Math334 HW #4 Solution

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1. Find the Wronskian of $e^{2t}, e^{-3t/2}$.

Solution.

$$W(e^{2t}, e^{-3t/2}) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{3}{2}e^{2t}e^{-3t/2} - 2e^{2t}e^{-3t/2} = -\frac{7}{2}e^{t/2}.$$

2. Find the Wronskian of $\cos t, \sin t$.

Solution.

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$$

3. Find the Wronskian of e^{-2t}, te^{-2t}.

Solution.

$$W(e^{-2t}, te^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-2t}(e^{-2t} - 2te^{-2t}) + 2te^{-2t}e^{-2t} = e^{-4t}.$$

4. Find the Wronskian of $e^t \sin t, e^t \cos t$.

Solution.

$$\begin{aligned} W(e^t \sin t, e^t \cos t) &= \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \sin t + e^t \cos t & e^t \cos t - e^t \sin t \end{vmatrix} \\ &= e^{2t}(\sin t \cos t - \sin^t - \sin t \cos t - \cos^2 t) \\ &= -e^{2t}(\sin^2 t + \cos^2 t) = -e^{2t}. \end{aligned}$$

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#1

$$\exp(2-3i) = e^2(\cos 3 - i\ \sin 3).$$

#2

$$e^{i\pi} = \cos\pi + i\ \sin\pi = \cos\pi = -1.$$

#4

$$\begin{aligned} 2^{1-i} &= (e^{\ln 2})^{1-i} = e^{\ln 2(1-i)} = e^{\ln 2 - i\ln 2} = e^{\ln 2}e^{-i\ln 2} \\ &= e^{\ln 2}(\cos(-\ln 2) + i\ \sin(-\ln 2)) = 2(\cos(\ln 2) - i\sin(\ln 2)). \end{aligned}$$

#12. Solve the IVP (no sketching).

$$y'' + 4y = 0, \quad y(0) = 0,\ y'(0) = 1.$$

Solution. The characteristic equation is

$$r^2 + 4 = 0$$

and the characteristic roots are $r_1 = 2i, r_2 = -2i$. The two linearly independent solutions are $y_1(x) = \cos(2x)$ and $y_2(x) = \sin(2x)$, and the general solution is

$$y(x) = c_1\cos(2x) + c_2\sin(2x).$$

To find c_1, c_2, we first differentiate $y(x)$

$$y'(x) = -c_1 2\sin(2x) + c_2 2\cos(2x).$$

Letting $x = 0$ in $y(x)$ and $y'(x)$

$$0 = y(0) = c_1, \quad 1 = y'(0) = 2c_2.$$

Therefore, $c_1 = 0, c_2 = \frac{1}{2}$, and the unique solution to the IVP is

$$y(x) = \frac{1}{2}\sin(2x).$$

#13. Solve the IVP (no sketching).

$$y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \ y'(\pi/2) = 2.$$

Solution. The characteristic equation is

$$r^2 - 2r + 5 = 0$$

and the characteristic roots are $r_1 = 1 + 2i, r_2 = 1 - 2i$. The two linearly independent solutions are

$$y_1(x) = e^x \cos(2x), \quad y_2(x) = e^x \sin(2x),$$

and the general solution is

$$y(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x).$$

To find c_1, c_2, we first differentiate $y(x)$

$$y'(x) = c_1(e^x \cos(2x) - e^x 2\sin(2x)) + c_2(e^x \sin(2x) + e^x 2\cos(2x)).$$

