Graph Theory (MATH 322): Assignment 3

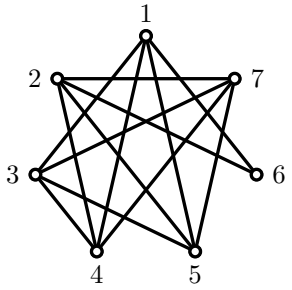
There are seven questions; answer them all. The assignment is due by 23:58 MDT on Friday 24th May.

1. Let G be a simple graph with n vertices, and suppose that $d(v) + d(w) \geq n - 1$ for all pairs of distinct non-adjacent vertices v, w. Show that any two vertices of G are connected by a path of length 2 or less. (In particular, G is connected.)

2. (a) (*i*) Is every Eulerian graph Hamiltonian? Provide a proof if so or a counterexample otherwise.

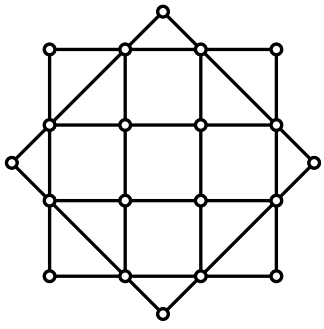
   (*ii*) Is every Hamiltonian graph Eulerian? Provide a proof if so or a counterexample otherwise.

   (b) Decide whether the graph below is Hamiltonian. Justify your answer.



3. (a) Let G be a Hamiltonian graph and S a set of $k \geq 1$ vertices in G. Show that $G - S$ has at most k connected components. (This fact was stated in class without proof.)

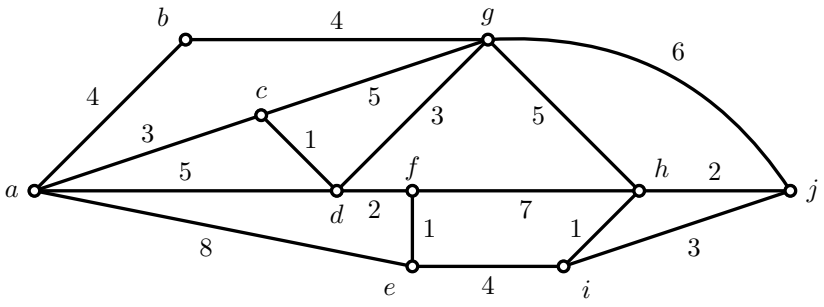
   (b) Use part (a) to show that the graph below is not Hamiltonian.



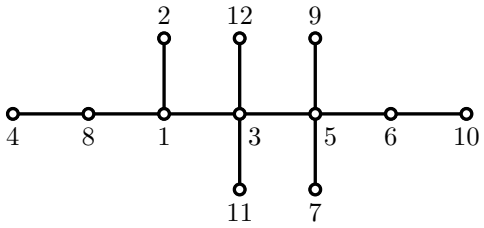
4. A simple graph G has degree sequence $(5, 5, 5, 5, 4, 3, 3)$. Show that G is Hamiltonian.

5. (a) In the weighted graph below, find the shortest distance from vertex a to every other vertex.

   (b) Write down a path of shortest length from a to j.



6. (a) (i) Find the Prüfer sequence of the following labelled tree.



   (I have encoded something in the Prüfer sequence of the above tree. There is a bonus mark for determining what is encoded there—besides the tree, of course. Please do not spend too long on it though!)

   (ii) Draw the labelled tree corresponding to the Prüfer sequence $(2, 3, 3, 3, 2, 4, 4, 4)$.

   (b) An end-vertex in a graph is, by definition, a vertex of degree 1. How many labelled trees are there on $n \geq 4$ vertices in which there are exactly three end-vertices?

7. Let G be a graph, and let T, T' be spanning trees in G. Show that if e is an edge in T, then there is an edge e' in T' such that the graph obtained by adding the edge e' to $T - e$ is again a spanning tree in G.