SCHOOL of ELECTRICAL ENGINEERING & COMPUTING
FACULTY of ENGINEERING & BUILT ENVIRONMENT
The UNIVERSITY of NEWCASTLE

Comp3320/6370 Computer Graphics

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LECTURE w03

Fractals

August 13, 2018

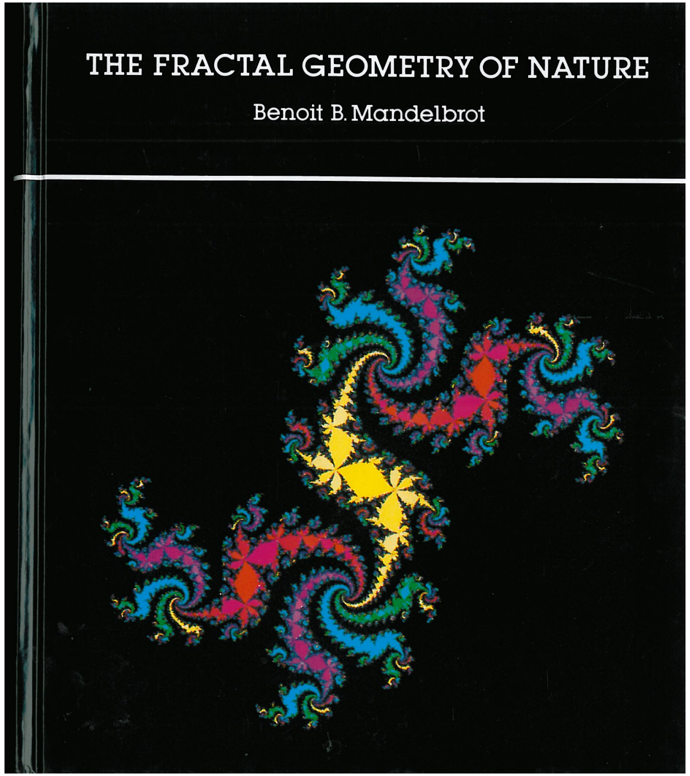
OVERVIEW

Overview

- Fractals are invariant sets (of dynamical systems) that have a self-similar structure.
- Interest in Fractal Geometry was inspired by Benoit Mandelbrot's book "Fractal Geometry of Nature" (Mandelbrot, 1983)
- We study how fractals can be used in computer graphics and image processing.

Fractal Geometry of Nature—The book

- Benoit Mandelbrot's book "Fractal Geometry of Nature" Mandelbrot (1983)
- Mandelbrot coined the word "fractal" in 1975.



Fractal Geometry—The research field

- Felix Hausdorff (1868-1942), Abram Besicovitch (1891-1970), Benoit Mandelbrot (1924-2010), as well as some others shaped a research field called "fractal geometry"
- The important feature of fractals is their independence of scaling.
- The fractal dimension describes the rate of scaling.
- Multifractals can be used for structures that have more than one scaling exponent. They can produce highly unusual shapes that can be important in applications.

Complex Numbers

Let $z = (x + iy)$ and $w = (u + iv)$ be two complex numbers, i.e. $x, y, u, v \in \mathbf{R}$ and $i = \sqrt{-1}$.
Then the product of two complex numbers is given by

$$\begin{aligned} z \cdot w &= (x + iy)(u + iv) \\ &= (xu - yv) + i(xv + yu) \end{aligned}$$

Their sum is given by componentwise addition

$$\begin{aligned} z + w &= (x + iy) + (u + iv) \\ &= (x + u) + i(y + v) \end{aligned}$$

The set $\mathbf{C} = \{z; z = (x + iy)$ where $x, y \in \mathbf{R}$ and $i = \sqrt{-1}\}$ together with the above two operations "$+$" and "$\cdot$" is a field (i.e. they satisfy the same algebraic axioms as real numbers).