Letting $x = \pi/2$ in $y(x)$ and $y'(x)$

$$0 = y(\pi/2) = -e^{\pi/2} c_1, \quad 1 = y'(\pi/2) = e^{\pi/2}[-c_1 - 2c_2].$$

Therefore, $c_1 = 0, c_2 = -\frac{1}{2}e^{-\pi/2}$, and the unique solution to the IVP is

$$y(x) = \frac{1}{2} e^{-\pi/2} e^x \sin(2x).$$

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#1 Find the general solution of

$$y'' - 2y' + y = 0.$$

Solution. The characteristic equation is

$$r^2 - 2r + 1 = 0$$

and the characteristic roots are $r_1 = r_2 = 1$. The two linearly independent solutions are $y_1(x) = e^x$ and $y_2(x) = xe^x$, and the general solution is

$$y(x) = c_1 e^x + c_2 x e^x.$$

#3 Find the general solution of

$$4y'' - 4y' - 3y = 0.$$

Solution. The characteristic equation is

$$4r^2 - 4r - 3 = 0$$

and the characteristic roots are $r_1 = 3/2, r_2 = -1/2$. The two linearly independent solutions are $y_1(x) = e^{3x/2}$ and $y_2(x) = e^{-x/2}$, and the general solution is

$$y(x) = c_1 e^{3x/2} + c_2 e^{-x/2}.$$

#10. Solve the initial value problem

$$y'' - 6y' + 9y = 0, \quad y(0) = 0,\ y'(0) = 2.$$

Solution. The characteristic equation is

$$r^2 - 6r + 9 = 0$$

and the characteristic roots are $r_1 = r_2 = 3$. The two linearly independent solutions are $y_1(x) = e^{3x}$ and $y_2(x) = xe^{3x}$, and the general solution is

$$y(x) = c_1 e^{3x} + c_2 x e^{3x}.$$

We also have

$$y'(x) = 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 x e^{3x}.$$

Using the initial conditions we obtain

$$0 = y(0) = c_1, \quad 2 = y'(0) = 3c_1 + c_2.$$

Therefore, $c_1 = 0,\ c_2 = 2$, and the unique solution to the IVP is

$$y(x) = 2xe^{3x}.$$

#11. Solve the initial value problem

$$y'' + 4y' + 4y = 0, \quad y(-1) = 2,\ y'(-1) = 1.$$

Solution. The characteristic equation is

$$r^2 + 4r + 4 = 0$$

and the characteristic roots are $r_1 = r_2 = -2$. The two linearly independent solutions are $y_1(x) = e^{-2x}$ and $y_2(x) = xe^{-2x}$, and the general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}.$$

We also have

$$y'(x) = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}.$$

Using the initial conditions we obtain

$$2 = y(-1) = c_1 e^2 + c_2(-1)e^2, \quad 1 = y'(-1) = -2c_1 e^2 + c_2 e^2 - 2c_2(-1)e^2.$$

Therefore, $c_1 = 7e^{-2},\ c_2 = 5e^{-2}$, and the unique solution to the IVP is

$$y(x) = 7e^{-2}e^{-2x} + 5e^{-2}xe^{-2x} = 7e^{-2(x+1)} + 5xe^{-2(x+1)}.$$

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#1 Solve

$$y'' - 2y' - 3y = 3e^{2t}.$$

Solution. The characteristic equation is

$$r^2 - 2r - 3 = 0,$$

which has two distinct real roots $r_1 = 3,\ r_2 = -1$. Two linearly independent solutions to the homogeneous equation are

$$y_1(t) = e^{3t},\ y_2(t) = e^{-t},$$

and thus the general solution of the homogeneous equation is

$$y_c(t) = c_1 e^{3t} + c_2 e^{-t}.$$

To find a particular solution $y_p(t)$ of the nonhomogeneous equation, we assume that

$$y_p(t) = Ae^{2t},$$

and substitute it into the DE to obtain

$$4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t}.$$

This gives $A = -1$, and $y_p(t) = -e^{2t}$. Therefore, the general solution to the DE is

$$y(t) = y_c(t) + y_p(t) = c_1e^{3t} + c_2e^{-t} - e^{2t}.$$

#2 Solve

$$y'' - y' - 2y = -2t + 4t^2.$$

Solution. The characteristic equation is

$$r^2 - r - 2 = 0,$$

which has two real roots $r_1 = -1$, $r_2 = 2$. The general solution of the homogeneous equation is

