

Analysis of Various Assets

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1 Background of Serial Correlation

1.1 Definition

Serial correlation, also known as autocorrelation, is the correlation of observations at different time point. Under the wide-sense stationary process assumption, the serial correlation of X between two time point t and s can be measured by the autocorrelation function as follows:

$$R(\tau) = R(s, t) = \frac{E[(X_t - \mu)(X_s - \mu)]}{\sigma^2} \quad (1)$$

where $\tau = t - s$.

1.2 Literature Review

Serial correlation is often associated with the violation of efficiency market and random walk hypothesis. The literature documenting empirical serial correlation is extensive in the late 1980's¹. In stock price, Lo and MacKinlay (1988) [?] argue that returns based on a horizon longer than one year show a significant mean reversion, while Poterba and Summers (1988) [?] detect a mean aversion for weekly and monthly returns. Lo and Mavkinlay (1988) [?], Conrad et al. (1991) [?] model the security returns using a positively autocorrelated common component, a idiosyncratic component and a white-noire component. More extensively, serial correlation has been documented in literature on nonsynchronous trading, which means assets are not traded simultaneously [?]. Mech and Timothy (1993) [?] present evidence that the autocorrelation is associated with the delaying in price adjustment caused by transaction costs. In hedge fund returns, Getmansky, Lo and Makarov (2004) argue that serial correlation is an out come of illiquidity exposure and smoothed returns, market inefficiencies, time-varying expected returns and leverage and incentive fees with high water marks.

Assuming a moving average representation of reported returns, Getmansky, Lo and Makarov (2004) [?] show that Sharp Ratio (SR) tends to be overstated and the market beta understated. Cesare, Stork and Vries (2014) [?] use the similar structure to demonstrate that the reported value-at-risk (VaR) and expected shortfall (ES) are always smaller than or equal to their actual values. Thus the risks of assets are easily underestimated using standard risk measures and the investment decisions may be misleading. Although based on serial correlation and smoothing feature of hedge

¹See Barucci and Emilio (2012, Section 6.5) [?] for a detailed review.

fund returns, their models are as well applicable to other assets with autocorrelated returns.

2 Asset Description

We examined 11 asset class in total which include bonds both in and outside the U.S.. We treat 3-Month U.S. Treasury Bill Index as the risk-free rate, and examine the returns and risk measures of the other 10.

- AGG: iShares Core US Aggregate Bond
Date ranges: 2003-09-29 to 2015-12-31
Components: US Treasuries (37.7%); US Agencies (2.5%); US Municipalities (0.8%); Corporates (24.2%); Non-Corporate Credit (4.4%); Mortgage-Backed Securities (MBS) (28.4%); Commercial Mortgage-Backed Securities (CMBS) (1.7%); Adjusted Rate Mortgages (ARMs) (0.2%)
Description: AGG provides access to 4,000+ bonds and offers exposure to 7 unique sectors as represented in the broad U.S. bond market.²
- HYG: iShares iBoxx \$ High Yield Corporate Bd
Date ranges: 2007-04-12 to 2015-12-31
Components: The sector breakdown data shows that four sectors occupied more than 10% high yield bond, those are: Communications(25.8%), Consumer Non-cyclical(14.2%), Energy(11.4%), Technology(10.7%)³
Description: The iShares iBoxx \$ High Yield Corporate Bond ETF seeks to track the investment results of an index composed of U.S. dollar-denominated, high yield corporate bonds.⁴
- TIP: iShares TIPS Bond
Date ranges: 2003-12-08 to 2015-12-31
Components: government bonds.
Description: Seeks to track the investment results of an index composed of inflation-protected U.S. Treasury bonds.⁵

²<https://www.ishares.com/us/literature/product-brief/ishares-core-us-aggregate-bond-etf-product-brief-en-us.pdf>

³<https://www.ishares.com/us/literature/product-brief/ishares-iboxx-high-yield-corporate-bond-etf-profile-en-us.pdf>

⁴<https://www.ishares.com/us/products/239565/ishares-iboxx-high-yield-corporate-bond-etf>

⁵<https://www.ishares.com/us/literature/fact-sheet/tip-ishares-tips-bond-etf-fund-fact-sheet-en-us.pdf>

- BCOM: Bloomberg Commodity Index
 Date ranges: 1991-01-03 to 2015-12-31
 Description: Bloomberg Commodity Index (BCOM) is calculated on an excess return basis and reflects commodity futures price movements. The index rebalances annually weighted 2/3 by trading volume and 1/3 by world production and weight-caps are applied at the commodity, sector and group level for diversification. Roll period typically occurs from 6th-10th business day based on the roll schedule.⁶
- G0O1: 3-Month U.S. Treasury Bill Index
 Date ranges: 1992-04-01 to 2015-12-31
 Description: The US 3-Month Treasury Bill Index is comprised of a single issue purchased at the beginning of the month and held for a full month. At the end of the month that issue is sold and rolled into a newly selected issue. The issue selected at each month-end rebalancing is the outstanding Treasury Bill that matures closest to, but not beyond, three months from the rebalancing date. To qualify for selection, an issue must have settled on or before the month-end rebalancing date. While the index will often hold the Treasury Bill issued at the most recent 3-month auction, it is also possible for a seasoned 6-month Bill to be selected.⁷
- MXEA: MSCI EAFE Index
 Date ranges: 1970-01-07 to 2015-12-31
 Description: The MSCI EAFE Index is an equity index which captures large and mid cap representation across Developed Markets countries* around the world, excluding the US and Canada. With 926 constituents, the index covers approximately 85% of the free float-adjusted market capitalization in each country.⁸
- MXEF: MSCI Emerging Markets Index
 Date ranges: 1988-01-01 to 2015-12-31
 Description: The MSCI Emerging Markets Index captures large and mid cap representation across 23 Emerging Markets (EM) countries*. With 838 constituents, the index covers approximately 85% of the free float-adjusted market capitalization in each country.⁹

⁶<http://www.bloomberg.com/quote/BCOM:IND>

⁷Merrill Lynch: <http://www.mlindex.ml.com/GISPublic/bin/getdoc.asp?fn=G0O1&source=indexrules>

⁸<https://www.msci.com/documents/10199/762896de-ebf3-49aa-89ec-e72c7592fd6b>

⁹<https://www.msci.com/documents/10199/10c3f32f-4565-4a92-aa1c-edf6f3a4e03f>

- RAY: Russell 3000 Index
 Date ranges: 1979-01-02 to 2015-12-31
 Description: The Russell 3000 Index is composed of 3000 large U.S. companies, as determined by market capitalization. This portfolio of Securities represents approximately 98% of the investable U.S. equity market. The Russell 3000 Index is comprised of stocks within the Russell 1000 and the Russell 2000 Indices. The index was developed with a base value of 140.00 as of December 31, 1986.¹⁰
- RMZ: MSCI US REIT Index
 Date ranges: 2005-06-20 to 2015-12-31
 Description: The MSCI US REIT Index is a free float-adjusted market capitalization index that is comprised of equity REITs. The index is based on MSCI USA Investable Market Index (IMI) its parent index which captures large, mid and small caps securities. With 151 constituents, it represents about 99% of the US REIT universe and securities are classified in the REIT sector according to the Global Industry Classification Standard (GICS). It however excludes Mortgage REIT and selected Specialized REITs.¹¹
- SPX: S&P 500 Index
 Date ranges: 1950-01-04 to 2015-12-31
 Description: Standard and Poor's 500 Index is a capitalization-weighted index of 500 stocks. The index is designed to measure performance of the broad domestic economy through changes in the aggregate market value of 500 stocks representing all major industries. The index was developed with a base level of 10 for the 1941-43 base period.¹²
- USGG10YR: US Generic Govt 10 Year
 Date ranges: 1962-01-03 to 2015-12-31
 Components: The index of US government bonds with a 10-year maturity (10-year bonds or in general 10-year treasuries). It measures the generic government 10-year yield for US issues of treasuries and provides the benchmark for various fixed-income instruments from corporate bonds to mortgages.
 Description: It is typically used to find out yield spreads for a host of fixed-income instruments with 10-year maturities.¹³

¹⁰<http://www.bloomberg.com/quote/RAY:IND>

¹¹<https://www.msci.com/documents/10199/7da6d18a-fdcb-47b6-b407-cec6cc4303bb>

¹²<http://www.bloomberg.com/quote/SPX:IND>

¹³<http://investment-and-finance.net/finance/u/usgg10yr.html>

3 Statistical Summary

In this section, we look into the basic summary and properties of different assets.

3.1 Statistics

Symbol explanation:

- i: represents different index.
- t: time period.

- Annualised return

Annualised return are calculated based on the daily returns.

$$R_a = (1 + R_d)^N - 1 \quad (2)$$

where R_a is the annualized returns, R_d is the daily returns, N is the number of trading days in one year ($N = 252$).

- Sharpe Ratio

$$\text{sharpe_ratio} = \frac{\bar{r}_i - R_f}{\sigma_i} \quad (3)$$

Here we let $R_f = 0$

- Standard deviation

$$\text{standard_deviation}_i = \sqrt{\frac{1}{n-1} \sum_{t=1}^n (r_i^t - \bar{r}_i)^2} \quad (4)$$

- Skewness

$$\text{skewness}_i = E_t \left[\left(\frac{r_i^t - \bar{r}_i}{\sigma_i} \right)^3 \right] \quad (5)$$

- Kurtosis

$$\text{kurtosis}_i = \frac{E_t \left[(r_i^t - \bar{r}_i)^4 \right]}{\left(E_t \left[(r_i^t - \bar{r}_i)^2 \right] \right)^2} \quad (6)$$

3.2 Summary

3.2.1 Returns

Figure 17 shows the returns over time for each asset (calculated based on their individual time range). We can see an obvious increase in volatilities during financial crises such as Black Monday in 1987 and the great recession in 2008. While GO01 (which is treated as the risk free rate in our analysis) has the smallest return, RMZ has the highest return as well as a highest volatility. Figure 18 show the empirical distribution of daily returns for each asset. The Kolmogorov tests show that the returns do not follow normal distribution ($p < 10^{-5}$). And the asset returns have a fatter tails than normal distribution. This result is unsurprising since fat tail is a quite common phenomenon in asset returns.

3.2.2 Statistical summary

Table 1 presents the statistical summary of each assets (calculated based on their individual time range). Note that here all asset returns have positive kurtosis (leptokurtic), and nonzero skewness (with both positive and negative values). This is consistent with the Kolmogorov test results in last subsection which show the asset return distribution generally have fat tails.

Table 1: Statistical Summary of Assets

Asset	Sharpe	Sd.	Skewness	Kurtosis
AGG	0.052	0.051	-2.51	81.31
HYG	0.025	0.134	0.87	36.71
TIP	0.040	0.065	0.10	6.48
BCOM	0.001	0.149	-0.27	4.33
GO01		0.002	0.69	26.76
MXEA	0.030	0.155	-0.32	10.74
MXEF	0.031	0.180	-0.39	7.71
RAY	0.036	0.173	-0.66	17.22
RMZ	0.016	0.366	0.36	13.68
SPX	0.035	0.153	-0.65	21.12
USGG10YR	0.003	0.201	0.12	8.81

4 Risk Diagnostics

In this section, we calculated the risk diagnostics based on the whole available data set of each asset. The value of risk diagnostics increases as the significant level increases.

4.1 VaR & ES

- VaR

Value at Risk (VaR) is a measure of the risk of investments. It estimates how much a set of investments might lose, given normal market conditions, in a set time period such as a day. VaR is typically used by firms and regulators in the financial industry to gauge the amount of assets needed to cover possible losses. The mathematical representation of VaR under α was shown below.¹⁴

$$VaR_\alpha(L) = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\} \quad (7)$$

- ES

Expected shortfall (ES) is a risk measure – a concept used in the field of financial risk measurement to evaluate the market risk or credit risk of a portfolio. The “expected shortfall at q% level” is the expected return on the portfolio in the worst q% of cases. ES is an alternative to Value at Risk that is more sensitive to the shape of the loss distribution in the tail of the distribution. The mathematical representation of ES was shown below.¹⁵

$$ES_\alpha(L) = E [L | L < VaR_\alpha(L)] \quad (8)$$

As shown in Table 2, here we calculate the VaR and ES based on different significance levels (0.9, 0.95, 0.99) for various assets. ES is generally bigger than VaR, and the value increases as the significant level increases. RMZ has largest VaR and ES thus the largest risk. AGG, HYG and TIP has smallest VaR and ES thus the smallest risk.

Note that all the risk diagnostics are calculated based on the empirical distribution of daily returns. Another commonly used method is based on a normal distribution assumption. However, in our case all asset returns have fat tails, it would be rather inappropriate to calculate use normal distribution. Figure 3 shows

¹⁴https://en.wikipedia.org/wiki/Value_at_risk

¹⁵https://en.wikipedia.org/wiki/Expected_shortfall

the VaR and ES calculated based on a normal assumption. It turns out VaR and ES is being overestimate at a lower confidence level and being underestimate at a higher confidence level. This discrepancy phenomenon is rather obvious when the return distribution has a fatter tail (a bigger kurtosis).

Table 2: Empirical VaR and ES under various probabilities

Asset	VaR(%)			ES(%)		
	0.90	0.95	0.99	0.90	0.95	0.99
AGG	0.29	0.40	0.69	0.50	0.66	1.23
HYG	0.62	1.03	2.50	1.41	2.03	4.01
TIP	0.44	0.62	1.01	0.72	0.91	1.47
BCOM	1.04	1.47	2.62	1.71	2.20	3.55
MXEA	1.02	1.46	2.59	1.74	2.26	3.76
MXEF	1.21	1.76	3.32	2.11	2.75	4.67
RAY	1.11	1.62	2.97	1.95	2.56	4.42
RMZ	1.91	3.00	7.56	3.99	5.62	9.99
SPX	0.99	1.43	2.58	1.71	2.23	3.80
USGG10YR	1.26	1.95	3.59	2.28	2.99	4.89

Table 3: Normal VaR and ES under various probabilities

Asset	VaR(%)			ES(%)		
	0.90	0.95	0.99	0.90	0.95	0.99
AGG	0.39	0.51	0.72	0.57	0.67	0.86
HYG	1.06	1.36	1.94	1.50	1.76	2.26
TIP	0.51	0.66	0.94	0.74	0.86	1.11
BCOM	1.20	1.54	2.18	1.65	1.94	2.50
MXEA	1.22	1.57	2.24	1.74	2.04	2.62
MXEF	1.42	1.83	2.61	2.03	2.38	3.06
RAY	1.36	1.75	2.49	1.95	2.29	2.94
RMZ	2.92	3.75	5.32	4.08	4.79	6.18
SPX	1.21	1.57	2.22	1.73	2.03	2.61
USGG10YR	1.62	2.08	2.95	2.23	2.62	3.39

5 Time Varying Risk Diagnostics

In this section, we calculate the time varying risk measures, which means to calculate risk measures based on rolling windows, such as 3 month rolling window. Then we get a series of risk measure. By examine how the series of risk measures vary over-time, we may discover some interesting facts about different risk measures.

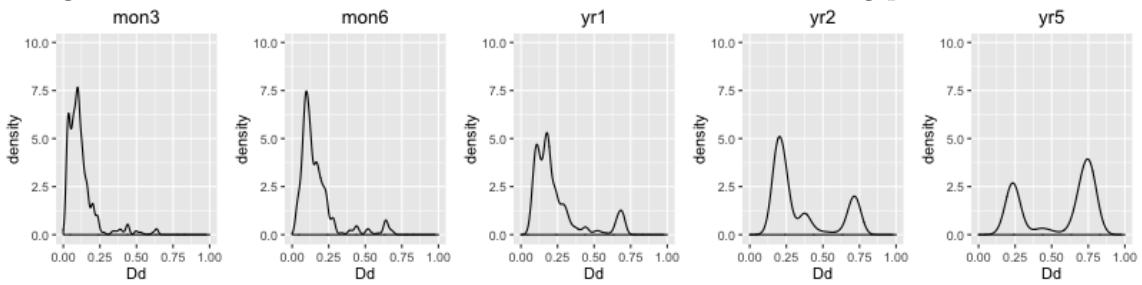
5.1 Time varying maximum drawdown

Maximum drawdown is the largest cumulative loss from peak to trough. In the return path of length n, the maximum drawdown is defined by

$$\mu(X_{T_n}) = \max_{1 < i < j \leq n} \max(X_{t_i} - X_{t_j}, 0) \quad (9)$$

Unlike VaR and ES, the empirical distribution of maximum drawdown is more sensitive to the time length of measurement. We calculate the maximum drawdown of various assets for different path length (3 months, 6 months, 1 year, 2 years, 5 years) separately. As shown in Figure 1, the maximum drawdown distribution tends to be multi-mode and centered around several specific values when we move to longer period. And the mean increases as the period length increases. Figure 19 shows the empirical distribution of maximum drawdown under 6 month rolling window for various assets. The maximum drawdown distribution tends to be negative skewed and the density curves seems to be unsmooth even under short rolling period as 6 month. Table 15 shows the tail mean of maximum drawdown distributions under different significant levels. Note that the tail mean of maximum drawdown increases as the rolling window increases.

