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# Supply function equilibria: Step functions and continuous representations

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**Abstract**

In most electricity markets generators must submit step-function offers to a uniform price auction. These markets are often modelled as simpler pure-strategy Supply Function Equilibria (SFE) with continuous supply functions. Critics argue that the discreteness and discontinuity of the required steps drastically change Nash equilibria, invalidating predictions of the SFE model. We prove that there are sufficient conditions, offered quantities can be continuously varied, offered prices are selected from a finite set, and the density of the additive demand shock is not too steep, where the resulting stepped SFE converges to the continuous SFE as the number of steps increases, reconciling the apparently very disparate approaches to modelling electricity markets.



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## 1. Introduction

This paper fills an increasingly embarrassing gap between theory and reality in multi-bid auction markets such as electricity wholesale markets, where costs are common knowledge. The leading equilibrium theory underpinning market analysis and most econometric estimations of strategic bidding behaviour in electricity auctions assume that generating companies offer a piecewise differentiable supply function, specifying the amounts they are willing to supply at each price. The market operator aggregates these supplies and clears the market at the lowest price at which supply is equal to demand – the Market Clearing Price (MCP). According to this theory, generators choose their offers by optimizing against the smooth residual demand, which gives well-defined first-order conditions. In reality, most wholesale markets require offers to take the form of a step function. This results in a partly non-differentiable residual demand, which critics have argued will drastically change Nash equilibria in the market [9].

Faced with this, economists have chosen either to model the market as a discrete unit auction [9], which typically leads to complex mixed-strategy equilibria, or have argued that with enough steps, the residual demand can be smoothed and then treated as differentiable [13]. The difference between these approaches appears dramatic, and it is the purpose of this paper to demonstrate that there are situations (sufficient conditions) where it is legitimate to approximate step functions by smooth differentiable functions, and hence to draw on the well-developed theory associated with continuous supply functions.

We consider a discrete version of Klemperer and Meyer's [20] Supply Function Equilibrium (SFE) model with additive demand shocks. Price levels are fixed and bidders chose one quantity per price level. By fixing the price levels, producers' payoffs become differentiable with respect to the remaining strategy variables that are quantities. We also show that for circumstances where an increasing pure-strategy differentiable SFE exists in the game with a continuous strategy space, there is a similar pure-strategy equilibrium in our discrete model with stepped offer functions that converges to the equilibrium of the continuous game, unless the density of demand shock is too steep. Thus under these circumstances, continuous SFE are robust to our discretization of the strategy space. Our novel argument relies on the convergence theory of numerical analysis for Ordinary Differential Equations (ODEs). Note that our purpose, methodology and results significantly depart from previous convergence studies of equilibria in multi-unit auctions by e.g., Reny [28], McAdams [24] and Kastl [19]. These approaches use existence of Nash equilibria for a discrete strategy space and equilibrium convergence to establish that there exists equilibria in divisible-good auctions (the limit game). In contrast to previous studies, we analyze the reverse problem. Our contribution is to provide a discretization of the strategy space together with conditions under which continuous equilibria are robust to this discretization, i.e., existence of equilibria in the limit game imply existence of a similar equilibrium in the discrete game.

### 1.1. Modelling electricity markets

Electricity liberalization creates electricity markets, and the two key markets to model are the day-ahead market and the balancing market (in the English Electricity Pool they were combined). In most such markets there is a separate auction for each delivery period, typically an hour. Normally, the post-2001 British balancing mechanism being an exception, the markets are organized as uniform price auctions. Rationing of excess supply at the clearing price may be necessary and so market designs must specify how rationing will take place, normally by pro-rata on-the-margin rationing [21]. Hence, only incremental supply at the clearing price is rationed

Table 1
Constraints on bids in various liberalized markets.

| Market | Installed capacity [MW] | Max steps | Price range [per MWh] | Tick size [per MWh] | Quantity multiple | No. quantities/ No. prices |
|---|---|---|---|---|---|---|
| Nord Pool spot | 90,000 | 64 per bidder | −€200–€2000 | €0.1 | 0.1 MWh | 41 |
| ERCOT balancing | 70,000 | 40 per bidder | −$1000–$1000 | $0.01 | 0.01 MWh | 35 |
| PJM | 160,000 | 10 per genset | 0–$1000 | $0.01 | 0.01 MWh | 160 |
| English Pool to 2001 | 80,000 | 5 per genset | 0–£2500 | £0.01 | 1 MW (half-hour) | 0.3 |
| Spain day-ahead | 46,000 | 5 per genset | 0–€180.3 | €0.01 | 0.1 MWh | 25 |
| Australia NEM | 56,000 | 10 per genset | −$1000–$12,500 | $0.01 | 1 MWh | 0.04 |

and the accepted share of each producer's incremental supply at this price is proportional to the size of its increment.

Producers in most such markets submit non-decreasing step function offers to the auction (and in some markets agents, normally retailers, may submit non-increasing demands). With its offer the producer states how much power it is willing to generate at each price. The successive offers specify a quantity that would be available at a fixed per unit price. The smallest step in the ladder is given by the number of allowed decimals in the offer. Thus all prices and quantities in an offer have to be a multiple of the price tick size and quantity multiple (or lot size), respectively. Table 1 summarizes these and other offer constraints for some liberalized electricity markets.

In particular most electricity markets have significantly more possible quantity levels compared with possible price levels.[1] In that sense, the quantity multiple is small relative to the price tick size. Fig. 1 presents an example of aggregated bid and offer ladders from the Amsterdam Power Exchange (APX).

Offers are submitted ahead of time (typically the day before) and may have to be valid for an extended period (e.g., 48 half-hour periods in the English Pool) during which demand can vary significantly. Plant may fail suddenly, requiring replacement at short notice, so the residual demand (i.e., the total demand less the supply accepted at each price from other generators) may shift suddenly, further increasing the range over which offers are required. Thus models of electricity auctions normally assume that an additive demand shock is realized after offers have been submitted. The production cost of a plant is primarily determined by fuel costs and its efficiency, which are well-known and common knowledge.

In Britain there are several hundred production units and until 2001 each unit could submit offers with up to 5 increments to the Electricity Pool.[2] In Australia each generator can submit up to 10 steps per production unit. Thus Green and Newbery [13] argued that the natural way to model such a market was to adapt Klemperer and Meyer's [20] (hereafter KM) Supply Function Equilibrium (SFE) formulation, in which costs are common knowledge and firms make offers to a uniform-price auction before the realization of an additive demand shock. Units of electricity are assumed to be divisible, so firms offer continuous supply functions (SFs) to the auction. Accordingly, residual demand is piecewise differentiable and, under some additional restrictions,

[1] As shown in Table 1 most auctions also have restrictions on the number of allowed steps per bidder. Kastl [19] analyzes how such restrictions influence equilibria in divisible-good auctions.

[2] In 2001 the Pool was replaced by bilateral contracting, leaving only the Balancing Mechanism as a bid and offer "market" with complex price rules. Under the Pool generators offered up to three monotonically increasing step supply functions for each generating unit.

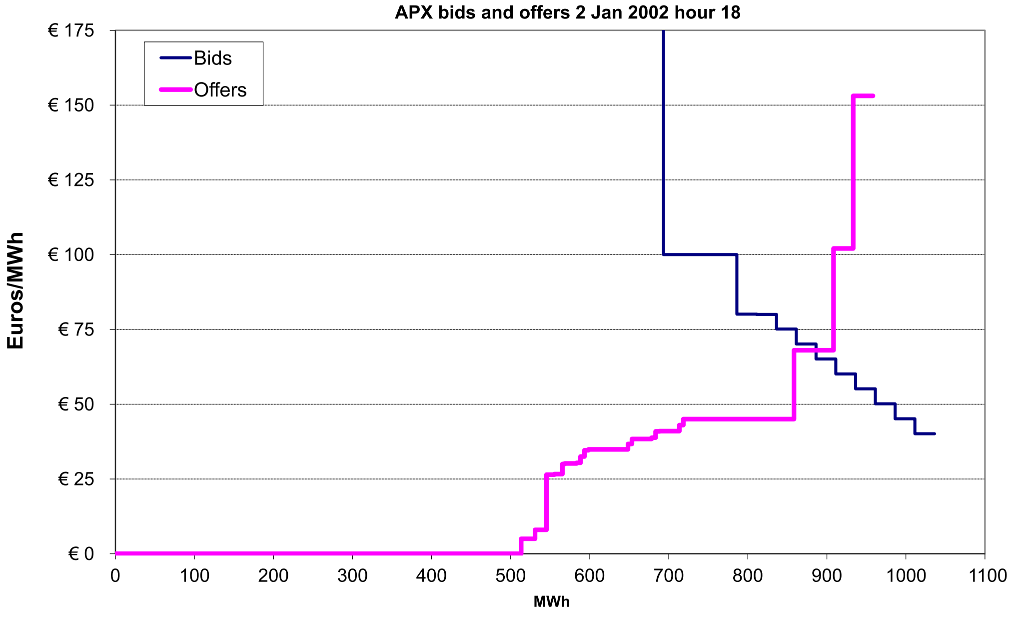


Fig. 1. Market clearing price detail from Amsterdam Power Exchange (APX) Hour 18, 2 January 2002.

each firm faces a non-increasing residual demand, which offers the prospect of a well-defined best response function at each point. An equilibrium is such that each firm ensures that given the supplies offered by all other firms, it maximizes its profits for each realization of demand, i.e., offers are ex-post optimal.

With a uniform price auction and a continuous SF the effect of lowering the price to capture the marginal unit lowers the price for the large quantity of inframarginal units (the 'price' effect) while only capturing an infinitesimal sale (the 'quantity' effect). The quantity effect is small if competitors' supply functions are close to inelastic and as a result very collusive supply function equilibria can be supported.

The first order conditions for the SFE are given by a set of linked differential equations. Analytical solutions can be found for the case of equal and constant marginal costs and linear marginal costs. Closed form solutions are also available for symmetric firms and perfectly inelastic demand [29,2]. With numerical algorithms SFE can be calculated for markets with asymmetric firms and general cost functions [14,4].

Green and Newbery [13] argued that the large number of possible steps meant that, given the uncertainty about, and variability of, demand, such steps could reasonably be approximated by continuous and piecewise differentiable functions. von der Fehr and Harbord [9], however, argued that the ladders were step functions that were not continuously differentiable, and it would be inappropriate to assume that they were. Instead, they modelled the electricity market as a multiple-unit uniform-price auction in which each generating unit submits a single bid from a continuum of prices (although in all existing electricity markets the set of prices is finite) for its entire capacity (supplies are chosen from a discrete set). With these assumptions, competition is almost everywhere in prices, with winner takes all over the whole step. Thus the 'price' effect, which can be made infinitesimally small in their model, of stealing some market is no longer larger than the now significant 'quantity' effect. If a producer is pivotal, i.e., competitors are not able to meet maximum demand without this producer, and minimum demand is sufficiently low,[3] then such Bertrand competition destroys any pure strategy equilibrium, leaving only a mixed-strategy equilibrium in which firms randomize over a distribution of possible prices. Choosing a mixed strategy in prices means that prices will be unstable, even under unchanged conditions; according to von der Fehr and Harbord [9], market designs that require stepped offers have an inherent price instability. Solving for such mixed strategy equilibrium is extremely difficult, so this result was rather pessimistic – existing supply function models were claimed to be flawed but suitable auction models were intractable when offer curves have more than one step and the demand uncertainty is sufficiently large.

If one can show that the continuous SFE model is a valid approximation, this would justify the common practice in empirical work of smoothing the residual demand, allowing a well-defined best response to be identified. Three empirical studies applying this approach to the balancing market in Texas (ERCOT) suggest that a continuous representation is consistent with profit-maximizing behaviour for the largest producers in this market [27,17,31]. Sweeting [32] similarly estimates best responses to smoothed residual demand schedules in the English Electricity Pool to characterize the exercise of market power.

Wolak [35] does not reject the hypothesis that generators bidding into the Australian market choose offers in order to maximize profits. He used observed bidding behaviour to back out the unobserved underlying cost and contract positions of generators. He notes that continuity of the SF gives a one-to-one mapping between the shocks and the market price and hence the best response does not depend on the distribution of shocks. Wolak smooths the ex post observed stepped residual demand schedule to find the best response supply, which is then compared with the actual supply (chosen before the residual demand was realized). Wolak [34] notes, however, that, unlike the continuous approximation, the choice of an optimal step function will depend on the distribution of the shocks. The restriction to stepped offers also explains why a firm's expected profit would theoretically be 11–17% higher for a best-response with continuous offers compared to the expected profit under the observed stepped offers in the Australian market.

### *1.2. Reconciling stepped and continuous supply function*

The central question raised by the von der Fehr and Harbord critique and the empirical applications is whether smoothing and/or increasing the number of steps in the ladder can reconcile the step function and continuous approaches to modelling electricity markets. Do markets with uncertain or variable demand and sufficiently finely graduated bidding ladders converge to supply function equilibria, or do they remain resolutely and significantly different? The central claim of this paper is that under well-defined conditions (the density of the demand shock must not be too steep), which provides an intellectually solid basis for accepting the SFE approach.

Fabra et al. [8] argue that the difference between the two approaches derives from the finite benefit of infinitesimal price undercutting in the ladder model. But this argument assumes that prices can be infinitely finely varied. In practice, the price tick size cannot be less than the smallest unit of account (e.g., 1 cent, 1 pence, normally per MWh), and might be further restricted, as in the multi-round California PX auction. In this case, the undercutting strategy is not necessarily profitable, because the price reduction cannot be made arbitrarily small. Whereas von der Fehr

[3] With pivotal producers and sufficiently small demand variation, von der Fehr and Harbord [9] show that there is an asymmetric NE, where one firm sets the market price by offering its capacity at the price cap and the other firm bids sufficiently low to avoid being undercut.

and Harbord [9] considered the extreme case when the set of quantities is finite and the set of prices is infinite, this paper considers the other extreme when the set of quantities is infinite and the set of prices is finite.

We show that there are circumstances when for sufficiently many allowed steps in the bid curves,[4] there is a pure-strategy NE with a set of step functions and a market-clearing price (MCP) that converge to the supply functions and price, respectively, predicted by the SFE model. We use a technique related to Dahlquist/Lax–Richtmyer's equivalence theorem to establish this. Although this is a standard technique for analyzing the convergence of numerical methods, we are apparently the first to apply this method as a crucial step when analyzing convergence of Nash equilibria. As in Dahlquist/Lax–Richtmyer's equivalence theorem [23], convergence of stationary solutions (solutions to the first-order conditions) requires that the discrete system is consistent with the continuous system – the first-order conditions of the two systems converge – and that the discrete solution is stable, i.e., the difference between the two stationary solutions does not grow too rapidly. Moreover, solutions should exist in both the discrete and continuous system. To get convergence of equilibria in the two systems, the converging stationary solutions must in addition be global profit maxima in their respective systems, which is a requirement beyond Dahlquist/Lax–Richtmyer.

If a producer is pivotal, the stationary solutions with the lowest mark-ups will typically not be equilibria, because such solutions give pivotal producers incentives to withhold output until the capacity constraints of the competitors bind, so that the market price can be significantly increased. Disregarding such solutions, assuming concave demand, convex costs and a density of the additive shock that does not have too steep a slope, we prove that remaining monotonically increasing[5] stationary solutions of the discrete and continuous systems are Nash equilibria if the number of price levels in the discrete system is sufficiently large. Hence, under these circumstances convergence of the Nash equilibria (NE) follows in a straightforward way from convergence of the stationary solutions. But it is not self-evident that continuous SFE are robust to our discretization of the strategy space. We show that in cases with constant marginal costs and steeply decreasing shock densities, a continuous SFE sometimes exist even if a corresponding stepped SFE does not exist in the discrete system for small tick-sizes.

Our model has parallels in the theoretical work by Anderson and Xu [3]. They analyze a duopoly model of the Australian electricity market, where each of two producers first chooses and discloses its price grid and later its offers at each price. They assume demand is random but inelastic, with an elastic outside supply at some price, which effectively sets a price ceiling. In the uniform-price/single-price auction, they show that, under certain conditions, the second stage has a pure strategy equilibrium in quantities, although the first stage only has mixed strategies in the choice of prices. Their second stage has similarities with our model, because prices are chosen from discrete sets in both models. On the other hand, generators' chosen price vectors generally differ as the declared prices are chosen by randomizing over a continuous range of prices. In our paper, however, the available price levels are given by the market design and accordingly are the same for all firms. Moreover, Anderson and Xu [3] do not compare their stepped equilibrium with a continuous SFE.

Wolak [36], in a path-breaking empirical paper, develops a similar model of the Australian market to that of Anderson and Xu, but Wolak observes the step function bids, the contract

[4] In practice bidders may not always use all allowed steps, as the effort required to submit a step may not be negligible [18]. Our study neglects such costs.

[5] A reasonable restriction, as partly decreasing offers are normally not allowed in electricity markets.

positions and the market clearing prices, and hence is able to construct the ex post residual demand facing any generator. Wolak derives first-order conditions that are satisfied on average for a firm that chooses a stepped offer curve in order to maximize its expected profit against a stepped and uncertain residual demand. The uncertainty in the residual demand could be caused by demand shocks, competitors' that randomize their offers or that have partly private costs. This allows the cost function to be identified, and to test whether on average there is any evidence to reject the maintained hypothesis that the generator selects stepped bids to maximize profits, given its contract position. The same model is used by Gans and Wolak [10] to assess the impact of vertical integration between a large electricity retailer and a large electricity generator in the Australian market.

Anderson and Hu [4] develop a numerical method for solving asymmetric supply function equilibria. To achieve this they approximate equilibria of the continuous system with piecewise linear supply functions and discretize the demand distribution. They show that equilibria of this approximation converge to equilibria in the original continuous model. The piecewise linear bid functions are carefully chosen to avoid the influence of kinks in the residual demand curves. These approximate bid curves are drawn so that all producers have locally well-defined derivatives in their residual demand curves for all possible discrete demand realizations. Anderson and Hu's discrete model is motivated by its computational properties. In contrast, as in real electricity markets, we deal with the worst kinks possible, i.e., steps, and we do so explicitly, because we want to analyze equilibrium convergence for a more problematic case where convergence has been disputed both empirically and theoretically.

Kastl [19] analyzes divisible-good auctions with certain demand and private values, i.e., bidders have incomplete information. This set-up, introduced by Wilson [33], is mainly used to analyze treasury auctions. Kastl assumes that both quantities and prices are chosen from continuous sets, but the maximum number of steps is restricted. He verifies consistency, i.e., that the first-order condition (the Euler condition) of the stepped bid curve converges to the first-order condition of a continuous bid-curve when the number of steps becomes unbounded. More generally, the convergence problem under study is related to the seminal paper by Dasgupta and Maskin [7] on games with discontinuous profits. But the purpose in Dasgupta and Maskin [7], and in related papers by Bagh [5], Gatti [11], Kastl [19], McAdams [24], Reny [28] and Simon [30] is different to ours. They want to use existence of equilibria in discrete approximations of the strategy space in order to prove existence of NE in the limit game, where the strategy space is continuous. In contrast, in response to the critique of von der Fehr and Harbord [9], we are interested in the inverse problem: when does an equilibrium in the limit game imply existence of a similar discrete NE when the strategy space is discretized? Or equivalently, when is the supply function equilibrium robust to discretizations of the strategy space?

## 2. The continuous supply function model

The aim of our study is to establish circumstances when existence of a Supply Function Equilibrium (SFE) in the continuous case, SFs, implies existence of a Stepped SFE (SSFE) that converges to this equilibrium. This section sets out the continuous Supply Function Model, following KM's approach as it would apply to an electricity market with uniform pricing. As in Klemperer and Meyer [20] (KM) we consider a uniform price auction where all accepted offers are paid the Market Clearing Price and calculate a pure strategy NE of a one-shot game, in which each generator, $i$, chooses a supply function that maximizes its profit. The following condition will be assumed to hold throughout the paper.

