Risk Measures and Serial Correlation

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Abstract

Conditional Expected Drawdown (CED), the tail mean of maximum drawdown distribution, is a newly proposed positive homogenous and convex risk measure. Since maximum drawdown is defined as the accumulative loss from peak to trough, we expect CED to be inherently path dependent and account for serial correlation. Most currently widely-used risk measures such as Value at Risk (VaR), Expected Shortfall (ES) and volatility are based on daily returns and do not account for consecutive losses. We compared CED with these risk measures and show that CED is more sensitive to serial correlation on empirical and theoretical perspective.

1 Introduction

Firms and regulators constantly quote risk measures such as VaR, ES and volatility to gauge the amount of asset needed for potential losses. These risk measures are usually calculated from the daily return distribution, thus only accounts for daily losses. However, during the events when consecutive losses happen such as 2008 financial crisis, these measures would become less informative due to the failure to consider the serial correlation of returns. Noticing this drawback of traditional risk measures, Goldberg and Mahmoud[1] developed Conditional Expected Drawdown (CED), a new risk measure defined over the empirical distribution of maximum drawdown. In this paper, we examine the relationship between serial correlation and various risk measures including CED.

1.1 Risk measures

In the rest of the paper, we mainly focus on the comparison of CED and three extensively studied risk measures including Value at Risk (VaR), Expected Shortfall (ES) and volatility.

- *Conditional Expected Drawdown (CED)* is defined as the tail mean of maximum drawdown distribution.

$$CED_\alpha(X_{T_n}) = \mathbb{E}(\boldsymbol{\mu}(X_{T_n})|\boldsymbol{\mu}(X_{T_n}) > DT_\alpha). \tag{1}$$

  where α is the significance level, DT_α is the α quantile of maximum drawdown distribution, $\boldsymbol{\mu}(X_{T_n})$ represents *Maximum drawdown*. In the return path of length n, the maximum drawdown is defined by

$$\boldsymbol{\mu}(X_{T_n}) = \max_{1<i<j\leq n}\max(X_{t_i} - X_{t_j}, 0). \tag{2}$$

  Maximum drawdown is interpreted as the largest accumulative loss from peak to trough. For the detailed description of CED and its properties including path-dependency, convexity, and positive homogeneity, we direct the interested reader to Goldberg and Mahmoud[1].

- *Value at Risk (VaR)* estimates the potential loss of financial investment in a period. VaR is widely used by investment industry to assess the amount of assets needed to cover possible losses. For a given significant level α, VaR is defined as the α quantile of the asset return distribution, which

suggests the probability that the amount of loss excess VaR(α) is less than α. The mathematical representation of VaR is:

$$VaR_\alpha(L) = inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} = inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}. \tag{3}$$

- *Expected shortfall (ES)* is a risk measure which resembles VaR but satisfies monotonicity, translation invariance, homogeneity and subadditivity. The ES of a financial asset is calculated as the tail mean of its return distribution. The Expected shortfall at level α is the expected value of loss which exceeds VaR(α). It is more sensitive to the shape of the loss distribution especially the tail of the distribution. The mathematical representation of ES is:

$$ES_\alpha(L) = E\left[L | L < VaR_\alpha(L)\right]. \tag{4}$$

- *Volatility* is measured by the standard deviation of asset returns. Higher volatility usually implies greater risk. Under the normality assumption of returns, volatility is proportional to VaR and ES. Although real world asset returns often have fatter tails than normal, volatility is often strongly correlated with VaR and ES, which we will see in later analysis.

1.2   Serial correlation

Serial correlation, also known as autocorrelation, is the correlation of observations at different time point. Under the wide-sense stationary process assumption, the serial correlation of X between two time point t and s can be measured by the autocorrelation function as follows:

$$R(\tau) = R(s,t) = \frac{E[(X_t - \mu)(X_s - \mu)]}{\sigma^2}. \tag{5}$$

where $\tau = t - s$.

Serial correlation is often associated with the violation of efficiency market and random walk hypothesis. The literature documenting empirical serial correlation is extensive in the late 1980's[1]. In stock price, Lo and MacKinlay (1988) [3] argue that returns based on the horizon longer than one year show a significant mean reversion, while Poterba and Summers (1988) [4] detect a mean aversion for weekly and monthly returns. Lo and Mavkinlay (1988) [3], Conrad et al. (1991) [5] model the security returns using a positively autocorrelated common component, an idiosyncratic component and a white-noise component. More extensively, the serial correlation has been documented in literature on nonsynchronous trading, which means assets are not traded simultaneously [6]. Mech and Timothy (1993) [7] present evidence that the autocorrelation is associated with the delaying in price adjustment caused by transaction costs. In hedge fund returns, Getmansky, Lo and Makarov (2004) argue that serial correlation is an outcome of illiquidity exposure and smoothed returns, market inefficiencies, time-varying expected returns and leverage and incentive fees with high water marks.

Assuming a moving average representation of reported returns, Getmansky, Lo and Makarov (2004) [8] show that Sharp Ratio (SR) tends to be overstated and the market beta understated. Cesare, Stork and Vries (2014) [9] use the similar structure to demonstrate that the reported value-at-risk (VaR) and expected shortfall (ES) are always smaller than or equal to their actual values. Thus, the risks of assets are easily underestimated using standard risk measures, and the investment decisions may be misleading. Although based on serial correlation and smoothing feature of hedge fund returns, their models are as well applicable to other assets with autocorrelated returns.

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[1]See Barucci and Emilio (2012, Section 6.5) [2] for a detailed review.

1.3 Synopsis

The plan of the paper is as follows. In *Part I: Empirical Analysis*, we present empirical studies analyzing daily returns of various assets. Section 2 provides overall and time-varying risk diagnostics stressing both correlation and the difference between CED and other risk measures. In Section 3, we relate serial correlation with risk measures by fitting time series models, includes AR, MA, ARMA and GARCH. In Section 4, we give an empirical analysis of risk contributions by constructing portfolios of different weights. Next in *Part II: Simulations*, we further illustrate properties of CED by generating returns from time series models. Section 5 contains the comparison of risk measures of various models. And we also give the analysis of the relationship between serial correlation and risk measures based on simulated models. Finally in Section 7, we show that higher serial correlation would result in higher drawdown risk concentrations.

Part I
Empirical Analysis

In this part, we provide the empirical analysis for various asset classes including S&P 500 Index (SPX), Russell 3000 Index (RAY), etc. Detailed descriptions of their date ranges, components, and summary statistics are given in Appendix A.

2 Risk diagnostics

2.1 Overall risk diagnostics

Table 1 shows the overall values of four risk measures for various assets over their time range. For each of VaR, ES and CED, we present results of two significance levels 90% and 95%. Volatility, ES and VaR are on daily-scale to allow comparison between risk measures in the future.

Note that VaR and ES are calculated based on the empirical distribution of daily returns. Another commonly used method would be based on Gaussian distribution assumption. However, in our case where all asset returns are fat-tailed distributed, applying Gaussian distribution would lead to erroneous results. VaR and ES will be overestimated at a lower confidence level and underestimated at a higher level. This discrepancy phenomenon is rather obvious when the return distribution has an extremely fat tail. We include results assuming Gaussian distribution in Appendix A for comparison.

