MATH 222

Assignment 1

- Logic Problems
- Coin Weighing Problems
- Modular Arithmetic
- The Pigeonhole Principle

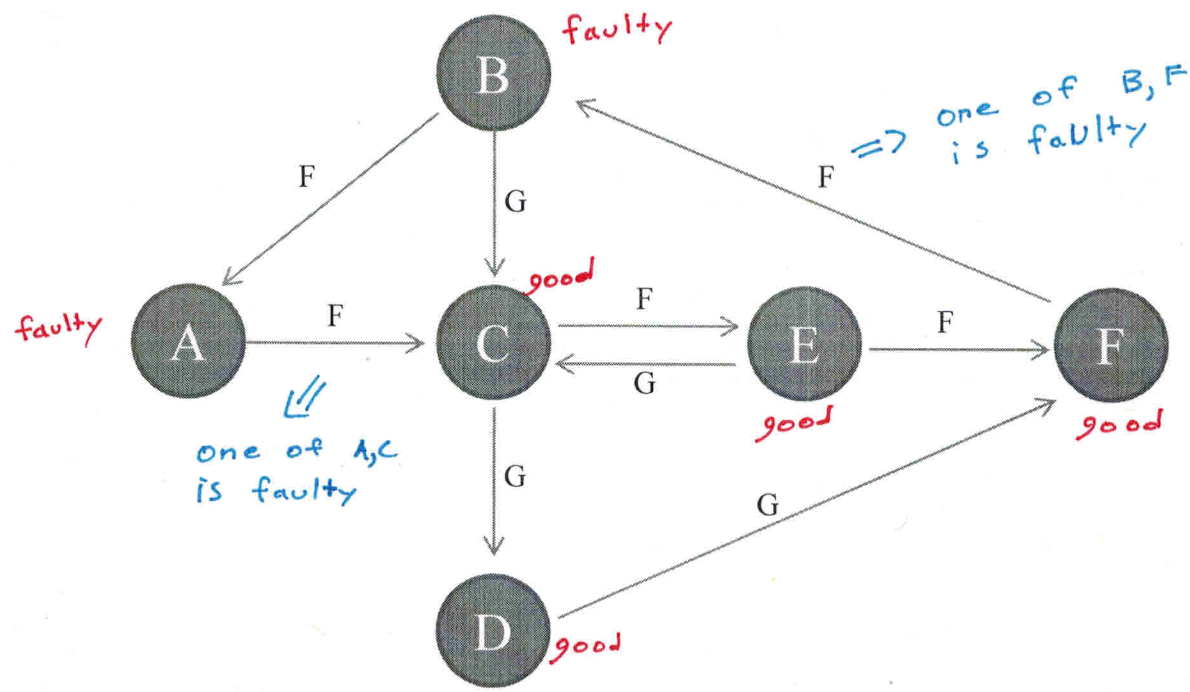
Group information (you may work by yourself, in a pair, or as a trio)

First Name	Last Name	ID
Doctor	Ecco	(mod 222)

1. Six robots are pointing fingers at one another.

A:	C is faulty.
B:	Either A is faulty or C is good.
C:	Either D is good or E is faulty.
D:	F is good.
E:	Either C is good or F is faulty.
F:	B is faulty.

A robot which is good will only tell the truth, but a robot which is faulty could be either be lying or telling the truth. What is the maximum number of robots that can be good? Hint: the following diagram will be helpful:



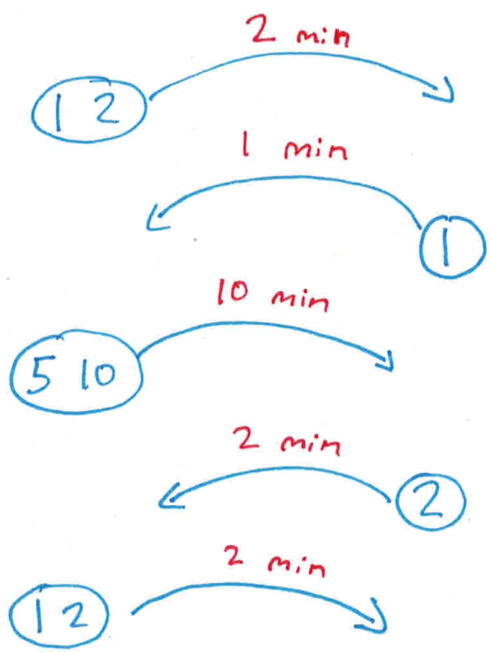
There can not be 5 Good robots since 2 of A, B, C, F are faulty.

