Math334 HW #2 Solution

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9. Solve IVP:

$$y' - y = 2te^{2t}, \quad y(0) = 1.$$

Solution. Step 1. Find the general solution. This is a linear DE, the IF $\mu(t) = e^{-t}$. Multiplying $\mu(t)$ both sides of the DE we obtain

$$e^{-t}y' - e^{-t}y = 2te^{t} \tag{1}$$

$$\frac{f}{dt}(e^{-t}y) = 2te^{t} \tag{2}$$

$$e^{-t}y = 2\int te^{t}dt = 2\int td(e^{t}) = 2(te^{t} - \int e^{t}dt) = 2te^{t} - 2e^{t} + c. \tag{3}$$

Therefore, the general solution is $y(t) = 2te^{2t} - 2e^{2t} + ce^{t}$.

Step 2. Find the value of c from initial condition by setting $t = 0$ in the general solution:

$$1 = y(0) = -2 + c, \quad \text{thus } c = 3.$$

The unique solution to the IVP is $y(t) = 2te^{2t} - 2e^{2t} + 3e^{t}$.

12. Solve

$$ty' + (t+1)y = t, \quad y(\ln 2) = 1,\ t > 0.$$

Solution. Rewrite the equation in standard form

$$y' + \frac{t+1}{t}y = 1.$$

The IF is

$$\mu(t) = e^{\int \frac{1+t}{t}dt} = e^{\ln t + t} = e^{\ln t}e^{t} = te^{t}.$$

Multiplying $\mu(t)$ to both sides of the standard form we obtain

$$te^{t}y' + (te^{t} + e^{t})y = te^{t} \tag{4}$$

$$(te^{t}y)' = te^{t} \tag{5}$$

$$te^{t}y = \int te^{t}dt = te^{t} - e^{t} + c. \tag{6}$$

Therefore the general solution is given as

$$y(t) = 1 - \frac{1}{t} + \frac{c}{t}e^{-t}.$$

To determine the value of c, we let $t = \ln 2$ in the general solution,

$$1 = y(\ln 2) = 1 - \frac{1}{\ln 2} + \frac{c}{\ln 2}e^{\ln 2},$$

this gives $c = 2$, and thus the unique solution to the IVP is

$$y(t) = 1 - \frac{1}{t} + \frac{2}{t}e^{-t}, \quad t > 0.$$

21. Consider the IVP

$$y' - \frac{3}{2}y = 3t + 2e^{t}, \quad y(0) = y_0.$$

Find the value of y_0 that separate solutions that grow positively as $t \to \infty$ from those that grow negatively.

Solution. Solve the IVP. Multiplying IF $e^{-3t/2}$ to both sides

$$e^{-3t/2}y' - \frac{3}{2}e^{-3t/2}y = 3te^{-3t/2} + 2e^{-t/2} \tag{7}$$

$$(ye^{-3t/2})' = 3te^{-3t/2} + 2e^{-t/2} \tag{8}$$

$$ye^{-3t/2} = 3\int te^{-3t/2}dt + 2\int e^{-t/2}dt = -2te^{-3t/2} - \frac{4}{3}e^{-3t/2} - te^{-t/2} + c. \tag{9}$$

Therefore, the general solution is

$$y(t) = -2t - \frac{4}{3} - 4e^{t} + ce^{3t/2}.$$

Setting $t = 0$, we obtain

$$y_0 = y(0) = -\frac{4}{3} - 4 + c,$$

and thus $c = y_0 + \frac{16}{3}$, and the solution to the IVP is

$$y(t) = -2t - \frac{4}{3} - 4e^{t} + (y_0 + \frac{16}{3})e^{3t/2}.$$

We can observe the following

1. If $y_0 + 16/3 > 0$, then, as $t \to \infty$, the solution $y(t)$ is dominated by $(y_0 + 16/3)e^{3t/2}$ and grows positively like $e^{3t/2}$.
2. If $y_0 + 16/3 < 0$, then the solution is negative for all t and grows negatively like $-e^{3t/2}$ as $t \to \infty$.