Figure 1: Maximum drawdown distribution of RMZ as rolling period increases



5.2 Time varying volatility, VaR and ES

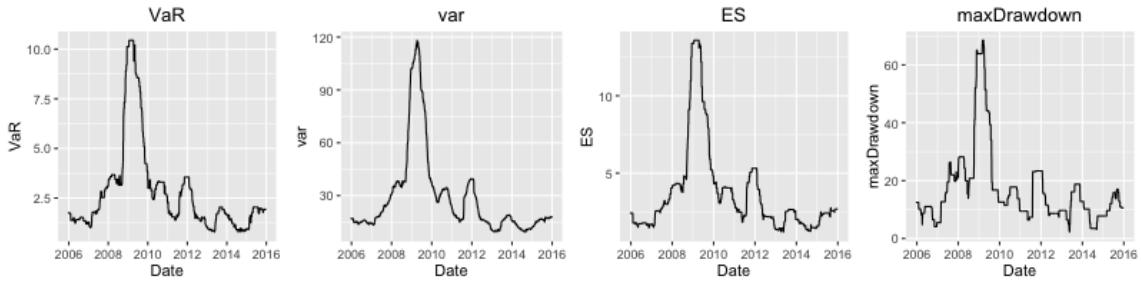
Figure 20, 21 and 22 show the time varying volatility, VaR and ES based on 6-month rolling window (significance level = 0.95) for different assets. For every single asset, three risk measures seems to have similar pattern with each other. We further look at this similarity by calculate the correlation between different risk measures.

Table 5 shows correlation between each pair of risk measures including VaR ($\alpha = 0.95$), ES ($\alpha = 0.95$), volatility and maximum drawdown. The correlation is calculated based on 6-month rolling risk measures and averaged over asset classes. Among them, VaR, ES and volatility are closely related with each other, with correlations over 95%. Maximum drawdown has strong but comparatively weaker correlation with the other three. In general, these risk measures are all closely correlated. Figure 2 shows the comparison of the four risk measures calculated based on rolling windows for asset RMZ. Their patters tend out to be very similar.

Table 4: Correlation between risk measures

Measures	Maximum drawdown	VaR	ES
VaR	0.85		
ES	0.81	0.96	
Volatility	0.86	0.96	0.97

Figure 2: Comparison of different risk measures of RMZ



5.3 CED

Maximum drawdown is the largest cumulative loss from peak to trough. Conditional Expected Drawdown (CED) is the tail mean of maximum drawdown distributions.¹⁶ Under confidence level α , the conditional expected drawdown is defined as:

$$CED_\alpha(X_{T_n}) = \mathbf{E}(\mu(X_{T_n}) | \mu(X_{T_n}) > DT_\alpha) \quad (10)$$

where $\mu(X_{T_n})$ is the maximum drawdown distribution over a finite path.

Here we calculated the CED using a 3-month-5-year rolling window (Figure 23). Note that here we use 3 month as the basic period. For empirical distribution of 3-month maximum drawdown, there are less multi-mode problems, and the tail value is more informative than using longer period. For longer path length such as 2-year and 5-year, there might be some missing values under large confidence levels. CED data tend to be lack of variance under long time period. In such cases, the empirical quantile no longer exists without some distribution or polynomial assumption of the tail. Since we do not familiar the performance of CED, it becomes inappropriate to make such assumptions.

CED is also closely related with other traditional risk measure. Table 5 shows the correlation between CED and other risk measures. Again, the correlation is calculated based on 6-month rolling risk measures and averaged over asset classes. Moreover, Figure 24 shows the plot of ES and CED, which indicates their positive correlated relationship.

Table 5: Correlation between CED (confidence level = 0.9) and other risk measures

Measures	Volatility	VaR	ES
CED	0.88	0.91	0.86

6 Regime Switching Model

Many economic time series data tend to behave differently in adjacent time period. For example, asset returns usually show large volatility during financial crisis. One common approach to model this abrupt change in regime is to use the Hamilton's Markov regime switching model. In our study, we use the following two-regime switching model to fit the returns:

¹⁶On a Convex Measure of Drawdown Risk. Lisa R. Goldberg, Ola Mahmoud

$$y_t - \mu_{s_t^*} = \phi_{s_t^*}(y_{t-1} - \mu_{s_{t-1}^*}) + \epsilon_t \quad (11)$$

where the number of autoregressive coefficient is set to 1. s_t^* is a two state Markov chain. $s_t^* = 1$ represent regime 1 and $s_t^* = 2$ represent regime 2. s_t^* depends on the past only through the most recent values:

$$P(s_t = j | s_{t-1}, s_{t-2}, \dots) = P(s_t = j | s_{t-1}) = p_{ij} \quad (12)$$

6.1 Comparison of basic summary of two regimes

Table 6 shows the autoregression coefficient ϕ of two regimes for various assets. One interesting fact is that asset returns are more likely (6 out of 10 assets) to show positive autocorrelation in low volatility regimes and negative autocorrelation in high volatility regimes.

Table 7 shows the standard deviation, skewness and kurtosis of various assets in two regimes separately. Note that regime 1 represents high volatility regime while regime 2 represents low volatility regime. In general, it is clear that returns of regime 1 has a larger standard deviation, skewness and kurtosis than that of regime 2. This indicates that the distribution in low volatility regimes are more “close” to normal distribution. Their mean returns are more close to zero and they have lighter tails than returns of high volatility regime. In contrast, in high volatility regimes, there are more extreme value of returns which means a fatter tail.

Table 6: ϕ of high and low volatility regimes for various assets

Asset	Regime 1		Regime 2
	High volatility	Low volatility	
AGG	-0.134	-0.114	
HYG	-0.010	0.025	
TIP	0.039	-0.026	
BCOM	-0.046	0.050	
MXEA	0.095	0.109	
MXEF	0.221	0.254	
RAY	-0.039	0.052	
RMZ	-0.244	0.007	
SPX	-0.018	0.113	
USGG10YR	-0.031	0.082	

Table 7: Summary statistics of two regimes for various assets

Asset	Volatility		Skewness		Kurtosis	
	Regime 1	Regime 2	Regime 1	Regime 2	Regime 1	Regime 2
AGG	0.141	0.036	-1.59	0.01	17.15	0.60
HYG	0.265	0.055	0.60	0.00	8.76	0.83
TIP	0.113	0.049	0.18	-0.06	2.50	0.16
BCOM	0.205	0.096	-0.25	-0.04	1.92	0.28
MXEA	0.253	0.101	-0.14	-0.02	4.26	0.25
MXEF	0.303	0.119	-0.08	-0.06	2.39	0.38
RAY	0.307	0.114	-0.39	-0.06	6.43	0.55
RMZ	0.661	0.159	0.29	-0.15	2.92	0.79
SPX	0.260	0.099	-0.43	-0.02	9.11	0.56
USGG10YR	0.314	0.098	0.09	-0.05	2.67	1.17

6.2 Detailed analysis (Take RMZ as an example)

6.2.1 Summary

In Figure 3, we use RMZ as an example of our volatility regime analysis. The upper panel show the daily returns and their corresponding regimes. The shadowed area represents the regime with high volatility and the white area low volatility. The X axis shows the indices of business days from 06/30/2005 to 12/31/2015. The lower panel shows the smoothed probability of Markov switching model corresponds with the high volatility regime. This model is consistent with the actual financial event in that the high volatility regime covered mainly from mid-2007 to 2010, which is roughly the period of 2008-09 financial crisis.

Table 8 shows the descriptive summary and risk diagnostics for RMZ of each regime. As we might expect, regime with high volatility also shows high risks in that they have larger ES and VaR values. Moreover, model of regime 1 only explains 26.3% of the observations, which means abrupt deviation is minority in total observations. By looking at assets with longer period, we find a similar proportion of high volatility regimes (SPX: 23.7%, RAY: 20.6%).

6.2.2 Risk diagnostics

In order to make a consistent comparison of risk diagnostics between two regimes, we ignore some short discontinuity and pick two longest single occurring episode for each regime. Both episode contain 530 trading days. The episode of regime 1 range

Table 8: Descriptive summary and risk diagnostics for RMZ of each regime

	Regime 1 High volatility	Regime 2 Low volatility
Mean	-0.046%	0.067%
Number of trading days	699	1954
Percentage of observations	26.3%	73.7%
Maximum daily return	18.8%	3.6%
Minimum daily return	-19.7%	-4.0%
Longest continuous days	381	546
VaR (empirical, p = 0.95)	6.8%	1.6%
ES (empirical, p = 0.95)	9.3%	2.2%

from 10/30/2007 to 12/07/2009, and the episode of regime 2 range from 06/20/2013 to 07/29/2015.

As shown in Table 9, The episode of regime 1 have significantly larger VaR, ES and CED value, which indicate a larger risk. This is consistent with the fact that returns of regime 1 have larger volatility. Figure 4 shows the maximum drawdown distribution of now-month rolling window. It is clear from the figure that regime 1 with higher volatility has larger maximum drawdown values than regime 2. For the case of RMZ, The regime with high risk also have a higher serial correlation for order one and order two.

Table 9: Risk diagnostics for RMZ of two equal-length episode of each regime

	Regime 1 High volatility	Regime 2 Low volatility
VaR (empirical, p = 0.95)	7.4%	1.5%
ES (empirical, p = 0.95)	9.9%	2.1%
CED (one-month, p = 0.9)	11.3%	3.1%
Serial correlation (order = 1)	-0.257	-0.026
Serial correlation (order = 2)	-0.023	0.008

Figure 3: Regime plot of RMZ returns and its smoothed probabilities

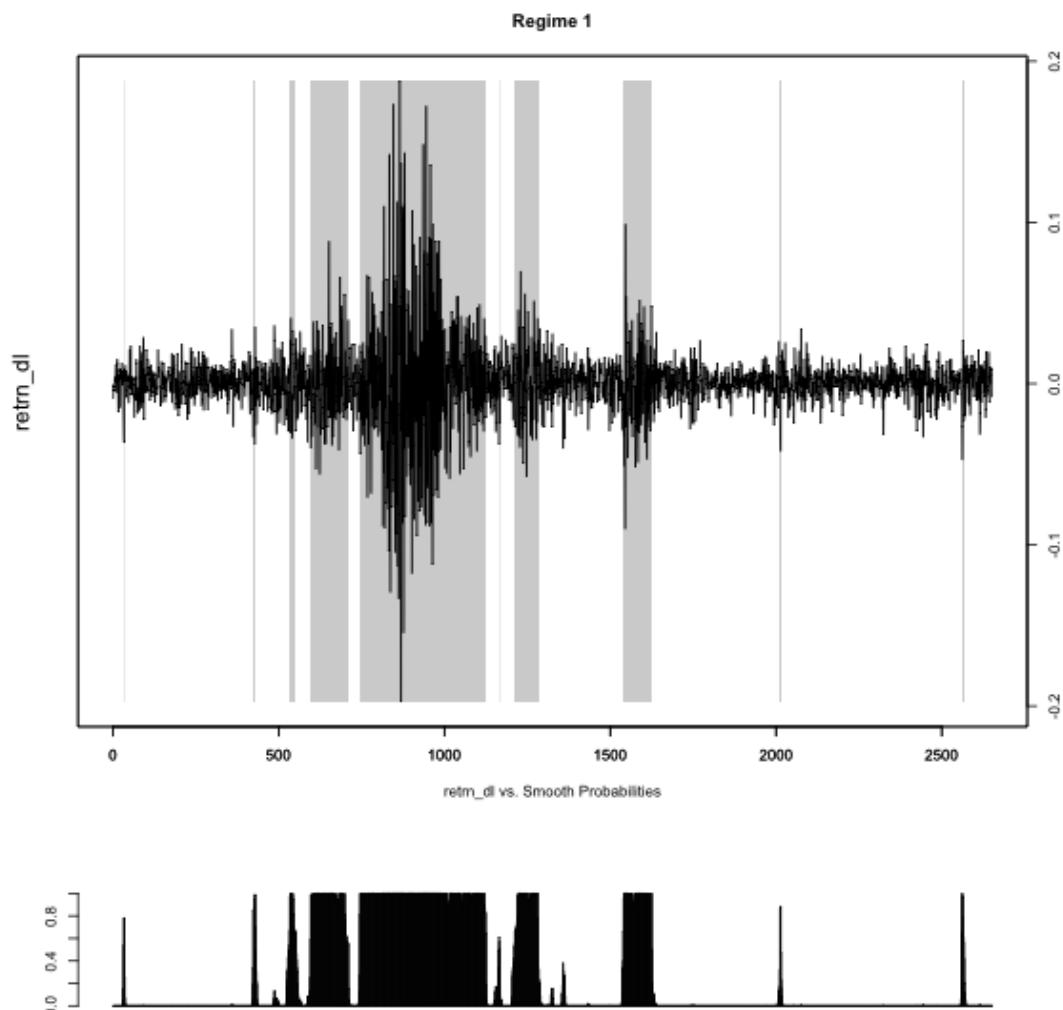
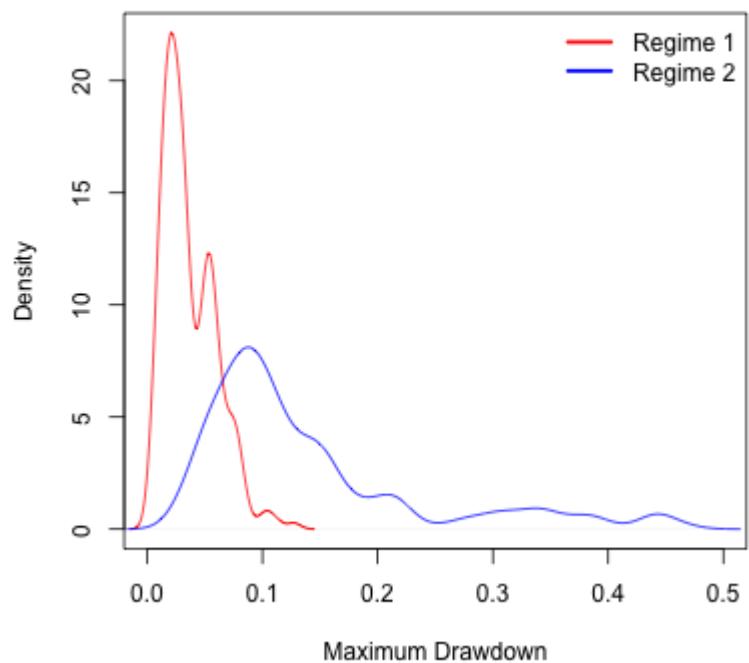


Figure 4: Maximum drawdown density of two regimes

Density of maximum drawdown distribution of two regim



7 Time Series – ARMA model

7.1 Exploratory Step

In this section, we have tried to explore the relationship between serial correlation with VaR, ES and CED using several simple time series model. They are AR(1), MA(1) and ARMA(1,1). The three models here are not necessarily the best one, or even the ones that all underlying assumptions are satisfied. This step is just try to have a general idea if the serial correlation of each model is related with risk measurements in some way. When it comes to serial correlation here, it refers to the first order autocorrelation of the time series models.

7.1.1 AR(1)

$$X_t = \phi X_{t-1} + \epsilon_t \quad (13)$$

AR(1) is the time series model utilized in the paper ON A CONVEX MEASURE OF DRAWDOWN RISK for exploring the relationship between serial correlation and CED of US Equity and US Government Bonds. Here we were trying to implement the same method but expand it to more assets (the eleven US equities). Moreover, we added scatter plots between κ and other risk measurements, which enable us to check if CED is more correlated with serial correlation of returns.

Figure 25, Figure 26 and Figure 27 show the relationship between the theoretical first-order serial correlation and VaR, ES, CED, separately. We can see that the correlation between first-order serial correlation and various risk measures are not so obvious. There are both positive (such as RMZ, BCOM, RAY, USGG10YR) and negative correlations (such as HYG, TIP). The correlation between first-order serial correlation ($\kappa(1)$) and various risk measures are listed in Table 10. We can see from the results that assets such as AGG (0.92) and RMZ (0.90) show high positive correlations, while HYG and TIP show significant negative correlation. Moreover, the correlations between VaR ,ES and $\kappa(1)$ are generally greater than the correlation between VaR and $\kappa(1)$.

