Q11. (a) $T(\underline{x}) = (x_1+x_2-x_3)\underline{u} = T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$

$T(\underline{y}) = (y_1+y_2-y_3)\underline{u} = T\left(\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\right)$

① $T(\underline{x}+\underline{y}) = T\left(\begin{bmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{bmatrix}\right) = [(x_1+y_1)+(x_2+y_2)-(x_3+y_3)]\underline{u} = ((x_1+x_2-x_3)+(y_1+y_2-y_3))\underline{u}$.

$= (x_1+x_2-x_3)\underline{u} + (y_1+y_2-y_3)\underline{u} = T(\underline{x}) + T(\underline{y}) \rightarrow$ true.

② $T(c\underline{x}) = T\left(\begin{bmatrix}cx_1\\cx_2\\cx_3\end{bmatrix}\right) = (cx_1+cx_2-cx_3)\underline{u} = c(x_1+x_2-x_3)\underline{u} = cT(\underline{x}) \rightarrow$ true.

$\therefore$ T is a linear transformation.

(b). $T(\underline{x}) = T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = (x_1+x_2-x_3)\underline{u} = 0$. $\because$ $\underline{u}$ is a fixed non-zero vector in R^3.

$\therefore$ $x_1+x_2-x_3=0 \Rightarrow x_3 = x_1+x_2$. so we have.

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_1+x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\\1\end{bmatrix} + x_2\begin{bmatrix}0\\1\\1\end{bmatrix}$ $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$ are linear independent. clearly.

So they form a basis for $\ker(T)$.

(c). $\dim(\ker(T)) = 2$ according to part (b). $\dim(R^3) = 3$.

$\dim(\text{range}(T)) = \dim(\text{im}(T)) = 3-2=1$.

(d). $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$ $T(\underline{x}) = T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = T\left(\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right)$

$= \text{span}\left(T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right), T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right), T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right) = \text{span}(\underline{u}, \underline{u}, -\underline{u})$

$= \text{span}(\underline{u})$. since $\underline{u}, \underline{u}, -\underline{u}$ are linear dependent,

This means that $\text{im}(T) = \text{range}(T) = \text{span}(\underline{u})$ $\therefore$ $\dim(\text{im}(T)) = 1$.

Q12. (a) $\dim(V) = 2 \times 2 = 4$

(b) ①. $F(A) = A^T - A \qquad F(B) = B^T - B$.

$F(A+B) = (A+B)^T - (A+B) = A^T + B^T - A - B = (A^T - A) + (B^T - B) = F(A) + F(B)$.

$\rightarrow$ true.

②. $F(A) = A^T - A$

$F(cA) = (cA)^T - cA = cA^T - cA = c(A^T - A) = c\,F(A) \rightarrow$ true.

$\therefore$ F is a linear map.

(c). $\ker(F)$: $F(A) = A^T - A = 0 \Rightarrow A^T = A$ $\therefore$ A is symmetric.

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = A^T = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} \Rightarrow a_2 = a_3 \Rightarrow A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix} = a_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$= a_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + a_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ And $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are clearly linear independent.

so, they form a basis for $\ker(F)$.

(d). $\dim(\ker(F)) = \dim(\mathrm{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix})) = 3.$

(e) $F(A) = A^T - A = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} - \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & a_3 - a_2 \\ a_2 - a_3 & 0 \end{bmatrix} = a_2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + a_3\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= a_2\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to 0 \Rightarrow$ not linear independent.

$\because A = \mathrm{span}(\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix})$

$F(A) = \mathrm{im}(F) = F(\mathrm{span}(\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix})) = \mathrm{span}(F(\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix})) = \mathrm{span}(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$

$= \mathrm{span}(\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}) = \mathrm{span}(\begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix})$

$\therefore \dim(\mathrm{im}(F)) = \dim(\mathrm{span}(\begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix})) = 1$

Q13. (a). $c_1\underline{u_1} + c_2\underline{u_2} + \cdots + c_n\underline{u_n} = \underline{0}$. $c_1, c_2, \ldots, c_n$ are scalars.