The choice of path length is crucial for CED calculation. As shown in Table 1, the value of CED is

Measures	Volatility	VaR(%)		ES(%)		CED(%) (3 month)		CED(%) (6 month)	
Levels		0.90	0.95	0.90	0.95	0.90	0.95	0.90	0.95
AGG	0.32	0.29	0.40	0.50	0.66	5.60	7.72	8.12	11.45
HYG	0.84	0.62	1.03	1.41	2.03	18.41	24.07	26.43	30.77
TIP	0.41	0.44	0.62	0.72	0.91	7.48	9.90	11.14	12.91
BCOM	0.94	1.04	1.47	1.71	2.20	18.14	22.54	26.61	33.66
MXEA	0.97	1.02	1.46	1.74	2.26	20.39	23.73	27.21	31.79
MXEF	1.13	1.21	1.76	2.11	2.75	26.21	30.80	36.35	43.30
RAY	1.09	1.11	1.62	1.95	2.56	20.65	25.64	27.81	34.08
RMZ	2.30	1.91	3.00	3.99	5.62	37.30	48.41	52.04	62.41
SPX	0.97	0.99	1.43	1.71	2.23	18.35	22.67	25.18	30.65
USGG10YR	1.27	1.26	1.95	2.28	2.99	23.28	28.11	32.78	39.00

Table 1: Overall risk measures for various assets

sensitive to path length. Here we present the CED with rolling three month and six month periods. CED is an increasing function of significance level and path length. We recommend period length no longer than six month for CED estimation. Please refer to an in-depth exploration of behaviour of maximum drawdown distribution and the choice of path length in next subsection. Due to the similarity in definition between CED and ES, they share many features in the calculation. Thus, many ES calculation technique could be implemented for CED estimation.

While volatility, VaR and ES are strongly correlated with each other across assets, CED has a comparatively weaker correlation with the other three. Note that the four risk measures do not give the same asset sequence from the largest to the least risky asset. Occasionally one risk measure indicates different order under different levels and path length. For example, CED indicates HYG has larger risk than SPX, but other three suggest the opposite comparison result. Moreover, the 6 month CED under 90% level gives the different relative risk between HYG and BCOM with its counterpart under 95% level. Notice that the reverse of relative risk suggested by same risk measure for distinct confidence level is more common for CED than for other risk measures.

Back to the economic implications of these risk values. Top three assets in Table 1 (AGG, HYG, TIP), which are comprised of US bonds, have the smallest risk among all asset class; RMZ, constructed by US equity REITs (Real Estate Investment Trust), have much greater risk than the others; MXEF (MSCI Emerging Market Index), reflecting the emerging market equities, have larger risk than MXEA (MSCI Developed Market Index).

2.2 Time-varying risk diagnostics

Time varying risk measures refer to risk measures based on fixed rolling windows. A series of risk measures is obtained. Risk at each time point is given by the past returns of a fixed length. Time-varying risk diagnostics enable us not only to compare risk across assets but to analyze risk for the same asset over time.

2.2.1 Maximum dradown distribution

Unlike VaR and ES, the empirical distribution of maximum drawdown is more sensitive to the time length of measurement. Figure 1 shows the maximum drawdown of various assets for different path length (3 months, 6 months, 1 year, 2 years, 5 years) separately. As revealed in Figure 1, maximum drawdown distribution tends to: a) have the larger mean and variance; b) be multi-mode; c) lack variability of values; d) center around several specific values when we move to longer periods.

For daily return data, we do not recommend window length greater than six months for CED estimation. Large drawdowns in real-world asset returns are usually associated with particular events during a short time, for example, the 2008-2009 financial crisis. When considering longer path such as two years or
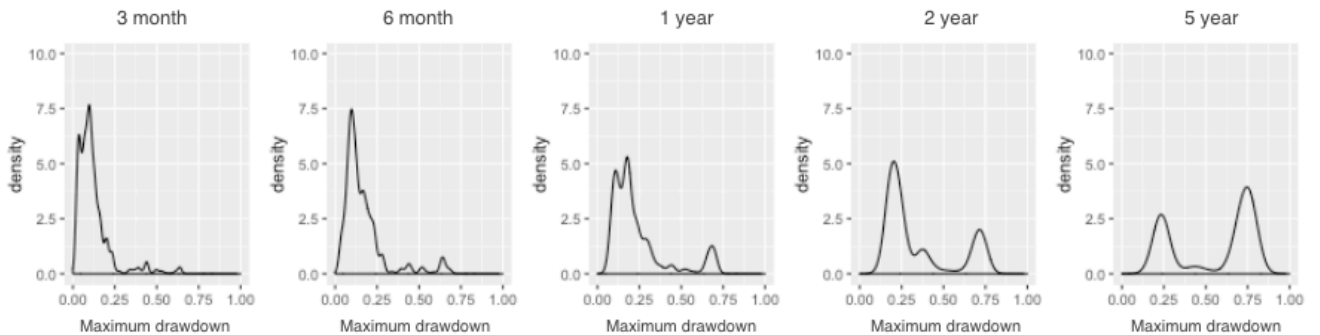

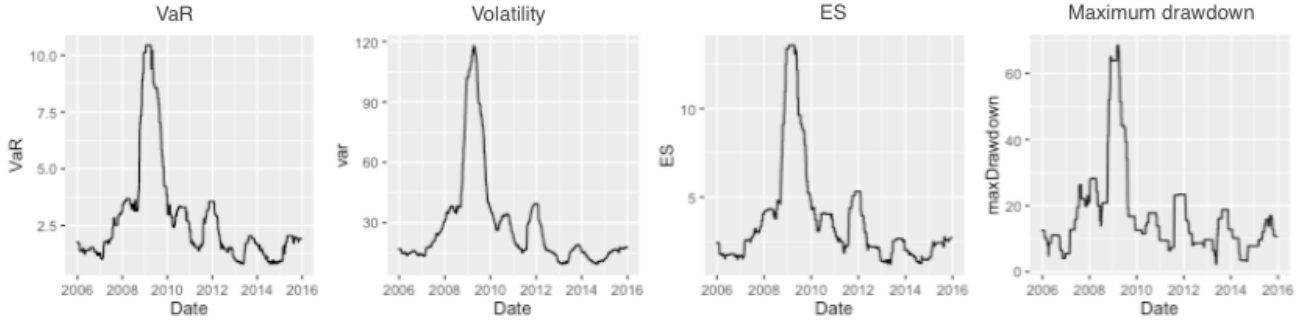Figure 1: Maximum drawdown distribution of RMZ as rolling period increases

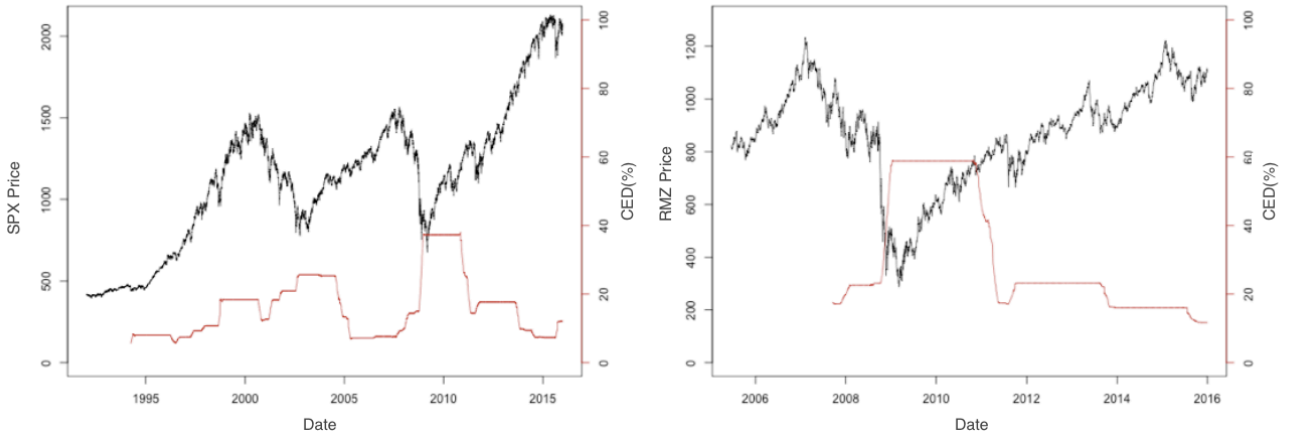Figure 2: Comparison of different risk measures of RMZ