The Table 10 is got from computing the correlation of $\kappa(1)$ and the risk measurement, VaR, ES and CED, which are calculated from the previous study. We also adjust them to be in the same time period. The time window used for get risk measurements and $\kappa(1)$ is 5 years, and the rolling window within 5 year for CED is 3 month. Consistent with what was showing in the plot, all the assets except MXEA and MXEF have the high correlation between $\kappa(1)$ and risk measurements. Within each asset, as expected, the correlation with VaR and ES are very closed, as VaR

is highly correlated with ES internally. However, compared with VaR and ES, the correlation with CED is relatively smaller, and sometimes the correlations are even with different sign. For example, for MXEF asset, the correlation with VaR and ES are positive, and though small, the correlation with CED is negative.

Table 10: Correlation between $\kappa(1)$ (calculated using AR(1)) and risk measures

Asset	VaR	ES	CED
AGG	0.92	0.95	0.94
HYG	-0.61	-0.56	-0.41
TIP	-0.67	-0.75	-0.73
BCOM	0.86	0.80	0.76
MXEA	0.26	0.33	0.08
MXEF	0.23	0.22	-0.05
RAY	0.79	0.76	0.49
RMZ	0.90	0.93	0.81
SPX	0.62	0.70	0.15
USGG10YR	0.67	0.64	0.66

7.1.2 MA(1)

MA(1) is another simple model that is widely used on the Financial time series. In fact, it is a very good one for fitting index **RMZ**, which will be further elaborated in next section. Since the plot of κ vs ES is similar to that with VaR, so only one plot is chosen to put on the report.

$$X_t = \epsilon_t + \theta\epsilon_{t-1} \quad (14)$$

For MA(1), there is a simple formula for calculating theoretical first-order serial correlation.

$$\kappa(h) = \begin{cases} \frac{\theta}{(1+\theta^2)} & \text{if } h = 1 \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

The plots are very similar with that of AR(1)

7.1.3 ARMA(1,1)

The ARMA(1, 1) has the following expression:

Table 11: Correlation between $\kappa(1)$ (calculated using MA(1)) and risk measures

Asset	VaR	ES	CED
AGG	0.94	0.96	0.94
HYG	-0.51	-0.45	-0.29
TIP	-0.65	-0.74	-0.73
BCOM	0.86	0.80	0.76
MXEA	0.01	0.11	0.15
MXEF	0.15	0.16	-0.12
RAY	0.79	0.77	0.50
RMZ	0.92	0.95	0.84
SPX	0.65	0.72	0.17
USGG10YR	0.65	0.64	0.68

$$X_t = \phi X_{t-1} + \epsilon_t + \theta \epsilon_{t-1} \quad (16)$$

The serial correlation calculated using the theoretical expression of fitted ARMA model has a more vague correlation with the various risk measures. The convergence problem occurred when fitting the model also indicate a potential model specification.

There is also a formula for calculating the first order correlation for ARMA(1,1) model.

$$\kappa(h) = \frac{(1 + \theta\phi)(\theta + \phi)}{1 + 2\theta\phi + \phi^2} \phi^{h-1}, \text{ if } h \geq 1 \quad (17)$$

Figure 30 shows the scatter plot of first-order serial correlation calculated using ARMA(1) model versus CED.

7.2 Model Selection

Things are becoming tricky when it comes to the model selection part. For the fact that it makes no sense to fit different model for each rolling time period within one asset, we tried to find a proper time series model for the all data for each asset. Unfortunately, observing from the ACF plot of returns, they are not stationary by nature without drift term, which means ARMA model is not a good selection for fitting them. It may also explain the reason why the previous section did not produce a expected result that CED is more correlated with serial correlation $\kappa(1)$.

Our criterion here for "better model" is the smaller AIC value. To confirm the fact that ARMA is not reasonable for almost all the indices, we tried sets of parameter

for ARMA and select the one with the smallest AIC value. Basically, we chose the combination of AR and MA parameters ranging from 1 to 5. Here is the results.

AIC short for Akaike information criterion, which is one of the most commonly used criterion for model selection. Here is the formula for AIC.

$$AIC = 2k - \ln(L)$$

k is the number of parameter in the model and L is the maximum value of the likelihood function for a certain model. Intuitively, AIC rewards goodness of fit (as assessed by the likelihood function), but it also includes a penalty that is an increasing function of the number of estimated parameters ¹⁷. Therefore, we want the model with smallest AIC value.

Table 12: Best ARMA Model Based on AIC Value

Asset	ARMA (p,q)
AGG	(5,5)
HYG	(5,5)
TIP	(5,4)
BCOM	(4,4)
MXEA	(5,5)
MXEF	(5,4)
RAY	(4,5)
RMZ	(3,3)
SPX	(5,4)
USGG10YR	(5,5)

From the Table 12 almost all the assets, the parameters are as high as 4 or 5, which indicates the time series are not stationary. It also involved some converging issue is when the parameters are getting larger.

For the next step of exploring reasonable model for those assets, several potential models is there can be taken in to consideration:

- **ARIMA model with $d = 1$.** We found the time series became much more stationary after first-order differencing.
- **ARCH/GARCH model.** Those are the models widely used for financial assets, in which variance of time series are more likely clustering.

¹⁷https://en.wikipedia.org/wiki/Akaike_information_criterion

7.2.1 EXAMPLE: RMZ data

Let us take RMZ for example. There are several reasons why we chose to use RMZ: 1) the RMZ has a shorter period, which means a smaller data set to test different time series models. 2) Larger serial correlation and ES, VaR comparing with other indices, implying it is more suitable for analysing the relationship between serial correlation and various risk measures. 3) There are less modes in Maximum drawdown of RMZ, indicating it is reasonable to get tail mean, which is CED.

We also compared RMZ with other index in models selection. From the ACF/PACF plots, RMZ seems the one most suitable for fit ARMA model. Others are not stationary even after a long lag.

First of all, we checked if the time series was stationary in long term. Here is the ACF/PACF plot of the ten-year data. It turns out to be good for using ARMA(0,0,1), MA(1) or ARMA(1,0,1) model, except several lags are a little bit higher than blue line ($|h| = 0.05$). We also took the first-order difference of the data. The time series become more stationary then before and ARIMA(0,1,1) seems to be a good model for it.

We also fitted in short time period (1 year period), and the time series became more stationary in each subperiods, and thus the three models mentioned seemed to be reasonable. Below we summarized the model we used for RMZ. We found that the MA(1) model has the smallest AIC within the three parameter sets we chose below.

- AR(1) — to be consistent with your paper
- MA(1) — produce least AIC among list 4 models when fitting the all data of RMZ
- ARMA(1,1)
- ARIMA(0,1,1) — Last two is chosen from the pattern of ACF, PACF plot.

Here are some diagnostics for MA(1):

From the Figure: 7 above, it is not surprising to see that MA(1) cannot capture the variance cluster in the return, resulting in some variance cluster left the residual plot. There still exists some correlation between residuals after more than 50 lags, but they are comparatively small after the fitting. Normal Q-Q shows that residuals are heavy-tailed. LjungBoxPierce Q-statistic, used for test grouped $\kappa_e(h)$, rejects H_0 after 5th lag, which also indicates the residual is not white noise after fitting.

In summary, although MA(1) seems to be the good model we could find in ARMA family, it is not enough to capture all the characters in the RMZ return.

Figure 5: ACF/PACF of Return for RMZ

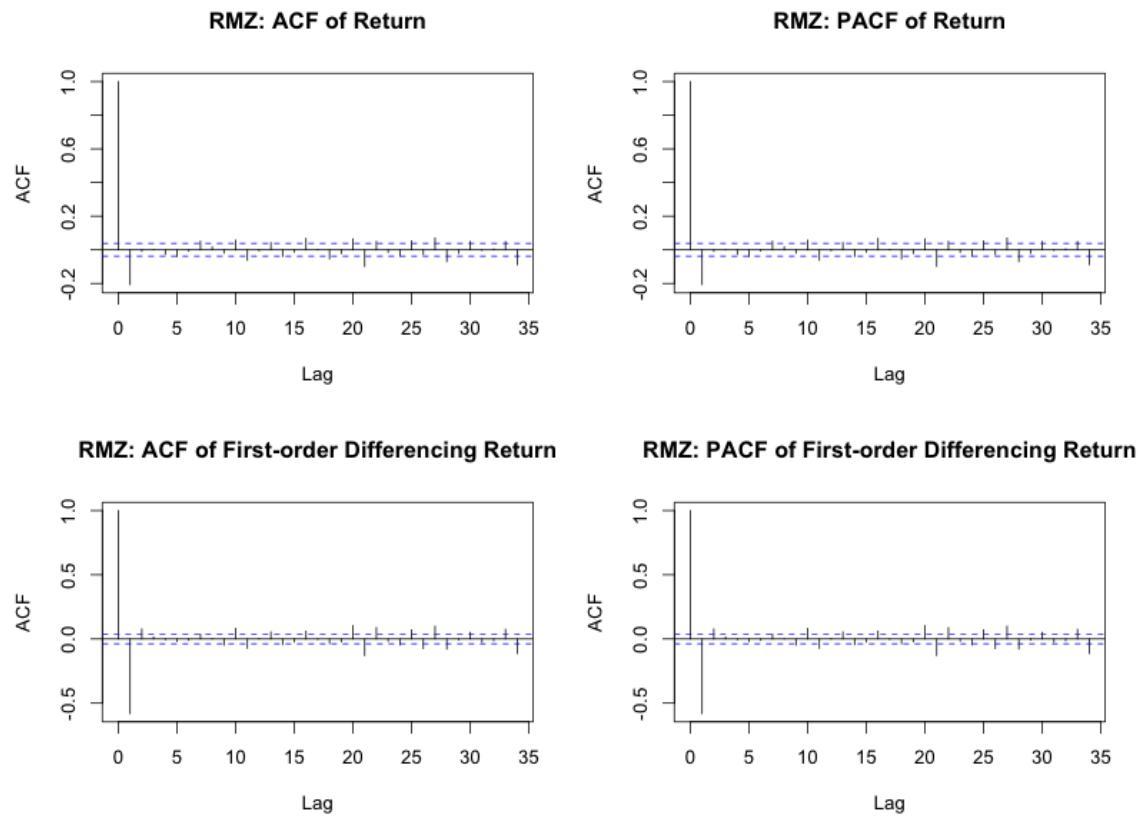
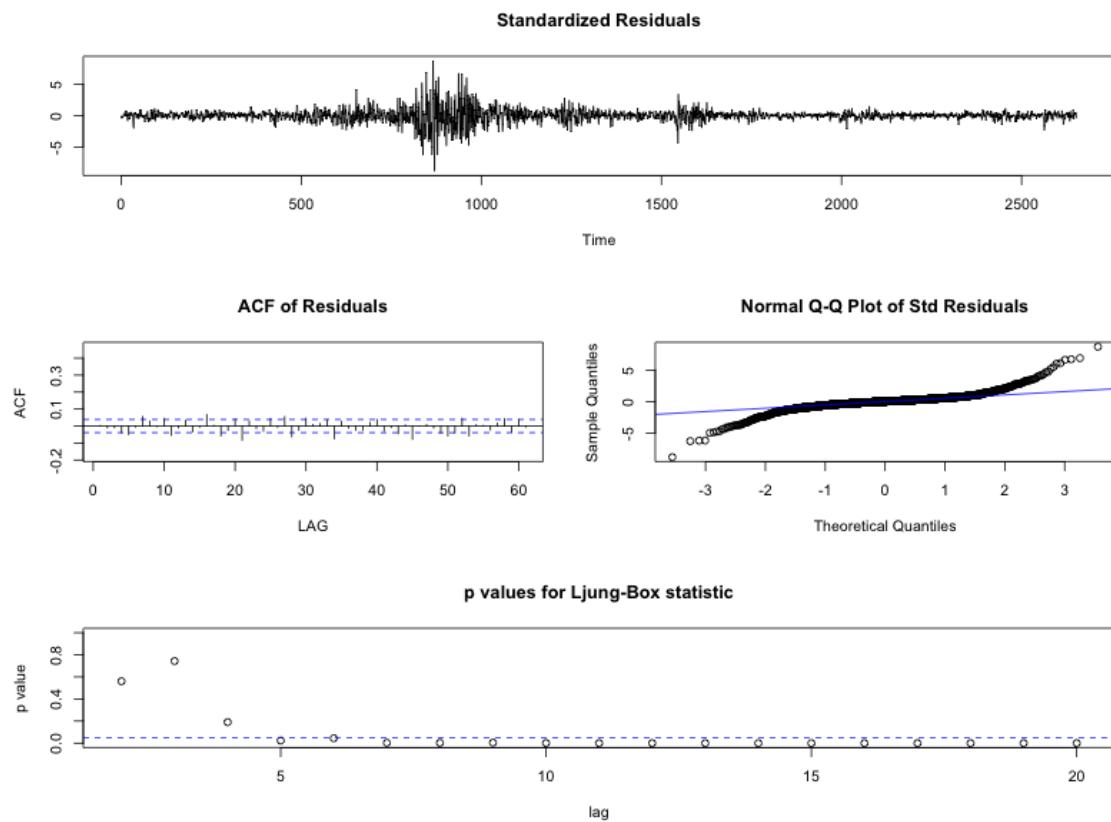


Figure 6: Diagnostic Plots of MA(1) for RMZ



8 Time Series – GARCH model

8.1 Introduction

As described in the previous report, for financial assets, the returns have strong dependent across the time, thus it is reasonable to fit time series to interpret and predict them. In the study before, the ARIMA model is used for fitting, while the diagnostics are infeasible for most assets. They cannot capture the variance cluster in the return, resulting in some variance cluster left the residual plot. In the example of RMZ before, we are still able to fit at least one “not too bad” ARIMA model. However, for other assets, it almost impossible to fit arima: the autocorrelations are still there even after a long lag, and after we adding order more than $p = 5, q = 5$, the convergence issue always happens. However, we still attach the best ARIMA for each assets we can find to the end.

In this report, we search to apply another widely used model, GARCH, to fit the returns, and also catch the variance clusters in the them. It turns out to have a much better performance in capturing the variance, and the diagnostic are also showing a delightful patterns

8.2 Fit Garch Model

In this section, three representative assets, RMZ and SPX, are chosen to explain the process of fitting GARCH. RMZ is selected to continue the study in the previous report; for the general descriptive tables and plot in the first report, SPX can represent feature for most of assets. SPX is also the most common asset and used in Ola’s report before, so we think it would be nice to study it individually.

GARCH(1,1) is the most widely used the model in financial assets, so we fit the residual in MA(1) to GARCH(1,1) here. Rather than ARIMA having constant variance assumption, GARCH(1,1) also model the heteroscedasiticy in the time series. For GARCH(1,1), the model is discribed as following:

$$\begin{aligned}y_t &= \sigma_t \epsilon_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2\end{aligned}$$

8.2.1 RMZ

In the previous study, we found the best ARIMA model for RMZ is MA(1), the diagnostic plot is attached below. Again, the variance clusters in the residual are not captured in the model.

