The University of Newcastle
School of Electrical Engineering and Computer Science

COMP2230/6230 Algorithms
Tutorial Week 2 Solutions

26-30 July 2021

Tutorial

1. For each of the following sequences, determine whether it is increasing, decreasing, nonincreasing or nondecreasing.

i. *2, 3, 88, 89, 100*

ii. *2, 3, 3, 88, 89, 100*

iii. *2, 3, 88, 89, 3, 100*

iv. *2*

v. *2, 1*

vi. *2, 2*

Solution:

i. increasing, nondecreasing
ii. nondecreasing
iii. none
iv. increasing, decreasing, nonincreasing, nondecreasing
v. decreasing, nonincreasing
vi. nonincreasing, nondecreasing

2. Which of the following Boolean expressions are in the conjunctive normal form and which are in the disjunctive normal form?

i. $p \wedge \overline{(q \wedge r)} \vee (\overline{p} \wedge s) \vee (q \wedge r)$

ii. $(p \vee q \vee r) \wedge (\overline{p} \vee s) \wedge (q \vee r) \wedge \overline{s}$

iii. $p \vee (\overline{q} \wedge r) \vee \overline{q}$

iv. $\overline{p}$

v. $p \vee q$

Solution:

i. neither
ii. CNF
iii. DNF
iv. both
v. both

3. Prove the two De Morgan's Laws:

i. $\overline{(p \wedge q)} = \overline{p} \vee \overline{q}$
ii. $\overline{(p \vee q)} = \overline{p} \wedge \overline{q}$

Solution:

i.

P	Q	$\overline{(p \wedge q)}$	$\overline{p} \vee \overline{q}$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

ii.

P	Q	$\overline{(p \vee q)}$	$\overline{p} \wedge \overline{q}$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

4. Simplify expressions (i.e., express without logarithms of products, powers or quotients), where $\log x$ denotes $\log_{10} x$, $\lg x$ denotes $\log_2 x$ and $\ln x$ denotes $\log_e x$.:

i. $\log \sqrt{1000}$

ii. $\log_3 3^{5x}$

iii. $e^{\ln 8}$

iv. $2^{x \lg 3}$

v. $\lg(x^3 2^x)$

vi. $\log(\frac{\sqrt{x} \sin x}{x+4})$

Solution:

i. $\frac{3}{2}$

ii. $5x$

iii. 8

iv. 3^x

v. $3\lg x + x$

vi. $\frac{1}{2}\log x + \log(\sin x) - \log(x+4)$

5. Prove that for all n ≥ 1, $\frac{1}{2n} \le \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$

Solution:

<u>Base Case</u>**:** *n* = *1*: $\frac{1}{2} \le \frac{1}{2}$ holds, so the inequality is true for *n* = 1.

<u>Inductive Assumption</u>**:** We assume that inequality holds for n=k. That is, we are assuming that

$$\frac{1}{2k} \le \frac{1 \times 3 \times \ldots \times (2k-1)}{2 \times 4 \times 6 \times \ldots \times 2k}$$

<u>Inductive Step:</u> We show that it then holds for *n=k+1*. That is, we show that

$$\frac{1}{2k+2} \le \frac{1 \times 3 \times \ldots \times (2k+1)}{2 \times 4 \times \ldots \times (2k+2)}$$

We start from the right-hand side of the inequality and by inductive assumption we have

$$\frac{1 \times 3 \ldots \times (2k+1)}{2 \times 4 \times \ldots \times (2k+2)} = \frac{1 \times 3 \ldots \times (2k-1)}{2 \times 4 \times \ldots \times 2k} \times \frac{2k+1}{2k+2} \ge \frac{1}{2k} \times \frac{2k+1}{2k+2}$$
$$= \frac{2k+1}{2k} \times \frac{1}{2k+2} \ge \frac{1}{2k+2}$$

as $\frac{2k+1}{2k} \ge 1$.

6. The Fibonacci sequence is defined as $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove by mathematical induction that

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-n}\right)$$

Solution:

Base Cases:

n = 0:

$f_0 = \frac{1}{\sqrt{5}}(1-1) = 0$,

Thus the equation holds for n = 0.

n = 1:

$$f_1 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{1} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-1}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}}\right) =$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}}\frac{1-\sqrt{5}}{1-\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$

Thus the equation also holds for $n = 1$.

