MATH1510 - Discrete Mathematics
Trees

University of Newcastle

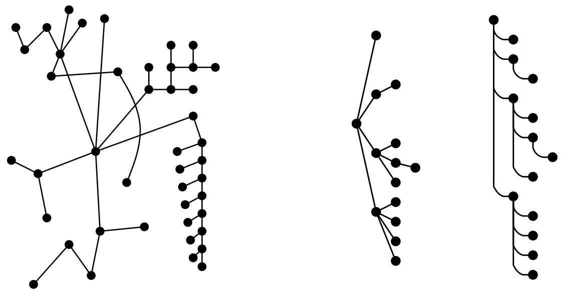
UoN

Trees

Definition

A tree is a connected graph without cycles.

Trees are used extensively in Computing (for data structures, file systems, databases, object class hierarchies, decision processes, etc.) and in other fields (organisational hierarchies, probability tree, genealogy, genetics, statistics, sport, . . .)



Some observations

- Our textbook defines them as graphs in which there is a unique path between any pair of vertices.
- Trees are simple and planar.
- A tree with $n \geqslant 1$ vertices has $n - 1$ edges. (Theorem later)
- Trees have leaves (vertices of degree 1).

Tree criterion (unique paths)

Theorem

A graph G is a tree if, and only if, there is a unique simple path between any two vertices.

Proof "$\Longrightarrow$"

- Suppose G is a tree.
- For any two vertices there is a path between them because G is connected. We will prove by contradiction that these paths are unique.
- Assume there exists at least one pair of vertices with distinct paths.
- Choose such a pair u and v with minimum distance.
- Let P be a path from u to v of length $\mathrm{dist}(u, v)$.
- By assumption there must be another path $Q \neq P$ between u and v.

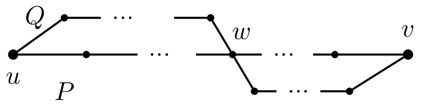
Tree criterion (unique paths)

Theorem

A graph G is a tree if, and only if, there is a unique simple path between any two vertices.

Proof "$\Longrightarrow$" (continued)

- P and Q will not have any vertices other than u or v in common:



$\Longrightarrow$ two paths between u and w and $\text{dist}(u, w) < \text{dist}(u, v)$.

- The path from u to v along P and back along Q is a simple cycle.

$\Rightarrow$ Contradiction, since G does not contain cycles because it is a tree.

- So our assumption (that there is a pair with distinct paths) is false, i.e. the paths between vertex pairs are unique.

Tree criterion (unique paths)

Theorem

A graph G is a tree if, and only if, there is a unique simple path between any two vertices.

Proof "$\Longleftarrow$".

Now suppose there is a unique simple path between any two vertices. Then G is connected by definition. If G had a cycle, then it would have a simple cycle (Lecture 3), and since a simple cycle gives two distinct simple paths between any vertices in it, G cannot have any cycles, so it is a tree. $\square$

Tree criterion (edge number)

Theorem

A tree with n vertices has $n-1$ edges.

Proof.

We proceed by induction on n.
Basis Step: $n=1$ works.
Inductive Step: Suppose true for all trees with k vertices: i.e., they all have $k-1$ edges. Let T be any tree having $k+1$ vertices. Now T must have some vertex of degree 1. Call that vertex x.
Remove this vertex x and its edge from T. The resulting graph, T', is a tree with k vertices, and therefore $k-1$ edges.
Consequently T has $k+1$ vertices and k edges. Therefore the statement is true for T.
So by induction, the statement is true for all positive integers n. □

Some corollaries

Corollary

An acyclic graph with n vertices and m connected components has $n-m$ edges.

Corollary

If an acyclic graph has n vertices and $n-1$ edges then it is a tree

Corollary

A connected graph with n vertices and at least one cycle has $\geqslant n$ edges

Corollary

If a connected graph has n vertices and $n-1$ edges then it is a tree.

Tree characterizations

These corollaries imply that a graph on n vertices is a tree if and only if it satisfies one (and thus all) of the following four equivalent conditions:

- Connected, no cycles (the definition here)
- Unique simple paths between any two vertices (Johnsonbaugh's definition)
- Connected, $n - 1$ edges
- No cycles, $n - 1$ edges

A tree must have leaves

Corollary

A tree must have at least 2 vertices of degree 1.

Sketch proof.

Let T be a tree with n vertices.

- What does the handshake theorem tell us?
- What is the minimum possible degree of vertices in a tree?
- What would happen if there were < 2 vertices of degree 1?

$\square$