Figure 3: Daily price of the S&P 500 (SPX) and US equity REITs index (RMZ) togther with their 2-year-3month rolling CED.

five years, maximum drawdown values tend to be dominant by these events. The empirical distribution would only be defined on several distinct values. In such cases, the empirical quantile no longer exists without distributional or polynomial assumptions of the tail. Thus, it becomes hard to calculate the tail mean (CED). Later in our analysis, we use three or six-month path length. Figure 12 shows the empirical distribution of maximum drawdown under the six-month rolling window for various assets.

2.2.2 Time varying risk measures

Under Gaussian distribution, VaR, ES, and volatility are linearly dependent. For empirical data where the return distribution has fat tails and distinct kurtosis, they are still strongly correlated. (With average correlation $> 95\%$ for six months rolling window) Figure 2 shows the VaR, volatility, ES, and maximum drawdown with rolling six-month periods. Four risk measures share a similar pattern of ups and downs, where the risk shot up during the 2008-2009 financial crisis.

CED requires more data compared with other risk measures and usually remain constant for a period. To empirically estimate the drawdown distribution, we often rolling a fixed path length. Credible quantile estimation requires hundreds of rollings. For example, two-year-three-month CED evaluates the three-month drawdown risk over the past two years. Figure 3 shows the daily price of the S&P 500 (SPX) and US equity REITs Index (RMZ) together with their two-year-three-month rolling CED. The CED series also reflect the sharp increase in economic depressions.

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CED is closely related to other risk measures. However, the correlation between CED and the other three risk measures are slightly weaker than the correlation among volatility, VaR, and ES. Table 8 shows the correlation between CED and other risk measures calculated based on six-month rolling risk measures for every asset.

3 Time series analysis

3.1 ARMA

ARMA is the simplest and most commonly used time series model, which is enable to capture the correlation between current time point and one or several lags in a stationary time series. Therefore, in the section 3.1, ten kinds of U.S indices are feeded into ARMA models, first simple chosen models from the family, then best model through selection steps. From there, one can take the first sight of the relationship between serial correlation and different risk measurements in real-world data.

3.1.1 Result of basic ARMA model fit

Started from the simplest ones, ten U.S indices have been feeded into 3 most basic ARMA models: AR(1), MA(1) and ARMA(1,1). The three models here are not necessary the best one, or even the ones that all underlying assumptions are satisfied. This step is just try to have a general idea if the serial correlation of each model is related with risk measurements in some way. When it comes to serial correlation here, it refers to the first order autocorrelation of the time series models.

Following is a short summary on these three ARMA model, including their formulus, description and the method for calculating the serial correlation.

1. **AR(1)**

    - Model:
    $$X_t = \phi X_{t-1} + \epsilon_t \quad s.t. \ |\phi| < 1 \tag{6}$$

    - AR(1) just has the AR component with lag 1 in the ARMA model, which characterized by only the previous term in the process and the noise term contributing to the output. When ϕ is closer to 0, the whole process is more like to be white noise.

    - The equation for calculating the serial correlation:
    $$\rho(h) = \phi^h, \ h = 0, 1, \ldots \tag{7}$$

2. **MA(1)**

    - Model:
    $$X_t = \epsilon_t + \theta \epsilon_{t-1} \tag{8}$$

    - MA(1) just has the MA component with lag 1 in the ARMA model, which impost the shock for the current period and q periods into the future; in contrast, in the AR model a shock affects X values infinitely far into the future.

    - The equation for calculating the serial correlation:
    $$\rho(h) = \begin{cases} \frac{\theta}{(1+\theta^2)} & \text{if } h = 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

3. **ARMA(1,1)**

- Model:

$$X_t = \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1} \tag{10}$$

- ARMA(1,1) has both AR and MA component with lag 1. ARMA(1,1) is the combination of AR(1) and MA(1), which ACF incorporate the characteristics from both AR and MA: both error term and data point are directely affected by that on the previous time points.

- The equation for calculating the serial correlation:

$$\rho(h) = \frac{(1 + \theta\phi)(\theta + \phi)}{1 + 2\theta\phi + \phi^2}\phi^{h-1}, \text{if } h \geq 1 \tag{11}$$

The Table2 shows the correlatio between serial correlation and different risk measures for AR(1) and ARMA(1,1). Figure13 and Figure 14 in the appendix shows a more straightforward relationship between first order serial correlation of AR(1) and risk measures. Consistent with what was showing in the plot, all the assets except MXEA and MXEF have the high correlation between $\kappa(1)$ and risk measurements. Within each asset, as expected, the correlation with VaR and ES are very closed, as VaR is highly correlated with ES internally. However, compared with VaR and ES, the correlation with CED is relatively smaller, and sometimes the correlations are even with different sign.

	AR(1)			MA(1)		
Asset	VaR	ES	CED	VaR	ES	CED
AGG	0.92	0.95	0.94	0.94	0.96	0.94
HYG	-0.61	-0.56	-0.41	-0.51	-0.45	-0.29
TIP	-0.67	-0.75	-0.73	-0.65	-0.74	-0.73
BCOM	0.86	0.80	0.76	0.86	0.80	0.76
MXEA	0.26	0.33	0.08	0.01	0.11	0.15
MXEF	0.23	0.22	-0.05	0.15	0.16	-0.12
RAY	0.79	0.76	0.49	0.79	0.77	0.50
RMZ	0.90	0.93	0.81	0.92	0.95	0.84
SPX	0.62	0.70	0.15	0.65	0.72	0.17
USGG10YR	0.67	0.64	0.66	0.65	0.64	0.68

Table 2: Correlation between $\rho(1)$ and risk measures. The time window used for get risk measurements and $\rho(1)$ is 5 years, and the rolling window within 5 year for CED is 3 month.

Both the table and figures are unable to tell the story what are the relationships between risk measures and serial correlation for the indices, since the model here are recklessly chosen and more complicated factors shocking the market everyday. However, the table provides an idea that relationship with CED is more closed to that with VaR and ES when this correlation is internally large.

3.1.2 Model selection

The model selection is accomplished by trying to find a proper time series model on the unit of all data for each asset, for the fact that it makes no sense to fit different model for each rolling time period within one financial asset. Unfortunately, observing from the ACF plot of returns, most of them are not stationary by nature without drift term, which means ARMA model is inherently not or just a partial a good selection for fitting them. It may also explain the reason why the previous section did not produce a expected result that CED is more correlated with serial correlation $\kappa(1)$.