Thus the critical value of y_0 is $-16/3$. Furthermore, when $y_0 = -16/3$, the solution grows negatively like $-e^t$.

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2. Solve

$$y' + y^2 \sin x = 0.$$

Solution. Rewrite as

$$\frac{dy}{dx} = -y^2 \sin x.$$

This is a separable DE. Assuming that $y \neq 0$, separate the variables and integrate

$$\int \frac{1}{y^2} dy = -\int \sin x dx \tag{10}$$

$$-\frac{1}{y} = \cos x + c \tag{11}$$

Therefore, the general solution is

$$y(x) = -\frac{1}{\cos x + c}.$$

Note: We can verify that $y \equiv 0$ is also a solution of the DE and it is not included in the general solution by choosing a value for c.

6. Solve the DE

$$y' = \frac{x^2}{1 + y^2}.$$

Solution. This is a separable DE. Separation of variables and integration

$$\int (1 + y^2) dy = \int x^2 dx \tag{12}$$

$$y + \frac{y^3}{3} = \frac{x^3}{3} + c. \tag{13}$$

The general solution is defined implicitly by the equation

$$x^3 - y^3 - 3y = c.$$

28. Solve the DE

$$\frac{dy}{dx} = \frac{4y - 3x}{2x - y}.$$

Solution. (a) The right-hand function

$$f(x, y) = \frac{4y - 3x}{2x - y}$$

is homogeneous of degree 0. In fact, for $\lambda \neq 0$,

$$f(\lambda x, \lambda y) = \frac{4\lambda y - 3\lambda x}{2\lambda x - \lambda y} = \frac{4y - 3x}{2x - y} = f(x, y).$$

(b). Make the change of variables $u = y/x$, and thus $y = xu$. Differentiating both sides $y' = u + xu'$. Substitution, we obtain a DE for u

$$u + x\frac{du}{dx} = \frac{4ux - 3x}{2x - ux} = \frac{4u - 3}{2 - u}. \tag{14}$$

$$x\frac{du}{dx} = \frac{4u - 3}{2 - u} - u = \frac{u^2 + 2u - 3}{2 - u}. \tag{15}$$

This is a separable equation. Separation of variables and integration

$$\int \frac{2 - u}{u^2 + 2u - 3} du = \int \frac{1}{x} dx \tag{16}$$

$$\int \frac{2 - u}{(u + 3)(u - 1)} du = \ln|x| + c \tag{17}$$

$$\int \Big[- \frac{5}{4(u + 3)} + \frac{1}{4(u - 1)} \Big] du = \ln|x| + c \tag{18}$$

$$-\frac{5}{4} \ln|u + 3| + \frac{1}{4} \ln|u - 1| = \ln|x| + c \tag{19}$$

$$\ln \frac{|u - 1|}{|u + 3|^5} = 4\ln|x| + c \tag{20}$$

$$\frac{|u - 1|}{|u + 3|^5} = cx^4 \ (c > 0) \tag{21}$$

Change back to y by substituting $u = y/x$

$$\frac{|y/x - 1|}{|y/x + 3|^5} = cx^4.$$

Page 67. **2.** Phase-line analysis for the autonomous DE

$$\frac{dy}{dt} = y(y-1)(y-2), \quad y_0 \geq 0.$$

Solution. The DE has three equilibria (constant solutions):

