# Risk Measures and Serial Correlation

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**Abstract**

Conditional Expected Drawdown (CED), the tail mean of maximum drawdown distribution, is a newly proposed positive homogenous and convex risk measure. Since maximum drawdown is defined as the accumulative loss from peak to trough, we expect CED to be inherently path dependent and account for serial correlation. Most currently widely-used risk measures such as Value at Risk (VaR), Expected Shortfall (ES) and volatility are based on daily returns and do not account for consecutive losses. We compared CED with these risk measures and show that CED is more sensitive to serial correlation on empirical and theoretical perspective.

## 1 Introduction

Firms and regulators constantly quote risk measures such as VaR, ES and volatility to gauge the amount of asset needed for potential losses. These risk measures are usually calculated from the daily return distribution, thus only accounts for daily losses. However, during the events when consecutive losses happen such as 2008 financial crisis, these measures would become less informative due to the failure to consider the serial correlation of returns. Noticing this drawback of traditional risk measures, Goldberg and Mahmoud[1] developed Conditional Expected Drawdown (CED), a new risk measure defined over the empirical distribution of maximum drawdown. In this paper, we examine the relationship between serial correlation and various risk measures including CED.

### 1.1 Risk measures

In the rest of the paper, we mainly focus on the comparison of CED and three extensively studied risk measures including Value at Risk (VaR), Expected Shortfall (ES) and volatility.

- *Conditional Expected Drawdown (CED)* is defined as the tail mean of maximum drawdown distribution.

  $$CED_\alpha(X_{T_n}) = \mathbb{E}(\boldsymbol{\mu}(X_{T_n})|\boldsymbol{\mu}(X_{T_n}) > DT_\alpha). \tag{1}$$

  where $\alpha$ is the significance level, $DT_\alpha$ is the $\alpha$ quantile of maximum drawdown distribution, $\boldsymbol{\mu}(X_{T_n})$ represents *Maximum drawdown*. In the return path of length n, the maximum drawdown is defined by

  $$\boldsymbol{\mu}(X_{T_n}) = \max_{1<\mathrm{i}<\mathrm{j}\leq\mathrm{n}}\max(X_{t_i} - X_{t_j}, 0). \tag{2}$$

  Maximum drawdown is interpreted as the largest accumulative loss from peak to trough. For the detailed description of CED and its properties including path-dependency, convexity, and positive homogeneity, we direct the interested reader to Goldberg and Mahmoud[1].

- *Value at Risk (VaR)* estimates the potential loss of financial investment in a period. VaR is widely used by investment industry to assess the amount of assets needed to cover possible losses. For a given significant level $\alpha$, VaR is defined as the $\alpha$ quantile of the asset return distribution, which suggests the probability that the amount of loss excess VaR($\alpha$) is less than $\alpha$. The mathematical representation of VaR is:

$$VaR_\alpha(L) = inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} = inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}. \tag{3}$$

- *Expected shortfall (ES)* is a risk measure which resembles VaR but satisfies monotonicity, translation invariance, homogeneity and subadditivity. The ES of a financial asset is calculated as the tail mean of its return distribution. The Expected shortfall at level $\alpha$ is the expected value of loss which exceeds VaR($\alpha$). It is more sensitive to the shape of the loss distribution especially the tail of the distribution. The mathematical representation of ES is:

$$ES_\alpha(L) = E\left[L | L < VaR_\alpha(L)\right]. \tag{4}$$

- *Volatility* is measured by the standard deviation of asset returns. Higher volatility usually implies greater risk. Under the normality assumption of returns, volatility is proportional to VaR and ES. Although real world asset returns often have fatter tails than normal, volatility is often strongly correlated with VaR and ES, which we will see in later analysis.

### 1.2 Serial correlation

Serial correlation, also known as autocorrelation, is the correlation of observations at different time point. Under the wide-sense stationary process assumption, the serial correlation of $X$ between two time point $t$ and $s$ can be measured by the autocorrelation function as follows:

$$R(\tau) = R(s,t) = \frac{E[(X_t - \mu)(X_s - \mu)]}{\sigma^2}. \tag{5}$$

where $\tau = t - s$.

Serial correlation is often associated with the violation of efficiency market and random walk hypothesis. The literature documenting empirical serial correlation is extensive in the late 1980's[1]. In stock price, Lo and MacKinlay (1988) [3] argue that returns based on the horizon longer than one year show a significant mean reversion, while Poterba and Summers (1988) [4] detect a mean aversion for weekly and monthly returns. Lo and Mavkinlay (1988) [3], Conrad et al. (1991) [5] model the security returns using a positively autocorrelated common component, an idiosyncratic component and a white-noise component. More extensively, the serial correlation has been documented in literature on nonsynchronous trading, which means assets are not traded simultaneously [6]. Mech and Timothy (1993) [7] present evidence that the autocorrelation is associated with the delaying in price adjustment caused by transaction costs. In hedge fund returns, Getmansky, Lo and Makarov (2004) argue that serial correlation is an outcome of illiquidity exposure and smoothed returns, market inefficiencies, time-varying expected returns and leverage and incentive fees with high water marks.

Assuming a moving average representation of reported returns, Getmansky, Lo and Makarov (2004) [8] show that Sharp Ratio (SR) tends to be overstated and the market beta understated. Cesare, Stork and Vries (2014) [9] use the similar structure to demonstrate that the reported value-at-risk (VaR) and expected shortfall (ES) are always smaller than or equal to their actual values. Thus, the risks of assets are easily underestimated using standard risk measures, and the investment decisions may be misleading. Although based on serial correlation and smoothing feature of hedge fund returns, their models are as well applicable to other assets with autocorrelated returns.

[1] See Barucci and Emilio (2012, Section 6.5) [2] for a detailed review.

### 1.3 Synopsis

The plan of the paper is as follows. In *Part I: Empirical Analysis*, we present empirical studies analyzing daily returns of various assets. Section 2 provides overall and time-varying risk diagnostics stressing both correlation and the difference between CED and other risk measures. In Section 3, we relate serial correlation with risk measures by fitting time series models, includes AR, MA, ARMA and GARCH. In Section 4, we give an empirical analysis of risk contributions by constructing portfolios of different weights. Next in *Part II: Simulations*, we further illustrate properties of CED by generating returns from time series models. Section 5 contains the comparison of risk measures of various models. And we also give the analysis of the relationship between serial correlation and risk measures based on simulated models. Finally in Section 7, we show that higher serial correlation would result in higher drawdown risk concentrations.

# Part I
# Empirical Analysis

In this part, we provide the empirical analysis for various asset classes including S&P 500 Index (SPX), Russell 3000 Index (RAY), etc. Detailed descriptions of their date ranges, components, and summary statistics are given in Appendix A.

## 2 Risk diagnostics

### 2.1 Overall risk diagnostics

Table 1 shows the overall values of four risk measures for various assets over their time range. For each of VaR, ES and CED, we present results of two significance levels 90% and 95%. Volatility, ES and VaR are on daily-scale to allow comparison between risk measures in the future.

**Our calculation of risk measures is based on empirical distribution. Assumptions of normal distribution would result in erroneous results.** Note that VaR and ES are calculated based on the empirical distribution of daily returns. Another commonly used method would be based on Gaussian distribution assumption. However, in our case where all asset returns are fat-tailed distributed, applying Gaussian distribution would lead to erroneous results. VaR and ES will be overestimated at a lower confidence level and underestimated at a higher level. This discrepancy phenomenon is rather obvious

| Measures | Volatility | VaR(%) | | ES(%) | | CED(%) (3 month) | | CED(%) (6 month) | |
|---|---|---|---|---|---|---|---|---|---|
| Levels | | 0.90 | 0.95 | 0.90 | 0.95 | 0.90 | 0.95 | 0.90 | 0.95 |
| AGG | 0.32 | 0.29 | 0.40 | 0.50 | 0.66 | 5.60 | 7.72 | 8.12 | 11.45 |
| HYG | 0.84 | 0.62 | 1.03 | 1.41 | 2.03 | 18.41 | 24.07 | 26.43 | 30.77 |
| TIP | 0.41 | 0.44 | 0.62 | 0.72 | 0.91 | 7.48 | 9.90 | 11.14 | 12.91 |
| BCOM | 0.94 | 1.04 | 1.47 | 1.71 | 2.20 | 18.14 | 22.54 | 26.61 | 33.66 |
| MXEA | 0.97 | 1.02 | 1.46 | 1.74 | 2.26 | 20.39 | 23.73 | 27.21 | 31.79 |
| MXEF | 1.13 | 1.21 | 1.76 | 2.11 | 2.75 | 26.21 | 30.80 | 36.35 | 43.30 |
| RAY | 1.09 | 1.11 | 1.62 | 1.95 | 2.56 | 20.65 | 25.64 | 27.81 | 34.08 |
| RMZ | 2.30 | 1.91 | 3.00 | 3.99 | 5.62 | 37.30 | 48.41 | 52.04 | 62.41 |
| SPX | 0.97 | 0.99 | 1.43 | 1.71 | 2.23 | 18.35 | 22.67 | 25.18 | 30.65 |
| USGG10YR | 1.27 | 1.26 | 1.95 | 2.28 | 2.99 | 23.28 | 28.11 | 32.78 | 39.00 |

Table 1: Overall risk measures for various assets

when the return distribution has an extremely fat tail. We include results assuming Gaussian distribution in Appendix A for comparison.

**The choice of path length is crucial for CED calculation.** As shown in Table 1, the value of CED is sensitive to path length. Here we present the CED with rolling three month and six month periods. CED is an increasing function of significance level and path length. We recommend period length no longer than six month for CED estimation. Please refer to an in-depth exploration of behaviour of maximum drawdown distribution and the choice of path length in next subsection. Due to the similarity in definition between CED and ES, they share many features in the calculation. Thus, many ES calculation technique could be implemented for CED estimation.

**While volatility, VaR and ES are strongly correlated with each other across assets, CED has a comparatively weaker correlation with the other three.** Note that the four risk measures do not give the same asset sequence from the largest to the least risky asset. Occasionally one risk measure indicates different order under different levels and path length. For example, CED indicates HYG has larger risk than SPX, but other three suggest the opposite comparison result. Moreover, the 6 month CED under 90% level gives the different relative risk between HYG and BCOM with its counterpart under 95% level. Notice that the reverse of relative risk suggested by same risk measure for distinct confidence level is more common for CED than for other risk measures.

**The economic implications of risk values are consistent with intuition.** Back to the economic implications of these risk values. Top three assets in Table 1 (AGG, HYG, TIP), which are comprised of US bonds, have the smallest risk among all asset class; RMZ, constructed by US equity REITs (Real Estate Investment Trust), have much greater risk than the others; MXEF (MSCI Emerging Market Index), reflecting the emerging market equities, have larger risk than MXEA (MSCI Developed Market Index).

## 2.2 Time-varying risk diagnostics

Time varying risk measures refer to risk measures based on fixed rolling windows. A series of risk measures is obtained. Risk at each time point is given by the past returns of a fixed length. Time-varying risk diagnostics enable us not only to compare risk across assets but to analyze risk for the same asset over time.

### 2.2.1 Maximum dradown distribution

**Unlike VaR and ES, the empirical distribution of maximum drawdown is more sensitive to the time length of measurement.** Figure 1 shows the maximum drawdown of various assets for different path length (3 months, 6 months, 1 year, 2 years, 5 years) separately. As revealed in Figure 1, maximum drawdown distribution tends to: a) have the larger mean and variance; b) be multi-mode; c) lack variability of values; d) center around several specific values when we move to longer periods.