Our criterion here for "better model" is the smaller AIC value. Appendix D provides a more detailed explanation on model select criteria. To confirm the fact that ARMA is not reasonable for

Table 3: Best ARIMA Model for the U.S. Assets

Asset	ARIMA (p,d,q)
AGG	(5,0,5)
HYG	(3,0,1)
TIP	(0,0,0)

almost all the indices, we tried sets of parameter for ARMA and select the one with the smallest AIC value. Essentially, we chose the combination of AR and MA parameters ranging from 1 to 5. Table 3 shows the best selection based on AIC criteria. The Table 3 indicates that the parameters are as high as 4 or 5 for almost all the assets, which suggests that the time series are not stationary. It also involved some converging issue is when the parameters are getting too larger.

To illustrate this point, the following is an example analysis on RMZ index. There are several reasons why RMZ is selected: 1) the RMZ has a shorter period, which means it includes less abnormal market shock such as financial crisis in 2008 and more close to the normal fluctuation in stock price. 2) Larger serial correlation and ES, VaR comparing with other indices, implying it is more suitable for analysing the relationship between serial correlation and various risk measures. 3) There are less modes in Maximum drawdown of RMZ, suggesting it is reasonable to get tail mean, which is CED.

3.1.3   RMZ: an example

3.2   GARCH

3.2.1   Why GARCH is better

3.2.2   Result of GARCH fit

3.2.3   RMZ: an example

3.3   Regime dependent analysis

Most economic time series data behave differently in the adjacent period. For example, asset returns usually show considerable volatility during a financial crisis. One standard approach to model abrupt changes in regime is to use the Hamilton's Markov regime switching model[10][11]. In our study, we use the following two-regime switching model to fit the returns:

$$y_t - \mu_{s_t^*} = \phi_{s_t^*}(y_{t-1} - \mu_{s_{t-1}^*}) + \epsilon_t \tag{12}$$

where the number of autoregressive coefficients is set to 1. s_t^* is a two-state Markov chain. $s_t^* = 1$ represent regime 1 and $s_t^* = 2$ represent regime 2. s_t^* depends on the past only through the most recent values:

$$P(s_t = j | s_{t-1}, s_{t-2}, \dots) = P(s_t = j | s_{t-1}) = p_{ij} \tag{13}$$

3.3.1   Comparison of basic summary of two regimes

Table 4 shows the estimated autoregression coefficient ϕ and standard deviation of the noise term of two regimes for various assets. We also provide the standard deviation, skewness, and kurtosis of different assets in Table 9 in the appendix. Note that regime 1 represents high volatility regime while regime 2 represents low volatility one. In general, it is clear that returns of regime 1 have a larger standard deviation, skewness, and kurtosis than that of regime 2. This difference in kurtosis indicates that the return distribution in low volatility regimes has a lighter tail. In

	Regime 1 High volatility		Regime 2 Low volatility	
	ϕ	Std	ϕ	Std
AGG	-0.134	0.009	-0.114	0.002
HYG	-0.010	0.016	0.025	0.004
TIP	0.039	0.007	-0.026	0.003
BCOM	-0.046	0.013	0.050	0.006
MXEA	0.095	0.015	0.109	0.006
MXEF	0.221	0.018	0.254	0.007
RAY	-0.039	0.019	0.052	0.007
RMZ	-0.244	0.040	0.007	0.010
SPX	-0.018	0.016	0.113	0.006
USGG10YR	-0.031	0.020	0.082	0.006

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contrast, in high volatility regimes, there are more extreme values of returns.

We observe that low volatility regimes are more likely (9 out of 10 assets) to show larger autocorrelation coefficient. Although we expect greater drawdown risk when serial correlation increase (which means more significant autocorrelation coefficients estimate here), we did not prove this through empirical results. Later we will show in the simulation study that given different serial correlation and error term variance of the time series model, the latter factor dominant the impact to CED values, which is the case in our empirical findings. But under the same level of noise term variance, the larger the serial correlation, the greater the CED.
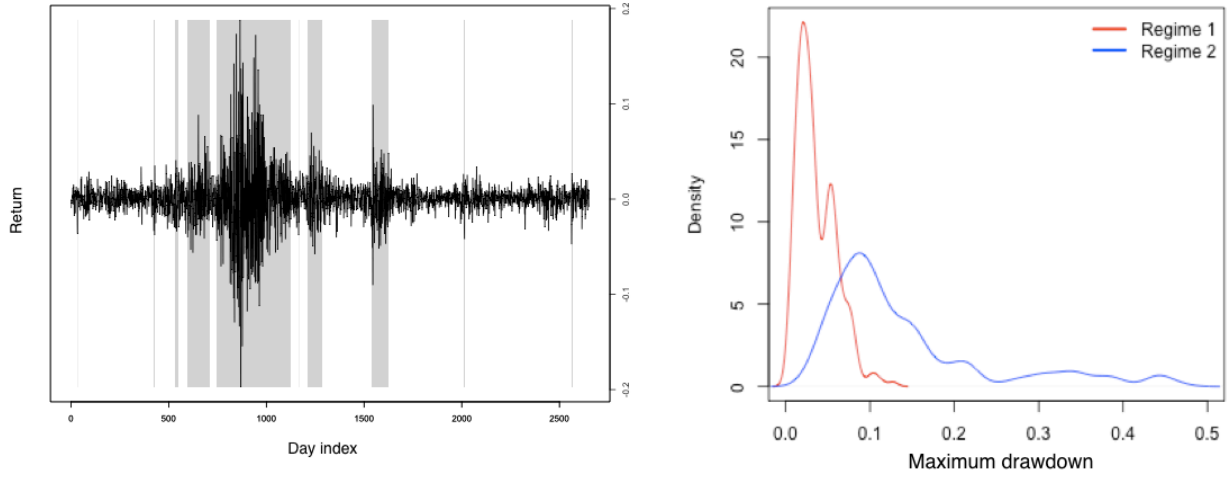

Figure 4: Left panel: Return plot of RMZ from June 20th, 2005 to December 31st, 2015. Shadowed areas represent regime 1 with high volatility, and white areas indicate regime 2 with low volatility. Right panel: one-month maximum drawdown distribution of regime 1 and regime 2 of RMZ.

	Regime 1 High volatility	Regime 2 Low volatility
VaR (empirical, p = 0.95)	7.4%	1.5%
ES (empirical, p = 0.95)	9.9%	2.1%
CED (one-month, p = 0.9)	38.4%	8.4%
Serial correlation (order = 1)	-0.257	-0.026
Serial correlation (order = 2)	-0.023	0.008

Table 5: Risk diagnostics for RMZ of two equal-length episode of each regime

3.3.2 RMZ: An example

In order to make a consistent comparison of risk diagnostics between two regimes, we ignore some short discontinuity and pick two longest single occurring episode for each regime. Here we use RMZ, the most risky asset in our data set as an example. Both episode contain 530 trading days. The episode of regime 1 range from October 30th, 2007 to December 7th, 2009, and the episode of regime 2 range from June 20th, 2013 to July 29th, 2015.

In Figure 4, the left panel shows the return of RMZ from June 20th, 2005 to December 31st, 2015. The

shadowed area represents the regime with high volatility and the white area low volatility. This model is consistent with the actual financial event in that the high volatility regime covered mainly from mid-2007 to 2010. As we might expect, regime with high volatility also shows high risks in that they have larger ES and VaR values. Moreover, model of regime 1 only explains 26.3% of the observations, which means abrupt deviation is a minority in total observations. By looking at assets with longer period, we find a similar proportion of high volatility regimes (SPX: 23.7%, RAY: 20.6%). The right panel shows the one-month maximum drawdown distribution calculated based on the continuous episode we selected. The maximum drawdown distribution in regime 1 has larger mean and variance as well as greater knewness and kurtosis.

