Math334 HW #5 Solution

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#1 Find a particular solution using variation of parameters.

$$y'' - 5y' + 6y = 2e^t.$$

Solution. Solving the homogeneous equation. The characteristic equation is

$$r^2 - 5r + 6 = 0$$

and the characteristic roots are $r_1 = 2, r_2 = 3$. The two linearly independent solutions are $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$. Find a particular solution $y_p(t)$ by assuming

$$y_p(t) = u_1 y_1 + u_2 y_2,$$

and u_1', u_2' satisfy

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}$$

Therefore,

$$u_1' = \frac{\begin{vmatrix} 0 & e^{3t} \\ 2e^t & 3e^{3t} \end{vmatrix}}{\begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix}} = \frac{-2e^{4t}}{e^{5t}} = -2e^{-t},$$

and thus

$$u_1(t) = -2 \int e^{-t} dt = 2e^{-t}.$$

Similarly,

$$u_2' = \frac{\begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & 2e^t \end{vmatrix}}{\begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix}} = \frac{2e^{3t}}{e^{5t}} = 2e^{-2t},$$

and thus

$$u_2(t) = 2 \int e^{-2t} dt = -e^{-2t}.$$

This gives

$$y_p(t) = 2e^{-t}e^{2t} - e^{-2t}e^{3t} = e^t.$$

#4 Find a particular solution using variation of parameters.

$$y'' + y = \tan t.$$

Solution. The characteristic equation for the homogeneous equation is

$$r^2 + 1 = 0$$

and the characteristic roots are $r_1 = i, r_2 = -i$. The two linearly independent solutions are $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

Find a particular solution $y_p(t)$ by assuming

$$y_p(t) = u_1 y_1 + u_2 y_2,$$

and u_1', u_2' satisfy

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan t \end{bmatrix}$$

Therefore,

$$u_1' = \frac{\begin{vmatrix} 0 & \sin t \\ \tan t & \cos t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = \frac{-\tan t \sin t}{1} = -\frac{\sin^2 t}{\cos t},$$

and thus

$$\begin{aligned} u_1(t) &= -\int \frac{\sin^2 t}{\cos t} dt = -\int \frac{1 - \cos^2 t}{\cos t} dt \\ &= -\int \sec t dt + \int \cos t dt = -\ln|\sec t + \tan t| + \sin t. \end{aligned}$$

Similarly,

$$u_2' = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = \frac{\cos t \tan t}{1} = \sin t,$$

and thus

$$u_2(t) = -\int \sin t dt = -\cos t.$$

This gives

$$y_p(t) = \cos t \ln|\sec t + \tan t|.$$

#6. Find a particular solution using variation of parameters.

$$y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0.$$

Solution. The characteristic equation of the homogeneous equation is

$$r^2 + 4r + 4 = 0$$

and the characteristic roots are $r_1 = r_2 = -2$. The two linearly independent solutions are $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$.

Find a particular solution $y_p(x)$ by assuming

$$y_p(t) = u_1 y_1 + u_2 y_2,$$

and u_1', u_2' satisfy

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t^{-2}e^{-2t} \end{bmatrix}$$

Therefore,

$$u_1' = \frac{\begin{vmatrix} 0 & te^{-2t} \\ t^{-2}e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}}{\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}} = \frac{-t^{-1}e^{-4t}}{e^{-4t}} = -\frac{1}{t},$$

and thus

$$u_1(t) = -\int \frac{1}{t} dt = -\ln t.$$

Similarly,

$$u_2' = \frac{\begin{vmatrix} e^{-2t} & 0 \\ -2e^{-2t} & t^{-2}e^{-2t} \end{vmatrix}}{\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}} = \frac{t^{-2}e^{-4t}}{e^{-4t}} = t^{-2},$$

and thus

$$u_2(t) = \int t^{-2} dt = -\frac{1}{t}.$$

This gives

$$y_p(t) = -e^{-2t} \ln t - e^{-2t},$$

or equivalently,

$$y_p(t) = e^{-2t} \ln t,$$

since e^{-2t} is a solution of the homogeneous solution.