There can be 4 good robots : A - faulty
B - faulty
C - good
D - good
E - good
F - good

The maximum number of good robots is: $\boxed{4}$

2. Four people are being pursued by a menacing beast. It is nighttime, and they need to cross a bridge to reach safety. It is pitch black, and only two can cross at once. They need to carry a lamp to light their way. The first person can cross the bridge in 10 minutes, the second in 5 minutes, the third in 2 minutes, and the fourth in 1 minute. If two cross together, the couple is only as fast as the slowest person. (That is, a fast person can't carry a slower person to save time, for example. If the 10-minute person and the 1-minute person cross the bridge together, it will take them 10 minutes.) The person or couple crossing the bridge needs the lamp for the entire crossing and the lamp must be carried back and forth across the bridge (no throwing, etc.) If they don't all get completely across in less than 18 ½ minutes, whoever is on the bridge or left behind will be eaten by the beast. Is it possible for all of them to get across?

Hint: you can save time by sending the 5 minute person with the 10 minute person.



Circle one:

Yes, everyone can get across in 17 minutes. No.

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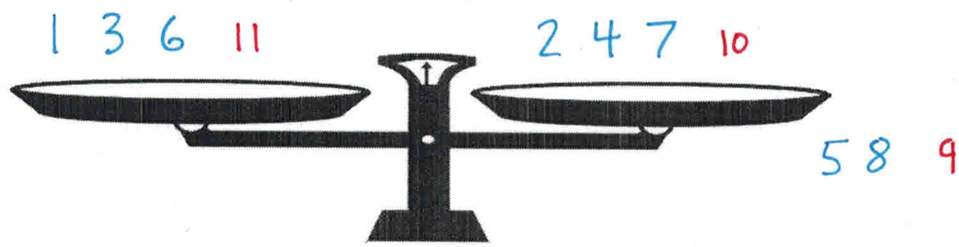
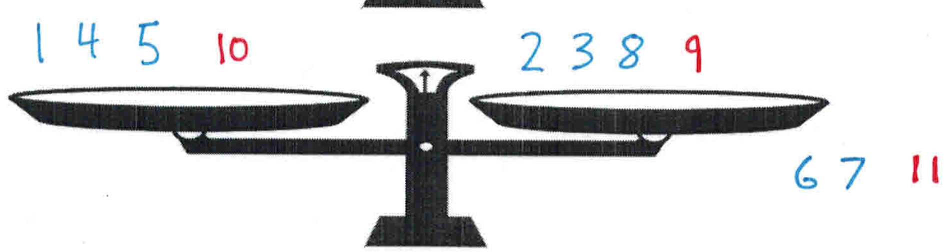
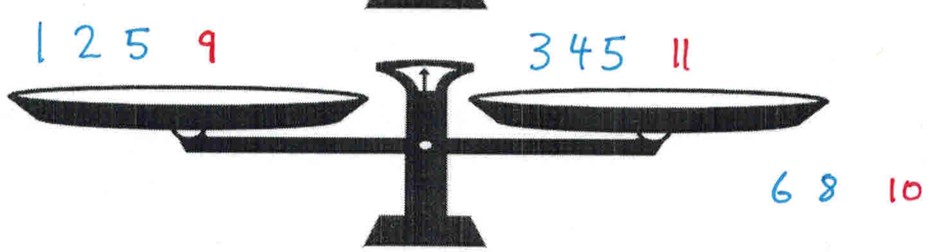
3. There are 11 coins, all identical except that one is counterfeit and is a different weight than the others. It is not known whether the counterfeit is heavier or lighter. Find a non-adaptive solution that can find the counterfeit in three weighings using a pan balance. You do not have to identify if the counterfeit coin is too heavy or too light. In your solution let "R" represent the scale tipping to the right, let "L" represent the scale tipping to the left, and let "B" represent when the scale is balanced. Note: this problem may take some time playing around with different coin arrangements on the 3 scales. Don't worry exam problems will always be based upon practice questions from the course; there won't be any brand new problems on the exam.