We may wrongly interpretate that larger serial correlation results in less drawdown risk. Notice from the results in later simulation section that variance of noice term dominant the influence for CED. And we can only obverve the positive correlation between serial correlation and CED under fixed variance level.

3.4 Other return frequencies

4 Empirical risk contribution

4.1 SPX & RMZ: An example

Part II
Simulations

In this part, we explore the relationship between serial correlation and risk measures by analyzing models from data generated by known time series. Real world financial data is usually more complicated and simultaneously influenced by multiple effects. Simulation allow us to examine one specific factor by fixing others. For example, we may change one coefficient of a time series model while keeping other at the same level, which enables us to examine the relationship of certain order of serial correlation and CED.

5 Serial correlation and risk measures

In this section, we simulate various time series models and examine the impact of serial correlation on risk measures.

5.1 ARMA model with normal noise

We started from the simplest model AR(1) and the normal noice term:

$$X_t = \kappa_1 X_{t-1} + \epsilon_t \tag{14}$$

We simulate AR(1) for various values of the autoregressive parameter for $\kappa_1 \in (-1, 1)$. Figure 5 shows the relationship between AR(1) coefficient and risk measures of interest including ES, VaR, volatility and CED. For AR(1) model, the order one serial correlation is the value of the autoregressive parameter.

CED shows a decreasing trend when $\kappa_1 \in (-1, -0.75)$ and an increasing trend when $\kappa_1 \in (-0.75, 1)$. As shown in Figure 5, it becomes feasible for us to distinguish negative and positive serial correlation using the CED values. However, the other 3 risk measures are all symmetric about $\kappa_1 = 0$, and they increase as the absolute values of κ_1 increases.

For VaR, ES and CED, the derivative of risk measure values to κ_1 approaches to zero as κ_1 goes to 0 and increase as κ_1 increase. This suggests that the value of this three risk measures can hardly reflect the change of serial correlation when the serial correlation is small. And they perform better when the serial correlation increases. The trend of derivatives reverses for CED. While the change of κ_1 has a comparatively larger influence around 0, the impact becomes weaker as we move to greater κ_1 values.

Figure 6 shows the maximum drawdown distribution for various κ_1. Same as revealed in Figure 5, the mean and tail mean of maximum drawdown distribution increases as we increase κ_1 from negative values to positive values.

As we move to higher order models, the relationship between serial correlation and risk measures becomes complicated. Due to the page limit, we omit the results for higher order models. The interested reader is welcome to refer to our simulation report.
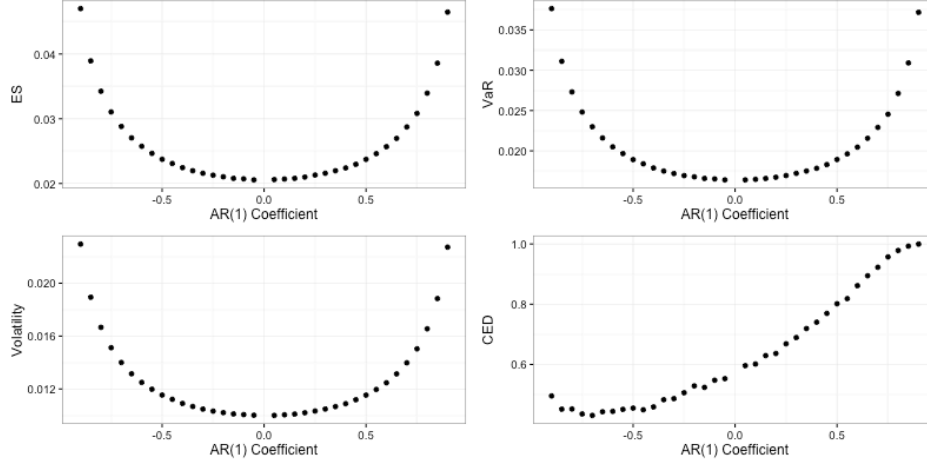

Figure 5: AR(1): Relationship between auto-correlation coefficients and risk measures
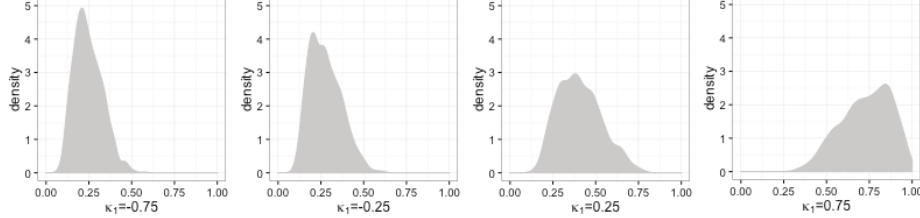(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

Figure 6: AR(1): Maximum drawdown distribution for various κ_1 values
(Empirical distribution, path length = 1000, sample size = 1000, $\epsilon_t \sim N(0, 0.0001)$)

5.2 ARMA model with fat tailed noise

Empirical asset returns are usually fat-tailed distributed. Extreme losses consistently occur, which lead us to consider noise with student t distribution.

Figure 7 shows the simulated AR(1) model and the corresponding risk measures. The relationship between serial correlation and risk measures revealed in this subsection resembles that of the last subsection where noise terms follow the normal distribution. There are slight differences due to the heavy-tail distribution. However, the points are not lies perfectly on a curve because the heavy tail introduces more randomness of the quantile and tail mean values. The simulated return distribution has mean and skewness close to zero. And they have high kurtosis which implies the fat tail. The simulated maximum drawdown distribution has larger the variance than that of the normal case.
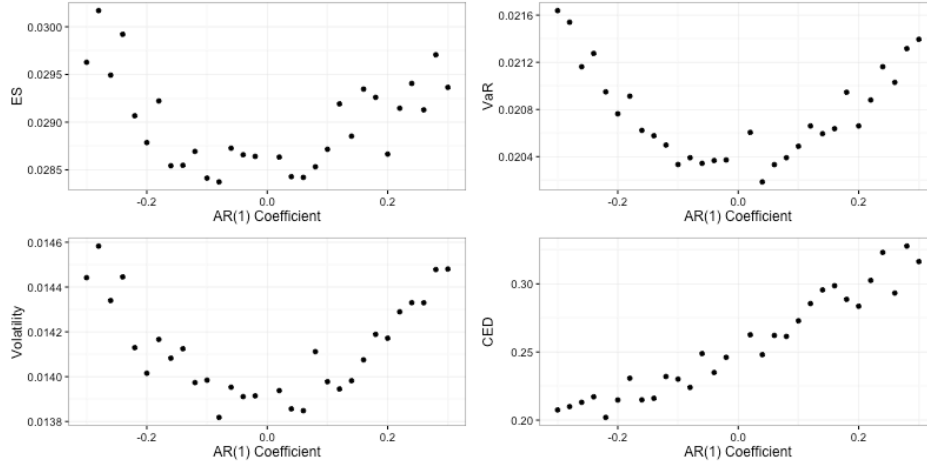
Figure 7: AR(1): Relationship between auto-correlation coefficients and risk measures (Simulation path length: 63, $\epsilon_t \sim 0.01T(df = 4)$)

5.3 ARMA model with standard error adjusted

So far our simulation studies are based on fixed distribution of the noise term. For example, for normal distribution noise we use $\epsilon \sim N(0, 0.0001)$ and for t-distribution we use $\epsilon \sim 0.01T(df = 4)$. In this section, we standardize the simulated time series by dividing certain factors such that the time series have same standard deviation.