Absolute value of complex numbers

Let $z = (x + iy)$ and $w = (u + iv)$ be two complex numbers,
i.e. $x, y, u, v \in \mathbf{R}$ and $i = \sqrt{-1}$.
Their absolute values are given by

$$|z|^2 = x^2 + y^2 \text{ and } |w|^2 = u^2 + v^2, \text{ .i.e.}$$

$$|z| = \sqrt{x^2 + y^2} \text{ and } |w| = \sqrt{u^2 + v^2}$$

Notes

1. The complex number $z = (x + iy) \in \mathbf{C}$ and the vector $\vec{z} = (x, y) \in \mathbf{R}^2$ have the same "magnitude" i.e. $|z| = \|\vec{z}\|$.
2. $|z \cdot w| = |z| \cdot |w|$, because
   $|z \cdot w|^2 = (xu - yv)^2 + (xv + yu)^2 = (x^2 + y^2)(u^2 + v^2)$.
3. What happens if w is a real number?
4. DEF. If $z = x + iy$ is a complex number then $\bar{z} = x - iy$ is called its conjugate.

Euler's Formula

Complex number on the unit circle can be described by

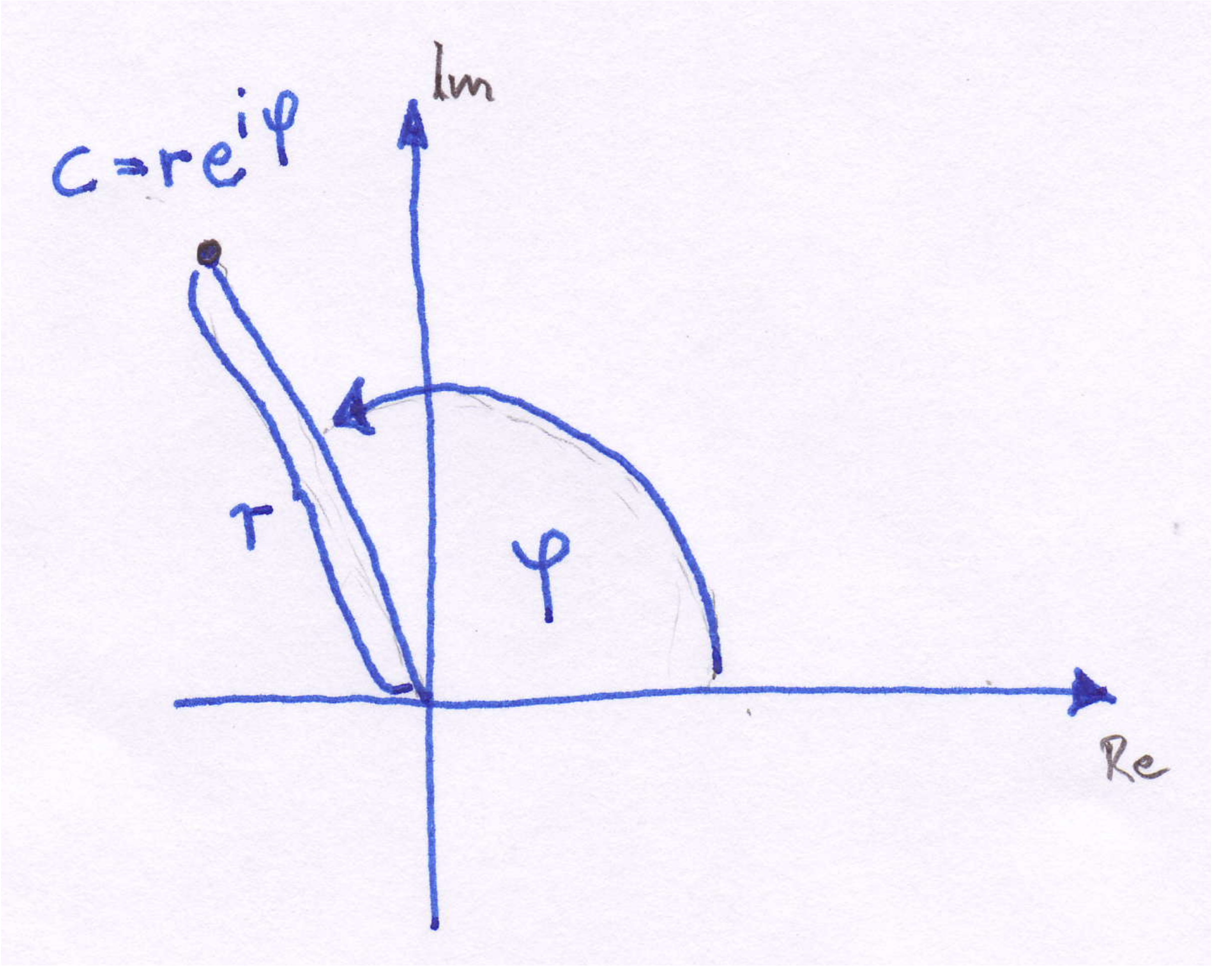
$$c = e^{ix} = \cos x + i \sin x$$

where $x \in [0, 2\pi]$.