Counterfeit Coin	How the Scale Responds
1	L L L or R R R
2	R R L or L L R
3	L R R or R L L
4	R L R or L R L
5	B L L or B R R
6	L B B or R B B
7	R B R or L B L
8	B R B or B L B
9	B R L or B L R
10	R L B or L R B
11	L B R or R B L

add 3 coins to a solution with 8 coins.

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4. Baskerhound herded Evangeline and her friends into a room containing twelve caskets. Baskerhound stated that sixty minutes after he leaves the room, eleven of the twelve caskets will disintegrate and release a poisonous gas. The other casket contains the key to the door. They must find the key before the hour is up to escape.

Baskerhound gave a clue: The key is in casket number $222^{222^{222}}$. This is possible because the caskets are counted in a funny way:

Casket 1 was 1, Casket 2 was 2, and so on, until Casket 12 which was 12. Then the kidnapper started counting backwards: Casket 11 was 13, Casket 10 was 14, and so on, until Casket 1 which was 23. Then the count reversed once more, and Casket 2 was 24, Casket 3 was 25, etc. Casket are counted in a loop of $2 \cdot 12 - 2 = 22$

Find the casket which contains the key. Hint: $222^{222^{222}} = 222^{(222^{222})}$.

$222^{222^{222}}$

$\equiv 2^{222^{222}}$

$\equiv 2^{(222^{222} \bmod 10)}$

$\equiv 2^{(2^{222} \bmod 10)}$

$\equiv 2^{(2^{(222 \bmod 4)} \bmod 10)}$

$\equiv 2^{(2^{2} \bmod 10)}$

$\equiv 2^{4}$

$\equiv 16 \pmod{22}$

mod 22

$2^0 \equiv 1$
$2^1 \equiv 2 \equiv 2^{11}$
$2^2 \equiv 4$
$2^3 \equiv 8$
$2^4 \equiv 16$
$2^5 \equiv 10$
$2^6 \equiv -2$
$2^7 \equiv -4$
$2^8 \equiv -8$
$2^9 \equiv -16$
$2^{10} \equiv 12$
$\therefore$ loop of 10

mod 10

$2^0 \equiv 1$
$2^1 \equiv 2 \equiv 2^5$
$2^2 \equiv 4$
$2^3 \equiv 8$
$2^4 \equiv 6$
$\therefore$ loop of 4

The casket which contains the key is: *(Circle one)*											
1	2	3	4	5	6	7	8	9	10	11	12
	22	21	20	19	18	17	16	15	14	13	

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5. A mathematician's mother was illiterate and never recorded his birthdate. Although the mathematician learned some details from his mother:

- He was born in April.
- "Ascension Thursday" was 8 days after he was born.
- The year he was born "Ascension Thursday" was on the second Thursday of May.
- He knows he was born in one of the four years: 1776, 1777, 1778, or 1779.

born April 30th on a Wednesday

What is the birthday of this mathematician? Who is this mathematician? To solve this problem, you will need to count the number of leap years. The three leap year rules are:

Rule 1: Every 4^{th} year is a leap year: 2020, 2024, 2028, ...
Rule 2: Every 100^{th} year is not a leap year: 2100, 2200, 2300, ... *(an exception to rule 1)*
Rule 3: Every 400^{th} year is a leap year: 2000, 2400, 2800, ... *(an exception to rule 2)*

# of leap days from 2016 to 1776 :
April 30 April 30

$$\underbrace{\frac{2016-1776}{4}}_{\text{Rule 1}} - \underbrace{\frac{2000-1700}{100}}_{\text{Rule 2}} + \underbrace{\frac{2000-1600}{400}}_{\text{Rule 3}} = 58$$

