

Drawdown Project - Simulation

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1 Noise term with normal distribution

To better understand the behaviour of conditional expected drawdown (CED) and the maximum drawdown distribution, we use some simple assumptions and parameters in this section as follows:

1. Noise term in the time series model follows Normal distribution with standard deviation of 0.01: $\epsilon \sim N(0, 0.0001)$.
2. Risk measures including volatility, VaR, ES and maximum drawdown are calculated based on simulated time series with path length 1000. The calculation of maximum drawdown is replicated 1000 times in order to obtain the maximum drawdown distribution and its tail mean (CED).
3. All time series parameters in this section range from -0.9 to 0.9. For time series with multiple parameters such as AR(2), ARMA(1, 1), the parameters are cartesian product of arithmetic progressions range from -0.9 to 0.9. Note that to take the stationary of AR and ARMA models into consideration (all moving averages are stationary, but the AR and ARMA model have to meet certain criteria to be stationary), not all parameters in the cartesian product are used in the simulation.

1.1 AR(1)

We started from the simplest model AR(1):

$$X_t = \kappa_1 X_{t-1} + \epsilon_t \quad (1)$$

We simulate AR(1) for various values of the autoregressive parameter for $\kappa_1 \in (-1, 1)$. Figure 1 shows the relationship between AR(1) coefficient and risk measures of interest including ES, VaR, volatility and CED. Note that for AR(1) model, the order one serial correlation is the value of autoregressive parameter.

CED shows a decreasing trend when $\kappa_1 \in (-1, -0.75)$ and an increasing trend when $\kappa_1 \in (-0.75, 1)$. As shown in Figure 1, it becomes feasible for us to distinguish negative and positive serial correlation using the CED values. However, the other 3 risk measures are all symmetric about $\kappa_1 = 0$, and they increase as the absolute values of κ_1 increases.

For VaR, ES and CED, the derivative of risk measure values to κ_1 approaches to zero as κ_1 goes to 0 and increase as κ_1 increase. This suggests that the value of this three risk measures can hardly reflect the change of serial correlation when the serial correlation is small. And they perform better when the serial correlation increases. The trend of derivatives reverse for CED. While the change of κ_1 has a comparatively larger influence around 0, the influence becomes weaker as we move to larger κ_1 values.

Figure 2 shows the maximum drawdown distribution for various κ_1 . Same as revealed in Figure 1, the mean and tail mean of maximum drawdown distribution increases as we increase κ_1 from negative values to positive values.

Table 1 shows the basic summary statistics of the simulated return sequences. The standard deviation increases as we increase the absolute values of the coefficient. The skewness and kurtosis are close to zeros. Kolmogorov-Smirnov Tests indicate that they all follow normal distribution.

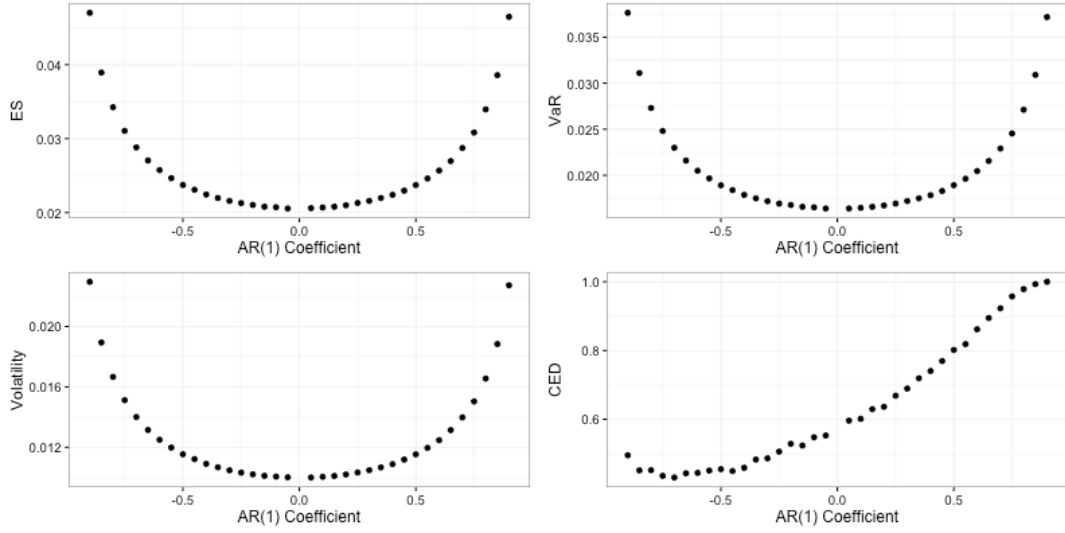


Figure 1: AR(1): Relationship between auto-correlation coefficients and risk measures (Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

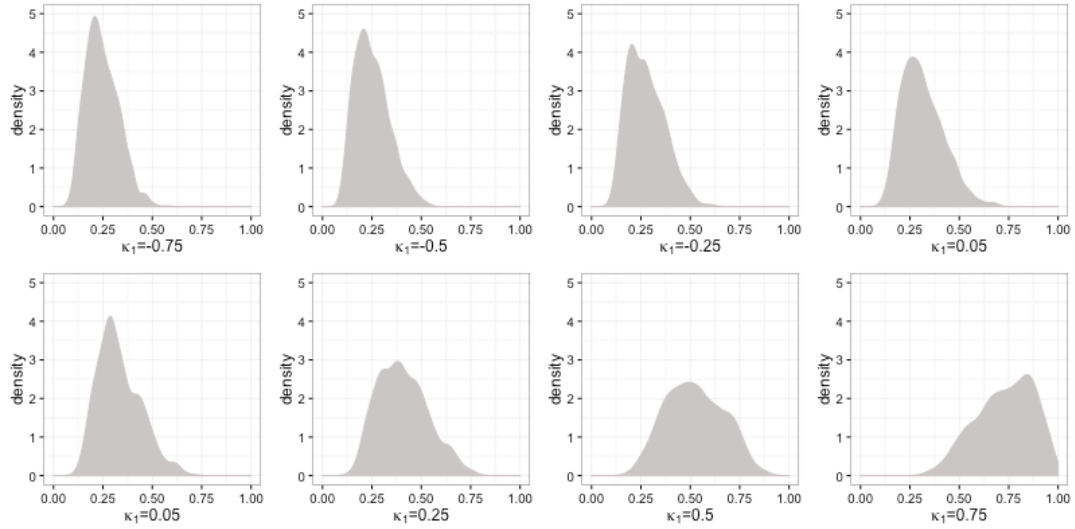


Figure 2: AR(1): Maximum drawdown distribution for various κ_1 values (Empirical distribution, path length = 1000, sample size = 1000, $\epsilon_t \sim N(0, 0.0001)$)

	Mean	Sd	Sknewness	Kurtosis
$\kappa_1 = -0.75$	0.0	0.015	0.009	0.032
$\kappa_1 = -0.5$	0.0	0.012	0.006	0.021
$\kappa_1 = -0.25$	0.0	0.010	0.001	0.010
$\kappa_1 = 0.25$	0.0	0.010	-0.009	0.004
$\kappa_1 = 0.5$	0.0	0.012	-0.012	-0.008
$\kappa_1 = 0.75$	0.0	0.015	0.013	0.016

Table 1: Statistics of simulated distribution of AR(1)

1.2 AR(2)

We then move to model AR(2):

$$X_t = \kappa_1 X_{t-1} + \kappa_2 X_{t-2} + \epsilon_t \quad (2)$$

Figure 3 and 4 shows the relationship between order one serial correlation ρ_1 and various risk measures. With fixed κ_2 , the relationship between serial correlation and various risk measures are close to AR(1) model. The curve shows some discontinuity mainly because the stationary condition for which we have to drop some possible combination of coefficients. Nevertheless, Figure 3 and 4 shows more complexity in that as we increase the κ_2 value, the values of all four risk measures increases.

Figure 7 and 8 shows the maximum drawdown distribution as we fix κ_2 to 0.2 and -0.2 respectively and change κ_1 . Again the maximum drawdown distribution shows monotonicity as we increase κ_1 from negative to positive for both $\kappa_2 = 0.2$ and $\kappa_2 = -0.2$. It would be the similar if we fix κ_1 and examine the maximum drawdown distribution for different κ_2 values.

Table 2 shows the summary statistics for the simulated return sequences. As indicated above, the standard deviation value increases as we increase the absolute values of the coefficient. The skewness and kurtosis are close to zeros.

