Math334 HW #3 Solution

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3. Determine if the DE is exact. If it is, solve the equation.

$$(3x^2 - 2xy + 2) + (6y^2 - x2 + 3)y' = 0.$$

Solution. Rewrite the equation in symmetric form

$$(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0.$$

Let

$$M(x, y) = 3x^2 - 2xy + 2, \quad N(x, y) = 6y^2 - x2 + 3.$$

Then

$$\frac{\partial M}{\partial y} = -2x = \frac{\partial N}{\partial x}$$

and the equation is exact. Find a potential function $f(x, y)$ by solving

$$\frac{\partial f}{\partial x} = M(x, y) = 3x^2 - 2xy + 2, \tag{1}$$

$$\frac{\partial f}{\partial y} = N(x, y) = 6y^2 - x^2 + 3. \tag{2}$$

Integrating equation (1) with respect to x we obtain

$$f(x, y) = x^3 - x^2y + 2x + m(y). \tag{3}$$

To find function $m(y)$, we substitute (3) into (2) and obtain

$$-x^2 + m'(y) = 6y^2 - x^2 + 3.$$

Therefore,

$$m'(y) = 6y^2 + 3,$$

and $m(y) = 2y^3 + 3y$. From (3), we obtain

$$f(x, y) = x^3 - x^2y + 2y^3 + 2x + 3y.$$

The general solution is defined by the equation

$$x^3 - x^2y + 2y^3 + 2x + 3y = c.$$

4. Determine if the DE is exact. If it is, solve the equation.

$$\frac{dy}{dx} = \frac{ax + by}{bx + cy}.$$

Solution. Rewrite the equation in symmetric form

$$(ax + by)dx + (bx + cy)dy = 0.$$

Let

$$M(x, y) = ax + by, \quad N(x, y) = bx + cy.$$

Then

$$\frac{\partial M}{\partial y} = b = \frac{\partial N}{\partial x},$$

and the equation is exact. Find a potential function $f(x, y)$ by solving

$$\frac{\partial f}{\partial x} = M(x, y) = ax + by, \tag{4}$$

$$\frac{\partial f}{\partial y} = N(x, y) = bx + cy. \tag{5}$$

Integrating equation (4) with respect to x we obtain

$$f(x, y) = ax^2 + bxy + m(y). \tag{6}$$

To find $m(y)$, we substitute (6) into (5) and obtain

$$bx + m'(y) = bx + cy,$$

Therefore,

$$m'(y) = cy, \quad m(y) = \frac{cy^2}{2}.$$

From (6), we obtain

$$f(x, y) = \frac{ax^2}{2} + bxy + \frac{cy^2}{2}.$$

The general solution is defined by

$$ax^2 + 2bxy + cy^2 = C,$$

where C is an arbitrary constant.

7. Determine if the DE is exact. If it is, solve the equation.

$$(e^x \sin y - 2y \sin x) + (e^x \cos y + 2 \cos x)y' = 0.$$

Solution. Rewrite the equation in symmetric form

$$(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2 \cos x)dy = 0.$$

Let

$$M(x,y) = e^x \sin y - 2y \sin x, \quad N(x,y) = e^x \cos y + 2 \cos x.$$

Then

$$\frac{\partial M}{\partial y} = e^x \cos y - 2 \sin x = \frac{\partial N}{\partial x},$$

and the equation is exact. Find a potential function $f(x,y)$ by solving

$$\frac{\partial f}{\partial x} = M(x,y) = e^x \sin y - 2y \sin x, \tag{7}$$

$$\frac{\partial f}{\partial y} = N(x,y) = e^x \cos y + 2 \cos x. \tag{8}$$

Integrating equation (7) with respect to x we obtain

$$f(x,y) = e^x \sin y + 2y \cos x + m(y). \tag{9}$$

To find $m(y)$, we substitute (9) into (8) and obtain

$$e^x \cos y + 2 \cos x + m'(y) = e^x \cos y + 2 \cos x.$$

Therefore, $m'(y) = 0$ and $m(y) = c$, and we can choose $m(y) = 0$. From (9), we obtain

$$f(x,y) = e^x \sin y + 2y \cos x.$$

The general solution is defined by

$$e^x \sin y + 2y \cos x = c.$$

10. Solve the initial value problem.

$$(9x^2 + y - 1) - (4y - x)y' = 0, \quad y(1) = 0.$$

Solution. Rewrite the equation in symmetric form

$$(9x^2 + y - 1)dx - (4y - x)dy = 0.$$

Let

$$M(x, y) = 9x^2 + y - 1, \quad N(x, y) = x - 4y.$$

Then,

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x},$$

and the equation is exact. Find a potential function $f(x, y)$ by solving

$$\frac{\partial f}{\partial x} = M(x, y) = 9x^2 + y - 1, \tag{10}$$

$$\frac{\partial f}{\partial y} = N(x, y) = x - 4y. \tag{11}$$

Integrating equation (10) with respect to x we obtain

$$f(x, y) = 3x^3 + xy - x + m(y). \tag{12}$$

To find $m(y)$, we substitute (12) into (11), we obtain

$$x + m'(y) = x - 4y.$$

Therefore, $m'(y) = -4y$ and $m(y) = -2y^2$. From (12), we obtain

$$f(x, y) = 3x^3 + xy - x - 2y^2.$$

The general solution is defined by

$$3x^3 + xy - x - 2y^2 = c.$$

To determine c using the initial condition, we let $x = 1, y = 0$ in the general solution, and obtain