April 30th 2016

# of years

leap days

$$6 - 365(\underbrace{2016-1776}_{\text{\# of years}}) - 58$$

$$\equiv 6 - (1)\ 240 - 58$$

$$\equiv 6 - 2 - 2 \equiv 2$$

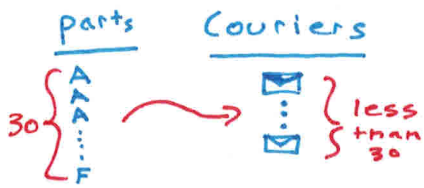
∴ April 30th, 1776 is a Tuesday
April 30th, 1777 is a Wednesday
April 30th, 1778 is a Thursday
April 30th, 1779 is a Friday

This Mathematician's name is: $\boxed{\text{Karl Friedrick Gauss}}$

and he was born: $\boxed{\text{Wednesday, April 30th 1777}}$

6. Suppose a general wants to send a message to his troops behind enemy lines. He has several couriers that he can use to carry the message; however, up to 4 of them may be caught by the enemy. To ensure the message gets across, the general makes 5 copies of the message. To ensure the enemy doesn't intercept the entire message, the general cuts each of the 5 messages into 6 pieces and sends each courier with some combination of the different pieces. What is the minimum number of couriers the general needs to send so the message gets across and the enemy cannot intercept the entire message?

Assume the minimum is less than 30:



$\therefore$ One courier holds 2 parts (no courier can have more than 2) say AB.

If A, B are together then C, D, E, F are apart.
$\therefore$ we need at least 21 couriers:

AB
C _
C _
C _
C _
C _
5 more
D
D
D
D
D
E
E
E
E
E
F
F
F
F
F

with 21 couriers there is space for only 5 more parts, but we need to send 8 more parts (AAAA BBBB)
$\therefore$ we need at least 23 couriers.

There is a solution with 23 couriers:

A B	
A B	
A B	
C A	E
C A	E
C B	E
C B	E
C	E
D	F
D	F
D	F
D	F
D	F

The minimum number of couriers needed is: $\boxed{23}$

7. Prove that it is possible to find two powers of two which differ by a multiple of 2020.

Hint: a more general version of this problem was given in our lecture on the pigeonhole principle. You may use the result of the more general problem to solve this problem. We do not need to find specific integers i, j, m such that $2^i - 2^j = m2020$, we only need to show that it is possible to find such integers.

Pick 2020 + 1 different powers of 2:

$$2^0, 2^1, \ldots, 2^{2020}$$

By Ex 4 of Lecture 3 we can find $2^i - 2^j = m\,2020$ is this list.

Therefore it is possible to find integers i, j, m such that $2^i - 2^j = m2020$.

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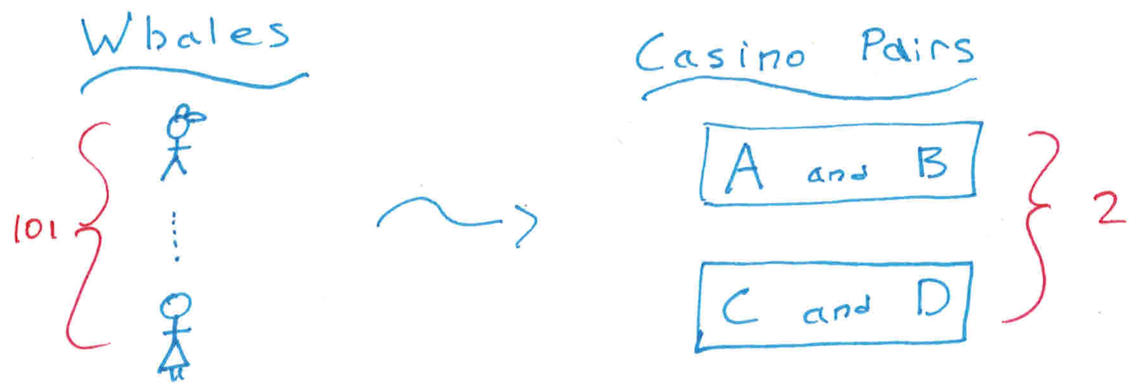
8. In Las Vegas on a particular weekend night there are 101 whales (a nickname given to high stakes gamblers) who randomly pick four different casinos to gamble at.

   a) Prove that there is a pair of casinos that will host a combined total of 51 or more whales. Hint: use the pigeonhole principle with 101 pigeons and 2 holes.