**Standing Assumption.** Continuous demand at $p$ is given by $d(p) + \varepsilon$, where $\varepsilon$ is an additive demand shock and $d(p)$ is the twice continuously differentiable concave deterministic non-increasing demand curve. The additive demand shock has a strictly positive probability density, $g(\varepsilon)$, with continuous and bounded first- and second-order derivatives on its support $[\underline{\varepsilon}, \overline{\varepsilon}]$. The cost function of firm $i$, $C_i(s_i)$, is an increasing, convex continuous function with bounded and continuous first-, second- and third-order derivatives up to the capacity constraint $k_i$. Costs are common knowledge.

These assumptions are similar to KM, the main difference is that we allow for capacity constraints, which are important in electricity markets. Let $s_i(p)$ be the quantity of piecewise differentiable supply offered by a risk-neutral electricity producer or generator, $i$ ($i = 1, \dots, N$), and $s_{-i}(p)$ denote competitors' collective quantity offers, both at price $p$. We consider cases where offers are strictly increasing. This monotonicity property is natural for electricity markets, where offers are restricted to be increasing, although not necessarily strictly increasing. As in KM we do not explicitly impose monotonicity constraints as that property will follow from our assumptions. KM show that the vector[6] of SFs, $\mathbf{s}(p)$, must maximize each generator's profit

$$p\big[\underbrace{d(p) + \varepsilon - s_{-i}(p)}_{s_i}\big] - C_i\big\{\underbrace{d(p) + \varepsilon - s_{-i}(p)}_{s_i}\big\}, \quad i = 1, 2, \dots, N,$$

for each realized price $p \in [a, b]$ – the range of eligible prices, where $b \leqslant P^M$ (the price cap). The resulting system of first-order conditions is given by:

$$-s_i(p) + \big[p - C_i'\big(s_i(p)\big)\big]\big(s_{-i}'(p) - d'(p)\big) = 0. \tag{1}$$

This system has one degree of freedom, and hence an infinite number of potential solutions. We can therefore index the continuum of continuous SFE by boundary conditions $\mathbf{s}(b) = \mathbf{q}$. Baldick and Hogan [6] show that the system of differential equations can be written in the standard form of an ordinary differential equation (ODE):

$$s_i'(p) = \frac{d'(p)}{N-1} - \frac{N-2}{N-1}\frac{s_i(p)}{p - C_i'(s_i(p))} + \frac{1}{N-1}\sum_{k \neq i}\frac{s_k(p)}{p - C_k'(s_k(p))}. \tag{2}$$

The first-order condition in (2) is not directly applicable to parts of the offer curve that are always or never accepted in equilibrium. We will assume that offers that are always accepted are perfectly inelastic and offers that are never accepted are perfectly elastic up to the capacity limit, $k_i$. This shape discourages competitors from deviating from a candidate SFE, and accordingly gives the largest set of solutions for the set of SFE:

$$s_i(p) = s_i(a) \quad \text{if } p < a, \quad \text{and} \quad s_i(p) = k_i \quad \text{if } p > b. \tag{3}$$

The next definition provides the notation for solutions to the continuous system.

**Definition 1.** A continuous stationary solution $\breve{\mathbf{s}}(p)$ is a set of continuous solutions to the system of the differential equations (2) on the price interval $[a, b]$ with boundary conditions $\mathbf{s}(b) = \mathbf{q}$. In addition, $\breve{\mathbf{s}}(p)$ is a segment of a continuous SFE if the set of strategies $\breve{\mathbf{s}}(p)$ formed by taking $\mathbf{s}(p) = \breve{\mathbf{s}}(a)$ if $p < a$, $\mathbf{s}(p) = \breve{\mathbf{s}}(p)$ if $p \in [a, b]$, and $\mathbf{s}(p) = \mathbf{k}$ if $p > b$, is an SFE.

[6] Vectors are in bold to distinguish them from scalars.

### 2.1. Positive mark-ups and monotonicity

To avoid singularities in the system of differential equations in (2), we restrict attention to cases where all producers have positive mark-ups at their clearing point, ensured by Assumptions 1 and 2 below. Otherwise it would not follow from Picard–Lindelöf's theorem that the continuous system has a unique solution for a given vector of boundary values $\mathbf{q}$, and non-uniqueness would cause problems for the analysis of convergence of equilibria in later sections.

The first assumption ensures that the minimum demand is sufficiently high. Let $n$ be the producer with the highest marginal cost at zero output, $C_n'(0)$. The KM equation (1) implies that its output can only be non-positive if the market clears at a price at or below its marginal cost. From the same equation we see that competitors, whose marginal costs at zero output are no higher than $C_n'(0)$, will offer positive outputs with positive mark-ups for prices above $C_n'(0)$. Now, if the lowest demand curve crosses their total marginal cost curve at a price above $C_n'(0)$ then producer $n$'s output must be positive at this outcome, motivating the following assumption on costs and shocks:

**Assumption 1.** Let $n$ be the producer with the highest marginal cost at zero output. The lowest shock is such that $C_{-n}'(\underline{\varepsilon} + d(C_n'(0))) > C_n'(0)$, where $C_{-n}(s_{-n})$ is the minimum (efficient) production cost of competitors producing $s_{-n}$ units.

Where there are several such producers, the assumption is satisfied for all producers that have the same highest marginal cost at zero output. In addition, electricity markets impose monotonicity requirements, motivating

**Assumption 2.** There is a continuous stationary solution $\breve{\mathbf{s}}(p)$ of (2) that is strictly increasing with a bounded slope on the interval $[a, b]$ for each firm $i = 1, \ldots, N$.

The next subsection shows that equilibria with this property can be found for a wide range of circumstances. From the two assumptions we can now establish that mark-ups are positive (all proofs are in Appendix A):

**Lemma 1.** *If Assumption 1 holds and a continuous stationary solution $\breve{\mathbf{s}}(p)$ satisfies Assumption 2, then the mark-up, $p - C_i'(\breve{s}_i(p))$, is bounded below by a positive constant that is independent of $i$ and $p \in [a, b]$.*

From twice continuous differentiability of demand and costs, and positive mark-ups bounded away from zero, it now follows from (2) that

**Corollary 1.** *If Assumption 1 holds and a continuous stationary solution $\breve{\mathbf{s}}(p)$ satisfies Assumption 2, then $\breve{\mathbf{s}}(p)$ is twice continuously differentiable on $(a, b)$.*

### 2.2. Existence

von der Fehr and Harbord [9] and others have justifiably questioned whether smooth SFE are robust to discretizations of the strategy set. Thus we are mainly interested in when SFE satisfying Assumption 2 are robust to the particular discretizations that we consider. Contrary to McAdams [24] and Reny [28] and related studies, we are not *per se* interested in establishing existence of an equilibrium in the limit game with a continuous strategy set. However, to ensure that our robustness analysis applies fairly generally, and not just to an empty or small set of cases, the brief extension below establishes that there exist a wide class of continuous SFE satisfying Assumption 2.

For a duopoly market with asymmetric firms, Assumption 1 puts a lower bound on the lowest demand shock. To avoid kinks in asymmetric SFE on the interval $[a, b]$, which may occur when firms' production capacities bind at different prices, the range of demand shocks must be further restricted.[7] Thus Proposition 1 below is conditional on an upper bound on the highest demand shock in the asymmetric case. This upper bound depends on two parameters, $s_1^*$ and $s_2^*$. It follows from Anderson [1] that they are uniquely defined by the implicit relations below, if demand is decreasing and concave and costs are increasing and convex. Anderson [1] proves that firm $i$'s capacity constraint binds at or below $p_i^*$, which is implicitly defined by

$$\left[p_i^* - C_i'(k_i)\right]\left(-d'\left(p_i^*\right)\right) = k_i, \quad i = 1, 2,$$

and that $s_j^*$, the lowest output of firm $j \neq i$ at $p_i^*$, provided its capacity constraint is not yet binding, satisfies

$$s_j^* - \left[p_i^* - C_i'\left(s_j^*\right)\right]\left(-d'\left(p_i^*\right)\right) = 0, \quad i, j = 1, 2 \text{ and } j \neq i.$$

Using these implicit definitions of $s_1^*$ and $s_2^*$, it follows from KM and Anderson [1]:

**Proposition 1.** *Continuous SFEs $\breve{\mathbf{s}}(p)$ satisfying Assumption 2 exist for markets with* (i) *multiple symmetric firms with strictly convex marginal costs facing concave demand, $d(p)$, or* (ii) *an asymmetric duopoly with convex marginal costs, and concave demand that satisfies $d'''(p) \leqslant 0$, $-2d''(p)\max(k_1, k_2) < (-d'(p))^2$, and additive demand shocks that are bounded below by Assumption 1 and above by $\min(k_1, k_2, s_1^*, s_2^*) > \bar{\varepsilon} + d(C'(\min(k_1, k_2, s_1^*, s_2^*)))$, where $C(s)$ is the minimum* (*efficient*) *production cost of the duopoly firms producing $s$ units in total.*

As an example, condition (ii) for asymmetric duopoly firms with convex marginal costs is always satisfied for linear demand, as long as the maximum demand shock is not too large and the minimum demand shock is not too small.

### 2.3. Sufficient conditions

Here we show that a non-decreasing solution of the continuous stationary conditions presented above must be an SFE if Assumption 3 below is satisfied. That is, the non-decreasing condition acts rather like a second-order condition in ensuring sufficiency. This is a result that will be useful when analyzing convergence of equilibria. Proposition 2 below extends KM's sufficiency result for symmetric producers to cases with asymmetric firms. We also generalize KM by considering capacity constraints, but at the cost of additional complexity, because competitors' capacity constraints introduce kinks and non-concavities in the producer's pay-off function. These kinks will start to influence the range of possible equilibria when a producer is pivotal at the highest realized price $b$, i.e., when competitors' total production capacity is not sufficient to meet market demand at this price [13,6,12,15,4]. Pivotal producers will find it profitable to deviate from the stationary solutions with the lowest mark-ups by withholding output to make competitors' capacities bind.

[7] We have no reason to believe that such kinks would change our convergence results; it is just that they would make our analysis significantly more complicated.

For example, the Bertrand equilibria in the model by von der Fehr and Harbord [9] can be ruled out as soon as one firm is pivotal. To rule out boundary conditions with too low mark-ups for a pivotal producer, i.e., when $b$ is too low, we only consider boundary conditions $\mathbf{q} = \breve{\mathbf{s}}(b)$, such that

**Assumption 3.**

$$bq_i - C_i(q_i) \geqslant p_d\left[\bar{\varepsilon} + d(p_d) - k_{-i}\right] - C_i\left[\bar{\varepsilon} + d(p_d) - k_{-i}\right], \quad \forall p_d \in \left(b, P^M\right] \text{ and } \forall i.$$

The left-hand side of the inequality is firm $i$'s profit at the boundary condition $p = b$. The right-hand side is the profit when supply is withheld until competitors' total capacities, $k_{-i}$, bind, so that the price can be increased, $p_d \in (b, P^M]$. If the assumption is not satisfied then there will always be some shock density $f(\varepsilon)$ (with sufficient probability mass near $\bar{\varepsilon}$), such that $\breve{\mathbf{s}}(p)$ with $\breve{\mathbf{s}}(b) = \mathbf{q}$ is not a segment of an SFE. Assumption 3 is always satisfied if $b$ is sufficiently close to the price cap $P^M$ or if producers are non-pivotal. Genc and Reynolds [12] provided a more detailed analysis of pivotal producer's influence on the existence of SFE. Given Assumption 3, it is straightforward to verify that the stationary solution will have sufficient mark-ups to deter deviations with $p_d > b$ by any pivotal producer for any shock outcome. (See Lemma A in Appendix A.) Proposition 2 below rules out deviations resulting in prices $p \leqslant b$. This is relevant for both pivotal and non-pivotal producers, to prove that non-decreasing, continuous stationary solutions satisfying Assumption 3 are Nash equilibria.

**Proposition 2.** *Let Assumption 3 hold and $\breve{\mathbf{s}}(p)$ be a continuous stationary solution. If $\breve{\mathbf{s}}(p)$ is a vector with non-decreasing supply functions on $[a, b]$, then $\breve{\mathbf{s}}(p)$ is a segment of a continuous SFE.*

## 3. The step supply function model

This section characterizes the step supply function model. Variables in the discrete model are distinguished by using capital letters where the continuous model used lower case (except where the variables are common to both).

As before, consider a uniform price auction where all accepted offers are paid the Market Clearing Price. In the pure strategy NE of a one-shot game, each generator, $i$, chooses a step supply function to maximize its expected profit $E$ in (6) below. There are $M$ price levels, $P^j$, $j = 1, 2, \dots, M$, with the price tick $\Delta P^j = P^j - P^{j-1}$, for the most part assumed equal and then denoted by $\Delta P$. Quantities can be continuously varied.

Generator $i$ $(i = 1, \dots, N)$ submits a supply vector $\mathbf{S}_i$ consisting of the non-negative maximum quantities $\{S_i^1, \dots, S_i^M\}$ it is willing to produce at each price level $\{P^1, \dots, P^M\}$. The step length $\Delta S_i^j = S_i^j - S_i^{j-1}$ is required to be non-negative in electricity markets, but this constraint can be ignored, as monotonicity will anyway follow from convergence of stepped SFE (SSFE) to the strictly monotonic continuous SFE that we consider. Let $\mathbf{S} = \{\mathbf{S}_1 \dots S_N\}$ and denote competitors' collective quantity offers at price $P^j$ as $S_{-i}^j$ and the total market offer as $S^j$.

Electricity consumers are non-strategic. Their demand is stepped and the minimum demand at each price is $D^j + \varepsilon$. Decremental demand is $\Delta D^j = D^j - D^{j-1} \leqslant 0$, with $\Delta D^j \geqslant \Delta D^{j+1}$, corresponding to the continuously differentiable concave deterministic demand curve, $d(p)$, in the continuous case. Demand in the discrete model is such that $D^j \to d(P^j)$ and $\frac{D^{j+1} - D^{j-1}}{P^{j+1} - P^{j-1}} \to d'(P^j)$ as $P^{j-1}, P^{j+1} \to P^j$.

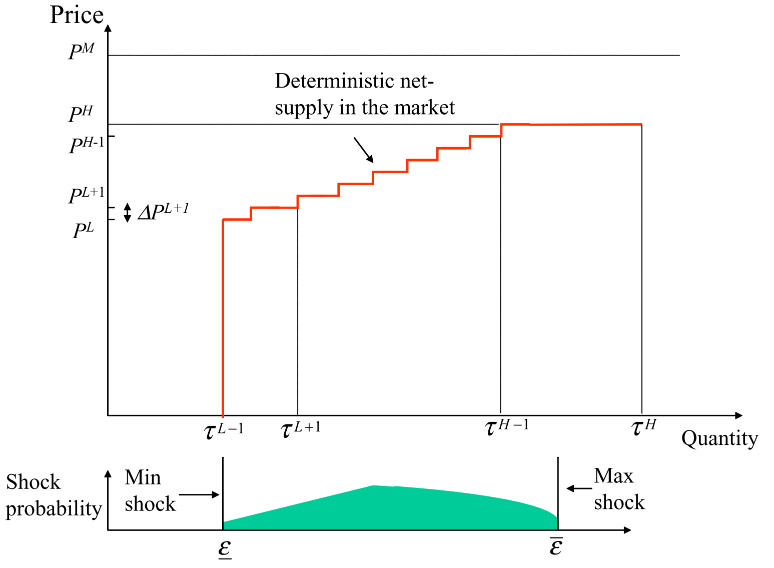


Fig. 2. Stepped supply, demand shocks and key price levels.

Let $\tau^j = S^j - D^j$ be the deterministic part of total net supply (excluding the stochastic shock) at price $P^j$, and define the increase in net supply from a positive increment in price as $\Delta\tau^j = \tau^j - \tau^{j-1}$. Similarly, the residual deterministic net supply is $\tau_{-i}^j = S_{-i}^j - D^j$ and its increase is $\Delta\tau_{-i}^j = \tau_{-i}^j - \tau_{-i}^{j-1}$.

The Market Clearing Price (MCP) is the lowest price at which the deterministic net-supply is larger than the stochastic demand shock. Thus the equilibrium price as a function of the additive demand shock is left-continuous. Given all players' chosen strategies, $P(\varepsilon) := P^j$ if $\varepsilon \in (\tau^{j-1}, \tau^j]$, determining the MCP for each demand shock in the interval $[\underline{\varepsilon}, \overline{\varepsilon}]$. The lowest and highest realized prices are denoted by $P^L$ and $P^H$, where $1 \leqslant L \leqslant H \leqslant M$. Both depend on the available number of price levels, $M$, as well as the boundary conditions, and these price levels are shown in Fig. 2 together with the support of the additive demand shock.

Given all players' chosen strategies, producer $i$'s accepted output, $S_i(\varepsilon)$, is a function of the additive demand shock. With the finite number of price levels in the market, the market will typically clear in the middle of the step at the clearing price, so that rationing is required. With pro-rata on-the-margin rationing, all supply offers below the MCP, $P^j$, are accepted, while offers at $P^j$ are rationed pro-rata on that step. Thus for $\varepsilon \in (\tau^{j-1}, \tau^j]$, $\varepsilon - \tau^{j-1}$ is excess demand at $P^{j-1}$, so the accepted supply of a generator $i$ is given by:

$$S_i(\varepsilon) = S_i^{j-1} + \frac{\Delta S_i^j(\varepsilon - \tau^{j-1})}{\Delta\tau^j} = \varepsilon - \tau_{-i}^{j-1} - \frac{\Delta\tau_{-i}^j(\varepsilon - \tau^{j-1})}{\Delta\tau^j}, \tag{4}$$

(making use of the fact that $\tau^j = \tau_{-i}^j + S_i^j$ and $\Delta\tau^j = \Delta\tau_{-i}^j + \Delta S_i^j$).[8]

[8] The demand side is also rationed pro-rata on the margin, so that accepted demand is given by $D(\varepsilon) = \varepsilon + D^{j-1} + \frac{\Delta D^j(\varepsilon - \tau^{j-1})}{\Delta\tau^j}$.

The contribution to the expected profit of generator $i$ from realizations $\varepsilon \in (\tau^{j-1}, \tau^j]$ is:

$$E_i^j = \int\limits_{\tau^{j-1}}^{\tau^j} \left[S_i(\varepsilon)P^j - C_i\left(S_i(\varepsilon)\right)\right]g(\varepsilon)\,d\varepsilon$$

$$= \int\limits_{S_i^{j-1}+\tau_{-i}^{j-1}}^{S_i^j+\tau_{-i}^j} \left[\left(\varepsilon - \tau_{-i}^{j-1} - \frac{\Delta\tau_{-i}^j(\varepsilon - \tau^{j-1})}{\Delta\tau^j}\right)P^j\right.$$

$$\left. - C_i\left(\varepsilon - \tau_{-i}^{j-1} - \frac{\Delta\tau_{-i}^j(\varepsilon - \tau^{j-1})}{\Delta\tau^j}\right)\right]g(\varepsilon)\,d\varepsilon, \tag{5}$$

where again $\tau^j = \tau_{-i}^j + S_i^j$. Generator $i$'s total expected profit is

$$E(\mathbf{S}) = \sum_{j=1}^{M} E_i^j\left(S_i^j, S_i^{j-1}\right). \tag{6}$$

### 3.1. First-order conditions

The problem facing each generator is to maximize (6) by choosing $S_i^j$, given its competitors' chosen stepped supply functions, $S_i^{j-1}$, termed SSFs. Taken together these constitute an SSFE, when each firm unilaterally maximizes its profit. Note that the expected profit contribution in (5) is differentiable in the only choice variables, $S_i^j$, as we have fixed the price levels, i.e., $\partial E(\mathbf{S})/\partial S_i^j \equiv \Gamma_i^j(\mathbf{S})$, is always well-defined.

Neglecting monotonicity constraints on offers, Lemma 2 gives the expression for the first-order conditions $\Gamma_i^j(\mathbf{S}) = 0$ that are necessary for optimality of an SSF. Lemma 2 changes variables to simplify the analysis.

