Lecture 12

PDE and Programming (3)

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Week 11

- **Multi-dimension**
  - Heat equation
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Week 12

- **Poisson equation**
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    - ✓ Gauss-Seidel method
    - ✓ Successive over relaxation method (SOR)
  - Non-linear PDEs

Poisson equation

- **Elliptical Partial Differential Equation**
  - Characteristics:
    - ➔ Elliptic equations usually arise from steady state or equilibrium physical problems
    - ➔ Boundary value problems where the solution is inter-related at all the points in the domain
    - ➔ A perturbation at one point can affect the solution instantly in the entire domain; information propagates at infinite speed
    - ➔ Boundary conditions should be specified.

Energy 가 점점 사라짐

$$\nabla^2\phi = 0,\ \text{Laplace equation}$$

Energy 가 사라지지 않음

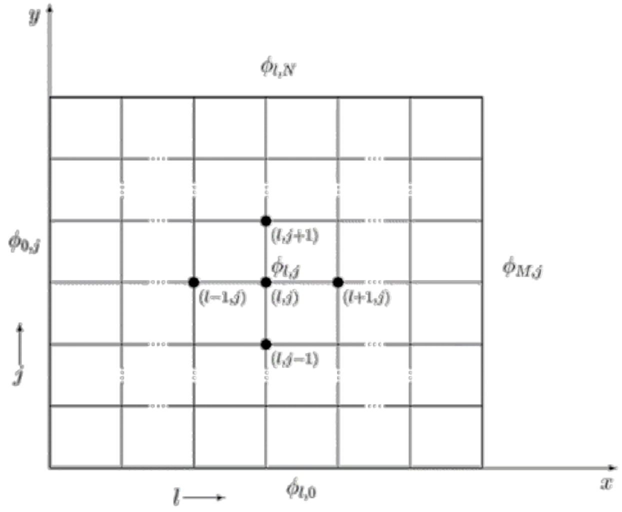
$$\nabla^2\phi = f,\ \text{Poisson equation}$$

$$\nabla^2\phi + \alpha^2\phi = 0,\ \text{Helmholtz equation}$$

$$c_1\phi + c_2\frac{\partial\phi}{\partial n} = h,\ \text{Boundary condition}$$

Where n is a normal direction coordinate

Poisson equation



- **Semi-discretization**
  - Consider the two-dimensional Poisson equation

$$\nabla^2\phi = f(x,y)$$

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j+1}}{\Delta y^2} = f_{i,j}$$

➔ Let consider the uniform spacing, $\Delta x = \Delta y = \Delta$, then

$$\phi_{i+1,j} - 4\phi_{i,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} = \Delta^2 f_{i,j}$$

➔ For $i = 1,2,\cdots,M-1$ and $j = 1,2,\cdots,N-1$

➔ $i = 0, M$ and $j = 0, N$ are the boundary conditions

➔ Ex1) For $i = 1,\ \phi_{0,j}$ is prescribed through Dirichlet boundary conditions,

$$\phi_{2,j} - 4\phi_{1,j} + \phi_{1,j+1} + \phi_{1,j-1} = \Delta^2 f_{1,j} - \phi_{0,j}$$

Poisson equation

$$\boxed{\phi_{i+1,j} - 4\phi_{i,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} = \Delta^2 f_{i,j}}$$

- **Semi-discretization**
  - ➔ EX2) At $i = 0$, is prescribed to be $\partial\phi/\partial x = g(y)$, Neumann boundary conditions,

$$\frac{-3\phi_{0,j} + 4\phi_{1,j} - \phi_{2,j}}{2\Delta} = g_j$$

$$\frac{2}{3}\phi_{2,j} - \frac{8}{3}\phi_{1,j} + \phi_{1,j+1} + \phi_{1,j-1} = \Delta^2 f_{1,j} - \frac{2}{3}\Delta g_j$$