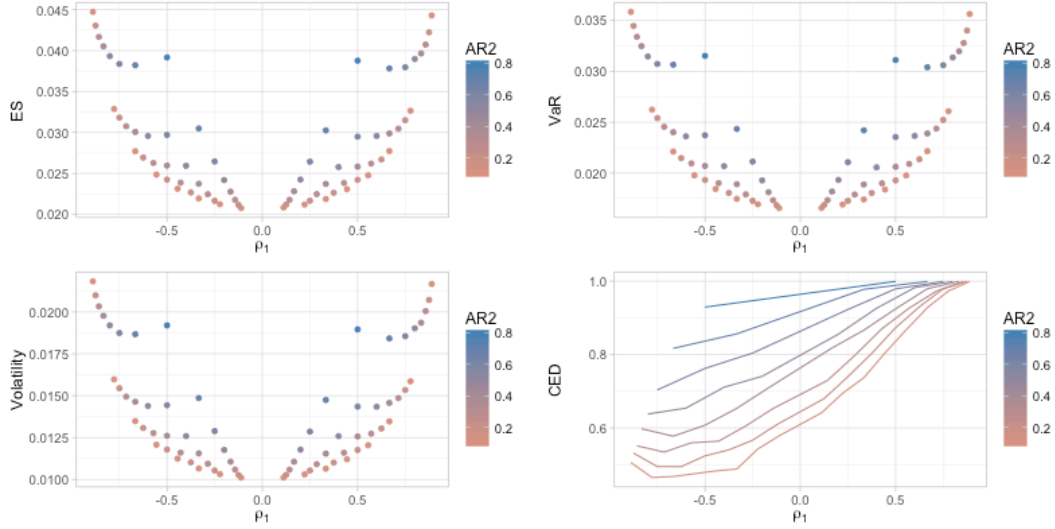


Figure 3: AR(2): Relationship between serial correlation ρ_1 and risk measures ($\kappa_2 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

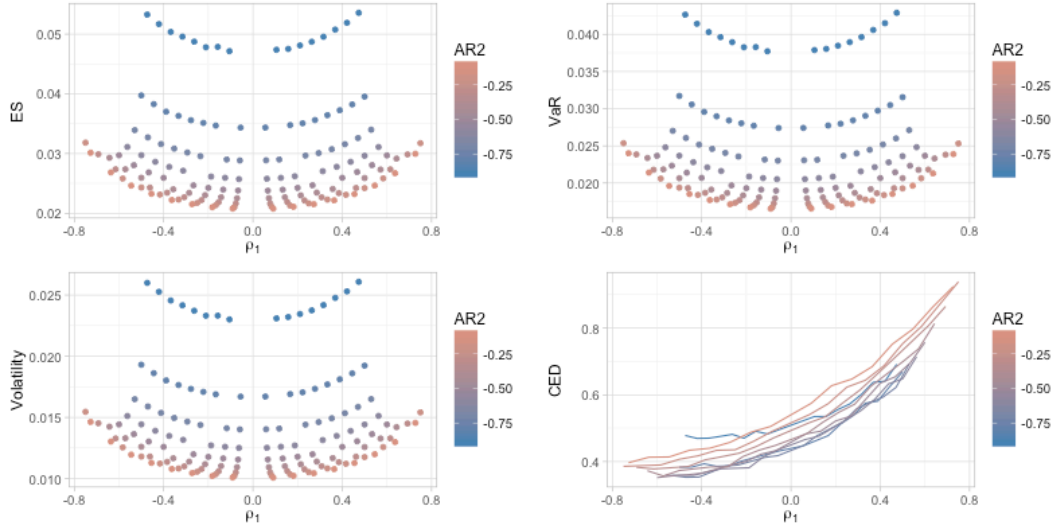


Figure 4: AR(2): Relationship between serial correlation ρ_1 and risk measures ($\kappa_2 < 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

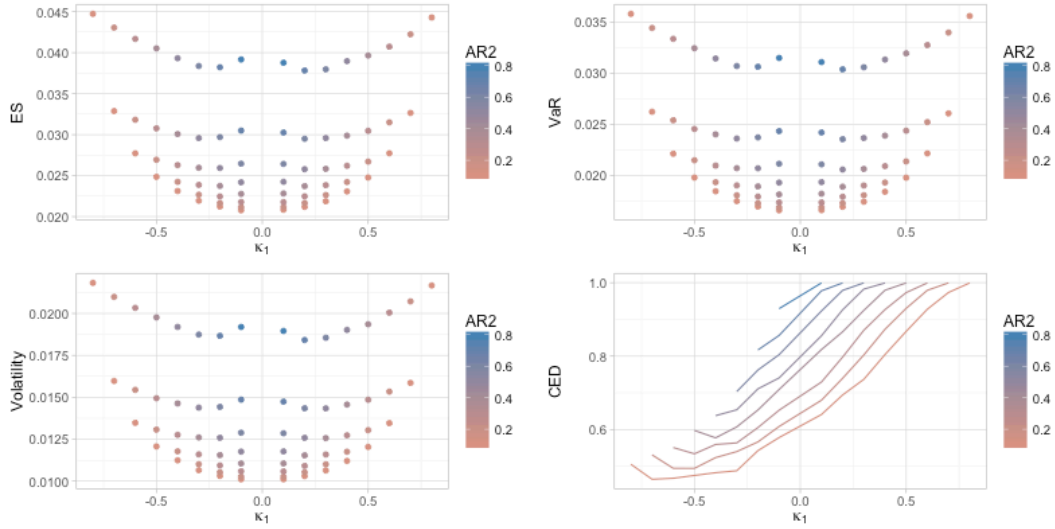


Figure 5: AR(2): Relationship between model coefficients and risk measures ($\kappa_2 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

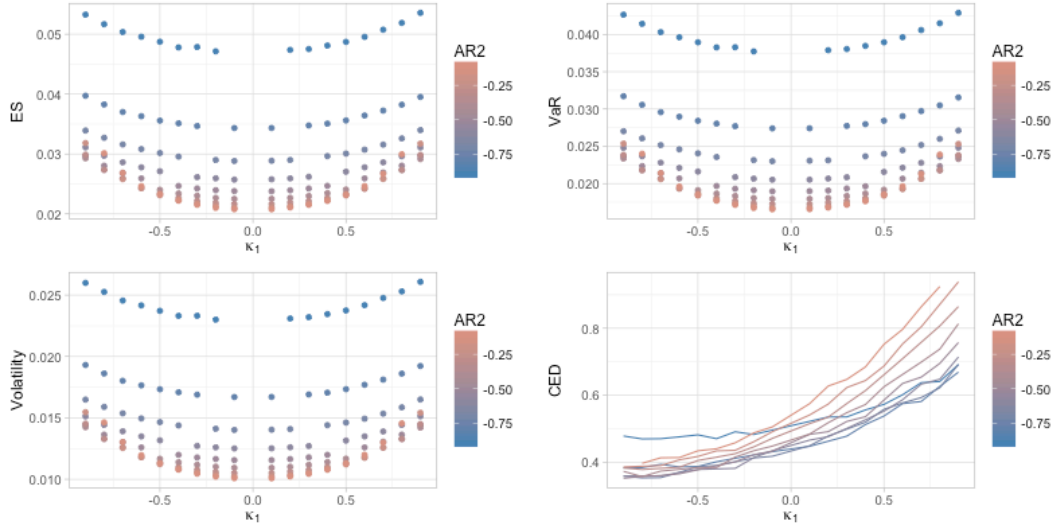


Figure 6: AR(2): Relationship between model coefficients and risk measures ($\kappa_2 < 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

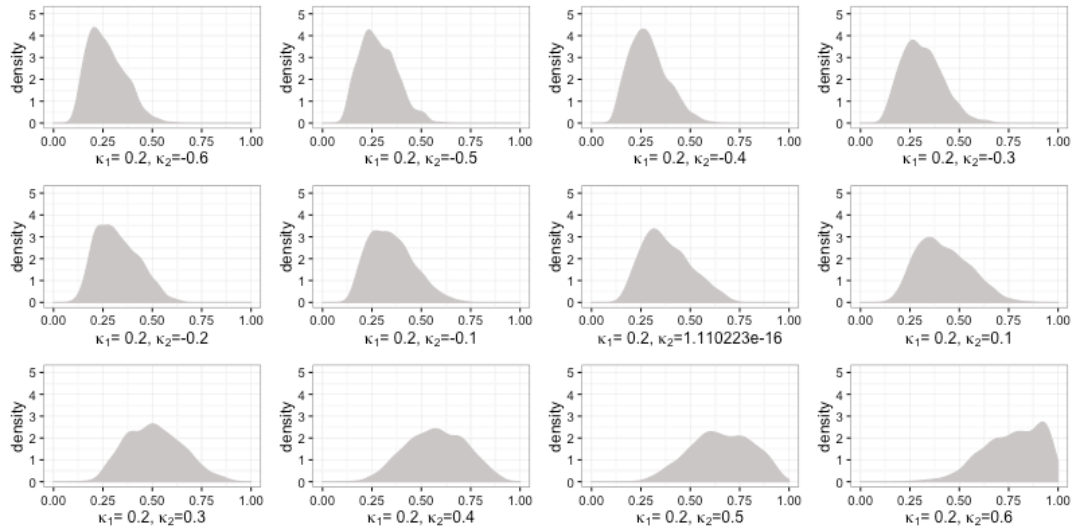


Figure 7: AR(2): Maximum drawdown distribution for $\kappa_2 = 0.2$
(Empirical distribution, path length = 1000, sample size = 1000, $\epsilon_t \sim N(0, 0.0001)$)

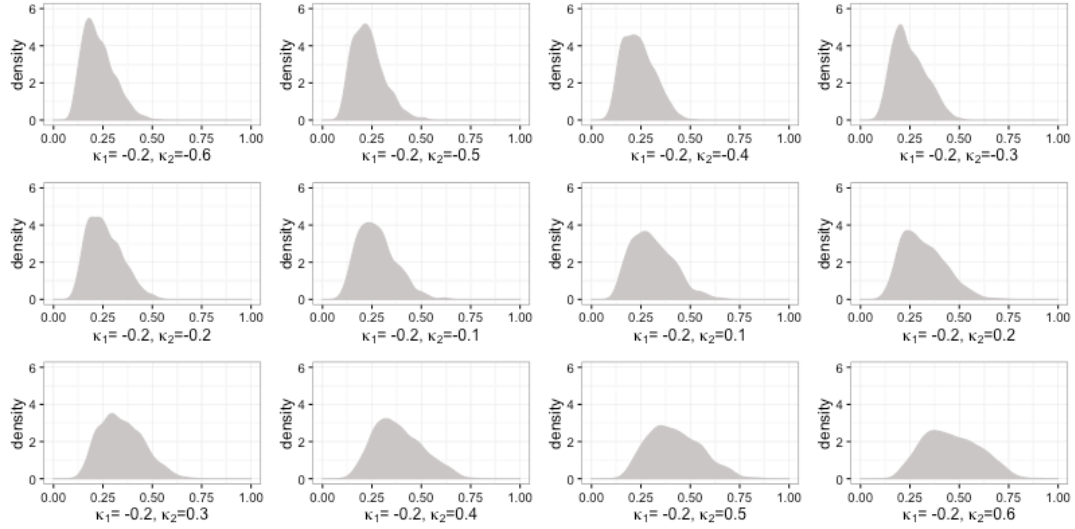


Figure 8: AR(2): Maximum drawdown distribution for $\kappa_{-} = -0.2$
(Empirical distribution, path length = 1000, sample size = 1000, $\epsilon_t \sim N(0, 0.0001)$)

