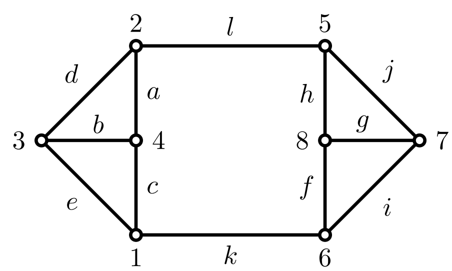
Graph Theory (MATH 322): Assignment 2

There are seven questions; answer them all. The assignment is due by 23:58 MDT on Tuesday 21st May.

1. (a) We have seen that if a simple graph has more than $\binom{n-1}{2}$ edges, where n is the number of vertices, then it must be connected. Show that the converse is false for every $n \geq 4$. That is, for each $n \geq 4$, find a connected simple graph on n vertices with no more than $\binom{n-1}{2}$ edges.

   (b) Show that if G is a simple graph with $n \geq 3$ vertices and every vertex has degree at least $n-2$, then G is connected.

2. Let G be the following graph, with vertices $1, \ldots, 8$ and edges $a, \ldots, l$:



   (a) Find a cutset of size 2 and a cutset of size 3.

   (b) Answer yes or no to each of the following. If your answer is "yes", then give an example. Otherwise, just say "no".

      (i) Is there a minimal separating set of size 1?

      (ii) Is there a minimal separating set of size 2?

      (iii) Is there a minimal separating set of size 3?

      (iv) Is there a minimal separating set of size 4?

   (c) Find $\kappa(G)$, $\lambda(G)$, and $\delta(G)$.

3. Draw a simple graph G with $\kappa(G) = 2$, $\lambda(G) = 3$, and $\delta(G) = 4$.

4. At a summer school on ancient languages, 10 participants want to play a translation game. One participant writes a short story in one ancient language and translates it into another ancient language. Then, a second participant takes the translated story and translates it into another ancient language, before passing it to a third participant, and so on. Once the last participant has done their translation, the game is over. Languages may be repeated during the game.

   Each participant knows only two ancient languages, as follows:

Participant	Languages known
Aidan	Hebrew, Sanskrit
Bede	Latin, Old Saxon
Cuthbert	Greek, Latin
Edward	Greek, Hebrew
George	Latin, Sanskrit
Hild	Old Saxon, Sanskrit
Josephine	Greek, Sanskrit
Mary	Hebrew, Old Saxon
Peter	Latin, Sanskrit
Thomas	Hebrew, Old Saxon

   Is it possible for everyone to take part in the game in such a way that each person does exactly one translation? If so, provide a sequence of the form

   1. <participant>, from <language> to <language>
   2. <participant>, from <language> to <language>
   3. <participant>, from <language> to <language>

   ⋮

   10. <participant>, from <language> to <language>

   Otherwise, explain why it is not possible.

5. In this question, all adjacency matrices are assumed to be taken with respect to some numbering of the vertices. The particular numbering does not matter.

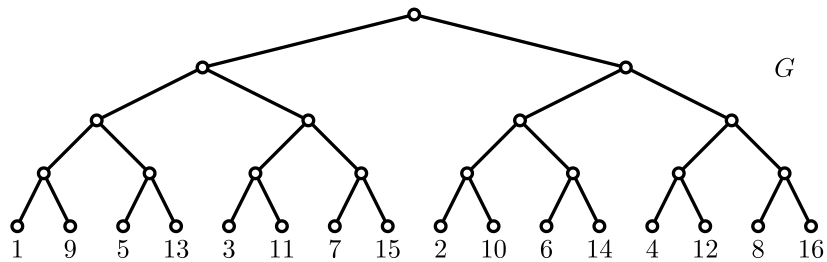
   (a) Let G be a disconnected graph and A its adjacency matrix. Show that for every positive integer k, the matrix A^k has at least one zero entry.

   (b) Let B be the adjacency matrix of the path graph P_n, where $n \geq 2$. Decide which of the following is true, with justification:

   (i) For all $k > 0$, the matrix B^k has at least one zero entry.

   (ii) There exists $k > 0$ such that every entry of B^k is positive.

6. Consider the graph G below. Choose a numbering of the vertices such that the 16 vertices of degree 1 are numbered $1, \ldots, 16$ as shown, and let A be the adjacency matrix of G with respect to such a numbering.

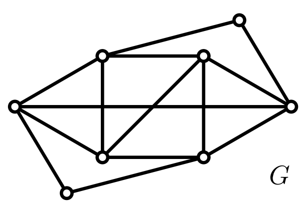


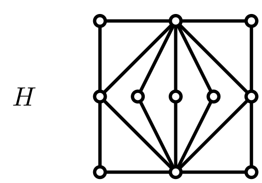
(a) Any non-zero integer a may be written uniquely in the form $a = 2^v b$ where $v \geq 0$ is an integer and b is an odd integer. The number v is called the 2-adic valuation of a and is denoted $v_2(a)$. For distinct integers $i, j \in \{1, \ldots, 16\}$, find the length of the shortest path from vertex i to vertex j in terms of the numbers i and j and the function v_2. *Hint: It may help to first subtract* 1 *from the number of each vertex.*

(b) What is the $(2, 15)$-entry of A^8?

(c) For distinct integers $i, j \in \{1, \ldots, 16\}$, let $f(i, j)$ be the smallest integer $m > 0$ such that the (i, j)-entry of A^m is positive. Find $f(4, 12)$, $f(6, 11)$, $f(7, 13)$, and $f(10, 14)$.

7. (a) For each of the following graphs G and H, decide whether it is Eulerian, briefly justifying your answer.





(b) Find two edge-disjoint trails in the following graph such that every edge of the graph is in one of the two trails.