MATH 222

Assignment 4

- Reccurence Relations
- Induction
- Strong Induction
- Tromino Problems

Group information (you may work by yourself, in a pair, or as a trio)

First Name	**Last Name**	**ID**
J	O	C
A	B	C
C	E	O

1. Let a_n be the number of words (strings) of length n that can be made using the digits $\{0,1,2,3\}$ with an odd number of twos. Find a recurrence relation for a_n and solve the recurrence. The first few values of a_n are calculated for you:

2

$a_1 = 1$

20 21 23 02 12 32

$a_2 = 6$

020 021 023 002 012 032
120 121 123 102 112 132
320 321 323 302 312 332
200 201 203 210 211 213 230 231 233 222

$a_3 = 28$

Solution:

Notice that there are a total of 4^n words of length n using the digits $\{0,1,2,3\}$. It follows that there are $4^n - a_n$ such words of length n with an even number of twos. There are two cases for the first digit in the word: the first digit is either a 2 or not.

If the first digit is not a 2, then it can be a 0, 1, or 3. Thus, there are $3a_{n-1}$ such words of length n with an odd number of twos. Further, if the first digit is a 2, then there are $4^{n-1} - a_{n-1}$ such words of length n with an odd number of twos. Therefore by adding the two cases together we get

$$\begin{aligned} a_n &= 3a_{n-1} + 4^{n-1} - a_{n-1} \\ &= 2a_{n-1} + 4^{n-1} \\ &= 2a_{n-1} + 2^{2n-2} \end{aligned}$$

This gives the recurrence relation:

The recurrence relation is:

$a_1 = 1$

$a_n = 2a_{n-1} + 2^{2n-2}$

$\overline{2}$

Solve by iteration:

$$a_1 = 1 = 2^0$$

$$a_2 = 2^1 + 2^2$$

$$a_3 = 2^2 + 2^3 + 2^4$$

$$a_4 = 2^3 + 2^4 + 2^5 + 2^6$$

$$\vdots$$

$$a_n = 2^{n-1} + 2^n + \cdots + 2^{2n-2}$$

$$= (2^0 + 2^1 + \cdots + 2^{n-2}) + 2^{n-1} + 2^n + \cdots + 2^{2n-2}$$

$$-(2^0 + 2^1 + \cdots + 2^{n-2})$$

$$= (2^{2n-1} - 1) - (2^{n-1} - 1) \quad \text{(by the closed form for geometric sequences)}$$

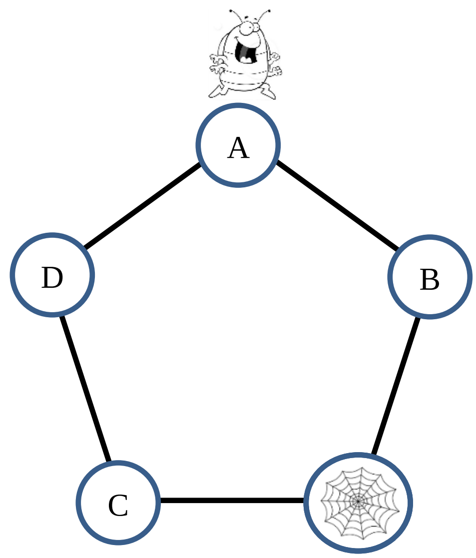
$$= 2^{2n-1} - 2^{n-1}$$

$$= 2^{n-1}(2^n - 1)$$

A solution to the recurrence relation is given by:

$$a_n = 2^{n-1}(2^n - 1)$$

2. A bug starts at vertex A of the pentagon below and each minute travels to an adjacent vertex. There is a spider web on one vertex; if the bug moves there it gets stuck. Let a_n be the number of different ways the bug can travel from vertex A to the spider web after n minutes. Find a recurrence relation for a_n; you do not have to solve this recurrence. Hint: consider two cases the 1st move is to D or B, in each case write down what a_n is in terms of a_{n-1} or a_{n-2}. The first four values of a_n are: $a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 2$.