	Mean	Sd	Skewness	Kurtosis
$\kappa_1 = -0.75, \kappa_2 = 0.2$	0.0	0.029	-0.005	0.039
$\kappa_1 = -0.5, \kappa_2 = 0.2$	0.0	0.013	0.000	0.004
$\kappa_1 = -0.25, \kappa_2 = 0.2$	0.0	0.011	0.003	-0.021
$\kappa_1 = 0.25, \kappa_2 = 0.2$	0.0	0.011	0.003	-0.008
$\kappa_1 = 0.5, \kappa_2 = 0.2$	0.0	0.013	0.012	-0.010
$\kappa_1 = 0.75, \kappa_2 = 0.2$	0.0	0.029	-0.001	-0.021

Table 2: Statistics of simulated distribution of AR(2)

1.3 MA(1)

For simplest moving average model MA(1):

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} \quad (3)$$

Figure 9 presents the relationship between four risk measures and serial correlation ρ_1 . Similar with autoregressive models, CED is a informative risk measures for distinguish positive and negative serial correlations. The curve of ρ_1 versus CED is convex for positive serial correlation and concave for negative serial correlation. With means the rate of change of CED to ρ_1 is smaller for small serial correlation values and increases for larger serial correlation values. Figure 10 shows the relationship between risk measures and the moving average coefficient θ_1 . Unlike Figure 9, the MA(1) coefficient shows a relationship close to linear with the change of CED.

Figure 11 shows the maximum drawdown distribution as we increase θ_1 . Same as revealed in the tail mean (CED) of the distribution. The maximum drawdown distribution monotonic increase as θ_1 increases.

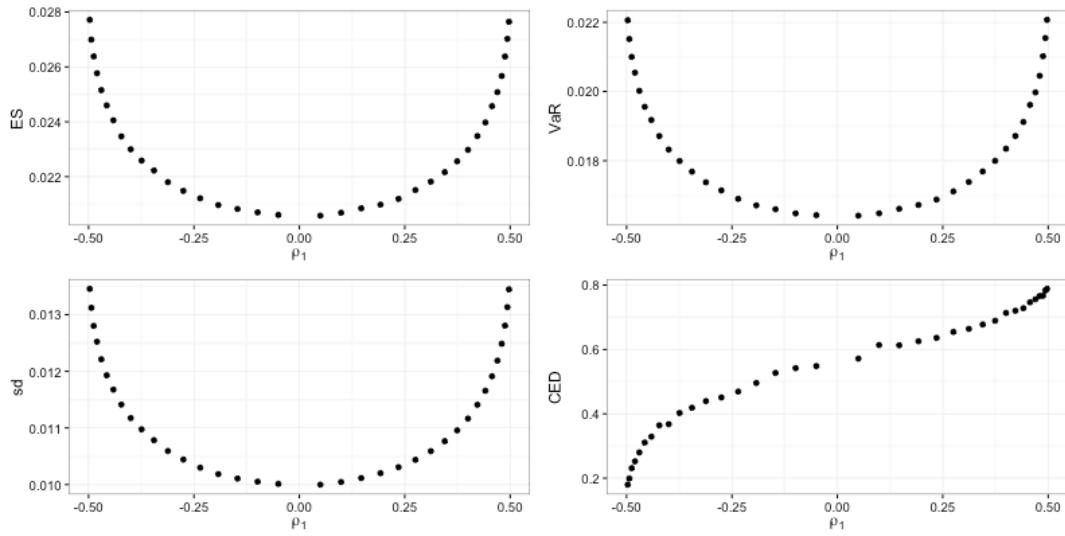


Figure 9: MA(1): Relationship between serial correlation ρ_1 and risk measures
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

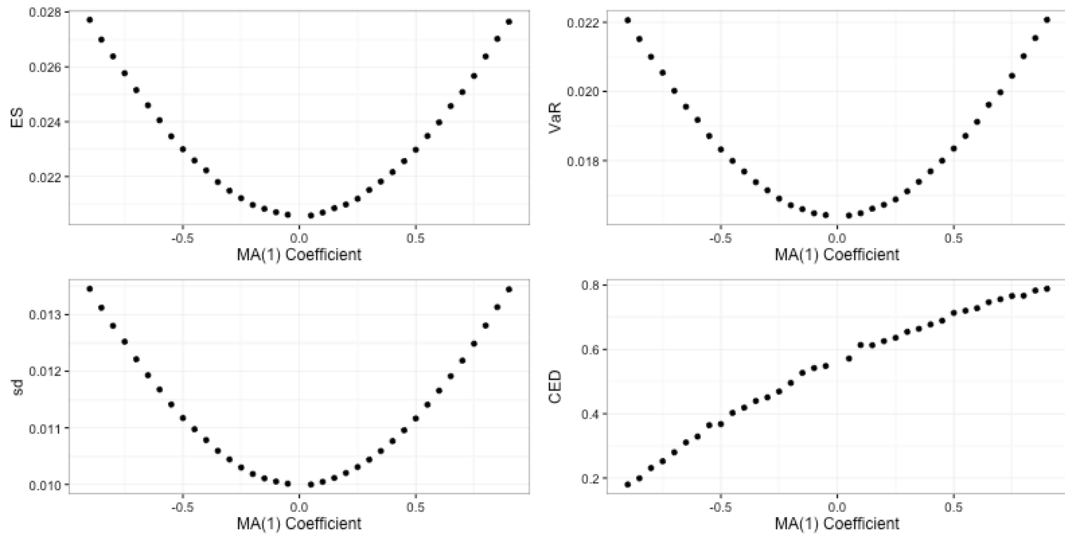


Figure 10: MA(1): Relationship between model coefficient and risk measures
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

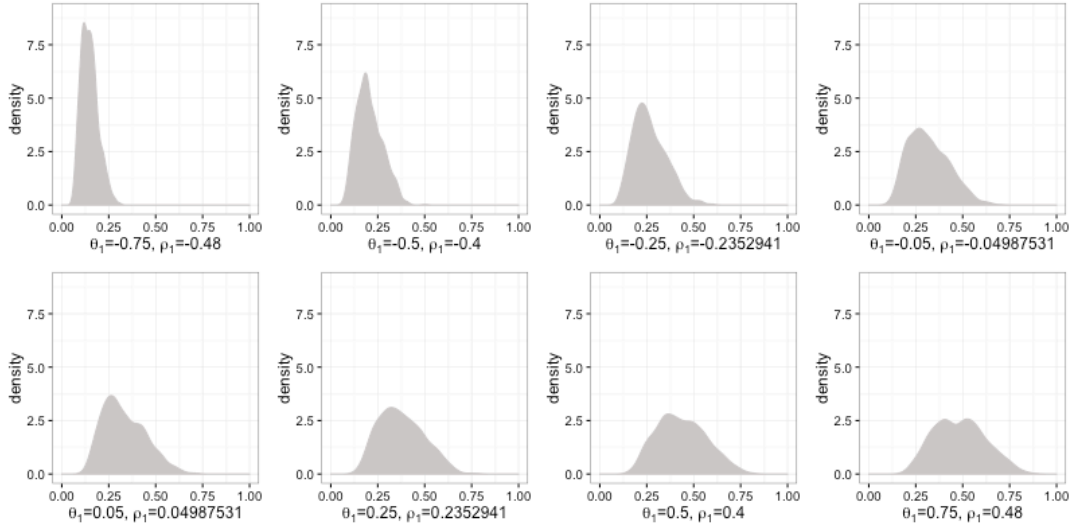


Figure 11: MA(1): Maximum drawdown distribution for various θ_1 values
(Empirical distribution, path length = 1000, sample size = 1000, $\epsilon_t \sim N(0, 0.0001)$)

	Mean	Sd	Skewness	Kurtosis
$\theta_1 = -0.75$	0.0	0.012	-0.002	-0.008
$\theta_1 = -0.5$	0.0	0.011	-0.004	-0.009
$\theta_1 = -0.25$	0.0	0.010	0.001	-0.020
$\theta_1 = 0.25$	0.0	0.010	0.011	0.023
$\theta_1 = 0.5$	0.0	0.011	0.000	0.006
$\theta_1 = 0.75$	0.0	0.013	-0.003	0.012

Table 3: Statistics of simulated distribution of MA(1)

1.4 MA(2)

The expression of MA(2) model is:

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} \quad (4)$$

As shown in Figure 12 and Figure 4, when fix θ_2 , the relationship between serial correlation ρ_1 and risk measures resembles that of the MA(1) model. When we increase the absolute values of ρ_2 , the value of ES, VaR and volatility increases. However, there are more complicated relationship between serial correlation and CED. For positive θ_2 , CED increase as θ_2 increases. The pattern seems less interpretable for negative θ_2 values.