$$3 - 1 = c,$$

and thus $c = 2$. The unique solution to the IVP is defined by the equation

$$3x^3 + xy - x - 2y^2 = 2.$$

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1. Solve

$$y'' + 2y' - 3y = 0.$$

Solution. The characteristic equation is

$$\lambda^2 + 2\lambda - 3 = 0,$$

which has two distinct real roots $\lambda_1 = -3,\ \lambda_2 = 1$. Two linearly independent solutions to the DE are

$$y_1(t) = e^{-3t},\ y_2(t) = e^t,$$

and thus the general solution is

$$y(t) = c_1 e^{-3t} + c_2 e^t.$$

3. Solve

$$6y'' - y' - y = 0.$$

Solution. The characteristic equation is

$$6\lambda^2 - \lambda - 1 = 0,$$

which has two distinct real roots

$$\lambda_1 = -\frac{1}{3},\ \lambda_2 = \frac{1}{2}.$$

Two linearly independent solutions are given by

$$y_1(t) = e^{-\frac{t}{3}}, \quad y_2(t) = e^{\frac{t}{2}},$$

and the general solution is

$$y(t) = c_1 e^{-\frac{t}{3}} + c_2 e^{\frac{t}{2}}.$$

10. Solve the IVP

$$2y'' + y' - 4y = 0, \quad y(0) = 0,\ y'(0) = 1.$$

Solution.

(1) Find the general solution. The characteristic equation of the DE is

$$2\lambda^2 + \lambda - 4 = 0,$$

which has two distinct roots $\lambda_1 = -1/4 + \sqrt{33}/4, \lambda = -1/4 - \sqrt{33}/4$. Two linearly independent solutions are

$$y_1(t) = e^{-(1-\sqrt{33})t/4},\ y_2(t) = e^{-(1+\sqrt{33})t/4},$$

and the general solution is

$$y(t) = c_1 e^{-(1-\sqrt{33})t/4} + c_2 e^{-(1+\sqrt{33})t/4}.$$

(2) Find constants c_1, c_2 from initial conditions. Let $t = 0$ in $y(t)$ and

$$y'(t) = -\frac{1-\sqrt{33}}{4} c_1 e^{-(1-\sqrt{33})t/4} - \frac{1+\sqrt{33}}{4} c_2 e^{-(1+\sqrt{33})t/4},$$

we obtain

$$0 = y(0) = c_1 + c_2, \quad 1 = -c_1 \frac{1-\sqrt{33}}{4} - c_2 \frac{1+\sqrt{33}}{4}.$$

Soving the system we obtain $c_1 = 2/\sqrt{33}$ and $c_2 = -2/\sqrt{33}$. The unique solution to the IVP is

$$y(t) = \frac{2}{\sqrt{33}} e^{-(1-\sqrt{33})t/4} - \frac{2}{\sqrt{33}} e^{-(1+\sqrt{33})t/4}.$$

12. Solve the IVP

$$4y'' - y = 0, \quad y(-2) = 1,\ y'(-2) = -1.$$

Solution.

(1) Find the general solution. The characteristic equation of the DE is

$$4\lambda^2 - 1 = 0,$$

which has two distinct roots $\lambda_1 = -1/2, \lambda = 1/2$. Two linearly independent solutions are

$$y_1(t) = e^{-\frac{t}{2}},\ y_2(t) = e^{\frac{t}{2}},$$

and the general solution is

$$y(t) = c_1 e^{-\frac{t}{2}} + c_2 e^{\frac{t}{2}}.$$

(2) Find constants c_1, c_2 from initial conditions. Let $t = -2$ in $y(t)$ and

$$y'(t) = -\frac{1}{2}c_1 e^{-\frac{t}{2}} + \frac{1}{2}c_2 e^{\frac{t}{2}},$$

we obtain

$$1 = y(-2) = c_1 e^1 + c_2 e^{-1}, \quad -1 = y'(-2) = -\frac{1}{2}c_1 e^1 + \frac{1}{2}c_2 e^{-1},$$

namely

$$c_1 e + c_2 \frac{1}{e} = 1, \quad -\frac{1}{2}c_1 e + \frac{1}{2}c_2 \frac{1}{e} = -1,$$

which give $c_1 = \frac{3}{2e}, c_2 = \frac{e}{2}$, and the unique solution to the IVP is

$$y(t) = \frac{3}{2e}e^{-\frac{t}{2}} - \frac{e}{2}e^{\frac{t}{2}} = \frac{3}{2}e^{-\frac{t}{2}-1} - \frac{1}{2}e^{\frac{t}{2}+1}.$$