- Consider the matrix equation,

➔ $$A\phi = b$$

- The matrix A is a block-tridiagonal matrix

➔ $$\left[\phi_{1,1}, \phi_{2,1}, \phi_{3,1}, \cdots, \phi_{M-1,1}, \phi_{1,2}, \phi_{2,2}, \phi_{3,2}, \cdots\right]^T$$

$$\begin{bmatrix}
-4 & 1 & & & & 1 & & & & & & & & & \\
1 & -4 & 1 & & & & 1 & & & & & & & & \\
 & 1 & -4 & 1 & & & & 1 & & & & & & & \\
 & & 1 & -4 & 1 & & & & 1 & & & & & & \\
 & & & 1 & -4 & & & & & 1 & & & & & \\
1 & & & & & -4 & 1 & & & & 1 & & & & \\
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 & & & & & & & & 1 & & & & 1 & -4 & 1 \\
 & & & & & & & & & 1 & & & & 1 & -4
\end{bmatrix}
\begin{bmatrix}
\phi_{11} \\ \phi_{21} \\ \phi_{31} \\ \phi_{41} \\ \phi_{51} \\ \phi_{12} \\ \phi_{22} \\ \phi_{32} \\ \phi_{42} \\ \phi_{52} \\ \phi_{13} \\ \phi_{23} \\ \phi_{33} \\ \phi_{43} \\ \phi_{53}
\end{bmatrix}
=
\begin{bmatrix}
\Delta^2 f_{11} - \phi_{01} - \phi_{10} \\
\Delta^2 f_{21} - \phi_{20} \\
\Delta^2 f_{31} - \phi_{30} \\
\Delta^2 f_{41} - \phi_{40} \\
\Delta^2 f_{51} - \phi_{61} - \phi_{50} \\
\Delta^2 f_{12} - \phi_{02} \\
\Delta^2 f_{22} \\
\Delta^2 f_{32} \\
\Delta^2 f_{42} \\
\Delta^2 f_{52} - \phi_{62} \\
\Delta^2 f_{13} - \phi_{03} - \phi_{14} \\
\Delta^2 f_{23} - \phi_{24} \\
\Delta^2 f_{33} - \phi_{34} \\
\Delta^2 f_{43} - \phi_{44} \\
\Delta^2 f_{53} - \phi_{63} - \phi_{54}
\end{bmatrix}$$

Iterative solution methods

- **Basic concepts**

$$A\phi = b$$

- Let $A = A_1 - A_2$, then,

$$A_1\phi = A_2\phi + b$$

$$A_1\phi^{(k+1)} = A_2\phi^{(k)} + b \;\Rightarrow\; \phi^{(k+1)} = A_1^{-1}\left(A_2\phi^{(k)} + b\right)$$

➔ For $k = 0, 1, 2, \cdots$, and initial guess for the solution $\phi^{(0)}$.

- Requirements

➔ A_1 should be easily “invertible.”

➔ Iterations should converge (hopefully rapidly), $\lim_{k\to\infty} \phi^{(k)} = \phi$

Iterative solution methods

$$\boxed{\phi^{(k+1)} = A_1^{-1}\left(A_2\phi^{(k)} + b\right)}$$

- **Convergence**

$$\varepsilon^{(k)} = \phi - \phi^{(k)}$$

$$A_1\, \varepsilon^{(k+1)} = A_2\varepsilon^{(k)}$$

$$\varepsilon^{(k+1)} = A_1^{-1}A_2\varepsilon^{(k)} \Rightarrow \ \varepsilon^{(k)} = \left(A_1^{-1}A_2\right)\varepsilon^{(0)}$$

- Let λ_i be the eigenvalues of the matrix, $A_1^{-1}A_2$, then, as $|\lambda_i|_{max} \leq 1$

$$\lim_{k\to\infty} \varepsilon^{(k)} = 0$$

Iterative solution methods

$$\phi^{(k+1)} = A_1^{-1}\left(A_2\phi^{(k)} + b\right)$$
$$\phi_{i+1,j} - 4\phi_{i,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} = \Delta^2 f_{i,j}$$

