MATH 222

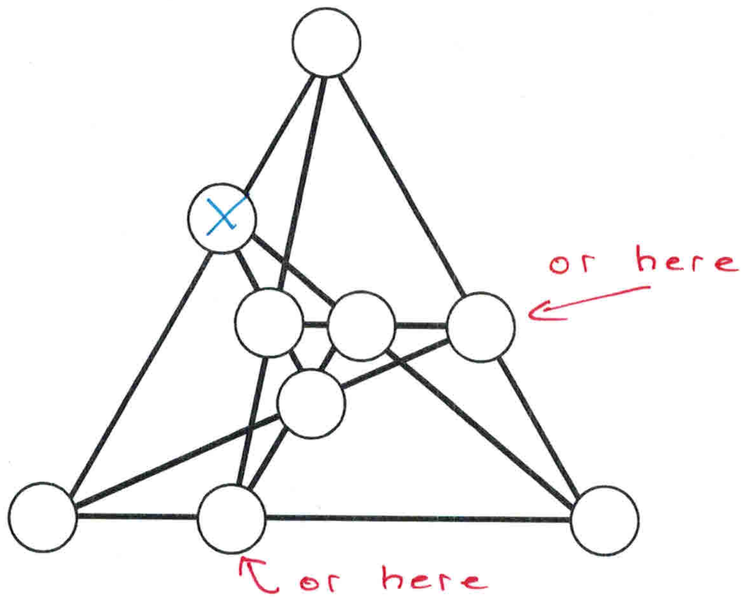
Assignment 2

- Two Player Games
- Nim
- Classic Nim
- Coding Theory
- Campers Problem
- Binary numbers
- Hamming Code

Group information (you may work by yourself, in a pair, or as a trio)

First Name	Last Name	ID
		222
	Ecco	
Jacob		

1. The following game is called Tri-Tic-Tack-Toe. How to play: Player X and Player O alternately put their symbol in an empty circle on their turn. Whoever makes three in a row on any straight line wins the game. Each player can only take four turns.



State whether Tri-Tic-Tack-Toe ends in a theoretical draw, the first player can always win, or the second player can always win. If the first player can always win, state the first move that he or she needs to make. If the second player can always win, state the first move he or she must make in response to every way the game may begin. **Hint**: The game board for Tri-Tic-Tack-Toe has a rotational symmetry of 120°. Therefore, there are only 3 different opening moves. Analyze all three cases by printing out the Tri-Tic-Tack-Toe template on the course web page (you do not have to hand in the template with your assignment).

If both players make the best possible moves during game play then (*circle one*):

- The game ends in a tie.
- The 1st player wins. (*mark in the correct first move on the game board above*)
- The 2nd player wins. (*describe the 2nd player first move below*)

