MATH 222

Assignment 4

- Reccurence Relations
- Induction
- Strong Induction
- Tromino Problems

Group information (you may work by yourself, in a pair, or as a trio)

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1. Let a_n be the number of words (strings) of length n that can be made using the digits {0,1,2,3} with an odd number of twos. Find a recurrence relation for a_n and solve the recurrence. The first few values of a_n are calculated for you:

$a_1 = 1$ 2

$a_2 = 6$ 20 21 23 02 12 32

020 021 023 002 012 032
120 121 123 102 112 132
320 321 323 302 312 332
200 201 203 210 211 213 230 231 233 222

$a_3 = 28$

CASE 1. The word starts with 0, 1, 3.

0 odd # of 2's
1 odd # of 2's
3 odd # of 2's

$\Big\} \; 3a_{n-1}$

n-1, n

CASE 2: The word start with 2.

2 Even # of 2's. $\rightarrow 4^{n-1} - a_{n-1}$

n-1, n

$a_n = 3a_{n-1} + 4^{n-1} - a_{n-1}$

$= 2a_{n-1} + 4^{n-1} = 2a_{n-1} + 2^{2n-2}$

$a_1 = 1$

$a_n = 2a_{n-1} + 2^{2n-2}$

The recurrence relation is:

$a_1 = 1$

$a_n = 2a_{n-1} + 2^{2n-2}$

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solve:

$$a_2 = 2(1) + 2^{2(2)-2} = 2\cdot 1 + 2^2$$

$$a_3 = 2^2\cdot 1 + 2^3 + 2^4.$$

$$a_4 = 2^3\cdot 1 + 2^4 + 2^5 + 2^6.$$

$$\vdots$$

$$a_n = 2^{n-1} + 2^n + 2^{n+1} \cdots + 2^{2n-2}$$

$$+ 2^0 + 2^1 + \cdots + 2^{n-1}$$

$$-(2^0 + 2^1 + \cdots + 2^{n-1})$$

$$= (2^{2n-1} - 1) - (2^{n-1} - 1)$$

$$= 2^{2n-1} - 2^{n-1}$$

$$= 2^n \cdot 2^{n-1} - 2^{n-1}$$

$$= 2^{n-1}(2^n - 1).$$

A solution to the recurrence relation is given by:

$a_n = 2^{n-1}(2^n - 1)$

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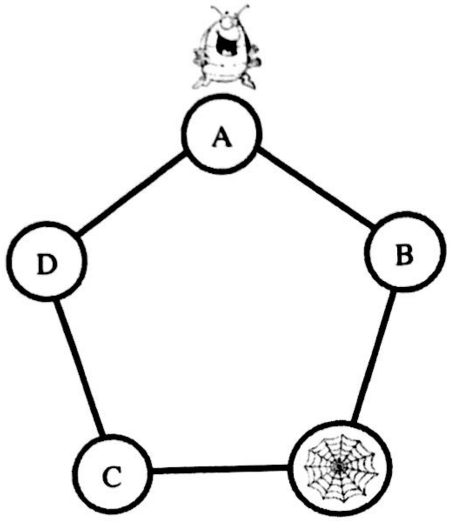
[redacted]

2. A bug starts at vertex A of the pentagon below and each minute travels to an adjacent vertex. There is a spider web on one vertex; if the bug moves there it gets stuck. Let a_n be the number of different ways the bug can travel from vertex A to the spider web after n minutes. Find a recurrence relation for a_n; you do not have to solve this recurrence. Hint: consider two cases the 1st move is to D or B, in each case write down what a_n is in terms of a_{n-1} or a_{n-2}. The first four values of a_n are: $a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 2$.