Inductive Assumption**:** We assume that inequality holds for $n=k$ and $n=k-1$. That is,

$$f_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k}\right)$$

and

$$f_{k-1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k+1}\right)$$

Inductive Step: We show that the equation then holds for $n=k+1$. That is, we show that

$$f_{k+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k-1}\right)$$

We start from the recursive definition of f_{k+1}.

$$f_{k+1} = f_k + f_{k-1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k+1}\right) =$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k \left(1+\frac{2}{1+\sqrt{5}}\right) - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k}\left(1+\frac{-1-\sqrt{5}}{2}\right)\right) =$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k \frac{3+\sqrt{5}}{1+\sqrt{5}}\frac{1-\sqrt{5}}{1-\sqrt{5}} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k}\frac{1-\sqrt{5}}{2}\right) =$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k \frac{1+\sqrt{5}}{2} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k}\frac{1-\sqrt{5}}{2}\frac{1+\sqrt{5}}{1+\sqrt{5}}\right) =$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k}\frac{2}{-1-\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{-1-\sqrt{5}}{2}\right)^{-k-1}\right)$$

7. Suppose that the only randomness which your algorithm has access to is a fair die (i.e., possible outcomes are 1, 2, 3, 4, 5 and 6, all with equal probabilities). Can you use this function to write an algorithm that implements a random fair coin toss?

Solution idea: Throw a die; consider the outcomes 1, 2 and 3 as heads, and 4, 5 and 6 as tails.

8. Show that the following algorithm shuffles a deck of cards (52 cards) in a fair manner, that is, every possible permutation of 1 through 52 is obtained with the same probability.

```
shuffle(a) {for i = 1 to 52
                swap(a[i], a[rand(i, 52)])}
```

Solution: The probability of every permutation is the same and is equal to $(n\cdot (n-1)\cdot (n-2)\ldots 2\cdot 1)^{-1} = (n!)^{-1}$. To see this, consider an arbitrary permutation $a_1, a_2, \ldots, a_{52}$, where a_i is the card in the position i. The probability that a_1 is selected by the algorithm as the first card is $(n)^{-1}$; the probability that a_1 is selected as the first card and a_2 as the second is $(n\cdot(n-1))^{-1}$ and so on.

9. If p(x) and q(x) are polynomials of degrees 10 and 5 respectively, what can you say about the degrees of p(x) + q(x) and p(x) - q(x) ?

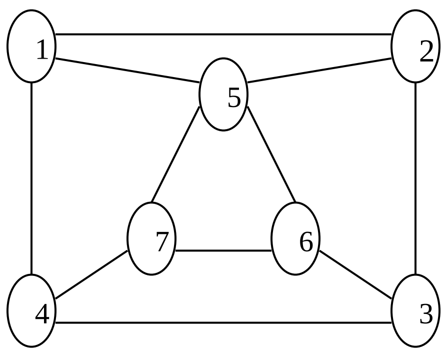
Solution: Both degrees are 10.

10. If A and B are two closed intervals such that $A \cap B = I \neq \varnothing$. Which of the following can interval I be:

i. a closed interval
ii. an open interval
iii. a half-open interval?

Solution: a closed interval

11. Given a graph G:



i. Find the degree of each vertex.
ii. Find the neighborhood of each vertex.
iii. Write an adjacency matrix of the graph G.
iv. Is G bipartite? Why?
v. Find all simple paths from vertex 1 to vertex 3.
vi. Does G have a Hamiltonian cycle? If yes, find it.
vii. Does G have Euler cycle? If yes, find it. If not, why not?

Solution:

i.

V	**δ(v)**
1	3
2	3
3	3

4	3
5	4
6	3
7	3

ii.

V	**N(v)**
1	{2,4,5}
2	{1,3,5}
3	{2,4,6}
4	{1,3,7}
5	{1,2,6,7}
6	{3,5,7}
7	{4,5,6}

iii.

	1	**2**	**3**	**4**	**5**	**6**	**7**
1	0	1	0	1	1	0	0
2	1	0	1	0	1	0	0
3	0	1	0	1	0	1	0
4	1	0	1	0	0	0	1
5	1	1	0	0	0	1	1
6	0	0	1	0	1	0	1
7	0	0	0	1	1	1	0

iv. No, the triangle in the center prevents it from being bipartite.

v. 1→2→3,
1→2→5→6→3,
1→2→5→6→7→4→3,
1→2→5→7→6→3,
1→2→5→7→4→3,
1→5→2→3,
1→5→6→3,
1→5→6→7→4→3,
1→5→7→6→3,
1→5→7→4→3,
1→4→3,
1→4→7→5→2→3,
1→4→7→5→6→3,
1→4→7→6→3,

$1 \rightarrow 4 \rightarrow 7 \rightarrow 6 \rightarrow 5 \rightarrow 2 \rightarrow 3$

vi. $1 \rightarrow 2 \rightarrow 3 \rightarrow 6 \rightarrow 5 \rightarrow 7 \rightarrow 4 \rightarrow 1$

vii. There are vertices with odd degree; therefore there can be no Eulerian cycle.

