114-1 電工實驗（通信專題）

Up/Down-Sampling, Filtering

Chia-Yi Yeh (葉佳宜)
ycyyeh@ntu.edu.tw

Department of Electrical Engineering
National Taiwan University

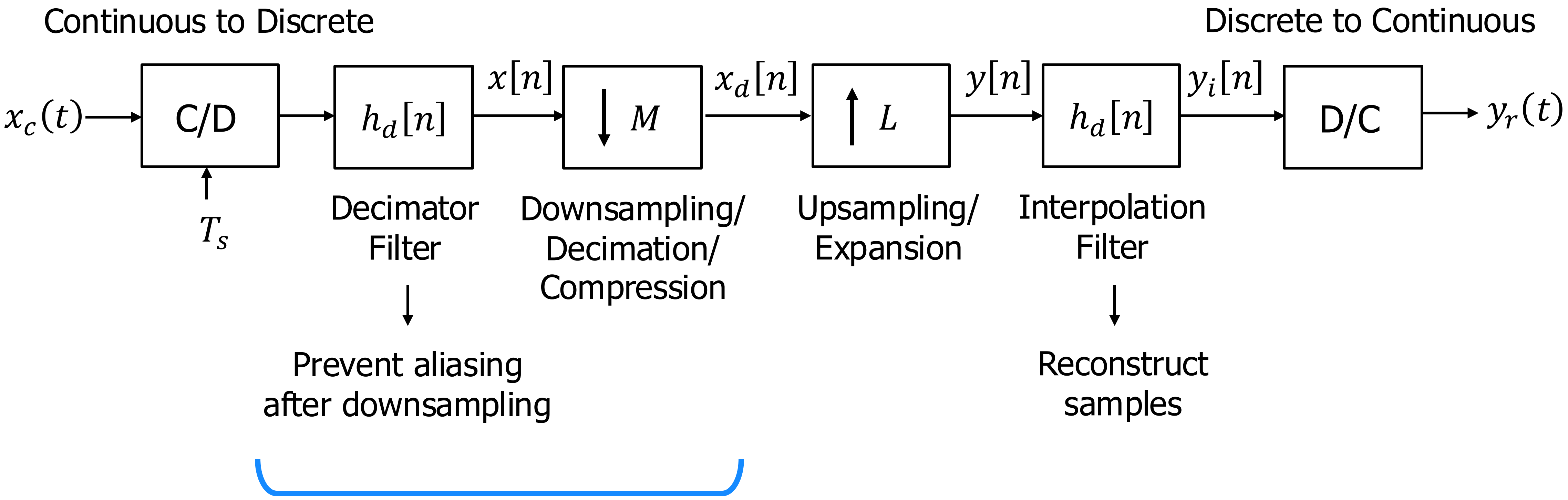
How was lab 1?

Up/Down-Sampling

Overview

Why change sampling frequency?

Compatibility between different devices



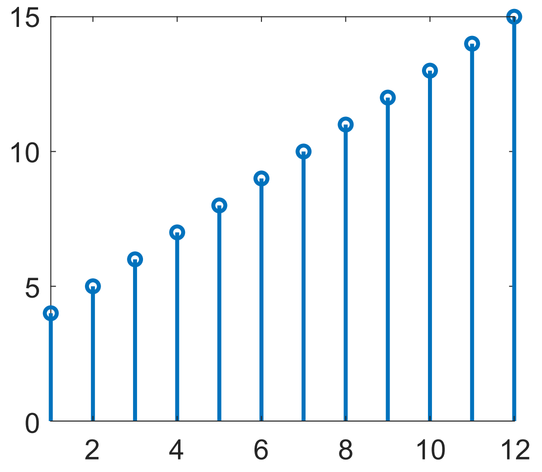
Downsampling

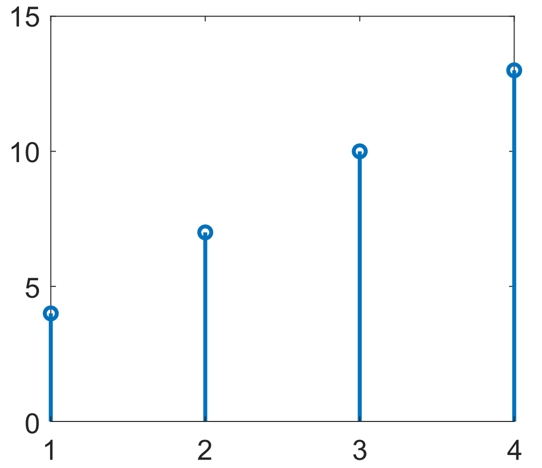
$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow y[n] = x[Mn]$$

$$f_s \rightarrow \boxed{\downarrow M} \rightarrow \left\lceil \frac{f_s}{M} \right\rceil$$