$a_1 = 0$ $a_2 = 1$

$a_3 = 1$ $a_4 = 2$

$a_5 = 3$



The bug's 1st move is to B or D.

to B: it must return if $n>2$

So, there are a_{n-2} ways the bug can reach the web for the first time

to D: There are a_{n-1} ways the bug can reach the web for the first time

$\therefore a_n = a_{n-2} + a_{n-1}$

$= a_{n-1} + a_{n-2} \quad (n>2)$

The recurrence relation is:

$a_1 = 0$

$a_2 = 1$

$a_n = a_{n-1} + a_{n-2} \quad n \geq 3.$

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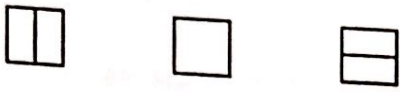
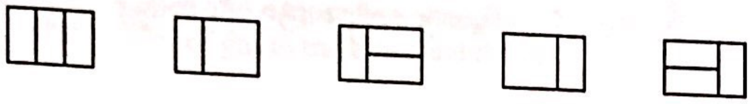
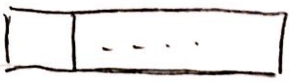
3. Let a_n be the number of ways can you tile a $2 \times n$ rectangle using dominoes and square tetrominoes. Dominoes are of size 2×1 and tetrominoes are of size 2×2. Find a recurrence relation for a_n; you do not have to solve this recurrence. The first few values of a_n are calculated for you.

$a_1 = 1$ 

$a_2 = 3$ 

$a_3 = 5$ 

 There are a_{n-1} ways.

 There are a_{n-2} ways.

 There are a_{n-2} ways.

$$a_n = a_{n-1} + a_{n-2} + a_{n-2}$$
$$= a_{n-1} + 2a_{n-2} \qquad n \geq 3.$$

The recurrence relation is:

$a_1 = 1$

$a_2 = 3$

$a_n = a_{n-1} + 2a_{n-2} \quad n \geq 3.$

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4. Conjecture a formula for the sum of the first n Fibonacci numbers with odd indices and prove your formula works by using mathematical induction. That is, find and prove a formula for

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1}$$

Note:

$$F_1 = 1,$$
$$F_2 = 1,$$
$$F_n = F_{n-1} + F_{n-2}.$$

P_n:

$F_1=1$ $F_2=1$ $F_3=2$ $F_4=3$ $F_5=5$ $F_6=8$ $F_7=13$ $F_8=21$ $F_9=34$ $F_{10}=55$.

$n=1$ $1=1 \to F_1 = F_2$.

$n=2$ $1+2=3 \to F_4 = F_{2\times2} = F_1+F_3$

$n=3$ $1+2+5=8 \to F_6 = F_{2\times3} = F_1+F_3+F_5$

$n=4$ $1+2+5+13=21 \to F_8 = F_{2\times4} = F_1+F_3+F_5+F_7$

$n=5$ $1+2+5+13+34=55 \to F_{10} = F_{2\times5} = F_1+F_3+F_5+F_7+F_9$.

BS: P_n: $F_1+F_3+\cdots+F_{2n-1} = F_{2n}$ ~~with odd indices~~.

$F_1 = F_{2\cdot1} = F_2 = 1$. $\to \therefore P_1$ is true.

IS: show $P_n \to P_{n+1}$

$F_1+F_3+\cdots+F_{2(n+1)-1}$

$= F_1+F_3+\cdots+F_{2n-1}+F_{2n+1}$

$= F_{2n}+F_{2n+1}$

$= F_{2n+2}$

$= F_{2(n+1)}$ $\therefore P_{n+1}$ is true.

$$\boxed{F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}}$$

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5. An equal number of open and closed gas stations are distributed an equal distance apart around a ring road of a city. It is not known in which pattern the open and closed gas stations are distributed (they could alternate open and then closed all the way around or there could be many open gas stations in a row followed by many closed gas stations in a row). Only 1 liter can be added from any open station; however we do not have to worry about a maximum volume limit for our gas tank. The gas tank is empty to start out with.

Define the following statement:

P_n : There is an open gas station to start at which allows one loop of travel clockwise around the city if

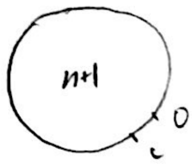
- there are n open and n closed gas stations.
- it takes n liters of gas to travel around the city

Show that P_n is true for $n \geq 1$.