Figure 7: Diagnostic Plots of MA(1) for RMZ

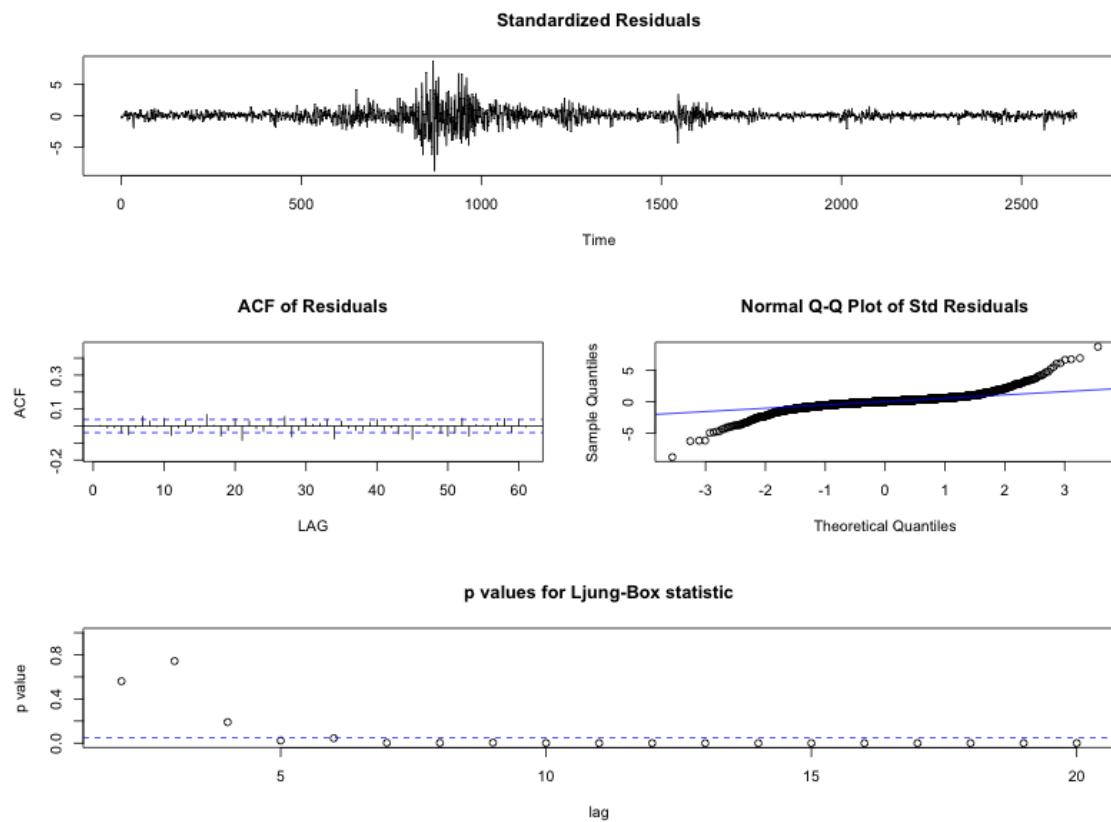
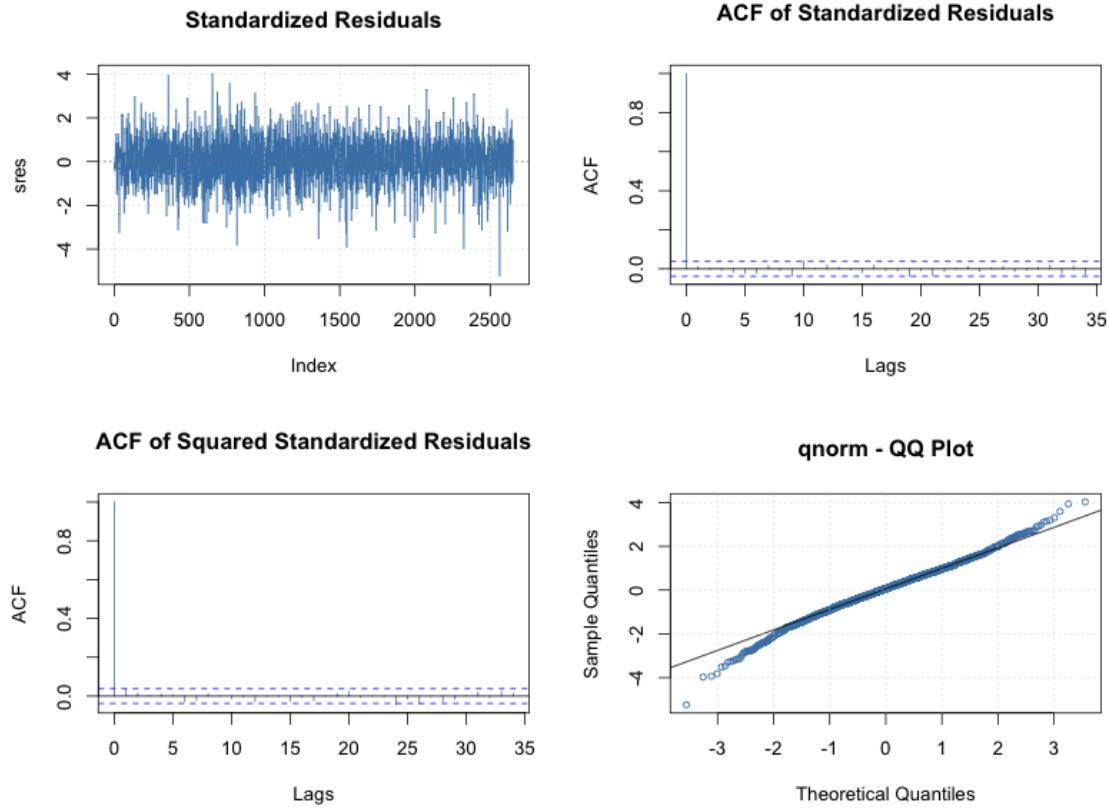


Figure 8: RMZ: Diagnostic Plots of GARCH(1,1) with Gaussian Conditonal Distribution



Fit the residual of MA(1) to GARCH(1,1), we got the follow model:

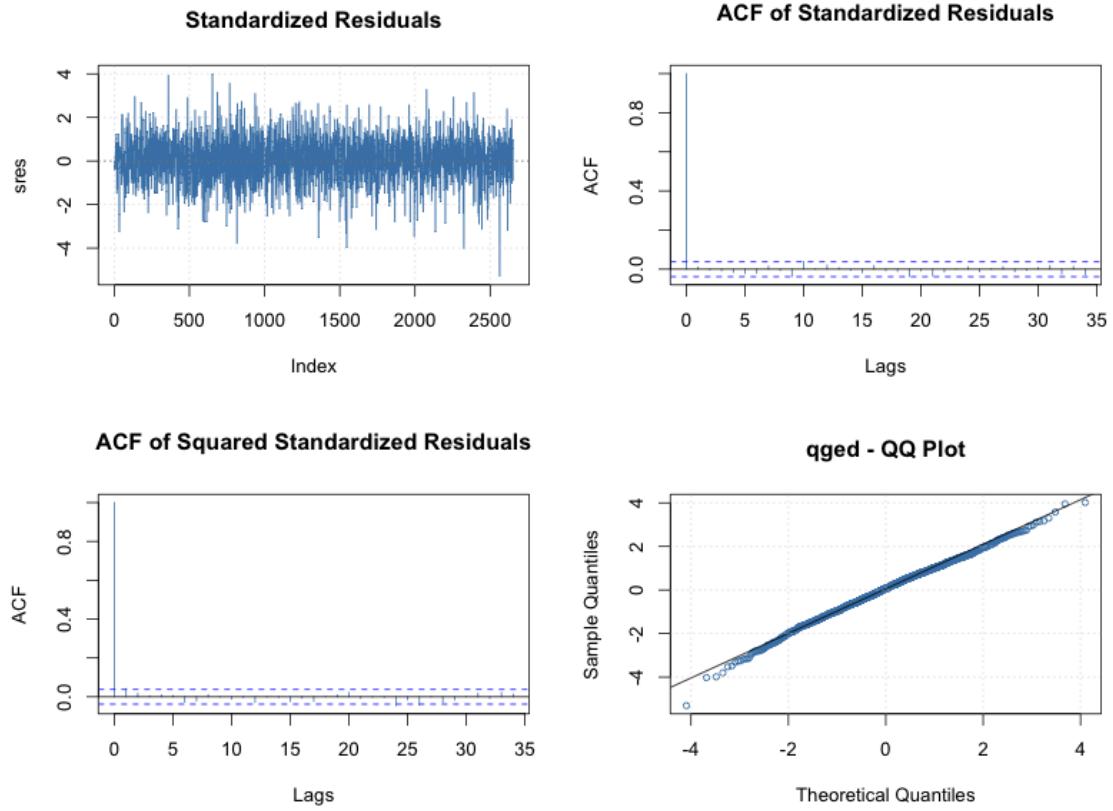
$$R_t = -4.1561 \times 10^{-2} \epsilon_{t-1}$$

$$\sigma_t^2 = 1.9955 \times 10^{-6} + 1.1083 \times 10^{-1} T_{t-1}^2 + 8.8506 \times 10^{-1} \sigma_{t-1}^2$$

The Figure: 8 are the plots of the standardized residuals.

After fitting GARCH, the residual left is almost white noise, which is a good sign for eliminating the correlations between points, as evidenced in the next two ACF plots. However, the heavy tail problem is still there. Several approach can be adopted to fix it, one of which is instead of normal, using Student t distribution as conditional distribution in the GARCH. Finding t distribution is not really solve the problem, we finally chose to use Generalize Normal Distribution (GED) as the

Figure 9: RMZ: Diagnostic Plots of GARCH(1,1) with GED Conditonal Distribution



conditional distribution. QMLE is another potential alternative. The Figure: 9 indicates that the heavy tail problem is almost solved in the by applying GED.

Figure 10 the plot compare the estimation of standard deviation using GARCH(1,1), with using naive estimate of standard deviation. The black continuous line is the estimate of the instantaneous conditional standard deviation of the GARCH(1,1). The large peak indicative of the huge standard deviation corresponding to the crisis of 2008. Red line is the empirical standard deviation of the entries of the series in the window containing the entries of the last 30 days. As shown in the plot, the naive estimator is smoother than GARCH, while they almost follow the sample trend.

Figure 11 show the last 120 residuals from MA(1), we used all but last 20 to fit the GARCH(1,1) model with GED conditional distribution and create an confident interval based on that. And it is nice to see that all the last 20 points lies in the

Figure 10: RMZ: Estimate of the Instantaneous Conditional Standard Deviation

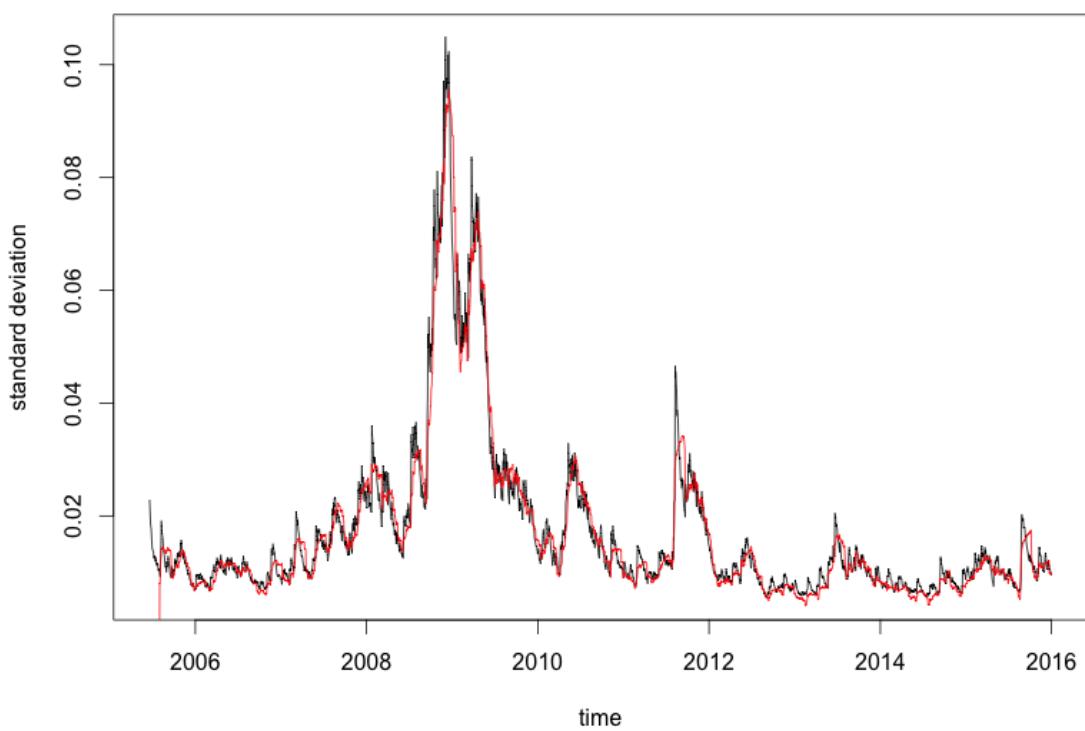
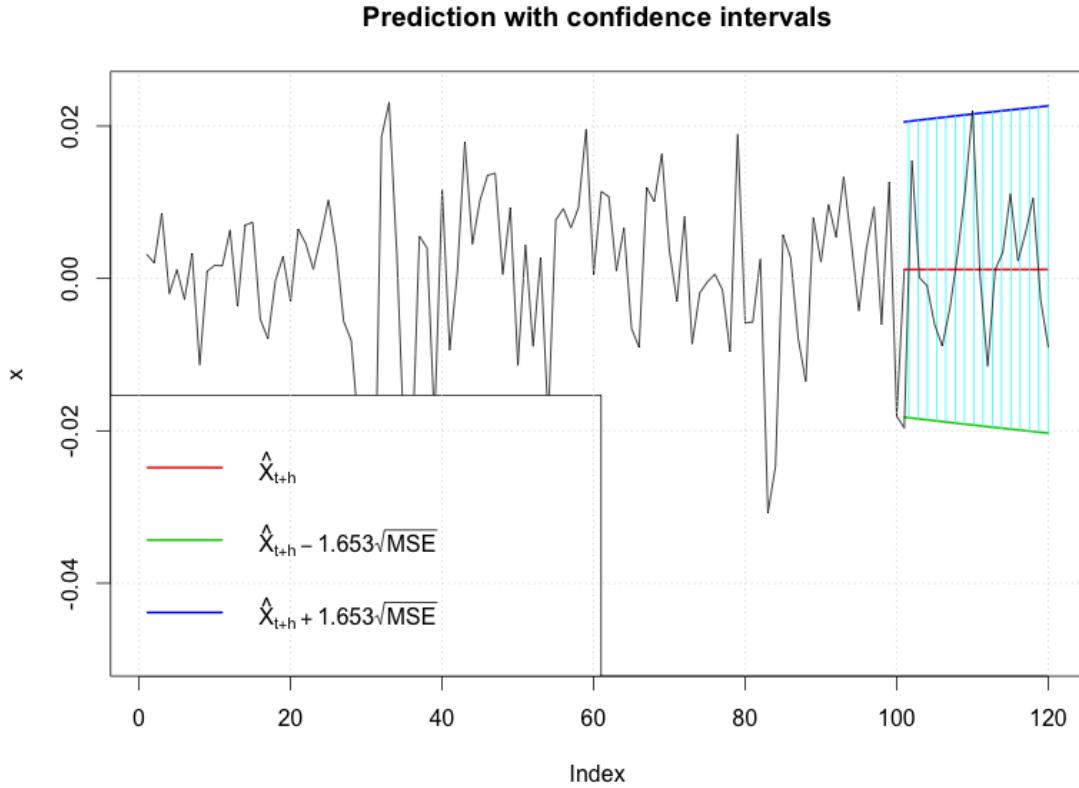


Figure 11: RMZ: Estimate of the Instantaneous Conditional Standard Deviation



interval.

In summary, instead of assuming constant variance using ARIMA, modeling the residuals from MA(1) using GARCH(1,1) seems to be more reasonable for RMZ.

8.2.2 SPX

After plot the monthly plot and text if exists any trends. It turns out that no seasonal component or trend exists in the time series. Fit SPX time series with a set of ARIMA model. According to AIC, BIC and AICc criterions, ARMA(2,2) is the most reasonable one for fitting SPX. Similar to RMZ, the Figure ?? shows a non-constant variance in the residual.