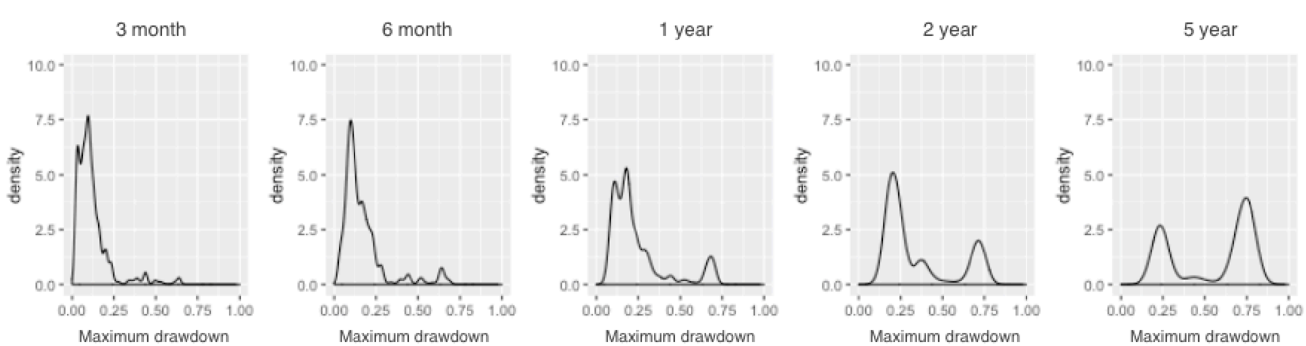



Figure 1: Maximum drawdown distribution of RMZ as rolling period increases

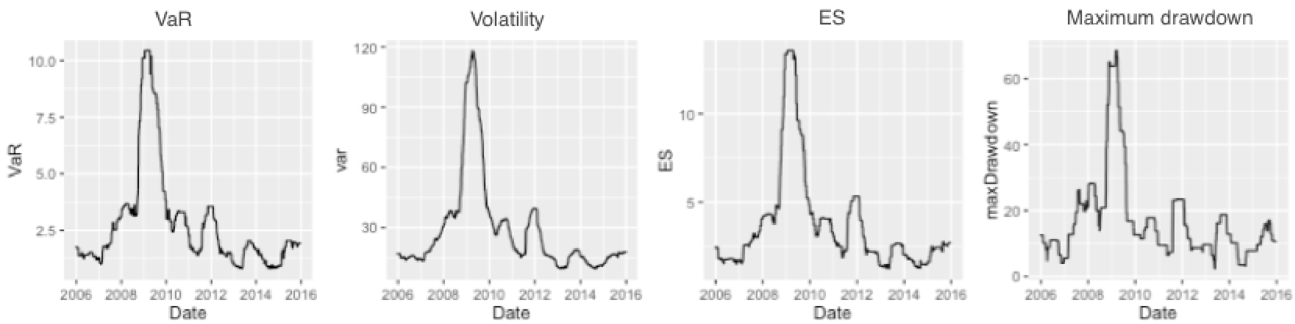



Figure 2: Comparison of different risk measures of RMZ

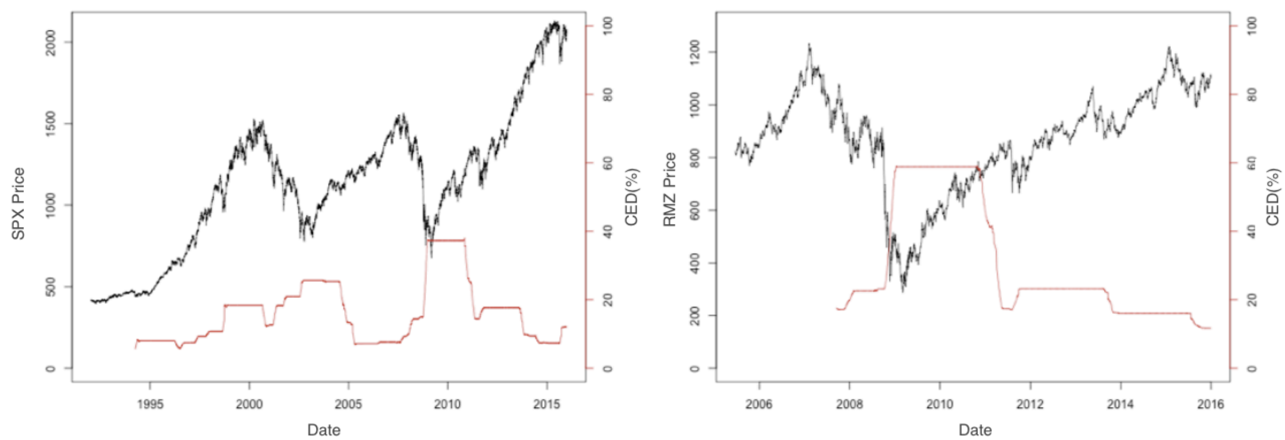



Figure 3: Daily price of the S&P 500 (SPX) and US equity REITs index (RMZ) togther with their 2-year-3month rolling CED.

**For daily return data, we do not recommend window length greater than six months for CED estimation.** Large drawdowns in real-world asset returns are usually associated with particular events during a short time, for example, the 2008-2009 financial crisis. When considering longer path such as two years or five years, maximum drawdown values tend to be dominant by these events. The empirical distribution would only be defined on several distinct values. In such cases, the empirical quantile no longer exists without distributional or polynomial assumptions of the tail. Thus, it becomes hard to calculate the tail mean (CED). Later in our analysis, we use three or six-month path length. Figure 22 shows the empirical distribution of maximum drawdown under the six-month rolling window for various assets.

### 2.2.2 Time varying risk measures

Under Gaussian distribution, VaR, ES, and volatility are linearly dependent. For empirical data where the return distribution has fat tails and distinct kurtosis, they are still strongly correlated. (With average correlation $> 95\%$ for six months rolling window) Figure 2 shows the VaR, volatility, ES, and maximum drawdown with rolling six-month periods. Four risk measures share a similar pattern of ups and downs, where the risk shot up during the 2008-2009 financial crisis.

**CED requires more data compared with other risk measures and usually remain constant for a period.** To empirically estimate the drawdown distribution, we often rolling a fixed path length. Credible quantile estimation requires hundreds of rollings. For example, two-year-three-month CED evaluates the three-month drawdown risk over the past two years. Figure 3 shows the daily price of the S&P 500 (SPX) and US equity REITs Index (RMZ) together with their two-year-three-month rolling CED. The

CED series also reflect the sharp increase in economic depressions.

**Time-varying risk measures are also closely related, and CED has a weaker correlation with the other three.** CED is closely related to other risk measures. However, the correlation between CED and the other three risk measures are slightly weaker than the correlation among volatility, VaR, and ES. Table 10 shows the correlation between CED and other risk measures calculated based on six-month rolling risk measures for every asset.

# 3 Time series analysis

## 3.1 ARMA

ARMA is the simplest and most commonly used time series model, which enable us to capture the correlation between current time point and one or several lags in a stationary time series. Therefore, in this section, ten assets returns are fitted into ARMA models, first simple chosen models from the family, then best model through selection steps. From there, one can take the first sight of the relationship between serial correlation and different risk measurements in real-world data.

### 3.1.1 Result of basic ARMA model fit

Started from the simplest ones, ten assets have been fitted into three most basic ARMA models: AR(1), MA(1) and ARMA(1,1). The three models here are not necessary the best one, or even the ones that all underlying assumptions are satisfied. This step is just to try to have a general idea if the serial correlation of each model is related with risk measurements in some way. When it comes to serial correlation here, it refers to the first order autocorrelation of the time series models.

Following is a summary of these three ARMA model, including their formulas and the method for calculating the serial correlation.

1. **AR(1)**

$$X_t = \phi X_{t-1} + \epsilon_t \quad s.t. \ |\phi| < 1 \tag{6}$$

$$\rho(h) = \phi^h, \ h = 0, 1, \ldots \tag{7}$$

2. **MA(1)**

$$X_t = \epsilon_t + \theta \epsilon_{t-1} \tag{8}$$

$$\rho(h) = \begin{cases} \frac{\theta}{(1+\theta^2)} & \text{if } h = 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

3. **ARMA(1,1)**

$$X_t = \phi X_{t-1} + \epsilon_t + \theta \epsilon_{t-1} \tag{10}$$

$$\rho(h) = \frac{(1+\theta\phi)(\theta+\phi)}{1+2\theta\phi+\phi^2}\phi^{h-1}, \text{if } h \geq 1 \tag{11}$$

Table 2 shows the correlation between serial correlation and different risk measures for AR(1) and ARMA(1,1). Figure 23 and Figure 24 in the appendix shows a more straightforward relationship between first order serial correlation of AR(1) and risk measures. Consistent with what was showing in the plot, all the assets except MXEA and MXEF have the high correlation between $\kappa(1)$ and risk measurements. Within each asset, as expected, the correlation with VaR and ES are very closed, as VaR is highly correlated with ES internally. However, compared with VaR and ES, the correlation with CED is relatively smaller, and sometimes the correlations are even with the different sign.

Both the table and figures are unable to tell the story what are the relationships between risk measures and serial correlation for the indices, since the model here are recklessly chosen and more complicated factors shocking the market every day. However, the table provides an idea that relationship with CED is more closed to that with VaR and ES when this correlation is internally significant.

| Asset | AR(1) | | | MA(1) | | |
|---|---|---|---|---|---|---|
| | VaR | ES | CED | VaR | ES | CED |
| AGG | 0.92 | 0.95 | 0.94 | 0.94 | 0.96 | 0.94 |
| HYG | -0.61 | -0.56 | -0.41 | -0.51 | -0.45 | -0.29 |
| TIP | -0.67 | -0.75 | -0.73 | -0.65 | -0.74 | -0.73 |
| BCOM | 0.86 | 0.80 | 0.76 | 0.86 | 0.80 | 0.76 |
| MXEA | 0.26 | 0.33 | 0.08 | 0.01 | 0.11 | 0.15 |
| MXEF | 0.23 | 0.22 | -0.05 | 0.15 | 0.16 | -0.12 |
| RAY | 0.79 | 0.76 | 0.49 | 0.79 | 0.77 | 0.50 |
| RMZ | 0.90 | 0.93 | 0.81 | 0.92 | 0.95 | 0.84 |
| SPX | 0.62 | 0.70 | 0.15 | 0.65 | 0.72 | 0.17 |
| USGG10YR | 0.67 | 0.64 | 0.66 | 0.65 | 0.64 | 0.68 |

Table 2: Correlation between $\rho(1)$ and risk measures. The time window used for get risk measurements and $\rho(1)$ is 5 years, and the rolling window within 5 year for CED is 3 month.

### 3.1.2 Model selection

The model selection is accomplished by trying to find a proper time series model on the unit of all data for each asset, for the fact that it makes no sense to fit different model for each rolling time period within one financial asset. Unfortunately, observing from the ACF plot of returns, most of them are not stationary by nature without drift term, which means ARMA model is inherently not or just a partial a good selection for fitting them. It may also explain the reason why the previous section did not produce an expected result that CED is more correlated with serial correlation $\kappa(1)$.

Our criterion here for "better model" is the smaller AIC value. Appendix D provides a more detailed explanation on model select criteria. To confirm the fact that ARMA is not reasonable for almost all the indices, we tried sets of parameters for ARMA and selected the one with the smallest AIC value. Mostly, we chose the combination of AR and MA parameters ranging from 1 to 5. Table 3 shows the best selection based on AIC criteria. The Table 3 indicates that the parameters are as high as 4 or 5 for almost all the assets, which suggests that the time series are not stationary. It also involved some converging issue is when the parameters are getting too larger.

Table 3: Best ARIMA Model for the U.S. Assets

| Asset | ARIMA (p,d,q) |
|---|---|
| AGG | (5,0,5) |
| HYG | (3,0,1) |
| TIP | (0,0,0) |
| BCOM | (0,0,0) |
| MXEA | (2,0,2) |
| MXEF | (4,0,2) |
| RAY | (2,0,2) |
| RMZ | (0,0,1) |
| SPX | (2,0,2) |
| USGG10YR | (0,0,0) |

To illustrate this point, the following is an example analysis of RMZ index. There are several reasons why RMZ is selected: 1) the RMZ has a shorter period, which means it includes less abnormal market shock such as the financial crisis in 2008 and more close to the normal fluctuation in stock price. 2) Larger serial correlation and ES, VaR comparing with other indices, implying it is more suitable for analyzing the relationship between serial correlation and various risk measures. 3) There are fewer modes in the maximum drawdown distribution of RMZ, suggesting it is reasonable to get tail mean, which is CED.

### 3.1.3 RMZ: an example

Grid searching of the best parameter set of ARIMA model is applied to RMZ index. By both AIC and BIC selection criteria (Appendix D), MA(1) turns out to be the best option.

However, the residuals after fitting do not convert into white noise. The factors of leading this result are complicated and beyond our discussion, as this is a 10-year-long time series and market shock varied

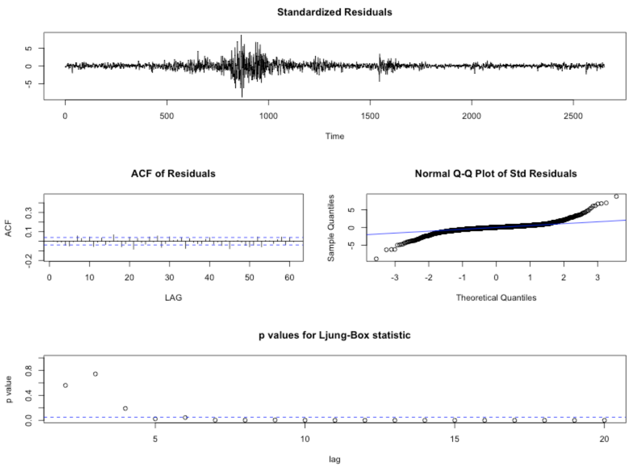



Figure 4: Diagnostic Plots of MA(1) for RMZ

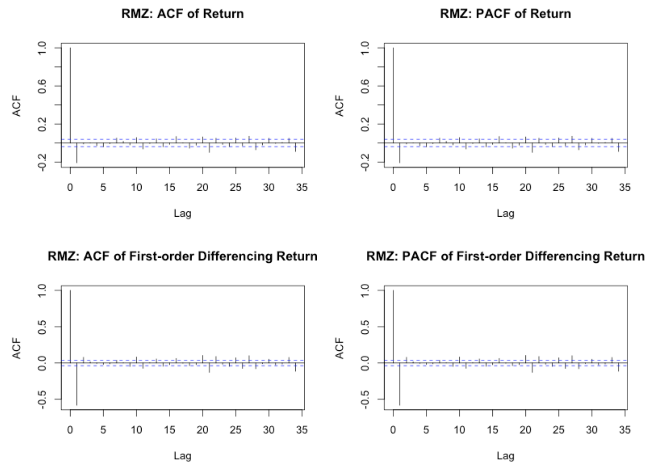



Figure 5: ACF/PACF of Return for RMZ

during this long time. Figure 4 shows the diagnostic results after the fitting. The variance of residuals are clustered across the time. After taking a closer look at the data, agreed with expectation, the high volatility period is corresponding to 2008 financial crisis.