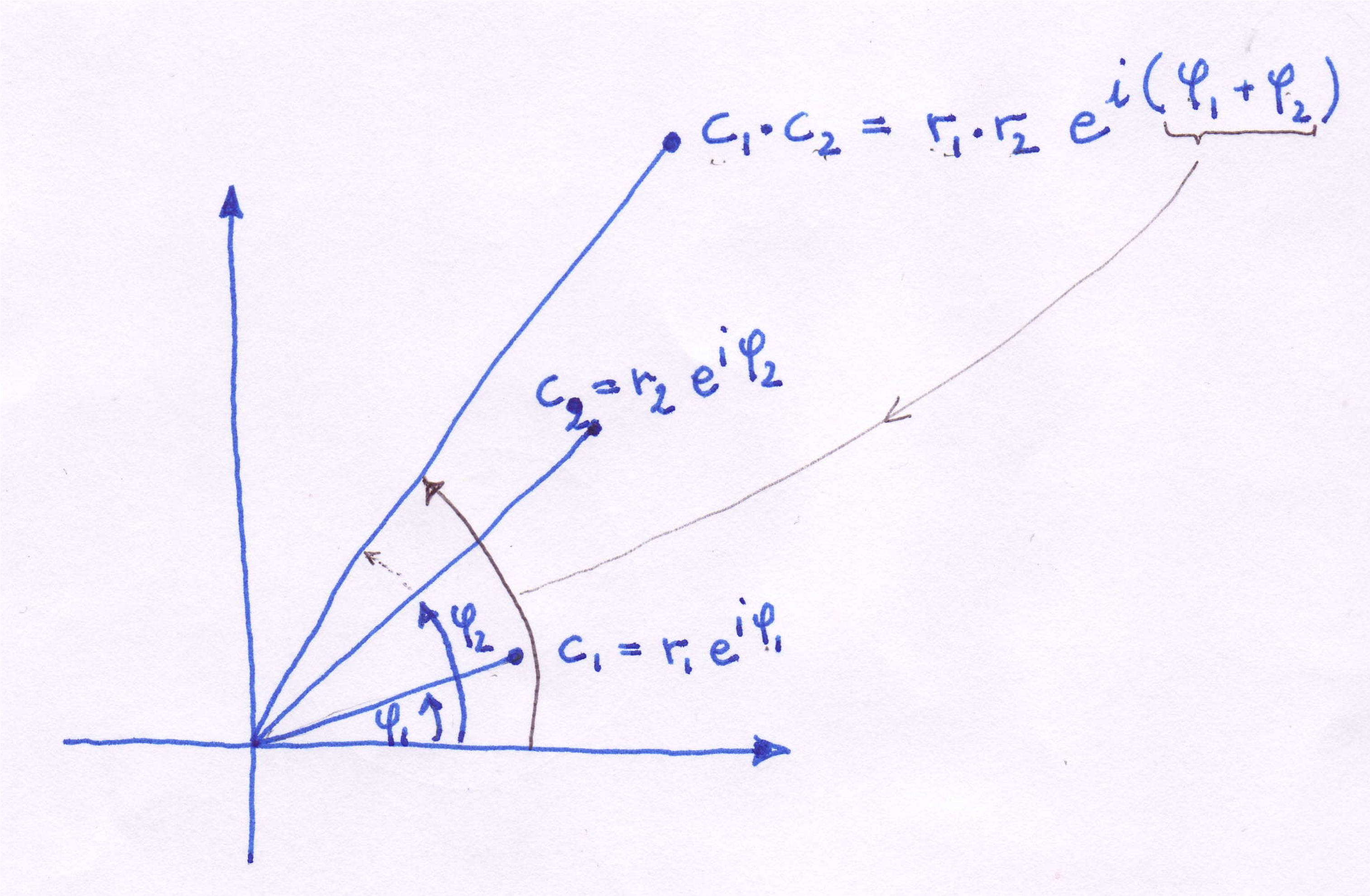
$$\cos x = Real(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos(-x)$$

$$\sin x = Im(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix}) = -\sin(-x)$$