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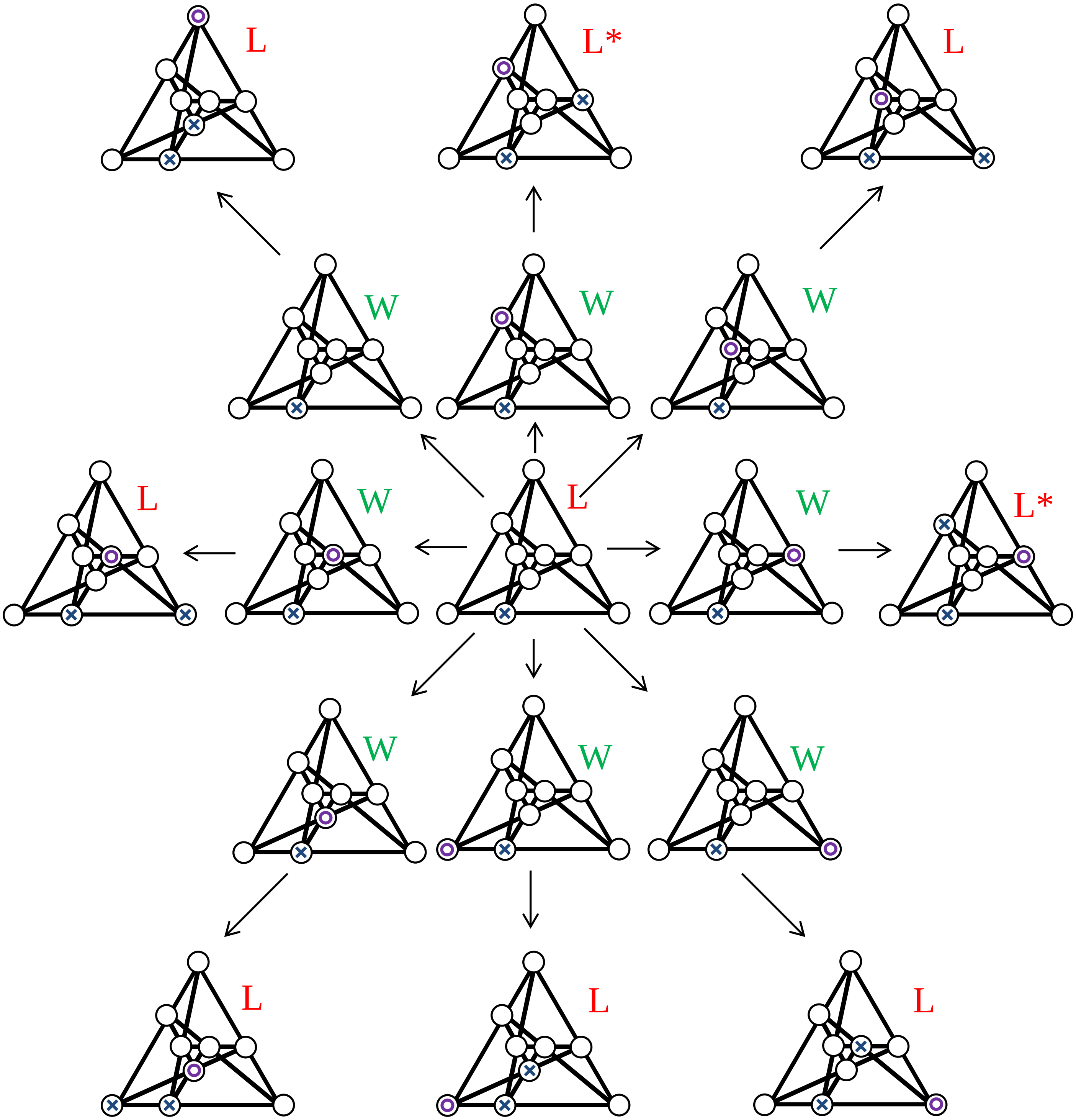
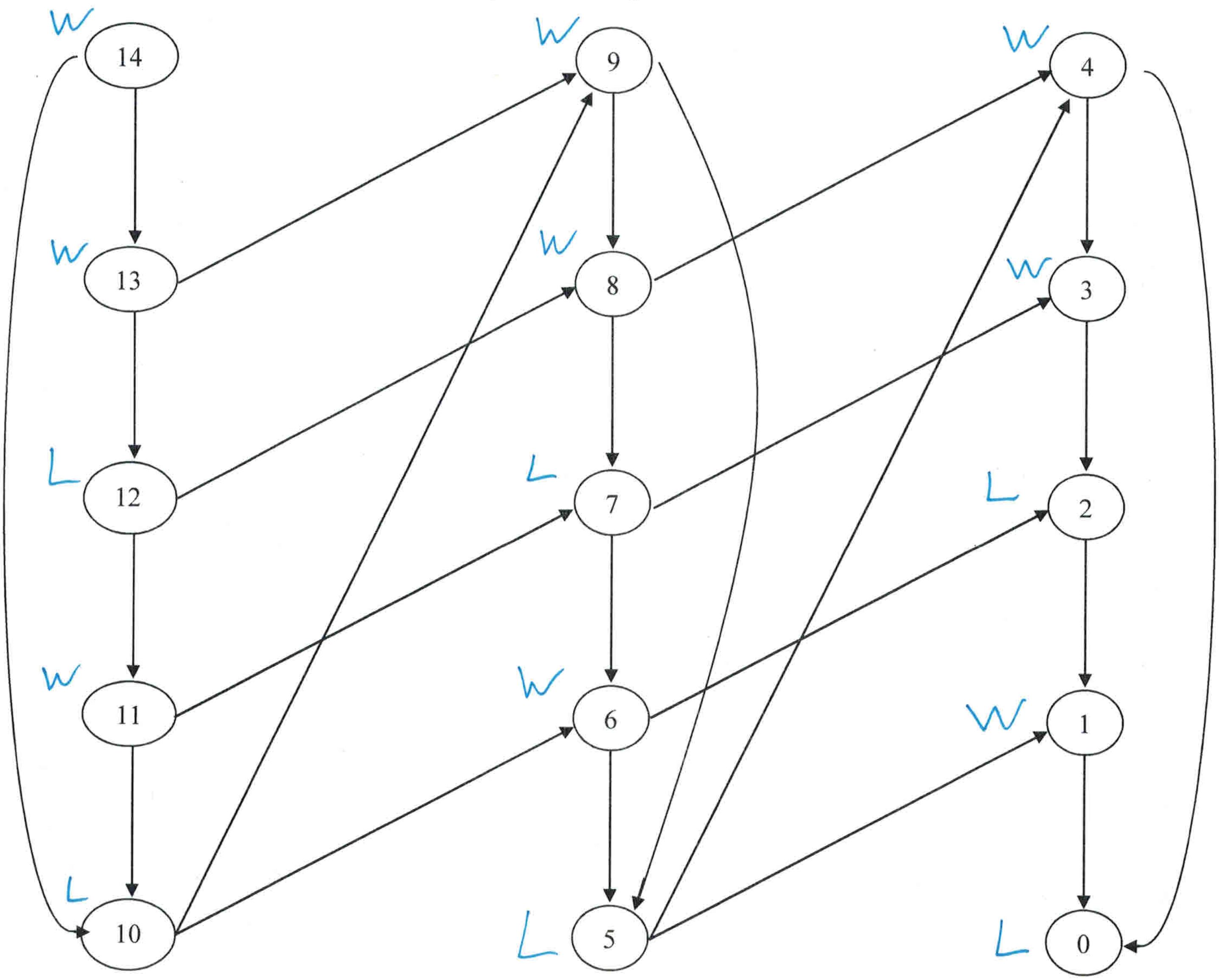