$$f_s \rightarrow \boxed{\downarrow M} \rightarrow \frac{f_s}{M}$$

If f_s is a multiple of M

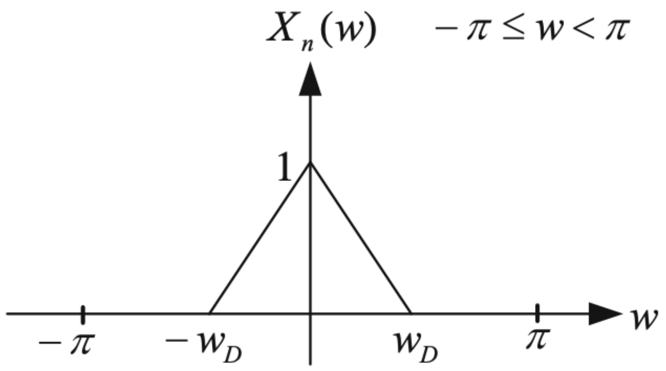


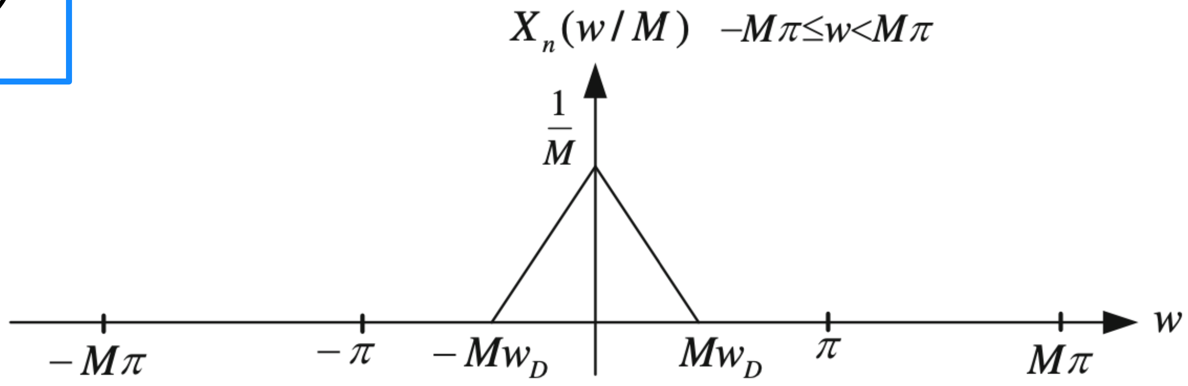


Downsampling – Frequency Domain

$$y[n] = x[Mn]$$

$$\boxed{Y_n(w) = \frac{1}{M}\sum_{k=0}^{M-1} X_n\left(\frac{w \pm k2\pi}{M}\right)}$$