Solution:

Number of Moves	Paths to the Web	Result
1	None	$a_1 = 0$
2	$A - B - Web$	$a_2 = 1$
3	$A - D - C - Web$	$a_3 = 1$
4	$A - B - A - B - Web$ $A - D - A - B - Web$	$a_4 = 2$
5	$A - B - A - D - C - Web$ $A - D - C - D - C - Web$ $A - D - A - D - C - Web$	$a_5 = 3$

Consider the bug's choice for the first move: the bug can either go to B or to D:

- If the bug goes to B, it must return to A (when $n \geq 3$). Therefore, in this case there are a_{n-2} ways the bug can reach the web for the first time.

- If the bug goes to D, use the symmetry of the pentagon to conclude there are a_{n-1} ways the bug can reach the web for the first time.

Adding together the two cases gives the recurrence relation:

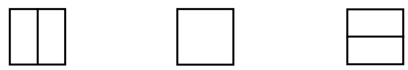
The recurrence relation is:

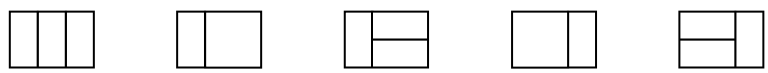
$a_1 = 0$
$a_2 = 1$
$a_n = a_{n-1} + a_{n-2} \qquad n \geq 3$

$\overline{2}$

3. Let a_n be the number of ways can you tile a $2 \times n$ rectangle using dominoes and square tetrominoes. Dominoes are of size 2×1 and tetrominoes are of size 2×2. Find a recurrence relation for a_n; you do not have to solve this recurrence. The first few values of a_n are calculated for you.

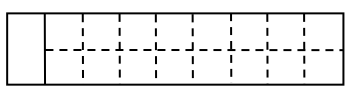
$a_1 = 1$ 

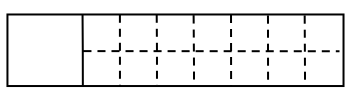
$a_2 = 3$ 

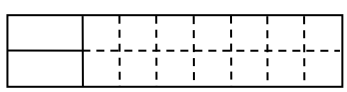
$a_3 = 5$ 

Solution:

Notice that there are three ways to tile the two most left squares in the $2 \times n$ rectangle:

 There are a_{n-1} ways to tile the rest of the

 There are a_{n-2} ways to tile the rest of the

 There are a_{n-2} ways to tile the rest of the

Adding together the cases gives the recurrence relation.

The recurrence relation is:

$a_1 = 1$

$a_2 = 3$

$a_n = a_{n-1} + 2a_{n-2}$

$\overline{\ \ }$
2

4. Conjecture a formula for the sum of the first n Fibonacci numbers with odd indices and prove your formula works by using mathematical induction. That is, find and prove a formula for

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1}$$

Note:

$$F_1 = 1,$$
$$F_2 = 1,$$
$$F_n = F_{n-1} + F_{n-2}.$$

Solution:

Recall, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$

The first few terms gives us:

$n = 1$: $1 = 1$
$n = 2$: $1 + 2 = 3$
$n = 3$: $1 + 2 + 5 = 8$
$n = 4$: $1 + 2 + 5 + 13 = 21$
$n = 5$: $1 + 2 + 5 + 13 + 34 = 55$

Perhaps the sum of the first n Fibonacci numbers with odd indices is F_{2n} .

Base case: $n = 1$

$F_1 = 1 = F_{2\cdot 1} = F_2$, so the base case holds.

Inductive Step:

Show: $F_1 + F_3 + F_5 + \cdots + F_{2k-1} = F_{2k}$
$\Rightarrow$ $F_1 + F_3 + F_5 + \cdots + F_{2k-1} + F_{2k+1} = F_{2(k+1)}$

$$F_1 + F_3 + F_5 + \cdots + F_{2k-1} + F_{2k+1} \underset{\boxed{\text{By induction}}}{=} F_{2k} + F_{2k+1} \underset{\boxed{\text{By Fibonacci}}}{=} F_{2k+2} = F_{2(k+1)}$$

$$\boxed{F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}}$$

$\overline{2}$

5. An equal number of open and closed gas stations are distributed an equal distance apart around a ring road of a city. It is not known in which pattern the open and closed gas stations are distributed (they could alternate open and then closed all the way around or there could be many open gas stations in a row followed by many closed gas stations in a row). Only 1 liter can be added from any open station; however we do not have to worry about a maximum volume limit for our gas tank. The gas tank is empty to start out with.