The first two moves of the winning strategy for the 1st player (player X) are outlined in the partial state diagram. In each "outer" losing position (except for the two marked with a *) player O is forced to block three X's in a row. Then player X plays so that there are two different ways to complete 3 X's in a row on the following turn; therefore, player X will win.

In the L* positions, player O can play anywhere; each play will force player X to block three O's in a row. In every case after blocking three O's in a row, player X has two different ways to get to get 3 X's in a row on the following turn; therefore, player X will win.

2. The Matchstick Game. How to play: from a pile of 222 matchsticks, two players take turns removing either 1 or 4 sticks. The player who removes the last stick is the winner. Which player can always win: the first player or the second player? **Hint**: Try to find a pattern of winning and losing positions among the first 15 positions:



$222 \equiv 2 \pmod{5}$

(The player with 2, 7, 12, 17, ..., 222 sticks will lose)

If both players make the best possible moves during game play then (*circle one*):

- The 1^{st} player wins.
- The 2^{nd} player wins.

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3. Let's play Nim. This version of the game will have 4 rows. The rows will have the following amount of counters:

Row 1: 30

Row 2: 31

Row 3: 42

Row 4: 50

Which player can always win: the first player or the second player? How many opening moves follow a winning strategy?

Find the nim-sum:

$s = 30 \oplus 31 \oplus 42 \oplus 50$

$(\cancel{16} + \cancel{8} + \cancel{4} + \cancel{2}) \oplus (\cancel{16} + \cancel{8} + \cancel{4} + \cancel{2} + 1) \oplus (\cancel{32} + 8 + \cancel{2})$

$\oplus (\cancel{32} + 16 + \cancel{2}) = 16 + 8 + 1 = 25$

$s \neq 0$ ∴ the 1st player can always win, by taking from row 1, 2, or 4 (since 16 is in these rows)

(The reason there are no other opening moves that follow a winning strategy is given in the bonus problem)

If both players make the best possible moves during game play then (*circle one*): • The 1st player wins. • The 2nd player wins.

