Problem 1

a) closed under union ($\cup$)

Let $\mathbb{L} = \{ L \mid \exists M \in \mathbb{M},\ L(M) \}$

$$\forall L_1 \in \mathbb{L}, \forall L_2 \in \mathbb{L} \Rightarrow \exists M_1 \in \mathbb{M},\ L_1(M_1)$$
$$\exists M_2 \in \mathbb{M},\ L_2(M_2)$$

$$\sim M_1(w) = \begin{cases} 1; & \text{if } w \in L_1 \\ 0, & \text{if } w \notin L_1 \end{cases},\quad M_2(w) = \begin{cases} 1; & \text{if } w \in L_2 \\ 0; & \text{if } w \notin L_2 \end{cases}$$

$$L = L_1 \cup L_2 = \{ w \mid \forall w \in \Sigma^*,\ w \in L_1 \wedge w \in L_2 \}$$

Construct $$D(w) = \begin{cases} 1; & \text{if } M_1(w) = 1 \vee M_2(w) = 1 \\ 0; & \text{otherwise} \end{cases}$$

$$\sim D(w) = \begin{cases} 1; & \text{if } w \in L_1 \vee w \in L_2 \\ 0; & \text{if } w \notin L_1 \wedge w \notin L_2 \end{cases}$$

$$\sim D(w) = \begin{cases} 1; & \text{if } w \in L \\ 0; & \text{if } w \notin L \end{cases} \Rightarrow L(D)$$

$$\Rightarrow L \in \mathbb{L}$$

b) closed under concatenation ($\langle\rangle$)

$\forall L_1 \in \mathbb{L}, \forall L_2 \in \mathbb{L} \Rightarrow \exists M_1 \in \mathbb{M}, L_1(M_1)$
$\exists M_2 \in \mathbb{M}, L_2(M_2)$

$L^* = \{ L \mid L = (w_1, \ldots, w_n), \exists i \in N,$
$(w_1, \ldots, w_i) \in L_1, (w_{i+1}, \ldots, w_n) \in L_2 \}$

$D(w) =$ Let $n = |w|$
in variable result = false
$i \leftarrow [0, 1, 2, \ldots, n]$
result = result $\vee$ $(M_1(w_1 \ldots w_i)$
$\wedge M_2(w_{i+1} \ldots w_n))$
return result

$L^*(D) \Rightarrow L^* \in \mathbb{L}$

c) complementation

$$\forall L_1 \in \mathbb{L} \Rightarrow \exists M \in \mathbb{M}, L_1(M_1)$$

$$\Rightarrow M(w) = \begin{cases} 1 ; & \text{if } w \in L_1 \\ 0 ; & \text{if } w \notin L_1 \end{cases}$$

$$L = \Sigma^* \setminus L_1$$

<u>Construct</u> $D(w) = \neg M(w) = \begin{cases} 1 ; & \text{if } w \notin L_1 \\ 0 ; & \text{if } w \in L_1 \end{cases}$

$$= \begin{cases} 1 ; & \text{if } w \in L \\ 0 ; & \text{if } w \notin L \end{cases}$$

$$\Rightarrow L(D) \Rightarrow L \in \mathbb{L}$$

d) closed under intersection ($\cap$)

$$\forall L_1 \in \mathbb{L}, \forall L_2 \in \mathbb{L} \Rightarrow \exists M_1 \in \mathbb{M}, L_1(M_1); \exists M_2 \in \mathbb{M}, L_2(M_2) \Rightarrow M_1(w) = \begin{cases} 1; & \text{if } w \in L_1 \\ 0; & \text{if } w \notin L_1 \end{cases} \quad M_2(w) = \begin{cases} 1; & \text{if } w \in L_2 \\ 0; & \text{if } w \notin L_2 \end{cases}$$

$$L = L_1 \cap L_2 = \{ w \mid \forall w \in \Sigma^*, w \in L_1 \wedge w \in L_2 \}$$

<u>Construct</u> $D(w) = \begin{cases} 1; & \text{if } M_1(w) = 1 \wedge M_2(w) = 1 \\ 0; & \text{otherwise} \end{cases}$

$$= \begin{cases} 1; & \text{if } w \in L_1 \wedge w \in L_2 \\ 0; & \text{if } w \notin L_1 \vee w \notin L_2 \end{cases}$$

$$= \begin{cases} 1; & \text{if } w \in L \\ 0; & \text{if } w \notin L \end{cases} \Rightarrow L(D) \Rightarrow L \in \mathbb{L}$$

Problem 2

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data TM_State = { state :: Q
                , tape :: [Γ]
                , position :: ℕ
                }
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Can mutate empty cells?

Problem 3

$$HALT = \{\langle M, w\rangle \mid \forall M \in \mathcal{M}, \forall w \in \Sigma^*, M(w) = 1\}$$

a) $BLANK\,HALT = \{\langle M\rangle \mid \forall M \in \mathcal{M}, \; M(\varepsilon) = 1\}$

HALT	Blank HALT
Given $\langle M, w\rangle$	Given $\langle M'\rangle$

reduction

construct M' on top of M, with w encoded in its FSA

behavior $\forall$ input $w' \in \Sigma^*$
- erase input w'
- write w to the tape
- reset pointer and state

$\}$ FSA

- simulate M (on w) $\Rightarrow \forall w' \in \Sigma^*, \; M'(w') = M(w)$

$\Rightarrow M' \in Blank\,HALT$

$\Rightarrow \forall \langle M, w\rangle \in HALT, \; \exists M' \in BlankHAT,$

if $M'(\varepsilon) = 1$, then $M(w) = 1$

b) $Exist = \{ \langle M \rangle \mid \forall M \in \mathcal{M}, \exists w \in \Sigma^*, M(w) = 1 \}$

$(L(M) \neq \phi$ or $|L(M)| \geq 1)$

the same reduction as (a) $\Rightarrow \forall w' \in \Sigma^*, M'(w') = M(w)$

$\Rightarrow \exists w' \in \Sigma^*, M'(w') = 1$

$\simeq M(w) = 1$

c) $Finite = \{ \langle M \rangle \mid \forall M \in \mathcal{M}, L(M)$ is finite $\}$

~ able to find an upper bound for $|L(M)|, \forall M \in \mathcal{M}$.

the same reduction as (a) with its answer flipped.

$\Rightarrow \forall w' \in \Sigma^*, M'(w') = \neg M(w)$

$\Rightarrow \underbrace{L(M') = \phi}_{\text{finite}} \simeq M(w) = 1$

$(\forall w' \in \Sigma^*, M'(w') = \neg M(w) = 0$ or run forever$)$

d) $L = \{ \langle M_1, M_2 \rangle \mid \forall M_1, M_2 \in \mathbb{M},$
$L(M_1) = L(M_2) \}$