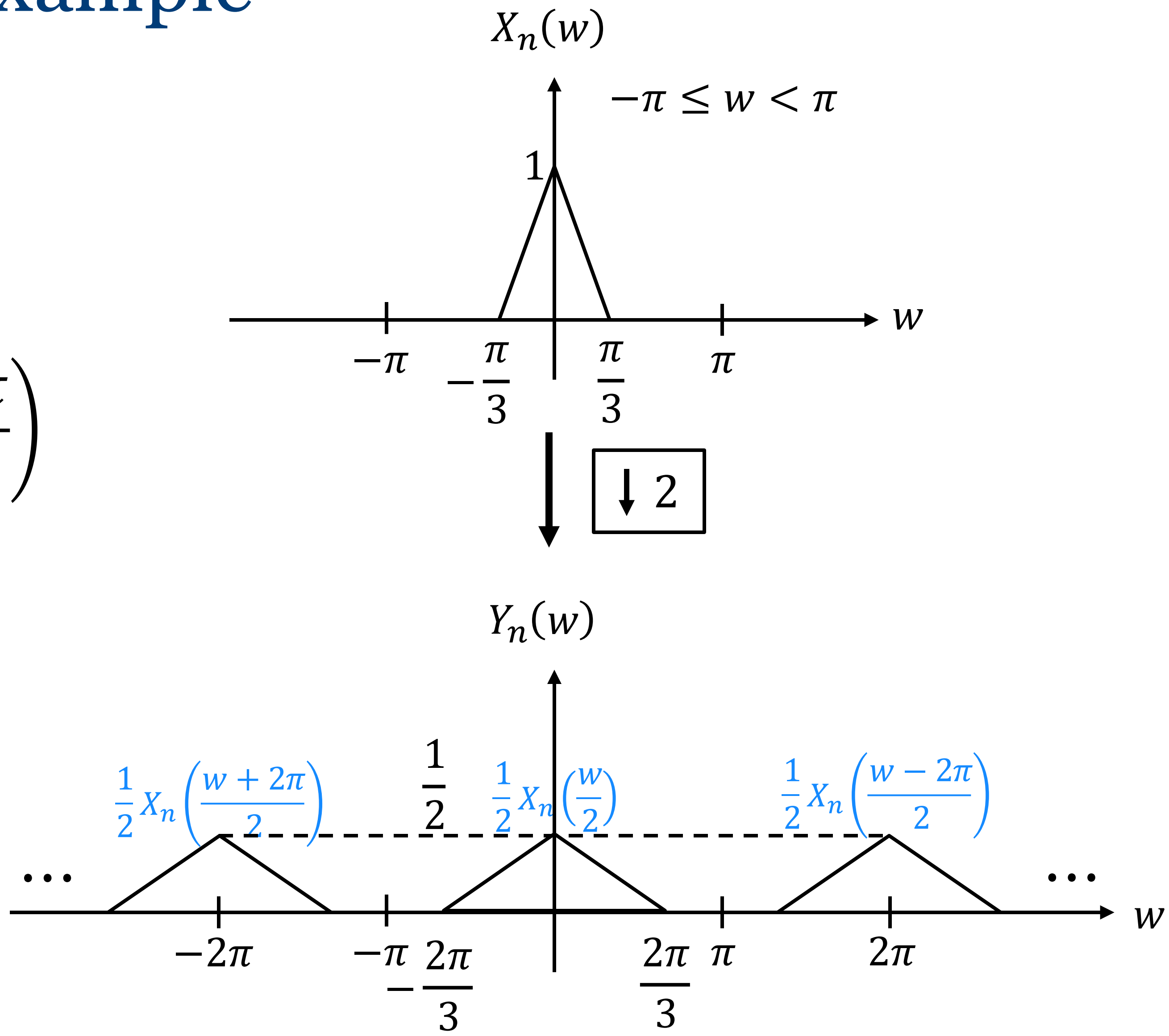
Signal compressed in the time domain ⇒ Spectrum expanded in the frequency domain

Downsampling - Example

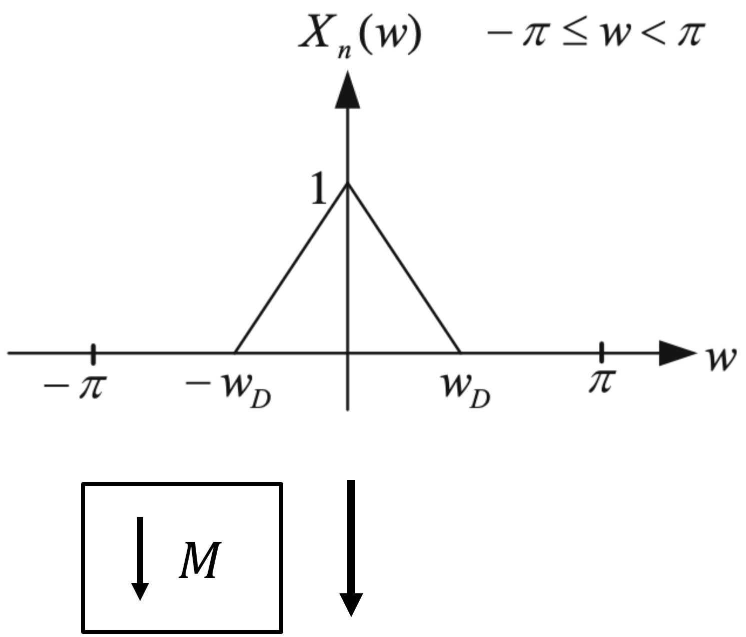
Example:

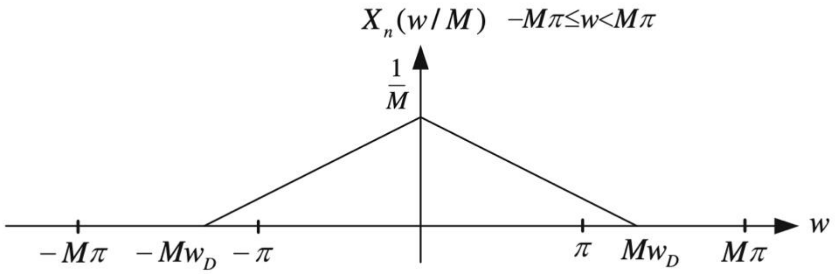
$$y[n] = x[2n]$$

$$Y_n(w) = \frac{1}{2}\sum_{k=0}^{M-1} X_n\left(\frac{w \pm k2\pi}{2}\right)$$

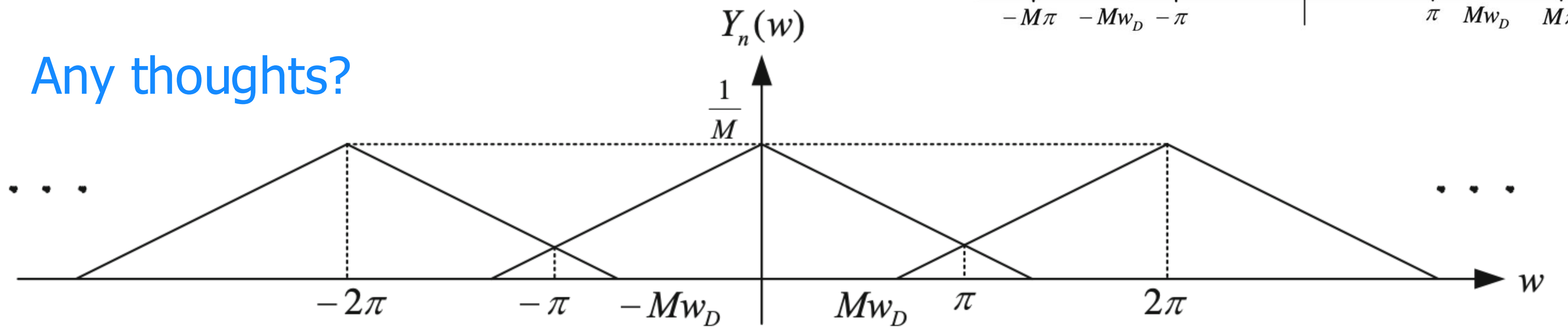


Aliasing in Downsampling

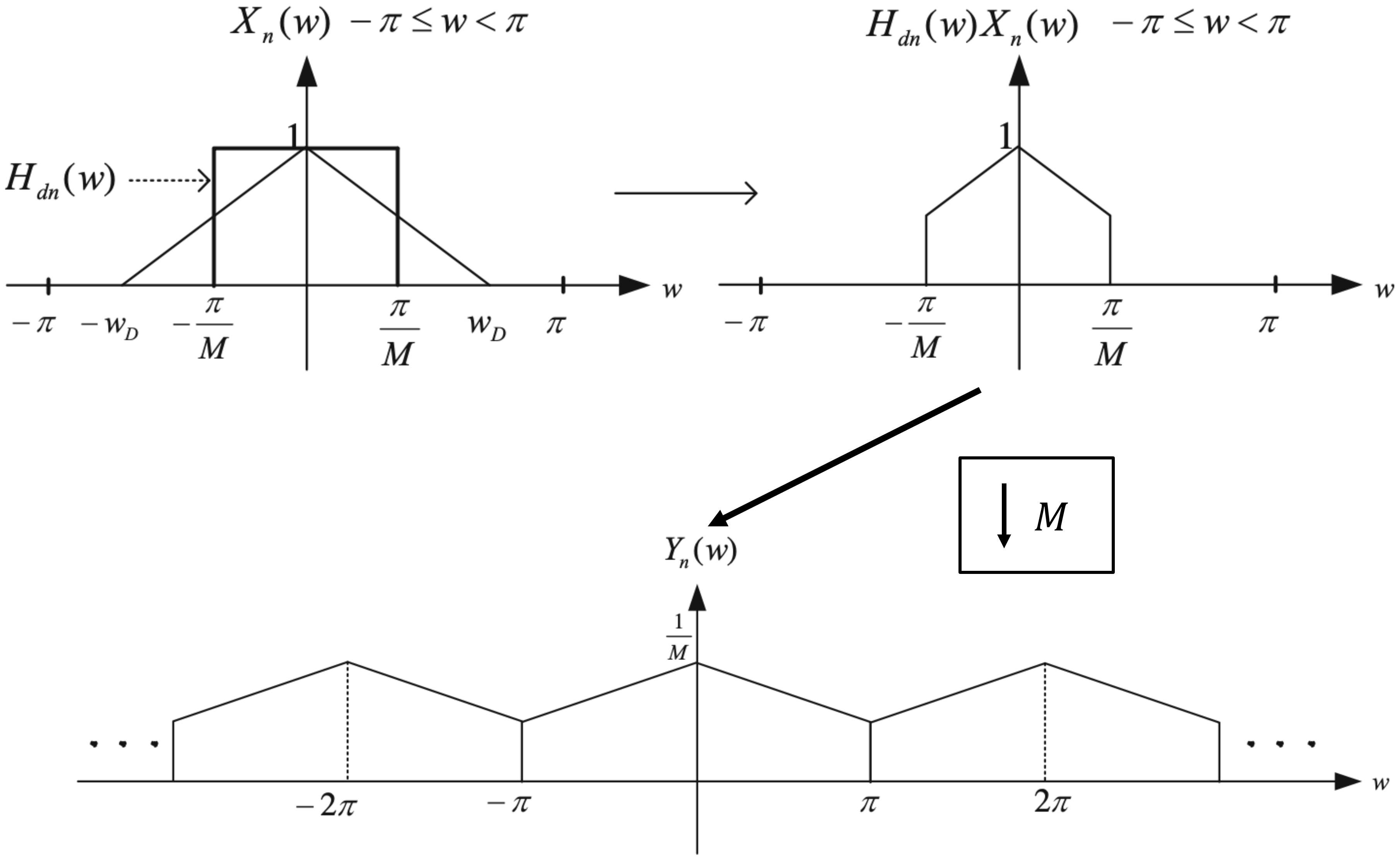




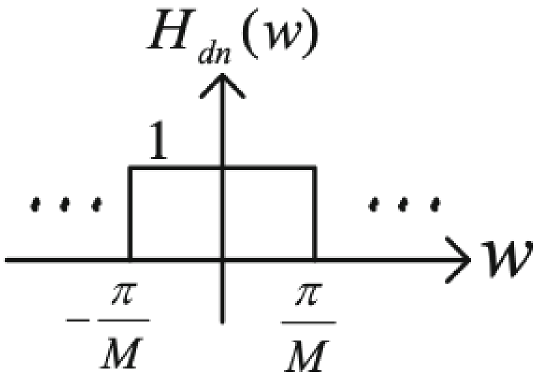