Homework

12. Which sequences are both increasing and decreasing?

Solution: Sequences consisting of a single element.

13. Which sequences are both nonincreasing and nondecreasing?

Solution: Sequences in which all elements have the same value.

14. Show that $p \leftrightarrow q$ and $(\overline{p} \vee q) \wedge (\overline{q} \vee p)$ are equivalent.

Solution:

p	q	$p \leftrightarrow q$	$(\overline{p} \vee q) \wedge (\overline{q} \vee p)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

15. Simplify (i.e., express without logarithms of products, powers or quotients):

i. $\ln(\sqrt{x}e^{4x})$

ii. $\ln \dfrac{x^2 - 1}{(x-1)^2}, x > 1$

Solution:

i. $\ln(\sqrt{x}e^{4x}) = \dfrac{1}{2}\ln x + 4x$

ii. $\ln \frac{x^2-1}{(x-1)^2} = \ln(x+1) - \ln(x-1)$

16. Solve each equation:

i. $100 = 50e^{-x}$

ii. $\frac{1}{5} = 5^{3x-2}$

iii. $\ln(2x+5) = 0$

iv. $\log_x 6 = \frac{1}{3}$

Solution:

i. $x = -\ln 2$

ii. $x = \frac{1}{3}$

iii. $x = -2$

iv. $x = 216$

17. Use induction to prove that each equation is true for every positive integer *n*.

i. $\sum_{i=1}^{n}(2i-1) = n^2$

ii. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

18. Suppose that the only randomness which your algorithm has access to is *rand(1,2),* that is, a random fair coin toss. Can you use this function to write an algorithm that implements throwing a fair die (i.e., the outcome should be 1, 2, 3, 4, 5 or 6, with equal probability)?

Solution idea: Toss a fair coin 3 times and write the outcomes as a binary number (e.g., head is 0 and tail is 1); if the number is 0 or 7, discard the result and toss the coin 3 times again.

19. If p(x) and q(x) are polynomials of degrees 10 and 5 respectively, what can you say about the degree of p(x) × q(x)?

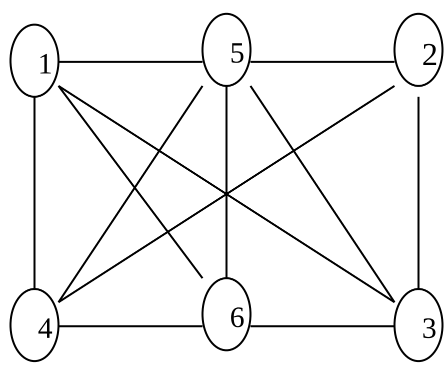
Solution: The degree of p(x) × q(x) is 15.

20. If A and B are two open intervals such that $A \cap B = I \neq \emptyset$. Which of the following can interval I be:

i. a closed interval
ii. an open interval
iii. a half-open interval?

Solution: an open interval

21. Given a graph G:



i. Find the degree of each vertex.
ii. Find the neighborhood of each vertex.
iii. Write an adjacency matrix of the graph G.
iv. Is G bipartite? Why?
v. Find all simple paths from vertex 1 to vertex 3.
vi. Does G have a Hamiltonian cycle? If yes, find it.
vii. Does G have Euler cycle? If yes, find it. If not, why not?

Solution:

i.

v	**δ(v)**
1	4
2	3
3	4
4	4
5	5
6	4

ii.

v	**N(v)**
1	{3,4,5,6}
2	{3,4,5}
3	{1,2,5,6}
4	{1,2,5,6}
5	{1,2,3,4,6}
6	{1,3,4,5}

iii.

	1	**2**	**3**	**4**	**5**	**6**
1	0	0	1	1	1	1
2	0	0	1	1	1	0
3	1	1	0	0	1	1
4	1	1	0	0	1	1
5	1	1	1	1	0	1
6	1	0	1	1	1	0

iv. No, there are several triangular structures (1,4,6 for example). In general, odd cycles of any length prevent a graph from being bipartite.

v. Numerous.

vi. 1→5→2→3→6→4→1

vii. There are vertices with odd degree; therefore there can be no Eulerian cycle.