GARCH(1,1) are selected from a set of GARCH models with lowest BIC value to

Figure 12: SPX: Residual Plot after ARIMA

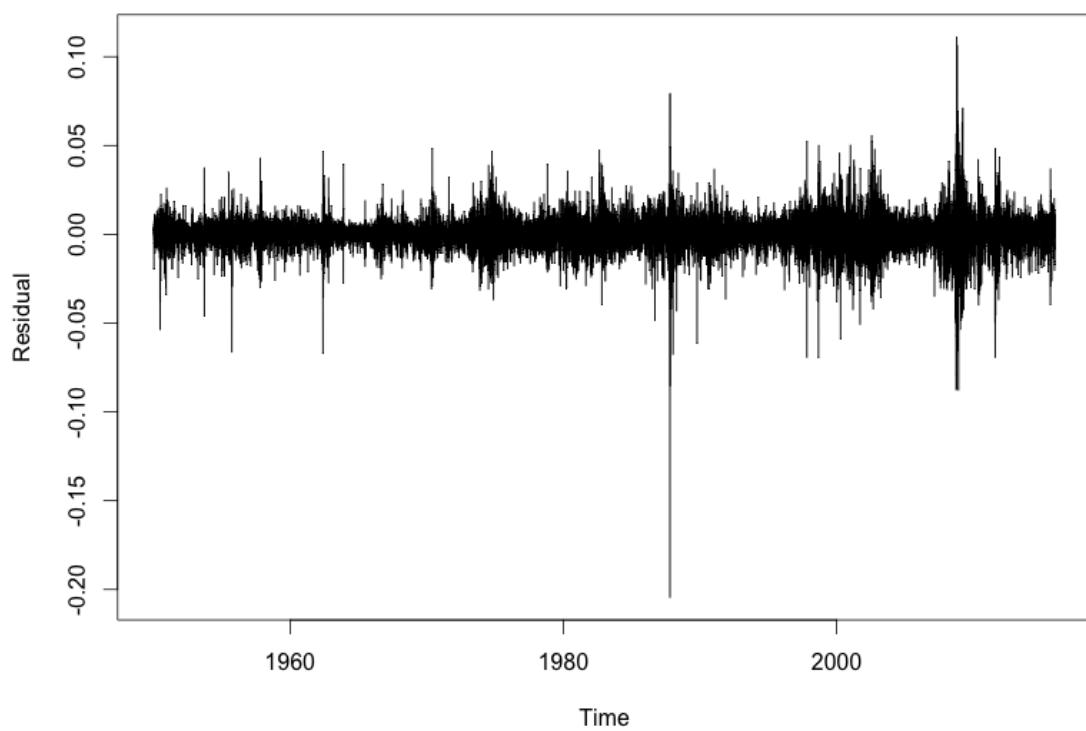
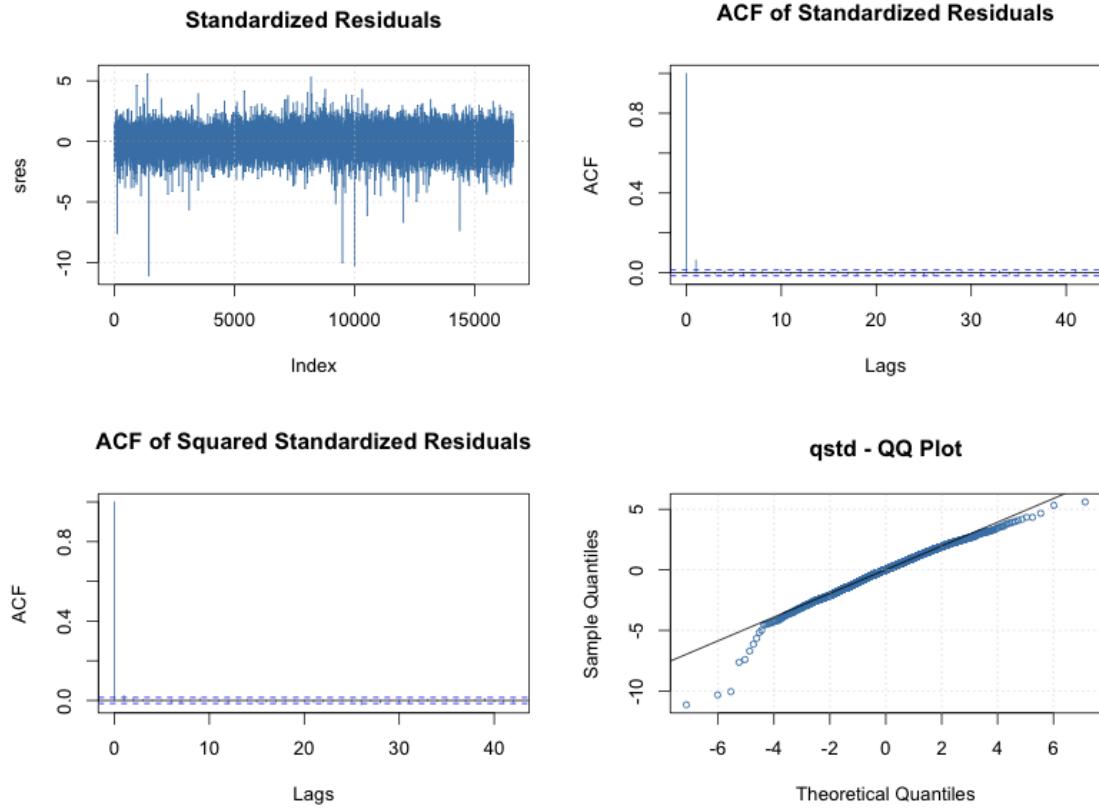


Figure 13: SPX: Diagnostic Plots of GARCH(1,1) with Student t Conditonal Distribution



model the residuals from ARMA(2,2). As GARCH tend to be overfitting, BIC thus is more precise compared with AIC or AICc. Instead of using Gaussian or Generalized Gaussian, we used Student t distribution to fix the heavy tail problem. However, Figure 13 still show issues in Q-Q plot, which caused by some extreme points in 1988 and 2008. We will just ignore them at current level.

It seems that we are not able to completely solve the problem with correlation of standardized residuals. The second-order differencing is a potential approach to fix this, while it may break the economical meaning of returns.

Similarly, Figure14 and Figure 15 shows that ARMA(2,2)+GARCH(1,1) is a really good model for characterize the SPX time series.

The Figure 16 gives the GARCH prediction of volotility of ARMA(2,2) residuals,

Figure 14: SPX: Estimate of the Instantaneous Conditional Standard Deviation

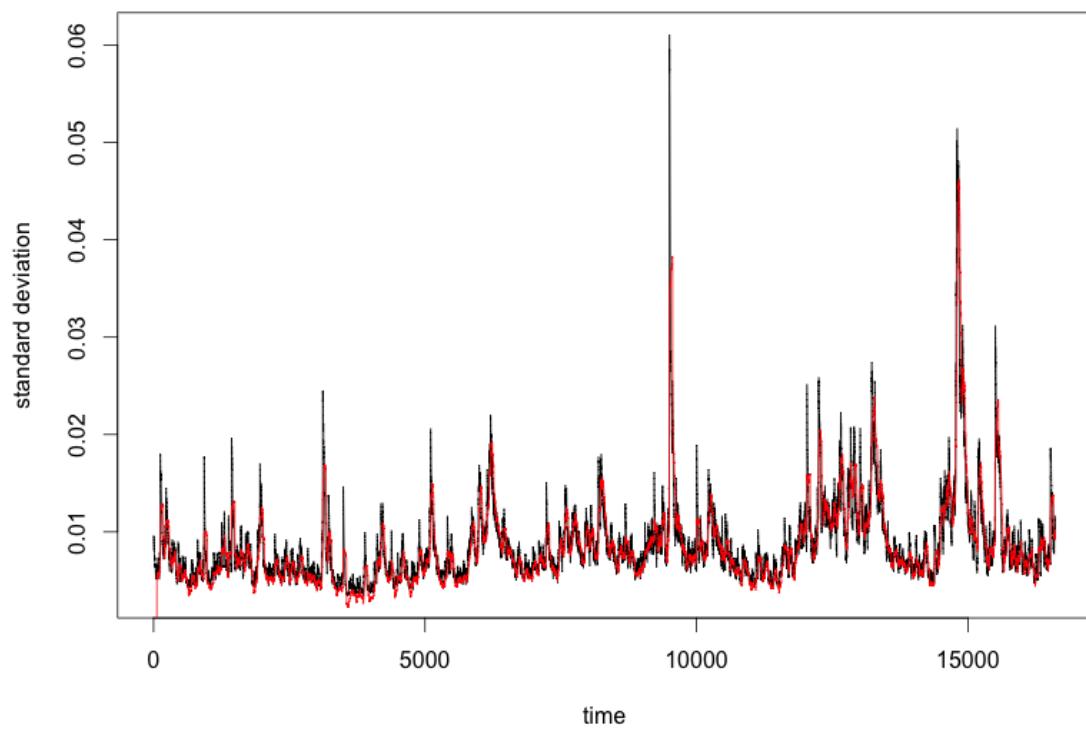


Figure 15: SPX: Estimate of the Instantaneous Conditional Standard Deviation

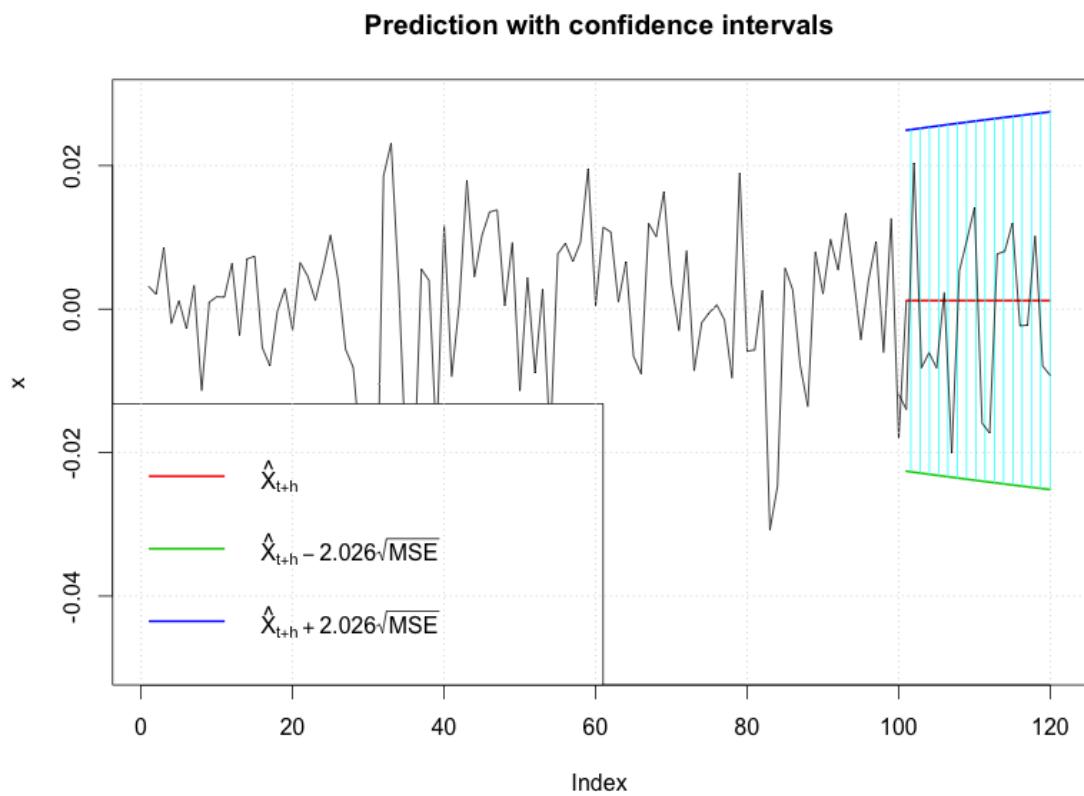
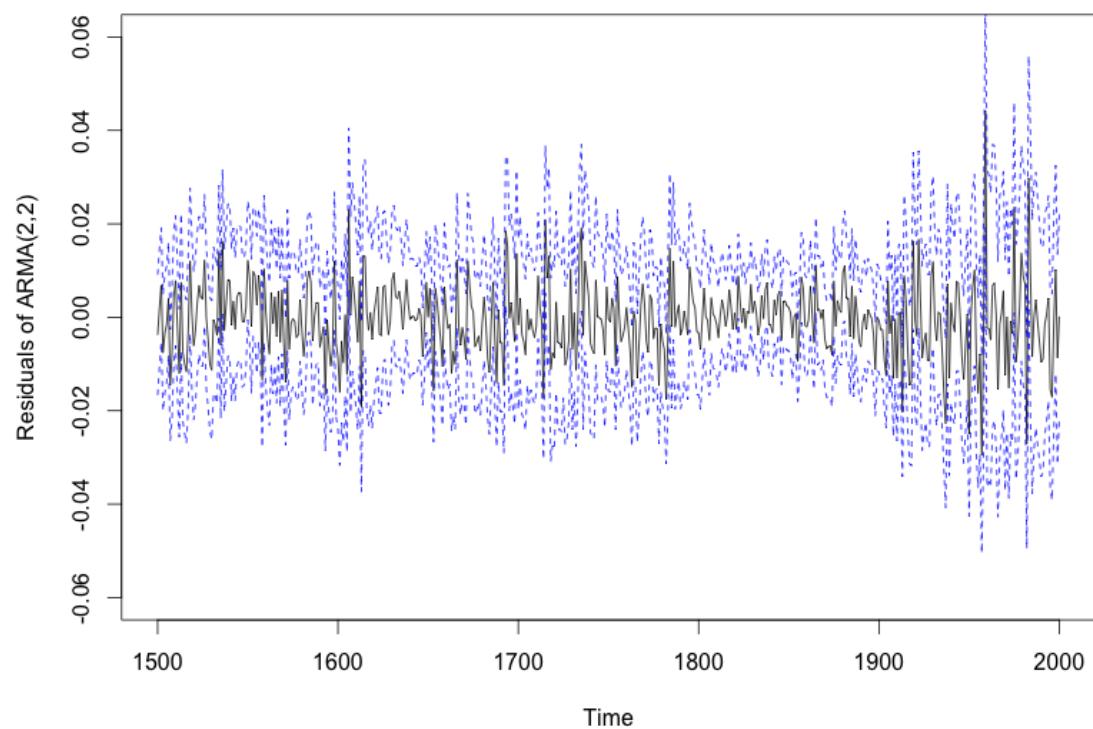


Figure 16: SPX: GARCH predictions of the volatility(standard deviation)



and its $\pm 2\hat{\sigma}_t$ interval.

8.2.3 Best ARIMA Model for all the U.S. Assets

According to AIC, BIC and AICc, we select the most reasonable models for each asset. From Table 13, again, we verify the argument that it does not make any sense

Table 13: Best ARIMA Model for the U.S. Assets

Asset	ARIMA (p,d,q)
AGG	(5,0,5)
HYG	(3,0,1)
TIP	(0,0,0)
BCOM	(0,0,0)
MXEA	(2,0,4)
MXEF	(4,0,2)
RAY	(2,0,2)
RMZ	(0,0,1)
SPX	(2,0,2)
USGG10YR	(0,0,0)

to use ARIMA model fit some assets, such as TIP, BCOM, AGG and USGG10YR.

8.3 Best GARCH Model for all the U.S. Assets

By using the BIC as the criterion, the following table shows the best GARCH model we selected for fitting residuals of each asset. Table 14 shows all the residuals are not homoscedastic, and need to be fitted using GARCH. Within all the assets, only two of them need GARCH with order higher than (1,1). They are MXEA and USGG10YR.

Table 14: Best GARCH Model for the residuals after Fitting ARIMA

Asset	ARIMA (p,d,q)+ GARCH(m,n)
AGG	ARMA(5,5)+GARCH(1,1)
HYG	ARMA(3,1)+GARCH(1,1)
TIP	GARCH(1,1)
BCOM	GARCH(1,1)
MXEA	ARMA(2,4)+ GARCH(1,2)
MXEF	ARMA(4,2) + GARCH(1,1)
RAY	ARMA(2,2) + GARCH(1,1)
RMZ	MA(1) + GARCH(1,1)
SPX	ARMA(2,2) +GARCH(1,1)
USGG10YR	GARCH(1,3)

9 Appendix: Plot and Tables

Table 15: tail mean of maximum drawdown distribution under 3-month, 6-month, 1-year and 2-year rolling window

Asset	3 month			6 month		
	0.9	0.95	0.99	0.9	0.95	0.99
AGG	5.60	7.72	12.84	8.12	11.45	12.84
HYG	18.41	24.07	29.67	26.43	30.77	32.26
TIP	7.48	9.90	13.10	11.14	12.91	14.39
BCOM	18.14	22.54	38.03	26.61	33.66	51.74
G0O1	0.10	0.145	0.26	0.14	0.23	0.26
MXEA	20.39	23.73	33.32	27.21	31.79	47.11
MXEF	26.21	30.80	48.03	36.35	43.30	59.63
RAY	20.65	25.64	35.95	27.81	34.08	45.08
RMZ	37.30	48.41	63.45	52.04	62.41	67.61
SPX	18.35	22.67	32.46	25.18	30.65	40.69
USGG10YR	23.28	28.11	41.83	32.78	39.00	49.28
Asset	1 year			2 year		
	0.9	0.95	0.99	0.9	0.95	0.99
AGG	12.11	12.84	12.84	12.84	12.84	12.83
HYG	33.15	34.20		34.24	34.25	
TIP	13.73	14.50	14.57	14.57	14.57	
BCOM	39.91	49.26	57.14	53.38	57.14	
G0O1	0.23	0.25	0.26	0.26	0.26	0.26
MXEA	36.15	42.61	56.70	50.21	58.08	61.85
MXEF	48.63	58.81	64.56	62.45	65.90	
RAY	37.56	44.06	52.42	47.95	54.33	
RMZ	67.09	69.86	70.02	73.70	74.56	74.92
SPX	34.07	39.15	50.30	44.62	50.47	56.77
USGG10YR	42.88	48.37	53.96	54.79	59.14	62.82