In our simulation, the standard deviation for each our simulated time series is different. Say for AR(1), the standard deviation of time series $X_t = 0.5X_{t-1} + \epsilon_t$ and $X_t = 0.1X_{t-1} + \epsilon_t$ is different since we use the same distribution for ϵ_t. The standard deviation for AR(1) model is :

$$SD(X_t) = \left(\frac{Var(\epsilon_t)}{1 - \kappa_1^2}\right)^{1/2} \tag{15}$$

Then what if we compare two time series with the different distribution of ϵ_t but the same standard deviation? Since standard deviation is one risk measures (volatility), this equals to fixing other risk measures to one level, say volatility = 1.2%, and to see how CED changes with the serial correlation.

Since our simulation is based on the normal distribution of the noise term, volatility would be proportional to ES and VaR, which means fixed volatility also means fixed volatility.

Figure 8 shows the result for AR(1) and MA(1) respectively. And Figure 9 shows the results for AR(2), MA(2) and ARMA(1, 1). There are clearer upward trends than the last two sections. These figures suggest that CED is a better metric in describing the returns with serial correlation. For the same volatility level, CED increases as the serial correlation increases, which means CED captures the changes in serial correlation while volatility, VaR and ES do not.

Figure 8: Left panel: first-order serial correlation calculated using AR(1) model versus CED. Right panel: First-order serial correlation calculated using AR(1) model versus VaR
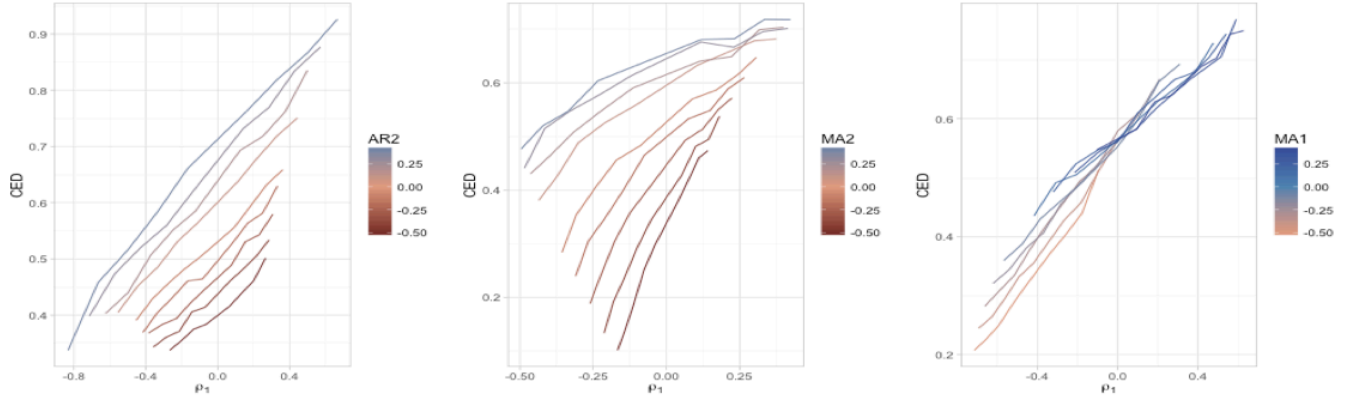
Figure 9: Relationship between serial correlation ρ_1 and CED with adjusted standard error (Simulation path length: 1000, from left to right: AR(2), MA(2), ARMA(1, 1), model coefficient $\in (-0.5, 0.5)$)

5.4 More complex noise: Garch model

Also started from AR(1) GARCH(1,1) model
Discribe simulation parameters of AR
Insert that table that compare with other ARMA models

5.5 Findings

In this section, we present findings summarized from the previous simulation study, and provide the corresponding theorial explainations.

1. CED distinguishes negative from positive autocorrelation

For time series with positive autocorrelation, the return is more likely to remain the same sign as the previous day. The likelihood increase as the serial correlation increase. As a consequence, returns tend to keep they current profit or loss status for a period. If the current status is loss, then, unfortunately, the price of the asset may continuously go down, which results in a large CED. Conversely, time series with negative autocorrelation tend to reverse the sign in the previous day. If the price of one asset goes down in one day, it is more likely that the price would bounce back in the next day, which results in alternating profit and loss status thus a smaller CED.

While CED captures the difference in serial correlation, other risk measures do not. VaR, ES, and volatility are obtained from the empirical distribution of returns. If we shuffle the return sequence, we will get the same value for VaR, ES, and volatility.

Figure 10 shows the comparison of simulated returns and prices when serial correlation is 0.7 and -0.7. The maximum drawdown of simulated series is 22.76% and 9.49% separately. And two return series have the same VaR, ES, and volatility.
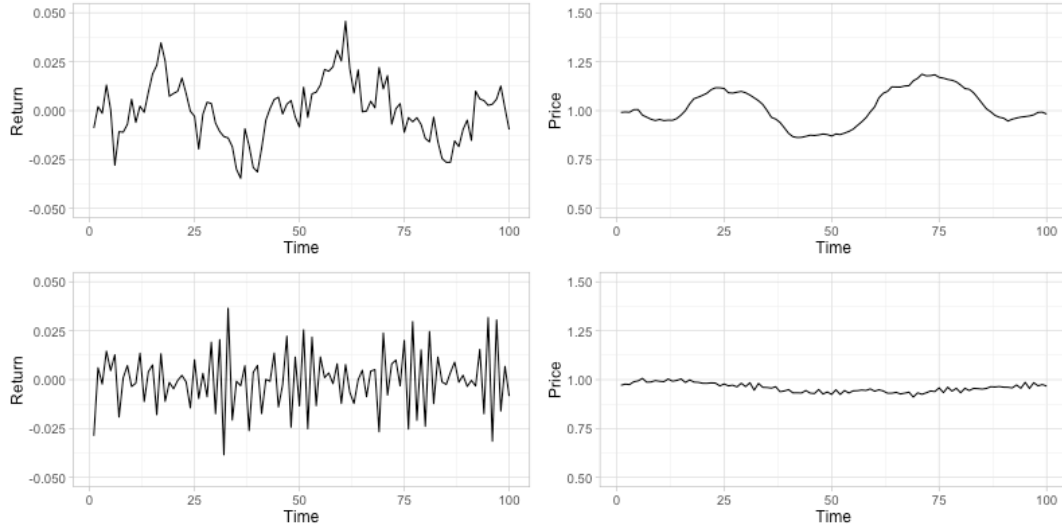
Figure 10: Comparison of returns with positive and negative serial correlation
(Simulation length: 100; upper panel: AR(1) with $\kappa = 0.7$; lower panel: AR(1) with $\kappa = -0.7$,
$\epsilon \sim N(0, 0.0001)$)