There are (*circle one*): 0 1 2 3 4 opening moves that follow a winning strategy.

4. Let's play **Classic** Nim. This version of the game will have 4 rows. The rows will have the following amount of counters:

Row 1: 1

Row 2: 1

Row 3: 10

Row 4: 10

Which player can always win: the first player or the second player? If the 1st player takes 9 counters from row 3 what should the 2nd player do to guarantee victory?

- Find the nim-sum:

$S = \cancel{1} \oplus \cancel{1} \oplus \cancel{10} \oplus \cancel{10} = 0$

- Row 1: 1
  Row 2: 1
  Row 3: 1
  Row 4: 10

There is one big row so the player should leave an odd amount of 1's.

If both players make the best possible moves during game play then (*circle one*):

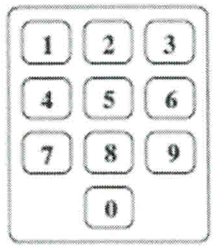
- The 1st player wins.
- The 2nd player wins.

When the 1st player takes 9 counters from row 3 the 2nd player should respond by taking: (*fill in the blanks*)

10 counters from row 4.

5. The telephone numbers in a town run from 00000 to 99999. There are a number of errors that commonly occur:

- The Diagonal Error: a common error in dialling on a standard keypad is to punch in a digit diagonally adjacent to the intended one. So on a standard dialling keypad, 4 could erroneously be entered as 2 or 8 (but not as 1, 5, or 7).



- The Switch Bug: After a correct number has been called, in transmission, one pair of adjacent digits gets swapped. For example the number $abcde$ could be called but whoever is at $\boldsymbol{abced}$ receives the phone call.

- The Smudged Digit: A single digit gets corrupted in transmission and it is known which digit got corrupted. For example: 38754 is dialed but in transmission, it becomes 387□4.

- The Digit Error: A single digit gets changed into another random digit during transmission and it is unknown which digit was changed. For example 38754 is dialed but 38757 receives the call.

It has been decided that a sixth digit will be added to each phone number. Given a phone number $abcde$, there are three different proposed schemes:

Code 1: $a + b + c + d + e + X \equiv 0 \pmod{10}$

Code 2: $2a + b + 2c + d + 2e + X \equiv 0 \pmod{10}$

Code 3: $6a + 5b + 4c + 3d + 2e + X \equiv 0 \pmod{10}$

Fill out the following chart with a Yes or a No (you DO NOT have to show your work).

	Can detect a Diagonal Error?	Can detect a Switch Bug?	Can correct a Smudged Digit?	Can detect a Digit Error?
Code 1	Yes	No	Yes	Yes
Code 2	Yes	Yes	No	No
Code 3	No	Yes	No	No

5a

Solution.

Code 1:

Given a phone number $abcde$ a sixth digit X is added so:

$$a + b + c + d + e + X \equiv 0 \pmod{10}$$

- Can detect a Diagonal Error? Yes, by the same reasoning Code 1 can detect a Digit Error (below).

- Can detect a Switch Bug? No, for example the switch bug could change 100009 into 010009 both of which are valid codewords.

- Can correct a Smudged Digit? Yes, for example if $a \leftrightarrow y$ then we can solve

$$y + b + c + d + e + X \equiv 0 \pmod{10}$$

To get

$$y \equiv -b - c - d - e - X \pmod{10}$$

The smudge y can be corrected to $a = -b - c - d - e - X$

- Can detect a Digit Error? Yes, for example if $a \leftrightarrow a'$

Case 1: $a' + b + c + d + e + X \not\equiv 0 \pmod{10}$

$\therefore$ in this case the digit error is detected.