$F(c_1\underline{u_1} + c_2\underline{u_2} + \cdots + c_n\underline{u_n}) = c_1F(\underline{u_1}) + c_2F(\underline{u_2}) + \cdots + c_nF(\underline{u_n})$

$= F(\underline{0}) = \underline{0}$ since F is a linear map.

$\because$ $F(\underline{u_1}), F(\underline{u_2}), \ldots, F(\underline{u_n})$ are linear independent.

$\therefore$ $c_1 = c_2 = \cdots = c_n = 0$.

$\therefore$ $\{\underline{u_1}, \underline{u_2}, \ldots, \underline{u_n}\}$ is a linear independent set in V.

Q14.

(a). $P^TQ = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+cg & af+ch \\ be+dg & bf+dh \end{bmatrix}$

(b). $\langle P, Q \rangle = (ae+cg)+(bf+dh) = ae+bf+cg+dh$.

(c) $P^TP = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$

$\langle P, P \rangle = a^2+b^2+c^2+d^2$.

(d). $\langle P, P \rangle = a^2+b^2+c^2+d^2=0$ $\therefore$ $a=b=c=d=0$. $\therefore$ P is a zero matrix.

Q15.

(a). $u_1(x) = 1$ and $u_2(x) = x$ are both in $V = P_2$.

$\langle u_1, u_2 \rangle = \int_{-1}^{1} u_1(x) u_2(x)\, dx = \int_{-1}^{1} x\, dx = \frac{1}{2}x^2 \Big|_{-1}^{1} = \frac{1}{2}(1^2 - (-1)^2) = 0.$

$\therefore$ $u_1(x)$ and $u_2(x)$ are orthogonal to each other.

(b). $\langle f, u_1 \rangle = \int_{-1}^{1} f(x) u_1(x)\, dx = \int_{-1}^{1} (ax^2 + bx + c)\, dx = \left(\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right)\Big|_{-1}^{1}$

$= \frac{a}{3}(1^3 - (-1)^3) + \frac{b}{2}(1^2 - (-1)^2) + c(1 - (-1)) = \frac{2a}{3} + 2c = 0$

$\langle f, u_2 \rangle = \int_{-1}^{1} f(x) u_2(x)\, dx = \int_{-1}^{1} (ax^2 + bx + c)(x)\, dx = \int_{-1}^{1} (ax^3 + bx^2 + cx)\, dx$

$= \left(\frac{a}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2\right)\Big|_{-1}^{1} = \frac{a}{4}(1^4 - (-1)^4) + \frac{b}{3}(1^3 - (-1)^3) + \frac{c}{2}(1^2 - (-1)^2) = \frac{2b}{3} = 0$

$\therefore$ $2a + 6c = 0 \Rightarrow a + 3c = 0.$ $\begin{cases} b = 0 \\ a = -3c \end{cases}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \cdot t$ t is a scalar. $t \in \mathbb{R}$.

(c). To find $W^\perp$ we need $V(x) = ax^2 + bx + c$ $\begin{cases} \langle V, 1\rangle = 0 \\ \langle V, x\rangle = 0 \end{cases}$

$$\int_{-1}^{1} V(x) \cdot 1 \, dx = \int_{-1}^{1} (ax^2+bx+c)\,dx = \left(\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right)\Big|_{-1}^{1} = \frac{a}{3}(1+1) + c(1+1)$$

$$= \frac{2a}{3} + 2c = 2\left(\frac{a}{3}+c\right) = 0. \qquad a = -3c$$

$$\int_{-1}^{1} V(x)\, x \, dx = \int_{-1}^{1} (ax^2+bx+c)x\,dx = \int_{-1}^{1} (ax^3+bx^2+cx)\,dx = \left(\frac{a}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2\right)\Big|_{-1}^{1}$$

$$= \frac{2b}{3} = 0 \quad \Rightarrow \quad b = 0.$$

$\therefore$ $V(x) = -3c\,x^2 + bx + c = (-3c)x^2 + c = c(1-3x^2) \quad c \in \mathbb{R}$

$\therefore$ $W^\perp = \{ c(1-3x^2) : c \in \mathbb{R} \} = \operatorname{span}\{1-3x^2\}$.