2. Change standard deviation but fix everything in time series simulation would result in linear change of risk measures

Simulation in Figure 11 for AR(1) model shows a linear relationship for all risk measures. We simulate the model $X_t = 0.3X_{t-1} + \epsilon_t$ and change the ϵ_t from 0.001 to 0.015.
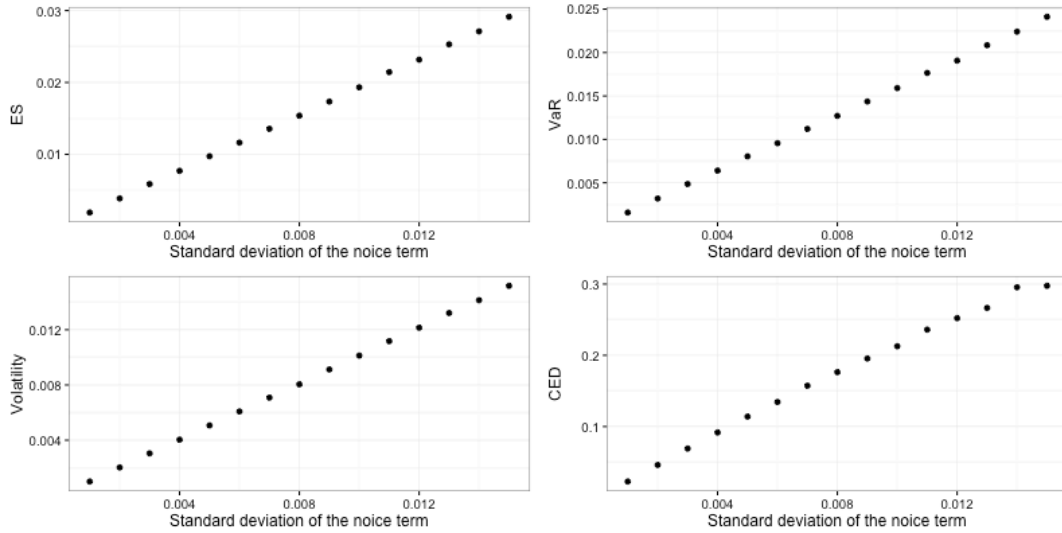

Figure 11: Comparison of returns with positive and negative serial correlation
(Simulation length: 100; upper panel: AR(1) with $\kappa = 0.7$; lower panel: AR(1) with $\kappa = -0.7$,
$\epsilon \sim N(0, 0.0001)$)

Fixing everything but changing the standard deviation of the noise term would simply just to multiplying the simulated time series model by a constant. For volatility, ES and VaR, it is evident. For accumulated returns in a path, the return is approximately the sum of all returns for each day in this time period when the return is small (which is often the case):

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$$R = \prod_{i=1}^{n}(1 + r_i) - 1 \simeq \prod_{i=1}^{n} e^{r_i} - 1 = e^{\sum_{i=1}^{n} r_i} - 1 \simeq \sum_{i=1}^{n} r_i \tag{16}$$

All the returns in every subpath would be multiplied by approximately a constant. Thus, the domain of the maximum drawdown distribution and CED would also be scaled by the same number.

6   Risk contribution

Started from AR(1), present an example how risk contribution varied for different risk measurements, different weight and different ARMA parameters.

High light difference between CED and traditional risk measurements.

Short discuss how CED useful in capture the risk of financial assets on risk contribution aspect.

A   Appendix A: Dataset

A.1   Description

Symbol	Name	Date range
AGG	iShares Core US Aggregate Bond	2003-09-29 to 2015-12-31
HYG	iShares iBoxx High Yield Corporate Bond	2007-04-12 to 2015-12-31
TIP	iShares TIPS Bond	2003-12-08 to 2015-12-31
BCOM	Bloomberg Commodity Index	1991-01-03 to 2015-12-31
G0O1	3-Month U.S. Treasury Bill Index	1992-04-01 to 2015-12-31
MXEA	MSCI Developed Markets Index	1970-01-07 to 2015-12-31
MXEF	MSCI Emerging Markets Index	1988-01-01 to 2015-12-31
RAY	Russell 3000 Index	1979-01-02 to 2015-12-31
RMZ	MSCI US REIT Index	2005-06-20 to 2015-12-31
SPX	S&P 500 Index	1950-01-04 to 2015-12-31
USGG10YR	US Generic Govt 10 Year	1962-01-03 to 2015-12-31

Table 6: Normal VaR and ES under various levels

A.2   Statistical summary

annualized return, Sharpe Ratio, standard deviation, skewness, kurtosis [Empirical Page 8]

A.3   Return

Return across time and Return distribution [Empirical Page 8]

A.4   Normal VaR and ES

	VaR(%)			ES(%)		
Asset	0.90	0.95	0.99	0.90	0.95	0.99
AGG	0.39	0.51	0.72	0.57	0.67	0.86
HYG	1.06	1.36	1.94	1.50	1.76	2.26
TIP	0.51	0.66	0.94	0.74	0.86	1.11
BCOM	1.20	1.54	2.18	1.65	1.94	2.50
MXEA	1.22	1.57	2.24	1.74	2.04	2.62
MXEF	1.42	1.83	2.61	2.03	2.38	3.06
RAY	1.36	1.75	2.49	1.95	2.29	2.94
RMZ	2.92	3.75	5.32	4.08	4.79	6.18
SPX	1.21	1.57	2.22	1.73	2.03	2.61
USGG10YR	1.62	2.08	2.95	2.23	2.62	3.39

Table 7: Normal VaR and ES under various levels

B Appendix B: Rolling risk diagnostics

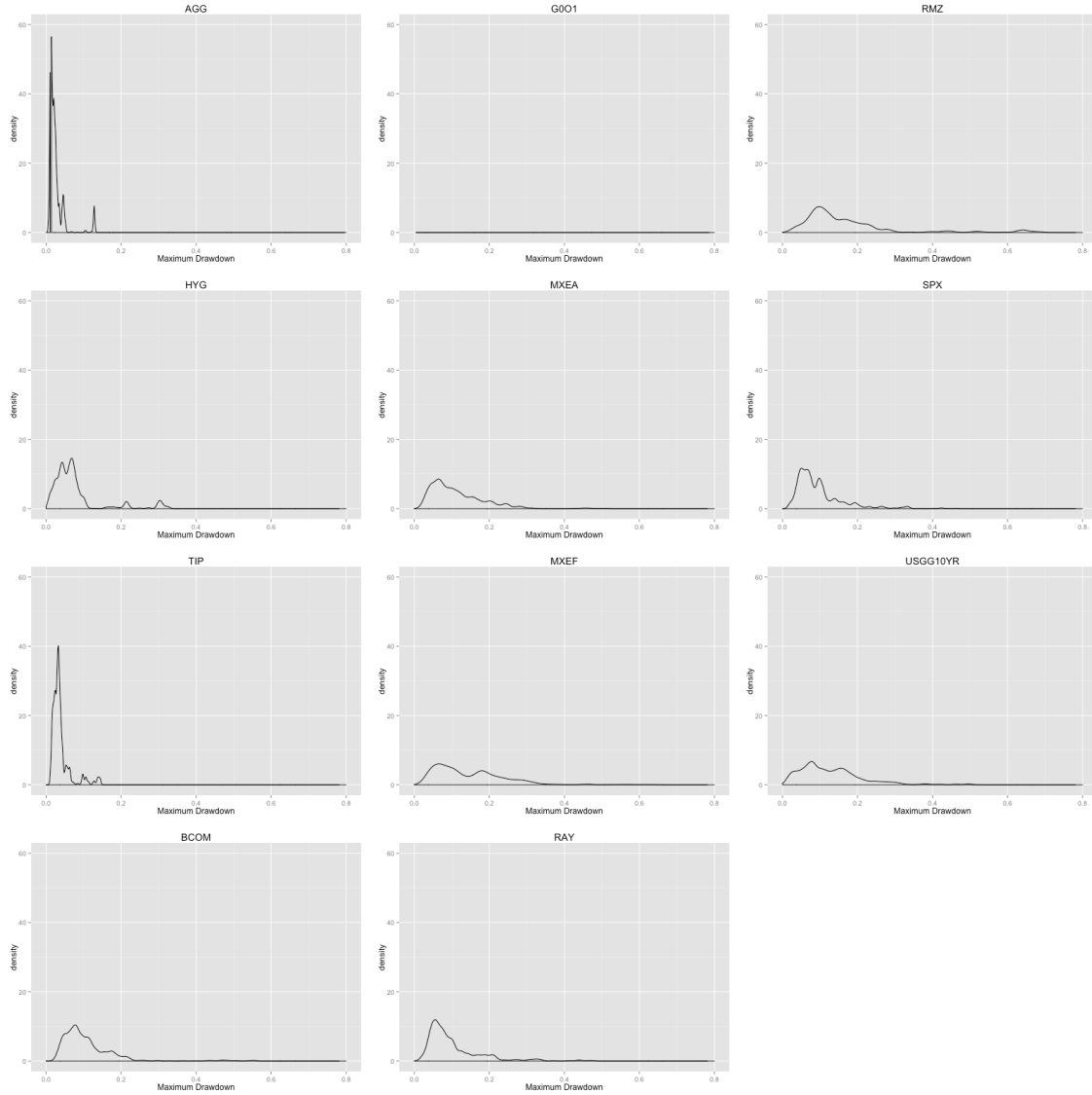
Figure 12: Empirical distribution of maximum drawdown under 6 month rolling window