Figure 17: Daily Returns

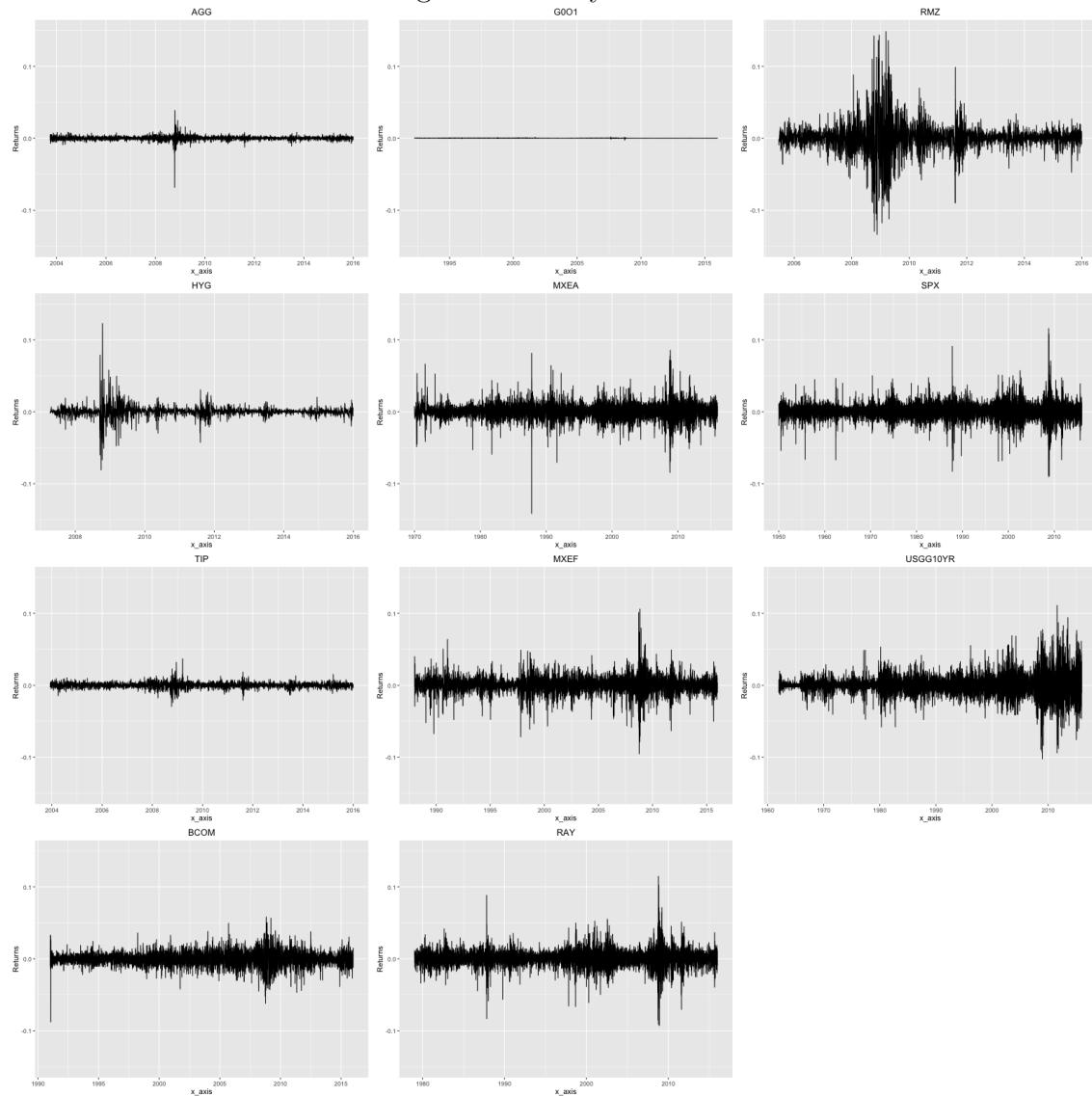


Figure 18: Empirical distribution of daily returns

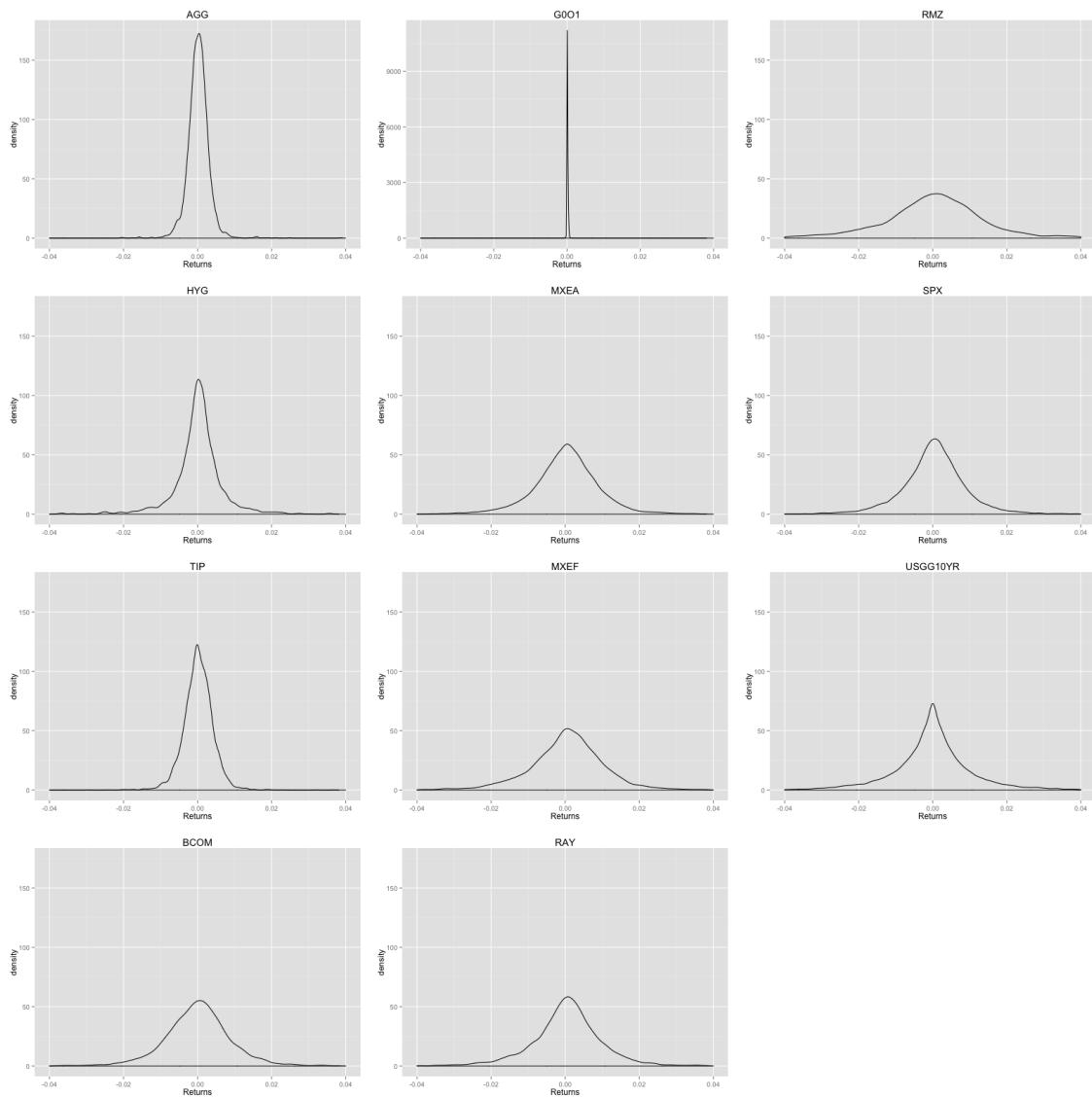


Figure 19: Empirical distribution of maximum drawdown under 6 month rolling window