- **Point Jacobi method**
  - Decompose the diagonally dominant matrix A into
  - Diagonal(D) and off-diagonal$(L+U)$ components. ⇒ $A = A_1 - A_2,\ \ A_1 = D,\ A_2 = L + U$

$$\phi^{(k+1)} = -\frac{1}{4}A_2\phi^{(k)} - \frac{1}{4}\boldsymbol{R}$$

$$\phi_{i,j}^{(k+1)} = \frac{1}{4}\left[\phi_{i+1,j}^{(k)} + \phi_{i-1,j}^{(k)} + \phi_{i,j+1}^{(k)} + \phi_{i,j-1}^{(k)}\right] - \frac{1}{4}R_{i,j}$$

$$\phi^{(k+1)} = -D^{-1}(L+U)\phi^{(k)} + D^{-1}\boldsymbol{b}$$

  - The eigenvalue is, for $m = 1,2,\cdots,M-1$ and for $n = 1,2,\cdots,N-1$,

$$\lambda_{mn} = \frac{1}{2}\left[\cos\frac{m\pi}{M} + \cos\frac{n\pi}{N}\right] = 1 - \frac{1}{4}\left[\frac{\pi^2}{M^2} + \frac{\pi^2}{N^2}\right] + \cdots$$

➔ $|\lambda_{mn}| < 1$ for all m and n, and the method converges. 사이즈가 클 수록 수렴이 오래 걸려!

➔ For large M and N, $|\lambda|_{max}$ is only slightly less than 1, the convergence is very slow.

Iterative solution methods

- **Gauss-Seidel method**
  - Decompose the diagonally dominant matrix A
  - Diagonal-lower-triangle$(D-L)$ and upper-triangle(U). $\Rightarrow$ $A = A_1 - A_2,\ A_1 = D - L,\ A_2 = U$

$$\phi_{i,j}^{(k+1)} = \frac{1}{4}\left[\phi_{i+1,j}^{(k)} + \phi_{i-1,j}^{(k+1)} + \phi_{i,j+1}^{(k)} + \phi_{i,j-1}^{(k+1)}\right] - \frac{1}{4}R_{i,j}$$

$$(D-L)\phi^{(k+1)} = U\phi^{(k)} + \boldsymbol{b}$$

  - The eigenvalue is, for $m = 1,2,\cdots,M-1$ and for $n = 1,2,\cdots,N-1$,

$$\lambda_{mn} = \frac{1}{4}\left[\cos\frac{m\pi}{M} + \cos\frac{n\pi}{N}\right]^2$$

➔ $|\lambda_{mn}| < 1$ for all m and n, and the method converges.

➔ Gauss-Seidel method converges twice as fast as the point Jacobi method.

Iterative solution methods

- **Successive Over Relaxation Scheme (SOR)**
  - To increase the rate of convergence of the Gauss-Seidel method,

$$(D-L)\phi^{(k+1)} = U\phi^{(k)} + \Delta^2 \boldsymbol{f}$$

  - Let define the change in the solution between two successive iterations, $\boldsymbol{d}$,

$$\boldsymbol{d} = \phi^{(k+1)} - \phi^{(k)}$$

$$\phi^{(k+1)} = \phi^{(k)} + \boldsymbol{d}$$

$$\Rightarrow \quad \phi^{(k+1)} = \phi^{(k)} + \omega\boldsymbol{d}$$

➔ $\omega > 1$ is the acceleration or "relaxation" parameter.

$$D\tilde{\phi}^{(k+1)} = L\phi^{(k+1)} + U\phi^{(k)} + \boldsymbol{b}$$

$$\phi^{(k+1)} = \phi^{(k)} + \omega\left(\tilde{\phi}^{(k+1)} - \phi^{(k)}\right)$$

$$\phi^{(k+1)} = \underbrace{(I-\omega D^{-1}L)^{-1}[(1-\omega)I + \omega D^{-1}U]}_{G_{SOR}}\phi^{(k)} + (1-\omega D^{-1}L)^{-1}\omega D^{-1}\boldsymbol{b}$$