Figure 5 also demonstrates this problem in ACF/PACF plot, in which some correlations exist between residuals after more than 50 lags, but they are comparatively small after the fitting.Normal Q-Q shows that residuals are heavy-tailed. LjungBoxPierce Q-statistic, used for test grouped $\kappa_e(h)$, rejects $H_0$ after $5^{th}$ lag, which also indicates the residual is not white noise after fitting. The diagnostics suggest that AR(1), MA(1) and ARMA(1,1) could all be good options while the time series is not stationary by nature. Therefore, a more complicated model should be adopted to capture the pattern within the variance of residual.

## 3.2 GARCH

For financial assets, the returns have high dependency at the time. Thus, it is reasonable to fit time series to interpret and predict them. As described in Section 3.1, the ARIMA models are infeasible for most assets from diagnostics plots, in which the variance cluster in the return could not be captured, resulting in some variance cluster left the residual plot. To solve the problems listed above, the GARCH then is attempted to fit the returns for various assets. It turns out to have a much better performance in capturing the variance, and hence, a generous diagnostic plot.

### 3.2.1 Why GARCH is better?

The GARCH is the most widely used the model in financial data fitting. Rather than ARIMA having constant variance assumption, GARCH also model the heteroscedasticity in the time series. For GARCH(1,1), the model is described as following:

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t \\
\sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2
\end{aligned}
$$

These set of formulas in indicates that the current variance of the noise is affected by the variance of error and the magnitude of return in the several previous periods. In this model, the clustered variance of residuals can be efficiently captured. Additionally, GARCH also inherits good properties of the ARMA in characterizing the correlation between consecutive time points. Considering all the benefits listed above, GARCH is an ideal model for improving the ARMA mentioned in Section 3.1.

### 3.2.2 Result of GARCH fit

As shown in the return plots for different assets, the clustered variance is a typical case in financial assets. Especially for long time series, such as *MXEA*, the market shock is even harder to interpret and predict. Therefore, for those assets with long time series, a longer lag of ARMA model is required. The best fit of GARCH model on all the selected assets is shown in Table 4. The parameters are selected using in-sample

Table 4: Best GARCH Model for the residuals after Fitting ARIMA

| Asset | ARIMA (p,q)+ GARCH(m,n) | BIC | $\kappa(1)$ |
|---|---|---|---|
| AGG | ARMA(5,5)+GARCH(1,1) | -6.8688 | -0.0698 |
| HYG | ARMA(3,1)+GARCH(1,1) | -6.8628 | -0.5756 |
| TIP | GARCH(1,1) | -8.3838 | 0 |
| BCOM | GARCH(1,1) | -6.7632 | 0 |
| MXEA | ARMA(2,2)+ GARCH(1,2) | -6.7857 | -0.2176 |
| MXEF | ARMA(4,2) + GARCH(1,1) | -6.5399 | -0.6328 |
| RAY | ARMA(2,2) + GARCH(1,1) | -6.6313 | 0.2526 |
| RMZ | MA(1) + GARCH(1,1) | -5.6962 | 0 |
| SPX | ARMA(2,2) +GARCH(1,1) | -6.8697 | 0.3488 |
| USGG10YR | GARCH(1,3) | -6.664873 | 0 |

error measure BIC. Details about BIC can be found in Appendix D. The mean difference between BIC and AIC is that BIC tends to penalize more on complexity and to result in less feature in the model. As GARCH models are very likely to be overfitting, choosing BIC is of greater reasonability.

The GARCH model highly reduces the correlation within the residual and ends up with a more intelligent diagnostics, while the predictivity of the model has been highly improved (6 is a case in point). However, there are still problems revealed in GARCH model and needs further improvement. The heavy tail is existing as the sign from the normal Q-Q plots, in which points deviate from the line at both ends. The failure in the modeling heavy tail is caused by the inappropriate assumption of the Gaussian conditional distribution in GARCH. Instead of Gaussian, Student-t could be used as the conditional distribution and can catch the variance on both tails. Considering convergence issue caused by t-Student distribution, Generalize Normal Distribution (GED) and QMLE are other potential options.

### 3.2.3 RMZ: an example

This section presents a model selection and assessment process of the RMZ example on GARCH method, showing a diagnostics of the model and the predictive capability of the model. Section 5.4 is highly related to this section, in which GARCH time series are simulated and fitted so that a more clear pattern is generated eliminating all the other market factors. There, we can see a more clear relationship between serial correlation and different risk measures.

First of all, The MA(1) GARCH(1,1) for RMZ is estimated as followed:

$$R_t = -4.1561 \times 10^{-2}\epsilon_{t-1}$$
$$\sigma_t^2 = 1.9955 \times 10^{-6} + 1.1083 \times 10^{-1} T_{t-1}^2 + 8.8506 \times 10^{-1}\sigma_{t-1}^2$$

Figure 6 shows the diagnostic after fitting GARCH(1,1) with a generalized normal distribution on the residuals of RMZ after the ARMA fitting. The resulting errors feasibly tend to be white noise, without any apparent correlation within them. After applying the heavy tail fixer on conditional distribution using GED, the resulting error is more Gaussian distribution, following almost exactly to the line in Q-Q plot.

Figure 7 indicates the capability of GARCH in one and several steps prediction. It shows the last 120 residuals from RMZ index, and all but last 20 are used to fit the GARCH(1,1) model with GED conditional distribution. A confident interval is also created based on the mean and standard deviation. And it is nice to see that almost all the last 20 points lie in the interval.

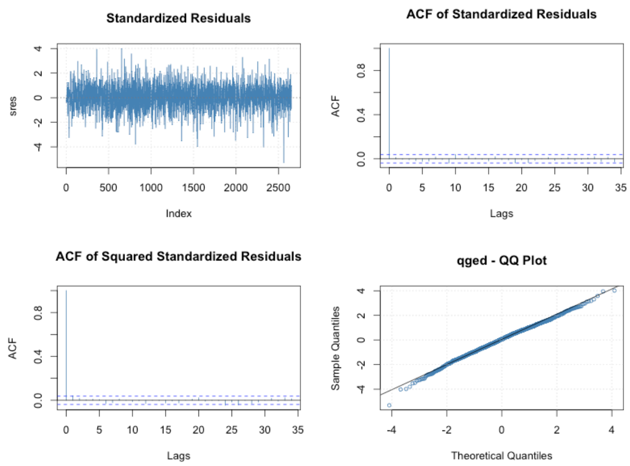



Figure 6: RMZ: Diagnostic Plots of GARCH(1,1) with GED Conditonal Distribution

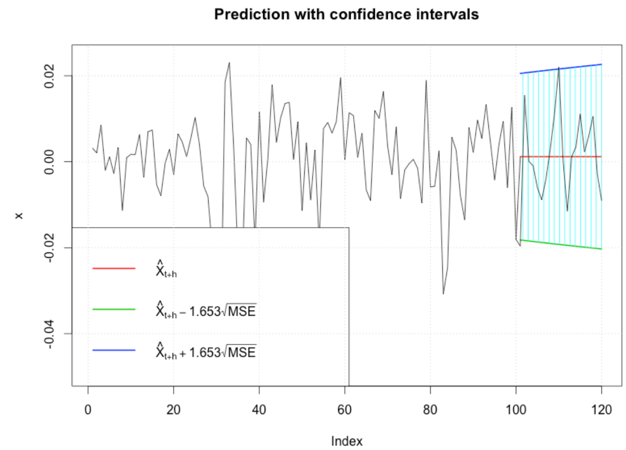



Figure 7: RMZ: Estimate of the Instantaneous Conditional Standard Deviation

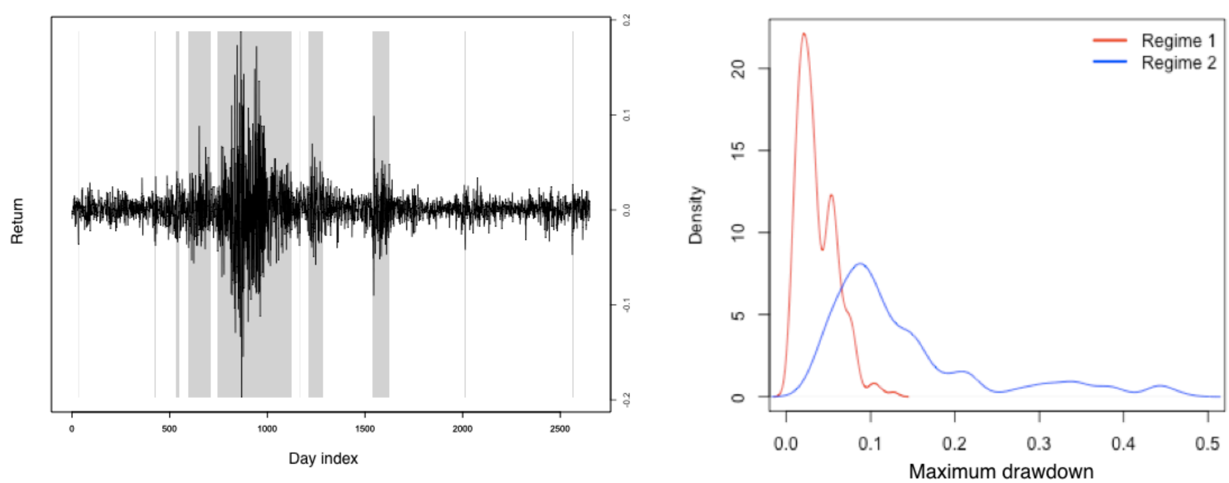



Figure 8: Left panel: Return plot of RMZ from June 20th, 2005 to December 31st, 2015. Shadowed areas represent regime 1 with high volatility, and white areas indicate regime 2 with low volatility. Right panel: one-month maximum drawdown distribution of regime 1 and regime 2 of RMZ.

### 3.3 Regime dependent analysis

Most economic time series data behave differently in the adjacent period. For example, asset returns usually show considerable volatility during a financial crisis. One standard approach to model abrupt changes in regime is to use the Hamilton's Markov regime switching model[10][11]. In our study, we use the following two-regime switching model to fit the returns:

$$y_t - \mu_{s_t^*} = \phi_{s_t^*}(y_{t-1} - \mu_{s_{t-1}^*}) + \epsilon_t \tag{12}$$

where the number of autoregressive coefficients is set to 1. $s_t^*$ is a two-state Markov chain. $s_t^* = 1$ represent regime 1 and $s_t^* = 2$ represent regime 2. $s_t^*$ depends on the past only through the most recent values:

$$P(s_t = j|s_{t-1}, s_{t-2}, \dots) = P(s_t = j|s_{t-1}) = p_{ij} \tag{13}$$

### 3.3.1 Comparison of basic summary of two regimes

Table 5 shows the estimated autoregression coefficient $\phi$ and standard deviation of the noise term of two regimes for various assets. We also provide the standard deviation, skewness, and kurtosis of different assets in Table 11 in the appendix. Note that regime 1 represents high volatility regime while regime 2 represents low volatility one. In general, it is clear that returns of regime 1 have a larger standard deviation, skewness, and kurtosis than that of regime 2. This difference in kurtosis indicates that the return distribution in low volatility regimes has a lighter tail. In contrast, in high volatility regimes, there are more extreme values of returns.

| | Regime 1 High volatility | | Regime 2 Low volatility | |
|---|---|---|---|---|
| | $\phi$ | Std | $\phi$ | Std |
| AGG | -0.134 | 0.009 | -0.114 | 0.002 |
| HYG | -0.010 | 0.016 | 0.025 | 0.004 |
| TIP | 0.039 | 0.007 | -0.026 | 0.003 |
| BCOM | -0.046 | 0.013 | 0.050 | 0.006 |
| MXEA | 0.095 | 0.015 | 0.109 | 0.006 |
| MXEF | 0.221 | 0.018 | 0.254 | 0.007 |
| RAY | -0.039 | 0.019 | 0.052 | 0.007 |
| RMZ | -0.244 | 0.040 | 0.007 | 0.010 |
| SPX | -0.018 | 0.016 | 0.113 | 0.006 |
| USGG10YR | -0.031 | 0.020 | 0.082 | 0.006 |

Table 5: Coefficient estimation of two regimes

We observe that low volatility regimes are more likely (9 out of 10 assets) to show larger autocorrelation coefficient. Although we expect greater drawdown risk when serial correlation increase (which means more significant autocorrelation coefficients estimate here), we did not prove this through empirical results. Later we will show in the simulation study that given different serial correlation and error term variance of the time series model, the latter factor dominant the impact to CED values, which is the case in our empirical findings. But under the same level of noise term variance, the larger the serial correlation, the greater the CED.

| | Regime 1 High volatility | Regime 2 Low volatility |
|---|---|---|
| VaR (empirical, p = 0.95) | 7.4% | 1.5% |
| ES (empirical, p = 0.95) | 9.9% | 2.1% |
| CED (one-month, p = 0.9) | 38.4% | 8.4% |
| Serial correlation (order = 1) | -0.257 | -0.026 |
| Serial correlation (order = 2) | -0.023 | 0.008 |

Table 6: Risk diagnostics for RMZ of two equal-length episode of each regime

### 3.3.2 RMZ: An example

In order to make a consistent comparison of risk diagnostics between two regimes, we ignore some short discontinuity and pick two longest single occurring episode for each regime. Here we use RMZ, the most risky asset in our data set as an example. Both episode contain 530 trading days. The episode of regime 1 range from October 30th, 2007 to December 7th, 2009, and the episode of regime 2 range from June 20th, 2013 to July 29th, 2015.