**Lemma 2.** *The first-order condition for the supply of firm $i$ at price level $j$, such that $\underline{\varepsilon} \leqslant \tau^j \leqslant \bar{\varepsilon}$, is given by*:

$$0 = \Gamma_i^j(\mathbf{S}) = \frac{\partial E(\mathbf{S})}{\partial S_i^j}$$

$$= -\Delta P^{j+1} S_i^j g\left(\tau^j\right) + \Delta\tau_{-i}^j \int\limits_0^1 \left[P^j - C_i'\left(\overline{S}_i^j(u)\right)\right]u g\left(\bar{\tau}^j(u)\right)du$$

$$+ \Delta\tau_{-i}^{j+1} \int\limits_0^{\overline{u}^{j+1}} \left[P^{j+1} - C_i'\left(\overline{S}_i^{j+1}(u)\right)\right](1-u) g\left(\bar{\tau}^{j+1}(u)\right)du, \tag{7}$$

*where $\overline{u}^{j+1} = 1$ for all $j$ except $j = H-1$, for which $\overline{u}^H = \frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H}$, and*

$$\overline{S}_i^j(u) := (1-u)S_i^{j-1} + uS_i^j,$$
$$\bar{\tau}^j(u) := (1-u)\tau^{j-1} + u\tau^j. \tag{8}$$

The first-order condition can be intuitively interpreted as follows. When calculating $\Gamma_i^j(\mathbf{S}) = \partial E(\mathbf{S})/\partial S_i^j$, supply is increased at $P^j$ while holding the supply at all other price levels constant. This implies that the offer price of one (infinitesimally small) unit of power is decreased from $P^{j+1}$ to $P^j$. This decreases the MCP for the event when the unit is price-setting, i.e., when $\varepsilon = \tau^j$. This event brings a negative contribution to the expected profit, which corresponds to the first term in the first-order condition (7). But because of the rationing mechanism, decreasing the price of one unit (weakly) increases the accepted supply for demand outcomes $\varepsilon \in (\tau^{j-1}, \tau^{j+1}]$. This brings a positive contribution to the expected profit – the two integrals in (7). The first integral covers $\varepsilon \in (\tau^{j-1}, \tau^j]$ when the MCP is $P^j$, and the second for $\varepsilon \in (\tau^j, \tau^{j+1}]$ when the MCP is $P^{j+1}$. The price level $P^H$ is special as the shock distribution is truncated at $\bar{\varepsilon}$, explaining the truncated integration limit $\bar{u}^H = \frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H}$.

The first-order condition in Lemma 2 is not directly applicable to parts of the offer curve that are always or never accepted in equilibrium. The appendix shows that, because of pro-rata on the margin rationing, a producer's profit is maximized if offers that are never accepted are offered with a perfectly elastic supply (until the capacity constraint binds) at $P^H$, so that $S_i^H = k_i$, and offers that are always accepted are offered below $P^L$, so that $\tau^{j-1} = \underline{\varepsilon}$. Apart from this, these inframarginal offers normally cannot be determined in equilibrium. However, inframarginal offers influence competitors' payoffs if they deviate from a potential NE. To ensure that offers are monotonic and to maximize the set of NE in the discrete model, set

$$S_i^j = S_i^{L-1} \quad \text{if } j < L - 1, \tag{9}$$

because this minimizes competitors' payoffs if they deviate by undercutting the price level $P^L$. In summary, equilibrium supply is constant for $P < P^L$, satisfies (7) for $P = P^L \dots P^{H-1}$, and jumps to $k_i$ at $P^H$.

The difference equation in (7) is of the second-order so that solutions, should they exist, would be indexed by two boundary conditions that could appear in a variety of forms, e.g., initial and final (boundary) values or, as here, two boundary values at the upper end of the interval. As argued above, one of the boundary conditions is pinned down by the capacity constraint $S_i^H = k_i$. This leaves each firm with one remaining free parameter, $S_i^{H-1}$, that will be tied down with a second boundary condition, $S_i^{H-1} = Q_i$, for some constant $Q_i$. This latter condition, which can be freely chosen within an interval, allowing for a continuum of possible SSFEs, corresponds to the single boundary condition needed for the continuous case.

Section 4.1 studies convergence of first-order solutions of the discrete system to first-order solutions of the continuous system, and links the discrete and continuous boundary conditions by requiring $\lim_{\Delta P \to 0} Q_i = q_i$, where $Q_i$ depends on $\Delta P$ or, equivalently, on $M$. Definition 2 gives the notation for a set of stepped solutions, meaning a list of simultaneous solutions, one for each player $i$ and price level $P^j$.

**Definition 2.** Let $\{\{\widehat{S}_i^j\}_{j=L}^{j=H}\}_{i=1}^N$ or $\widehat{\mathbf{S}}$ denote a set of solutions to the system of difference equations (7) given two boundary conditions $\widehat{S}_i^H = k_i$ and $\widehat{S}_i^{H-1} = Q_i$, for some constant $Q_i$. We call this a stepped stationary solution and say this set is a segment of an SSFE if the set of strategies $\widehat{\mathbf{S}}$ formed by taking $S_i^j = \widehat{S}_i^{L-1}$ if $j < L$, $S_i^j = \widehat{S}_i^j$ if $L \leqslant j \leqslant H$, and $S_i^j = k_i$ if $j > H$ is an SSFE for the discrete game.

### 3.2. Sufficient conditions for SSFE

The next lemma gives a sufficiency condition for stepped equilibria that is the equivalent of Proposition 2 for continuous equilibria. The second-order condition is trivial to verify when marginal costs are constant and demand is uniformly distributed, because then $\partial E(\mathbf{S})/\partial S_i^j$ is independent of firm $i$'s offers at other price levels, see (7). Lemma 3 also establishes that the second-order condition is satisfied if

**Assumption 4.** The additive demand shock is such that (i) $\frac{\partial}{\partial x}\{xg(x+z)\} \geqslant 0$ and (ii) for each $i = 1, \ldots, N$, $\frac{\partial}{\partial x}\{[P^M - C_i'(x)]g(x+z)\} \leqslant 0$ whenever $x \in (0, k_i)$ and $x + z \in (\underline{\varepsilon}, \overline{\varepsilon})$.

Let $g_{\min}$ and $g'_{\min}$ be the infima of $g$ and $g'$ on $(\underline{\varepsilon}, \overline{\varepsilon})$, and $g'_{\max}$ be the supremum of $g'$ on $(\underline{\varepsilon}, \overline{\varepsilon})$. Then Assumption 4 is always satisfied if $g'_{\max} \leqslant 0$ and $k_i g'_{\min} + g_{\min} \geqslant 0$ for each $i = 1, \ldots, N$, i.e., when $g(x)$ is constant or decreasing at a sufficiently low rate on $(\underline{\varepsilon}, \overline{\varepsilon})$. Strictly convex production costs allow for a larger $g'_{\max}$ and smaller $g'_{\min}$ and $g_{\min}$, and thus for more general probability distributions, including truncated exponential, normal and Pareto distributions providing their densities are sufficiently flat on the support set $(\underline{\varepsilon}, \overline{\varepsilon})$. Lemma 3 also relies on the requirement that no producer would find it profitable to deviate and undercut $P^L$ or withhold supply in order to push the price above $P^H$. Moreover, stepped stationary solutions are assumed to be non-decreasing with positive mark-ups. In the next section we will show that for a sufficiently small tick-size, these properties are inherited from the corresponding continuous stationary solution, because of convergence of stationary solutions.

**Lemma 3.** *Consider a set $\widehat{\mathbf{S}}$ of non-decreasing stepped stationary solutions to the system of difference equations in (7) under the usual boundary conditions $\widehat{\mathbf{S}}^H = \mathbf{k}$ and $\widehat{\mathbf{S}}^{H-1} = \mathbf{Q}$. Let $\Delta P^j = \Delta P$ and suppose that $P^j - C_i'(\widehat{S}_i^j) \geqslant 0$ for all price levels $L \leqslant j \leqslant H-1$ and each $i = 1, \ldots, N$. If Assumption 4 holds, then $\widehat{\mathbf{S}}$ is a segment of an SSFE, unless there are profitable deviations involving undercutting $P^L$ or withholding supply in order to increase the price above $P^H$. On the other hand, if marginal costs $c_i = C_i'$ are constant, the tick-size is sufficiently small (or $M$ is sufficiently large), $P^j - c_i > 0$, $-\widehat{S}_i^j g'(\widehat{\tau}^j) > 3g(\widehat{\tau}^j)$ and $-\widehat{S}_i^{j+1} g'(\widehat{\tau}^{j+1}) > 3g(\widehat{\tau}^{j+1})$ for some $i = 1, \ldots, N$ and some $j \in \{L, \ldots, H-2\}$, then $\widehat{\mathbf{S}}$ is not a segment of an SSFE.*

The next section shows that well-behaved continuous SFE are robust to this type of discretization when the shock density is not too steep, as defined by Assumption 4. However, robustness is not self-evident. It follows from Proposition 1 that there exists a continuous SFE satisfying Assumption 2, also for constant marginal costs and steeply decreasing shock densities. From Lemma 1 that under Assumption 1, such SFE must have strictly positive mark-ups that are uniformly bounded away from zero at the clearing prices. Thus the last sentence of Lemma 3 implies that there are circumstances when well-behaved continuous SFE exists that are not robust to these discretizations, i.e., in those cases there is no SSFE that converges to the well-behaved continuous SFE.

## 4. Convergence analysis

This section establishes robustness of well-behaved continuous SFE by showing that, under sufficient conditions, there exists a stepped SFE (SSFE) that converges to such continuous SFE.

Section 4.1 establishes convergence of stepped stationary solutions to continuous stationary solutions. Section 4.2 draws this result and previous results for SFE and SSFE together in our main result, Theorem 1: given a well-behaved SFE, an SSFE exists under sufficient conditions and converges to the continuous SFE as $\Delta P \to 0$.

### *4.1. Convergence of stationary solutions*

This section states (and Appendix A proves) that for a market for which a continuous stationary solution satisfying Assumption 2 exists, there also normally exists a stepped stationary solution that converges to the continuous solution as $\Delta P \to 0$, which is non-obvious given the results in von der Fehr and Harbord [9]. The steps in the convergence proof are related to the steps in the proof of Dahlquist's equivalence theorem[9] for discrete approximations of ODEs [23]. It is not standard to approximate ODEs by systems that are both non-linear and implicit (since solving an approximating system then requires an iterative procedure at each step of the integration). Nevertheless the convergence proof has to deal with systems of difference equations in Lemma 2 that are implicit and non-linear and that requires an extension to the framework of LeVeque [23].

The proof strategy and the convergence result require continuous solutions to be related to solutions of the discrete system (7). The first step in proving convergence is to verify that the discrete system of stationary conditions in Lemma 2 is consistent with the stationary conditions for continuous SFE written as the ODE (2). Lemma C of Appendix A shows this to be the case. That the discrete system is a consistent approximation of the continuous system implies the former set of equations converges to the latter as the number of price steps $M$ goes to infinity. Thus as $M \to \infty$, the second-order difference equation in (7) converges to a differential equation of the first-order, which corresponds to the KM equation (1). Let the error be the difference between the continuous and stepped solutions, then the convergence of the first-order conditions ensures that the local error that is introduced over a short fixed price interval is reduced as the number of steps becomes larger. However, this does not ensure that a stepped stationary solution will exist nor, if it does, that it will converge to the continuous solution, because accumulated errors may still grow at an unbounded rate along the considered price interval as the size of the price ticks becomes smaller and the number of price levels increases. Thus even if the discrete system is consistent with the continuous one, the accumulated error could explode when the number of steps becomes large – the unstable case. Hence the second step in the convergence analysis is to establish existence and stability.

Lemma E in Appendix A shows that a solution at the price level $H-2$ (the first step working backwards from the boundary condition) is ensured, if the boundary conditions $\mathbf{S}^{H-1} = \mathbf{Q}$ and $\mathbf{S}^H = \mathbf{k}$ satisfy:

$$\begin{aligned} 0 &\leqslant \Delta P S_i^{H-1} g\left(\tau^{H-1}\right) - \Delta\tau_{-i}^H \int_0^{(\bar{\varepsilon}-\tau^{H-1})/\Delta\tau^H} \left[P^H - C_i'\left(\bar{S}_i^H(u)\right)\right](1-u) g\left(\bar{\tau}^H(u)\right) du \\ &< \frac{3[P^{H-1} - C_i'(S_i^{H-1})]^2 g_{\min}^2}{2(C''_{\max} g_{\max} + [P^{H-1} - C_i'(0)]\hat{g}'_{\max})}, \end{aligned} \tag{10}$$

[9] The more general Lax–Richtmyer equivalence theorem applies to partial differential equations.

where $\Delta P$ is the uniform tick-size, $C''_{\max}$ is the maximum slope of the marginal cost curves, $g_{\max}$ and $g_{\min}$ are the largest and smallest densities and $g'_{\max}$ is the largest slope in the support of $g(\varepsilon)$. Recall that $g_{\min}$ is assumed to be strictly positive. To simplify the inequality above, we have used $\hat{g}'_{\max} = \max(0, g'_{\max})$. With strictly positive mark-ups, the term on the right is always bounded from below by some positive constant, independent of $\Delta P$. Hence the condition (10) is always satisfied for some large but finite $M$ (and small $\Delta P$) for $Q_i$ sufficiently close to a continuous solution with strictly positive mark-ups. For example, the integral is zero when $S_i^{H-1} = Q_i$ is chosen such that $\tau^{H-1} = \bar{\varepsilon}$. Thus as long as $M$ is sufficiently large, there is always a boundary condition such that a solution at the price level $H-2$ exists.

Provided $\mathbf{Q}$ is such that the inequality above is satisfied, minimum demand is not too low, the shock density does not have too steep slopes, and a well-behaved continuous stationary solution satisfying Assumption 2 exists, Proposition 3 states that the stepped stationary solution exists and is stable for sufficiently large $M$. Moreover, the proposition shows that the solution converges to the continuous stationary solution as $M \to \infty$. Recall that $P^L$ and $P^H$ are the lowest and highest realized prices, and that the indices $L$ and $H$ vary with $M$ (and the boundary conditions).

**Proposition 3.** *Make Assumption 1 and Assumption 4, and let $\breve{\mathbf{s}}(p)$ be a continuous stationary solution on the interval $[a,b]$ that satisfies Assumption 2. Consider the discrete stationary system of difference equations (in Lemma 2) with $\Delta P^j = \Delta P$ and boundary conditions $\widehat{\mathbf{S}}^H = \mathbf{k}$ and $\widehat{\mathbf{S}}^{H-1} = \mathbf{Q}$ satisfying (10) for each $i$. If as $M \to \infty$ we have $P^H \to b$ and $\mathbf{Q}$ converges to $\mathbf{q} = \breve{\mathbf{s}}(b)$, then for sufficiently large $M$ there exists a unique stepped stationary solution $\widehat{\mathbf{S}}$. As the number of steps grows ($M \to \infty$), $\widehat{\mathbf{S}}$ converges to $\breve{\mathbf{s}}(p)$ in the interval $[a,b]$.*

The meaning of convergence in this result is that if $j$ is chosen to depend on $M$ such that $P^j \to p \in [a,b]$ as $M \to \infty$, then $\widehat{S}_i^j \to \breve{s}_i(p)$ as $M \to \infty$ for each $i$. The proof of Proposition 3 in Appendix A also provides an expression that is helpful in estimating a bound on the difference between the stepped and continuous stationary solution for a given tick-size $\Delta P$, once the continuous stationary solution is available. It follows from (55) that for sufficiently many steps[10]

$$\left|\widehat{S}_i^j - \breve{s}_i(p)\right| \leqslant e^{\lambda N(b-a)}(b-a)\vartheta \Delta P, \tag{11}$$

where $\vartheta$ and $\lambda$ are constants that satisfy the inequalities (12) for all $p \in (a,b)$ and all $i \in \{1,\dots,N\}$. The inequalities are stated in (39) and (49), respectively. The product $\vartheta \Delta P$ will bound the local truncation error and $\lambda$ will bound the rate at which the error grows in our model.

$$\begin{aligned}
\vartheta &> \frac{2N-3}{N-1}\left|\frac{(\breve{s}'_{-i}(p) - d'(p))}{2[p - C'_i(\breve{s}_i(p))]}\right|, \\
\lambda &> \left|\frac{3(p - C'_i(\breve{s}_i(p))) + 2\breve{s}_i(p)C''_i(\breve{s}_i(p))}{[p - C'_i(\breve{s}_i(p))]^2}\right|.
\end{aligned} \tag{12}$$

Eq. (11) confirms that convergence of solutions occurs as long as $\lambda$ and $\vartheta$ are finite constants. However, the inequalities (12) indicate that convergence of solutions is expected to be particularly slow at prices where firms have elastic supply and/or a low mark-up, especially when there are many firms in the market. This is related to the numerical instability for low mark-ups and many firms that has been observed when continuous SFE are calculated by means of standard numerical integration methods [6,14].

[10] It is assumed that accumulated local errors dominate initial errors at the boundary.

### 4.2. Convergence of stepped and continuous SFE

This section states (and Appendix A proves) the central result of the paper: that for a market for which a continuous SFE satisfying Assumptions 1–3 exists and slopes in the demand density are not too steep, then an SSFE also exists and converges to the continuous SFE as $\Delta P \to 0$. Thus under these sufficient conditions the continuous SFE is robust to the discretization of the strategy space that we consider. Section 4.1 proved convergence of stationary solutions, using techniques normally applied to ODEs. We now depart from the theory of ODEs in order to prove convergence of the equilibria themselves. Fortunately this turns out to follow relatively easily from convergence of the stationary solutions. We use the observation that a stationary solution of the continuous system is actually an NE strategy if it is increasing in price: see Proposition 2. The convergence of the stepped stationary solution to the continuous one, proved in Proposition 3, ensures that for a sufficiently small tick size, the stepped stationary solution inherits three important properties from the continuous stationary solution: 1) the stepped stationary solution is increasing (i.e., it satisfies the monotonicity constraint enforced in many electricity markets), 2) mark-ups are strictly positive, and 3) the stationary solution has sufficiently high mark-ups that any pivotal producer will not wish to deviate to a price $P_d > P^H \to b$ and concave demand ensures that it will not be profitable to undercut $P^L$. Together with restrictions on the demand shock stated in Assumption 4, these properties are enough to ensure that the stepped stationary solution is an NE, and the proof of Theorem 1 is complete.

**Theorem 1.** *Let Assumptions 1–3 hold, then*:

(a) $\breve{\mathbf{s}}(p)$ *is a segment of a continuous SFE.*
(b) *In addition, suppose* $\Delta P^j = \Delta P$ *and that boundary conditions* $\widehat{S}_i^{H-1} = Q_i$ *and* $\widehat{S}_i^H = k_i$ *satisfy (10) for each* $i$, *and that Assumption 4 holds. If as* $M \to \infty$ *we have that* $P^H$ *converges to* $b$ *and* $Q_i$ *converges to* $q_i = \breve{s}_i(b)$, *then for sufficiently large* $M$ *there exists a unique stepped stationary solution* $\widehat{\mathbf{S}}$ *that is a segment of an SSFE. Moreover as the number of steps grows* ($M \to \infty$), $\widehat{\mathbf{S}}$ *converges to* $\breve{\mathbf{s}}(p)$ *in the interval* $[a, b]$.

Recall that $P^L$ and $P^H$ are the lowest and highest realized prices, and that the indices $L$ and $H$ vary with $M$ (and the boundary conditions). Note that the convergence result is valid for general convex cost functions and asymmetric producers and probability densities of the additive demand shock that do not have too steep slopes. Lemma 2 shows that the probability density of the demand shock influences the discrete difference equation for a finite number of steps, but apparently this dependence disappears in the limit when the stepped solution converges to the continuous one, which does not depend on the shock density.