Figure 16 shows the change of maximum drawdown distribution when we fix $\theta_1 = 0.2$, and Figure 17 shows the maximum drawdown distribution when we fix $\theta_2 = 0.2$. When fixing one parameter and change the other parameter value from negative to positive, the maximum drawdown distributions are monotonic increasing, which indicates that the tail mean of the distribution (CED) increases.

Same as the AR(1), AR(2), and MA(1) model above, the mean, skewness and kurtosis is small for our simulated return sequences. The standard deviation increase as we increase the absolute value of the coefficient, which can be explained by the theoretical calculation.

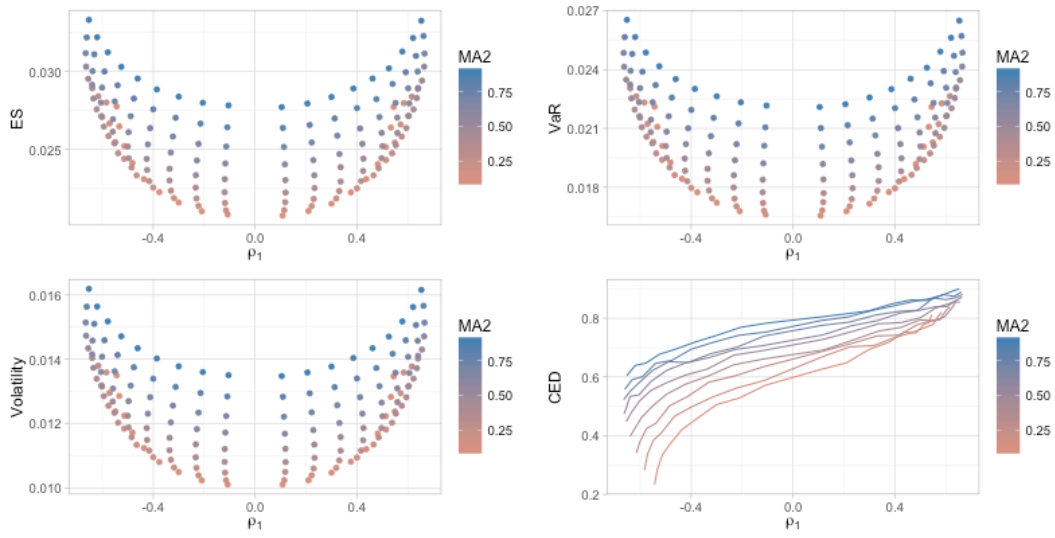


Figure 12: MA(2): Relationship between serial correlation ρ_1 and risk measures ($\theta_2 > 0$) (Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

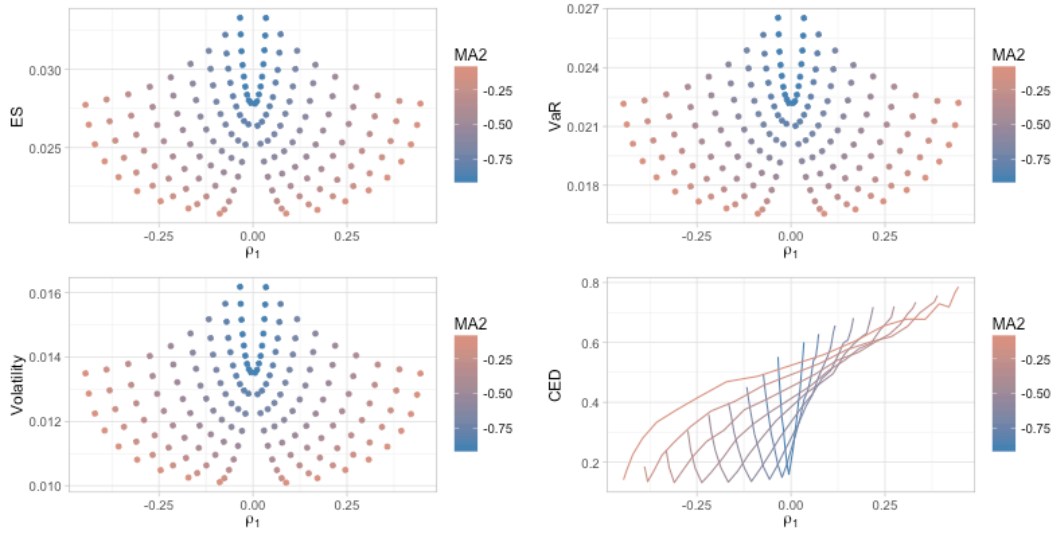


Figure 13: MA(2): Relationship between serial correlation ρ_1 and risk measures ($\theta_2 < 0$) (Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

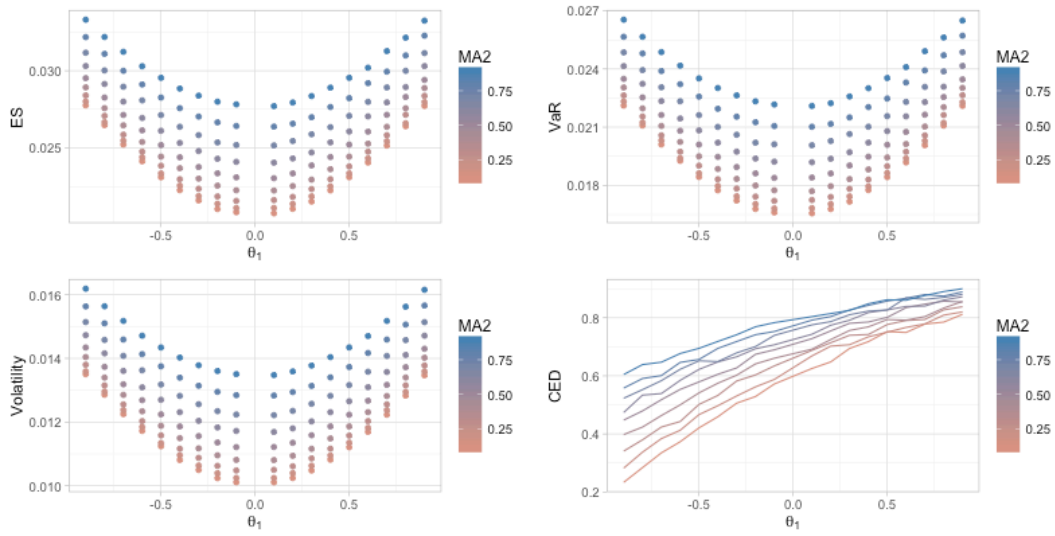


Figure 14: MA(2): Relationship between model coefficients and risk measures ($\theta_2 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

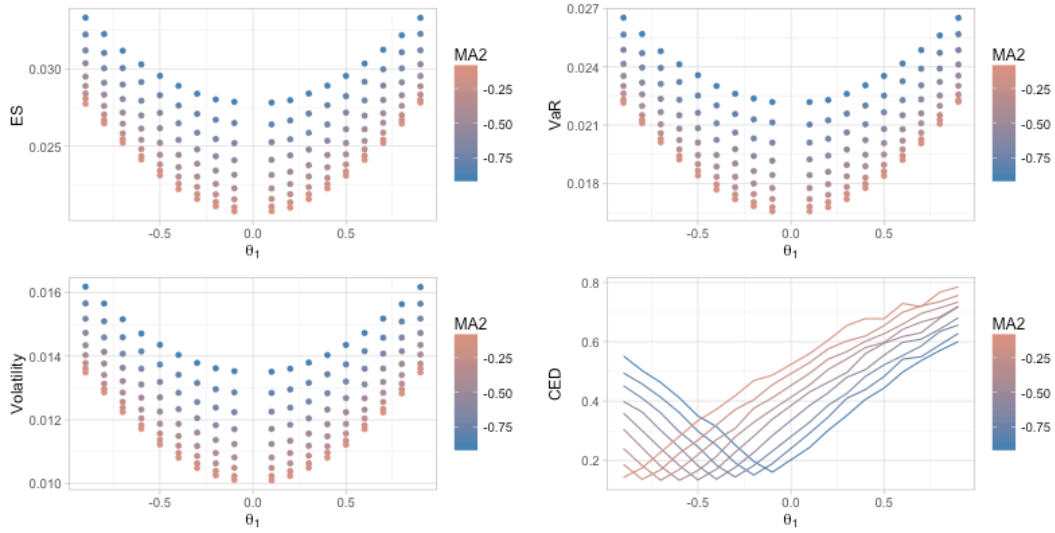


Figure 15: MA(2): Relationship between model coefficients and risk measures ($\theta_2 < 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

