1. Simplify the following expressions.

   (a) $\log_{10}(1000)$

   (b) $\log_2\left(\sqrt{128}\right)$

   (c) $\log_e\left(e^{100}\right)$

   (d) $\log_{10}\left(\frac{\sqrt{x}\sin(x)}{x+4}\right)$

   (a) 3
   (b) $\frac{7}{2}$
   (c) 100
   (d) $\frac{1}{2}\log(x) + \log(\sin(x)) - \log(x+4)$

2. Solve each of the following equations for $x$.

   (a) $100 = 50e^{-x}$

   (b) $\frac{1}{5} = 5^{3x-2}$

   (c) $\log(2x+5) = 0$

   (d) $\log_x(6) = \frac{1}{3}$

   (a) $x = -\log_2(2)$
   (b) $x = \frac{1}{3}$
   (c) $x = -2$
   (d) $x = 216$

3. Find the particular solution to the Fibonacci recurrence relation, that is

$$F_n = F_{n-1} + F_{n-2}$$

with $F_1 = 1$ and $F_2 = 1$. Be careful with the square roots and negative signs, it will get messy!

First we recognise that $F_n = F_{n-1} + F_{n-2}$ is a linear, homogeneous recurrence relation, so we can use theorem 9 from the lecture notes. So we need to construct the characteristic equation. The degree of the recurrence relation is 2 so the degree of our characteristic equation is also 2. The characteristic equation is then

$$r^2 - r - 1 = 0.$$

The roots of this quadratic are

$$r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}.$$

Now we use theorem 9 to find the general solution of the recurrence relation, which is

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We now use our initial conditions, $F_1 = 1$ and $F_2 = 2$ to find $c_1$ and $c_2$. When $n = 1$ we have

$$1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^1$$

and when $n = 2$ we have

$$1 = c_1 1 \left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^2.$$

These are two equations in two unknowns, so we can solve this set of simultaneous equations and find

$$c_1 = \frac{1}{\sqrt{5}}$$

and

$$c_2 = -\frac{1}{\sqrt{5}}.$$

Finally, substituting these two values back into our general solution, we find the particular solution is

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

4. Prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

using induction.

We want to show

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We will use induction to do this.

Our base case is when $n = 1$. The left hand side is $1^2 = 1$. The right hand side is $\frac{1(1+1)(2\times1+1)}{6} = \frac{6}{6} = 1$ so our left and right side are the same, so the base case is true.

Now we assume that the statement is true for $n = k$, that is, we can take it as true that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

Given our assumption, we want to show that $n = k + 1$ is true. We want to arrive at

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = (1^2 + 2^2 + 3^2 + \ldots + k^2) + (k+1)^2.$$

The first set of brackets on the right hand side groups the terms that are part of our assumption, and the second set of brackets is the new stuff we've added in the $k + 1$ term. So we can replace the first set of brackets with our assumption to get

$$\sum_{i=0}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1).$$

Now we will expand this out to get

$$\sum_{i=1}^{k+1} = \frac{2k^3 + 9k^2 + 13k + 6}{6}.$$

In theory, we can factorise this to get to where we want, in practice that's much harder. So we will expand out the expression we want to get to, which is

$$\frac{(k+1)(k+2)(2k+3)}{6} = \frac{2k^3 + 9k^2 + 13k + 6}{6}$$

and because this is the same as what we had on the last line, we get that

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

as required.

5. Prove that

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

using induction.

We wish to show that

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

using induction.

Our base case is when $n = 0$, so the left hand side is $2^0 = 1$ and our right hand side is $2^2 - 1 = 1$, so our base case is true.

We will now assume that

$$\sum_{k=0}^{m} 2^k = 2^{m+1} - 1.$$

Note that we had to use a different variable name here because we had already defined $k$.

We now wish to show that

$$\sum_{k=0}^{m+1} 2^k = 2^{m+2} - 1.$$

So expanding out the left hand side we have

$$\sum_{k=0}^{m+1} 2^k = (2^0 + 2^1 + 2^2 + \cdots + 2^m) + (2^{m+1})$$

where the first set of brackets are the terms from our assumption, and the second set of brackets gives the newly added terms that aren't part of our assumption. So we can replace the first set of brackets with the assumption to give

$$\sum_{k=0}^{m+1} 2^k = (2^{m+1} - 1) + (2^{m+1}).$$

We can now drop the brackets and rearrange a little bit to get

$$\sum_{k=0}^{m+1} 2^k = 2^{m+1} + 2^{m+1} - 1$$

and now we have two lots of $2^{m+1}$, so we can write that as $2 \times 2^{m+1} = 2^{m+2}$, so we have

$$\sum_{k=0}^{m+1} = 2^{m+2} - 1$$

as we wanted.

6. Prove that, for all $n \geq 1$,

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

While this question looks like it can be done with induction (and it can), lets do it more directly.