The working paper [16, Proposition 5] proves a result that reverses the implication of Theorem 1 to show that if a stepped stationary solution is non-decreasing and converges to a set of smooth functions (one per player) with positive mark-ups, then the limiting set of functions is a continuous SFE.[11] That is, if the probability density of the additive demand shock does not have

[11] This result is not surprising as expected profits $E(\mathbf{S})$ are differentiable with respect to the strategy variables in our setting, where firms choose quantities at fixed price levels. Thus expected profit is continuous in the discrete strategy variables, and convergence of equilibria in the discrete game to equilibria in the continuous game should be non-problematic [7].

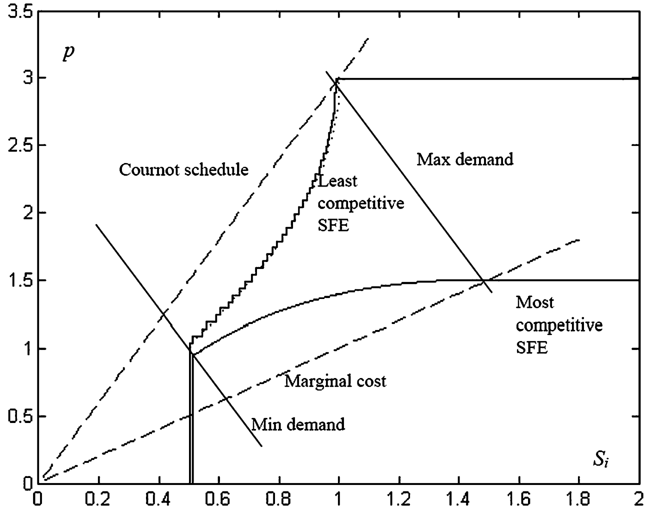


Fig. 3. The most and least competitive continuous SFE (dotted) and their discrete approximations (solid) when $c = 1$. The discrete approximations have a tick-size of $\Delta p = 0.05$ (non-competitive case) and $\Delta p = 0.001$ (competitive case).

too steep slopes, then the family of increasing SFE satisfying Assumptions 1–3 are asymptotically in one-to-one correspondence with the family of corresponding well-behaved SSFE.

## 5. Example

Consider a market with two symmetric firms that have infinite production capacity. Each producer has linear increasing marginal costs $C_i' = cs_i$. Continuous demand at each price level is by assumption given by $d(p) + \varepsilon = \varepsilon - 0.5p$. The additive demand shock, $\varepsilon$, is assumed to be uniformly distributed on the interval $[1.5, 3.5]$, i.e., $g(\varepsilon) = 0.5$ in this range.

In the continuous case, there is a continuum of symmetric stationary solutions to the differential equation in (1). The chosen solution depends on the end-condition. KM and Green and Newbery [13] show that in the continuous case, the symmetric solution slopes upwards between the marginal cost curve and the Cournot schedule, while it slopes downwards (or backwards) outside this wedge. The Cournot schedule is the set of Cournot solutions that would result for all possible realizations of the demand shock, and the continuous SFE is vertical at this line (with price on the $y$-axis). In the other extreme, when price equals marginal cost the solution becomes horizontal. Infinite production capacities ensure that Assumption 3 is satisfied and in this case a continuous symmetric solution constitutes an SFE if and only if the solution is within the wedge for all realized prices. Fig. 3 plots the most and least competitive continuous SFE. All solutions of the differential equations (1) or (2) in-between the most and least competitive continuous cases are also continuous SFE.[12]

For the marginal cost and demand curves assumed in this example, the difference equation in Proposition 2 can be simplified to:

[12] The dotted continuous SFs are very close to the stepped SF and for the most competitive case are essentially indistinguishable.

$$-\Delta P S_i^j + \frac{1}{2}\left(P^j - \frac{c}{3}\left(S_i^{j-1} + 2S_i^j\right)\right)\Delta\tau_{-i}^j$$
$$+\frac{1}{2}\left(P^{j+1} - \frac{c}{3}\left(2S_i^j + S_i^{j+1}\right)\right)\Delta\tau_{-i}^{j+1} = 0. \tag{13}$$

In a symmetric duopoly equilibrium with $\Delta D = -0.5\Delta P$, $\Delta\tau_{-i}^j = S_i^j - S_i^{j-1} + 0.5\Delta P$. Thus the first-order condition can be written as:

$$-\Delta P S_i^j + \frac{1}{2}\left(P^j - \frac{c}{3}\left(S_i^{j-1} + 2S_i^j\right)\right)\left(S_i^j - S_i^{j-1} + 0.5\Delta P\right)$$
$$+\frac{1}{2}\left(P^{j+1} - \frac{c}{3}\left(2S_i^j + S_i^{j+1}\right)\right)\left(S_i^{j+1} - S_i^j + 0.5\Delta P\right) = 0.$$

In Fig. 3 the stepped stationary solutions are plotted. As producers are non-pivotal and demand is uniformly distributed, it follows from Lemma 3 that these solutions are SSFE, and so are all stepped non-decreasing stationary solutions in-between them. Our experience is that we need a much smaller tick-size in the most competitive case compared to the least competitive case in order to get a monotonic solution, which is consistent with the bound on the difference between the stationary solutions given by (11) and (12).

## 6. Concluding remarks

Green and Newbery [13], and Newbery [26] assume that the allowed number of steps in the supply function bids of electricity auctions is so large that equilibrium bids can be approximated by continuous SFE. This is a very attractive assumption, because it implies that a pure-strategy equilibrium can be calculated analytically for simple cases and numerically for general cost functions and asymmetric producers. The pure-strategy equilibrium also justifies empirical models of strategic bidding with smooth supply functions in electricity auctions, such as Wolak [35] who is able to deduce contract positions, marginal costs and the price-cost mark-up from observed bids.

von der Fehr and Harbord [9], however, argued that as long as the number of steps is finite, then continuous SFE are not a valid representation of bidding in electricity auctions. Under the extreme assumption that prices can be chosen from a continuous distribution so that the price tick size is negligible, von der Fehr and Harbord [9] show that uniform price electricity auctions have an inherent price instability; firms randomize their offers in equilibrium, so that prices vary also for unchanged conditions. If demand variation is sufficiently large, so that no producer is pivotal at minimum demand and at least one firm is pivotal at maximum demand, then there is no pure strategy NE, only mixed strategy NE. The intuition behind the non-existence of pure strategy NE is that producers slightly undercut each other's step bids until mark-ups are zero. Whenever producers are pivotal they have profitable deviations from such an outcome.

We claim that the von der Fehr and Harbord result is not driven by the stepped form of the supply functions, but rather by their discreteness assumption. We consider the other extreme in which the price tick size is significant and the quantity multiple is negligible. We show that in this case step equilibria converge to continuous supply function equilibria, as long as the probability density of the demand shock does not have too steep slopes. The intuition for the existence of pure strategy stepped supply function equilibria is that with a significant price tick size, it is not necessarily profitable to undercut perfectly elastic segments in competitors' bids.

Our results imply that the concern that the standard design of electricity auctions has an inherent price instability and that they cannot be modelled by continuous SFE is not necessarily correct. We also claim that this potential problem can be avoided if tick sizes are such that the number of price levels is small compared to the number of possible quantity levels, which is the case in many electricity markets. To avoid risking price instability, we also recommend that restrictions in the number of steps should be as lax as possible, even if some restrictions are probably administratively necessary. Restricting the number of steps increases each producer's incremental supply offered at each step, encouraging price randomization.

Our recommendation to have small quantity multiples contrasts with that of Kremer and Nyborg [22] who recommend a large minimum quantity increment relative to the price tick size to encourage competitive bidding. Their recommendation is correct for markets in which bidders are non-pivotal for all demand realizations, because in such markets pure strategy equilibria with very low mark-ups are possible. For example, von der Fehr and Harbord's [9] model has a Bertrand equilibrium in this case. However, when one or several producers are pivotal for some demand realization, encouraging producers to undercut competitors' bids can lead to non-existence of pure strategy NE and not necessarily lower average mark-ups [9].

Because of a singularity at zero mark-up, equilibrium bid-curves tend to be numerically unstable and easily non-monotonic near such points [6,14]. We have the same experience with our stepped offer curves. The policy implication is that smaller tick-sizes, and even smaller quantity multiples, are needed in competitive markets with small mark-ups in order to get stable prices.

An electricity market may still fail to have a pure-strategy NE if quantity multiples are relatively large and producers are pivotal. In this case we conjecture that if firms can choose a sufficiently large number of steps (and most firms have a large number of individual generating sets), then the range over which each price is randomized may shrink as the number of possible price choices increases. In future research, it may be possible to demonstrate convergence of step SFEs to the continuous SFEs even when the possible price steps are smaller than the quantity steps. If so, the price instability at any level of demand would be small, and errors in using continuous representations also small. For similar reasons, local non-convexities in production costs may destroy pure-strategy NE, but when such irregularities are limited to short quantity intervals, then we would conjecture that the continuous SFE would still be a valid approximation of the randomized offers caused by such irregularities.

We show that never-accepted out-of-equilibrium bids of rational producers are perfectly elastic at the highest realized market price in uniform-price procurement auctions with SFs and pro-rata on-the-margin rationing. This theoretical prediction, which should not depend on the size of tick-sizes and quantity multiples, restrictions on the number of steps, or on whether costs are private information, can be used to empirically test whether producers in electricity auctions believe that some of their offers are rejected with certainty, which is assumed in many theoretical models of electricity auctions. Another by-product of our analysis is the result that any set of, not necessarily symmetric, solutions to KM's system of differential equations constitute a continuous SFE if supply functions are increasing for all realized prices, demand is concave, and if a pivotal producer does not have a profitable deviation at the highest demand outcome.

Finally, we would not claim that the apparent tension between tractable but unrealistic continuous SFEs and realistic but intractable step SFEs is the only, or even the main, problem in modelling electricity markets. First, there are multiple SFE if some offers are always accepted or never accepted. Then under reasonable conditions, there is a continuum of continuous SFE bounded by (in the short run) a least and most profitable SFE. Second, the position of the SFEs depends on the contract position of all the generators, and determining the choice of contracts and their impact on the spot market is a hard and important problem. The greater the extent of contract cover, the less will be the incentive for spot market manipulation [25], and as electricity demand is very inelastic and markets typically concentrated, this is an important determinant of market performance. Newbery [26] argued that these can be related, in that incumbents can choose contract positions to keep both the contract and average spot price at the entry-deterring level, thus simultaneously solving for prices, contract positions, and embedding the short-run SFE within a longer run investment and entry equilibrium. A full long-run model of the electricity market should also be able to investigate whether some market power is required for (or inimical to) adequate investment in reserve capacity to maintain adequate security of supply. With such a model one could also make a proper assessment of how many competing generators are needed to deliver a workably competitive but secure electricity market.

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## Appendix A. Proofs of propositions

### A.1. Continuous supply function model

**Proof of Lemma 1.** Demand is non-increasing with respect to the price and as the stationary solutions satisfy Assumption 2, it follows that $\breve{s}_{-i}{}'(p) - d'(p)$ is strictly positive and bounded. The market design only allows for non-negative outputs. Hence, mark-ups must be non-negative according to (1). Thus with continuous marginal costs, increasing offer curves and non-negative mark-ups, it follows from Assumption 1 that competitors to firm $n$ must offer strictly less than $\underline{\varepsilon} + d(C_n'(0))$ at the price $C_n'(0)$, and according to (1) the output of firm $n$ is zero at this price. The demand curve is differentiable and the considered supply functions are continuous. Thus the market must clear at a price strictly larger than $C_n'(0)$ for the lowest demand outcome. According to (1) this implies that all firms have strictly positive outputs at the lowest price. Stationary solutions are assumed to be increasing with respect to price, so the output must be strictly positive for all shock outcomes.

Thus from equation (1), strict positivity of $\breve{s}_i(p)$ implies that the mark-up, equal to $\breve{s}_i(p)/(\breve{s}_{-i}{}'(p) - d'(p))$, must be strictly positive. Moreover, continuity of $C_i(\cdot)$ and Assumption 2 ensure that the mark-up is bounded below by a positive constant for all $p$ in the compact set $[a, b]$. The smallest of these constants over all $i$ furnishes the result. $\square$

**Proof of Proposition 1.** The proof under condition (i) follows directly from Klemperer and Meyer [20].

Provided marginal costs are convex and demand satisfies $d'''(p) \leqslant 0$ and $-2d''(p)\max(k_1, k_2) < (-d'(p))^2$, it follows from Anderson [1] that an SFE exists in a duopoly market. But these equilibria may have non-smooth segments when one of the firm's capacity constraints or zero-output constraints are binding or close to binding. Thus it is necessary to introduce bounds on the demand shock to ensure that $\{\breve{s}_i(p)\}_{i=1}^2$ satisfies the properties in Assumption 2 in the price interval $[a,b]$. Anderson proves that the equilibria that exist have different properties depending on the order in which the firm's capacity constraints bind. Nevertheless one can deduce from Anderson [1] that existing equilibria are smooth and have bounded slopes in the range $[a,b]$ if both firms' outputs are strictly larger than zero, which is ensured by Assumption 1, and if the firms' total output is strictly less than $\min(k_1,k_2,s_1^*,s_2^*)$. The latter is ensured by the condition

$$\min(k_1,k_2,s_1^*,s_2^*) > \bar{\varepsilon} + d\big(C'\big(\min(k_1,k_2,s_1^*,s_2^*)\big)\big), \tag{14}$$

as it is never profitable to make offers with negative mark-ups. It follows from the set of continuous solutions to the system of the differential equations (2) for duopoly firms and Lemma 3 in Anderson [1] that as soon as the continuous stationary solution of one firm has $s_i'(p_0)=0$, then this solution will have $s_i'(p_0+\delta)<0$ for all positive $\delta$ that are sufficiently small. Such solutions cannot be part of the non-decreasing SFE that Anderson considers. Thus SFE where firms have $s_i'(p_0)=0$ can only occur at the end of the smooth segments in Anderson's setting, but (14) ensures that all such points are outside the price interval $[a,b]$. Thus under the stated assumptions, SFE will exist that satisfy properties stated in Assumption 2 in the price interval $[a,b]$. □

Lemma A shows that if Assumption 3 holds, i.e., it is not profitable to deviate from the continuous stationary solution by withholding production until the capacity constraints of the competitors bind when the highest shock outcome is realized, then this particular type of deviation will not be profitable for lower shock outcomes either.

**Lemma A.** *If Assumption 3 is satisfied then*

$$p(\varepsilon)\breve{s}_i\big(p(\varepsilon)\big) - C_i\big(\breve{s}_i\big(p(\varepsilon)\big)\big) > p_d\big[\varepsilon + d(p_d) - \bar{s}_{-i}\big] - C_i\big[\varepsilon + d(p_d) - \bar{s}_{-i}\big],$$
$$\forall p_d \in \big(b, P^M\big] \text{ and } \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}).$$

**Proof.** Let

$$\chi(\varepsilon) = p\breve{s}_i(p) - C_i\big(\breve{s}_i(p)\big) - p_d\big[\varepsilon + d(p_d) - k_{-i}\big] + C_i\big[\varepsilon + d(p_d) - k_{-i}\big]$$

where $p = p(\varepsilon)$. Now differentiate $\chi$ with respect to $\varepsilon$:

$$\chi'(\varepsilon) = \big[p - C_i'\big(\breve{s}_i(p)\big)\big]\breve{s}_i'(p)p'(\varepsilon) + \breve{s}_i(p)p'(\varepsilon) - \big[p_d - C_i'\big(\varepsilon + d(p_d) - k_{-i}\big)\big].$$

From (1) that $\breve{s}_i(p) = [p - C_i'(\breve{s}_i(p))](\breve{s}_{-i}'(p) - d'(p))$, so

$$\chi' = \big[p - C_i'\big(\breve{s}_i(p)\big)\big]\big[\breve{s}'(p) - d'(p)\big]p'(\varepsilon) - \big[p_d - C_i'\big(\varepsilon + d(p_d) - k_{-i}\big)\big].$$

But $\breve{s}(p) - d(p) \equiv \varepsilon$, so $[\breve{s}'(p) - d'(p)]p'(\varepsilon) = 1$. Thus

$$\chi' = \big[p - C_i'\big(\breve{s}_i(p)\big)\big] - \big[p_d - C_i'\big(\varepsilon + d(p_d) - k_{-i}\big)\big] < 0,$$

because $p_d > b \geqslant p(\varepsilon)$ and $\breve{s}_i(p) > \varepsilon + d(p_d) - k_{-i}$. From Assumption 3 $\chi(\bar{\varepsilon}) \geqslant 0$, so the above proves that $\chi(\varepsilon) > 0$ $\forall \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon})$, or equivalently that

$$p(\varepsilon)\breve{s}_i\big(p(\varepsilon)\big) - C_i\big(\breve{s}_i\big(p(\varepsilon)\big)\big) > p_d\big[\varepsilon + d(p_d) - k_{-i}\big] - C_i\big[\varepsilon + d(p_d) - k_{-i}\big] \quad \forall \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon})$$

if Assumption 3 is satisfied. □

Proposition 2 states that a set of non-decreasing stationary solutions to the continuous first-order conditions is a sufficient condition for a supply function equilibrium if Assumption 3 is satisfied, i.e., no producer is sufficiently pivotal.

**Proof of Proposition 2.** Consider an arbitrary firm $i$. Assume that its competitors follow the strategy implied by the continuous stationary solution. The question is whether it will be a best response of firm $i$ to do the same. The profit $\pi_i$ of producer $i$ for the outcome $\varepsilon$ is given by

$$\pi_i(p,\varepsilon) = \big(\varepsilon + d(p) - \breve{s}_{-i}(p)\big)p - C_i\big(\varepsilon + d(p) - \breve{s}_{-i}(p)\big).$$

Hence

$$\frac{\partial \pi_i(p,\varepsilon)}{\partial p} = \big[d'(p) - \breve{s}'_{-i}(p)\big]\big[p - C_i'\big(\varepsilon + d(p) - \breve{s}_{-i}(p)\big)\big] + \varepsilon + d(p) - \breve{s}_{-i}(p). \tag{15}$$

From the first-order condition in (1) it is known that

$$\big[d'(p) - \breve{s}'_{-i}(p)\big]\big[p - C_i'\big(\breve{s}_i(p)\big)\big] + \breve{s}_i(p) = 0, \quad \forall p \in (a,b).$$

Subtracting this expression from (15) yields:

$$\frac{\partial \pi_i(p,\varepsilon)}{\partial p} = \big[\breve{s}'_{-i}(p) - d'(p)\big]\big[C_i'\big(\underbrace{\varepsilon + d(p) - \breve{s}_{-i}(p)}_{s_i}\big) - C_i'\big(\breve{s}_i(p)\big)\big] + \big(\underbrace{\varepsilon + d(p) - \breve{s}_{-i}(p)}_{s_i} - \breve{s}_i(p)\big), \quad \forall p \in (a,b). \tag{16}$$

Let $\breve{p}(\varepsilon)$ be the clearing price implied by the stationary solution and $p_d$ is the clearing price when producer $i$ deviates and sells $s_i$ units at the shock $\varepsilon$ rather than $\breve{s}_i$ units. First consider the case when $p_d \in [a,b]$. Offers are non-decreasing and demand non-increasing, so residual demand is non-increasing. Thus it follows that $\breve{s}'_{-i}(p) - d'(p) \geqslant 0$ and that

$$p_d \leqslant p \quad \Leftrightarrow \quad s_i \geqslant \breve{s}_i(p) \quad \Leftrightarrow \quad C_i'(s_i) \geqslant C_i'\big(\breve{s}_i(p)\big).$$

Thus it follows from (16) that

$$\left.\frac{\partial \pi_i(p,\varepsilon)}{\partial p}\right|_{p=p_d} \geqslant 0 \quad \text{if } a < p_d \leqslant \breve{p}(\varepsilon) < b, \quad \text{and}$$
$$\left.\frac{\partial \pi_i(p,\varepsilon)}{\partial p}\right|_{p=p_d} \leqslant 0 \quad \text{if } a < \breve{p}(\varepsilon) \leqslant p_d < b. \tag{17}$$

The assumptions in (3) imply that all supply functions of the potential equilibrium are perfectly inelastic below $a$. This assumption and concavity of the demand curve imply that $\frac{d\{d'(p) - \breve{s}'_{-i}(p)\}}{dp} \leqslant 0$ if $p < a$. Thus it follows from (15) and (17) that $\frac{\partial \pi_i(p,\varepsilon)}{\partial p}|_{p=p_d} \geqslant 0$ if $p_d < a$. Hence, given $\breve{s}_{-i}(p)$ and $\varepsilon$, the profit of firm $i$ is pseudo-concave for prices up to $b$. We know from (3) that the price can never be higher than $b$ unless the capacity constraints of the competitors bind, but such deviations are never profitable according to Assumption 3 and Lemma A.