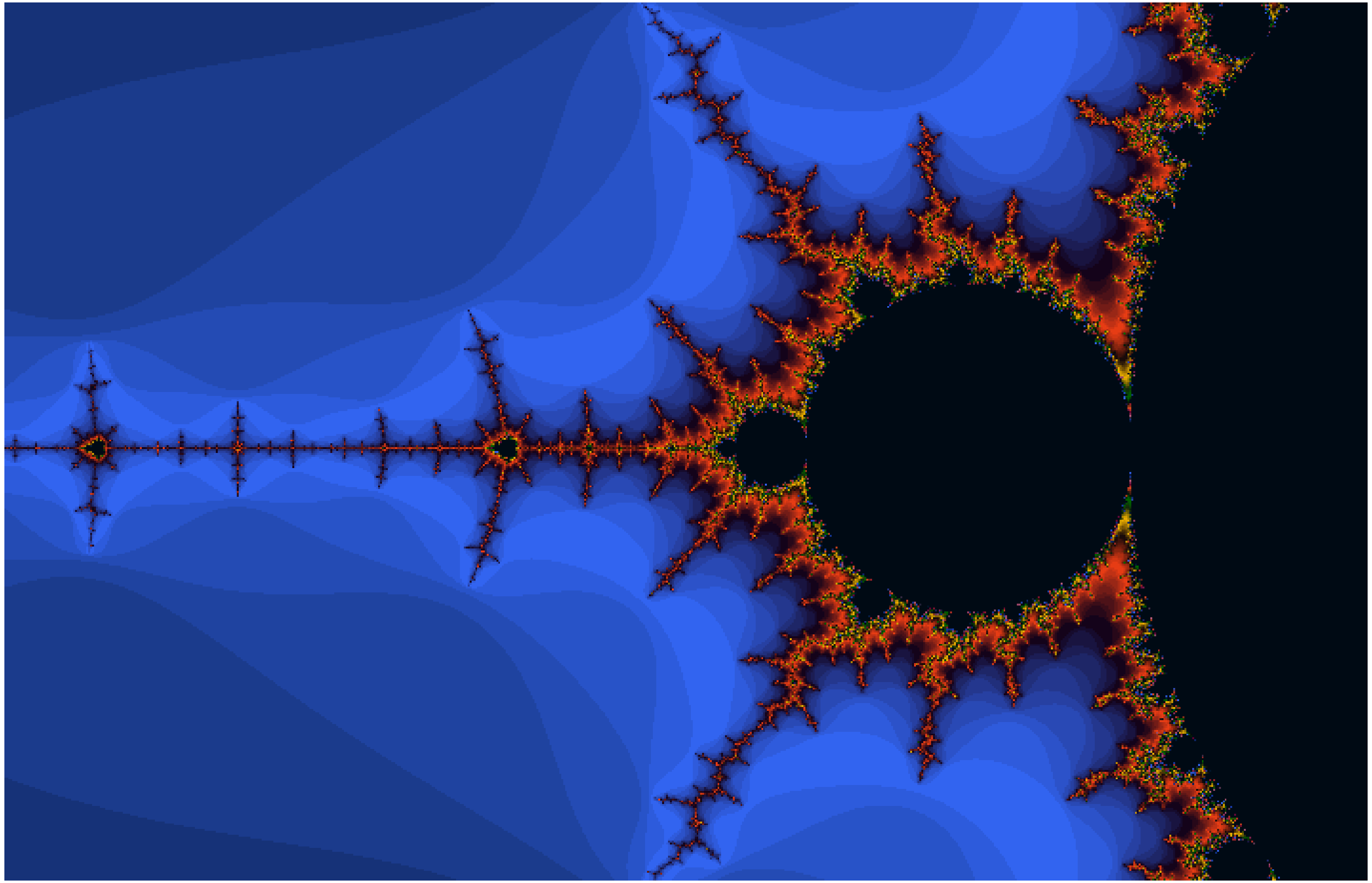
Geomteric Interpretation



Geomteric Interpretation, cont.



