Q6:

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ -2 & -1 & 1 & 2 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 3 & -1 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1+2x_2-x_3-x_4=0 \\ 3x_2-x_3=0 \end{matrix}$$

$\therefore$ $x_3=3x_2$

$x_1=x_3+x_4-2x_2=3x_2+x_4-2x_2=x_2+x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2+x_4 \\ x_2 \\ 3x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$$

suppose $c_1\begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3-3R_2]{R_2-R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow c_1=c_2=0.$$

$\therefore$ $\begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent.

So. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is the basis of $W$.

Q7:

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ -1 & 1 & -4 & 2 \\ 0 & -2 & -2 & 1 \\ 2 & 0 & 6 & 2 \end{bmatrix} \xrightarrow[R_2+R_1]{R_4-2R_1} \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & -2 & -2 & 1 \\ 0 & -4 & -4 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2]{R_4-2R_3} \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & -1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{3R_3+R_2} \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 0 & -2 & \frac{7}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(W) = 3$.

Q8.

① $\underline{u} = \frac{1}{2}[(\underline{u}+\underline{v}) + (\underline{u}-\underline{v})] = \frac{1}{2}(\underline{u}+\underline{v}) + \frac{1}{2}(\underline{u}-\underline{v})$

$\underline{v} = \frac{1}{2}(\underline{u}+\underline{v}) - \frac{1}{2}(\underline{u}-\underline{v})$

since $\underline{u}, \underline{v}$ can be expressed by $(\underline{u}+\underline{v}), (\underline{u}-\underline{v})$, $V = \text{span}\{(\underline{u}+\underline{v}), (\underline{u}-\underline{v})\}$.

② suppose $c_1(\underline{u}+\underline{v}) + c_2(\underline{u}-\underline{v}) = 0$.

$(c_1+c_2)\underline{u} + (c_1-c_2)\underline{v} = 0$.

$\begin{cases} c_1+c_2=0 \\ c_1-c_2=0 \end{cases}$ since $a\underline{u}+b\underline{v}=0$ $a=b=0$.

$\therefore$ $c_1 = c_2 = 0$ $\rightarrow$ $(\underline{u}+\underline{v}), (\underline{u}-\underline{v})$ are linearly independent.

So, $\{\underline{u}+\underline{v}, \underline{u}-\underline{v}\}$ is also a basis for $V$.

由 扫描全能王 扫描创建

Q9. $[\underline{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$  $\underline{x} = c_1 \underline{u_1} + c_2 \underline{u_2} + c_3 \underline{u_3} + \cdots + c_n \underline{u_4}$,

$[\underline{y}]_B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$  $\underline{y} = b_1 \underline{u_1} + b_2 \underline{u_2} + \cdots + b_n \underline{u_n}$

$\underline{x} + \underline{y} = (c_1 + b_1) \underline{u_1} + (c_2 + b_2) \underline{u_2} + (c_3 + b_3) \underline{u_3} + \cdots + (c_n + b_n) \underline{u_n}$

$\therefore [\underline{x} + \underline{y}]_B = [\underline{x}]_B + [\underline{y}]_B. = \begin{bmatrix} c_1 + b_1 \\ c_2 + b_2 \\ \vdots \\ c_n + b_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Q10.

a). $A\underline{u} = \begin{bmatrix}12 & -2\\ -2 & 15\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}24-2\\-4+15\end{bmatrix} = \begin{bmatrix}22\\11\end{bmatrix} = 11\begin{bmatrix}2\\1\end{bmatrix}$

$A\underline{v} = \begin{bmatrix}12 & -2\\ -2 & 15\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-12-4\\2+30\end{bmatrix} = \begin{bmatrix}-16\\32\end{bmatrix} = 16\begin{bmatrix}-1\\2\end{bmatrix}$

$A\underline{w} = \begin{bmatrix}-12 & -2\\ -2 & 15\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-12-2\\-2+15\end{bmatrix} = \begin{bmatrix}-14\\13\end{bmatrix} \neq$ multiple of $\begin{bmatrix}1\\1\end{bmatrix}$, so not an eigenvector.

$\therefore$ Eigenvalues are 11, 16;

Eigenvectors are $\underline{u}=\begin{bmatrix}2\\1\end{bmatrix}$, $\underline{v}=\begin{bmatrix}-1\\2\end{bmatrix}$.

b). A collection of $X=\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\}$ of vectors in $R^2$ is a basis for $R^2$ if

① $X$ is a spanning set for $R^2$. So $R^2 = \text{span}(X)$

which is true for $R^2 = \text{span}\left(\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right)$

②. $X$ is linearly independent.

$\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}$, for $c_1\underline{u} + c_2\underline{v} = 0$.

$$\left[\begin{array}{cc|c}2 & -1 & 0\\ 1 & 2 & 0\end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c}0 & -5 & 0\\ 1 & 2 & 0\end{array}\right] \xrightarrow[-\frac{1}{5}R_1]{R_1\leftrightarrow R_2} \left[\begin{array}{cc|c}1 & 2 & 0\\ 0 & 1 & 0\end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\end{array}\right]$$

$c_1 = c_2 = 0$.

$\therefore$ $\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}$ are linearly independent.

So, $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\}$ can form a basis for $R^2$.

c). Suppose $c_1\underline{u} + c_2\underline{v} = \underline{b}$

$$\left[\begin{array}{cc|c}2 & -1 & 3\\ 1 & 2 & 4\end{array}\right] \xrightarrow[R_1\leftrightarrow R_2]{R_1-2R_2} \left[\begin{array}{cc|c}1 & 2 & 4\\ 0 & -5 & -5\end{array}\right] \xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{cc|c}1 & 2 & 4\\ 0 & 1 & 1\end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c}1 & 0 & 2\\ 0 & 1 & 1\end{array}\right]$$

$\therefore$ $c_1 = 2$ $c_2 = 1$.

$\underline{b} = \begin{bmatrix}3\\4\end{bmatrix} = 2\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}4-1\\2+2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$. $\underline{b} = 2\underline{u} + 1\cdot\underline{v}$

d). The formular for $\underline{x}(t)$ is $\underline{x}(t) = c_1 e^{11t}\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{16t}\begin{bmatrix}-1\\2\end{bmatrix}$

put $t=0$, we can get $\underline{x}(0) = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\2\end{bmatrix}$ Since $e^0 = 1$

$\underline{x}(0) = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\2\end{bmatrix} = \underline{b} = \begin{bmatrix}3\\4\end{bmatrix}$.

$\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} \Rightarrow \left[\begin{array}{cc|c}2 & -1 & 3\\ 1 & 2 & 4\end{array}\right]$ we can solve to get $\begin{cases}c_1 = 2\\ c_2 = 1\end{cases}$