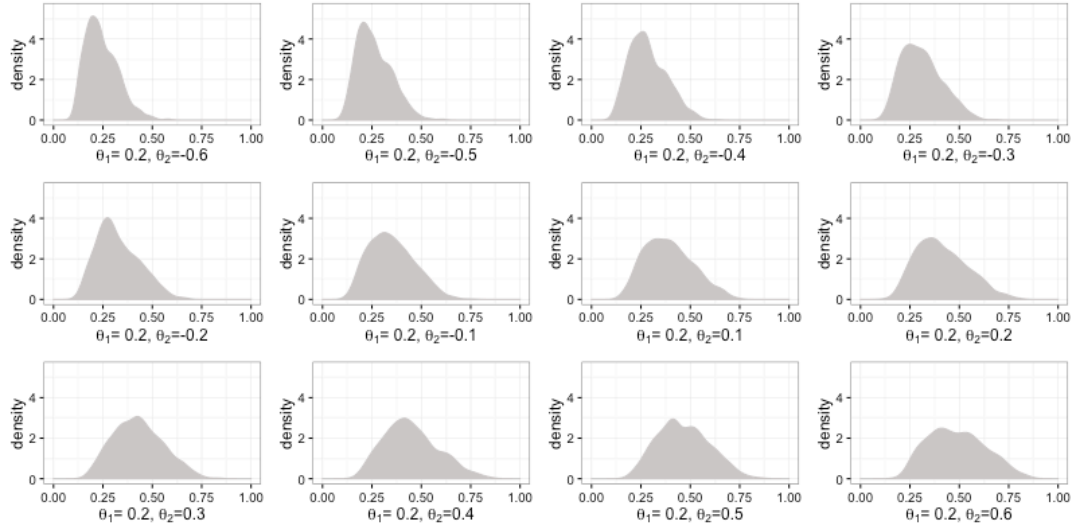


Figure 16: MA(2): Maximum drawdown distribution for $\theta_1 = 0.2$
(Empirical distribution, path length = 1000, sample size = 1000, $\epsilon_t \sim N(0, 0.0001)$)

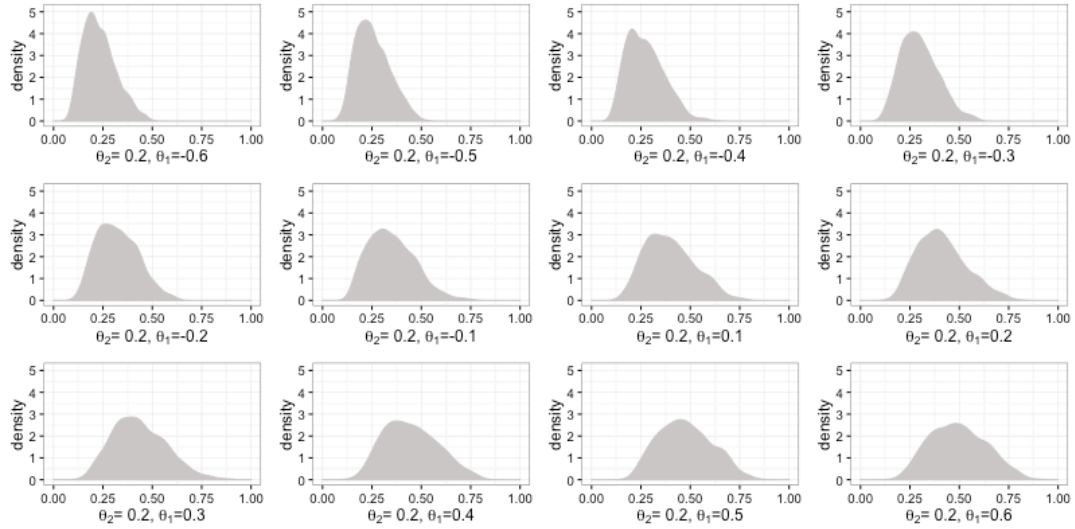


Figure 17: MA(2): Maximum drawdown distribution for $\theta_2 = 0.2$
(Empirical distribution, path length = 1000, sample size = 1000, $\epsilon_t \sim N(0, 0.0001)$)

	Mean	Sd	Skewness	Kurtosis
$\theta_1 = -0.75, \theta_2 = 0.2$	0.0	0.013	0.004	-0.015
$\theta_1 = -0.5, \theta_2 = 0.2$	0.0	0.011	-0.006	-0.002
$\theta_1 = -0.25, \theta_2 = 0.2$	0.0	0.011	-0.004	-0.012
$\theta_1 = 0.25, \theta_2 = 0.2$	0.0	0.010	-0.020	-0.007
$\theta_1 = 0.5, \theta_2 = 0.2$	0.0	0.011	0.009	0.012
$\theta_1 = 0.75, \theta_2 = 0.2$	0.0	0.013	-0.015	0.025

Table 4: Statistics of simulated distribution of MA(2)

1.5 ARMA(1, 1)

For the most simple ARMA model:

$$X_t = \kappa_1 X_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} \quad (5)$$

Figure 18 and Figure 19 shows the relationship between serial correlation ρ_1 and risk measures for positive and negative θ_1 respectively. For positive θ_1 , the time series is more likely to be positively correlated while for negative θ_1 it is more likely to be negative correlated. Interestingly, for positive θ_1 values, the relationship between CED and serial correlation ρ_1 is not influenced by the value of ρ_1 . CED shows an increasing trend as we increase the value of serial correlation. Similar as the above subsections for other time series models, CED is a good tool to distinguish the positive and negative serial correlation.

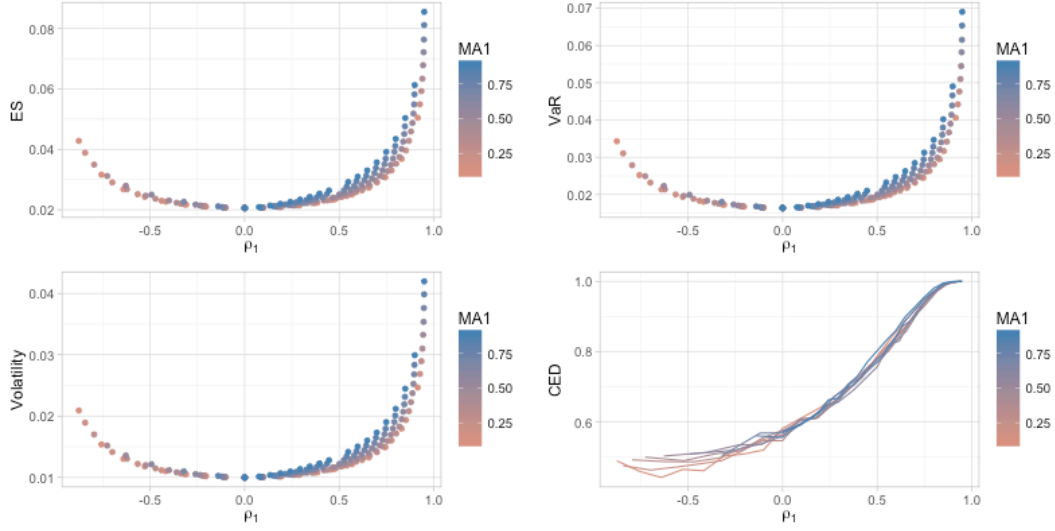


Figure 18: ARMA(1, 1): Relationship between serial correlation ρ_1 and risk measures ($\theta_1 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

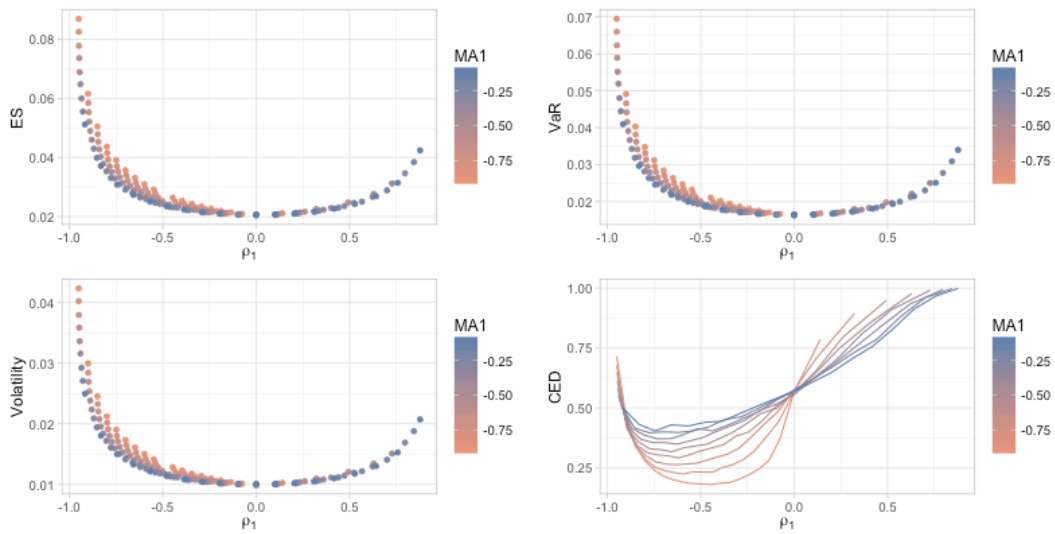


Figure 19: ARMA(1, 1): Relationship between serial correlation ρ_1 and risk measures ($\theta_1 < 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

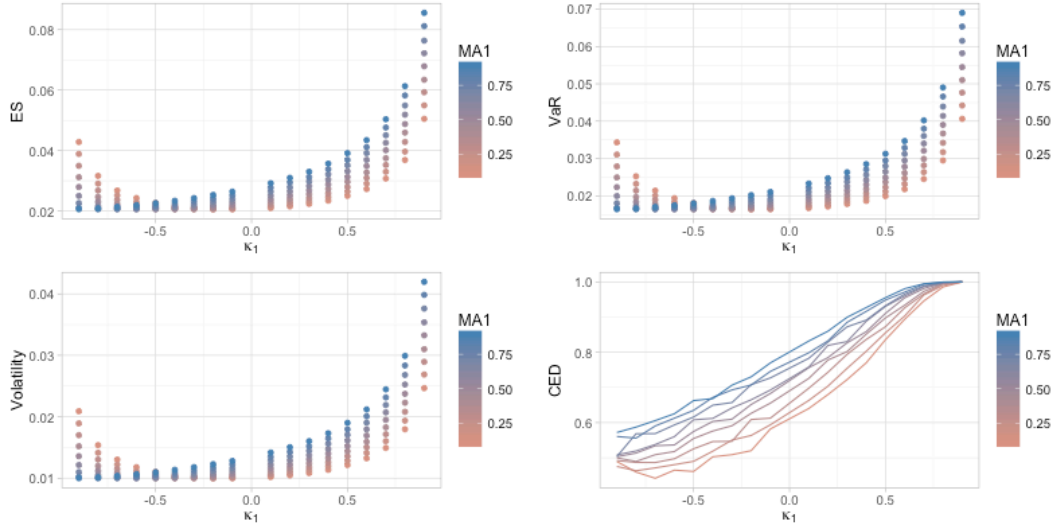


Figure 20: ARMA(1, 1): Relationship between model coefficient and risk measures ($\theta_1 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

