Q21.

(a). $P^TP = I = PP^T$ $\quad Q^TQ = I = QQ^T$

$$(PQ)^T PQ = (Q^T P^T) PQ = Q^T(P^TP)Q = Q^TIQ = Q^TQ = I$$

$$PQ(PQ)^T = PQ(Q^TP^T) = P(QQ^T)P^T = PIP^T = PP^T = I$$

$\therefore (PQ)^T PQ = I = PQ(PQ)^T$ $\quad PQ$ is orthogonal matrix

$$(P^{-1})^T(P^{-1}) = (IP^{-1})^T(IP^{-1}) = (P^TPP^{-1})^T(P^TPP^{-1}) = (P^T)^T(P^T) = PP^T = I.$$

$$(P^{-1})(P^{-1})^T = (IP^{-1})(IP^{-1})^T = (P^TPP^{-1})(P^TPP^{-1})^T = P^T(P^T)^T = P^TP = I$$

$\therefore (P^{-1})^T(P^{-1}) = (P^{-1})(P^{-1})^T = I$ $\quad P^{-1}$ is orthogonal matrix

(b) $P^TP = I = PP^T$

$$\det(P)^2 = \det(P)\det(P) = \det(P^T)\det(P) = \det(P^TP) = \det(I) = 1$$

Since P and P^T are all $n \times n$ matrix

$\therefore \det(P)^2 = 1.$

Q22. Need to show $\langle \underline{u}, \underline{v} \rangle = 0$.

$\langle A\underline{u}, \underline{v} \rangle = \langle a\underline{u}, \underline{v} \rangle = a \langle \underline{u}, \underline{v} \rangle$

$\langle A\underline{u}, \underline{v} \rangle = \langle \underline{u}, A^T\underline{v} \rangle = \langle \underline{u}, A\underline{v} \rangle = \langle \underline{u}, b\underline{v} \rangle = b \langle \underline{u}, \underline{v} \rangle$

$\therefore \quad a \langle \underline{u}, \underline{v} \rangle = b \langle \underline{u}, \underline{v} \rangle$

$(a-b) \langle \underline{u}, \underline{v} \rangle = 0 \quad \because \ a \neq b \quad \therefore \ \langle \underline{u}, \underline{v} \rangle = 0$.

$a - b \neq 0$

$\therefore$ $\underline{u}$ and $\underline{v}$ are orthogonal.

Q23. $B^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$

$B^TB = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$\begin{vmatrix} -\lambda+4 & 0 & 2 \\ 0 & -\lambda+4 & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} = 4\lambda + (-\lambda+1)(-\lambda+4)^2 - 16 = 0.$

$\lambda_1 = 0$ $\lambda_2 = 4$ $\lambda_3 = 5$ are eigenvalues for B.

so singular values of B are.

$\sigma_1 = \sqrt{5} \approx 2.24$ $\sigma_2 = \sqrt{4} = 2$ $\sigma_3 = \sqrt{0} = 0$

Q25. $A = USV^T$ $\qquad A^+ = VS^+U^T$

$$PAQ = PUSV^TQ = (PU)\,S\,(V^TQ)$$

$$(PAQ)^+ = (V^TQ)^T S^+ (PU)^T$$

$$= Q^T V S^+ U^T P^T = Q^T (VS^+U^T) P^T$$

$$= Q^T A^+ P^T.$$

Q24. (a). $M = USV^T$ $M^+ = VS^+U^T$

$MM^+ = (USV^T)(VS^+U^T) = US(V^TV)S^+U^T = USS^+U^T$

$\because$ $SS^+S = S$ $\therefore$ $S^+S = I$

$\therefore$ $USS^+U^T = UIU^T = UU^T = I$, which is a symmetric matrix

So, MM^+ is a symmetric matrix

(b) According to part (a) $BB^+ = I$.

$BR = \frac{1}{10}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 0 \end{bmatrix}$ which can not be computed.

So, R is not B^+.

$$BS = \frac{1}{10}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 2 & 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8+2 & 0 \\ 0 & 10 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So, S ~~是~~ may be the B^+.

$$BT = \frac{1}{10}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 3 & 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8+3 & 0 \\ 0 & 10 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 11 & 0 \\ 0 & 10 \end{bmatrix}$$

So T can not be the B^+.

So, S may ~~是~~ be B^+.