MATH1510 - Discrete Mathematics
Graphs

University of Newcastle

Recap: Planar graphs

- A graph G is planar if a representation of it can be drawn on the plane without any edges crossing. (Such a drawing we call a planar configuration of G.)
- When a planar graph is drawn with no edge crossings, its edges divide the plane into regions called faces. The area *outside* the graph is also a face.
- If a graph G is in a planar configuration, the dual graph G^* has
  - a vertex of G^* for each face of G,
  - an edge between two vertices of G^* for each edge separating two faces of G
  - a face of G^* for each vertex of G.

Euler's formula

Theorem (Euler, 1750)

If G is a connected planar graph with v vertices, e edges and f faces, then $f + v = e + 2$.

Proof

We proceed by induction on e, the number of edges.

Basis Step: For $e = 1$, there are two cases to check:

- a loop at a single vertex: $v = 1$, $e = 1$, $f = 2$
- and and edge connecting two distinct vertices: $v = 2$, $e = 1$, $f = 1$

Proof (continued)

Inductive Step: Suppose the formula is true for all connected planar graphs with k edges. Let G be any connected planar graph with $k + 1$ edges. We will remove an edge giving a new graph G' with only k edges.

Case 1. If G has a vertex of degree 1, remove it and the incident edge to get a graph G': $v' = v - 1$, $e' = e - 1$, $f' = f$.

Case 2. If G has no vertices of degree 1, then G contains a cycle. Let x be an edge in this cycle, then x separates a face inside the cycle from a face outside the cycle. Remove x to get a graph G': $v' = v$, $e' = e - 1$, $f' = f - 1$.

In both cases the inductive hypothesis $v' + f' = e' + 2$ implies $v + f = e + 2$, and so the inductive step is proven. $\square$

Upper bounds on edges

Using Euler's theorem on a planar simple graph, upper bounds on the number of edges in the graph can be obtained.

Corollary (Upper bound on edges)

If G is a connected, simple planar graph with at least 3 vertices, then $e \leqslant 3v - 6$

Proof.

Let G^* be the dual graph of G then since G is simple $\delta(v_i^*) \geqslant 3$ for each vertex v_i^* if G^*.
Consequently, the sum of $\delta(v_i^*)$ over all dual vertices v_i^* is at least $3|V^*| = 3f$.
But $f = 2 + e - v$, so $2e \geqslant 3(2 + e - v)$ giving $e \leqslant 3v - 6$. □

Proving non-planarity (the first example)

Upper bound on edges

G connected, simple, planar, $v \geqslant 3 \implies e \leqslant 3v - 6$

Corollary

K_5 is not planar.

Proof.

$$e = 10 > 9 = 3 \cdot 5 - 6 = 3v - 6.$$

□

Q: Can we ever get $e = 3v - 6$?

A: Yes, if and only if all vertices v^* in the dual graph G^* have $\delta(v^*) = 3$, meaning all faces in G are triangles — G is a triangulation of the plane.

Proving non-planarity (the second example)

Corollary (Upper bound on edges for bipartite graph)

If a connected, simple planar graph contains no triangular faces, then $e \leqslant 2v - 4$.

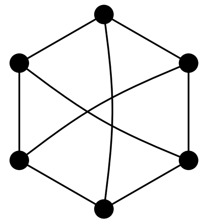
Note: *Bipartite graphs* have no triangular faces, so they must have $e \leqslant 2v - 4$ if they are to be planar.

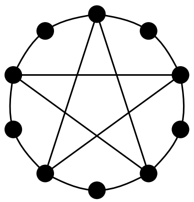
Corollary

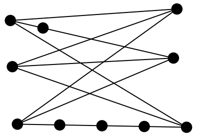
$K_{3,3}$ is not planar.

Graphs containing K_5 and $K_{3,3}$

Q: How do we tell if a graph is *planar* without actually finding a way of drawing it without crossings?
Certainly, it can't contain a subgraph which is a K_5 or a $K_{3,3}$. Nor can it contain subgraphs that are trivial modifications of these:







A: The surprising fact is that this is as bad as it gets: If there is no obstruction in form of a K_5 or $K_{3,3}$ inside our graph G, then G can be drawn in the plane without crossings.

Series reduction

Removing a vertex of degree 2 and joining the edges is called a series reduction. If a graph G_2 is obtained from a graph G_2 by a sequence of series reductions, we say G_1 and G_2 are homeomorphic to each other. More precisely, we have the following

Definition

Two graphs G_1 and G_2 are homeomorhic if there is a graph H which can be obtained from both G_1 and G_2 by a sequence of series reductions.

Kuratowski's Theorem

Theorem (Kuratowski's Theorem, 1935)

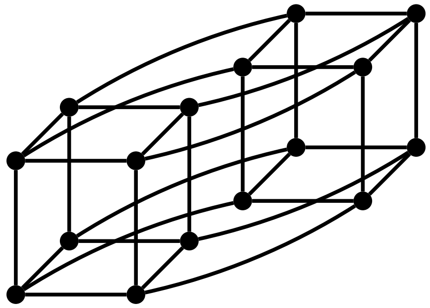
A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

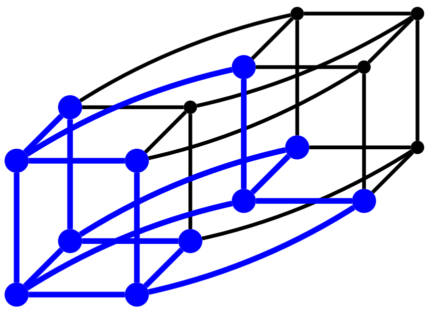
"Only if" is obvious. "If" is difficult, beyond MATH1510.