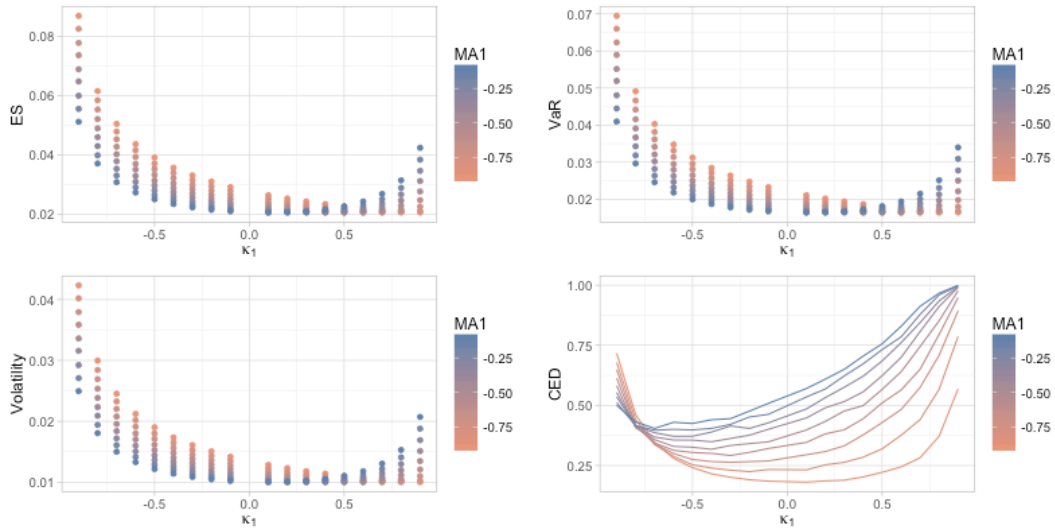


Figure 21: ARMA(1, 1): Relationship between model coefficient and risk measures ($\theta_1 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

2 Noise term with student t distribution

In this section, parameters used are close to what we've encountered in the empirical studies.

1. White noise terms are simulated using t-distribution (degree of freedom = 4). Financial return usually have fat-tail distributions.
2. Set path length to 63, which is the number of trading days in three month.
3. Time series parameters all range from -0.3 to 0.3, which is similar to the financial time series data in the real world. Usually financial data does not show strong serial correlation. If we could narrow down our scope to smaller serial correlations we may found that the serial correlation is more closely related with various risk measures.

The relationship between serial correlation and risk measures revealed in this section resembles that of the last section where noise terms follow normal distribution. There are some differences due to the heavy-tail distribution. We illustrate more in the specific model part.

2.1 AR(1)

Models using time series with t-distribution white noise show more randomness as shown in Figure 5. The relationship between AR(1) coefficients (which is the serial correlation ρ_1) and risk measures is similar to the model with normal distribution. However, the points are not lies prefectly in a curve probably because the heavy tail introduce more randomness of the quantile and tail mean values.

Table 5 shows the summary statisctics of the simulated return distribution. Their mean and skewness are close to zeor. And they have high kurtosis which implies the fat tail.

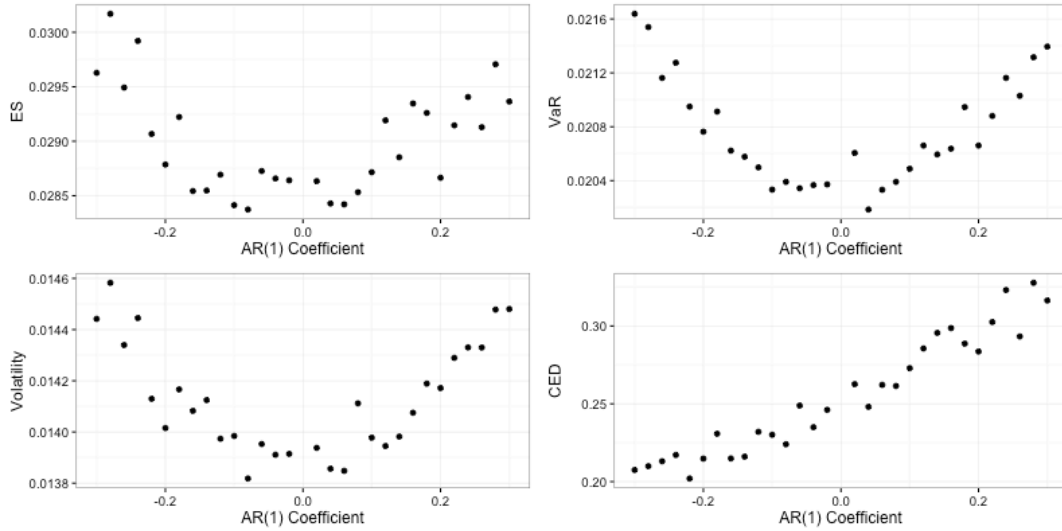


Figure 22: AR(1): Relationship between auto-correlation coefficients and risk measures (Simulation path length: 63, $\epsilon_t \sim 0.01T(df = 4)$)

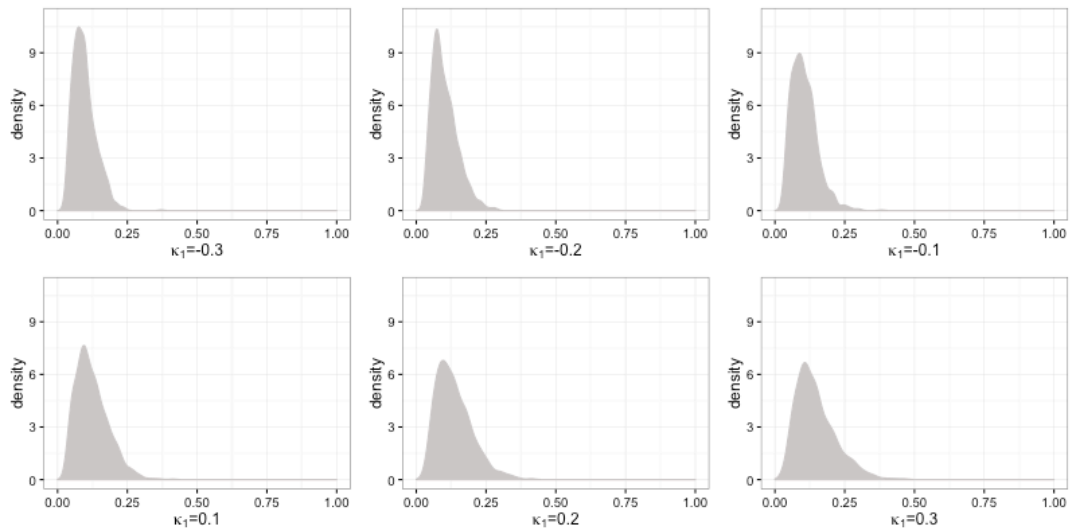


Figure 23: AR(1): Maximum drawdown distribution for various κ_1 values (Empirical distribution, path length = 63, sample size = 1000, $\epsilon_t \sim 0.01T(df = 4)$)

	Mean	Sd	Skewness	Kurtosis
$\kappa_1 = -0.3$	0.0	0.015	-0.53	18.8
$\kappa_1 = -0.2$	0.0	0.014	-0.13	6.4
$\kappa_1 = -0.1$	0.0	0.014	-0.05	6.3
$\kappa_1 = 0.1$	0.0	0.014	-0.30	15.8
$\kappa_1 = 0.2$	0.0	0.014	0.13	7.0
$\kappa_1 = 0.3$	0.0	0.015	-0.06	7.4

Table 5: Statistics of simulated distribution of AR(1)

2.2 AR(2)

Same as presented in the last section, ES, VaR and volatility increases as we increase the absolute values of κ_2 . And CED increases as we change κ_2 from negative to positive. This shows that CED values are inherently not symmetric not only for the serial correlation but for the coefficients. The summary statistics in Table 6 also indicate the fat tail distribution.