Mandelbrot Set



(Java code by Josef Jelinek, http://java.rubikscube.info/)

“Escape time” Algorithm for the Mandelbrot Set

Select a suitable `area` on the screen, e.g. [-2.5, 1], [-1.25, 1.25].
Interpret points/pixels in this area as complex numbers c.
Choose a constant `MAX` and a suitable function `f` that maps time to a colour gradient.

```
For each complex number c in area LOOP{
     z[0] := 0 and time := 0
     WHILE (|z[time]| < 2 and time < MAX) LOOP{
          z[time + 1] := z[time] ^ 2 + c
          time++;
     }END

     IF (|z[time]| < 2)
     THEN
         colour = black
     ELSE
         colour = f(time)

     PLOT(c,color)
}END
```

Some Alternatives and Variations

- $z \mapsto z^2 + c$ (basic Mandelbrot set)
- $z \mapsto z^p + c$, for $|p| \geq 2$ (Multibrot sets)

Julia Sets

- $z \mapsto z^2 + (.99 + .14i)z$ (a closed curve Julia set)
- $z \mapsto z^2 + (-.765 + .12i)$ (a totally disconnected Julia set)
- $z \mapsto z^2 + i$ (a "dendrite" Julia set)
- $z \mapsto z^2 - 1.75488$ (an "airplane" Julia set)
- $z \mapsto z^3 + \frac{12}{25}z + \frac{116}{125}i$
- $z \mapsto z^2 + e^{2\pi i \frac{3}{7}}z$
- $z \mapsto z^2 + z$
- $z \mapsto z^3 - iz^2 + z$

See Milnor (1990) or Milnor (1999).

Mandelbox

- A box-shaped fractal discovered by Tom Lowe in 2010.
- It is a multifractal that can be defined in any dimension.
- http://images.math.cnrs.fr/Mandelbox.html

Self-Similarity and Dimension

For any geometric object which can be subdivided into several pieces the following relationship can be considered

$$a = \frac{1}{s^D}$$

where a is the number of pieces, s is the reduction factor, and D is a real number which is called the *self-similarity dimension*. D can be calculated as

$$D = \frac{\log a}{\log \frac{1}{s}}$$

which for non-fractal structures will be an integer number Bovill (1996).

Hausdorff measure

There are several different ways to define the term fractal dimension Edgar (1993).

DEF. The s-dimensional Hausdorff measure of a set F is defined as

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \inf\{\sum_{i=1}^{\infty} |U_i|^s;\ \{U_i\} \text{ is a } \delta\text{-cover of } F\}$$

and a δ-cover is a countable collection of sets U_i such that $F \subset \cup_{i=1}^{\infty} U_i$ with $0 < \sup\{|x-y|;\ x, y \in U_i\} \leq \delta\ \forall i$.

Box dimension

The box-dimension is defined as,

$$D(S) = \lim_{\epsilon \to 0} (\log(N_\epsilon(S)) / \log \epsilon),$$

where S is a given set of points and $N_\epsilon(S)$ is the number of boxes in an overlayed lattice of boxes with edge length ϵ which intersect with S Bouligand (1929).
A discrete approximation of the box-dimension can be achieved via box-counting, a method which has been applied in architectural image analysis Bovill (1996); Ostwald and Tucker (2007); Chalup et al. (2009); Ostwald et al. (2009). Some of our research group's papers on fractal analysis are available at Nova

`http://hdl.handle.net/1959.13/809051`
`http://hdl.handle.net/1959.13/37944`

Related notions of dimension

- Hausdorff dimension (sometimes difficult to calculate)
- Correlation dimension
- Information dimension
- Similarity dimension
- Fractal dimension (a notion describing all of the above—assuming they coincide on good examples)
- Caratheodory dimension Pesin (1997)

Exercise

Experimentally explore some regions of the Mandelbrot set.
You can use the Java code by Josef Jelinek, http://java.rubikscube.info/.

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