Case 2: $a' + b + c + d + e + X \equiv 0 \pmod{10}$

Then

$$\begin{array}{rl} & a' + b + c + d + e + X \equiv 0 \pmod{10} \\ - & a + b + c + d + e + X \equiv 0 \pmod{10} \\ \hline & a' - a \qquad\qquad\qquad\quad \equiv 0 \pmod{10} \end{array}$$

$\Rightarrow a' \equiv a \pmod{10}$

$\Rightarrow a' = a$ since $a, a' \in \{0,1,2,\dots,9\}$

$\therefore$ in this case there was no error.

Code 2:

Given a phone number $abcde$ a sixth digit X is added so:

$$2a + b + 2c + d + 2e + X \equiv 0 \pmod{10}$$

- Can detect a Diagonal Error? Yes. There are two situations:

  1. When a digit with a coefficient of 2 gets corrupted for example $a \leftrightarrow a'$.
  2. When a digit with a coefficient of 1 gets corrupted for example $b \leftrightarrow b'$.

  The same reasoning code 1 can detect a Digit Error (above) shows that code 2 can detect when $b \leftrightarrow b'$. We are left to discuss when $a \leftrightarrow a'$.

Case 1: $2a' + b + 2c + d + 2e + X \not\equiv 0 \pmod{10}$

$\therefore$ in this case the Diagonal Error is detected.

Case 2: $2a' + b + 2c + d + 2e + X \equiv 0 \pmod{10}$

Then

$$\begin{array}{rl} & 2a' + b + 2c + d + 2e + X \equiv 0 \pmod{10} \\ - & 2a + b + 2c + d + 2e + X \equiv 0 \pmod{10} \\ \hline & 2a' - 2a \equiv 0 \pmod{10} \end{array}$$

$\Rightarrow 2(a' - a) \equiv 0 \pmod{10}$

$\Rightarrow (a' - a) \equiv 0$ or $5 \pmod{10}$

On the other hand, consider all the possibilities for a Diagonal Error on the keypad:

$$a' - a = \pm 2, \pm 4, \pm 7, \pm 9$$

$\Rightarrow (a' - a) \equiv 1$ or 2 or 3 or 4 or 6 or 7 or 8 or $9 \pmod{10}$ 

$\therefore$ case 2 results in a contradiction and does not occur

5c

- Can detect a Switch Bug? Yes. For example if $a \leftrightarrow b$

Case 1: $2b + a + 2c + d + 2e + X \not\equiv 0 \pmod{10}$

$\therefore$ in this case the digit error is detected.

Case 2: $2b + a + 2c + d + 2e + X \equiv 0 \pmod{10}$

Then

$$\begin{array}{rl} & 2b + a + 2c + d + 2e + X \equiv 0 \pmod{10} \\ - & 2a + b + 2c + d + 2e + X \equiv 0 \pmod{10} \\ \hline & b - a \qquad\qquad\qquad\qquad \equiv 0 \pmod{10} \end{array}$$

$\Rightarrow b \equiv a \pmod{10}$

$\Rightarrow b = a$ since $a, a' \in \{0,1,2,\dots,9\}$

$\therefore$ in this case there was no error.

- Can correct a Smudged Digit? No, for example the smudged phone number □00000 could have been correctly dialed as 000000 or 500000; both of which are valid codewords.

- Can detect a Digit Error? No, the valid codewords 000000 or 500000 have a Hamming distance of 1. Therefore, the minimum Hamming distance of Code 2 is one, which means this code can't detect a Digit Error.

Code 3:

Given a phone number $abcde$ a sixth digit X is added so:

$$6a + 5b + 4c + 3d + 2e + X \equiv 0 \pmod{10}$$

- Can detect a Diagonal Error? No, for example a Diagonal Error could change 020000 into 040000 both of which are valid codewords.

- Can detect a Switch Bug? Yes. For example if $a \leftrightarrow b$

Case 1: $6b + 5a + 4c + 3d + 2e + X \not\equiv 0 \pmod{10}$

$\therefore$ in this case the digit error is detected.