#12. (a). Verify that $y_1 = 1 + t$ and $y_2 = e^t$ are solutions of

$$ty'' - (1+t)y' + y = 0, \quad t > 0,$$

and (b) find the general solution of the nonhomogeneous equation

$$ty'' - (1+t)y' + y = t^2 e^{2t}, \quad t > 0,$$

Solution. Part (a) is straightforward. We compute the Wronskian of y_1, y_2 to show that they are linearly independent.

$$W(y_1, y_2)(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} (1+t) & e^t \\ 1 & e^t \end{bmatrix} = te^t \neq 0 \quad \text{for } t > 0.$$

Convert the DE into the standard form:

$$y'' - (\frac{1}{t} + 1)y' + \frac{1}{t} y = te^{2t}, \quad t > 0,$$

Therefore the homogeneous term is $f(t) = te^{2t}$.

Find a particular solution $y_p(t)$ by assuming

$$y_p(t) = u_1 y_1 + u_2 y_2,$$

and u_1', u_2' satisfy

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ te^{2t} \end{bmatrix}$$

Therefore,

$$u_1' = \frac{\begin{vmatrix} 0 & e^t \\ te^{2t} & e^t \end{vmatrix}}{W(y_1, y_2)(t)} = \frac{-te^{3t}}{te^t} = -e^{2t},$$

and thus

$$u_1(t) = -\int e^{2t}dt = -\frac{1}{2}e^{2t}.$$

Similarly,

$$u_2' = \frac{\begin{vmatrix}(1+t) & 0 \\ 1 & te^{2t}\end{vmatrix}}{W(y_1, y_2)(t)} = \frac{t(1+t)e^{2t}}{te^t} = (1+t)e^t,$$

and thus

$$u_2(t) = \int (1+t)e^t dt = \int (1+t)d(e^t) = (1+t)e^t - e^t = te^t.$$

This gives

$$y_p(t) = -\frac{1}{2}(1+t)e^{2t} + te^{2t} = -\frac{1}{2}e^{2t} + \frac{1}{2}te^{2t}.$$

and the general solution is

$$y(t) = c_1(1+t) + c_2e^t - \frac{1}{2}e^{2t} + \frac{1}{2}te^{2t}.$$

#13. (a). Verify that $y_1 = x^2$ and $y_2 = x^2 \ln x$ are solutions of

$$x^2y'' - 3xy' + 4y = 0, \quad x > 0,$$

and (b) find the general solution of the nonhomogeneous equation

$$x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0,$$

Solution. Part (a) is straightforward. We compute the Wronskian of y_1, y_2 to show that they are linearly independent.

$$W(y_1, y_2)(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} x^2 & x^2 \ln x \\ 2x & 2x\ln x + x \end{bmatrix} = x^3 \neq 0 \quad \text{for } x > 0.$$

Convert the DE into the standard form:

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln x, \quad t > 0,$$

Therefore the homogeneous term is $f(x) = \ln x$.

Find a particular solution $y_p(x)$ by assuming

$$y_p(x) = u_1 y_1 + u_2 y_2,$$

and u_1', u_2' satisfy

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \ln x \end{bmatrix}$$

Therefore,

$$u_1' = \frac{\begin{vmatrix} 0 & x^2 \ln x \\ \ln x & 2x \ln x + x \end{vmatrix}}{W(y_1, y_2)(x)} = \frac{-x^2(\ln x)^2}{x^3} = -\frac{1}{x}(\ln x)^2,$$

and thus

$$u_1(x) = -\int \frac{1}{x}(\ln x)^2 dx = -\int (\ln x)^2 d(\ln x) = -\frac{1}{3}(\ln x)^3$$

Similarly,

$$u_2' = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & \ln x \end{vmatrix}}{W(y_1, y_2)(t)} = \frac{x^2 \ln x}{x^3} = \frac{\ln x}{x},$$

and thus

$$u_2(x) = \int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2.$$

This gives

$$y_p(x) = -\frac{1}{3}x^2(\ln x)^3 + \frac{1}{2}x^2(\ln x)^3 = \frac{1}{6}x^2(\ln x)^3.$$

and the general solution is

$$y(x) = c_1 x^2 + c_2 x^2 \ln x + \frac{1}{6}x^2(\ln x)^3.$$