Any thoughts?



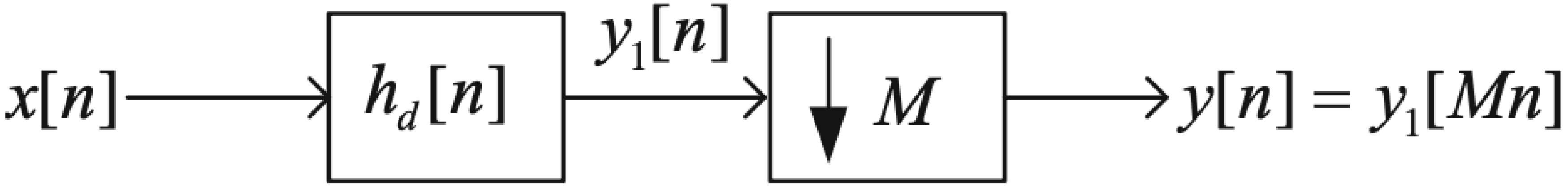
Prevent Aliasing: Lowpass filter before downsampling



Decimator System

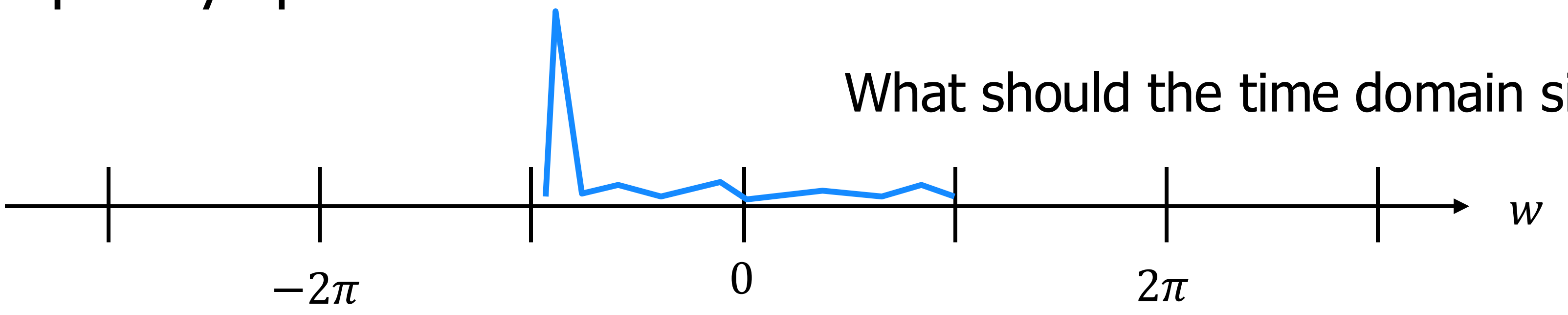


$$H_{dn}(w) = \begin{cases} 1 & \text{if} \quad |w| < \frac{\pi}{M} \\ 0 & \text{if} \quad \frac{\pi}{M} < |w| < \pi. \end{cases}$$