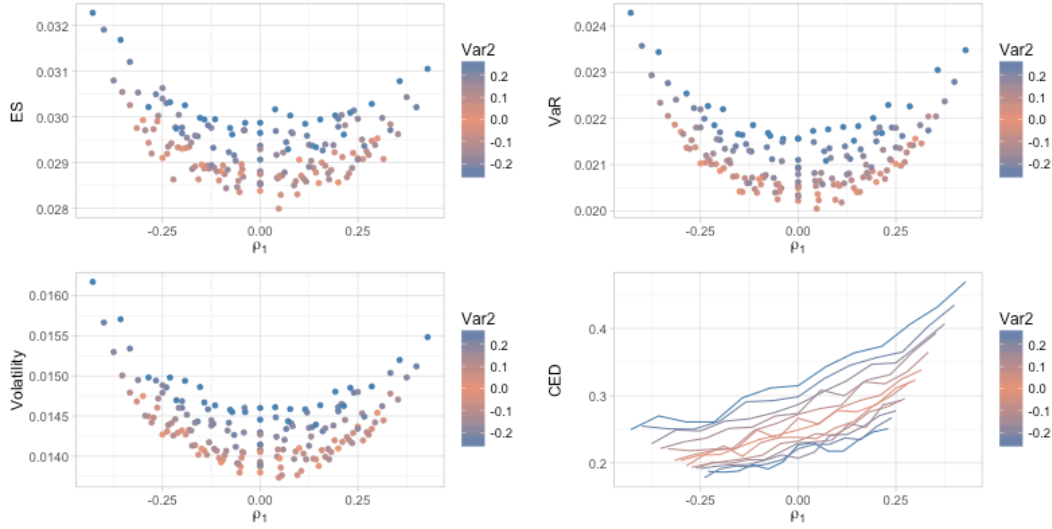


Figure 24: AR(2): Relationship between auto-correlation coefficients and risk measures (Simulation path length: 63, $\epsilon_t \sim 0.01T(df = 4)$)

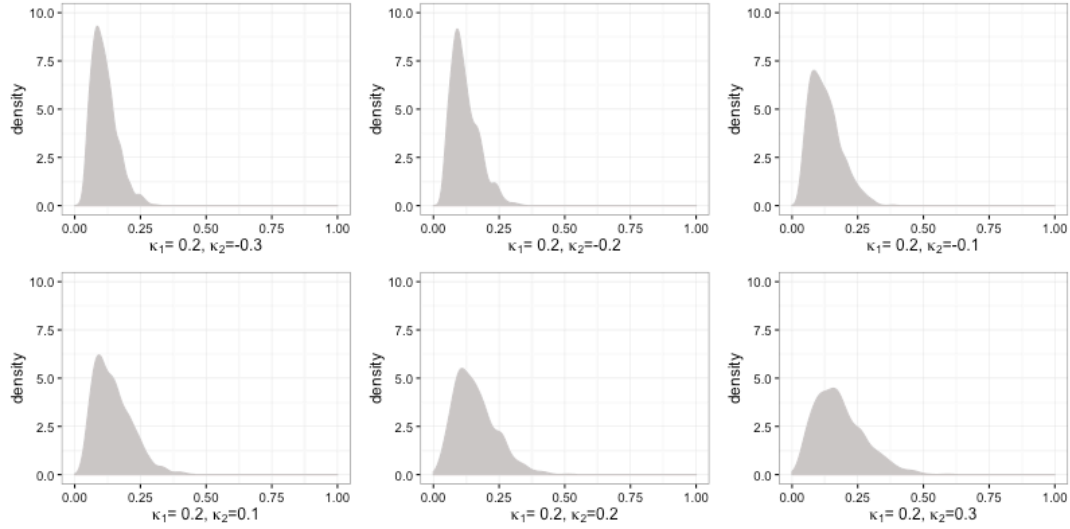


Figure 25: AR(2): Maximum drawdown distribution for various κ_2 values
(Empirical distribution, path length = 63, sample size = 1000, $\epsilon_t \sim 0.01T(df = 4)$)

	Mean	Sd	Skewness	Kurtosis
$\kappa_1 = 0.1, \kappa_2 = -0.3$	0.0	0.015	-0.024	16.7
$\kappa_1 = 0.1, \kappa_2 = -0.2$	0.0	0.014	-0.030	8.0
$\kappa_1 = 0.1, \kappa_2 = -0.1$	0.0	0.014	-0.261	16.7
$\kappa_1 = 0.1, \kappa_2 = 0.1$	0.0	0.014	-0.461	18.2
$\kappa_1 = 0.1, \kappa_2 = 0.2$	0.0	0.014	0.028	6.0
$\kappa_1 = 0.1, \kappa_2 = 0.3$	0.0	0.015	-0.009	5.8

Table 6: Statistics of simulated distribution of AR(2)

2.3 MA(1)

Again the relationship between serial correlation and risk measures for MA(1) model greatly resembles the relationship in last section for the normal distributed noise term. The difference lies in the large randomness caused by the fat tail.

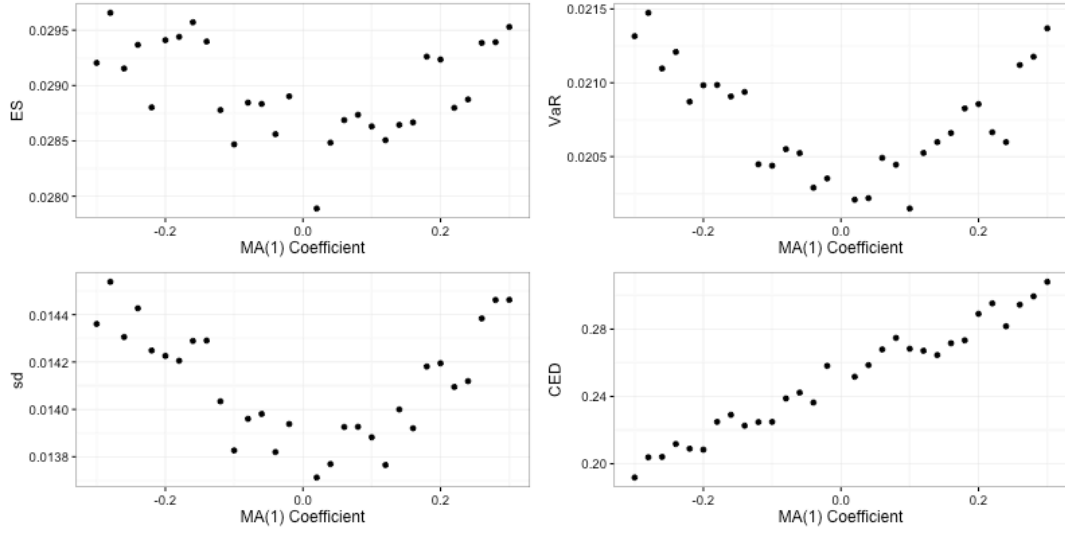


Figure 26: MA(1): Relationship between serial correlation ρ_1 and risk measures
(Simulation path length: 63, $\epsilon_t \sim 0.01T(df = 4)$)

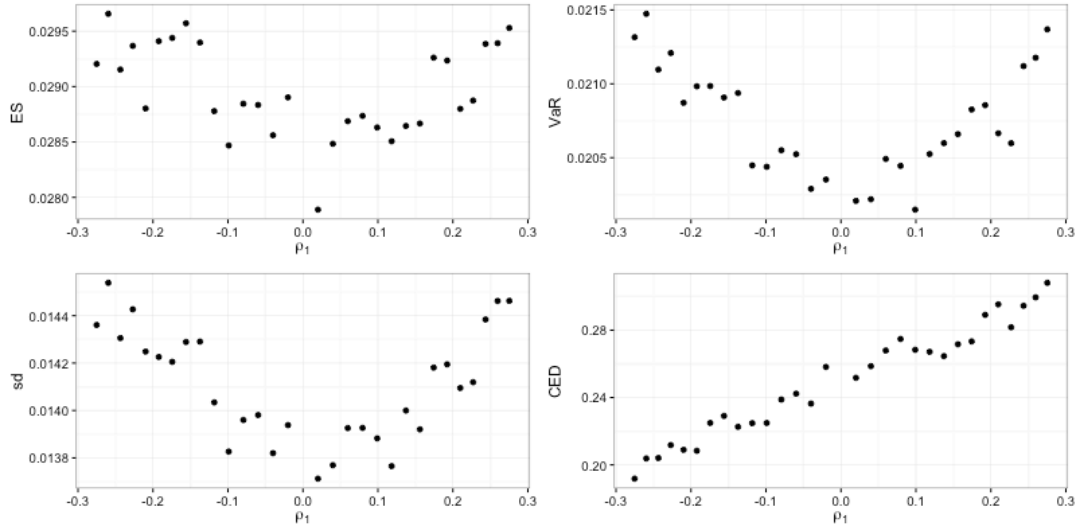


Figure 27: MA(1): Relationship between model coefficient and risk measures
(Simulation path length: 63, $\epsilon_t \sim 0.01T(df = 4)$)

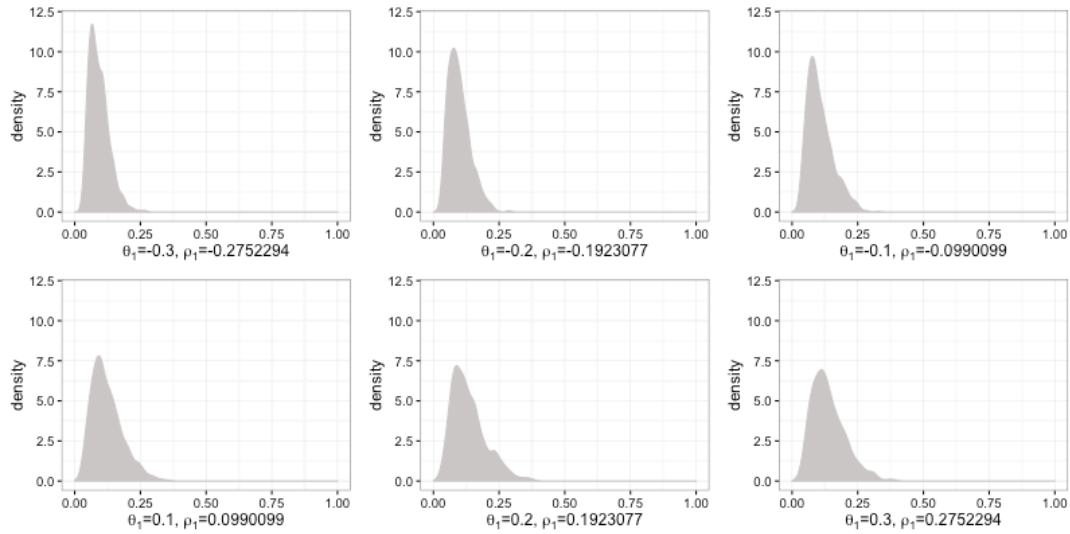


Figure 28: MA(1): Maximum drawdown distribution for various κ_1 values (Empirical distribution, path length = 63, sample size = 1000, $\epsilon_t \sim 0.01T(df = 4)$)