In Figure 8, the left panel shows the return of RMZ from June 20th, 2005 to December 31st, 2015. The shadowed area represents the regime with high volatility and the white area low volatility. This model is consistent with the actual financial event in that the high volatility regime covered mainly from mid-2007 to 2010. As we might expect, regime with high volatility also shows high risks in that they have larger ES and VaR values. Moreover, model of regime 1 only explains 26.3% of the observations, which means abrupt deviation is a minority in total observations. By looking at assets with longer period, we find a similar proportion of high volatility regimes (SPX: 23.7%, RAY: 20.6%). The right panel shows the one-month maximum drawdown distribution calculated based on the continuous episode we selected. The maximum drawdown distribution in regime 1 has larger mean and variance as well as greater knewness and kurtosis.

We may wrongly interpretate that larger serial correlation results in less drawdown risk. Notice from the results in later simulation section that variance of noice term dominant the influence for CED. And we can only obverve the positive correlation between serial correlation and CED under fixed variance level.

# 4 Empirical risk contribution

In this section, we present the empirical analysis of portfolio risk using two asset class, SPX (S&P 500 Index) and RMZ (MSCI US REIT Index) in 60/40, 50/50, 40/60 allocation separately. For each portfolio, we calculate the risk contributions of each asset component for CED, ES, VaR and volatility. Figure 10 shows the time-varying risk contribution of 50/50 allocation.

**The four measures indicate similar risk of portfolios over time, but large differences when it comes to risk contributions.** The variation of risk contributions for ES, VaR and volatility resemble each other under fixed asset allocation weights. For asset RMZ, while its contribution to ES, VaR and volatility maximized during 2010, the contribution to CED reached its peak during 2014. In the case of 40/60 asset allocation, SPX contributes to over 98% of the total CED in 2014. S&P 500 Index had a continuously rising trend during 2013-2014. However, RMZ became volatile during this period, and several downward trend occurs, which lead to the major risk contribution for CED.

The left panel of Figure 9 shows the scenario when a portfolio has equal risk contributions with respect to one risk measure but show difference for other risk measures. We calculated the fractional risk contributions of SPX in two asset portfolio in three types of risk parity cases: CED parity, ES parity and volatility parity. The right panel of Figure 9 shows the overall risk contributions of SPX in SPX and RMZ portfolio for weights of 50/50, 60/40 and 70/30. Among the risk measures, VaR shows the greatest contribution for SPX and CED the second. As we increase the weight of SPX in the portfolio, the risk contribution increases for all risk measures.

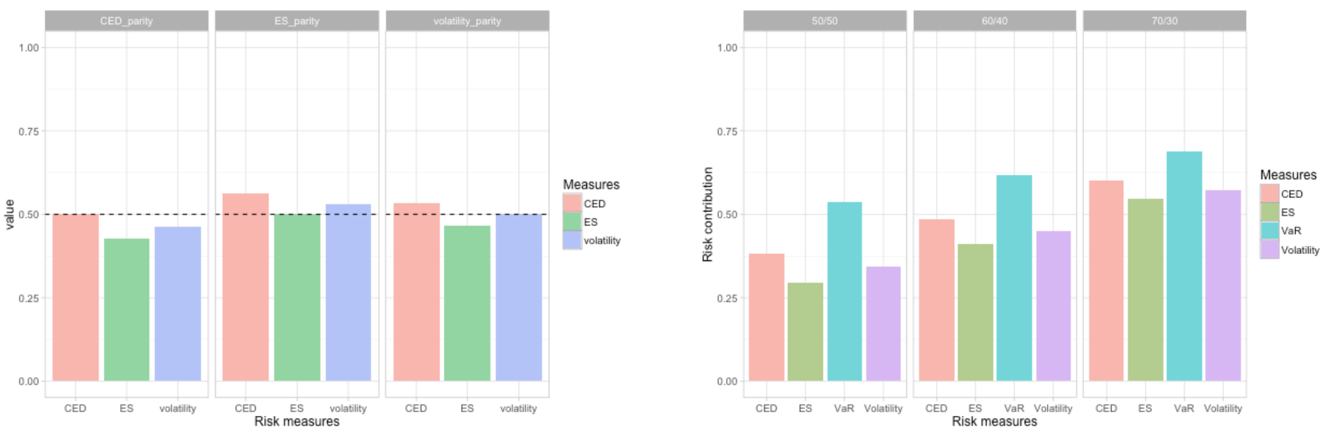



Figure 9: Left panel: Fractional Risk Contributions of SPX constructed based on CED Parity, ES Parity and Volatility Parity. Right panel: Fractional Risk Contributions of SPX for the weights of 50%, 60% and 70%. Protfolios are based on SPX and RMZ over the period January 3, 2006 to December 31, 2015. ES and CED are calculated at the 90% confidence level. Maximum drawdown distribution is calculated based on 3-month rolling period.

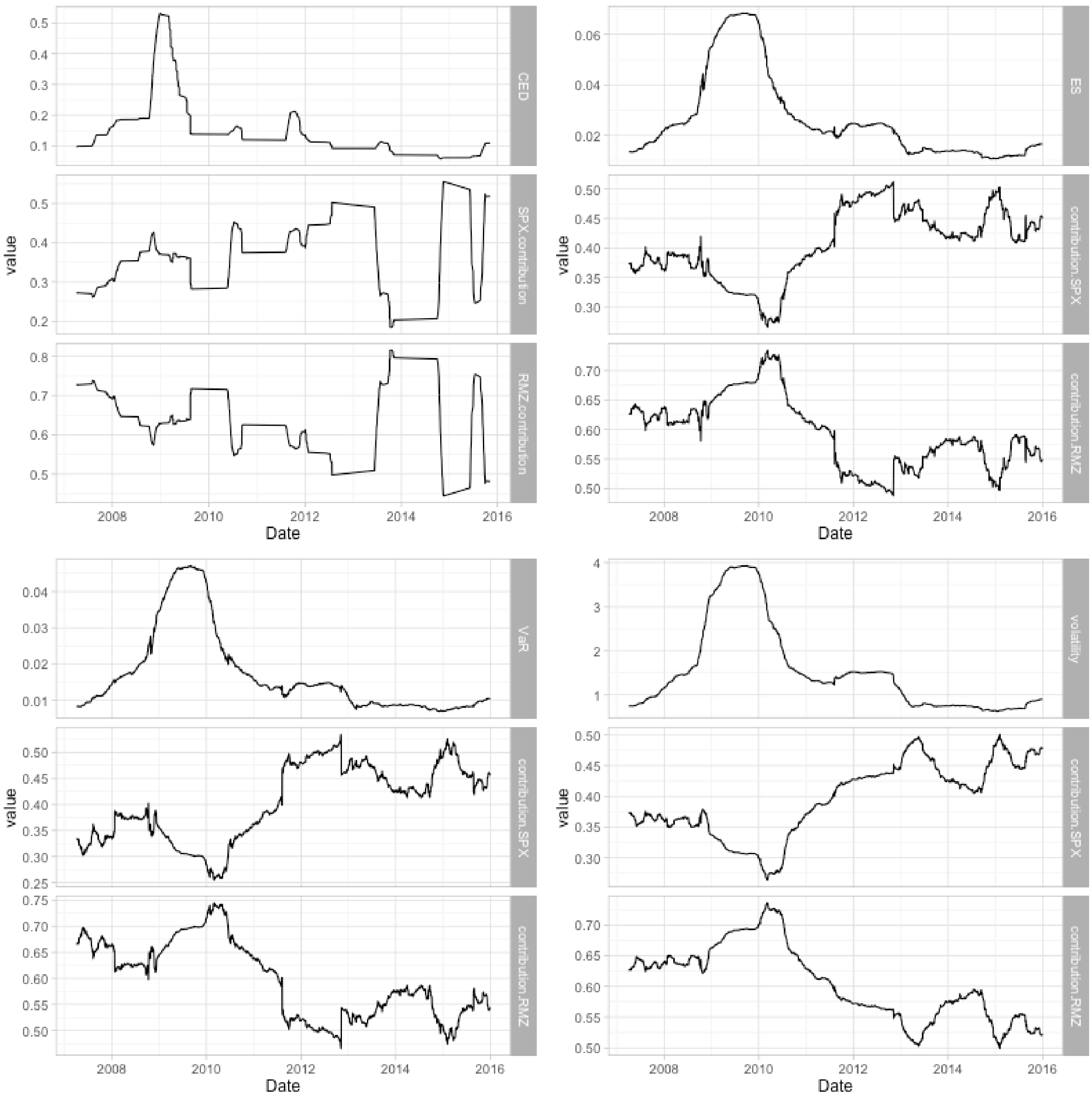



Figure 10: 3-month 1-year rolling risk contribution for CED, ES, VaR and volatility (portfolio is constructed using two asset classes (SPX and RMZ) in the balanced 50/50 allocation, probability level = 0.9)

# Part II
# Simulations

In this part, we explore the relationship between serial correlation and risk measures by analyzing models from data generated by known time series. Real world financial data is usually more complicated and simultaneously influenced by multiple effects. Simulation allow us to examine one specific factor by fixing others. For example, we may change one coefficient of a time series model while keeping other at the same level, which enables us to examine the relationship of certain order of serial correlation and CED.

## 5 Serial correlation and risk measures

In this section, we simulate various time series models and examine the impact of serial correlation on risk measures.

### 5.1 ARMA model with normal noise

We started from the simplest model AR(1) and the normal noice term:

$$X_t = \kappa_1 X_{t-1} + \epsilon_t \tag{14}$$

We simulate AR(1) for various values of the autoregressive parameter for $\kappa_1 \in (-1, 1)$ . Figure 11 shows the relationship between AR(1) coefficient and risk measures of interest including ES, VaR, volatility and CED. For AR(1) model, the order one serial correlation is the value of the autoregressive parameter.

CED shows a decreasing trend when $\kappa_1 \in (-1, -0.75)$ and an increasing trend when $\kappa_1 \in (-0.75, 1)$. As shown in Figure 11, it becomes feasible for us to distinguish negative and positive serial correlation using the CED values. However, the other 3 risk measures are all symmetric about $\kappa_1 = 0$, and they increase as the absolute values of $\kappa_1$ increases.

For VaR, ES and CED, the derivative of risk measure values to $\kappa_1$ approaches to zero as $\kappa_1$ goes to 0 and increase as $\kappa_1$ increase. This suggests that the value of this three risk measures can hardly reflect the change of serial correlation when the serial correlation is small. And they perform better when the serial correlation increases. The trend of derivatives reverses for CED. While the change of $\kappa_1$ has a comparatively larger influence around 0, the impact becomes weaker as we move to greater $\kappa_1$ values.

Figure 12 shows the maximum drawdown distribution for various $\kappa_1$. Same as revealed in Figure 11, the mean and tail mean of maximum drawdown distribution increases as we increase $\kappa_1$ from negative values to positive values.

As we move to higher order models, the relationship between serial correlation and risk measures becomes complicated. Due to the page limit, we omit the results for higher order models. The interested reader is welcome to refer to our simulation report.

### 5.2 ARMA model with fat tailed noise

Empirical asset returns are usually fat-tailed distributed. Extreme losses consistently occur, which lead us to consider noise with student t distribution.

Figure 13 shows the simulated AR(1) model and the corresponding risk measures. The relationship between serial correlation and risk measures revealed in this subsection resembles that of the last subsection where noise terms follow the normal distribution. There are slight differences due to the heavy-tail distribution. However, the points are not lies perfectly on a curve because the heavy tail introduces more randomness of the quantile and tail mean values. The simulated return distribution has mean and skewness close to zero. And they have high kurtosis which implies the fat tail. The simulated maximum drawdown distribution has larger the variance than that of the normal case.