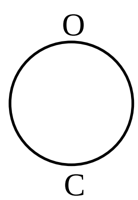
   Define the following statement:

   P_n : There is an open gas station to start at which allows one loop of travel clockwise around the city if
   - there are n open and n closed gas stations.
   - it takes n liters of gas to travel around the city

   Show that P_n is true for $n \geq 1$.

Solution:

<u>Base case:</u> $n = 1$

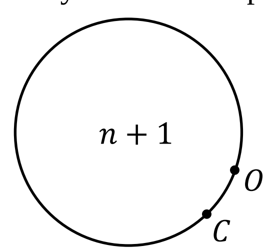


When there is only one open and one closed gas station we can start at the open gas station:

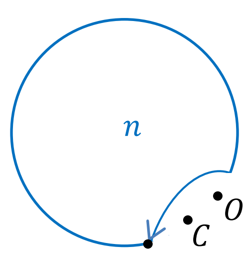
<u>Inductive Step:</u>

Show: if it possible to find a starting position to travel around the city with n open and n closed gas stations then it possible to find a starting position to travel around the city with $n+1$ open and $n+1$ closed gas stations.

Consider the ring road with $n+1$ open and $n+1$ closed gas stations and search for an open gas station O which is followed by a closed gas station C.



Notice that during travel around the city if one adds a gallon at O one can always travel one gas station past C. Noting that reduces the problem to finding a starting position among n open and n closed gas stations:



and by our assumption there is a starting position.

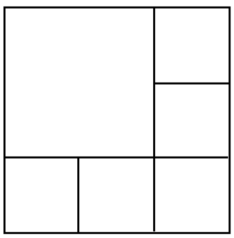
$\overline{2}$

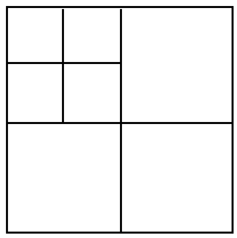
6. Consider the following statement.

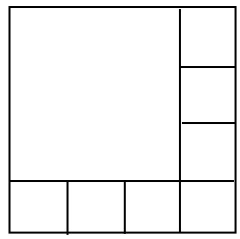
   P_n: A square can be dissected into n smaller squares .
   (The smaller squares do not all have to be the same size)

   Professor Scarlett attempted strong inductive proof to show P_n is true for $n \geq 6$. Unfortunately, Professor Scarlett forgot two of the base cases. Complete the missing base cases to complete the proof.

<u>Base Cases:</u> Show: P_6, P_7, P_8 are true.





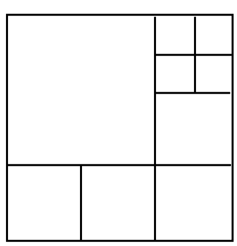


Therefore P_6, P_7, and P_8 are true.

<u>Inductive Step:</u>

Show: $P_n \Rightarrow P_{n+3}$.

Notice that if a square can be dissected into n smaller squares, then it can be dissected into $n + 3$ smaller squares. This is done by taking one of the existing squares and dissecting it into four squares of equal size. For example, using the dissection of 6 squares, the following is a dissection into 9 squares:



Therefore if P_n is true then P_{n+3} is also true.

7. For this question define the statement for natural numbers:

$$P_n\text{: Any deficient } n \times n \text{ board can be tiled with trominoes.}$$

The objective of this question to prove that P_n is true for all natural numbers other than 5 and multiples of three. That is:

$$P_n \iff n \neq 5 \text{ and } n \not\equiv 0 \pmod 3.$$

In our lecture notes we have shown that P_n is true when:

$$n = 2,8$$
$$n \equiv 1 \pmod 3$$

and that P_n is false when:

$$n = 5$$

To finish the proof of this theorem complete parts a) and b).

a) Show that if $n \equiv 0 \pmod 3$ then P_n is false. Hint: Show that the area of a deficient $n \times n$ board is not a multiple of 3.