C Appendix C: Simulations specifics

C.1 ARMA model with normal noise

We use some simple assumptions and parameters for simulation with normal distribution as follows:

1. Noise terms in the time series model follow Normal distribution with standard deviation of 0.01: $\epsilon \sim N(0, 0.0001)$.

2. Risk measures including volatility, VaR, ES and maximum drawdown are calculated based on simulated time series with path length 1000. The calculation of maximum drawdown is replicated 1000 times in order to obtain the maximum drawdown distribution and its tail mean (CED).

3. All time series parameters in this section range from -0.9 to 0.9. For time series with multiple parameters such as AR(2), ARMA(1, 1), the parameters are cartesian product of arithmetic progressions

Table 8: Correlation between CED (confidence level = 0.9) and other risk measures

Measures	Volatility	VaR	ES
AGG	0.94	0.89	0.95
HYG	0.98	0.97	0.97
TIP	0.77	0.85	0.85
BCOM	0.84	0.89	0.89
MXEA	0.84	0.83	0.86
MXEF	0.91	0.91	0.93
RAY	0.92	0.85	0.92
RMZ	0.96	0.96	0.97
SPX	0.84	0.81	0.84
USGG10YR	0.91	0.93	0.95

Table 9: Summary statistics of two regimes for various assets

Asset	Volatility		Skewness		Kurtosis	
	Regime 1	Regime 2	Regime 1	Regime 2	Regime 1	Regime 2
AGG	0.141	0.036	-1.59	0.01	17.15	0.60
HYG	0.265	0.055	0.60	0.00	8.76	0.83
TIP	0.113	0.049	0.18	-0.06	2.50	0.16
BCOM	0.205	0.096	-0.25	-0.04	1.92	0.28
MXEA	0.253	0.101	-0.14	-0.02	4.26	0.25
MXEF	0.303	0.119	-0.08	-0.06	2.39	0.38
RAY	0.307	0.114	-0.39	-0.06	6.43	0.55
RMZ	0.661	0.159	0.29	-0.15	2.92	0.79
SPX	0.260	0.099	-0.43	-0.02	9.11	0.56
USGG10YR	0.314	0.098	0.09	-0.05	2.67	1.17

range from -0.9 to 0.9. Note that to take the stationary of AR and ARMA models into consideration (all moving averages are stationary, but the AR and ARMA model have to meet certain criteria to be stationary), not all parameters in the cartesian product are used in the simulation.

C.2 ARMA model with fat tailed noise

In the simulation with fat tailed noise, parameters used are close to what we've encountered in the empirical studies.

1. White noise terms are simulated using t-distribution (degree of freedom = 4). Financial return usually have fat-tail distributions.

2. Set path length to 63, which is the number of trading days in three month.

3. Time series parameters all range from -0.3 to 0.3, which is similar to the financial time series data in the real world. Usually financial data does not show strong serial correlation. If we could narrow down our scope to smaller serial correlations we may found that the serial correlation is more closely related with various risk measures.

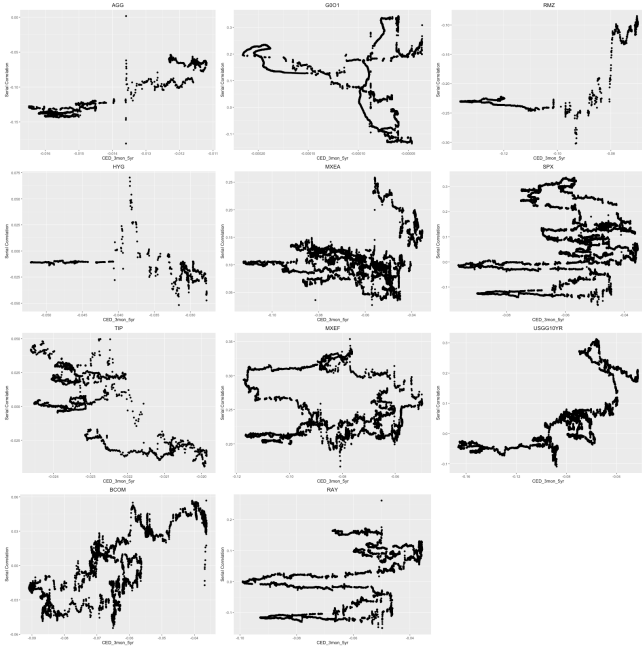## D   Appendix D: Empirical Study Results

Figure 13: First-order serial correlation calculated using AR(1) model versus CED
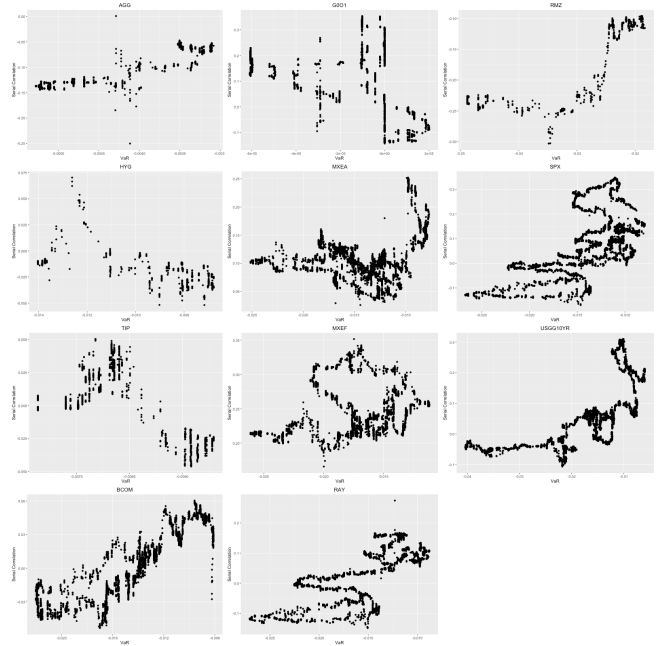
Figure 14: First-order serial correlation calculated using AR(1) model versus VaR

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