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 2.$$

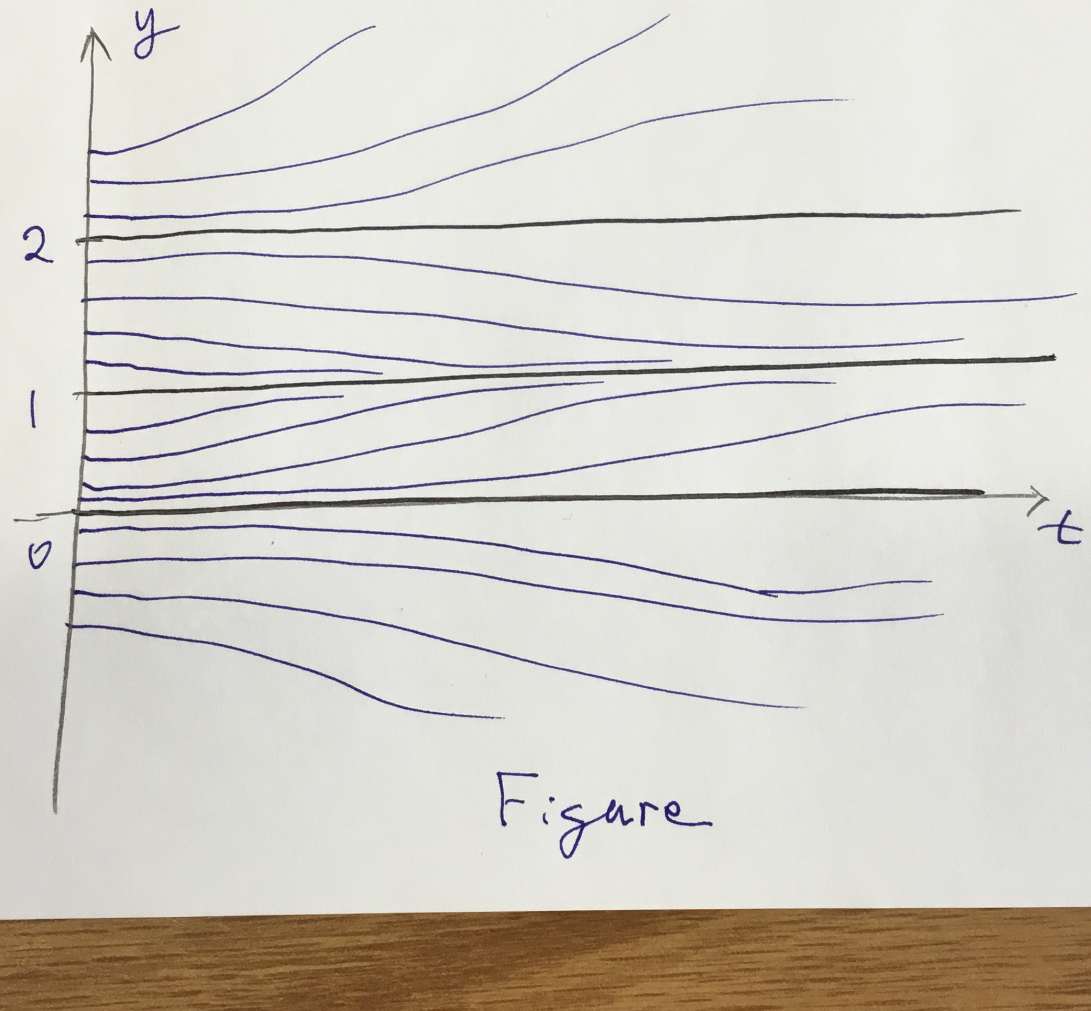
The graph of the vector field $f(y) = y(y-1)(y-2)$ is shown in Figure, and we see that

$$f(y) \text{ is } \begin{cases} > 0, & \text{if } 2 < y \\ < 0, & \text{if } 1 < y < 2 \\ > 0, & \text{if } 0 < y < 1 \\ < 0, & \text{if } y < 0. \end{cases}$$

Therefore,

1. solutions near $y = 2$ moves away from $y = 2$ and the equilibrium $y = 2$ is unstable;
2. solutions near $y = 1$ moves towards $y = 1$ and the equilibrium $y = 1$ is stable;
3. solutions near $y = 0$ moves away from $y = 0$ and the equilibrium $y = 0$ is unstable;

The sketches of solutions are shown in Figure.



Figure