Thus $\breve{s}_i(p)$ must be a best response to $\breve{s}_{-i}(p)$. This is true for any firm and we can conclude that the stationary solution is an equilibrium. □

### A.2. The step supply function model

**Proof of Lemma 2.** To find an equilibrium we need to determine the best response of firm $i$ given its competitors' offers. The best response necessarily satisfies a first-order condition for each price level, found by differentiating (5) with respect to $S_i^j$ and $S_i^{j-1}$, noting that the limits are functions of $S_i^j$ and $S_i^{j-1}$, as $\tau^j = \tau_{-i}^j + S_i^j$:

$$\frac{\partial E_i^j}{\partial S_i^j} = \int_{\tau^{j-1}}^{\tau^j} \big(P^j - C_i'(\cdot)\big)\frac{\Delta\tau_{-i}^j(\varepsilon - \tau^{j-1})}{(\Delta\tau^j)^2} g(\varepsilon)\,d\varepsilon + \big[P^j S_i^j - C_i\big(S_i^j\big)\big]g\big(\tau^j\big), \tag{18}$$

and

$$\frac{\partial E_i^j}{\partial S_i^{j-1}} = \int_{\tau^{j-1}}^{\tau^j} \big(P^j - C_i'(\cdot)\big)\Delta\tau_{-i}^j\frac{(\tau^j - \varepsilon)}{(\Delta\tau^j)^2} g(\varepsilon)\,d\varepsilon - \big[P^j S_i^{j-1} - C_i\big(S_i^{j-1}\big)\big]g\big(\tau^{j-1}\big).$$

From the last expression it follows that:

$$\begin{aligned}\frac{\partial E_i^{j+1}}{\partial S_i^j} = {} & \int_{\tau^j}^{\tau^{j+1}} \big[P^{j+1} - C_i'(\cdot)\big]\Delta\tau_{-i}^{j+1}\frac{(\tau^{j+1} - \varepsilon)}{(\Delta\tau^{j+1})^2} g(\varepsilon)\,d\varepsilon \\ & - \big[P^{j+1} S_i^j - C_i\big(S_i^j\big)\big]g\big(\tau^j\big).\end{aligned} \tag{19}$$

Combining (18) and (19) gives the first-order condition for step supply functions:

$$\begin{aligned}\frac{\partial E(\mathbf{S})}{\partial S_i^j} &= \frac{\partial E_i^j}{\partial S_i^j} + \frac{\partial E_i^{j+1}}{\partial S_i^j} \\ &= -\Delta P^{j+1} S_i^j g\big(\tau^j\big) + \int_{\tau^{j-1}}^{\tau^j} \big[P^j - C_i'\big(S_i(\varepsilon)\big)\big]\frac{\Delta\tau_{-i}^j(\varepsilon - \tau^{j-1})}{(\Delta\tau^j)^2} g(\varepsilon)\,d\varepsilon \\ &\quad + \int_{\tau^j}^{\tau^{j+1}} \big[P^{j+1} - C_i'\big(S_i(\varepsilon)\big)\big]\Delta\tau_{-i}^{j+1}\frac{(\tau^{j+1} - \varepsilon)}{(\Delta\tau^{j+1})^2} g(\varepsilon)\,d\varepsilon = 0,\end{aligned} \tag{20}$$

where $S_i(\varepsilon)$ is given by (4) if $\varepsilon \in [\tau^{j-1}, \tau^j]$. To simplify the equation, rewrite it using the variable substitution $\varepsilon = \tau^{j-1} + u\Delta\tau^j$ in the first integral and the variable substitution $\varepsilon = \tau^j + u\Delta\tau^{j+1}$ in the second integral.

$$\begin{aligned}\frac{\partial E(\mathbf{S})}{\partial S_i^j} &= -\Delta P^{j+1} S_i^j g\big(\tau^j\big) \\ &\quad + \Delta\tau_{-i}^j \int_0^1 \big[P^j - C_i'\big((1-u)S_i^{j-1} + uS_i^j\big)\big]ug\big((1-u)\tau^{j-1} + u\tau^j\big)\,du\end{aligned}$$

$$+\Delta\tau_{-i}^{j+1}\int_{0}^{\overline{u}^{j+1}}\left[P^{j+1}-C_i'\big((1-u)S_i^j+uS_i^{j+1}\big)\right]$$
$$\times(1-u)g\big((1-u)\tau^j+u\tau^{j+1}\big)\,du, \qquad (21)$$

where $\overline{u}^{j+1}=1$ for all $j$ except $j=H-1$ for which $\overline{u}^H=\frac{\overline{\varepsilon}-\tau^{H-1}}{\Delta\tau^H}$, because the shock density is truncated at $\varepsilon=\overline{\varepsilon}$. Finally, it must be the case that $\frac{\partial E(\mathbf{S})}{\partial S_i^j}=0$ for an equilibrium offer – a standard result of calculus – as monotonicity constraints have not been imposed. □

The first-order condition in Lemma 2 is not directly applicable to parts of the offer curves that are never accepted in equilibrium, i.e., for price levels $P^j$ such that $\tau^j>\overline{\varepsilon}$. Recall that $P^H$ is the highest price level that is realized with a positive probability. By differentiating the expected profit in (6), one can show that

$$0<\frac{\partial E(\mathbf{S})}{\partial S_i^H}=\int_{\tau^{H-1}}^{\overline{\varepsilon}}\left[P^H-C_i'\big(S_i(\varepsilon)\big)\right]\frac{\Delta\tau_{-i}^H(\varepsilon-\tau^{H-1})}{(\Delta\tau^H)^2}g(\varepsilon)\,d\varepsilon, \qquad (22)$$

because $g(\varepsilon)=0$ for $\varepsilon>\overline{\varepsilon}$. Thus to maximize its expected profit a firm should offer all of its remaining capacity at $P^H$. The intuition for this result is as follows: due to pro-rata on-the-margin rationing, maximizing the supply at $P^H$ maximizes the firm's share of the accepted supply at $P^H$, and, because of the bounded range of demand shocks, there is no risk that an increased supply at $P^H$ will lead to a lower price for any realized event. Hence $S_i^H=k_i$. The discreteness and uncertainty assumptions should not be critical for this result. Intuitively, never-accepted offers should be perfectly elastic in any uniform price auction with SSFs and pro-rata on the margin rationing.

Now, consider offers that are always inframarginal. Recall that $P^L$ is the lowest price that is realized with positive probability. Differentiate expected profit in (6):

$$0<\frac{\partial E(\mathbf{S})}{\partial S_i^{L-1}}=\int_{\tau^{L-1}}^{\tau^L}\left[P^L-C_i'\big(S_i(\varepsilon)\big)\right]\Delta\tau_{-i}^L\left(\frac{\tau^L-\varepsilon}{(\Delta\tau^L)^2}\right)g(\varepsilon)\,d\varepsilon \quad \text{if } \tau^{L-1}<\underline{\varepsilon}, \qquad (23)$$

because $g(\varepsilon)=0$ for $\varepsilon<\underline{\varepsilon}$. Hence $\tau^{L-1}=\underline{\varepsilon}$. This result makes sense intuitively. To increase the accepted supply with pro-rata on-the-margin rationing at the price level $P^L$, inframarginal offers that are never price-setting should be offered below $P^L$ rather than at $P^L$, because bids at $P^L$ are rationed for the lowest shock outcome. Again, always-accepted offers are offered below $P^L$ in any uniform price auction with SSFs and a pro-rata on the margin rationing mechanism. Lemma B below derives a Taylor expansion and other properties of the discrete first-order condition which is useful in showing that SSFE converge to continuous SFE.

**Lemma B.** *We can make the following statements for any price level $j$, such that $L\leqslant j\leqslant H-1$, $P^j-C_i'(S_i^j)>0$, $P^{j+1}-C_i'(S_i^{j+1})>0$, $\Delta\tau_{-i}^j\geqslant 0$ and $\Delta\tau_{-i}^{j+1}\geqslant 0$ for all firms $i=1,\dots,N$:*

1. *For solutions to the system of first-order conditions in (7), the difference $\Delta S_i^j$ is of the order $\overline{\Delta P}^j$ if $L\leqslant j\leqslant H-1$ and $\overline{\varepsilon}-\tau^{H-1}$ is of order $\overline{\Delta P}^{H-1}$, where $\overline{\Delta P}^j=\max(\Delta P^j,\Delta P^{j+1})$.*

2. *The discrete first-order condition in (7) can be approximated by the following Taylor series expansions: if $L \leqslant j < H-1$ then $\Gamma_i^j(\mathbf{S}) \equiv \frac{\partial E(\mathbf{S})}{\partial S_i^j}$ can be written as*

$$
\begin{aligned}
\Gamma_i^j(\mathbf{S}) = {} & -\Delta P^{j+1} S_i^j g(\tau^j) + \frac{\Delta\tau_{-i}^j [P^j - C_i'(S_i^j)] g(\tau^j)}{2} \\
& + \frac{\Delta\tau_{-i}^{j+1} [P^{j+1} - C_i'(S_i^j)] g(\tau^j)}{2} + \frac{\Delta\tau_{-i}^j C''(S_i^j) \Delta S_i^j g(\tau^j)}{6} \\
& - \frac{\Delta\tau_{-i}^j [p^j - C_i'(S_i^j)] g'(\tau^j) \Delta\tau^j}{6} - \frac{\Delta\tau_{-i}^{j+1} \Delta S_i^{j+1} C''(S_i^j) g(\tau^j)}{6} \\
& + \frac{\Delta\tau_{-i}^{j+1} \Delta\tau^{j+1} [P^{j+1} - C_i'(S_i^j)] g'(\tau^j)}{6} + O\big((\overline{\Delta P}^j)^3\big)
\end{aligned}
$$

*and $\Gamma_i^{H-1}(\mathbf{S}) \equiv \frac{\partial E(\mathbf{S})}{\partial S_i^{H-1}}$ satisfies*

$$
\begin{aligned}
\Gamma_i^{H-1}(\mathbf{S}) = {} & -\Delta P^H S_i^{H-1} g(\tau^{H-1}) + \frac{\Delta\tau_{-i}^{H-1} [P^{H-1} - C_i'(S_i^{H-1})] g(\tau^{H-1})}{2} \\
& + \frac{\Delta\tau_{-i}^{H-1} C''(S_i^{H-1}) \Delta S_i^{H-1} g(\tau^{H-1})}{6} \\
& - \frac{\Delta\tau_{-i}^{H-1} [P^{H-1} - C_i'(S_i^{H-1})] g'(\tau^{H-1}) \Delta\tau^{H-1}}{6} \\
& + \frac{\Delta\tau_{-i}^{H} [P^{H} - C_i'(S_i^{H-1})] g(\tau^{H-1})}{2} \left( \frac{2(\bar{\varepsilon} - \tau^{H-1})}{\Delta\tau^H} - \left( \frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H} \right)^2 \right) \\
& - \frac{\Delta\tau_{-i}^{H} \Delta S_i^H C''(S_i^{H-1}) g(\tau^{H-1})}{6} \left( 3\left( \frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H} \right)^2 - 2\left( \frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H} \right)^3 \right) \\
& + \frac{\Delta\tau_{-i}^{H} \Delta\tau^H [P^{H} - C_i'(S_i^{H-1})] g'(\tau^{H-1})}{6} \\
& \times \left( 3\left( \frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H} \right)^2 - 2\left( \frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H} \right)^3 \right) + O\big((\overline{\Delta P}^{H-1})^3\big).
\end{aligned}
\tag{24}
$$

**Proof.** As mark-ups are positive and $\Delta\tau_{-i}^j \geqslant 0$, the first-order condition in (7) has two non-negative integrals. Each of them must be $O(\Delta P^{j+1})$, otherwise the first-order condition in (7) cannot be satisfied for small $\Delta P^{j+1}$, since $S_i^j g(\tau^j)$ is bounded. The first integral is

$$
I_1 := \Delta\tau_{-i}^j \int_0^1 \big[P^j - C_i'\big((1-u)S_i^{j-1} + uS_i^j\big)\big] u g\big((1-u)\tau^{j-1} + u\tau^j\big)\, du.
$$

Assume for constants $c_1, c_2 > 0$ and $\varepsilon \in [\tau^{j-1}, \tau^j]$ that

$$
\begin{aligned}
& \big[P^j - C_i'\big(S_i(\varepsilon)\big)\big] \geqslant c_1, \\
& g(\varepsilon) \geqslant c_2.
\end{aligned}
$$

Thus $I_1 = O(\Delta P^{j+1}) \geqslant \frac{c_1 c_2}{2} \Delta\tau_{-i}^j$. Now $\Delta\tau_{-i}^j = \Delta S_{-i}^j - \Delta D^j$ and $\Delta D^j = O(\overline{\Delta P}^j)$ is bounded by continuous differentiability of $d(p)$, where $\overline{\Delta P}^j = \max(\Delta P^j, \Delta P^{j+1})$. Together this implies another bound $O(\overline{\Delta P}^j) \geqslant |\Delta S_{-i}^j| \geqslant |\Delta S_k^j|$ for each $i$, each $k \neq i$ and each $j = L \dots H-1$. Thus $O(\overline{\Delta P}^j) \geqslant |\Delta S_i^j|$ for each $i$ and each $j = L \dots H-1$. Similar analysis of the second integral gives the uniform bound $O(\overline{\Delta P}^j) \geqslant |\Delta S_i^{j+1}|$ for each $i$ and each $j = L \dots H-2$.

The Taylor expansions of the first-order condition can now be derived. The shock density and marginal cost functions are smooth, so it follows from the results above that[13]

$$
\begin{aligned}
&g\big((1-u)\tau^{j-1} + u\tau^j\big) = g\big(\tau^j\big) - (1-u)g'\big(\tau^j\big)\Delta\tau^j + O\big(\big(\overline{\Delta P}^j\big)^2\big),\\
&g\big((1-u)\tau^{j} + u\tau^{j+1}\big) = g\big(\tau^j\big) + g'\big(\tau^j\big)\Delta\tau^{j+1}u + O\big(\big(\overline{\Delta P}^j\big)^2\big),\\
&C_i'\big((1-u)S_i^{j-1} + uS_i^j\big) = C_i'\big(S_i^j\big) - (1-u)C''\big(S_i^j\big)\Delta S_i^j + O\big(\big(\overline{\Delta P}^j\big)^2\big),\\
&C_i'\big((1-u)S_i^{j} + uS_i^{j+1}\big) = C_i'\big(S_i^j\big) + uC''\big(S_i^j\big)\Delta S_i^{j+1} + O\big(\big(\overline{\Delta P}^j\big)^2\big),
\end{aligned}
$$

if $j = L \dots H-2$. Note that the second-order terms of the above Taylor expansions are bounded by the assumptions that $C_i'''$ and $g''$ are continuous and uniformly bounded. Thus it follows from the first-order condition in (7) and integration that

$$
\begin{aligned}
\Gamma_i^j(\mathbf{S}) = {} & -\Delta P^{j+1} S_i^j g\big(\tau^j\big) + \frac{\Delta\tau_{-i}^j [P^j - C_i'(S_i^j)] g(\tau^j)}{2} + \frac{\Delta\tau_{-i}^j C_i''(S_i^j)\Delta S_i^j g(\tau^j)}{6}\\
& - \frac{\Delta\tau_{-i}^j [P^j - C_i'(S_i^j)] g'(\tau^j)\Delta\tau^j}{6} + \frac{\Delta\tau_{-i}^{j+1} [P^{j+1} - C_i'(S_i^j)] g(\tau^j)}{2}\\
& - \frac{\Delta\tau_{-i}^{j+1}\Delta S_i^{j+1} C_i''(S_i^j) g(\tau^j)}{6} + \frac{\Delta\tau_{-i}^{j+1}\Delta\tau^{j+1}[P^{j+1} - C_i'(S_i^j)] g'(\tau^j)}{6}\\
& + O\big(\big(\overline{\Delta P}^j\big)^3\big),
\end{aligned}
\tag{25}
$$

if $j = L \dots H-2$. Note that the third-order term of the above Taylor expansion is bounded by the assumptions that $C_i''$, $C_i'''$, $g'$ and $g''$ are continuous and uniformly bounded.

The case when $j = H-1$ is special, as $\Delta\tau_{-i}^{H-1}$ is normally not bounded by $O(\Delta P^H)$. In case it is, then this step can be analyzed as above. Below assume that $\Delta\tau_{-i}^{H-1}$ is not bounded by $O(\Delta P^H)$. Then

$$
I_2 := \Delta\tau_{-i}^H \int_0^{\frac{\bar\varepsilon - \tau^{H-1}}{\Delta\tau^H}} \big[P^H - C_i'\big(\overline{S}_i^H(u)\big)\big](1-u) g\big(\bar\tau^H(u)\big)\,du.
$$

As $I_2 = O(\Delta P^H)$ it follows that $\bar\varepsilon - \tau^{H-1} \leqslant O(\Delta P^H)$. For $u \in [0, \frac{\bar\varepsilon - \tau^{H-1}}{\Delta\tau^H}]$

$$
\begin{aligned}
&g\big((1-u)\tau^{H-1} + u\tau^H\big) = g\big(\tau^{H-1}\big) + g'\big(\tau^{H-1}\big)\Delta\tau^H u + O\big(\big(\Delta P^H\big)^2\big),\\
&C_i'\big((1-u)S_i^{H-1} + uS_i^H\big) = C_i'\big(S_i^{H-1}\big) + uC''\big(S_i^{H-1}\big)\Delta S_i^H + O\big(\big(\Delta P^H\big)^2\big),
\end{aligned}
$$

as $(\frac{\bar\varepsilon - \tau^{H-1}}{\Delta\tau^H}) = O(\Delta P^H)$. Thus it follows from the first-order condition in (7) and integration that

[13] Note that $(1-u)x + uz = z + (1-u)(x-z) = x + u(z-x)$.

$$
\begin{aligned}
\Gamma_i^{H-1}(\mathbf{S}) = {} & -\Delta P^H S_i^{H-1} g\left(\tau^{H-1}\right) + \frac{\Delta\tau_{-i}^{H-1}[P^{H-1} - C_i'(S_i^{H-1})]g(\tau^{H-1})}{2} \\
& + \frac{\Delta\tau_{-i}^{H-1} C_i''(S_i^{H-1})\Delta S_i^{H-1} g(\tau^{H-1})}{6} \\
& - \frac{\Delta\tau_{-i}^{H-1}[P^{H-1} - C_i'(S_i^{H-1})]g'(\tau^{H-1})\Delta\tau^{H-1}}{6} \\
& + \frac{\Delta\tau_{-i}^{H}[P^{H} - C_i'(S_i^{H-1})]g(\tau^{H-1})}{2}\left(\frac{2(\bar{\varepsilon} - \tau^{H-1})}{\Delta\tau^H} - \left(\frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H}\right)^2\right) \\
& - \frac{\Delta\tau_{-i}^{H}\Delta S_i^H C_i''(S_i^{H-1})g(\tau^{H-1})}{6}\left(3\left(\frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H}\right)^2 - 2\left(\frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H}\right)^3\right) \\
& + \frac{\Delta\tau_{-i}^{H}\Delta\tau^H[P^{H} - C_i'(S_i^{H-1})]g'(\tau^{H-1})}{6} \\
& \times \left(3\left(\frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H}\right)^2 - 2\left(\frac{\bar{\varepsilon} - \tau^{H-1}}{\Delta\tau^H}\right)^3\right) + O\left(\left(\overline{\Delta P}^{H-1}\right)^3\right). \qquad \square \qquad (26)
\end{aligned}
$$

Lemma 3 shows that being non-decreasing is also a sufficient condition for an SSFE (so the non-decreasing condition acts rather like a second-order condition in ensuring sufficiency), unless the slope of the shock density is too steep. Note that only profitable deviations in the range $\{P^L, \dots, P^H\}$ are ruled out in this proof. In Theorem 1 profitable deviations outside this range are ruled out by properties inherited from the continuous stationary solution.