Iterative solution methods

- **Successive Over Relaxation Scheme (SOR)**
  - The eigenvalues of the matrix G_{SOR},

non-linearity가 심하면 구하기 힘들 수 있음

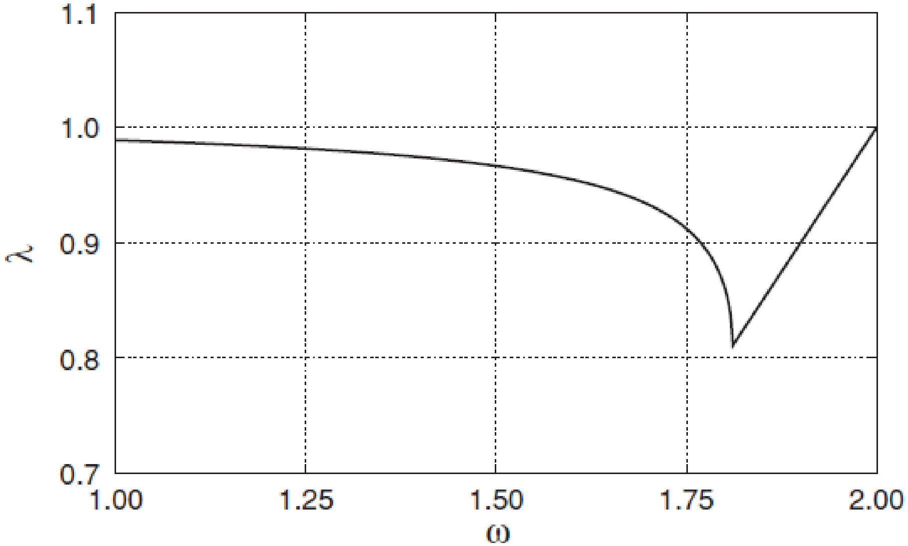
$$\lambda^{\frac{1}{2}} = \frac{1}{2}\left(\pm|\mu|\omega \pm \sqrt{\mu^2\omega^2 - 4(\omega - 1)}\right)$$

➔ μ is an eigenvalue of the point Jacobi matrix, $G_J = D^{-1}(L+U)$.

  - To optimize convergence, one should select the relaxation parameter ω to minimize the largest eigenvalue λ.

$\mu_{\max} = 0.9945.$

31×31 mesh



$$\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \mu_{\max}^2}}$$

$\Delta t = 0.05$ and $M = N = 20$

Method	Iterations
Point Jacobi	749
Gauss–Seidel	375
SOR ($\omega = 1.8$)	45

Non-linear PDEs

- **Burgers' equation**
  - Consider the one-dimensional Burgers' equation, Crank Nicolson이 시점이 언제로 되어있는가?

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

  - This equation has a non-linear convection-like term and a linear diffusion term

$$u^{(n+1)} - u^{(n)} = -\frac{\Delta t}{2}\underbrace{\left(3u^{(n)}\frac{\partial u^{(n)}}{\partial x} - u^{(n-1)}\frac{\partial u^{(n-1)}}{\partial x}\right)}_{\text{Adams–Bashforth method}} + \underbrace{\frac{\nu \Delta t}{2}\left(\frac{\partial^2 u^{(n+1)}}{\partial x^2} + \frac{\partial^2 u^{(n)}}{\partial x^2}\right)}_{\text{Crank–Nicolson method}}$$

$$\frac{\nu}{2}\frac{\partial^2 u^{(n+1)}}{\partial x^2} - \frac{u^{(n+1)}}{\Delta t} = -\frac{u^{(n)}}{\Delta t} + \frac{1}{2}\left(3u^{(n)}\frac{\partial u^{(n)}}{\partial x} - u^{(n-1)}\frac{\partial u^{(n-1)}}{\partial x}\right) - \frac{\nu}{2}\frac{\partial^2 u^{(n)}}{\partial x^2}$$

Q&A Thanks for listening