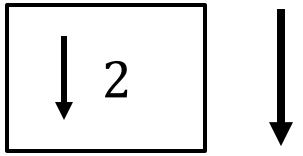


Thought Experiment: Aliasing

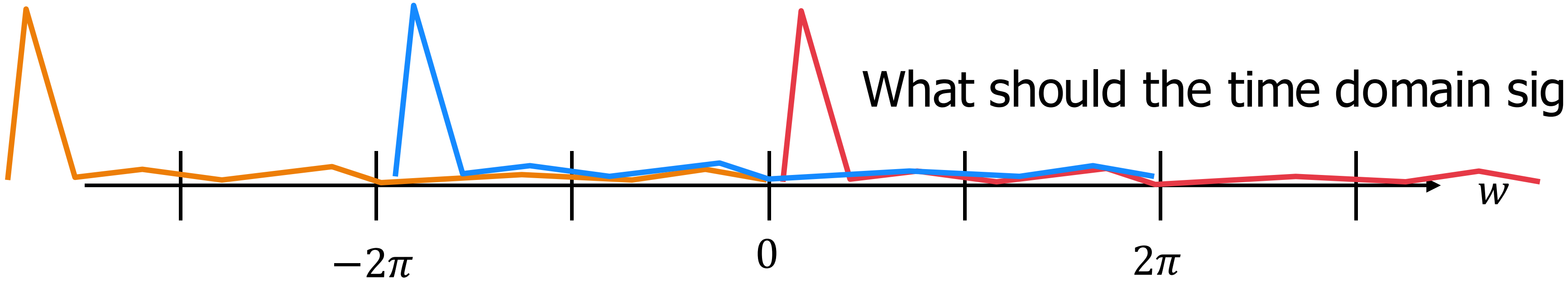
Assume frequency spectrum:



What should the time domain signal look like?

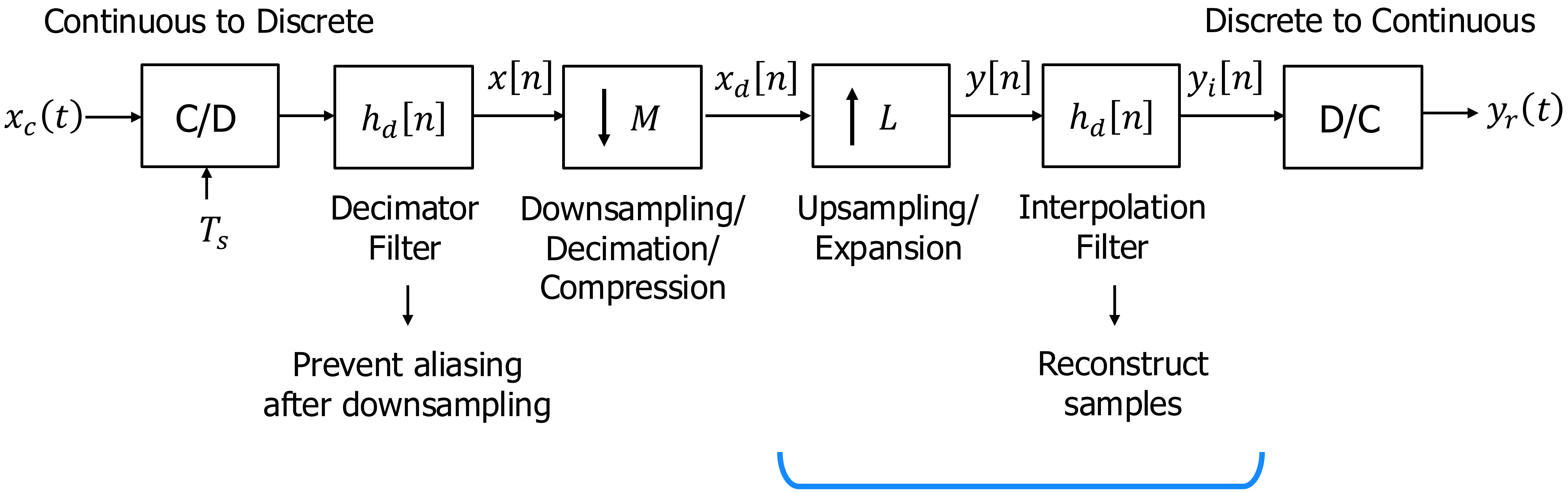


What should the frequency spectrum look like?



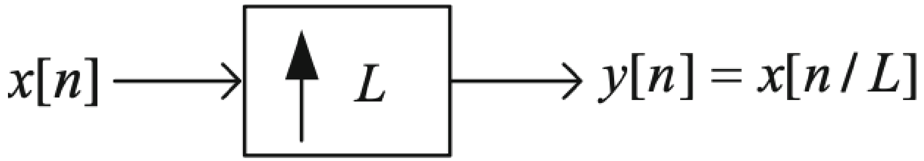
What should the time domain signal look like?