**Proof of Lemma 3.** Consider a set of stepped stationary solutions $\widehat{\mathbf{S}} = \{\widehat{\mathbf{S}}_1, \dots, \widehat{\mathbf{S}}_N\}$. The shock distribution is such that $P^L$ and $P^H$ are the lowest and highest realized prices. Consider an arbitrary firm $i$ and assume that competitors stick to $\widehat{\mathbf{S}}_{-i} = \{\widehat{S}_{-i}^1, \dots, \widehat{S}_{-i}^M\}$. For notational convenience set $\mathbf{S}_{-i} = \widehat{\mathbf{S}}_{-i}$. Below it is shown that for any $\mathbf{S}_i = \{S_i^1, \dots, S_i^M\}$ that differs from $\widehat{\mathbf{S}}_i$, the profit can (under the stated assumptions) always be improved by adjusting offers in the direction of the vector $\widehat{\mathbf{S}}_i - \mathbf{S}_i = \{\widehat{S}_i^1 - S_i^1, \dots, \widehat{S}_i^M - S_i^M\}$. Hence, it follows that $\widehat{\mathbf{S}}_i = \{\widehat{S}_i^1, \dots, \widehat{S}_i^M\}$ is a globally best response to $\widehat{\mathbf{S}}_{-i} = \{\widehat{S}_{-i}^1, \dots, \widehat{S}_{-i}^M\}$. The result holds for any firm and accordingly $\widehat{\mathbf{S}} = \{\widehat{\mathbf{S}}_1, \dots, \widehat{\mathbf{S}}_N\}$ constitutes an NE under the stated assumptions.

Under the assumptions stated in the lemma, it is not profitable for pivotal producers to withhold production and push the price above $P^H$ and it is not profitable to undercut the lowest realized price $P^L$. Thus it follows from (22) that it is always optimal to set $S_i^H = \widehat{S}_i^H = k_i$ and from (23) that it is always optimal to have $\tau^{L-1} = \underline{\varepsilon} = \widehat{\tau}^{L-1}$. As competitors' total supply is given by $\widehat{\mathbf{S}}_{-i}$, it must be that $S_i^{L-1} = \widehat{S}_i^{L-1}$. Thus consider an arbitrary offer $\mathbf{S}_i = \{S_i^1, \dots, S_i^M\}$ that satisfies $S_i^H = \widehat{S}_i^H$ and $S_i^{L-1} = \widehat{S}_i^{L-1}$. By definition $\frac{\partial E(\widehat{\mathbf{S}})}{\partial S_i^j} = 0$, so $\frac{\partial E(\mathbf{S})}{\partial S_i^j} = \frac{\partial E(\mathbf{S})}{\partial S_i^j} - \frac{\partial E(\widehat{\mathbf{S}})}{\partial S_i^j}$. It now follows from (7) that

$$
\begin{aligned}
\frac{\partial E(\mathbf{S})}{\partial S_i^j} = {} & -\Delta P S_i^j g\left(\tau^j\right) + \Delta P \widehat{S}_i^j g\left(\widehat{\tau}^j\right) \\
& + \Delta\widehat{\tau}_{-i}^j \int_0^1 \left[P^j - C_i'\left((1-u)S_i^{j-1} + uS_i^j\right)\right] u g\left((1-u)\tau^{j-1} + u\tau^j\right) du
\end{aligned}
$$

$$- \Delta\widehat{\tau}_{-i}^{j} \int_0^1 \left[P^j - C_i'\left((1-u)\widehat{S}_i^{j-1} + u\widehat{S}_i^j\right)\right] u g\left((1-u)\widehat{\tau}^{j-1} + u\widehat{\tau}^j\right) du$$
$$+ \Delta\widehat{\tau}_{-i}^{j+1} \int_0^{\overline{u}^{j+1}} \left[P^{j+1} - C_i'\left((1-u)S_i^j + uS_i^{j+1}\right)\right](1-u)$$
$$\times g\left((1-u)\tau^j + u\tau^{j+1}\right) du$$
$$- \Delta\widehat{\tau}_{-i}^{j+1} \int_0^{\overline{u}^{j+1}} \left[P^{j+1} - C_i'\left((1-u)\widehat{S}_i^j + u\widehat{S}_i^{j+1}\right)\right](1-u)$$
$$\times g\left((1-u)\widehat{\tau}^j + u\widehat{\tau}^{j+1}\right) du. \tag{27}$$

Now marginally adjust offers in the direction of $\widehat{\mathbf{S}}_i - \mathbf{S}_i$, i.e.,

$$\mathbf{dS}_i \equiv (\widehat{\mathbf{S}}_i - \mathbf{S}_i)\, dt, \tag{28}$$

where $dt$ is a small positive number. Hence, it follows from (27) that the marginal change in the expected profit is given by:

$$\sum_{j=L}^{H-1} \frac{\partial E(\mathbf{S})}{\partial S_i^j} dS_i^j$$
$$= \sum_{j=L}^{H-1} \left(-\Delta P S_i^j g\left(\tau^j\right) + \Delta P \widehat{S}_i^j g\left(\widehat{\tau}^j\right)\right) dS_i^j$$
$$+ \Delta\widehat{\tau}_{-i}^{L} \int_0^1 \left[P^L - C_i'\left((1-u)\widehat{S}_i^{L-1} + uS_i^L\right)\right] u g\left((1-u)\widehat{\tau}^{L-1} + u\tau^L\right) du\, dS_i^L$$
$$- \Delta\widehat{\tau}_{-i}^{L} \int_0^1 \left[P^L - C_i'\left((1-u)\widehat{S}_i^{L-1} + u\widehat{S}_i^L\right)\right] u g\left((1-u)\widehat{\tau}^{L-1} + u\widehat{\tau}^L\right) du\, dS_i^L$$
$$+ \sum_{j=L}^{H-2} \Delta\widehat{\tau}_{-i}^{j+1} \int_0^1 \left[P^{j+1} - C_i'\left((1-u)S_i^j + uS_i^{j+1}\right)\right]\left((1-u)\, dS_i^j + u\, dS_i^{j+1}\right)$$
$$\times g\left((1-u)\tau^j + u\tau^{j+1}\right) du$$
$$- \sum_{j=L}^{H-2} \Delta\widehat{\tau}_{-i}^{j+1} \int_0^1 \left[P^{j+1} - C_i'\left((1-u)\widehat{S}_i^j + u\widehat{S}_i^{j+1}\right)\right]\left((1-u)\, dS_i^j + u\, dS_i^{j+1}\right)$$
$$\times g\left((1-u)\widehat{\tau}^j + u\widehat{\tau}^{j+1}\right) du$$
$$+ \Delta\widehat{\tau}_{-i}^{H} \int_0^{\frac{\overline{\varepsilon} - \tau^{H-1}}{\Delta\tau^H}} \left[P^H - C_i'\left((1-u)S_i^{H-1} + u\widehat{S}_i^H\right)\right](1-u)$$

$$\times g\big((1-u)\tau^{H-1}+u\widehat{\tau}^{H}\big)\,du\,dS_i^{H-1}$$
$$-\Delta\widehat{\tau}_{-i}^{H}\int_0^{\frac{\bar{\varepsilon}-\widehat{\tau}^{H-1}}{\Delta\widehat{\tau}^{H}}}\big[P^H-C_i'\big((1-u)\widehat{S}_i^{H-1}+u\widehat{S}_i^{H}\big)\big](1-u)$$
$$\times g\big((1-u)\widehat{\tau}^{H-1}+u\widehat{\tau}^{H}\big)\,du\,dS_i^{H-1}, \tag{29}$$

where we have used that $\frac{\partial E(\mathbf{S})}{\partial S_i^j}$ and $\frac{\partial E(\mathbf{S})}{\partial S_i^{j+1}}$ have a similar term for the price level $P^{j+1}$ in common, for $j=L\ldots H-2$.

Before analyzing the above expression in more detail, it is helpful to establish five relations. The assumption $0\leqslant\frac{\partial}{\partial x}\{xg(x+z)\}$ and that competitors' offers are fixed imply that i)

$$\widehat{\tau}^L\geqslant\tau^L \quad \text{iff} \quad \widehat{S}_i^j\geqslant S_i^j$$

and ii)

$$\widehat{S}_i^j g\big(\widehat{\tau}^j\big)\geqslant S_i^j g\big(\tau^j\big) \quad \text{if } \widehat{S}_i^j\geqslant S_i^j,$$
$$\widehat{S}_i^j g\big(\widehat{\tau}^j\big)\leqslant S_i^j g\big(\tau^j\big) \quad \text{if } \widehat{S}_i^j\leqslant S_i^j.$$

Similarly, iii)

$$(1-u)\widehat{\tau}^j+u\widehat{\tau}^{j+1}\geqslant(1-u)\tau^j+u\tau^{j+1} \quad \text{if } (1-u)\widehat{S}_i^j+u\widehat{S}_i^{j+1}\geqslant(1-u)S_i^j+uS_i^{j+1},$$
$$(1-u)\widehat{\tau}^j+u\widehat{\tau}^{j+1}\leqslant(1-u)\tau^j+u\tau^{j+1} \quad \text{if } (1-u)\widehat{S}_i^j+u\widehat{S}_i^{j+1}\leqslant(1-u)S_i^j+uS_i^{j+1}.$$

Costs are convex and $\frac{\partial}{\partial x}\{[P^M-C_i'(x)]g(x+z)\}\leqslant 0$, so $\frac{\partial}{\partial x}\{[P^j-C_i'(x)]g(x+z)\}\leqslant 0$ always holds for $j=L,\ldots,H$, unless $P^j-C_i'(x)<0$. Thus as $P^{j+1}-C_i'(\widehat{S}_i^{j+1})\geqslant 0$, we have iv)

$$-\big[P^{j+1}-C_i'\big((1-u)\widehat{S}_i^j+u\widehat{S}_i^{j+1}\big)\big]g\big((1-u)\widehat{\tau}^j+u\widehat{\tau}^{j+1}\big)$$
$$\geqslant-\big[P^{j+1}-C_i'\big((1-u)S_i^j+uS_i^{j+1}\big)\big]g\big((1-u)\tau^j+u\tau^{j+1}\big)$$

if $(1-u)\widehat{S}_i^j+u\widehat{S}_i^{j+1}\geqslant(1-u)S_i^j+uS_i^{j+1}$ and[14]

$$-\big[P^{j+1}-C_i'\big((1-u)\widehat{S}_i^j+u\widehat{S}_i^{j+1}\big)\big]g\big((1-u)\widehat{\tau}^j+u\widehat{\tau}^{j+1}\big)$$
$$\leqslant-\big[P^{j+1}-C_i'\big((1-u)S_i^j+uS_i^{j+1}\big)\big]g\big((1-u)\tau^j+u\tau^{j+1}\big)$$

if $(1-u)\widehat{S}_i^j+u\widehat{S}_i^{j+1}\leqslant(1-u)S_i^j+uS_i^{j+1}$. Recall that $S_i^H=\widehat{S}_i^H$ and $S_i^{L-1}=\widehat{S}_i^{L-1}$, so $(1-u)\widehat{S}_i^{L-1}+u\widehat{S}_i^L\geqslant(1-u)S_i^{L-1}+uS_i^L$ is equivalent with $\widehat{S}_i^L\geqslant S_i^L$ and $(1-u)\widehat{S}_i^{H-1}+u\widehat{S}_i^H\geqslant(1-u)S_i^{H-1}+uS_i^H$ is equivalent with $\widehat{S}_i^{H-1}\geqslant S_i^{H-1}$. At the final step we also have that v):

$$\frac{\bar{\varepsilon}-\widehat{\tau}^{H-1}}{\Delta\widehat{\tau}^H}\leqslant\frac{\bar{\varepsilon}-\tau^{H-1}}{\Delta\tau^H} \quad \text{if } \widehat{S}_i^{H-1}\geqslant S_i^{H-1},$$
$$\frac{\bar{\varepsilon}-\widehat{\tau}^{H-1}}{\Delta\widehat{\tau}^H}\geqslant\frac{\bar{\varepsilon}-\tau^{H-1}}{\Delta\tau^H} \quad \text{if } \widehat{S}_i^{H-1}\leqslant S_i^{H-1},$$

as $\bar{\varepsilon}\leqslant\tau^H=\widehat{\tau}^H$.

[14] Note that the inequality is obvious when $S_i^j$ and $S_i^{j+1}$ are so large so that $P^{j+1}-C_i'((1-u)S_i^j+uS_i^{j+1})<0$.

Now consider (29). For each price level calculate the difference between two similar terms, one originating from $\frac{\partial E(\mathbf{S})}{\partial S_i^j}$ and the other from $\frac{\partial E(\hat{\mathbf{S}})}{\partial S_i^j}$. From the five relations established above and (28), each of these differences is given by the product of two terms with the same sign, so each difference must be positive. It follows that $\sum_{j=L}^{H-1} \frac{\partial E(\mathbf{S})}{\partial S_i^j} dS_i^j \geqslant 0$ and that $\widehat{\mathbf{S}}_i$ must be a globally best response to $\widehat{\mathbf{S}}_{-i}$.

Finally, we want to analyze cases when the local second-order conditions (necessary for optimality) are violated at a stepped stationary solution. For constant marginal costs $c_i = C_i'$ it follows from (7) that

$$\begin{aligned}\frac{\partial^2 E(\mathbf{S})}{(\partial S_i^j)^2} &= -\Delta P g(\tau^j) - \Delta P S_i^j g'(\tau^j) \\ &\quad + \Delta \tau_{-i}^j \int_0^1 (P^j - c_i) u^2 g'((1-u)\tau^{j-1} + u\tau^j)\, du \\ &\quad + \Delta \tau_{-i}^{j+1} \int_0^1 (P^{j+1} - c_i)(1-u)^2 g'((1-u)\tau^j + u\tau^{j+1})\, du. \end{aligned} \tag{30}$$

By assumption $g''$ is continuous and uniformly bounded, so from strictly positive mark-ups for considered steps and the first property of Lemma B it follows that

$$g'((1-u)\widehat{\tau}^{j-1} + u\widehat{\tau}^j) = g'(\widehat{\tau}^j) + O(\Delta P).$$

Thus

$$\begin{aligned}\left.\frac{\partial^2 E(\mathbf{S})}{(\partial S_i^j)^2}\right|_{S_i^j = \widehat{S}_i^j} &= -\Delta P g(\widehat{\tau}^j) - \Delta P \widehat{S}_i^j g'(\widehat{\tau}^j) \\ &\quad + \frac{\Delta \widehat{\tau}_{-i}^j (P^j - c_i) g'(\widehat{\tau}^j)}{3} + \frac{\Delta \widehat{\tau}_{-i}^{j+1} (P^{j+1} - c_i) g'(\widehat{\tau}^{j+1})}{3} \\ &\quad + O((\Delta P)^2). \end{aligned} \tag{31}$$

From the second property of Lemma B and strictly positive mark-ups for considered steps it now follows that:

$$\begin{aligned}\left.\frac{\partial^2 E(\mathbf{S})}{(\partial S_i^j)^2}\right|_{S_i^j = \widehat{S}_i^j} &= \frac{\Delta \widehat{\tau}_{-i}^j (P^j - c)(-3g(\widehat{\tau}^j)/\widehat{S}_i^j - g'(\widehat{\tau}^j))}{6} \\ &\quad + \frac{\Delta \widehat{\tau}_{-i}^{j+1} (P^{j+1} - c)(-3g(\widehat{\tau}^{j+1})/\widehat{S}_i^{j+1} - g'(\widehat{\tau}^{j+1}))}{6} \\ &\quad + O((\Delta P)^2). \end{aligned} \tag{32}$$

Thus the Hessian is not negative semi-definite and hence $\widehat{\mathbf{S}}_i$ is not a best response to $\widehat{\mathbf{S}}_{-i}$, if marginal costs are constant, the tick-size is sufficiently small, $-\widehat{S}_i^j g'(\widehat{\tau}^j) > 3g(\widehat{\tau}^j)$ and $-\widehat{S}_i^{j+1} g'(\widehat{\tau}^{j+1}) > 3g(\widehat{\tau}^{j+1})$ for some $j \in \{L, \ldots, H-2\}$. $\square$

*A.3. Convergence analysis*

**Lemma C.** *Under Assumption 1, the difference equation in Lemma 2 for price levels $j = L, \ldots, H-2$ is consistent with the continuous equation in (2) if $\{\breve{s}_i(p)\}_{i=1}^N$ satisfies Assumption 2, and $\Delta P^j = \Delta P$.*

**Proof.** A discrete approximation of an ordinary differential equation is consistent if the local truncation error is infinitesimally small when the step length is infinitesimally small [23]. The local truncation error is the discrepancy between the continuous slope and its discrete approximation when values $S_i^j$ in the discrete system are replaced with samples of the continuous solution $\breve{s}_i(P^j)$. Under Assumptions 1 and 2, Lemma 1 implies that the continuous solution is monotonic and has strictly positive mark-ups, so the Taylor approximation from Lemma B and $\Delta P^j = \Delta P$ can be used to approximate the difference equation in (7) for price levels $j = L, \ldots, H-2$:

$$0 = -\Delta P S_i^j g(\tau^j) + \frac{\Delta\tau_{-i}^j [P^j - C_i'(S_i^j)] g(\tau^j)}{2} + \frac{\Delta\tau_{-i}^{j+1} [P^{j+1} - C_i'(S_i^j)] g(\tau^j)}{2} + \Lambda_i^{j+1}, \tag{33}$$

where

$$\Lambda_i^{j+1} := \frac{\Delta\tau_{-i}^j C''(S_i^j) \Delta S_i^j g(\tau^j)}{6} - \frac{\Delta\tau_{-i}^j [P^j - C_i'(S_i^j)] g'(\tau^j) \Delta\tau^j}{6} - \frac{\Delta\tau_{-i}^{j+1} \Delta S_i^{j+1} C''(S_i^j) g(\tau^j)}{6} + \frac{\Delta\tau_{-i}^{j+1} \Delta\tau^{j+1} [P^{j+1} - C_i'(S_i^j)] g'(\tau^j)}{6} + O\big((\Delta P)^3\big). \tag{34}$$

From Corollary 1 stationary supply functions are smooth (twice differentiable) when Assumptions 1 and 2 are satisfied. Thus for samples of the continuous solution $\Delta S_i^{j+1} - \Delta S_i^j = O((\Delta P)^2)$, and in this case the whole term $\Lambda_i^{j+1}$ is bounded by $O((\Delta P)^3)$. As $g$ is assumed to be bounded away from zero, (33) can be written as follows:

$$0 = -\Delta P S_i^j + \frac{\Delta\tau_{-i}^j [P^j - C_i'(S_i^j)]}{2} + \frac{\Delta\tau_{-i}^{j+1} [P^{j+1} - C_i'(S_i^j)]}{2} + \frac{\Lambda_i^{j+1}}{g(\tau^j)}. \tag{35}$$

Samples of the continuous solution have positive mark-ups. Hence, (35) can be rewritten as:

$$0 = \frac{-\Delta P S_i^j}{P^j - C_i'(S_i^j)} + \frac{\Delta\tau_{-i}^j + \Delta\tau_{-i}^{j+1}}{2} + \frac{\Delta\tau_{-i}^{j+1} \Delta P}{2[P^j - C_i'(S_i^j)]} + \frac{\Lambda_i^{j+1}}{g(\tau^j)(P^j - C_i'(S_i^j))}. \tag{36}$$