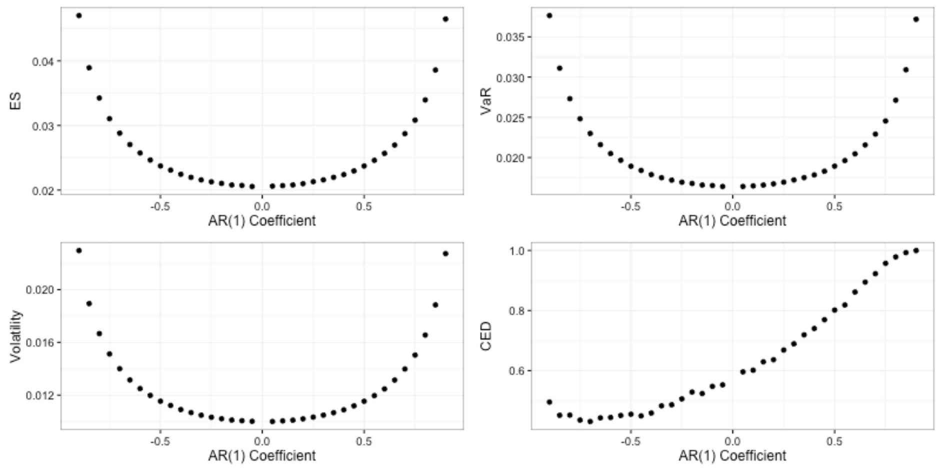



Figure 11: AR(1): Relationship between auto-correlation coefficients and risk measures (Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

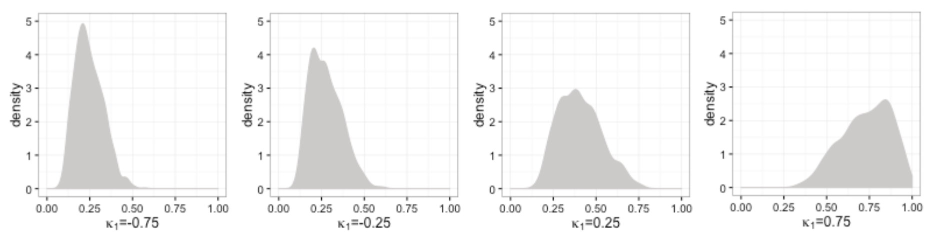



Figure 12: AR(1): Maximum drawdown distribution for various $\kappa_1$ values (Empirical distribution, path length = 1000, sample size = 1000, $\epsilon_t \sim N(0, 0.0001)$ )

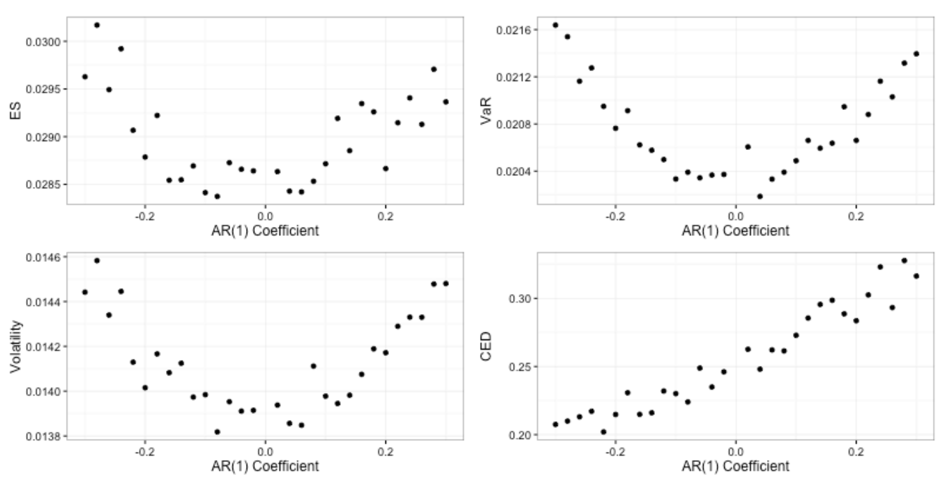



Figure 13: AR(1): Relationship between auto-correlation coefficients and risk measures (Simulation path length: 63, $\epsilon_t \sim 0.01T(df = 4)$)

## 5.3 ARMA model with standard error adjusted

So far our simulation studies are based on fixed distribution of the noise term. For example, for normal distribution noise we use $\epsilon \sim N(0, 0.0001)$ and for t-distribution we use $\epsilon \sim 0.01T(df = 4)$. In this section, we standardize the simulated time series by dividing certain factors such that the time series have same standard deviation.

In our simulation, the standard deviation for each our simulated time series is different. Say for AR(1), the standard deviation of time series $X_t = 0.5X_{t-1} + \epsilon_t$ and $X_t = 0.1X_{t-1} + \epsilon_t$ is different since we use the same distribution for $\epsilon_t$. The standard deviation for AR(1) model is :

$$SD(X_t) = \Big(\frac{Var(\epsilon_t)}{1 - \kappa_1^2}\Big)^{1/2} \tag{15}$$

Then what if we compare two time series with the different distribution of $\epsilon_t$ but the same standard deviation? Since standard deviation is one risk measures (volatility), this equals to fixing other risk measures to one level, say volatility = 1.2%, and to see how CED changes with the serial correlation.

Since our simulation is based on the normal distribution of the noise term, volatility would be proportional to ES and VaR, which means fixed volatility also means fixed volatility.

Figure 14 shows the result for AR(1) and MA(1) respectively. And Figure 15 shows the results for AR(2), MA(2) and ARMA(1, 1). There are clearer upward trends than the last two sections. These figures suggest that CED is a better metric in describing the returns with serial correlation. For the same volatility level, CED increases as the serial correlation increases, which means CED captures the changes in serial correlation while volatility, VaR and ES do not.

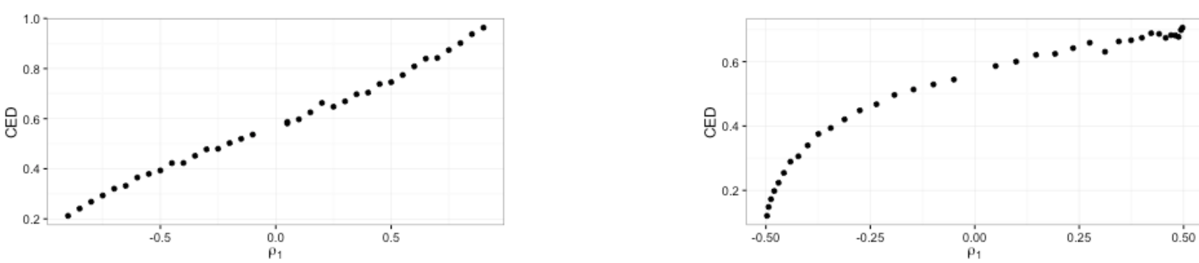



Figure 14: Left panel: first-order serial correlation calculated using AR(1) model versus CED. Right panel: First-order serial correlation calculated using AR(1) model versus VaR

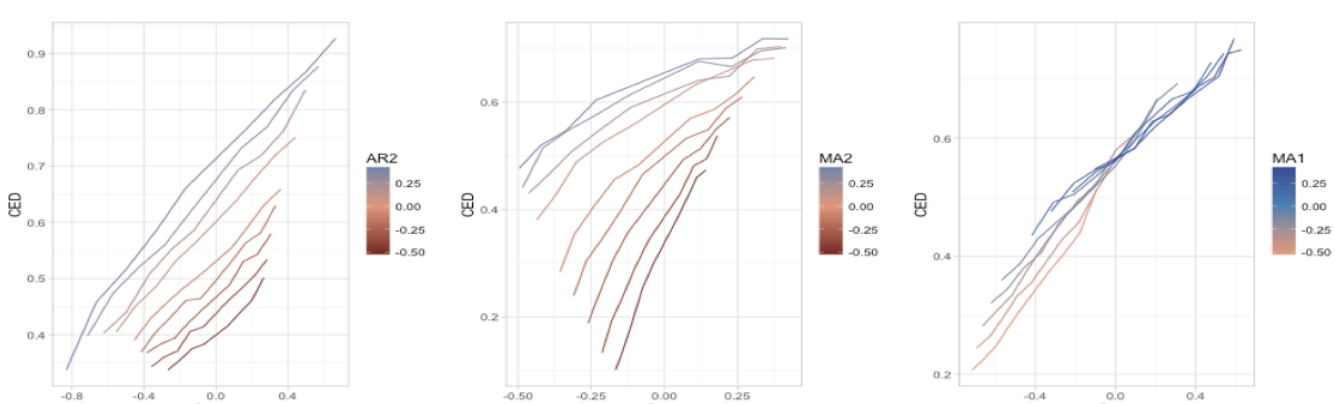



Figure 15: Relationship between serial correlation $\rho_1$ and CED with adjusted standard error (Simulation path length: 1000, from left to right: AR(2), MA(2), ARMA(1, 1), model coefficient $\in (-0.5, 0.5)$)

### 5.4 More complex noise: Garch model

The several previous sections have already stated general ideas that how the magnitude of risk measurements correlated with serial correlation and ARMA coefficients in AR(1), MA(1) and ARMA(1,1) models. However, the exciting results should not encourage us to ignore the complexity of the market: it always does not produce such regular time series. The occasional events shock the market and the returns always shows clustered variance. As has been analyzed in the Section 3.2, a popular way to handle the complex variance in the return is using generalized autoregressive conditional heteroskedasticity (GARCH) model. In this section, ARMA time series are simulated with GARCH(1,1) noise term, to which we expect analysis the extent of the effect that involved noise term is having on the relationship between serial correlation and risk measurements.

Several universal assumptions around GARCH term in this section should be stated at this point. The GARCH model is by the Gaussian conditional distribution. Although in the empirical study with thirteen American indices, it shows that a heavy tail conditional distributions, such as t-Student, are more likely to approximate the true market, Gaussian enables us to analyze more properties in the return without losing the generality. For GARCH(1,1), recall that:

$$\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2$$

The choose of potential parameters is on account of the study of the American financial indices. In summary, we choose $\omega \in \{10^{-7}, 10^{-8}\}$, $\alpha \in \{0.05, 0.08, 0.11, 0.14, 0.17, 0.20\}$, $\beta \in \{0.1, 0.2, \ldots, 0.7\}$.

#### 5.4.1 AR(1)∼GARCH(1,1)

Started from the simplest model AR(1)∼GARCH(1,1), the simulation results provide us a general idea how risk measures changes with $\alpha$ and serial correlation, namely, $\kappa(1)$. Two questions are focused here 1) which parameter within $\omega$, $\alpha$ and $\beta$ are more responsible for the change in risk measures? 2) Comparing the extent that the risk measures changing with serial correlation or coefficients, does $\alpha$ and $\beta$ counts a lot?

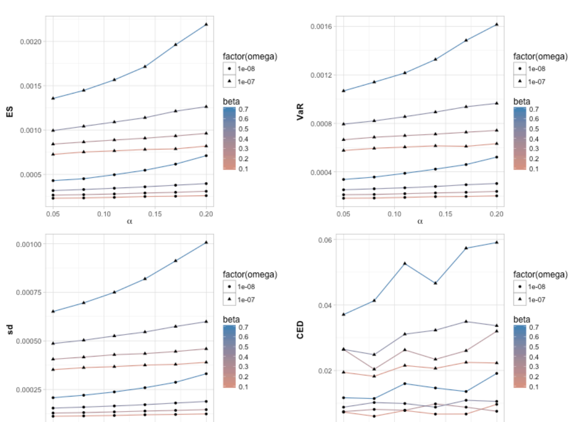



Figure 16: AR(1): Relationship between $\alpha$ parameter in the GARCH(1,1) and risk measures. The plots are produced under $\kappa_1 = -0.25$.

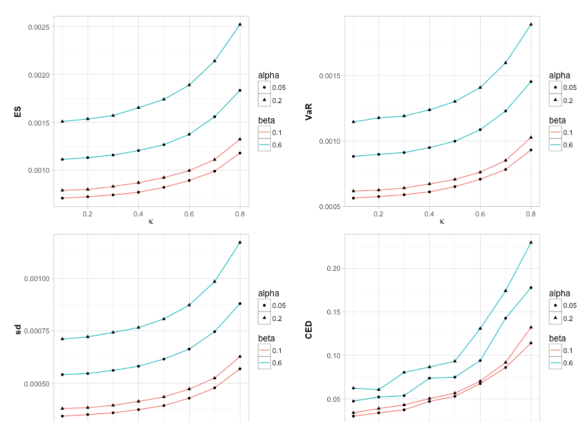



Figure 17: AR(1): Relationship between serial correlation in the GARCH(1,1) and risk measures. $\kappa_1 \in \{0.1, 0.2, \ldots, 0.8\}$

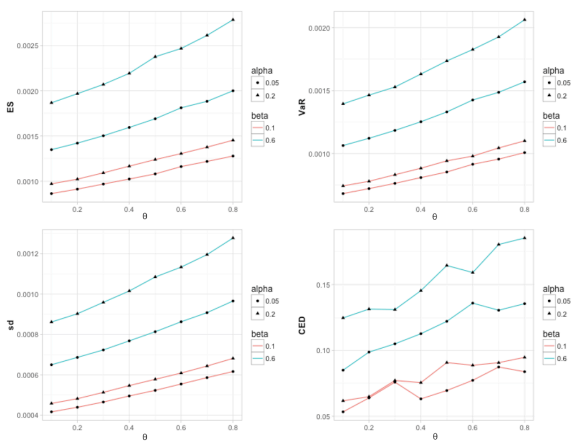
TThe Figure 16 verifies the guess. The higher $\omega$, $\alpha$ or $\beta$ are all partially responsible for high risk measures, in general. The eight lines within the plots are clearly separate into two groups, upper and lower. The more top four curves own a higher $\omega$, which indicates a general magnitude of variance. It is saved to conclude that $\omega$ is dominant among parameters.

ES, VaR, and Volatility are monotonically increasing with $alpha$. When the $\beta$ is relatively small, this relationship is almost linear, while when the $\beta$ is large, the relationship is more likely to be a higher order polynomial shape. In general, the CED has a similar trend with other risk measures, while it is less smooth. Especially, when the $\beta$ is low, $\alpha's$ effect is even more unclear. Indicating in the plot is the lowest three lines are twisting together.