Overview

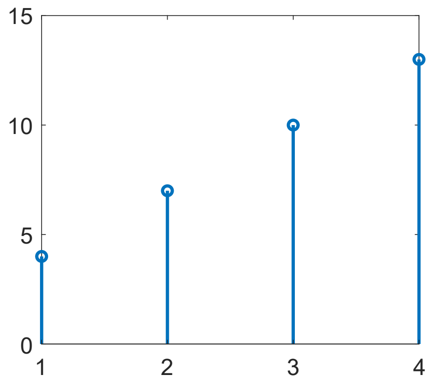

Continuous to Discrete
Discrete to Continuous
$x_c(t)$
C/D
T_s
$h_d[n]$
$x[n]$
$\downarrow M$
$x_d[n]$
$\uparrow L$
$y[n]$
$h_d[n]$
$y_i[n]$
D/C
$y_r(t)$
Decimator Filter
Downsampling/ Decimation/ Compression
Upsampling/ Expansion
Interpolation Filter
Prevent aliasing after downsampling
Reconstruct samples

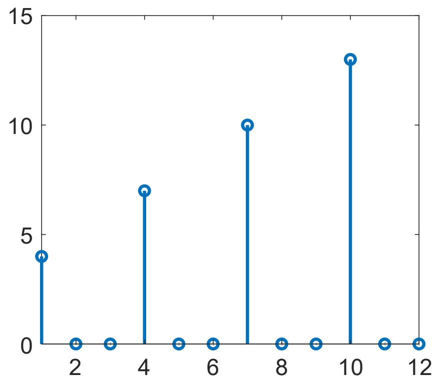
Upsampling

1. Signal expansion: insert zeros



$$y[n] = \begin{cases} x\left[\frac{n}{L}\right] & n = 0, \pm L, \pm 2L, \ldots \\ 0 & otherwise. \end{cases}$$



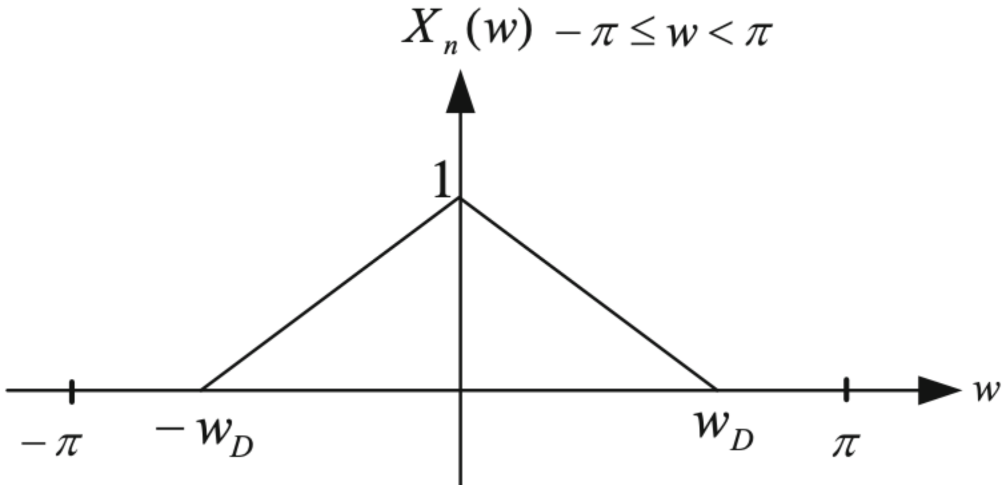


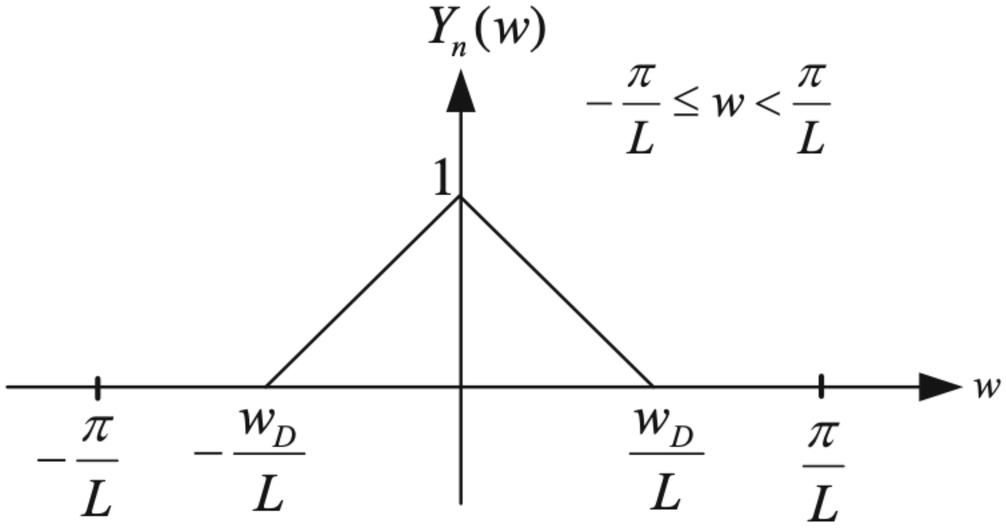
2. Interpolation

Upsampling – Frequency Domain

$$y[n] = \begin{cases} x\left[\frac{n}{L}\right] & n = 0, \pm L, \pm 2L, \ldots \\ 0 & otherwise. \end{cases}$$