Algorithmic complexity

Planarity testing can be done in time that grows linear in the number of vertices.

K_5 in Q_4





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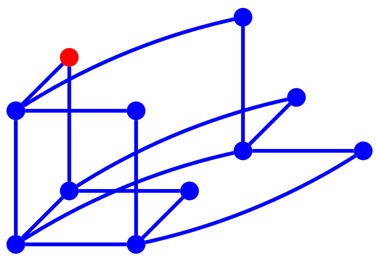

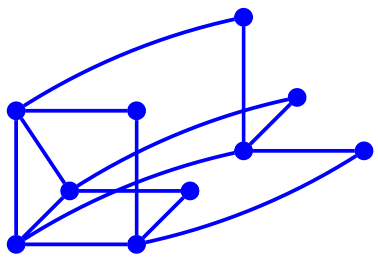

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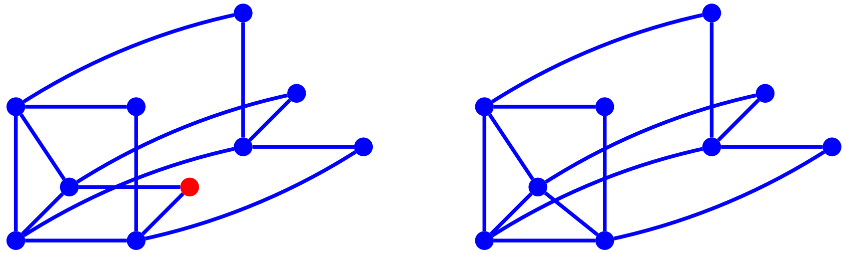

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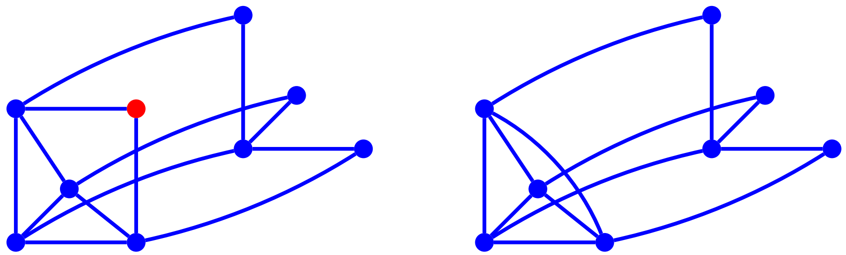

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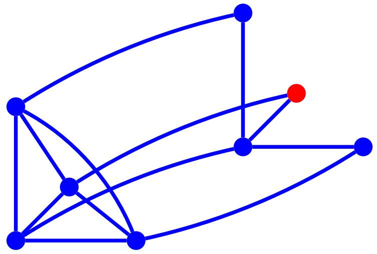

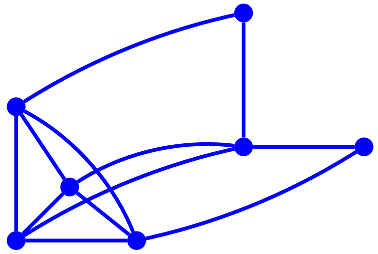

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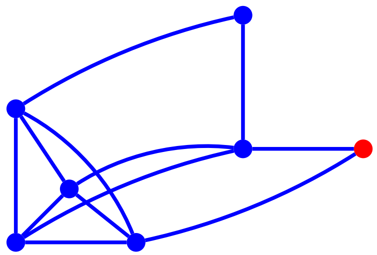

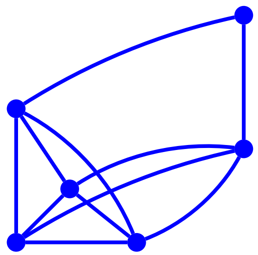

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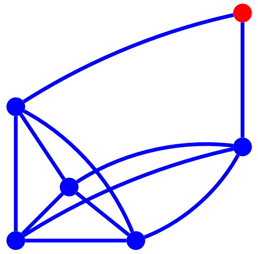

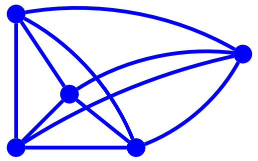

Summary

Planar graphs. can be drawn without crossings in the plane (or on the sphere)

Dual graphs. vertices $\longleftrightarrow$ faces

Euler's formula. $f + v = e + 2$

Series reduction. remove vertex of degree 2 and connect its neighbours by a new edge

Homeomorphic. two graphs that have a common series reduction

Kuratowski's theorem. planar $\iff$ no subgraph homeomorphic to K_5 or $K_{3,3}$