HW#4

Q16. $B = \{ w_1(x), w_2(x), w_3(x) \}$ Find the projection of $f(x) = x^5$.

For c_1:

$$\frac{\langle f(x), w_1(x) \rangle}{\langle w_1(x), w_1(x) \rangle} = \frac{\int_0^1 x^5 dx}{\int_0^1 1 dx} = \frac{\frac{1}{6}x^6 \Big|_0^1}{x \Big|_0^1} = \frac{\frac{1}{6}}{1} = \frac{1}{6}.$$

For c_2:

$$\frac{\langle f(x), w_2(x) \rangle}{\langle w_2(x), w_2(x) \rangle} = \frac{\int_0^1 x^5 (x-\frac{1}{2}) dx}{\int_0^1 (x-\frac{1}{2})^2 dx} = \frac{\int_0^1 (x^6 - \frac{1}{2}x^5) dx}{\int_0^1 (x^2 - x + \frac{1}{4}) dx}$$

$$= \frac{(\frac{1}{7}x^7 - \frac{1}{12}x^6) \Big|_0^1}{(\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x) \Big|_0^1} = \frac{\frac{1}{7} - \frac{1}{12}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} = \frac{5}{7}.$$

For c_3:

$$\frac{\langle f(x), w_3(x) \rangle}{\langle w_3(x), w_3(x) \rangle} = \frac{\int_0^1 x^5 (x^2 - x + \frac{1}{6}) dx}{\int_0^1 (x^2 - x + \frac{1}{6})^2 dx} = \frac{\int_0^1 (x^7 - x^6 + \frac{1}{6}x^5) dx}{\int_0^1 (x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}) dx}$$

$$= \frac{(\frac{1}{8}x^8 - \frac{1}{7}x^7 + \frac{1}{36}x^6) \Big|_0^1}{(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{9}x^3 - \frac{1}{6}x^2 + \frac{1}{36}x) \Big|_0^1} = \frac{\frac{1}{8} - \frac{1}{7} + \frac{1}{36}}{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}} = \frac{25}{14}.$$

$$\text{Proj}_W (x) = \frac{1}{6} w_1(x) + \frac{5}{7} w_2(x) + \frac{25}{14} w_3(x)$$

$$= \frac{1}{6} + \frac{5}{7}(x - \frac{1}{2}) + \frac{25}{14}(x^2 - x + \frac{1}{6}).$$

$$= \frac{25}{14}x^2 - \frac{15}{14}x + \frac{3}{28}$$

Q17. we assume that $\beta = \{\underline{w_1}, \underline{w_2}, \cdots, \underline{w_n}\}$ is an orthogonal basis of W. then $\text{proj}_W(\underline{x}) = c_1\underline{w_1} + c_2\underline{w_2} + \cdots + c_n\underline{w_n}$

where $c_i = \dfrac{\langle \underline{x}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle}$ $\quad i = 1, 2, \cdots, n$.

①. $F(\underline{x}+\underline{y}) = \text{proj}_W(\underline{x}+\underline{y}) = c_1\underline{w_1} + c_2\underline{w_2} + \cdots + c_n\underline{w_n}$ $\quad c_i = \dfrac{\langle \underline{x}+\underline{y}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle}$

$F(\underline{x}) = \text{proj}_W(\underline{x}) = a_1\underline{w_1} + a_2\underline{w_2} + \cdots + a_n\underline{w_n}$ $\quad a_i = \dfrac{\langle \underline{x}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle}$

$F(\underline{y}) = \text{proj}_W(\underline{y}) = b_1\underline{w_1} + b_2\underline{w_2} + \cdots + b_n\underline{w_n}$ $\quad b_i = \dfrac{\langle \underline{y}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle}$

$\because \dfrac{\langle \underline{x}+\underline{y}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle} = \dfrac{\langle \underline{x}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle} + \dfrac{\langle \underline{y}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle}$

$\therefore \quad c_i = a_i + b_i \quad i = 1, 2, \cdots, n$.

So. $F(\underline{x}+\underline{y}) = \text{proj}_W(\underline{x}+\underline{y}) = c_1\underline{w_1} + c_2\underline{w_2} + \cdots + c_n\underline{w_n}$

$= (a_1+b_1)\underline{w_1} + (a_2+b_2)\underline{w_2} + \cdots + (a_n+b_n)\underline{w_n}$

$= (a_1\underline{w_1} + a_2\underline{w_2} + \cdots + a_n\underline{w_n}) + (b_1\underline{w_1} + b_2\underline{w_2} + \cdots + b_n\underline{w_n})$

$= \text{proj}_W(\underline{x}) + \text{proj}_W(\underline{y}) = F(\underline{x}) + F(\underline{y}) \rightarrow$ true.