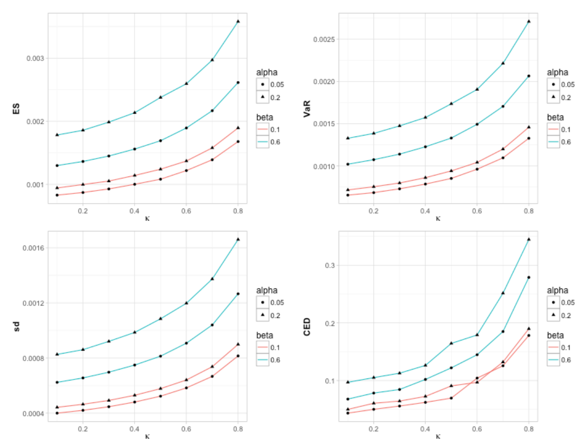
Figure 17 explores the how large $\omega$, $\alpha$ or $\beta$ affect the risk measurements comparing with changing in serial correlation. As proved, we only choose the positive coefficients to see the clear trend. Within $\alpha$ and $\beta$, the $\beta$ is dominant, especially in the first three plots, the 0.5 change in $beta$ is larger than 0.7 change in $\kappa$, fixed $\alpha$. A more exciting pattern shown in the CED plot. It shows that the complex variance, $\alpha$ or $\beta$ less affect CED, comparing the other risk measurements. In general, the curve in last plot is steeper, suggesting that CED is more sensitive to$\kappa$ in AR(1) $\sim$GARCH(1,1) model.

### 5.4.2 ARMA(1,1) $\sim$GARCH(1,1)

Similar to AR(1)$\sim$GARCH(1,1) model, we generalize it to ARMA(1,1)$\sim$GARCH(1,1), a more popular model in financial area. It indeed generates an indistinguishable pattern for traditional risk measures. Considering the complexity of the ARMA part, we find that AR and MA term have different impacts on the changing of risk measures.




(a) $\kappa_1 = 0.5$, varied $\theta$




(b) $\theta_1 = 0.5$, varied $\kappa$

Figure 18: ARMA(1,1): Relationship between AR and MA coefficients in the GARCH(1,1) and risk measures. $\theta_1 \in \{0.1, 0.2, \ldots, 0.8\}$, $\kappa_1 \in \{0.1, 0.2, \ldots, 0.8\}$. The $\omega$ is fixed as $10^{-7}$.

Figure 18 shows the ARMA parameters effects on the risk measurements, and also compare the magnitude of the GARCH parameters. For ES, VaR, and volatility, the plots suggest that the they are linearly increasing with the $\theta_1$ while non-linearly (probably quadratic, positive first derivative) increasing with the $\kappa_1$. Compared with traditional risk measures, the plots of CED are more jagged. Similarly to the previous conclusions, CED seems to be more sensitive to $\kappa$ and less sensitive to $\theta$. Moreover, when $\omega$, or the general trend, is small, the CED is undifferentiable for different alpha.

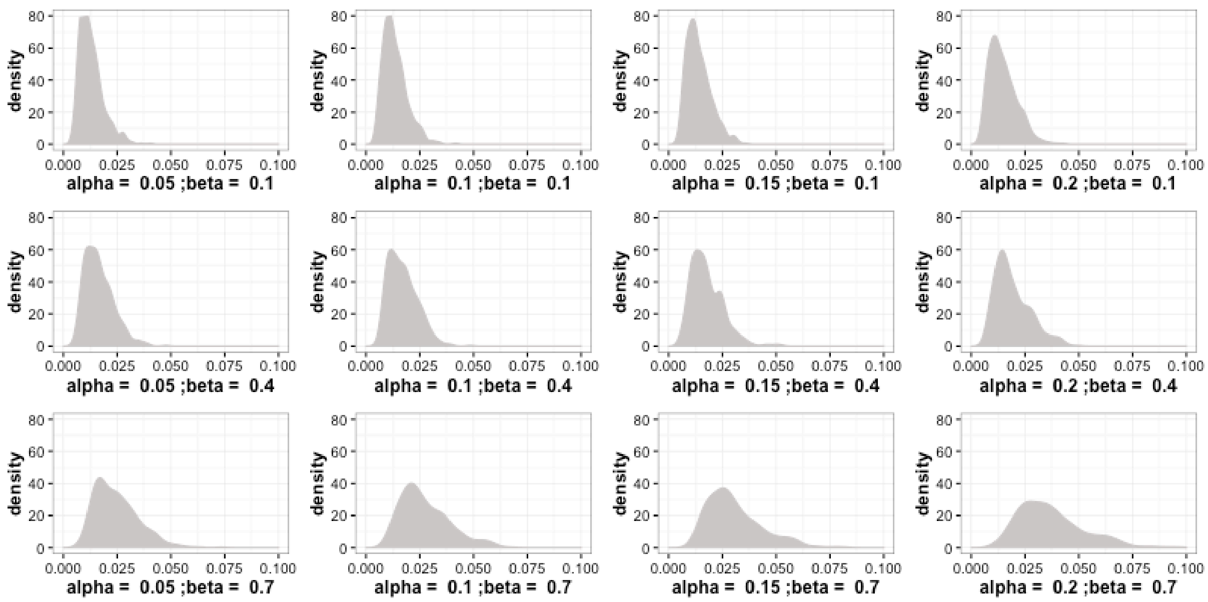



Figure 19: ARMA(1): maximum drawdown distributions under different $\alpha$ and $\beta$. $\theta = 0.25$, $\omega = 10^{-7}$, $\kappa_1 = 0.25$,$\theta_1 = -0.25$ is fixed

Figure 19 show the maximum drawdown distribution's change with the different $\alpha$ and $\beta$. Clearly a higher $\beta$ responsible for a more spread-out distribution, while $\alpha$ is less determinant. In Appendix C, Table 14 further confirms this point.

### 5.4.3 A short summary about GARCH

This section is targeted in analyzing the impact of complex variance on the evaluation relationship of serial correlation($\kappa$ or $\rho$) and risk measures. It is more straightforward to see this effects through the simulation study other than tedious derivations. Through the analysis above, we have several conclusions as followed.

1. The larger value of $\omega$, $\alpha$ and $\beta$ are all partially responsible for higher risk measures, in general, given the ARMA model.

2. Among three parameters, the effect of $\omega$ dominates. As $\omega$ indicates the general change of variance of the model, which is a counterpart of $\sigma^2$ in the pure ARMA.

3. When the $\beta$ is small, the effect of $\alpha$ on risk measurements are weak. Especially for CED, it is insensitive to the change of $\alpha$. However, when the $\beta$ value is large, the ES, VaR, and Volatility increase sharply with $\alpha$. For CED, it has an upward trend, but more indented than other risk measurements.

4. From the plots with $\kappa$ as an axis, we found that in the time series with a GARCH variance, $\alpha$ and $\beta$ less affect CED comparing with other risk measurements.

5. Last but not least, we found that $\alpha$ value of the GARCH model does not change the shape of maximum drawdown distribution a lot compare with $\beta$ values. Therefore, in summary, for CED, the ranking magnitude of effect from GARCH parameters are: $\alpha < \beta < \omega$

## 5.5 Findings summary

In this section, we present findings summarized from the previous simulation study, and provide the corresponding theorial explainations.

**1. CED distinguishes negative from positive autocorrelation**

For time series with positive autocorrelation, return is more likely to remain the same sign as the previous day. The likelihood increase as the serial correlation increase. As a consequence, returns tend to keep they current profit or loss status for a period. If the current status is loss, then unfortunately, the price of the asset may continuously go down, which results in a large CED. Conversely, time series with negative autocorrelation tend to reverse the sign in the previous day. If the the price of one assets goes down in one day, it is more likely that the price would bounce back in the next day, which results in alternating profit and loss status thus a smaller CED.

While CED captures the difference in serial correlation, other risk measures do not. VaR, ES and volatility are obtained from the empirical distribution of returns. If we shuffle the return sequence, we would get the same value for VaR, ES, and volatility.

Figure 20 shows the comparison of simulated returns and prices when serial correlation is 0.7 and -0.7. The maximum drawdown of simulated series is 22.76% and 9.49% separately. And two return series have the same VaR, ES and volatility.

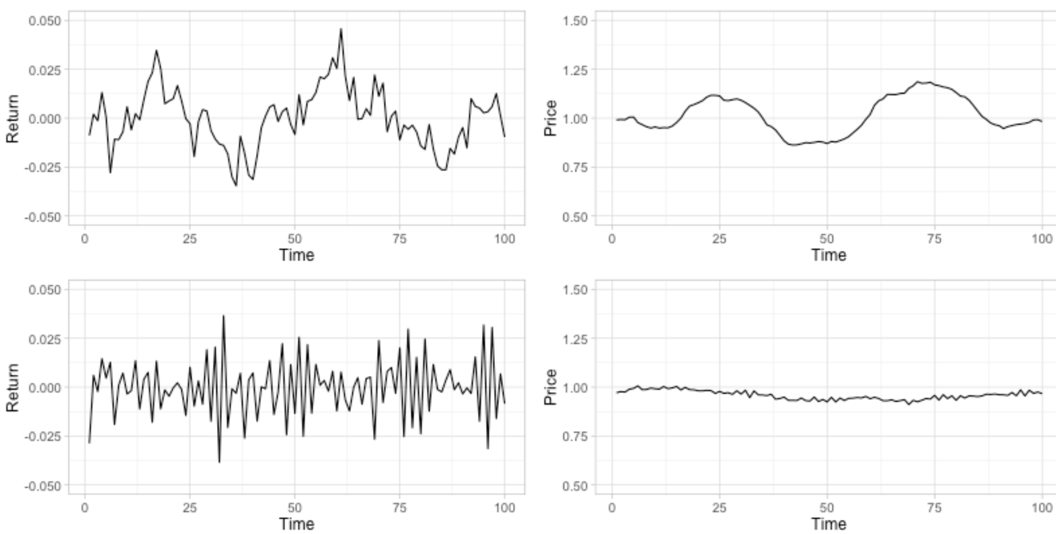



Figure 20: Comparison of returns with positive and negative serial correlation
(Simulation length: 100; upper panel: AR(1) with $\kappa = 0.7$; lower panel: AR(1) with $\kappa = -0.7$, $\epsilon \sim N(0, 0.0001)$)

**2. Change standard deviation but fix everything in time series simulation would result in linear change of risk measures**

Simulation in Figure **??** for AR(1) model shows a linear relationship for all risk measures. We simulate the model $X_t = 0.3X_{t-1} + \epsilon_t$ and change the $\epsilon_t$ from 0.001 to 0.015.

Fixing everything but changing the standard deviation of the noise term would simply equals to multiplying the simulated time series model by a constant. For volatility, ES and VaR, it is obvious. For accumulated returns in a path, the return is approximately the sum of all returns for each day in this time period when the return is small (which is often the case):

$$R = \prod_{i=1}^{n}(1 + r_i) - 1 \simeq \prod_{i=1}^{n} e^{r_i} - 1 = e^{\sum_{i=1}^{n} r_i} - 1 \simeq \sum_{i=1}^{n} r_i \tag{16}$$

All the returns in every subpath would be multiplied by approximately a constant. Thus, the domain of the maximum drawdown distribution and CED would also be scaled by the same number.

**3. Fatter tail of return distributions result in larger variance of the CED value**

Please refer to the comparison between Section 5.1 and Section 5.2. Since CED focus on the conditional tail distribution, fat tails indicate more volatile tails. Thus, fatter tail of return distributions result in larger variance of the CED value.

**4. Expanation about why empirical results of the relationship between serial correlation and CED is not significant.**

As shown in Appendix D.3, when we move to more sophisticated time series models, the relationship between serial correlation and CED becomes more obscure. The value of CED is simultaneously influenced by various factors, including noise term volatility and multiple serial correlation lag term. Real world data usually show time-varying volatility and could be fitted to various time series model. Then the internal correlation between serial correlation and CED may be veiled by multiple factors.

## 6 Risk contribution

In this section, we show that higher serial correlation result in higher drawdown risk concentrations compared with risk concentrations along the other risk measures.

We construct two-asset portfolios from simulated time series data. Figure 21 shows the risk contribution for volatility, ES and CED. Asset returns are simulated from AR(1) model. We fixed serial correlation of one asset and examine the relationship between the serial correlation and risk contributions of another asset. For each pair of simulated assets, we construct 50/50, 60/40, and 70/30 portfolios.

While the serial correlation to risk contribution is symmetric for volatility and ES. The contribution for CED increase as the serial correlation increase from -0.7 to 0.7. As we increase the asset weight in the portfolio, the risk contribution increases. Our findings in this section is consistent with that of the last simulation section. Since the CED risk is positively correlated with serial correlation, assets with higher serial correlation contribute to more CED when other conditions fixed at a same level.