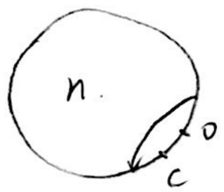
BS/: 

$\rightarrow$ P_1 is true

IS: show $P_n \rightarrow P_{n+1}$.



there are n+1 open and n+1 closed gas stations



It takes $n+1$ liters of gas to travel around. as it take n liters when there are n open and n closed gas stations.

$\therefore$ P_{n+1} is true.

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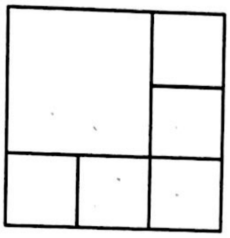
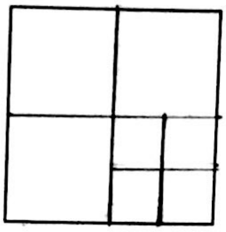
6. Consider the following statement.

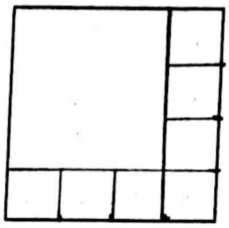
P_n: A square can be dissected into n smaller squares .
(The smaller squares do not all have to be the same size)

Professor Scarlett attempted strong inductive proof to show P_n is true for $n \geq 6$. Unfortunately, Professor Scarlett forgot two of the base cases. Complete the missing base cases to complete the proof.

<u>Base Cases:</u> Show: P_6, P_7, P_8 are true.





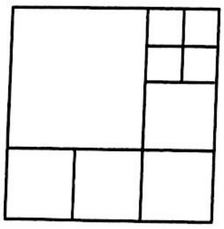


Therefore P_6, P_7, and P_8 are true.

<u>Inductive Step:</u>

Show: $P_n \Rightarrow P_{n+3}$.

Notice that if a square can be dissected into n smaller squares, then it can be dissected into $n+3$ smaller squares. This is done by taking one of the existing squares and dissecting it into four squares of equal size. For example, using the dissection of 6 squares, the following is a dissection into 9 squares:



Therefore if P_n is true then P_{n+3} is also true.

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7. For this question define the statement for natural numbers:

P_n: Any deficient $n \times n$ board can be tiled with trominoes.

The objective of this question to prove that P_n is true for all natural numbers other than 5 and multiples of three. That is:

$$P_n \iff n \neq 5 \text{ and } n \not\equiv 0 \pmod 3.$$

In our lecture notes we have shown that P_n is true when:

$$n = 2,8$$
$$n \equiv 1 \pmod 3$$

and that P_n is false when:

$$n = 5$$

To finish the proof of this theorem complete parts a) and b).

a) Show that if $n \equiv 0 \pmod 3$ then P_n is false. Hint: Show that the area of a deficient $n \times n$ board is different from the area formed by trominoes.

if $n \equiv 0 \pmod 3$

$n^2 - 1 \equiv 0^2 - 1 \equiv 2 \not\equiv 0 \mod 3$.

$\therefore$ if $n \equiv 0 \pmod 3$, then P_n is false.