Summing the corresponding expressions of all firms and then dividing by $N-1$ yields:

$$0 = \frac{S^{j+1} - S^{j-1}}{2} - \frac{N}{N-1} \frac{D^{j+1} - D^{j-1}}{2} - \frac{1}{N-1} \sum_k \frac{\Delta P S_k^j}{P^j - C_k'(S_k^j)} + \frac{1}{N-1} \sum_k \frac{\Delta\tau_{-k}^{j+1} \Delta P}{2[P^j - C_k'(S_k^j)]} + \frac{1}{N-1} \sum_k \frac{\Lambda_k^{j+1}}{g(\tau^j)(P^j - C_k'(S_k^j))}. \tag{37}$$

Subtract (36) from (37) and rearrange to give:

$$\begin{aligned}\frac{S_i^{j+1}-S_i^{j-1}}{2\Delta P} &= \frac{1}{N-1}\frac{D^{j+1}-D^{j-1}}{2\Delta P} - \frac{S_i^j}{P^j - C_i'(S_i^j)} + \frac{1}{N-1}\sum_k \frac{S_k^j}{P^j - C_k'(S_k^j)} \\ &\quad + \frac{\Delta\tau_{-i}^{j+1}}{2[P^j - C_i'(S_i^j)]} - \frac{1}{N-1}\sum_k \frac{\Delta\tau_{-k}^{j+1}}{2[P^j - C_k'(S_k^j)]} \\ &\quad + \frac{\Lambda_i^{j+1}}{g(\tau^j)(P^j - C_i'(S_i^j))\Delta P} \\ &\quad - \frac{1}{N-1}\sum_k \frac{\Lambda_k^{j+1}}{\Delta P g(\tau^j)(P^j - C_k'(S_k^j))}. \end{aligned} \tag{38}$$

This gives a discrete estimate of the slope of the continuous solution, when $S_i^j$ and $S_k^j$ are samples of the continuous solution $\breve{s}_i(P^j)$ and $\breve{s}_k(P^j)$. To calculate the local truncation error, $\upsilon_i^j$, subtract this discrete estimate of the slope from the slope of the continuous solution, as given in (2). Recall that $d'(P^j) = \frac{D^{j+1}-D^{j-1}}{2\Delta P}$, so

$$\begin{aligned}\boldsymbol{\upsilon}_i^j &= \frac{N-2}{N-1}\frac{(\breve{s}'_{-i}(P^j) - d'(P^j))\Delta P}{2[P^j - C_i'(\breve{s}_i(P^j))]} - \frac{1}{N-1}\sum_{k\neq i}\frac{(\breve{s}'_{-k}(P^j) - d'(P^j))\Delta P}{2[P^j - C_k'(\breve{s}_k(P^j))]} \\ &\quad + O\big((\Delta P)^2\big), \end{aligned} \tag{39}$$

because as argued above we have $\Lambda_i^{j+1} = O((\Delta P)^3)$ under these circumstances. From Assumption 2 and Lemma 1 the continuous stationary solution has a bounded slope and mark-ups bounded away from zero. Hence, it follows from (39) that $\lim_{\Delta P\to 0}\upsilon_i^j = 0$. Thus the discrete system is a consistent approximation of the continuous system. □

Lemma E below states that the system of first-order conditions implied by Lemma 2 has a unique solution for the price level $P^{j-1}$ if Assumption 4 holds and if supplies for the two previous steps, $P^j$ and $P^{j+1}$, are known and satisfy certain properties. In particular, it is necessary to assume that mark-ups in the previous steps are strictly positive to avoid the same singularity problem that arises for the continuous solution. Lemma E can then be used iteratively in the proof of Proposition 3 to ensure that there exist unique solutions to the discrete first-order condition for multiple price levels under given boundary conditions and other specified circumstances. To prove Lemma E we need a technical result which extends the implicit function theorem.

**Lemma D.** *Let $F : R^m \times R^n \to R^m$ be a continuously differentiable function and $F(\mathbf{x_0}, \mathbf{y_0}) = \mathbf{0}$, where $\mathbf{x_0}$ is a (globally) unique solution to this equation when $\mathbf{y} = \mathbf{y_0}$ and the partial Jacobian $D_x F(\mathbf{x}_0, \mathbf{y}_0)$ is invertible. For every $\mathbf{y} \in R^n$ let $V(\mathbf{y})$ be a (possibly empty) set that contains $\{\mathbf{x} \mid F(\mathbf{x}, \mathbf{y}) = \mathbf{0}\}$.*

1. *Let $\mathbf{y}_1 \in R^n$ and $\mathbf{p}$ be a continuous path from $\mathbf{y}_0$ to $\mathbf{y}_1$, i.e., a continuous function from $[0, 1]$ to $R^n$ with $\mathbf{p}(0) = \mathbf{y}_0$ and $\mathbf{p}(1) = \mathbf{y}_1$. If $\{\mathbf{x} \mid F(\mathbf{x}, \mathbf{p}(t)) = \mathbf{0}\colon t \in [0, 1]\}$ is bounded then for all small positive $t$ there exists a globally unique solution $\mathbf{x}$ to $F(\mathbf{x}, \mathbf{p}(t)) = \mathbf{0}$.*
2. *Let $W$ be a bounded set in $R^m$ and $U$ be a path connected set in $R^n$ containing $y_0$ such that $W \supset V(\mathbf{y})$ for every $\mathbf{y} \in U$. If $D_x F(\mathbf{x}, \mathbf{y})$ is invertible for every $\mathbf{y} \in U$ and $x \in V(\mathbf{y})$ then there is a globally unique solution $\mathbf{x}$ to $F(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ for every $\mathbf{y} \in U$.*

**Proof.** 1. The implicit function theorem gives neighbourhoods $V_0$ and $U_0$ of $\mathbf{x}_0$ and $\mathbf{y}_0$, respectively, and a continuously differentiable function $G : U_0 \to V_0$ such that $G(\mathbf{y}_0) = \mathbf{x}_0$ and, for each $\mathbf{y} \in U_0$, $\mathbf{x} = G(\mathbf{y})$ is the unique solution in $V_0$ of $F(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. For a contradiction of our statement suppose there is a sequence $\{t_k\} \in [0, 1]$ with $t_k \to 0$ such that for each $k$, there are two distinct solutions $\mathbf{x} = \mathbf{x}_k$ and $\mathbf{x} = \mathbf{x}'_k$ of $F(\mathbf{x}, \mathbf{p}(t_k)) = \mathbf{0}$. Since $\mathbf{x}_k, \mathbf{x}'_k \in V(\mathbf{p}(t_k))$, the sequences $\{\mathbf{x}_k\}, \{\mathbf{x}'_k\}$ are bounded and hence, by taking subsequences if necessary, it can be assumed that these sequences converge to the limits $\bar{\mathbf{x}}$ and $\bar{\mathbf{x}}'$, respectively. Note $F(\bar{\mathbf{x}}, \mathbf{y}_0) = \lim_{k\to\infty} F(\mathbf{x}_k, \mathbf{p}(t_k)) = \mathbf{0}$ and likewise $F(\bar{\mathbf{x}}', \mathbf{y}_0) = \lim_{k\to\infty} F(\mathbf{x}'_k, \mathbf{p}(t_k)) = \mathbf{0}$, thus $\bar{\mathbf{x}} = \bar{\mathbf{x}}' = \mathbf{x}_0$ by global uniqueness of $\mathbf{x}_0$. Therefore for large enough $k$, the points $\mathbf{x}_k$ and $\mathbf{x}'_k$ lie in $V_0$ and, hence, coincide with $G(\mathbf{p}(t_k))$ – contradicting the assumption $\mathbf{x}_k \neq \mathbf{x}'_k$.

2. Let $U$ be as specified in part 2 of the lemma and $\mathbf{y}_1 \in U$, so there is a continuous path $\mathbf{p}$ from $\mathbf{y}_0$ to $\mathbf{y}_1$. From part 1, there is a $T \in (0, 1]$ such that the following condition holds.

Condition $T$: $F(\mathbf{x}, \mathbf{p}(t)) = \mathbf{0}$ has a unique solution $\mathbf{x}$ for each $t \in [0, T)$.

Define $\overline{T} = \sup\{T \in (0, 1] \mid \text{Condition } T \text{ holds}\}$ and observe that $\overline{T} \in (0, 1]$ and that condition $\overline{T}$ obviously holds. The proof strategy is to show that $F(\mathbf{x}, \mathbf{p}(\overline{T})) = \mathbf{0}$ also has a unique solution.

Write $\bar{\mathbf{y}} = \mathbf{p}(\overline{T})$. Let $t_k \to \overline{T}$ with $\{t_k\} \subset [0, \overline{T})$, and $\mathbf{x} = \mathbf{x}_k$ be the unique solution (from Condition $\overline{T}$) to $F(\mathbf{x}, \mathbf{p}(t_k)) = \mathbf{0}$. The sequence $\{\mathbf{x}_k\}$ lies in $W$, hence is bounded and has a limit point $\bar{\mathbf{x}}$, which, by continuity of $F$ and $\mathbf{p}$, is a solution of $F(\mathbf{x}, \bar{\mathbf{y}}) = \mathbf{0}$. Consider any, possibly different, solution $\mathbf{x} = \hat{\mathbf{x}}$ of the last equation. By assumption, we can apply the implicit function theorem around each of $\bar{\mathbf{x}}$ and $\hat{\mathbf{x}}$ to find continuous local solution mappings $\bar{G} : \bar{U} \to \bar{V}$ and $\hat{G} : \hat{U} \to \hat{V}$ where $\bar{U}$ and $\hat{U}$ are neighbourhoods of $\bar{\mathbf{y}}$; $\bar{V}$ and $\hat{V}$ are neighbourhoods of $\bar{\mathbf{x}}$ and $\hat{\mathbf{x}}$ respectively; $\bar{G}(\bar{\mathbf{y}}) = \bar{\mathbf{x}}$ and $\hat{G}(\bar{\mathbf{y}}) = \hat{\mathbf{x}}$; and $F(\bar{G}(\mathbf{y}), \mathbf{y}) = F(\hat{G}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$ for $\mathbf{y} \in \bar{U} \cap \hat{U}$. Since $\bar{U} \cap \hat{U}$ is a neighbourhood of $\bar{\mathbf{y}}$, then $\mathbf{p}(t_k) \in \bar{U} \cap \hat{U}$ for large $k$. For such $k$, $\mathbf{x}_k$, by its uniqueness, must be equal to both $\bar{G}(\mathbf{p}(t_k))$ and $\hat{G}(\mathbf{p}(t_k))$. Thus, by continuity of $\bar{G}$, $\hat{G}$ and $\mathbf{p}$, the limit point $\bar{\mathbf{x}}$ must be equal to both $\bar{G}(\bar{\mathbf{y}})$ and $\hat{G}(\bar{\mathbf{y}})$. That is, $\bar{\mathbf{x}}$ is the only solution to $F(\mathbf{x}, \bar{\mathbf{y}}) = \mathbf{0}$.

Now apply part 1 to $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ instead of $\mathbf{x}_0$ and $\mathbf{y}_0$. In particular, take $\bar{\mathbf{p}}(t) = \mathbf{p}(\overline{T} + t(1 - \overline{T}))$ which is a continuous path from $\bar{\mathbf{y}}$ to $\mathbf{y}_1$ such that $\{\mathbf{x} \in V(\bar{\mathbf{p}}(t))\colon t \in [0, 1]\} = \{\mathbf{x} \in V(\mathbf{p}(t))\colon t \in [\overline{T}, 1]\} \subset W$, which is bounded. Thus part 1 shows that $F(\mathbf{x}, \bar{\mathbf{p}}(t)) = \mathbf{0}$ has a unique solution for each positive $t$ near 0, hence $F(\mathbf{x}, \mathbf{p}(t)) = \mathbf{0}$ has a unique solution for all $t \in [0, 1]$ near $\overline{T}$.

Thus, if $\overline{T} < 1$ it can be extended to some $T > \overline{T}$ while maintaining Condition $T$. The construction of $\overline{T}$, as a supremum, therefore gives $\overline{T} = 1$. It is only left to note that $F(\mathbf{x}, \mathbf{y}_1) = \mathbf{0}$ has a unique solution, $\bar{\mathbf{x}}$, because $\mathbf{y}_1 = \mathbf{p}(1) = \mathbf{p}(\overline{T}) = \bar{\mathbf{y}}$. □

In the following, $C''_{\max}$ is the highest slope of the marginal cost curves, $g_{\max}$ and $g_{\min}$ are the largest and smallest densities and $g'_{\max}$ is the largest slope in the support of $g(\varepsilon)$. Recall that we have made the assumption that $g_{\min}$ is strictly positive. We set $\hat{g}'_{\max} = \max(0, g'_{\max})$.

**Lemma E.** *Assume that Assumption 4 holds and that $\{S_i^{j+1}\}_{i=1}^N$ and $\{S_i^j\}_{i=1}^N$ are such that $P^j - C'_i(S_i^{j+1}) > 0$ and $S_i^{j+1} \geqslant S_i^j$ $\forall i = 1 \ldots N$, and that they satisfy*

$$0 \leqslant \Delta\widetilde{P} S_i^j g(\tau^j) - \Delta\tau_{-i}^{j+1} \int_0^{\bar{u}^{j+1}} \left[P^{j+1} - C'_i\left(\bar{S}_i^{j+1}(u)\right)\right](1-u) g\left(\bar{\tau}^{j+1}(u)\right) du$$

$$< \frac{3[P^j - C_i'(S_i^j)]^2 g_{\min}^2}{2(C''_{\max} g_{\max} + [P^j - C_i'(0)]g'_{\max})} \tag{40}$$

*for local tick-sizes $\Delta P^j = \Delta P^{j+1} = \Delta \widetilde{P}$. Under these circumstances, there exists a unique solution $\{S_i^{j-1}\}_{i=1}^N$ that together with $\{S_i^j\}_{i=1}^N$, $\{S_i^{j+1}\}_{i=1}^N$ and $\Delta P^j = \Delta P^{j+1} = \Delta \widetilde{P}$ satisfy the first-order condition $\{\Gamma_i^j\}_{i=1}^N$ in Lemma 2.*

**Proof.** For given values of $D^{j-1}$, $D^j$, $D^{j+1}$, $P^j$, $P^{j+1}$, $\{S_i^j\}_{i=1}^N$ and $\{S_i^{j+1}\}_{i=1}^N$ the aim is to solve for $\{S_i^{j-1}\}_{i=1}^N$ using the implicit function $\boldsymbol{\Gamma}$, defined by the first-order condition in Lemma 2. Lemma D is used to prove this. It is straightforward to verify that the functions $\Gamma_1^j \dots \Gamma_N^j$ are continuously differentiable in $D^{j-1}$, $D^j$, $D^{j+1}$, $P^j$, $P^{j+1}$, $\mathbf{S}^{j-1}$, $\mathbf{S}^j$ and $\mathbf{S}^{j+1}$. Below take $F = \boldsymbol{\Gamma}$ and let $V = V(\underbrace{\mathbf{S}^j, \mathbf{S}^{j+1}, D^{j-1}, D^j, D^{j+1}, P^j, P^{j+1}}_{\mathbf{y}})$ be a set that includes all $\mathbf{x} = \{S_i^{j-1}\}_{i=1}^N$ that are solutions to the first-order condition for a given input $\mathbf{y}$. Denote the set of inputs satisfying (40) by $U$. The first step of the proof is to prove that there is a fix point (base point) $(\mathbf{x}_0, \mathbf{y}_0)$, where $\mathbf{y}_0 \in U$ and for which the system of first-order conditions has a globally unique solution $\mathbf{x}_0$. Then for every $\mathbf{y} \in U$ it can be proved that there is a set $V(\mathbf{y})$ containing all solutions $\mathbf{x}$, such that the Jacobian of $\Gamma$ with respect to $\{S_i^{j-1}\}_{i=1}^N$ is invertible for $\mathbf{y} \in U$ and $\mathbf{x} \in V(\mathbf{y})$. Note that a bounded set $W$ can be found such that $V(\mathbf{y}) \subset W$ for every $\mathbf{y} \in U$. The last step is to prove that $U$ is path-connected, and thus it follows from Lemma D that each input $\mathbf{y} \in U$ has a unique global solution to the first-order condition in Lemma 2.

The first step is to identify a base point $(\mathbf{x}_0, \mathbf{y}_0)$ at which (42) is satisfied for all firms. For simplicity consider the point where $\mathbf{S}^j = \mathbf{S}^{j+1}$, $D^{j-1} = D^j = D^{j+1}$ and $\Delta \widetilde{P} = 0$. Hence, $\Delta \tau_{-i}^{j+1} = 0$, so the first-order condition in Lemma 2 can be written as:

$$0 = \Delta \tau_{-i}^j \int_0^1 \left[P^j - C_i'\left((1-u)S_i^{j-1} + uS_i^j\right)\right] u g\left((1-u)\tau^{j-1} + u\tau^j\right) du.$$

Mark-ups and the demand density are strictly positive, so the only possible solution is $\Delta \tau_{-i}^j = 0$ for all $i = 1, \dots, N$. Then

$$S_i^j - S_i^{j-1} = \Delta S_i^j = \Delta \tau^j - \Delta \tau_{-i}^j = \frac{1}{N-1} \sum_{i=1}^N \Delta \tau_{-i}^j - \Delta \tau_{-i}^j, \tag{41}$$

for $i = 1, \dots, N$, so $\mathbf{S}^{j-1} = \mathbf{S}^j$ is a unique global solution to (42) when $\mathbf{S}^j = \mathbf{S}^{j+1}$, $D^{j-1} = D^j = D^{j+1}$ and $\Delta \widetilde{P} = 0$. It is also straightforward to establish that the inputs $\mathbf{S}^j = \mathbf{S}^{j+1}$, $D^{j-1} = D^j = D^{j+1}$ and $\Delta \widetilde{P} = 0$ satisfy (40), so $\mathbf{y}_0 \in U$.