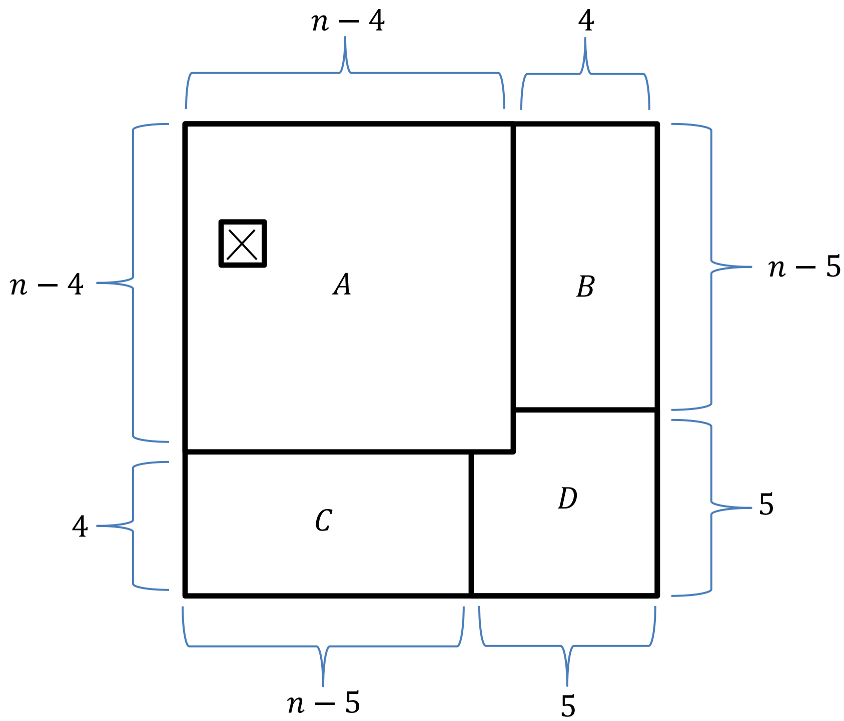
Solution:

In order to tile any board with trominoes its area must be divisible by 3 but if $n \equiv 0 \bmod 3$ then:

$$n^2 - 1 \equiv 0^2 - 1 \equiv 1 \not\equiv 0 \bmod 3.$$

Therefore, if $n \equiv 0 \bmod 3$ a deficient $n \times n$ board cannot be tiled with trominoes.

b) Show that when $n \equiv 2 \pmod 3$ and $n \geq 11$ then P_n is true by explaining how each of the four sections below can be tiled. Note: by rotational symmetry the deficiency is always in section A.



Solution:
Now by symmetry we only need to consider the case when the missing square is in section A.
Each section can be tiled with trominoes for the following reasons:

- $n - 4 \equiv 2 - 4 \equiv 2 - 1 \equiv 1 \bmod 3 \quad \xRightarrow{\textit{Lecture } 16 \;\; \textit{Prop.}2}$ Section A can be tiled with trominoes.
- $n - 5 \equiv 2 - 5 \equiv 2 + 1 \equiv 0 \bmod 3$, $4 \equiv 0 \bmod 2 \quad \xRightarrow{\textit{Lecture } 16 \;\; \textit{Prop.}1}$ Sections B and C can both be tiled with trominoes.
- Section D is a deficient 5×5 board missing its top left corner. $\xRightarrow{\textit{Lecture } 16 \;\; \textit{Ex}2}$ Section D can be tiled with trominoes.

Therefore, if $n \equiv 2 \pmod 3$ and $n \neq 5$ a deficient $n \times n$ board can be tiled with trominoes.

$\overline{2}$

Bonus.

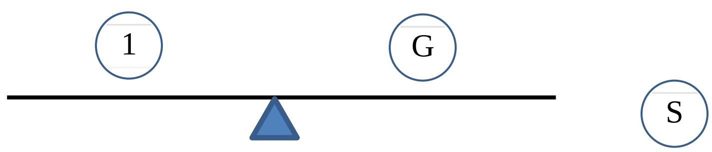
You are given $\left\lceil \frac{3^n}{2} \right\rceil + 1$ coins one of which is counterfeit. It is not known whether the counterfeit is slightly heavier or lighter than the others. Additional information is known about two of the coins. One of them is not the counterfeit. The other one is either known not to be too light or known not to be too heavy. Show that it is possible to find the counterfeit in n weighings $(n \geq 1)$ while identifying if the counterfeit is too light or too heavy.

