CSE5004 Scientific Computation with Python

HW6. Stokes second problem

Due date: May 31, 2023

Consider an infinitely extended flat wall carrying out harmonic oscillation in its own plane (see Figure 1). Due to the no-slip condition, the flow velocity at the wall (where $y = 0$) is $u(0,t) = U_0 \cos(nt)$.

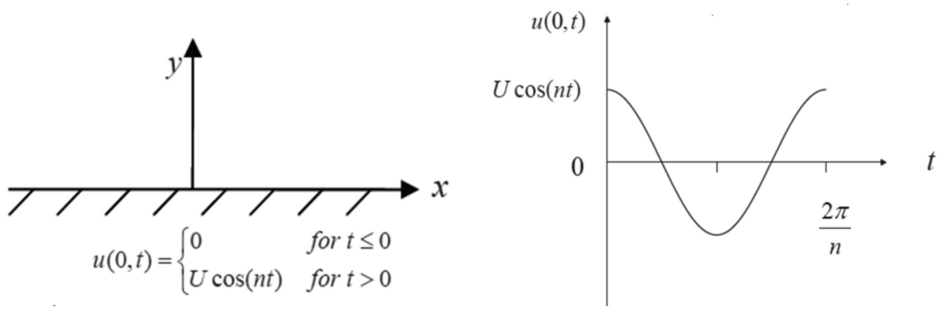

Figure 1: Schematic diagram of Stokes second problem.

1. (Analytic solution)

   (1) Starting from the three-dimensional Navier-Stokes equations, derive the simplified governing equation for the Stokes second problem. Provide all of the assumptions that are used in your derivation.

   $$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

   (2) Show that the solution to the Stokes second problem is

   $$u(y,t) = U_0 e^{-\eta_s} \cos(nt - \eta_s), \quad \text{where} \quad \eta_s = \sqrt{\frac{n}{2\nu}}\, y. \tag{2}$$

2. (Numerical analysis)

   Consider two infinitely long plates placed at $y = 0$ and $y = L$. The bottom plate ($y = 0$) is oscillating with $u(0,t) = \cos(nt)$, while the top plate ($y = L$) is stationary. We aim to obtain velocity profiles $u(y,t)$ between two plates by solving Eq. 1 under the assumptions $\nu = 1$, $n = 2$, $U_0 = 1$ and $L = 10$.

   (1) Solve Eq.(1) numerically using the first-order forward difference in time and second-order central difference in space (FTCS scheme). Plot the velocity profiles at $nt = 0, \pi/2, \pi, 3\pi/2, 2\pi$. Also plot the quasi-steady state velocity profiles at $(nt - T) = 0, \pi/2, \pi, 3\pi/2, 2\pi$. Note that T represents the transient period required to reach the quasi-steady state solutions. You may use $T = 10\pi$.

   (2) Repeat (1) using the Crank-Nicolson (C-N) scheme in time.

   (3) Discuss the convergence rates for the two different time schemes. In particular, verify that the orders of accuracy for the FTCS and C-N schemes are $O(\Delta t)$ and $O(\Delta t^2)$, respectively. Note that the solution in Eq.(2) for the Stokes second problem is only valid for a single oscillating wall (infinite in the x-direction), but you may use the solution of the Stokes second problem in Eq.(2) as an exact solution.

   (4) Repeat (2) with $L = 2$. Discuss the effect of the gap distance between the two plates on the velocity profiles.