More Exercises

22. Write an expression equivalent to $p \rightarrow q$ involving only $\wedge$ and $^{-}$ operators.

Solutions: $\overline{p \wedge \bar{q}}$

23. Solve each equation:

i. $\ln(0.8x + 0.03) = 0.01$

ii. $0.25 = e^{-0.4x}$

iii. $\lg(\log_3 x) = 4$

iv. $21 \times 2^{x+1} \times 3^{1-x} = 56$

Solutions:

i. $x = 1.23$

ii. $x = -\dfrac{\ln 0.25}{0.4} = 3.466$

iii. $x = 3^{16}$

iv. $x = 2$

24. Show that if *k* and *n* are positive integers satisfying $2^{k-1} < n < 2^k$, then $k - 1 < \lg n < k$, where $\lg n = \log_2 n$.

Solution: Recall that if $b > 1$ and $x > y > 0$, then $log_b\, x > log_b\, y$. Since $2 > 1$, we can take log of all sides of the inequality and we get:

$$\lg 2^{k-1} < \lg n < \lg 2^k$$

We then have $k - 1 < \lg n < k$.

25. Prove that for real numbers a and $r \neq 1$, $\sum_{i=0}^{n} ar^i = \dfrac{a(r^{n+1} - 1)}{r - 1}$ for all $n \geq 0$.

26. Use induction to prove that each equation is true for every positive integer *n*.

i. $$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

ii. $$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

iii. $$\sum_{i=1}^{n} (-1)^{i+1} i^2 = \frac{(-1)^{n+1} n(n+1)}{2}$$

iv. $$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

27. Mr Con R. Tiste suggests the following modification of the above shuffle algorithm for deck *a* of cards:

```
new_shuffle(a) {
        for i = 1 to 52
                swap(a[i], a[rand(1, 52)])
}
```

Does *new_shuffle(a)* produce every possible permutation of 1 through 52 with the same probability? Prove your claim.

Solution idea: There are 52^{52} outcomes that are all equally likely, but not all distinct. As there are only 52! distinct outcomes (permutations), they cannot possibly be equally likely as $52^{52}/(52!)$ is not an integer.

28. If p(x) and q(x) are polynomials, both of degree 5, what can you say about the degrees of p(x) + q(x), p(x) - q(x) and p(x) × q(x)?

Solution: The degrees of p(x) + q(x) and p(x) - q(x) can be anything in the interval [0,5]; the degree of p(x) × q(x) is 10.

29. If A and B are two half-open intervals such that $A \cap B = I \neq \emptyset$. Which of the following can interval I be:

i. a closed interval
ii. an open interval
iii. a half-open interval?

Solution: All of the above.

30. Locker doors *(A. Levitin, 2007).* There are n lockers in a hallway numbered sequentially from 1 to n. Initially, all the locker doors are closed. You make n passes by the lockers, each time starting with locker #1. On the i^{th} pass, i =1, 2, ..., n, you toggle the door of every i^{th} locker: if the door is closed, you open it, if it is open, you close it. For example, after the first pass every door is open; on the second pass you only toggle the even-numbered lockers (#2, #4, ...) so that after the second pass the even doors are closed and the odd ones are opened; the third time through you close the door of locker #3 (opened from the first pass), open the door of locker #6 (closed from the second pass), and so on. After the last pass, which locker doors are open and which are closed? How many of them are open?

Solution *(A. Levitin, 2007):* Since all the doors are initially closed, a door will be open after the last pass if and only if it is toggled an odd number of times. Door i ($1 \le i \le n$) is toggled on pass j ($1 \le j \le n$) if and only if j divides i. Hence, the total number of times door i is toggled is equal to the number of its divisors. Note that if j divides i, i.e. i = jk, then k divides i too. Hence all the divisors of i can be paired (e.g., for i = 12, such pairs are 1 and 12, 2 and 6, 3 and 4) unless i is a perfect square (e.g., for i = 16, 4 does not have another divisor to be matched with). This implies that i has an odd number of divisors if and only if it is a perfect square, i.e., $i = j^2$. Hence doors that are in the positions that are perfect squares and only such doors will be open after the last pass. The total number of such positions not exceeding n is equal to $\lfloor \sqrt{n} \rfloor$: these numbers are the squares of the positive integers between 1 and $\lfloor \sqrt{n} \rfloor$ inclusively.