The second step verifies that the Jacobian $\{\frac{\partial \Gamma_i^j}{\partial S_k^{j-1}}\}$ is invertible for relevant arguments. As $\Delta P^{j+1} = \Delta P^j = \Delta \widetilde{P}$ the first-order condition in Lemma 2 can be written as:

$$\begin{aligned}
0 &= \Gamma_i^j\left(\mathbf{S}^{j-1}, \mathbf{S}^j, \mathbf{S}^{j+1}, D^{j-1}, D^j, D^{j+1}, P^j, P^{j+1}\right) \\
&= -\Delta \widetilde{P} S_i^j g\left(\tau^j\right) + \Delta \tau_{-i}^j \int_0^1 \left[P^j - C_i'\left((1-u)S_i^{j-1} + uS_i^j\right)\right] u g\left((1-u)\tau^{j-1} + u\tau^j\right) du
\end{aligned}$$

$$+\Delta\tau_{-i}^{j+1}\int_0^{\bar{u}^{j+1}}\left[P^{j+1}-C_i'\left((1-u)S_i^j+uS_i^{j+1}\right)\right](1-u)$$
$$\times g\left((1-u)\tau^j+u\tau^{j+1}\right)du. \tag{42}$$

Note that $\tau^{j+1}$, $\tau^j$, $\tau_{-i}^{j+1}$ and $\tau_{-i}^j$ are given by the inputs, whereas $\tau^{j-1}$ and $\tau_{-i}^{j-1}$ depend on the unknown vector $\mathbf{S}^{j-1}$. By differentiating (42), it is straightforward to show that:

$$\left.\frac{\partial\Gamma_i^j}{\partial\tau_{-i}^{j-1}}\right|_{\text{fixed }\tau^{j-1}}=-\int_0^1\left[P^j-C_i'\left(\bar{S}_i^j(u)\right)\right]ug\left(\bar{\tau}^j(u)\right)du$$
$$+\Delta\tau_{-i}^j\int_0^1 C_i''\left(\bar{S}_i^j(u)\right)(1-u)ug\left(\bar{\tau}^j(u)\right)du$$
$$\leqslant\frac{\Delta\tau_{-i}^j C''_{\max}g_{\max}}{6}-\frac{[P^j-C_i'(S_i^j)]g_{\min}}{2} \tag{43}$$

and that

$$\alpha_i=\frac{\partial\Gamma_i^j}{\partial S_i^{j-1}}=\left.\frac{\partial\Gamma_i^j}{\partial\tau^{j-1}}\right|_{\text{fixed }\tau_{-i}^{j-1}}$$
$$=-\Delta\tau_{-i}^j\int_0^1 C_i''\left(\bar{S}_i^j(u)\right)(1-u)ug\left(\bar{\tau}^j(u)\right)du$$
$$+\Delta\tau_{-i}^j\int_0^1\left[P^j-C_i'\left(\bar{S}_i^j(u)\right)\right]u(1-u)g'\left(\bar{\tau}^j(u)\right)du$$
$$\leqslant\frac{[P^j-C_i'(0)]\Delta\tau_{-i}^j\hat{g}'_{\max}}{6}, \tag{44}$$

so it follows from (43) and (44) that

$$\beta_i=\frac{\partial\Gamma_i^j}{\partial S_{k\neq i}^{j-1}}=\left.\frac{\partial\Gamma_i^j}{\partial\tau^{j-1}}\right|_{\text{fixed }\tau_{-i}^{j-1}}+\left.\frac{\partial\Gamma_i^j}{\partial\tau_{-i}^{j-1}}\right|_{\text{fixed }\tau^{j-1}}$$
$$\leqslant\frac{\Delta\tau_{-i}^j C''_{\max}g_{\max}}{6}+\frac{[P^j-C_i'(0)]\Delta\tau_{-i}^j\hat{g}'_{\max}}{6}-\frac{[P^j-C_i'(S_i^j)]g_{\min}}{2}. \tag{45}$$

The working paper [16] proves that the Jacobian $\{\frac{\partial\Gamma_i^j}{\partial S_k^{j-1}}\}$ is invertible whenever $\beta_i<\alpha_i$ and $\beta_i<0$. It follows from (43)–(45) that this is true when

$$-\phi<\Delta\tau_{-i}^j<\frac{3[P^j-C_i'(S_i^j)]g_{\min}}{C''_{\max}g_{\max}+[P^j-C_i'(0)]\hat{g}'_{\max}}, \tag{46}$$

where $\phi$ is some arbitrary finite positive number introduced to bound the space. Eq. (41) gives a relation between $\{\Delta\tau_{-i}^j\}_{i=1}^N$ and $\{S_i^{j-1}\}_{i=1}^N$. Thus for any input $D^{j-1}$, $D^j$, $D^{j+1}$, $P^j$, $P^{j+1}$,

$\{S_i^j\}_{i=1}^N$ and $\{S_i^{j+1}\}_{i=1}^N$, the Jacobian $\{\frac{\partial \Gamma_i^j}{\partial S_k^{j-1}}\}$ is invertible for any $\{S_i^{j-1}\}_{i=1}^N$ in the bounded space $V(\mathbf{S}^j, \mathbf{S}^{j+1}, D^{j-1}, D^j, D^{j+1}, P^j, P^{j+1})$ defined as the set of all $\{S_i^{j-1}\}_{i=1}^N$ that satisfy (46). Prices are uniformly bounded by $P^M$ and $P^1$ and outputs are in bounded intervals $[0, k_i]$, so a bounded set $W$, such that $V(y) \subset W$ for all $\mathbf{y} \in U$, can always be found.

The third step makes sure that the space $V(\mathbf{y})$ includes all possible solutions to the first-order condition for a given input $\mathbf{y} \in U$. Consider the space where all potential solutions to the system of first-order conditions in Lemma 2 are located for inputs satisfying the inequality in (40). It follows from the latter inequality that $\{S_i^{j+1}\}_{i=1}^N$ and $\{S_i^j\}_{i=1}^N$ are such that

$$-\Delta \widetilde{P} S_i^j g(\tau^j) + \Delta \tau_{-i}^{j+1} \int_0^{\bar{u}^{j+1}} \left[P^{j+1} - C_i'\left(\bar{S}_i^{j+1}(u)\right)\right](1-u) g\left(\bar{\tau}^{j+1}(u)\right) du \leqslant 0.$$

From this inequality it follows that the first integral in the stepped first-order condition in Lemma 2 must be non-negative. By assumption $P^j > C_i'(S_i^j)$, so all potential solutions to the system of first-order conditions must be located in the space where $\Delta \tau_{-i}^j \geqslant 0$. It also follows from (42) that

$$\Delta \widetilde{P} S_i^j g(\tau^j) - \Delta \tau_{-i}^{j+1} \int_0^{\bar{u}^{j+1}} \left[P^{j+1} - C_i'\left(\bar{S}_i^{j+1}(u)\right)\right](1-u) g\left(\bar{\tau}^{j+1}(u)\right) du$$
$$\geqslant \frac{[P^j - C_i'(S_i^j)] g_{\min} \Delta \tau_{-i}^j}{2}.$$

Thus given that inputs $D^{j-1}$, $D^j$, $D^{j+1}$, $P^j$, $P^{j+1}$, $\{S_i^j\}_{i=1}^N$ and $\{S_i^{j+1}\}_{i=1}^N$ satisfy the inequality in (40), all solutions to the system of first-order conditions in Lemma 2 must be located in the space where

$$\begin{aligned} 0 &\leqslant \Delta \tau_{-i}^j \\ &\leqslant \frac{2\Delta \widetilde{P} S_i^j g(\tau^j) - 2\Delta \tau_{-i}^{j+1} \int_0^{\bar{u}^{j+1}} [P^{j+1} - C_i'(\bar{S}_i^{j+1}(u))](1-u) g(\bar{\tau}^{j+1}(u))\, du}{[P^j - C_i'(S_i^j)] g_{\min}}. \end{aligned} \tag{47}$$

As long as inputs satisfy (40), it follows from (47) that the condition in (46) is satisfied, i.e., for any input $\mathbf{y} \in U$ all possible solutions $\{S_i^{j-1}\}_{i=1}^N$ to the first-order condition are inside $V(\mathbf{y})$.

The final step establishes that inputs satisfying (40) are path connected. Let

$$\begin{aligned} T_i &= \Delta \widetilde{P} S_i^j g(\tau^j) - \Delta \tau_{-i}^{j+1} \int_0^{\bar{u}^{j+1}} \left[P^{j+1} - C_i'\left((1-u) S_i^j + u S_i^{j+1}\right)\right](1-u) \\ &\quad \times g\left((1-u)\tau^j + u\tau^{j+1}\right) du \\ &= \left(P^{j+1} - P^j\right) S_i^j g(\tau^j) + \int_0^{\bar{u}^{j+1}} \left[P^{j+1} - C_i'\left((1-u) S_i^j + u S_i^{j+1}\right)\right]\left(-\Delta \tau_{-i}^{j+1}(1-u)\right) \\ &\quad \times g\left(\tau^{j+1} - (1-u)\Delta \tau^{j+1}\right) du. \end{aligned}$$

Given Assumption 4, $T_i$ is increasing relative to the right-hand side of (40) if $S_k^j$ increases, $D^{j+1}$ increases or if $P^j$ decreases. Thus starting with an input satisfying (40) it is always possible to find a smooth path by continuously increasing $S_k^j$, continuously increasing $D^{j+1}$ and continuously increasing $P^j$ in such a manner that (40) is satisfied. We have $D^{j+1} \leqslant D^j$, so eventually the point where $\mathbf{S}^j = \mathbf{S}^{j+1}$ and $D^j = D^{j+1}$ is reached so that $\Delta\tau_{-i}^{j+1} = 0$ and $T_i = \Delta\widetilde{P} S_i^j g(\tau^j)$. Thus $P^j$ can be continuously increased and (40) will be satisfied until the base point where $P^j = P^{j+1}$ is reached, which completes the proof. □

The consistency and uniqueness properties of the first-order condition in Lemmas C and E are used when proving convergence below. Recall that $L$ and $H$ are the lowest and highest price indices, $j$, such that price $P^j$ occurs with positive probability, and varies with $M$ (and the boundary conditions).

**Proof of Proposition 3.** From Assumption 1 and 2 and Lemma 1 the mark-ups of the continuous stationary solution are uniformly bounded away from zero, and the relevant stationary solution is increasing with a bounded slope. Lemma C therefore applies and states that the discrete difference equation is a consistent approximation of the continuous differential equation. To show that the stepped stationary solution converges to the continuous stationary solution, it is necessary to prove that the stepped stationary solution exists and is stable, i.e., the error grows at a finite rate over a finite price range. The proof is inspired by LeVeque's [23] convergence proof for general one-step methods. Define the vector of global errors at the price $P^j$, $\mathbf{E}^j = \mathbf{S}^j - \breve{\mathbf{s}}(P^j)$. It follows from (39) that the local truncation error is of the order $\Delta P$.

It is useful to introduce a Lipschitz constant $\lambda$ [23] that will bound the growth of errors. For the cases considered, it follows from the implicit function theorem argument in Lemma E that the vector $\mathbf{S}^{j-1}$ can be written as a continuously differentiable function of $\mathbf{S}^j$. This is sufficient to establish that there exists a finite Lipschitz constant that uniformly bounds $|\frac{\partial S_i^{j-1}}{\partial S_k^j}|$ in the range considered. When discrete values are samples of the continuous stationary solution it can be shown by means of (38) that[15]:

$$\begin{aligned}\left|\frac{\partial S_i^{j-1}}{\partial S_k^j}\right| &\leqslant \frac{2(N-2)}{N-1}\left\|\frac{p - C_i'(\breve{s}_i(p)) + \breve{s}_i(p)C_i''(\breve{s}_i(p))}{[p - C_i'(\breve{s}_i(p))]^2}\right\|_\infty \Delta P \\ &\quad + \frac{2}{N-1}\left\|\frac{p - C_k'(\breve{s}_k(p)) + \breve{s}_k(p)C_k''(\breve{s}_k(p))}{[p - C_k'(\breve{s}_k(p))]^2}\right\|_\infty \Delta P \\ &\quad + \frac{(N-2)\Delta P}{N-1}\left\|\frac{1}{[p - C_i'(\breve{s}_i(p))]}\right\|_\infty + \frac{\Delta P}{N-1}\left\|\frac{1}{[p - C_i'(\breve{s}_i(p))]}\right\|_\infty \\ &\quad + O\big((\Delta P)^2\big) \\ &= \left\|\frac{3(p - C_i'(\breve{s}_i(p))) + 2\breve{s}_i(p)C_i''(\breve{s}_i(p))}{[p - C_i'(\breve{s}_i(p))]^2}\right\|_\infty \Delta P + O\big((\Delta P)^2\big).\end{aligned} \tag{48}$$

The required Lipschitz constant $\lambda$ satisfies the inequality

$$\lambda > \left\|\frac{3(p - C_i'(\breve{s}_i(p))) + 2\breve{s}_i(p)C_i''(\breve{s}_i(p))}{[p - C_i'(\breve{s}_i(p))]^2}\right\|_\infty \quad \forall p \in (a, b). \tag{49}$$

[15] Note that $\| \|_\infty$ is the max-norm, i.e., $\|E^j\|_\infty = \max_{1\leqslant i\leqslant N} |E^j|$ [23].

Hence, if discrete values are sufficiently close to the continuous stationary solution and the tick-size is sufficiently small, it follows that $|\frac{\partial S_i^{j-1}}{\partial S_k^j}| < \lambda \Delta P$. Introduce a second constant

$$\mu < \left\| \frac{3[p - C_i'(\breve{s}_i(p))]g_{\min}^2}{2(C''_{\max} g_{\max} + [p - C_i'(0)]\hat{g}'_{\max})} \right\|_\infty \quad \forall p \in (a, b), \tag{50}$$

which bounds the right-hand side of (40) from below. Such a constant can always be found as both mark-ups and the shock density are bounded away from zero. Again this inequality is also satisfied for the discrete system when discrete values are sufficiently close to the continuous stationary solution.

It has been assumed that boundary conditions are chosen such that the inequality in (10) is satisfied, so it follows from Lemma E that $\{S_i^{H-2}\}_{i=1}^N$ can be uniquely determined and from Lemma B that the solution $\Delta S_i^{H-1}$ is of the order $\Delta P$. The next step verifies that $\{S_i^{H-3}\}_{i=1}^N$ can be uniquely determined. First, note that it follows from the consistency property and (1) that

$$\lim_{\Delta P \to 0} \frac{[P^{j+1} - C_i'(S_i^{j+1})]\Delta \tau_{-i}^{j+1}}{2\Delta P} = \frac{[P^j - C_i'(\breve{s}_i(P^j))](\breve{s}'_{-i}(P^j) - d'(P^j))}{2} = \frac{\breve{s}_i(P^j)}{2}$$
$$= \lim_{\Delta P \to 0} \frac{S_i^j}{2}$$

when discrete values are samples of the continuous stationary solution. The result above and (50) ensures that:

$$\begin{aligned} 0 &\leqslant \Delta P S_i^j g(\tau^j) - \frac{[P^{j+1} - C_i'(S_i^{j+1})]\Delta \tau_{-i}^{j+1} g(\tau^j)}{2} + O((\Delta P)^2) \\ &< \mu < \frac{3[P^j - C_i'(S_i^j)]^2 g_{\min}^2}{2(C''_{\max} g_{\max} + [P^j - C_i'(0)]\hat{g}'_{\max})} \end{aligned} \tag{51}$$

if $L \leqslant j < H - 1$ and $\Delta P$ is sufficiently small. As $\Delta P S_i^j g(\tau^j) - \frac{[P^{j+1} - C_i'(S_i^{j+1})]\Delta \tau_{-i}^{j+1} g(\tau^j)}{2} + O((\Delta P)^2)$ is a Taylor expansion of

$$\Delta P S_i^j g(\tau^j) - \Delta \tau_{-i}^{j+1} \int_0^1 [P^{j+1} - C_i'((1-u)S_i^j + uS_i^{j+1})](1-u)$$
$$\times g((1-u)\tau^j + u\tau^{j+1})\, du,$$

the above ensures that the existence condition in (40) is satisfied for sufficiently small $\Delta P$. Thus it follows from Lemma E that $\{S_i^{H-3}\}_{i=1}^N$ can be uniquely determined for $M$ larger than some sufficiently large finite number. It follows from (49) and (38) that the global error satisfies the following inequality at this point:

$$\|\mathbf{E}^{H-3}\|_\infty = \|\mathbf{S}^{H-3} - \breve{\mathbf{s}}(P^{H-3})\|_\infty$$
$$\leqslant \|\mathbf{E}^{H-1}\|_\infty + \lambda N \Delta P \|\mathbf{E}^{H-2}\|_\infty + \Delta P \|\boldsymbol{\upsilon}^{H-2}\|_\infty \tag{52}$$
$$\leqslant (1 + \lambda N \Delta P)\|\mathbf{E}_{\max}^{H-2}\|_\infty + \Delta P \|\boldsymbol{\upsilon}^{H-2}\|_\infty, \tag{53}$$

where $\upsilon^{H-2}$ is the local truncation error and $\|\mathbf{E}_{\max}^{H-2}\|_\infty = \max(\|\mathbf{E}^{H-2}\|_\infty, \|\mathbf{E}^{H-1}\|_\infty)$. Thus if $\Delta P$ is sufficiently small, so that the initial errors $\|\mathbf{E}^{H-1}\|_\infty$ and $\|\mathbf{E}^{H-2}\|_\infty$ and the local truncation error $\|\boldsymbol{\upsilon}^{H-3}\|\Delta P$ are small enough, then $\|\mathbf{E}^{H-3}\|_\infty$ is sufficiently small. It now follows from the assumed properties of the continuous solution that $S_i^{H-2} - S_i^{H-3} \geqslant 0$ and that $P^{H-3} - C_i'(S_i^{H-2})$ is bounded away from zero $\forall i = 1 \dots N$.

If $\Delta P$ is sufficiently small, then the above argument for the vector $\mathbf{S}^{H-3}$ can be repeated iteratively to prove that the vector $\mathbf{S}^k$ $\forall k = L, \dots, H-4$ can be uniquely determined and that

$$\|\mathbf{E}^k\|_\infty = \|\mathbf{S}^k - \breve{\mathbf{s}}(P^k)\|_\infty \leqslant \|\mathbf{E}^{k+2}\|_\infty + \lambda N \Delta P \|\mathbf{E}^{k+1}\|_\infty + \Delta P \|\boldsymbol{\upsilon}^{k+1}\|_\infty. \tag{54}$$

Let $\upsilon_{\max}^k = \max\{\|\boldsymbol{\upsilon}^n\|_\infty\}_{n=k}^{H-1}$. From the inequality in (54), we can show by induction that:

$$\begin{aligned}
\|\mathbf{E}^k\|_\infty &\leqslant (1+\lambda N\Delta P)^{H-2-k}\|\mathbf{E}_{\max}^{H-2}\|_\infty + \Delta P \sum_{m=k+1}^{H-2} \|\upsilon^m\|_\infty (1+\lambda N \Delta P)^{m-k-1} \\
&< (1+\lambda N\Delta P)^{H-2-k}\big(\|\mathbf{E}_{\max}^{H-2}\|_\infty + (H-k-2)\Delta P \upsilon_{\max}^{k+1}\big) \\
&\leqslant (1+\lambda N\Delta P)^{H-L}\big(\|\mathbf{E}_{\max}^{H-2}\|_\infty + (H-L)\Delta P \upsilon_{\max}^{L}\big) \\
&\leqslant e^{\lambda N(H-L)\Delta P}\big(\|\mathbf{E}_{\max}^{H-2}\|_\infty + (H-L)\Delta P \upsilon_{\max}^{L}\big),
\end{aligned} \tag{55}$$

where the last inequality follows from the Taylor expansion of the exponential function. The number of equidistant steps $H-L$ covers the distance $b-a$, so that $(H-L)\Delta P \leqslant b-a+\Delta P$. The consistency property established in Lemma C ensures that the truncation error $\upsilon_{\max}^{L+1}$ can be made arbitrarily small by decreasing $\Delta P$. Thus the global error at each price level can be bounded by choosing a sufficiently small $\Delta P$. This and the properties of the continuous solution now imply that there will always be some sufficiently large but finite $M_0$, such that the condition for a unique solution in (51) is satisfied for $L+1 \leqslant j \leqslant H-2$ and $M \geqslant M_0$. In the limit as $\Delta P \to 0$ then $\|\mathbf{E}_{\max}^{H-2}\|_\infty \to 0$, $\upsilon_{\max}^{L+1} \to 0$ and $(H-L)\Delta P \to b-a$. Thus from (55), $\|\mathbf{E}^k\|_\infty \to 0$ when $\Delta P \to 0$, proving that the stepped stationary solution converges to the continuous one. □

### *A.4. Convergence of equilibria*

Theorem 1 can now be proved by given Propositions 2 and 3 and Lemma 3:

**Proof of Theorem 1.** Part (a) is a restatement of Proposition 2. The next step is to show part (b). Proposition 3 ensures that the stepped stationary solution exists for sufficiently large $M$, and that it will converge to the continuous solution. The continuous stationary solution satisfies Assumption 2, and together with consistency (proved in Lemma C) this implies that the stepped solution is non-decreasing for $M$ larger than some sufficiently large finite number of price levels. Moreover, convergence of the stationary solutions and Lemma 1 ensure that mark-ups in the stepped stationary solution are strictly positive for $M$ larger than some sufficiently large finite number of price levels. Finally, convergence of competitors' supply curves implies that the difference between a producer's profits in the discrete and continuous system will converge to zero, and this is also true for all possible deviations of the producer. In the continuous system, Lemma A ensures that it is strictly unprofitable for a firm to withhold output in order to push the price above $b$. Moreover, it follows from the argument in the proof of Proposition 2 that it is strictly unprofitable to undercut the price $a$ when stationary supply curves are increasing. These properties will be inherited by the stepped stationary solution for some sufficiently large finite $M$, so that there are no profitable deviations from the stepped stationary solution involving undercutting of $P^L$

or pushing the price above $P^H$. It follows that for sufficiently large $M$ the stepped stationary solution satisfies all conditions in Lemma 3, so that it becomes a segment of an SSFE. □

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