Problems

1. Prove the commutative, distributive, and associative properties of the convolution integral.

a. $f * g = g * f$
b. $f * (g_1 + g_2) = f * g_1 + f * g_2$
c. $f * (g * h) = (f * g) * h$

2. Find an example different from the one in the text showing that $(f * 1)(t)$ need not be equal to $f(t)$.

3. Show, by means of the example $f(t) = \sin t$, that $f * f$ is not necessarily nonnegative.

In each of Problems 4 through 6, find the Laplace transform of the given function.

4. $f(t) = \displaystyle\int_0^t (t-\tau)^2 \cos(2\tau)\, d\tau$

5. $f(t) = \displaystyle\int_0^t e^{-(t-\tau)} \sin\tau\, d\tau$

6. $f(t) = \displaystyle\int_0^t \sin(t-\tau) \cos\tau\, d\tau$

In each of Problems 7 through 9, find the inverse Laplace transform of the given function by using the convolution theorem.

7. $F(s) = \dfrac{1}{s^4(s^2+1)}$

8. $F(s) = \dfrac{s}{(s+1)(s^2+4)}$

9. $F(s) = \dfrac{1}{(s+1)^2(s^2+4)}$

10. **a.** If $f(t) = t^m$ and $g(t) = t^n$, where m and n are positive integers, show that

$$f * g = t^{m+n+1} \int_0^1 u^m (1-u)^n\, du.$$

b. Use the convolution theorem to show that

$$\int_0^1 u^m (1-u)^n\, du = \frac{m!\,n!}{(m+n+1)!}.$$

c. Extend the result of part b to the case where m and n are positive numbers but not necessarily integers.

In each of Problems 11 through 15, express the solution of the given initial value problem in terms of a convolution integral.

11. $y'' + \omega^2 y = g(t); \quad y(0) = 0, \quad y'(0) = 1$

12. $4y'' + 4y' + 17y = g(t); \quad y(0) = 0, \quad y'(0) = 0$

13. $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t); \quad y(0) = 1, \quad y'(0) = -1$

14. $y'' + 3y' + 2y = \cos(\alpha t); \quad y(0) = 1, \quad y'(0) = 0$

15. $y^{(4)} + 5y'' + 4y = g(t); \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$

16. Consider the equation

$$\phi(t) + \int_0^t k(t-\xi)\phi(\xi)\, d\xi = f(t),$$

in which f and k are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an **integral equation**; in particular, it belongs to a class of integral equations known as **Volterra integral equations**[7]. Take the Laplace transform of the given integral equation and obtain an expression for $\mathcal{L}\{\phi(t)\}$ in terms of the transforms $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{k(t)\}$ of the given functions f and k. The inverse transform of $\mathcal{L}\{\phi(t)\}$ is the solution of the original integral equation.

17. Consider the Volterra integral equation (see Problem 16)

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\, d\xi = \sin(2t). \tag{30}$$

a. Solve the integral equation (30) by using the Laplace transform.

b. By differentiating equation (30) twice, show that $\phi(t)$ satisfies the differential equation

$$\phi''(t) + \phi(t) = -4\sin(2t).$$

Show also that the initial conditions are

$$\phi(0) = 0, \quad \phi'(0) = 2.$$

c. Solve the initial value problem in part b, and verify that the solution is the same as the one in part a.

In each of Problems 18 and 19:

a. Solve the given Volterra integral equation by using the Laplace transform.

b. Convert the integral equation into an initial value problem, as in Problem 17b.

c. Solve the initial value problem in part b, and verify that the solution is the same as the one in part a.

18. $\phi(t) + \displaystyle\int_0^t (t-\xi)\phi(\xi)\, d\xi = 1$

19. $\phi(t) + 2\displaystyle\int_0^t \cos(t-\xi)\phi(\xi)\, d\xi = e^{-t}$

There are also equations, known as **integro-differential equations**, in which both derivatives and integrals of the unknown function appear. In each of Problems 20 and 21:

a. Solve the given integro-differential equation by using the Laplace transform.

b. By differentiating the integro-differential equation a sufficient number of times, convert it into an initial value problem.

c. Solve the initial value problem in part b, and verify that the solution is the same as the one in part a.

20. $\phi'(t) + \displaystyle\int_0^t (t-\xi)\phi(\xi)\, d\xi = t, \quad \phi(0) = 0$

21. $\phi'(t) - \dfrac{1}{2}\displaystyle\int_0^t (t-\xi)^2\phi(\xi)\, d\xi = -t, \quad \phi(0) = 1$

[7] See the footnote about **Vito Volterra** in Section 9.5.