Solution.

Let the statement be P_n and note that: $\left\lceil \frac{3^n}{2} \right\rceil + 1 = \frac{3^n+1}{2} + 1 = \frac{3^n+3}{2} = 3 \cdot \frac{3^{n-1}+1}{2}$. Let the good coin be coin "G". Let the coin that is either known to be not too light or known to be not too heavy be coin "S". Label the rest of the coins from 1 to $\frac{3^n+1}{2} - 1$.

<u>Base Case</u>

Show P_1 is true. Weigh the $\frac{3^1+3}{2} = 3$ coins in the following pattern:



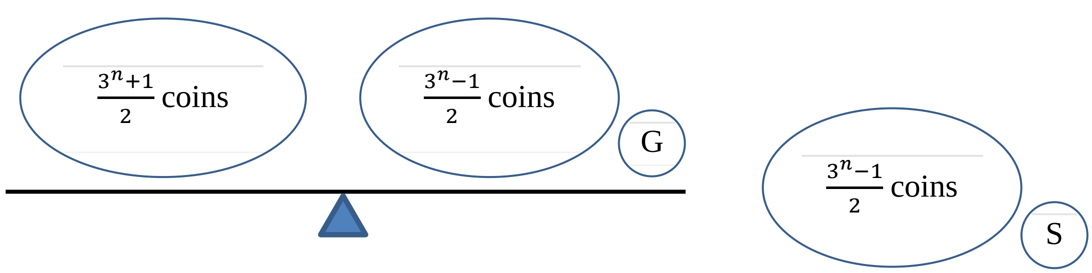
There are three cases:
CASE 1: The scale tips to the left. In this case Coin 1 will be too heavy.

CASE 2: The scale tips to the right. In this case Coin 1 will be too light.

CASE 3: The scale balances. Coin S is too heavy if it is known to be not too light. Otherwise, coin S is too light if it is known to be not too heavy.

<u>Inductive Step</u>

Show: $P_n \Rightarrow P_{n+1}$. Weigh the $3 \cdot \frac{3^n+1}{2}$ coins in the following pattern:



$\overline{2}$

There are three cases:

CASE 1: The scale tips to the left. In this case the counterfeit is on the left side of the scale because it is too heavy or on the right side of the scale because it is too light. Pair each of the coins on the left hand side of the scale with a coin on the right hand side of the scale. Each pair will form a theoretical coin made of two original coins. Take two coins that were left off of the scale and form one more theoretical coin. In total we have:

$$\frac{\overbrace{\frac{3^n+1}{2}}^{\text{Coins from the left}}+\overbrace{\frac{3^n+1}{2}}^{\text{Coins from the right}}}{2}+\underbrace{1}_{\text{2 coins left off the scale}}$$

$$=\frac{2\frac{3^n+1}{2}}{2}+1=\frac{3^n+1}{2}+1=\frac{3^n+3}{2}$$

theoretical coins. One of them is not the counterfeit since it was made from two coins that were left off of the scale. In addition, one of them is known not to be too light since it was made by a coin from the left hand side and coin “G”. The other $\frac{3^n+3}{2}-2$ theoretical coins could be slightly heavier or lighter than they should be; since, they were each made with a coin from the left side and a coin from the right side of the scale. Therefore the counterfeit can be found among these coins in n more weighings by P_n. In total $n+1$ weighings were used so P_{n+1} is true.

CASE 2: The scale tips to the right. This case is symmetric to case 1 and therefor the same procedure applies.

CASE 3: The scale balances. In this case the counterfeit is among the $\frac{3^n+1}{2}$ coins left off of the scale. The counterfeit can be found among these coins together with coin “G” by P_n. In total $n+1$ weighings were used so P_{n+1} is true.