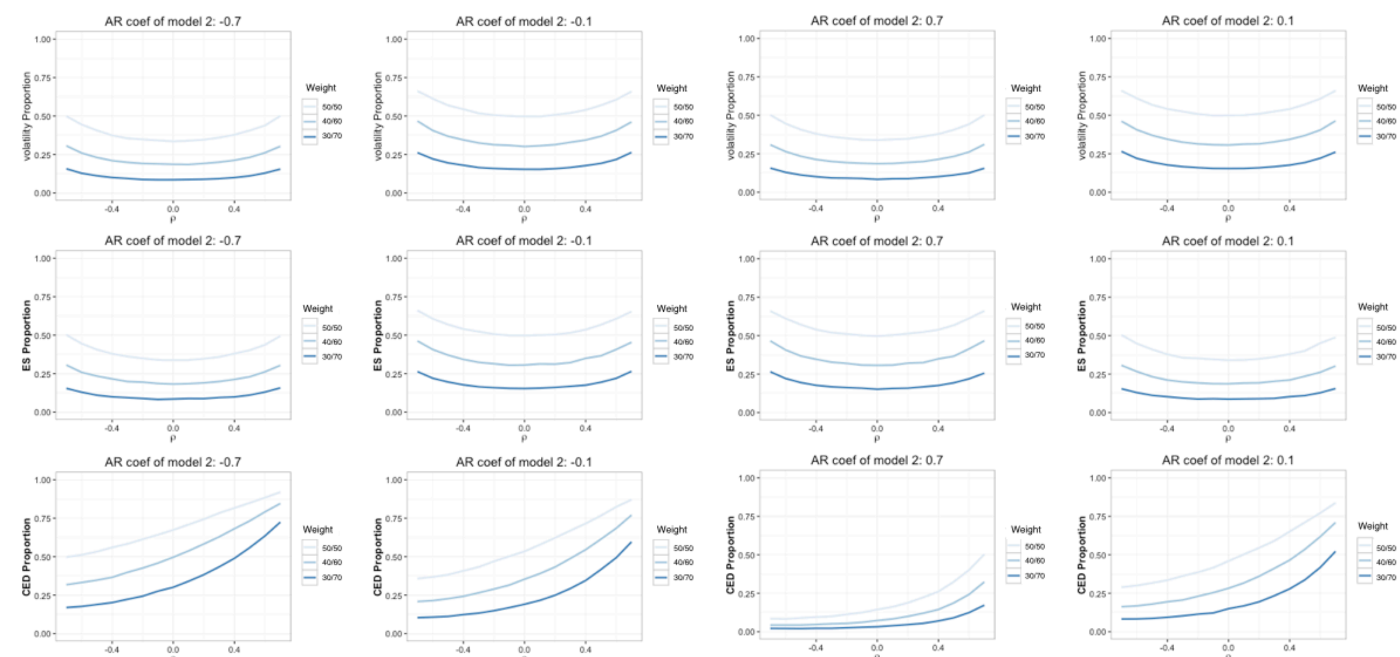



Figure 21: Risk contributin for volatility, ES and CED. Portfolios are constructed from two time series simulated using AR(1) model. In each subplot, we fix autoregression coefficients (which is the serial correlation in AR(1)) of asset 2 (at -0.7, -0.1, 0.1 and 0.7 respectively) and change the coefficient of asset 1. For each pair of simulated assets, we construct 50/50, 60/40, and 70/30 portfolios. The plots shows the asset 1 serial correlation versus asset one risk contribution.

## References


[1] Lisa R Goldberg and Ola Mahmoud. On a convex measure of drawdown risk. *Available at SSRN 2430918*, 2014.

[2] Emilio Barucci. *Financial markets theory: Equilibrium, efficiency and information*. Springer Science & Business Media, 2012.

[3] Andrew W Lo and A Craig MacKinlay. Stock market prices do not follow random walks: Evidence from a simple specification test. *Review of financial studies*, 1(1):41–66, 1988.

[4] James M Poterba and Lawrence H Summers. Mean reversion in stock prices: Evidence and implications. *Journal of financial economics*, 22(1):27–59, 1988.

[5] Jennifer Conrad, Gautam Kaul, and Mahendrarajah Nimalendran. Components of short-horizon individual security returns. *Journal of Financial Economics*, 29(2):365–384, 1991.

[6] Andrew W Lo and A Craig MacKinlay. An econometric analysis of nonsynchronous trading. *Journal of Econometrics*, 45(1):181–211, 1990.

[7] Timothy S Mech. Portfolio return autocorrelation. *Journal of Financial Economics*, 34(3):307–344, 1993.

[8] Mila Getmansky, Andrew W Lo, and Igor Makarov. An econometric model of serial correlation and illiquidity in hedge fund returns. *Journal of Financial Economics*, 74(3):529–609, 2004.

[9] Antonio Di Cesare, Philip A Stork, and Casper G De Vries. Risk measures for autocorrelated hedge fund returns. *Journal of Financial Econometrics*, page nbu023, 2014.

[10] James Douglas Hamilton. *Time series analysis*, volume 2. Princeton university press Princeton, 1994.

[11] James D Hamilton. Analysis of time series subject to changes in regime. *Journal of econometrics*, 45(1):39–70, 1990.

# A Appendix A: Dataset

## A.1 Description

| Symbol | Name | Date range |
|---:|---|---|
| AGG | iShares Core US Aggregate Bond | 2003-09-29 to 2015-12-31 |
| HYG | iShares iBoxx High Yield Corporate Bond | 2007-04-12 to 2015-12-31 |
| TIP | iShares TIPS Bond | 2003-12-08 to 2015-12-31 |
| BCOM | Bloomberg Commodity Index | 1991-01-03 to 2015-12-31 |
| G0O1 | 3-Month U.S. Treasury Bill Index | 1992-04-01 to 2015-12-31 |
| MXEA | MSCI Developed Markets Index | 1970-01-07 to 2015-12-31 |
| MXEF | MSCI Emerging Markets Index | 1988-01-01 to 2015-12-31 |
| RAY | Russell 3000 Index | 1979-01-02 to 2015-12-31 |
| RMZ | MSCI US REIT Index | 2005-06-20 to 2015-12-31 |
| SPX | S&P 500 Index | 1950-01-04 to 2015-12-31 |
| USGG10YR | US Generic Govt 10 Year | 1962-01-03 to 2015-12-31 |

Table 7: Normal VaR and ES under various levels

## A.2 Statistical summary

| Asset | Sharpe | Sd. | Skewness | Kurtosis |
|:---:|---|---|---:|---:|
| AGG | 0.052 | 0.051 | -2.51 | 81.31 |
| HYG | 0.025 | 0.134 | 0.87 | 36.71 |
| TIP | 0.040 | 0.065 | 0.10 | 6.48 |
| BCOM | 0.001 | 0.149 | -0.27 | 4.33 |
| MXEA | 0.030 | 0.155 | -0.32 | 10.74 |
| MXEF | 0.031 | 0.180 | -0.39 | 7.71 |
| RAY | 0.036 | 0.173 | -0.66 | 17.22 |
| RMZ | 0.016 | 0.366 | 0.36 | 13.68 |
| SPX | 0.035 | 0.153 | -0.65 | 21.12 |
| USGG10YR | 0.003 | 0.201 | 0.12 | 8.81 |

Table 8: Statistical Summary of Assets

### A.3 Normal VaR and ES

| Asset | VaR(%) 0.90 | VaR(%) 0.95 | VaR(%) 0.99 | ES(%) 0.90 | ES(%) 0.95 | ES(%) 0.99 |
|---|---|---|---|---|---|---|
| AGG | 0.39 | 0.51 | 0.72 | 0.57 | 0.67 | 0.86 |
| HYG | 1.06 | 1.36 | 1.94 | 1.50 | 1.76 | 2.26 |
| TIP | 0.51 | 0.66 | 0.94 | 0.74 | 0.86 | 1.11 |
| BCOM | 1.20 | 1.54 | 2.18 | 1.65 | 1.94 | 2.50 |
| MXEA | 1.22 | 1.57 | 2.24 | 1.74 | 2.04 | 2.62 |
| MXEF | 1.42 | 1.83 | 2.61 | 2.03 | 2.38 | 3.06 |
| RAY | 1.36 | 1.75 | 2.49 | 1.95 | 2.29 | 2.94 |
| RMZ | 2.92 | 3.75 | 5.32 | 4.08 | 4.79 | 6.18 |
| SPX | 1.21 | 1.57 | 2.22 | 1.73 | 2.03 | 2.61 |
| USGG10YR | 1.62 | 2.08 | 2.95 | 2.23 | 2.62 | 3.39 |

Table 9: VaR and ES calculated using normal distribution assumption various levels

# B Appendix B: Rolling risk diagnostics

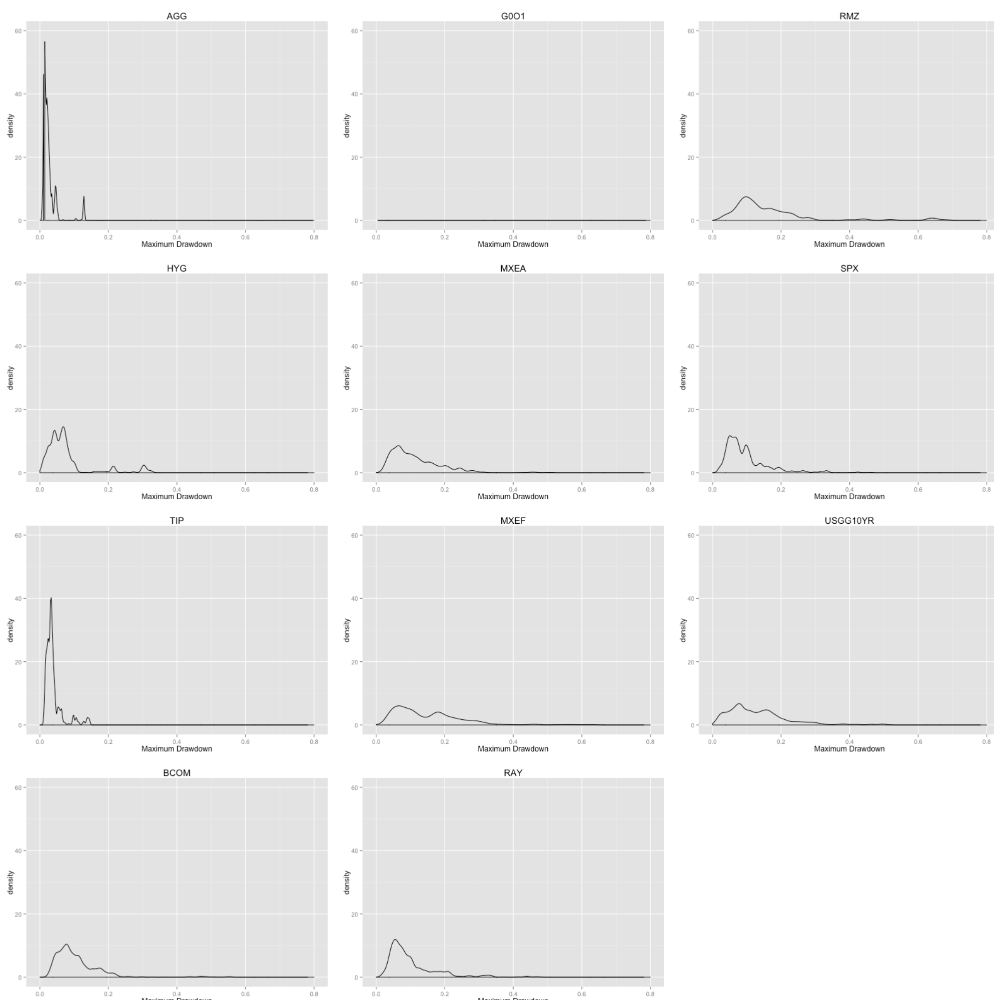



Figure 22: Empirical distribution of maximum drawdown under 6 month rolling window

Table 10: Correlation between CED (confidence level = 0.9) and other risk measures

| Measures | Volatility | VaR | ES |
|---|---|---|---|
| AGG | 0.94 | 0.89 | 0.95 |
| HYG | 0.98 | 0.97 | 0.97 |
| TIP | 0.77 | 0.85 | 0.85 |
| BCOM | 0.84 | 0.89 | 0.89 |
| MXEA | 0.84 | 0.83 | 0.86 |
| MXEF | 0.91 | 0.91 | 0.93 |
| RAY | 0.92 | 0.85 | 0.92 |
| RMZ | 0.96 | 0.96 | 0.97 |
| SPX | 0.84 | 0.81 | 0.84 |
| USGG10YR | 0.91 | 0.93 | 0.95 |

# C Appendix C: Empirical time series study

## C.1 Model Selection Criteria

In Section 3.1 and Section 3.2, two in-sample error measures, AIC and BIC, are used for selecting models. Following is a short introduction about these two criterions.

AIC shorts for Akaike information criterion, which is one of the most commonly used criterion for model selection. Here is the formula for AIC.

$$AIC = 2k - 2ln(L)$$

The k is the number of parameter in the model and L is the maximum value of the likelihood function for an individual model. Intuitively, AIC rewards goodness of fit (as assessed by the likelihood function), but it also includes a penalty that is an increasing function of the number of estimated parameters. Therefore, we want the model with smallest AIC value.

BIC shorts for Bayesian information criterion (BIC) or Schwarz criterion. Here is the formula for BIC.

$$BIC = k \times ln(n) - 2ln(L)$$

The only difference between AIC and BIC is the scalar for the number of features, k. In most of the cases when n is larger than 7, $log(n) > 2$. Therefore, BIC tends to penalize more on the complexity of the model.