Case 2: $6b + 5a + 4c + 3d + 2e + X \equiv 0 \pmod{10}$

Then

$$\begin{array}{rl} & 6b + 5a + 4c + 3d + 2e + X \equiv 0 \pmod{10} \\ - & 6a + 5b + 4c + 3d + 2e + X \equiv 0 \pmod{10} \\ \hline & b - a \qquad\qquad\qquad\qquad\quad \equiv 0 \pmod{10} \end{array}$$

$\Rightarrow b \equiv a \pmod{10}$

$\Rightarrow b = a$ since $a, a' \in \{0,1,2,\ldots,9\}$

$\therefore$ in this case there was no error.

- Can correct a Smudged Digit? No, for example the smudged phone number $0000\square0$ could have been correctly dialed as 000000 or 000050; both of which are valid codewords.

- Can detect a Digit Error? No, the valid codewords 000000 or 000050 have a Hamming distance of 1. Therefore, the minimum Hamming distance of Code 3 is one, which means this code can't detect a Digit Error.

6. A counselor has $3n+2$ campers with her at a junction in a hiking trail. She knows their camp is twenty minutes down one of four possible paths. It will be dark in one hour and the group must find their camp before dark. n of the $3n+2$ campers sometimes lie, and unfortunately the counselor doesn't know which n they are. She checks path 4, leaving paths 1, 2, and 3 for the $3n+2$ campers. She sends $n+1$ campers down each of paths 1 and 2 and sends n campers down path #3. Then she designs a code containing three codewords to deduce the location of the camp if she doesn't find it down path 4. Each codeword will represent the camp being down a particular path. Each bit (1 or 0) in the codeword will represent the answer given by a camper, a 1 will represent "yes" and a 0 will represent "no". Here is the counselor's code:

(C1)	Codeword #1:	The camp is down path 1 and no one lies:	$11\cdots11$	$00\cdots00$	$00\cdots0$
(C2)	Codeword #2:	The camp is down path 2 and no one lies:	$00\cdots00$	$11\cdots11$	$00\cdots0$
(C3)	Codeword #3:	The camp is down path 3 and no one lies:	$00\cdots00$	$00\cdots00$	$11\cdots1$
			Responses from the $n+1$ campers sent down path #1	Responses from the $n+1$ campers sent down path #2	Responses from the n campers sent down path #3

a) Find the minimum hamming distance of the counselor's code.

$\boxed{\min H(x,y) = 2n+1}$

b) Is the hamming distance large enough to correct n corrupted bits and therefore deduce the location of the camp?

(Circle one)

Yes   No

c) The counselor has an even number of campers (so n is even) with her and needs to figure out if the camp is down path 1, 2, or 3. When she gets back to the junction she asks everyone what was found down their path. First asking everyone that went down path 1, then path 2, and finally path 3. The responses were recorded with bits and listed in order:

X: $101010\cdots101 \quad 010101\cdots010 \quad 000000\cdots00$

Where is the camp? $H(x,C1)=n \quad H(x,C2)=n+2 \quad H(x,C3)=2n+1$

The camp is down path: *(Circle one of the four options)*

1   2   3   this scenario will never occur

(the code can correct the n liars)

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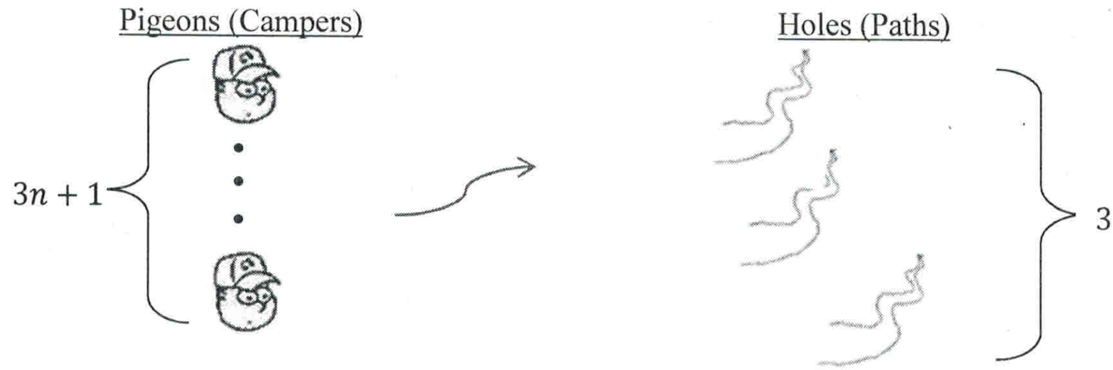
d) Prove that with only $3n + 1$ campers the counsellor cannot solve her problem. Start by using the pigeonhole principle as indicated below. **Hint**: follow the proof showing the campers' problem cannot be solved with only 7 campers. The proof needed here is almost the same except in this proof we require the variable n.