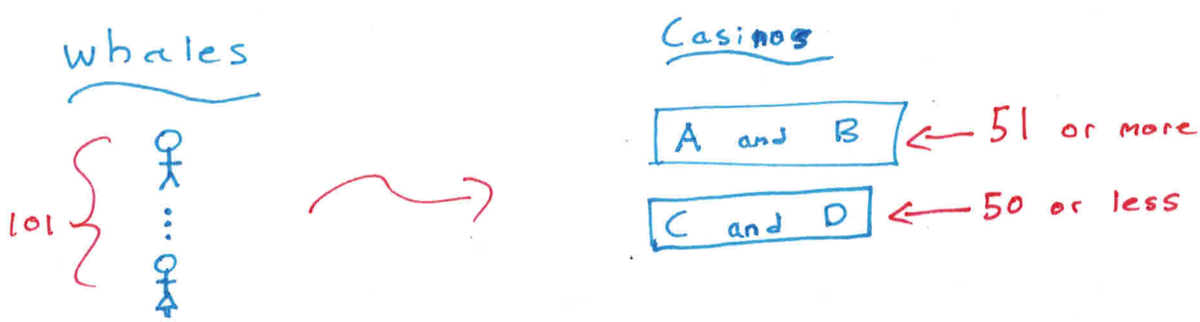
$$\left\lceil \frac{101}{2} \right\rceil = 51$$

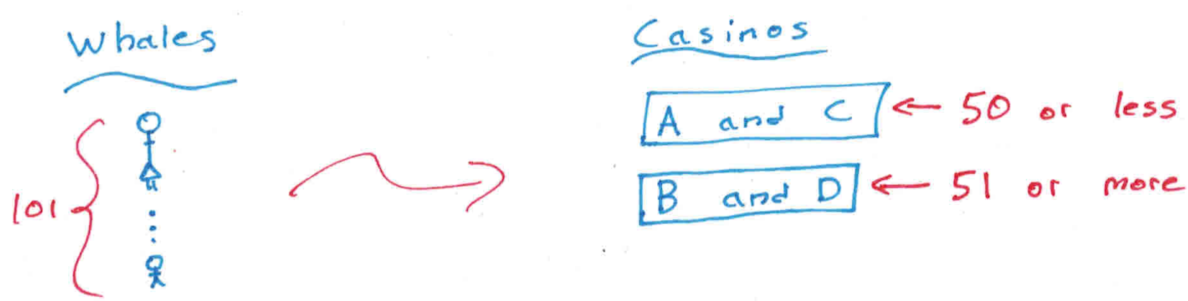
Therefore there is a pair of casinos that will host a combined total of at least 51 whales.

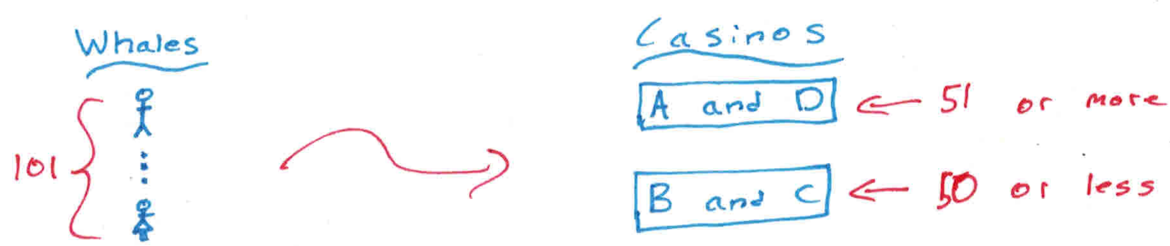
$\frac{\quad}{2}$

b) Out of the four casinos we can make a total of 6 pairs of casinos. How many pairs out of the 6 pairs will host a total of 51 or more whales? Your answer **must** include one of the quantifiers "at least", "exactly" or "at most". Hint: notice that there are 3 different ways of picking the holes in part a).