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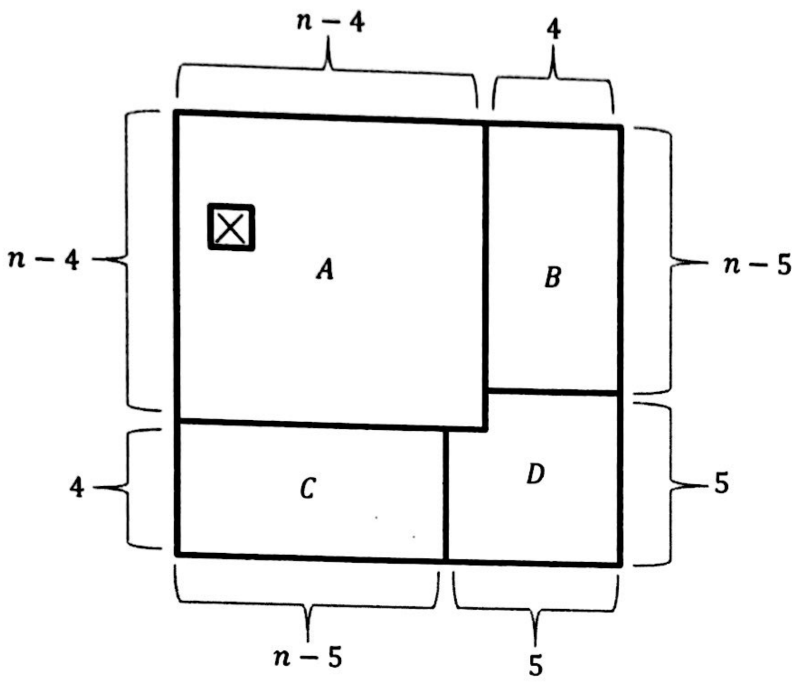
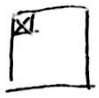
b) Show that when $n \equiv 2 \pmod 3$ and $n \geq 11$ then P_n is true by explaining how each of the four sections below can be tiled. Note: by rotational symmetry the deficiency is always in section A.



$\because n \equiv 2 \pmod 3$

A: $n-4 \equiv 2-4 \equiv 1 \pmod 3$ $\rightarrow \therefore$ A can be tiled by removing one square.

B&C: $n-5 \equiv 2-5 \equiv 0 \pmod 3$
$4 \equiv 0 \pmod 2$ $\Big\}$ $\therefore$ B.&C can be tiled.

D: is a 5×5 board missing the left top corner square.  $\rightarrow$ D can be tiled.

$\therefore$ if $n \equiv 2 \pmod 3$ and $n \geq 11$, P_n is true

2

You are given $\left\lceil \frac{3^n}{2} \right\rceil + 1$ coins one of which is counterfeit. It is not known whether the counterfeit is slightly ~~heavier or~~ lighter than the others. Additional information is known about two of the coins. One of them is not the counterfeit. The other one is either known not to be too light or known not to be too heavy. Show that it is possible to find the counterfeit in n weighings ($n \geq 1$) while identifying if the counterfeit is too light or too heavy.

$$\left\lceil \frac{3^n}{2} \right\rceil + 1 = \frac{3^n+1}{2} + 1 = \frac{3^n+3}{2} = 3 \cdot \frac{3^{n-1}+1}{2}$$

the normal coins would be $\frac{3^n+1}{2} - 1$, and two other coins.

one is not counterfeit and the other one is G or S.

BS/ $\frac{3^1+3}{2} = 3$ $\therefore$ 3 coins

case 1: the scale tips to the left, coin 1 is too heavy

case 2: the scale tips to the right, coin 1 is too light

case 3: the scale keeps balance. Coin S is too heavy or too light.

IS/ show $P_n \to P_{n+1}$

Case 1: the scale tips to the left. as left is too heavy or right is too light. Pair each of the coins on the left hand side of the scale with a coin on the right scale. Each pair will form a theoretical coin made of two original coins. Take 2 coins that were left off the scale and form one more theoretical coin. In total we have

$$\frac{\frac{3^n+1}{2} + \frac{3^n+1}{2}}{2} + 1 = \frac{2\,\frac{3^n+1}{2}}{2} + 1 = \frac{3^n+1}{2} + 1 = \frac{3^n+3}{2}$$

One of them is known not to be too light since it was made by a ~~coin~~ coin from left and coin G. The rest coins $\frac{3^n+3}{2} - 2$ could be slightly heavier or lighter than they should be, since they were each made with a coin from left and a coin from right ~~Therefore~~ Therefore the counterfeit

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can be found amoung these coins in n more ~~we~~ weighings by P_n.

In total n+1 weighings were used so P_{n+1} is true.

CASE 2: The scale tips to the right. The same as CASE 1.

CASE 3: Balance. In this case, counterfeit is amoung the $\frac{3^n+1}{2}$ coins left off of the scale. The counterfeit can be found amoung these coins together with coin G by P_n. In total n+1 weighing were used.

So. P_{n+1} is true.