Clearly $\frac{3}{2} > 1$, $\frac{5}{4} > 1$ and so on for any fraction of the form $\frac{k+1}{k}$. Now if we construct the product

$$\frac{3}{2}\frac{5}{4} \cdots \frac{(2n-1)}{(2n-2)}$$

we are just multiplying a lot of numbers together that are greater than 1, so the product is also greater than or equal to 1, that is

$$\frac{3}{2}\frac{5}{4} \cdots \frac{(2n-1)}{(2n-2)} \geq 1.$$

Now if we multiply the left hand side by $\frac{2n}{2n} = 1$, we won't change the value of the product because we are just multiplying by 1 in a cleaver way, so we have

$$1 \leq \frac{3}{2}\frac{5}{4} \cdots \frac{(2n-1)}{(2n-2)}\frac{(2n)}{(2n)}$$

and we can always multiply by 1 in a naive way, so

$$1 \leq 1\frac{3}{2}\frac{5}{4} \cdots \frac{(2n-1)}{(2n-2)}\frac{(2n)}{(2n)}.$$

Putting this all together gives

$$1 \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n)}{2 \cdot 4 \cdots (2n-2)(2n)}.$$

Finally, dividing both sides by $2n$ gives

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots (2n-2)(2n)}$$

as required.

7. For each of the following sequences, determine if they are *increasing*, *decreasing*, *non increasing*, *non decreasing*, or none of them.

   (a) $2, 3, 88, 89, 100$

   (b) $2, 3, 3, 88, 89, 100$

   (c) $2$

   (d) $2, 1$

   (e) $2, 1, 3, 4, 7, 11, 18$

(f) $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 2$ and $a_2 = 1$

First note, that increasing implies non decreasing, and decreasing implies non increasing, so we don't need to specify the non decreasing, or non increasing part if we specify either increasing or decreasing.

(a) Increasing

(b) Non decreasing

(c) None. The definitions for increasing and decreasing require two consecutive terms.

(d) Decreasing

(e) None

(f) None

8. For each of the following sequences, determine if they are *eventually increasing*, *eventually decreasing*, or neither.

   (a) $a_n = 3a_{n-1}$ with $a_1 = 1$

   (b) $a_n = 3a_{n-1} + 2a_{n-2}$ with $a_1 = 3$ and $a_2 = 2$

   (c) $a_n = (-1)^n 3a_{n-1}$

   (d) $a_n = \log(n) - n^{\frac{5}{4}} \sin\left(\frac{1}{n}\right)$

   (a) Eventually increasing

   (b) Eventually increasing

   (c) Neither

   (d) Eventually decreasing

9. What type of sequences are both *non increasing* and *non decreasing*?

   The constant sequences are neither increasing or decreasing. That is, the sequence of the form $a_n = a_{n-1}$.

10. Verify the two De Morgan's Laws for logic by using a truth table.

   (a) $\overline{(p \wedge q)} = \overline{p} \vee \overline{q}$

   (b) $\overline{(p \vee q)} = \overline{p} \wedge \overline{q}$

   (a)

| $p$ | $q$ | $\overline{p \wedge q}$ | $\overline{p} \vee \overline{q}$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |

   (b)

| $p$ | $q$ | $\overline{p \vee q}$ | $\overline{p} \wedge \overline{q}$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 |

11. Using only `OR`, `AND`, and $\neg$, construct a logic expression for two Boolean variables, $p$ and $q$ that is the same as the following truth table. It is called `XOR` and generally denoted as $\oplus$.

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 1 | 0 |

$p \oplus q = (p \vee q) \wedge \neg(p \wedge q)$. We can verify this by using a truth table.

| $p$ | $q$ | $p \oplus q$ | $(p \vee q) \wedge \neg(p \wedge q)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |

12. Write the double sum from theorem 9 in the lecture notes,

$$a_n = \sum_{i=1}^{k} \sum_{j=0}^{m_i - 1} c_{ij} n^j r_i^n,$$

as a nested set of loops, similar to the single loop for the single sum. Assume that there are variables `int[] m`, `int[][] c`, `int[] r`, and they are all filled with the correct data and are of the correct length.

```
int sum = 0;
for(int i = 0; i <= k - 1; i++) {
    for(int j = 0; j <= m[i] - 1; j++}
        sum += c[i][j]*pow(n,j)*pow(r[i],n);
    }
}
```

13. Solve the following recurrence relations.

(a) $a_n = 3a_{n-1} - 2a_{n-2}$

(b) $b_n = 4b_{n-1} - 4b_{n-2}$

(c) $c_n = 8c_{n-1} - 21c_{n-2} + 18c_{n-3}$. It might be useful to know that $x^3 - 8x^2 + 21x - 18 = (x-2)(x-3)^2$.