②. $F(c\underline{x}) = \text{proj}_W(c\underline{x}) = u_1\underline{w_1} + u_2\underline{w_2} + \cdots + u_n\underline{w_n}$ $\quad u_i = \dfrac{\langle c\underline{x}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle}$

$F(\underline{x}) = \text{proj}_W(\underline{x}) = a_1\underline{w_1} + a_2\underline{w_2} + \cdots + a_n\underline{w_n}$ $\quad a_i = \dfrac{\langle \underline{x}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle}$

$\because \dfrac{\langle c\underline{x}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle} = \dfrac{c\langle \underline{x}, \underline{w_i} \rangle}{\langle \underline{w_i}, \underline{w_i} \rangle}$

$\therefore \quad u_i = c \cdot a_i$

So. $F(c\underline{x}) = \text{proj}_W(c\underline{x}) = u_1\underline{w_1} + u_2\underline{w_2} + \cdots + u_n\underline{w_n}$

$= ca_1\underline{w_1} + ca_2\underline{w_2} + \cdots + ca_n\underline{w_n}$

$= c(a_1\underline{w_1} + a_2\underline{w_2} + \cdots + a_n\underline{w_n}) = c \cdot \text{proj}_W(\underline{x}) = cF(\underline{x})$

$\rightarrow$ true.

So, F is a linear transformation.

Q18.

$$\langle \underline{u}+\underline{v}, \underline{u}-\underline{v} \rangle = \langle \underline{u}+\underline{v}, \underline{u} \rangle - \langle \underline{u}+\underline{v}, \underline{v} \rangle$$

$$= \langle \underline{u}, \underline{u} \rangle + \langle \underline{v}, \underline{u} \rangle - (\langle \underline{u}, \underline{v} \rangle + \langle \underline{v}, \underline{v} \rangle)$$

$$= \|u\|^2 + \langle \underline{v}, \underline{u} \rangle - \langle \underline{u}, \underline{v} \rangle - \|v\|^2.$$

$$= \|u\|^2 - \|v\|^2$$

Q19. $g(x) = ax^2 + bx + c$.

x	$f(x)$	$g(x)$
1	1	$a+b+c$
2	3	$4a+2b+c$
3	-1	$9a+3b+c$
4	1	$16a+4b+c$
5	0	$25a+5b+c$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix}^+ = \begin{bmatrix} \frac{1}{7} & -\frac{1}{14} & -\frac{1}{7} & -\frac{1}{14} & \frac{1}{7} \\ -\frac{37}{35} & \frac{23}{70} & \frac{6}{7} & \frac{37}{70} & -\frac{23}{35} \\ \frac{9}{5} & 0 & -\frac{4}{5} & -\frac{3}{5} & \frac{3}{5} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix}^+ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & -\frac{1}{14} & -\frac{1}{7} & -\frac{1}{14} & \frac{1}{7} \\ -\frac{37}{35} & \frac{23}{70} & \frac{6}{7} & \frac{37}{70} & -\frac{23}{35} \\ \frac{9}{5} & 0 & -\frac{4}{5} & -\frac{3}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 2 \end{bmatrix}. \quad \begin{cases} a = 0 \\ b = -\frac{2}{5} \\ c = 2 \end{cases}$$

$$\therefore\ g(x) = -\frac{2}{5}x + 2 = -0.4x + 2.$$

Q20.

(a). $< \underline{u}, \underline{v} > = \underline{u}^T \underline{v} = 0$

$< P\underline{u}, P\underline{v} > = < P^T P \underline{u}, \underline{v} > = < I \underline{u}, \underline{v} > = < \underline{u}, \underline{v} > = \underline{u}^T \underline{v} = 0.$

So, $P\underline{u}$ and $P\underline{v}$ are also orthogonal to each other.

(b) $< \underline{x}, \underline{x} > = \| \underline{x} \|^2.$

$< P\underline{x}, P\underline{x} > = < P^T P \underline{x}, \underline{x} > = < I \underline{x}, \underline{x} > = < \underline{x}, \underline{x} > = \| \underline{x} \|^2$

$= \| P \underline{x} \|^2$

$\therefore \quad \| x \| = \| P \underline{x} \|.$