CSE5004 Scientific Computation with Python

HW8. Poisson equations

Due date: June 14, 2023

Two-dimensional Poisson equation is written as

$$\nabla^2 u(x,y) = f(x,y) \quad \text{for} \quad (x,y) \in \Omega$$

and on the boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$

$$u(x,y) = g(x,y) \quad \text{on} \quad \partial\Omega_D \quad \text{and} \quad \partial u/\partial n = h(x,y) \quad \text{on} \quad \partial\Omega_N$$

Note that n is the normal to the boundary, $\partial\Omega_D$ is the Dirichlet boundary, and $\partial\Omega_N$ is the Neumann boundary.

1. (Iterative Poisson solver) Let's consider the Poisson equation in the square domain $[0,1] \times [0,1]$ with homogenous boundary conditions $u = 0$ at all boundaries and $f(x,y) = \sin(\pi x)\sin(\pi y)$.

   (1) Develop iterative Poisson solvers based on 1) Jacobi method, 2) Gauss-Seidel methods, 3) Gauss-Seidel method with successive over-relaxation (SOR). You may consider the equation is discretized on a uniform grid using the five-point scheme.

   (2) Show the performance of the iteration methods. You may consider the norm of residual, errors, computational time, etc for the performance evaluation.

2. (Linearity) Let's consider a following Poisson equation

$$\nabla^2 u(x,y) = f_1(x,y) + f_2(x,y) \quad \text{for} \quad (x,y) \in \Omega$$

   and $u(x,y) = 0$ on the boundary $\partial\Omega$ of the square domain $\Omega \equiv [0,1] \times [0,1]$.

   (1) Find $u(x,y)$, the solution of Poisson equation, $\nabla^2 u(x,y) = f_1(x,y) + f_2(x,y)$ using Gauss-Seidel SOR. The forcing functions are defined as

$$f_1(x,y) = \sin(\pi x)\sin(\pi y)$$

$$f_2(x,y) = \exp(-100.0((x-0.5)^2 + (y-0.5)^2))$$

   (2) Find $u_2(x,y)$ the solution of Poisson equation, $\nabla^2 u_2(x,y) = f_2(x,y)$ using Gauss-Seidel SOR.

   (3) Discuss the solution $u(x,y)$ by comparing with the solutions $u_2(x,y)$ and $u_1(x,y)$ that is obtained in Problem 1.