COMS 4236: Computational Complexity (Fall 2018) Problem Set #2

Wenbo Gao - `wg2313@columbia.edu`

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Problem 4

Problem. Problem 8.15 on page 303 of TC (5 points if you can show the following problem is in PSPACE; 10 points if you can show if it is in P!): The cat-and-mouse game is played by two players, "CAT" and "Mouse", on an arbitrary undirected graph. At a given point each player occupies a node of the graph. The players take turns moving to a node adjacent to the one that they currently occupy. A special node of the graph is called "Hole". Terminal conditions of the game:

- **Cat wins** if the two players ever occupy the same node.
- **Mouse wins** if it reaches the Hole before the preceding happens.
- The game is a **draw** if the two players ever simultaneously reach positions that they previously occupied.

Let

$$\begin{aligned}\text{HAPPY-CAT} = \{(G, c, m, h) \mid\ & \text{G,c,m,h are respectively a graph, and} \\ & \text{positions of the Cat, Mouse, and Hole, such that} \\ & \text{Cat has a wining strategy, if Cat moves first.}\}\end{aligned}$$

Show that HAPPY-CAT is in PSPACE (and in P for more points).

Solution. *HAPPY-CAT is in P.*
Given undirected graph $G(V, E)$, where $|V| = n$.
The state space of the game has $2n^2$ vertices:

$$\begin{aligned}\text{State} = & \\ \{ & \quad \text{Cat\_position} :: V \\ , & \quad \text{Mouse\_position} :: V \\ , & \quad \text{turn\_player} :: \{\text{Cat}, \text{Mouse}\} \\ \} &\end{aligned}$$

$$\# \text{ of states} = |V| * |V| * 2 = n * n * 2 = 2n^2$$

The edges between states are valid state transitions, the number of which is at most $8n^4$.

Each vertex takes $log_2 n$ bits and each edge takes $2log_2 n$ bits, so the entire state graph takes $(2n^2 log_2 n + 16n^4 log_2 n)$ bits which is $O(n^5)$.

Annotate the states by Cat's win condition:

$$\forall V_0 \in V, (\{V_0, V_0, \text{Cat}\}, \text{Cat\_Win}) \text{ and } (\{V_0, V_0, \text{Mouse}\}, \text{Cat\_Win})$$

HAPPY-CAT then can be formalized as a reachability problem:
Given Cat's initial position $V_{Cat\_init} \in V$, Mouse's initial position $V_{Mouse\_init} \in V$, and initial turn player Cat, whether there's a state annotated by Cat_Win can be reached from this initial state.

We can use BFS traversing the state graph backwards from Cat's win states, until reach the initial state and accept, or exhaust all the reachable edges without hitting the initial state and reject. The runtime is bounded by the number of edges which is $O(n^4)$.

Therefore, HAPPY-CAT is in P. $\square$