(a) $a_n = 3a_{n-1} - 2a_{n-2}$. First we recognise that it is a degree 2 linear homogeneous recurrence relation. So our characteristic equation is

$$r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0.$$

So our roots are $r_1 = 1$ and $r_2 = 2$. So substituting it our roots into the double sum in theorem 9 gives

$$a_n = c_1 1^n + c_2 2^n = c_1 + c_2 2^n.$$

(b) $b_n = 4b_{n-1} - 4b_{n-2}$. Again, this is a degree 2 linear homogeneous recurrence relation. So our characteristic equation is

$$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0.$$

We have a repeated root of 2, with multiplicity 2, so using theorem 9 we get

$$b_n = c_1 2^n + c_2 n 2^n.$$

(c) $c_n = 8c_{n-1} - 21c_{n-2} + 18c_{n-3}$. This is a degree 3 linear homogeneous recurrence relation. Our characteristic equation is

$$r^3 - 8r^2 + 21r - 18 = 6$$

and using the equation in the question gives the factorisation, and hence the roots $r_1 = 2$ and $r_2 = 3$, with $r_2$ having multiplicity 2. So using theorem 9 again, we get that

$$c_n = c_1 2^n + c_2 3^n + c_3 n 3^n.$$

14. Prove the following

(a)
$$\sum_{i=1}^{n} (2i-1) = n^2$$

(b)
$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

(c)
$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

(a) We will prove
$$\sum_{i=1}^{n} (2i-1) = n^2$$
by induction. The base case is for $n = 1$ so we have the left hand side being $2 \times 1 - 1 = 1$. The right hand side is just $1^2 = 1$.
We will assume
$$\sum_{i=1}^{k} (2i-1) = k^2.$$

Now we wish to show that
$$\sum_{i=1}^{k+1} 2i - 1 = (k+1)^2.$$

So we have
$$\sum_{i=1}^{k+1} = [(2(1)-1) + (2(2)-1) + \cdots + (2(k)-1)] + (2(k+1)-1).$$

The terms between the [ and ] are all the terms that are involved in our assumption, so we can replace them with our assumption, and at the same time we will expand out the new term, to get
$$\sum_{i=1}^{k+1} = k^2 + 2k + 1.$$

Now we can factor the last expression to get

$$\sum_{i=1}^{k+1} = (k+1)^2$$

which is what we wanted.

(b) We will use a different approach to prove that

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

First we will split this into

$$\sum_{i=1}^{n} i^2 + i$$

which we can then further split into

$$\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i.$$

We have seen both of these sums already,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

and

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We can use these now to get

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}.$$

Putting the right hand side over a common denominator of 6 gives

$$\sum_{i=1}^{n} = \frac{3n(n+1) + n(n+1)(2n+1)}{6}.$$

Factoring the $n+1$ from the numerator gives

$$\sum_{i=1}^{n} = \frac{(n+1)(3n + n(2n+1))}{6}$$

and expanding the second set of brackets gets us to

$$\sum_{i=1}^{n} = \frac{(n+1)(2n^2+3n+n)}{6} = \frac{(n+1)(2n^2+4n)}{6}.$$

Now from the second set of brackets in the numerator and canceling with the denominator we get

$$\sum_{i=1}^{n} = \frac{(n+1)(2n^2+4n)}{6} = \frac{(n+1)2(n^2+2n)}{6} = \frac{(n+1)(n^2+2n)}{3},$$

and further factoring the second set of brackets gives us

$$\sum_{i=1}^{n} = \frac{(n+1)n(n+2)}{3} = \frac{n(n+1)(n+2)}{3}$$

which is what we wanted to show.

(c) We want to show

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

using induction.

First our base case is when $n = 1$. The left hand side is $1(1!) = 1$, and the right hand side is $(1+1)! - 1 = 1$.

Now we can assume that

$$\sum_{i=1}^{k} i(i!) = (k+1)! - 1,$$

and we want to show that

$$\sum_{i=1}^{k+1} i(i!) = (k+2)! - 1.$$

Now we expand our left hand side to get

$$\sum_{i=1}^{k+1} i(i!) = \left(\sum_{i=1}^{k} i(i!)\right) + (k+1)(k+1)!$$

and this time we have written the old stuff in a more compact way. We can replace the old terms with our assumption to get

$$\sum_{i=1}^{k+1} i(i!) = (k+1)! - 1 + (k+1)(k+1)!.$$

We can factor out a $(k+1)!$ to give

$$\sum_{i=1}^{k+1} = (k+1)!(1+k+1) - 1.$$

Simplifying the brackets gives

$$\sum_{i=1}^{k+1} i(i!) = (k+2)(k+1)! - 1$$

and $(k+2)(k+1)! = (k+2)!$, so we have

$$\sum_{i=1}^{k+1} i(i!) = (k+2)! - 1$$

as required.