$\therefore$ two paths have a combined total of

$$3n+1 - \left\lceil \frac{3n+1}{3} \right\rceil = 3n+1-(n+1) = 2n \text{ or less campers}$$

Now usign any code we will have:

$$\begin{array}{lll}
\text{path } 1: & 11\cdots1 & 0\cdots0 \quad 0\cdots0 \\
\text{path } 2: & 00\cdots0 & 1\cdots1 \quad 0\cdots0 \\
\text{path } 3: & \underbrace{00\cdots0}_{\substack{n+1 \\ \text{or more}}} & \underbrace{0\cdots0 \quad 1\cdots1}_{2n \text{ or less}}
\end{array}$$

$\therefore H(\text{path } 2, \text{path } 3) \leq 2n$

$\therefore \min H(x,y) \leq 2n$

$\therefore$ any code we design (or distribution of campers) will not be able to correct n errors (liars); because correcting n errors requires $\min H(x,y) = 2n+1$.

Therefore $3n + 1$ campers is not enough to solve the campers' problem

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7. In the problems below the 1st digit in the binary expansion appears in the rightmost position while the 6th digit in the binary expansion appears in the leftmost position:

a) Numbers from 1 to 63 are placed on 6 cards according to the following 7 rules:

1. The 1st digit in the binary expansion of each number on card 1 is a one.
2. The 2nd digit in the binary expansion of each number on card 2 is a one.
3. The 3rd digit in the binary expansion of each number on card 3 is a one.
4. The 4th digit in the binary expansion of each number on card 4 is a one.
5. The 5th digit in the binary expansion of each number on card 5 is a one.
6. The 6th digit in the binary expansion of each number on card 6 is a one.
7. There are 32 numbers on each of the six cards.

Professor Scarlett is looking at a number that appears on cards 2, 3, and 4 but not on cards 1, 5, and 6. What number are you looking at?

0 0 1 1 1 0

8 + 4 + 2 = 14

Professor Scarlett is looking at: 14

b) Numbers from 1 to 63 are placed on 6 cards according to the following 6 rules:

1. The 1st digit in the binary expansion of each number on card 1 is a one.
2. The 2nd digit in the binary expansion of each number on card 2 is a one.
3. The 3rd digit in the binary expansion of each number on card 3 is a one.
4. The 4th digit in the binary expansion of each number on card 4 is a one.
5. The 5th digit in the binary expansion of each number on card 5 is a one.
6. The 6th digit in the binary expansion of each number on card 6 is a one.

Dr. Ecco found a twin prime that appears on cards 2, 3, and 4 but not on cards 1, 5, and 6. Even though the amount of numbers printed on each card is unknown it is still possible to find Dr. Ecco's number. What is his number?

Definition: A prime number p is a *twin prime* if $p + 2$ or $p - 2$ is also a prime number. For example, 19 is a twin prime number since 17 is a prime number; while 23 is a prime number that is not twin since both 21 and 25 are not prime numbers.

? ? 1 1 1 ?

Dr. Ecco is looking at: 14, 15, 30, 31, 46, 47, 62, 63 of which only 31 is a twin prime.

Dr. Ecco is looking at: 31

8. Dr. Ecco is thinking of a number between 0 and 15. You ask him to write the digit in binary form (using 4 digits) and ask the following seven questions.

   Is the 1st (rightmost) digit in the binary expansion of your number a one?