### C.2 Relationship between serial correlation and risk measures

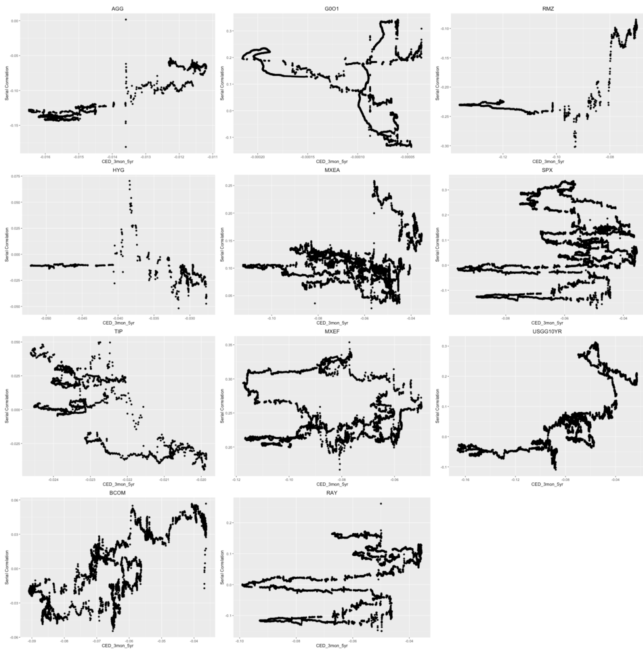


Figure 23: First-order serial correlation calculated using AR(1) model versus CED

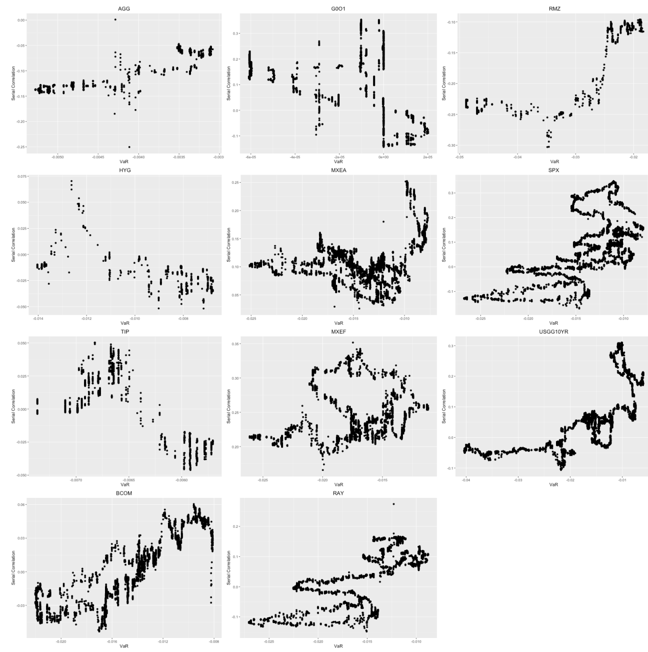


Figure 24: First-order serial correlation calculated using AR(1) model versus VaR

### C.3 Regime switching model results

Table 11: Summary statistics of two regimes for various assets

| | Volatility | | Skewness | | Kurtosis | |
|---|---|---|---|---|---|---|
| Asset | Regime 1 | Regime 2 | Regime 1 | Regime 2 | Regime 1 | Regime 2 |
| AGG | 0.141 | 0.036 | -1.59 | 0.01 | 17.15 | 0.60 |
| HYG | 0.265 | 0.055 | 0.60 | 0.00 | 8.76 | 0.83 |
| TIP | 0.113 | 0.049 | 0.18 | -0.06 | 2.50 | 0.16 |
| BCOM | 0.205 | 0.096 | -0.25 | -0.04 | 1.92 | 0.28 |
| MXEA | 0.253 | 0.101 | -0.14 | -0.02 | 4.26 | 0.25 |
| MXEF | 0.303 | 0.119 | -0.08 | -0.06 | 2.39 | 0.38 |
| RAY | 0.307 | 0.114 | -0.39 | -0.06 | 6.43 | 0.55 |
| RMZ | 0.661 | 0.159 | 0.29 | -0.15 | 2.92 | 0.79 |
| SPX | 0.260 | 0.099 | -0.43 | -0.02 | 9.11 | 0.56 |
| USGG10YR | 0.314 | 0.098 | 0.09 | -0.05 | 2.67 | 1.17 |

## D Appendix D: Simulations specifics

### D.1 Noise term with normal distribution

We use some simple assumptions and parameters for simulation with normal distribution as follows:

1. Noise terms in the time series model follow Normal distribution with standard deviation of 0.01: $\epsilon \sim N(0, 0.0001)$.

2. Risk measures including volatility, VaR, ES and maximum drawdown are calculated based on simulated time series with path length 1000. The calculation of maximum drawdown is replicated 1000 times in order to obtain the maximum drawdown distribution and its tail mean (CED).

3. All time series parameters in this section range from -0.9 to 0.9. For time series with multiple parameters such as AR(2), ARMA(1, 1), the parameters are cartesian product of arithmetic progressions range from -0.9 to 0.9. Note that to take the stationary of AR and ARMA models into consideration (all moving averages are stationary, but the AR and ARMA model have to meet certain criteria to be stationary), not all parameters in the cartesian product are used in the simulation.

## D.2 Noise term with student t distribution

We use some assumptions and parameters for simulation with student t distribution as follows:

1. White noise terms are simulated using t-distribution (degree of freedom = 4). Financial return usually have fat-tail distributions.

2. Set path length to 63, which is the number of trading days in three month.

3. Time series parameters all range from -0.3 to 0.3, which is similar to the financial time series data in the real world. Usually financial data does not show strong serial correlation. If we could narrow down our scope to smaller serial correlations we may found that the serial correlation is more closely related with various risk measures.

## D.3 Simulation results of other time series model

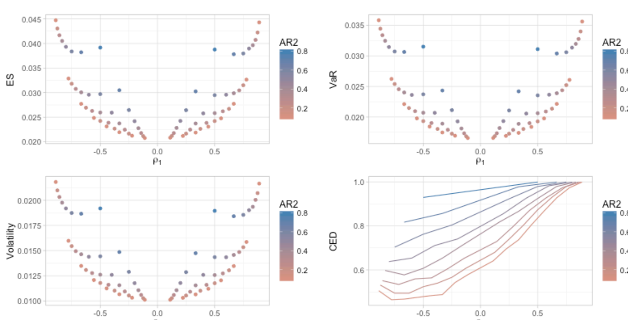



Figure 25: AR(2): Relationship between serial correlation $\rho_1$ and risk measures ($\kappa_2 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

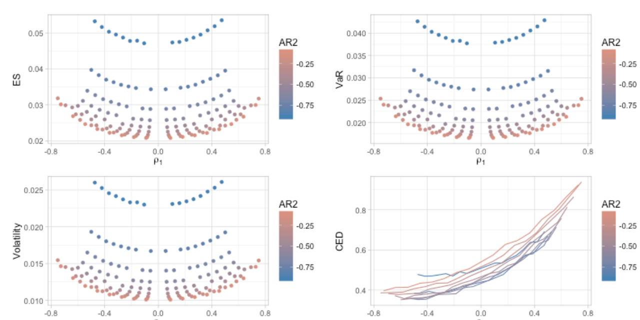



Figure 26: AR(2): Relationship between serial correlation $\rho_1$ and risk measures ($\kappa_2 < 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

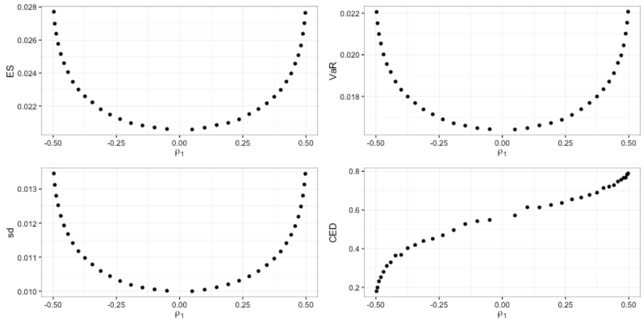



Figure 27: MA(1): Relationship between serial correlation $\rho_1$ and risk measures
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

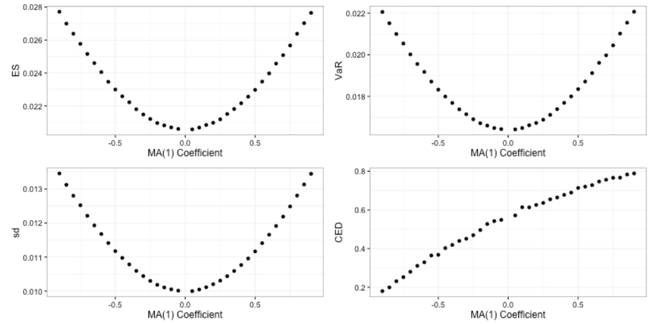



Figure 28: MA(1): Relationship between model coefficient and risk measures
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

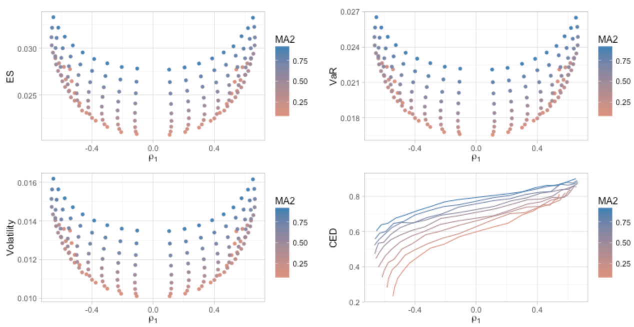



Figure 29: MA(2): Relationship between serial correlation $\rho_1$ and risk measures ($\theta_2 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

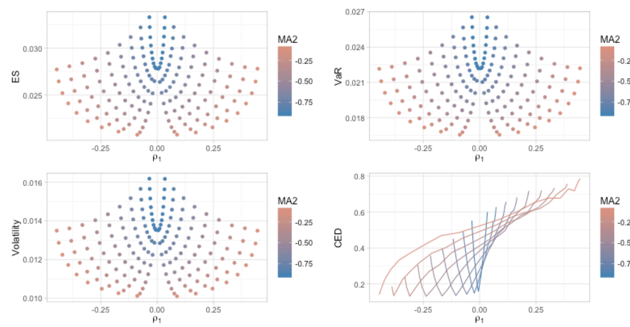



Figure 30: MA(2): Relationship between serial correlation $\rho_1$ and risk measures ($\theta_2 < 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

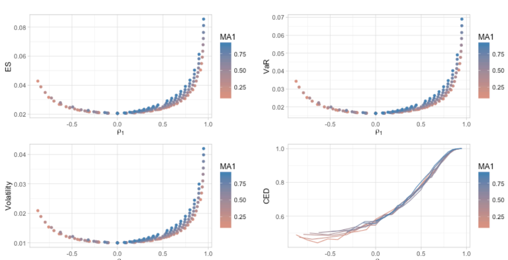



Figure 31: ARMA(1, 1): Relationship between serial correlation $\rho_1$ and risk measures ($\theta_1 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

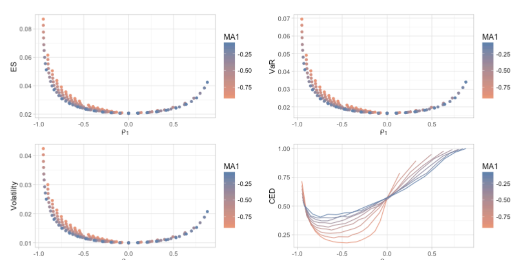



Figure 32: ARMA(1, 1): Relationship between serial correlation $\rho_1$ and risk measures ($\theta_1 > 0$)
(Simulation path length: 1000, $\epsilon_t \sim N(0, 0.0001)$)

### D.4 Maximum drawdown distribution descriptive statistics

| | Mean | Sd | Skewness | Kurtosis |
|---|---|---|---|---|
| $\kappa_1 = -0.75$ | 0.0 | 0.015 | 0.009 | 0.032 |
| $\kappa_1 = -0.5$ | 0.0 | 0.012 | 0.006 | 0.021 |
| $\kappa_1 = -0.25$ | 0.0 | 0.010 | 0.001 | 0.010 |
| $\kappa_1 = 0.25$ | 0.0 | 0.010 | -0.009 | 0.004 |
| $\kappa_1 = 0.5$ | 0.0 | 0.012 | -0.012 | -0.008 |
| $\kappa_1 = 0.75$ | 0.0 | 0.015 | 0.013 | 0.016 |

Table 12: Statistics of simulated distribution of AR(1)

| | Mean | Sd | Skewness | Kurtosis |
|---|---|---|---|---|
| $\kappa_1 = -0.3$ | 0.0 | 0.015 | -0.53 | 18.8 |
| $\kappa_1 = -0.2$ | 0.0 | 0.014 | -0.13 | 6.4 |
| $\kappa_1 = -0.1$ | 0.0 | 0.014 | -0.05 | 6.3 |
| $\kappa_1 = 0.1$ | 0.0 | 0.014 | -0.30 | 15.8 |
| $\kappa_1 = 0.2$ | 0.0 | 0.014 | 0.13 | 7.0 |
| $\kappa_1 = 0.3$ | 0.0 | 0.015 | -0.06 | 7.4 |

Table 13: Statistics of simulated distribution of AR(1) (student t noise)

| | Mean | Sd | Skewness | Kurtosis |
|---|---|---|---|---|
| $\alpha = 0.05$ | 0.0 | 0.029 | -0.003 | 0.088 |
| $\alpha = 0.08$ | 0.0 | 0.013 | 0.001 | -0.033 |
| $\alpha = 0.11$ | 0.0 | 0.011 | -0.000 | -0.001 |
| $\alpha = 0.14$ | 0.0 | 0.011 | -0.004 | 0.006 |
| $\alpha = 0.17$ | 0.0 | 0.013 | -0.014 | -0.004 |
| $\alpha = 0.20$ | 0.0 | 0.029 | 0.005 | 0.054 |

Table 14: Statistics of simulated return distribution of AR(1)+ GARCH(1,1)