	Mean	Sd	Skewness	Kurtosis
$\kappa_1 = 0.1, \kappa_2 = -0.3$	0.0	0.015	-0.401	22.9
$\kappa_1 = 0.1, \kappa_2 = -0.2$	0.0	0.014	-0.312	12.3
$\kappa_1 = 0.1, \kappa_2 = -0.1$	0.0	0.014	-0.204	8.5
$\kappa_1 = 0.1, \kappa_2 = 0.1$	0.0	0.014	-0.009	9.7
$\kappa_1 = 0.1, \kappa_2 = 0.2$	0.0	0.014	-0.080	7.2
$\kappa_1 = 0.1, \kappa_2 = 0.3$	0.0	0.015	-0.136	12.5

Table 7: Statistics of simulated distribution of MA(1)

2.4 MA(2)

Since we have narrowed down our scope to time series coefficients which range between -0.3 and 0.3. Here we can see that for fixed value of θ_2 , the relationship between serial correlation and CED is approximately a line. And for the other three risk measures, the relationship seems to have more randomness than using normal distribution.

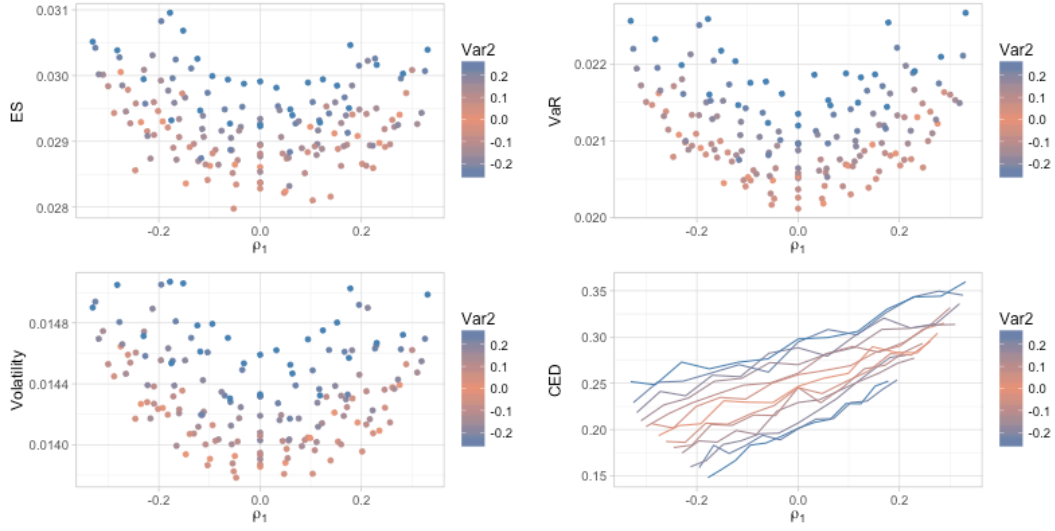


Figure 29: MA(1): Relationship between serial correlation ρ_1 and risk measures (Simulation path length: 63, $\epsilon_t \sim 0.01T(df = 4)$)

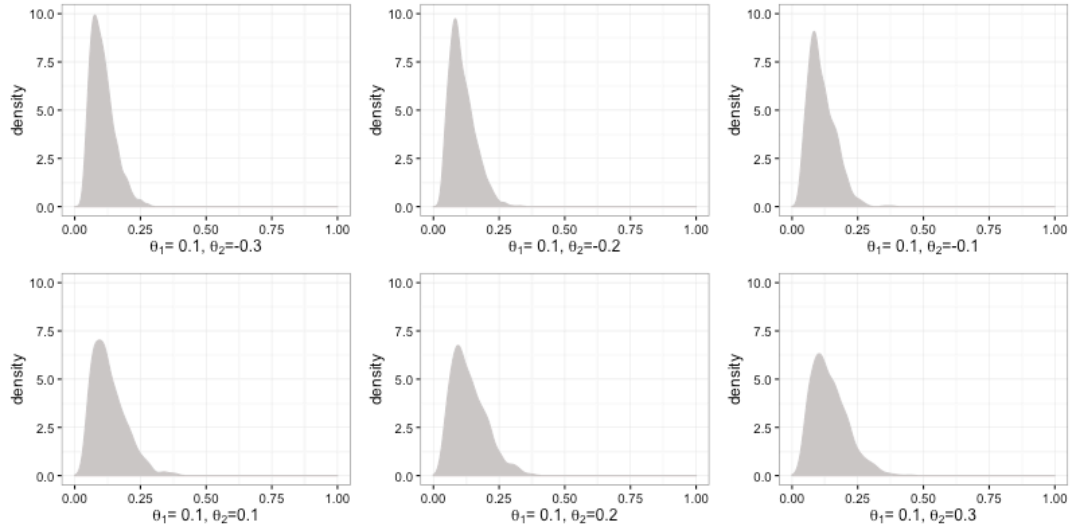


Figure 30: MA(1): Maximum drawdown distribution for various κ_1 values (Empirical distribution, path length = 63, sample size = 1000, $\epsilon_t \sim 0.01T(df = 4)$)

	Mean	Sd	Skewness	Kurtosis
$\kappa_1 = 0.1, \kappa_2 = -0.3$	0.0	0.015	-0.204	10.7
$\kappa_1 = 0.1, \kappa_2 = -0.2$	0.0	0.015	0.162	8.9
$\kappa_1 = 0.1, \kappa_2 = -0.1$	0.0	0.014	-0.229	6.2
$\kappa_1 = 0.1, \kappa_2 = 0.1$	0.0	0.014	-0.220	13.3
$\kappa_1 = 0.1, \kappa_2 = 0.2$	0.0	0.015	0.117	9.7
$\kappa_1 = 0.1, \kappa_2 = 0.3$	0.0	0.015	-0.097	9.9

Table 8: Statistics of simulated distribution of MA(2)

2.5 ARMA(1, 1)

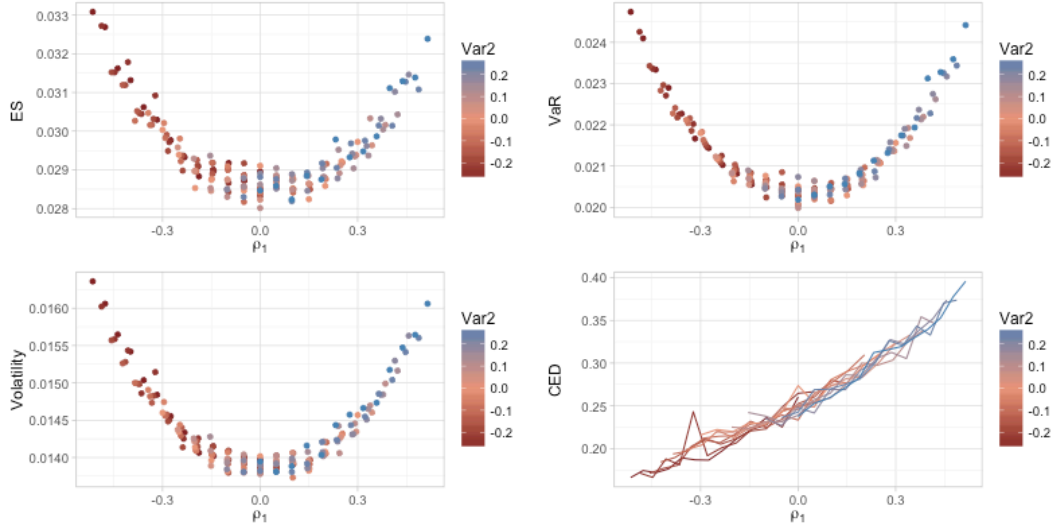


Figure 31: ARMA(1, 1): Relationship between serial correlation ρ_1 and risk measures
(Simulation path length: 63, $\epsilon_t \sim 0.01T(df = 4)$)

3 Thoughts

The relationship between serial correlation is complicated even for simple models such as AR(2), MA(2) and ARMA(1, 1). We tried several ways to present the relationship. Plots in this report are the best we came up with.

We did not move to more sophisticated time series models with higher orders. The patterns in some sense explained why we hardly get informative relationship in our empirical data analysis. Real word data sets are usually more complex than simulation.

Potential efforts:

1. Add Garch model.
2. Standardize the simulated time series by deviding certain factors such that the time series have same standard deviation. Currently our simulation studies are based on fixed distribution of the noice term. For example, for normal distribution noice we use $\epsilon \sim N(0, 0.0001)$ and for t-distribuiton we use $\epsilon \sim 0.01T(df = 4)$. We could also do our simulation based on fixed standard deviation of the simulated time series sequences.
3. So far we have been working on the first order serial correlation ρ_1 , we have also looked at the relationship between higher order serial correlation and risk measures. There are some complex relationship for different time series models. We may look into that in the future.