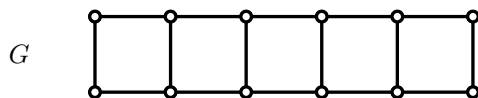


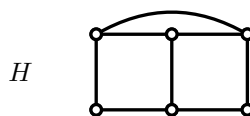
Graph Theory (MATH 322): Assignment 6

There are seven questions; answer them all. The assignment is due by 23:58 MDT on Wednesday 12th June.

1. (a) A graph has chromatic polynomial $k^8 - 9k^7 + 34k^6 - 69k^5 + 79k^4 - 48k^3 + 12k^2$. How many vertices, edges, and components does the graph have?
- (b) Find the chromatic polynomial of the following graph G . Give your answer in factorized form.



- (c) Find the chromatic polynomial of the following graph H . Give your answer in expanded form, $k^6 - 8k^5 + \dots$.



2. For $n \geq 4$, let W_n be the wheel graph on n vertices.
- (a) Use the equation $P_G(k) = P_{G-e}(k) - P_{G \setminus e}(k)$ together with the “triangle trick” to show that

$$P_{W_n}(k) = k(k-1)(k-2)^{n-2} - P_{W_{n-1}}(k)$$

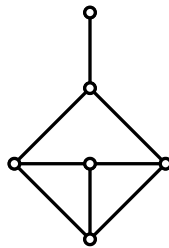
for all $n \geq 5$.

- (b) Using part (a), show by induction on $n \geq 4$ that

$$P_{W_n}(k) = k(k-2)((k-2)^{n-2} - (-1)^n).$$

3. Find the smallest positive integer n such that there are non-isomorphic simple graphs on n vertices that have the same chromatic polynomial. Explain carefully why the n you give as your answer is indeed the smallest.

4. Let $n \geq 3$ be an integer, let H_n be the cycle graph on $2n$ vertices $1, \dots, 2n$, with the vertices ordered such that $(1, 2, 3, \dots, 2n, 1)$ is a cycle in H_n , and let G_n be the graph obtained from H_n by adding an edge between vertex 1 and vertex $n + 1$.
- (a) Show that $P_{G_n}(k)$ is equal to
- $$(k-1)((k-1)^{2n-1} + 1) - \frac{1}{k}(k-1)^2((k-1)^{n-1} + (-1)^n)^2.$$
- (b) Find $P'_{G_n}(0)$, the derivative of P_{G_n} evaluated at 0. *Hint: What is the coefficient of k in $P_{G_n}(k)$? Try binomial expansion.*
5. (a) Let G be a simple graph with at least two edges. Show that if G has an end-vertex (a vertex of degree 1), then $P'_G(1) = 0$, where P'_G denotes the derivative of the polynomial P_G .
- (b) Show that the converse is false, that is, a simple graph with at least two edges may satisfy $P'_G(1) = 0$ but have no end-vertices.
6. Let $n \geq 3$. Show that the cycle graph C_n on n vertices is the only simple graph with its chromatic polynomial.
7. (a) Find the chromatic index χ' of the following graph, justifying your answer. Also provide a χ' -edge colouring.



- (b) Find the chromatic index χ' of the following graph, justifying your answer. Also provide a χ' -edge colouring.

