weak BPP: polynomial $q(n)$, $D(x,y) \in$ all DTMs

$x \in L \Rightarrow P[D(x,y)=1] \geq \frac{1}{2} + \frac{1}{q(n)}$

$x \notin L \Rightarrow P[D(x,y)=1] \leq \frac{1}{2} - \frac{1}{q(n)}$

Strong BPP:

$x \in L \Rightarrow P[D(x,y)=1] \geq 1 - e^{-q(n)}$

$x \notin L \Rightarrow P[D(x,y)=1] \leq e^{-q(n)}$

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show weak BPP = strong BPP

to fix it to an odd number so majority poll never ties.

Construct D' from D in weak BPP

$m = poly(n)$, input $x, \{y_1, y_2, \ldots, y_{2m+1}\}$

Chernoff Bound

n R.V.s

$\{X_1, X_2, \ldots, X_n\} \sim$ I.D. $\{0,1\}$ $\begin{cases} Pr[X_i=1] = P_i \\ Pr[X_i=0] = 1-P_i \end{cases}$

Let $X = \sum_{i=1}^{n} X_i$

$\mu = E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P_i$

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for $\Delta > 0$,

$Pr[X - \mu \geq \Delta]$?

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① $Pr[X - \mu \geq \Delta] \leq e^{-\frac{2\Delta^2}{n}}$

or $Pr[|X - \mu| \geq \Delta] \leq 2e^{-\frac{2\Delta^2}{n}}$

$E[X] = \sum_{y \in \{0,1\}} y \cdot P[X=y] = p$

$Var[X] = E[X^2] - (E[X])^2 = p - p^2$

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② when $P_1 = P_2 = \cdots = P$

$\Rightarrow \{X_1, X_2, \ldots, X_n\} \sim$ I.I.D. $(P, P-P^2)$

$\mu = n \cdot p$

for $\delta > 0$,

$Pr[X - \mu \geq \delta\mu] \leq e^{-\frac{\delta^2\mu}{4}}$

$Pr[X - \mu \leq -\delta\mu] \leq e^{-\frac{\delta^2\mu}{2}}$

when $x \notin L$, $p = \Pr_{y \in Y}[D(x,y)=1] \leq \frac{1}{2} - \frac{1}{q(n)}$

$\Pr[D'(x, y_1, y_2, \ldots, y_{2m+1}) = 1]$

$= \Pr[\underbrace{\sum_{i=1}^{2m+1} D(x, y_i)}_{\triangleq X} \geq m+1]$

$\mu = E[\underbrace{\sum_{i=1}^{2m+1} D(x, y_i)}_{\triangleq X}] = \sum_{i=1}^{2m+1} E[D(x, y_i)] = (2m+1) \cdot p$

$\leq (2m+1) \cdot (\frac{1}{2} - \frac{1}{q(n)})$

$\Rightarrow m+1 \geq \frac{1}{2}\left(\frac{1}{1-\frac{1}{q(n)}} \cdot \mu\right)$

$= \Pr[X - \mu \geq m+1-\mu]$

$= \Pr[X - \mu \geq \underbrace{m+1-(2m+1) \cdot p}_{}]$

$= \Delta \geq m+1-(2m+1) \cdot (\frac{1}{2} - \frac{1}{q(n)})$

$\leq e^{\frac{-2\Delta^2}{(2m+1)}}$

$= m+1-m-\frac{1}{2}+\frac{2m+1}{q(n)}$

$\leq e^{-\Omega\left(\frac{m}{q^2(n)}\right)}$

$= \frac{2m+1}{q(n)} + \frac{1}{2}$

$\boxed{\sim e^{-P(n)} \text{ in Strong BPP}} \Rightarrow m = P(n)\, q^2(n) \sim$ polynomial

$BPP \subseteq \Sigma_2^P \cap \Pi_2^P$