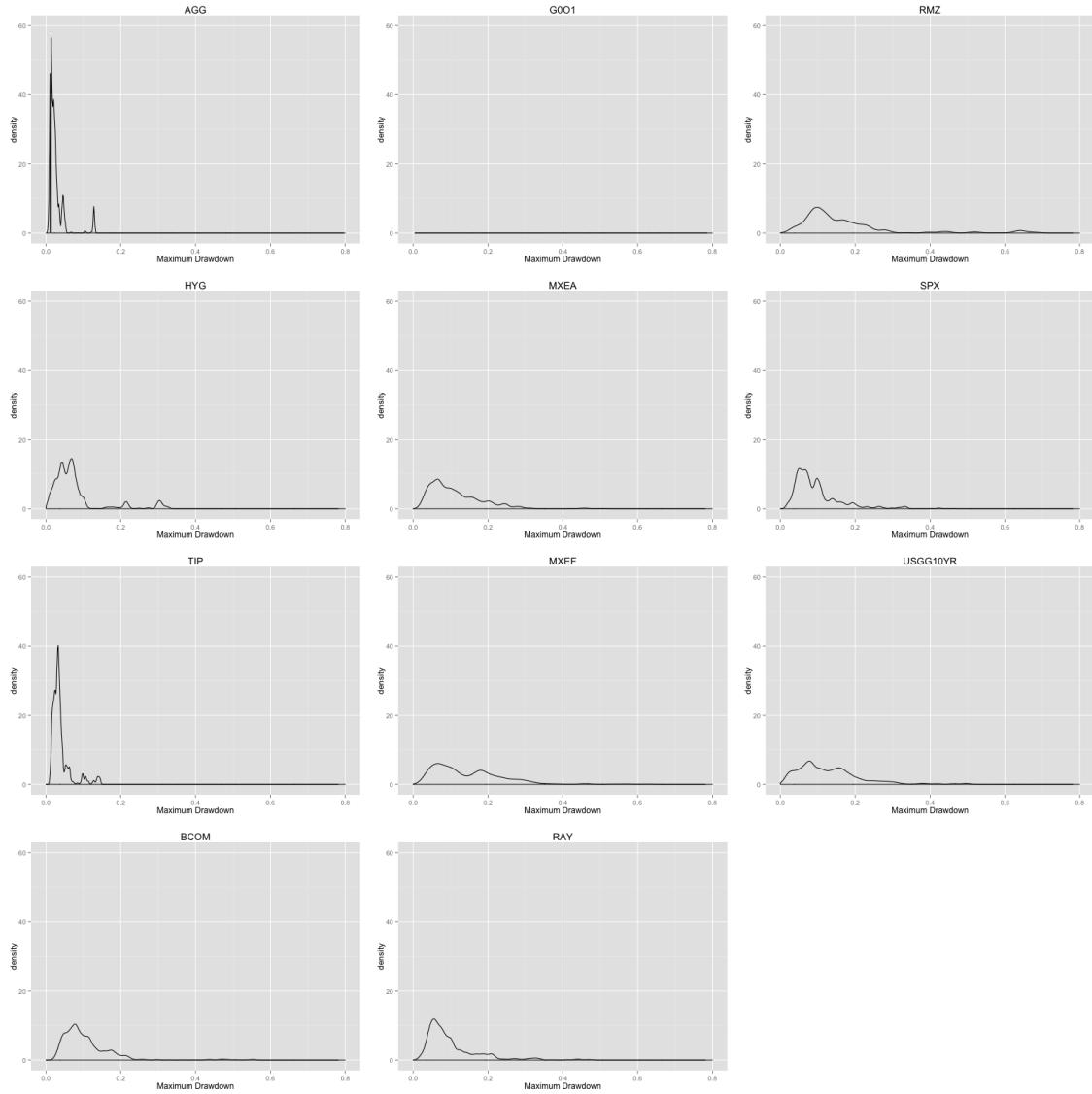


Figure 20: Volatility under 6-month rolling window

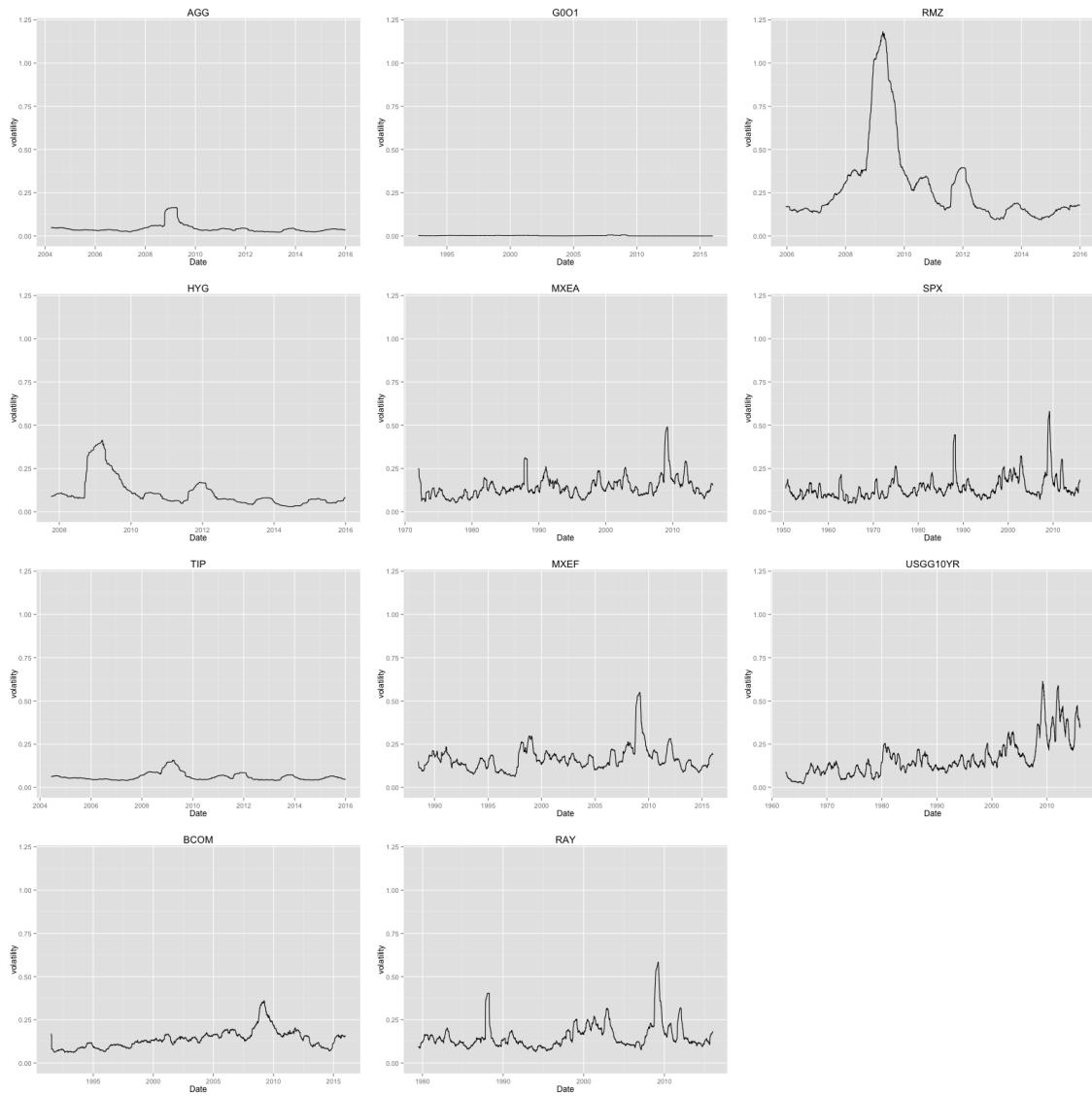


Figure 21: VaR(%) (significance level = 0.95) under 6-month rolling window

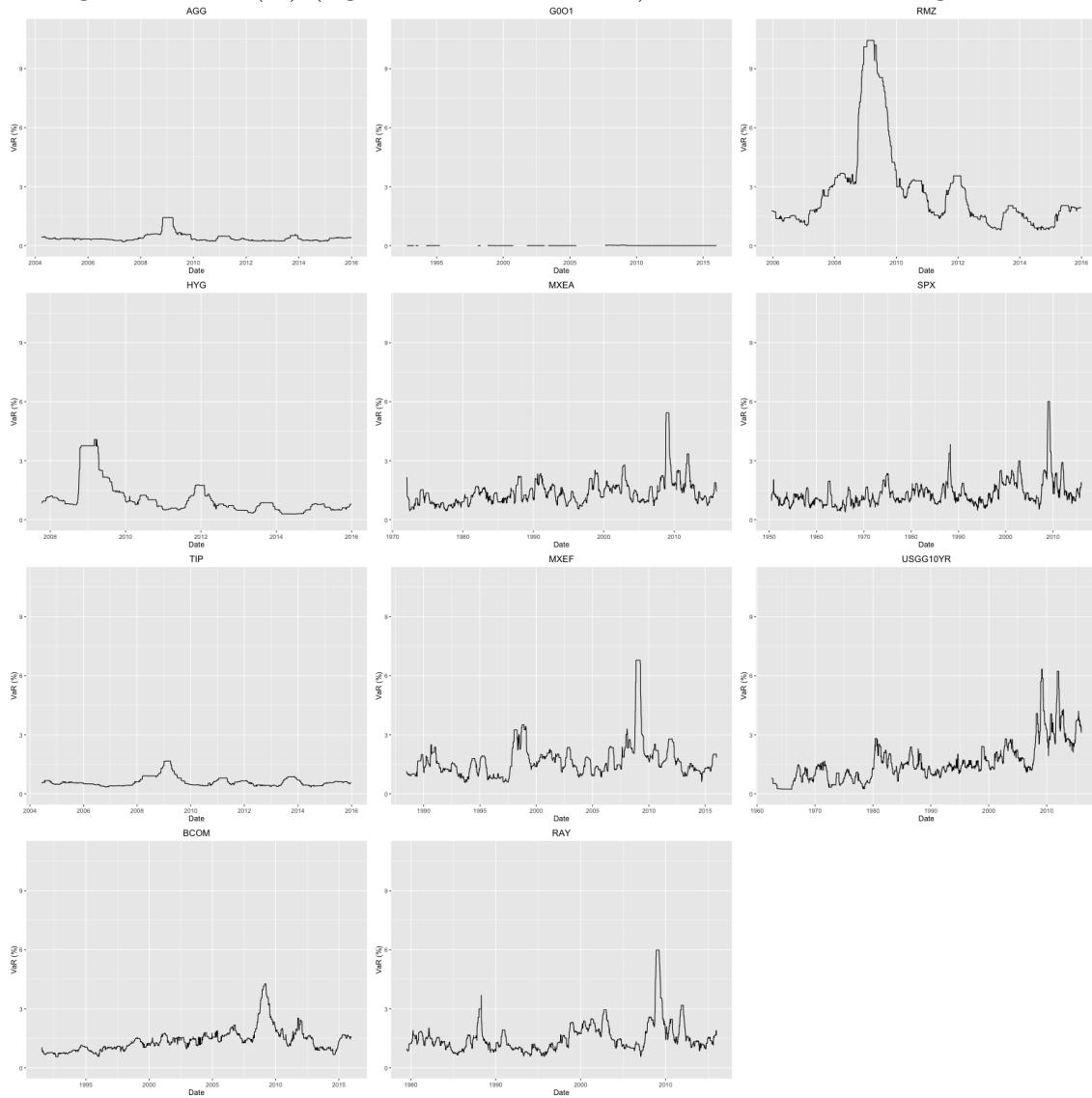


Figure 22: ES(%) (significance level = 0.95) under 6-month rolling window

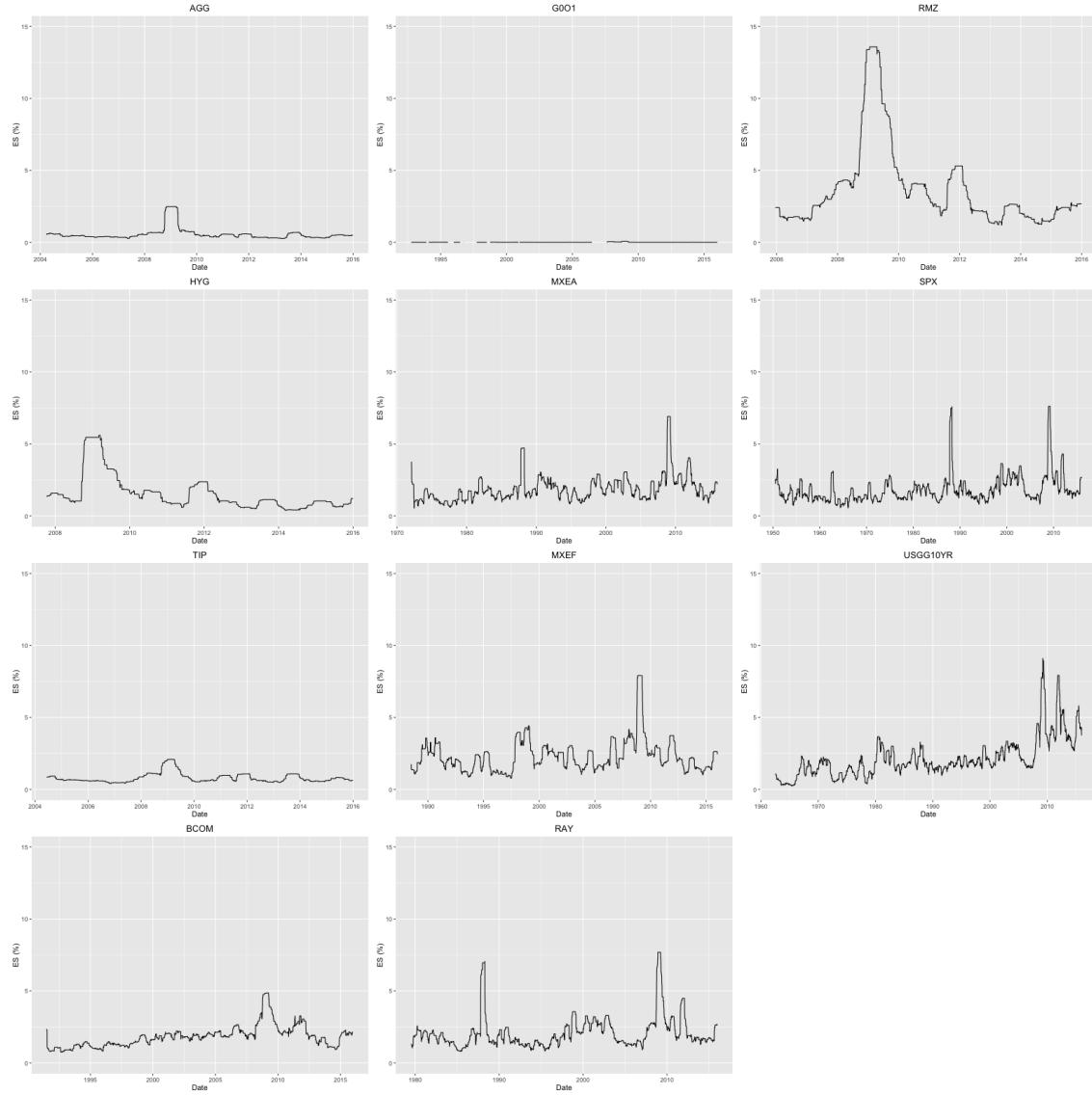


Figure 23: CED under 3-month-5-year Rolling Window (confidence level = 0.95)

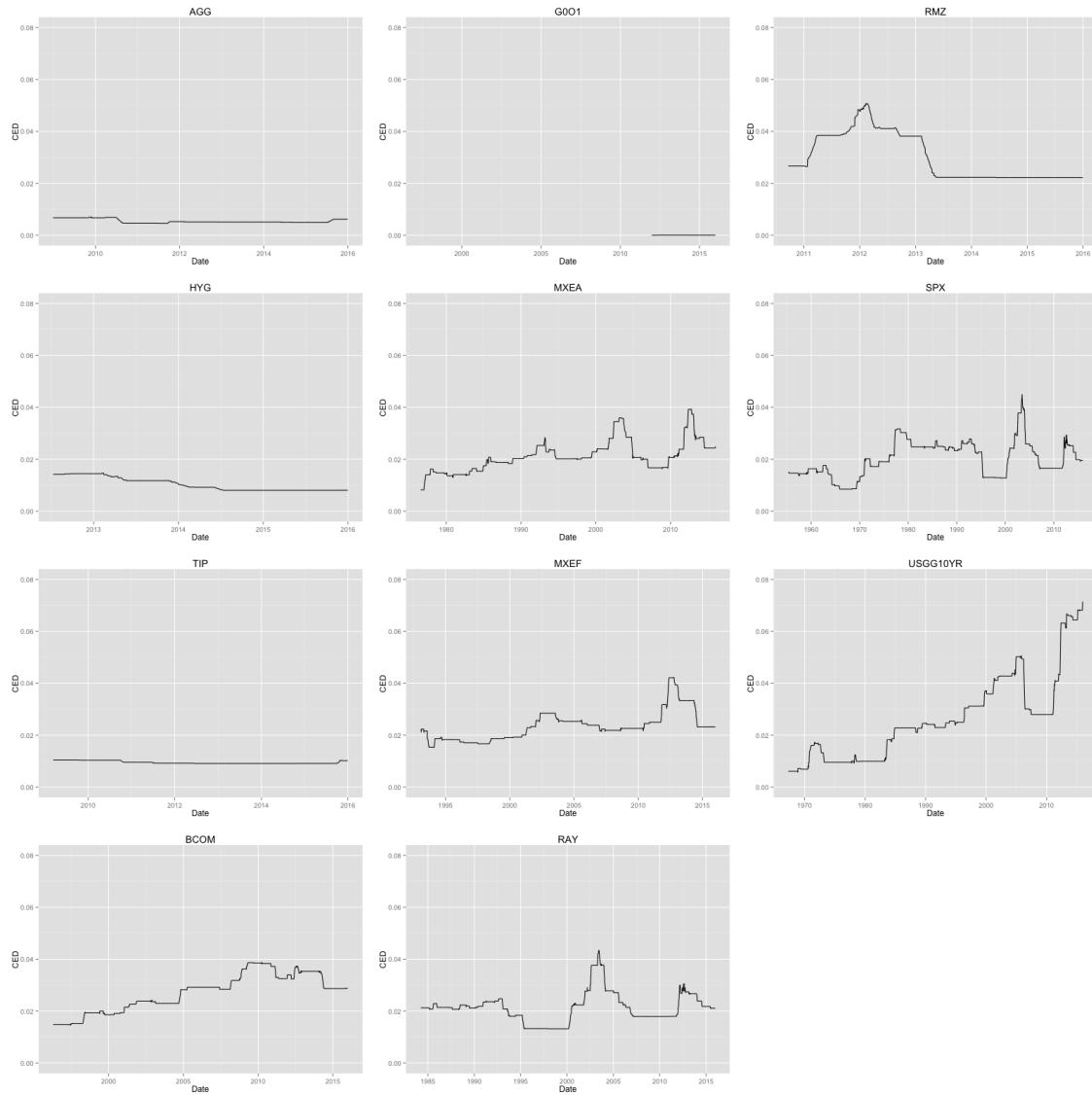


Figure 24: CED (confidence level = 0.9) versus ES under 3-month-5-year Rolling Window

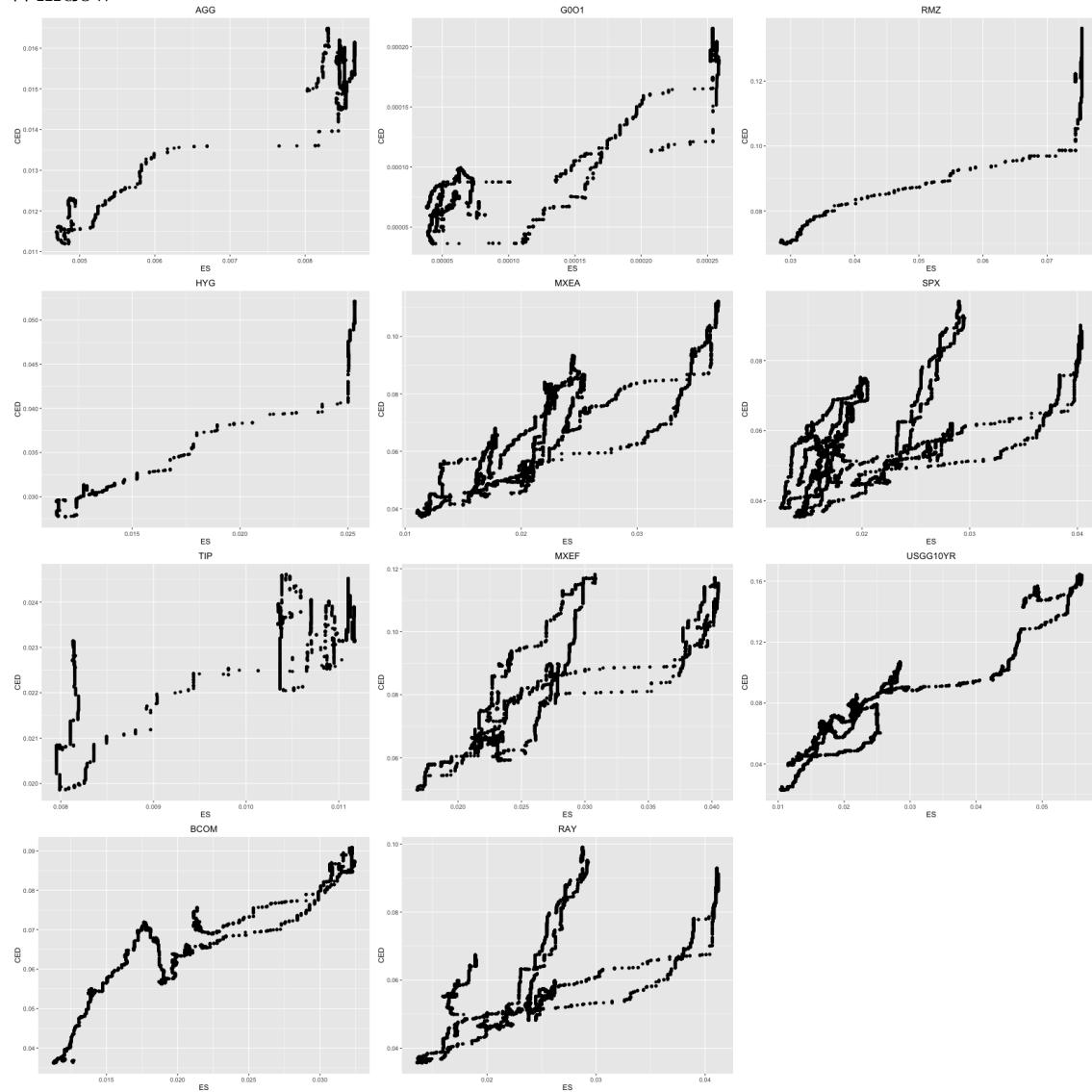


Figure 25: First-order serial correlation calculated using AR(1) model versus VaR

