Monoid on set

Monoid(hom-set, $(\circ)$)

binary operator/morphism: $(\times) :: a \to b \to a \times b$

unit object : $()$ ↖ terminal object

<u>prove</u> terminal object is the unit object under product $(\times)$.

<u>proof</u>

$$
\begin{tikzcd}
 & b \arrow[d, dashed, "p"] \arrow[ddl, bend right, "p"'] \arrow[ddr, bend left, "unit_b"] & \\
 & a \arrow[dl, "id_a"'] \arrow[dr, "unit_a"] & \\
a & & ()
\end{tikzcd}
\cong
\begin{tikzcd}
 & b \arrow[d, dashed, "p \times unit_b"] \arrow[ddl, bend right, "p"'] \arrow[ddr, bend left, "unit_b"] & \\
 & a \times () \arrow[dl, "fst"'] \arrow[dr, "snd"] & \\
a & & ()
\end{tikzcd}
$$

<u>Goal</u> prove a is isomorphic to the product of a and $()$.

<u>by definition of terminal object</u>

There is a unique morphism from any object to terminal object:

$unit :: \text{forall } a.\ a \to ()$.

<u>by definition of Category</u>

There is a unique morphism from any object to itself:

$id :: \text{forall } a.\ a \to a$.

forall candidate object b which has an arrow p to a, an arrow $unit_b$ to $()$.

There is a unique morphism from b to a : p.

$p = id_a \circ p$, $unit_b = unit_a \circ p$.

Coproduct

$$\begin{tikzcd}
a \arrow[dr, "left"] \arrow[ddr, bend right, "p"'] & & b \arrow[dl, "right"'] \arrow[ddl, bend left, "q"] \\
& a \mid b \arrow[d, dashed, "p \mid q"] & \\
& c &
\end{tikzcd}$$

pattern matching

case a|b of

$a \longrightarrow p\ a$

$b \longrightarrow q\ b$

$$\begin{tikzcd}
a \arrow[dr, "id_a"] \arrow[ddr, bend right, "p"'] & & () \arrow[dl, "constant_a"'] \arrow[ddl, bend left, "constant_b"] \\
& a \arrow[d, dashed, "p"] & \\
& b &
\end{tikzcd}
\quad \cong \quad
\begin{tikzcd}
a \arrow[dr, "left"] \arrow[ddr, bend right, "p"'] & & () \arrow[dl, "right"'] \arrow[ddl, bend left, "constant_b"] \\
& a \mid () \arrow[d, dashed, "p \mid constant_b"] & \\
& b &
\end{tikzcd}$$

Both Product and Coproduct are Bifunctors: $C \times C \to C$

(x) (|)

and have a unit object : the terminal object of C.

⇒ Both Product Category and CoProduct Category are Monoidal Category.

Monoid Category is a Category of one object, which is a set. append, mempty	Monoidal Category is a Category of one object, which is a category. $\otimes$, 1