Ch. 20 – CI for a Population Mean

- common situation is that σ is unknown, so the sample data must estimate it.

Recall $Z = \dfrac{\bar{Y} - \mu}{\sigma / \sqrt{n}}$. We now have $Z \neq \dfrac{\bar{Y} - \mu}{s / \sqrt{n}} = t$, where t is a diff. standardized variable.

The value of s may not be all that close to σ, especially when n is small. Consequently, there is extra variability and the distribution of t is more spread out than the z curve.

t-distributions:

As with normal curves, there exists a family of t-curves. The normal distribution has 2 parameters: μ and σ; the t-distribution has a single parameter: *degrees of freedom* (*df*).
Range of t: similar to range of z; range of *df*: 1, 2, 3, …, ∞

Properties of the t-distribution:

1. The t-curve, with any fixed *df*, is bell-shaped and centered at 0 (just like z-curve).
2. Each t-curve is more spread out than the z-curve. $t_{\alpha/2} > z_{\alpha/2}$
3. As *df* increases, the spread of corresponding t-curve decreases.
4. As *df* increases, the corresponding sequence of t-curves approaches the z-curve.

Let $y_1, y_2, \ldots, y_n$ constitute a random sample from a normal population distribution. Then the probability distribution of the standardized variable

$$t = \frac{\bar{Y} - \mu}{s / \sqrt{n}} \sim t_{df} \qquad (df = n - 1)$$

One-Sample t Confidence Interval:

Assumptions:

1. Sample is random.
2. The population distribution is normal *OR* the sample size is large ($n \geq 30$).
3. σ, the population standard deviation, is unknown.

$$\bar{y} \pm (t_{\alpha/2,\, n-1}) \times \left(\frac{s}{\sqrt{n}} \right)$$

Table T gives critical values appropriate for the most common confidence levels.

		Sample size (n)	
		$n \geq 30$	$n < 30$
Normality	(approx.) normal	OK	OK
	NOT normal	OK	DON'T use t!

Ex20.1) A naïve astrophysicist wants to determine the mean radius of all of Jupiter's numerous moons. Only taking the 4 largest moons, he discovers a sample mean of 2103.75 km with a standard deviation of 495 km. Construct and interpret a 90% confidence interval for the population mean radius of Jupiter's moons. (Note: changes were made in class to this example to make the assumptions hold.)

Choosing n for a population mean:

Consider the CI as $\bar{y} \pm ME$, where $ME = t_{\alpha/2}\dfrac{s}{\sqrt{n}} \approx z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}$

Solving for n here gives

$$n \approx \left(\frac{z_{\alpha/2}\sigma}{ME}\right)^2$$

Round up n to next integer. Replace σ by a prior estimate or, for a population that is not too skewed, use $\sigma \approx$ (range)/6 since 99.7% of the data is approximately within 3s of the sample mean. (This situation is rare.)

Ex20.2) You wish to estimate the mean cost of textbooks so that the margin of error is within \$7 of the true population mean. Most textbook prices are between \$100 and \$220. What is the minimum sample size required to be 99% confident in achieving the mentioned level of accuracy? Assume textbook cost is normally distributed.

Significance Tests about Means

As with intervals, it is rare to know σ, but for the test statistic t, the appropriate P-value can be found under the t-curve with $df = n - 1$.

Assumptions:

1. Sample is random.
2. The population distribution is normal *OR* the sample size is large ($n \geq 30$).
3. σ, the population standard deviation, is unknown.

Ex20.3) A website claims the average length of an episode is 21.104 min. A bored student takes a random sample of 7 episodes and found their mean length to be 21.16 min with a standard deviation of 0.20 min. It is known that the length of all episodes has an approximate normal distribution. Test the above claim using both approaches and $\alpha = 0.01$ for the SLA approach.

The sample is random. n is small, yet the population is normally distributed. Consequently, we may use a t-distribution; specifically, t_6.

H_0: $\mu = 21.104$ $\quad\quad$ H_A: $\mu \neq 21.104$

$$t = \frac{\overline{Y} - \mu_0}{s/\sqrt{n}} = \frac{21.16 - 21.104}{0.20/\sqrt{7}} = 0.741$$

Using the row of $df = 6$ in the t-table,

$0 \quad < t = 0.741 < 1.440$

$0.5 > \quad\quad\quad\quad > 0.10$

$1 \quad > \quad P\text{-value} > 0.20$ $\quad\quad$ → P-value range is (0.20, 1)

In fact, software produces an exact P-value of 0.487 (try it in StatCrunch).

Using the "judgment approach", the P-value is between 0.1 and 1, so we have weak evidence against H_0 and do not reject H_0.

Using the "significance level approach", $\alpha = 0.01 < P$-value, so we do not reject H_0.

Either way, we have insufficient evidence that the claim is invalid.

(Try one-tailed tests with this example as well. Another example using both two-tailed and one-tailed tests was discussed in class. More quick examples to understand the t-distribution were also done in class.)