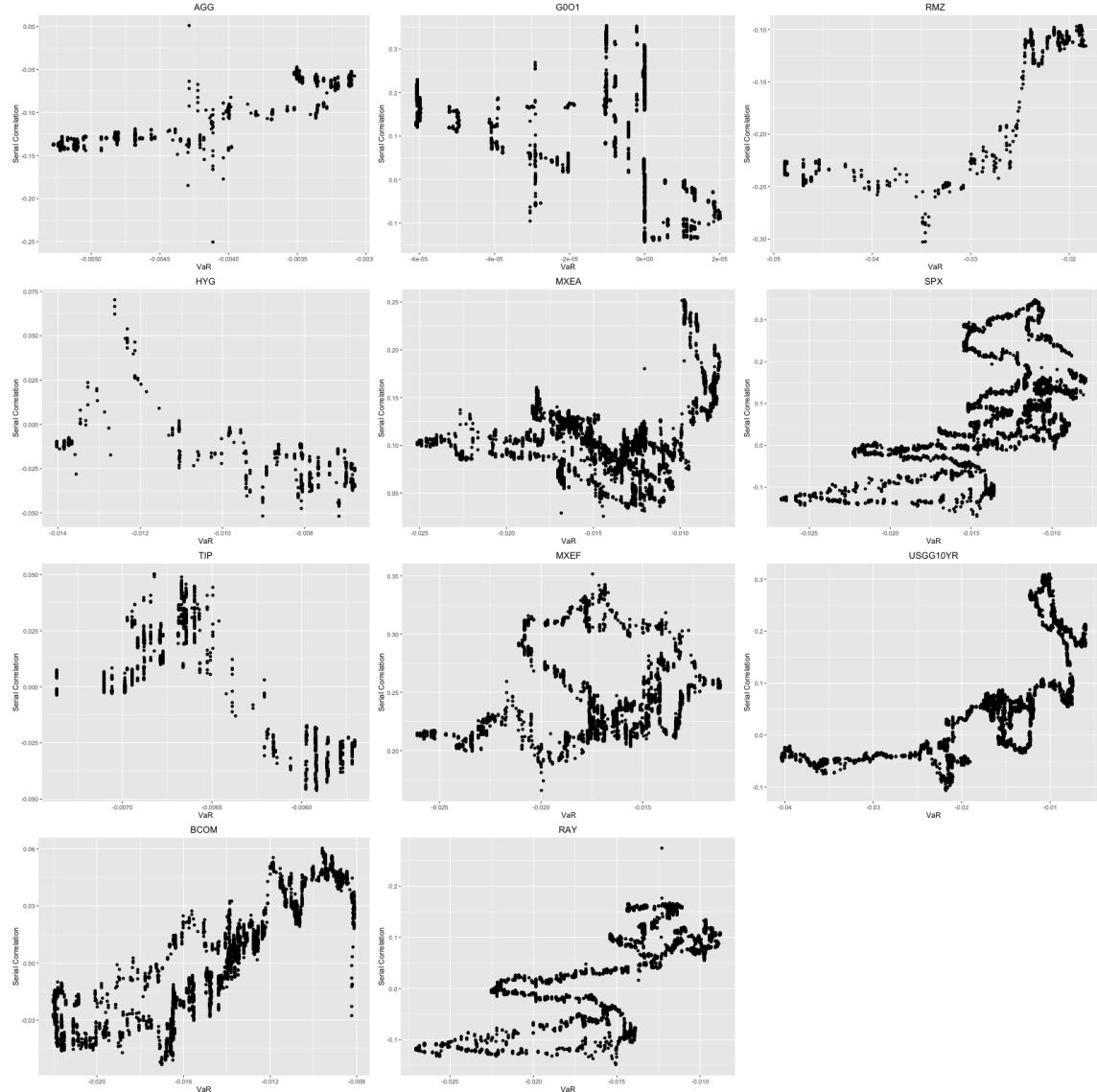


Figure 26: First-order serial correlation calculated using AR(1) model versus ES

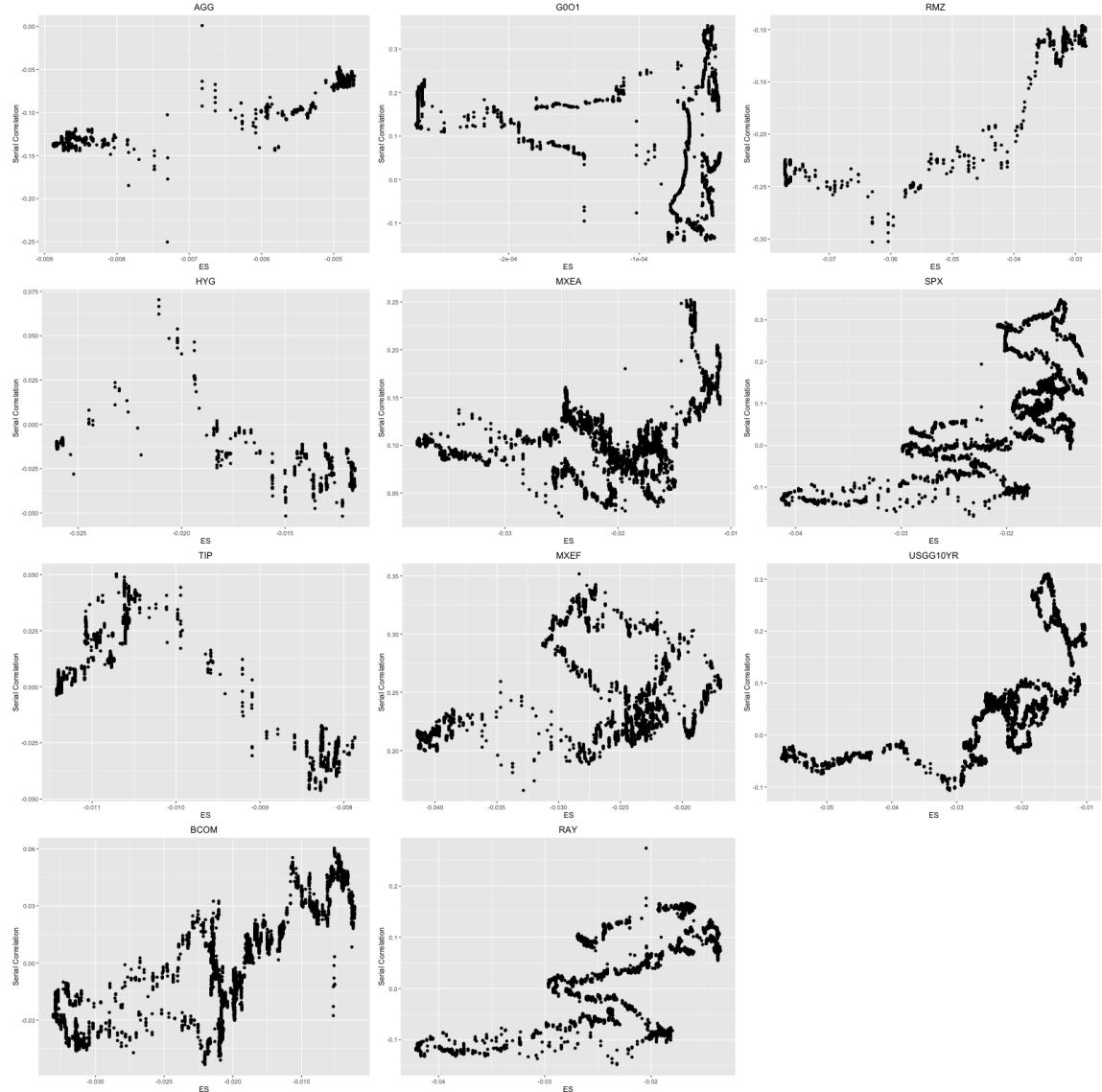


Figure 27: First-order serial correlation calculated using AR(1) model versus CED

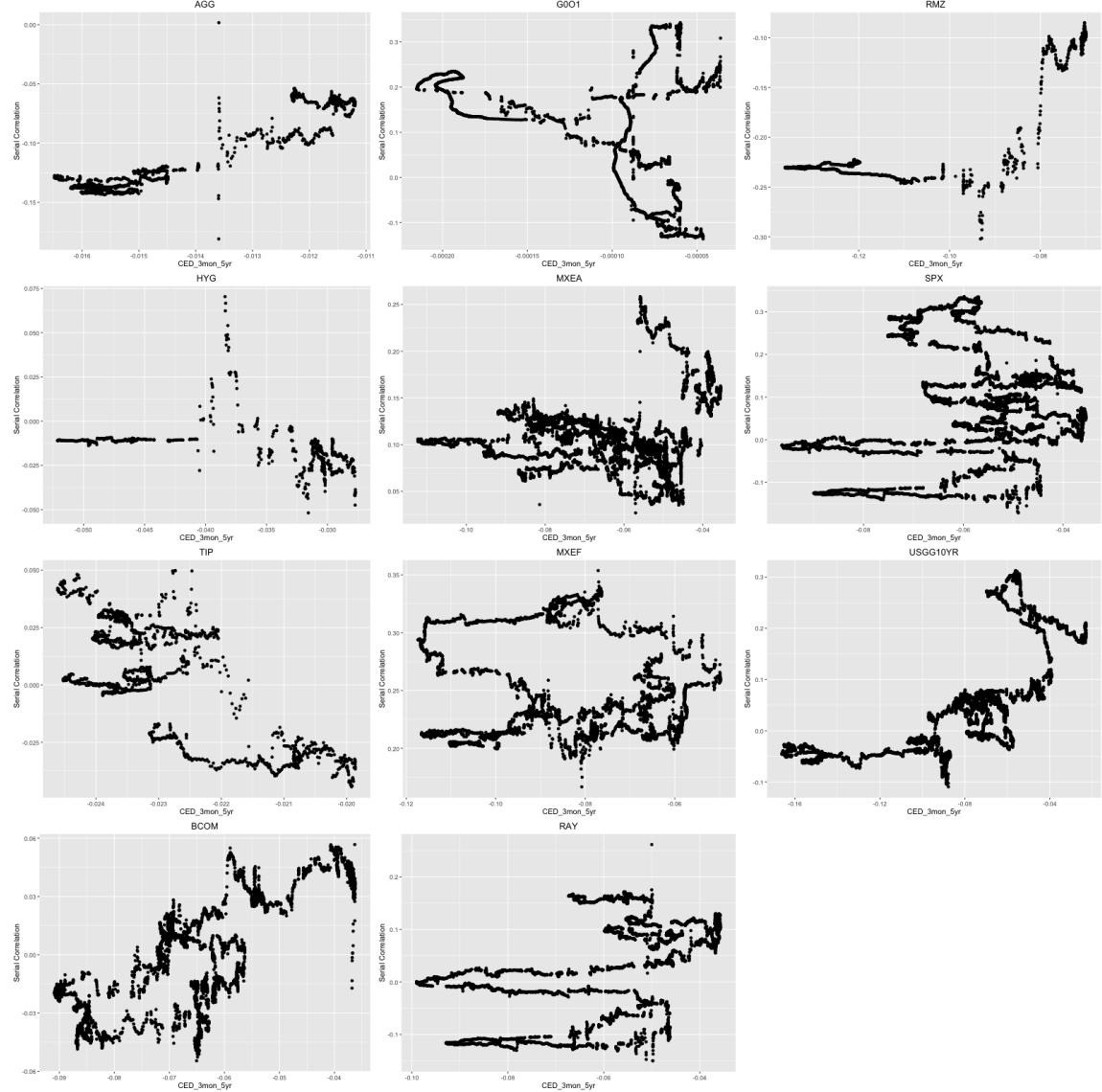


Figure 28: First-order serial correlation calculated using MA(1) model versus VaR

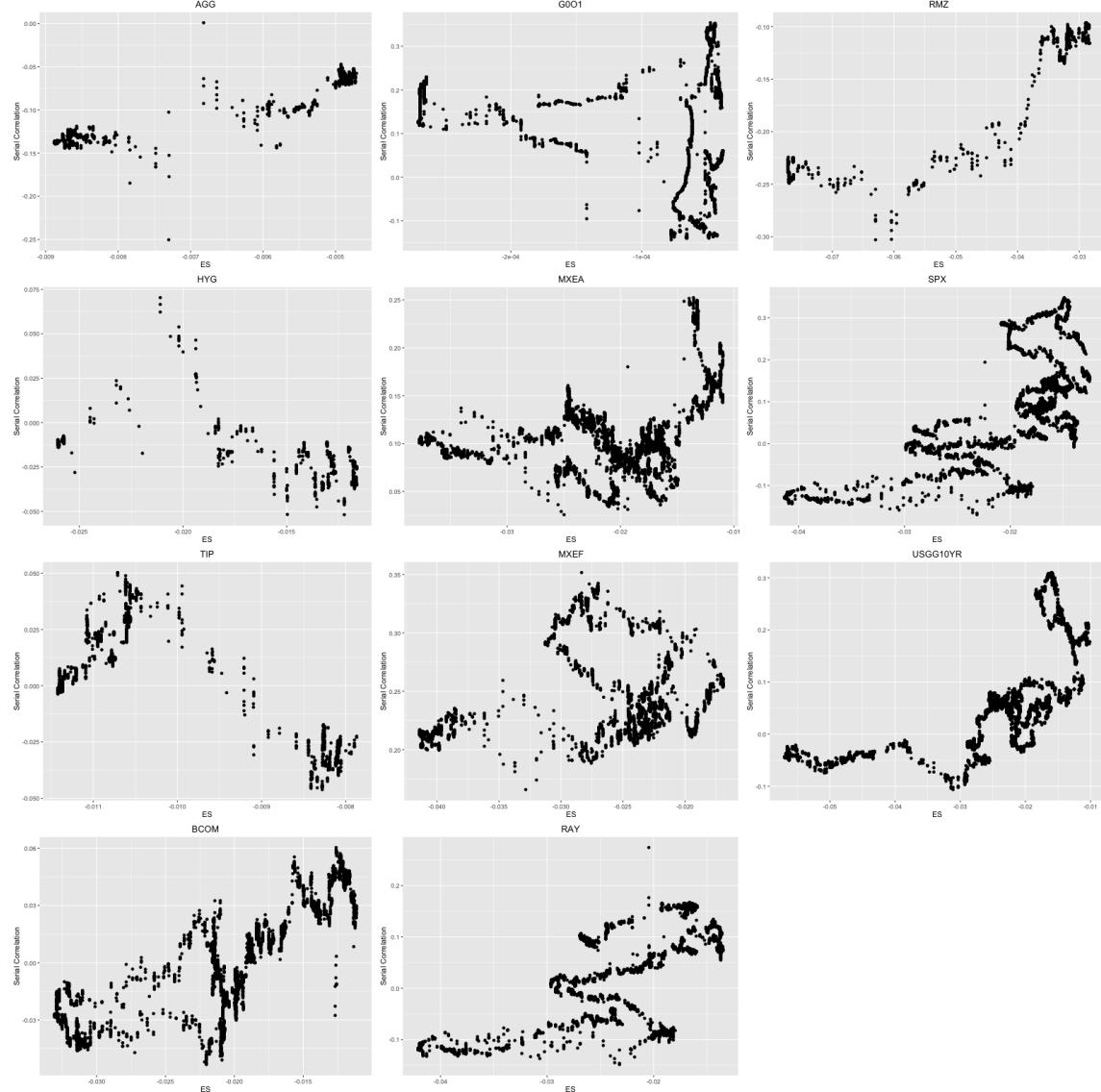


Figure 29: First-order serial correlation calculated using MA(1) model versus CED

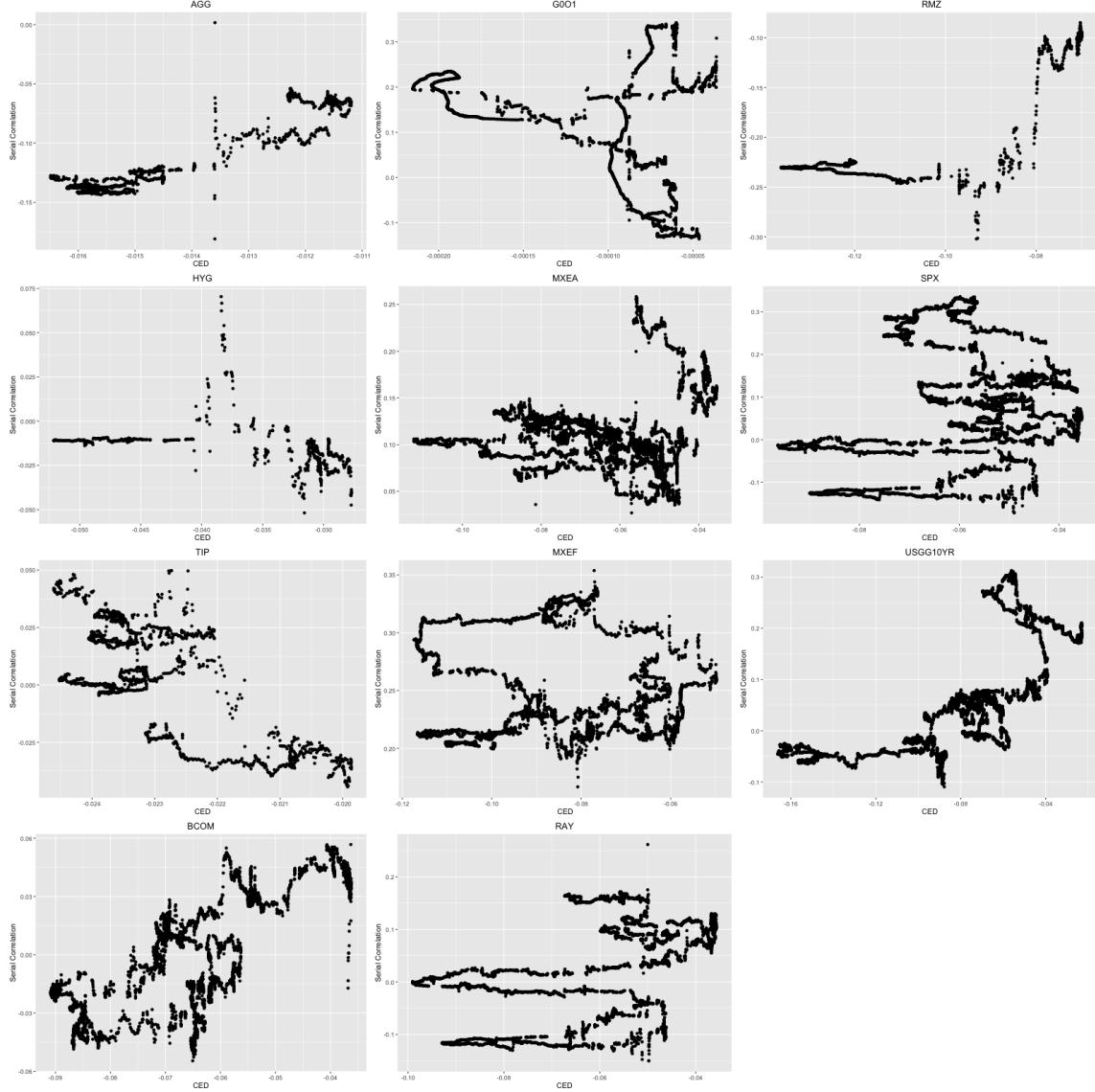


Figure 30: First-order serial correlation calculated using ARMA(1) model versus CED