There are 3 ways to do part a) in each way there is exactly on pair that has 51 or more whales:







Therefore:

at least exactly at most *(Circle one)*

1 2 3 4 5 6 *(Circle one)*

pair(s) will host a total of 51 or more whales.

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Bonus Problem

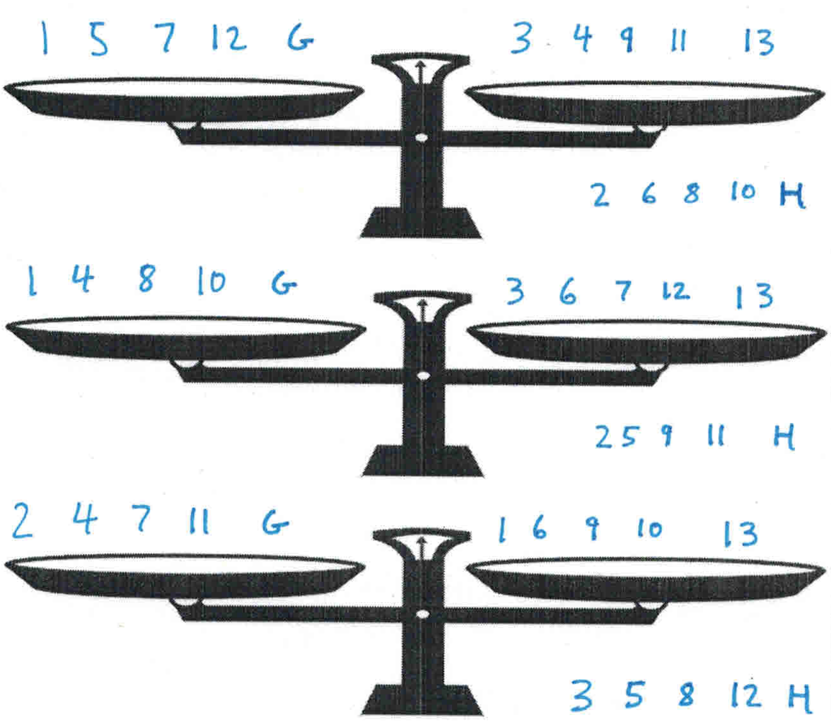
Suddenly a stern knock on Dr. Ecco's door, and in walked Michael Monetary. The man worked inside the government and was in charge of a coin factory. He stated his problem: "one of my coin making machines is not working properly. It produces coins in batches of 15; exactly one out of every batch has an incorrect weight. The first coin from a batch is always perfect, the second coin, if of incorrect weight, is too heavy, and otherwise the bad coin could end up being heavier or lighter than it's supposed to be. I would like my workers to quickly find and remove the bad coin from each batch using a regular pan balance in 3 weighings. Anymore weighings and I will surely lose my job over production losses. Finally, I would like to keep track of the bad coins being too heavy or to light; this statistic could help fix the machine."

After a few minutes Ecco handed Mr. Monetary a scrap of paper and said "here is your solution in three weighings, it's a good thing the first coin from each batch is always perfect."

How did Ecco solve Mr. Monetary's problem?

Your solution must be non-adaptive and correctly identify the bad coin as too light or too heavy. Also:

- Label G as the (first) coin that is always perfect.
- Label H as the (second) coin that could be too heavy but not too light.
- Label the other coins: 1,2,3,…,13



Heavy vs light

Counterfeit	How the Scale Responds
1	LLR vs RRL
2	BBL vs BBR
3	RRB vs LLB
4	RLL vs LRR
5	LBB vs RBB
6	BRR vs BLL
7	LRL vs RLR
8	BLB vs BRB
9	RBR vs LBL
10	BLR vs BRL
11	RBL vs LBR
12	LRB vs RLB
13	RRR vs LLL
H	BBB

2