$$y_c(t) = c_1e^{-t} + c_2e^{2t}.$$

To find a particular solution $y_p(t)$ of the nonhomogeneous equation, we assume that

$$y_p(t) = At^2 + Bt + C$$

and substitute it into the DE to obtain

$$(2A) - (2At + B) - 2(At^2 + Bt + C) = -2t + 4t^2,$$

Comparing coefficients of different powers of t, we obtain

$$2A - B - 2C = 0, \quad -2A - 2B = -2, \quad -2A = 4,$$

and thus $A = -2$, $B = 3$ and $C = -7/2$. Therefore

$$y_p(t) = -2t^2 + 3t - 7/2.$$

Therefore, the general solution to the DE is

$$y(t) = y_c(t) + y_p(t) = c_1e^{-t} + c_2e^{2t} - 2t^2 + 3t - \frac{7}{2}.$$

#7 Solve

$$y'' + y = 3\sin 2t + t\cos(2t).$$

Solution. The characteristic equation is

$$r^2 + 1 = 0,$$

which has two complex roots $r_1 = i,\ r_2 = -i$. Two linearly independent solutions to the homogeneous equation are

$$y_1(t) = \cos t,\ y_2(t) = \sin t,$$

and thus the general solution of the homogeneous equation is

$$y_c(t) = c_1\cos t + c_2\sin t.$$

To find a particular solution $y_p(t)$ of the nonhomogeneous equation, we assume that

$$y_p(t) = A\cos 2t + B\sin 2t + t(C\cos 2t + D\sin 2t),$$

and substitute it into the DE to obtain

$$\begin{aligned}(A - 4A + 4D)\cos 2t + (B - 4B - 4C)\sin 2t + (C - 4C)t\cos 2t \\ + (D - 4D)t\sin 2t = 3\sin 2t + t\cos 2t\end{aligned}$$

This gives

$$\begin{aligned} -3A + 4D &= 0, \\ -3B - 4C &= 3, \\ -3C &= 1, \\ -3D &= 0. \end{aligned}$$

Thus, $A = 0$, $B = -5/9$, $C = -1/3$, and $D = 0$. Therefore,

$$y_p(t) = -\frac{5}{9}\sin 2t - \frac{1}{3}t\cos 2t.$$

Therefore, the general solution to the DE is

$$y(t) = y_c(t) + y_p(t) = c_1\cos t + c_2\sin t - \frac{5}{9}\sin 2t - \frac{1}{3}t\cos 2t.$$

#11 Solve the IVP

$$y'' + y' - 2y = 2t, \quad y(0) = 0,\ y'(0) = 1.$$

Solution. The characteristic equation is

$$r^2 + r - 2 = 0,$$

which has two real roots $r_1 = 1,\ r_2 = -2$. The general solution of the homogeneous equation is

$$y_c(t) = c_1 e^t + c_2 e^{-2t}.$$

To find a particular solution $y_p(t)$ of the nonhomogeneous equation, we assume that

$$y_p(t) = At + B$$

and substitute it into the DE to obtain

$$(0) + (A) - 2(At + B) = 2t,$$

Comparing coefficients of different powers of t, we obtain

$$-2A = 2, \quad A - 2B = 0,$$

and thus $A = -1$ and $B = -1/2$. Therefore

$$y_p(t) = -t - 1/2.$$

Therefore, the general solution to the DE is

$$y(t) = y_c(t) + y_p(t) = c_1 e^t + c_2 e^{-2t} - t - \frac{1}{2}.$$

We also obtain

$$y'(t) = c_1 e^t - 2c_2 e^{-2t} - 1.$$

Using initial conditions we obtain

$$0 = y(0) = c_1 + c_2 - \frac{1}{2}, \quad 1 = y'(0) = c_1 - 2c_2 - 1.$$

This gives $c_1 = 1,\ c_2 = -1/2$, and thus the unique solution to the IVP is

$$y(t) = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2}.$$