with your intuition?

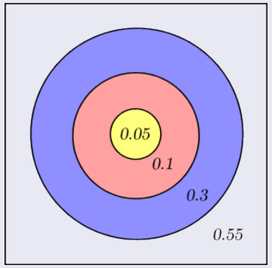
Exercise

Roll 2 dice. Calculate the expected value of the sum. How does this relate to the first example?

Expected Values – Examples

Example

You enter a darts game which pays if you hit certain areas of the target below. From your practice you estimate that your chances of hitting each of these regions are given by the indicated probabilities. If yellow scores 10, red scores 3 and blue scores 1, what is your expected return from a single throw?



Binomial Experiments

Many experiments have two outcomes which can be described as the 'success' and 'failure' of some objective.

Definition

Repeated independent experiments are called binomial (or Bernoulli), if there are only two possible outcomes for each experiment and their probabilities remain the same.

Example

Flip a coin 5 times. What is the probability of getting 3 tails?

We want the probability of 3 successes *and* 2 failures, each with probability $\frac{1}{2}$. The 3 successes can occur *anywhere* within the 5 experiments, so there are $C(5,3)$ ways of arranging them.

Binomial Distribution

For a binomial experiment with probability of success p, the probability of obtaining k successes out of n trials (experiments) follows the binomial distribution:

$$P(k) = C(n,k)p^k(1-p)^{n-k}.$$

Theorem

If X is a random variable for a binomial distribution with parameters n and p, then $E(X) = np$.

Example

A multiple-choice quiz has 15 questions with 5 possible answers for each. If you guess every answer,

- what is the probability you get fewer than 2 right?
- what is the expected value of correct answers?

Proof:

$$
\begin{aligned}
E(X) &= \sum_{k=0}^{n} kP(k) \\
&= \sum_{k=1}^{n} k\frac{n!}{k!(n-k)!}p^k q^{n-k} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!}p^k q^{n-k} \\
&= np\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}p^{(k-1)} q^{n-k} \\
&= np\sum_{k=1}^{n} C(n-1,k-1)p^{k-1}q^{(n-1)-(k-1)} \\
&= np\sum_{j=0}^{m} C(m,j)p^j q^{m-j} \qquad (\text{let } m=n-1, j=k-1) \\
&= np \qquad (\text{Binominal theorem, recall } p+q=1)
\end{aligned}
$$

Binomial Distribution – Examples

Examples

- Write down the probability function for the number of heads obtained in 5 flips of a coin.
- Write down the probability function for the number of heads obtained in 6 flips of a coin.
- Suppose the coin is not fair – the probability of a head is $\frac{2}{3}$. Write down the probability function for the number of heads obtained in 3 flips of the coin.
- Find the expected value of the number of tails for each of the above.

Binomial Distribution – Examples

Example

Suppose that some aeroplane engines operate independently and fail with probability $q = \frac{1}{5}$. If a plane needs at least half of its engines to fly, are you better off in a 2-engine or 4-engine plane?

Example

A baseball player has probability of hitting (batting average) of 0.33.

- What is his expected number of hits in 30 official balls?
- What is the probability that he will get at least one hit in his next three official balls?

For the four engine aeroplane, the probability of a crash is

$$
\begin{aligned}
P(0)+P(1) &= C(4,0)\left(\frac{4}{5}\right)^0\left(\frac{1}{5}\right)^4 + C(4,1)\left(\frac{4}{5}\right)^1\left(\frac{1}{5}\right)^3 \\
&= \frac{1}{5^4}+\frac{16}{5^4} \\
&= \frac{17}{5^4} \\
&< \frac{1}{25}.
\end{aligned}
$$

Applications of Probability

Example

It is commonly cited that 'you are more likely to be attacked by a shark closer to the beach'. This is nonsense. How might this fallacy arise from not understanding probability?

Example

It is often claimed that the life expectancy of a person living 2000 years ago was 35, giving the illusion that the average person died around that age, which is completely misleading.
This is because the expected value is not always a good indicator of the average. The child (under 5) mortality rate was very high in the past, around 30%.
Let A be the life expectancy after surviving past the age of 5, then $2.5 \times 0.3 + A \times 0.7 = 35$, so A is around 50.

Textbook exercises

Exercises Section 6.6:

- 28-30, 44-56