   Is the 2nd digit in the binary expansion of your number a one?

   Is the 3rd digit in the binary expansion of your number a one?

   Is the 4th digit in the binary expansion of your number a one?

   Is there an odd number of ones in the 2nd, 3rd, and 4th positions?

   Is there an odd number of ones in the 1st, 3rd, and 4th positions?

   Is there an odd number of ones in the 1st, 2nd, and 4th positions?

   His answers to your questions in the order you asked them are:

   No, Yes, Yes, No, Yes, No, No.

   However, he may have lied once. What is his number?

4.	3.	2.	1.	5.	6.	7.
a	a	a		a		
b	b		b		b	
c		c	c			c
0	1	1	0	1	0	0

a's − 3
b's − 1
c's − 1

there is an error under a, b, c.

1110
$= 8 + 4 + 2$

Dr. Ecco's number is: 14

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Bonus Problem. Consider a game of Nim with 3 rows. Prove that there is **at most** one move on row 1 that follows a winning strategy. In this game, is it possible to have a turn where there are **exactly** two ways to follow a winning strategy? Prove that the answer is no, or provide an example to prove the answer is yes.

Solution:
Suppose the game is in the state:

Row 1: a_1
Row 2: a_2
Row 3: a_3

and has a nim-sum of s.

<u>Part One</u>

Show that there is **at most** one move on row 1 that follows a winning strategy:

If $s = 0$ there are no winning strategies to follow, so we look at the case when $s \neq 0$. Suppose changing row one to a_1' follows a winning strategy, then:

$$0 = a_1' \oplus a_2 \oplus a_3 \qquad \& \qquad s = a_1 \oplus a_2 \oplus a_3$$

$$\Rightarrow 0 \oplus s = (a_1' \oplus a_2 \oplus a_3) \oplus (a_1 \oplus a_2 \oplus a_3)$$

$$\Rightarrow s = a_1' \oplus a_1$$

$$\Rightarrow a_1' = s \oplus a_1$$

Therefore there is exactly one choice for a_1'. Therefore there is at most one move on row 1 that follows a winning strategy. Note: if $s \oplus a_1 \geq a_1$ it is not possible to move so row one is $s \oplus a_1$. In part 2 we will discuss when it is possible to change a_1 to $s \oplus a_1$.

Part Two

Prove that it is not possible to have a turn where there are **exactly** two ways to follow a winning strategy

Let 2^k be the largest power of 2 in s. Start by showing the move made in part a) is valid only when 2^k is in a_1. Suppose that 2^k is not in a_1, then:

$$
\begin{aligned}
a_1 \oplus s &= (\overbrace{2^{l_a} + \cdots + 2^{l_1}}^{\text{The powers larger than } 2^k \text{ do not cancel}} \overset{\text{Skipping } 2^k}{\qquad} + \overbrace{2^{s_b} + \cdots + 2^{s_1}}^{\text{The powers smaller than } 2^k \text{ might cancel}}) \oplus (2^k + 2^{t_c} + \cdots + 2^{t_1}) \\
&\geq 2^{l_a} + \cdots + 2^{l_1} + 2^k \\
&= 2^{l_a} + \cdots + 2^{l_1} + 2^{k-1} + 2^{k-2} + \cdots + 2^0 + 1 \\
&> 2^{l_a} + \cdots + 2^{l_1} + 2^{k-1} + 2^{k-2} + \cdots + 2^0 \\
&\geq 2^{l_a} + \cdots + 2^{l_1} + 2^{s_b} + \cdots + 2^{s_1} \\
&= a_1
\end{aligned}
$$

Therefore changing a_1 to $a_1 \oplus s$ is not a valid move when 2^k is not in a_1. On the other hand, (by lecture notes) it is a valid move when 2^k is in a_1. The same argument can be used on the other two rows. Since 2^k did not cancel when calculating $a_1 \oplus a_2 \oplus a_3$ it must have appeared an odd number of times in a_1, a_2, a_3. This means there are exactly 1 or 3 ways to change the nim-sum to zero. In conclusion there cannot be exactly two ways to follow a winning strategy