$$\boxed{Y_n(w) = X_n(Lw)}$$





Signal expanded in the time domain ⇒ Spectrum compressed in the frequency domain

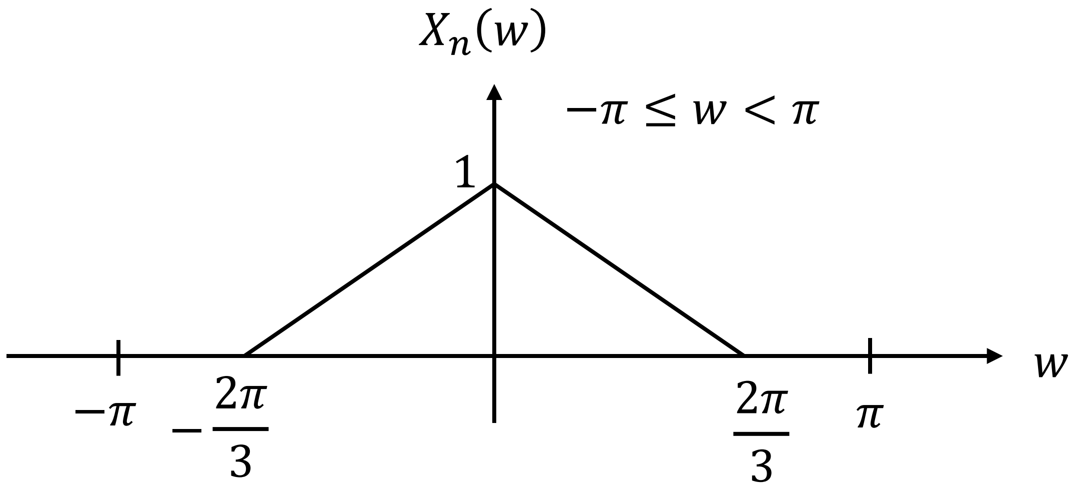
Signal compressed in the time domain ⇒ Spectrum expanded in the frequency domain

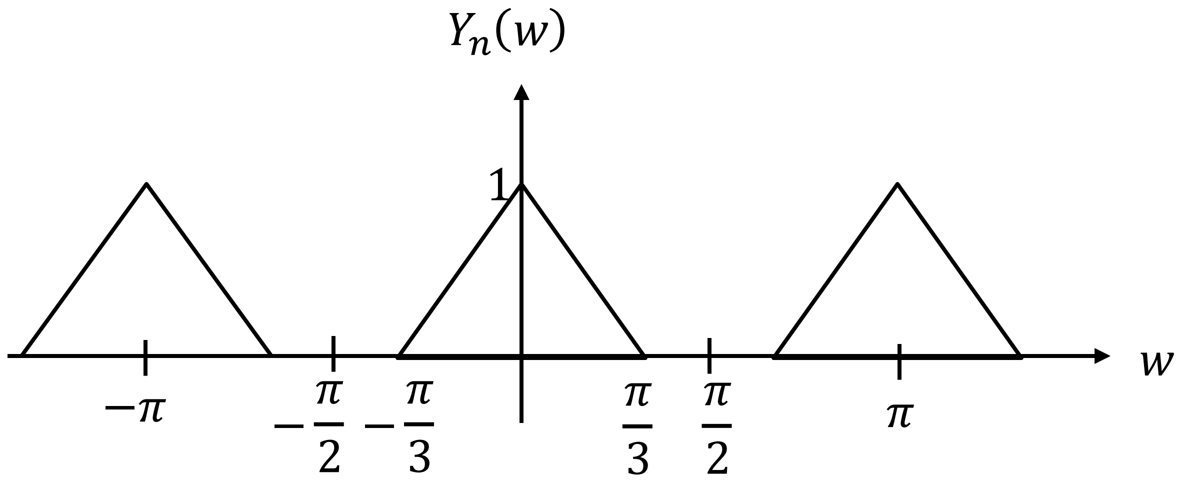
Upsampling - Example

Example:

$$y[n] = x\left[\frac{n}{2}\right]$$

$$Y_n(w) = X_n(2w)$$



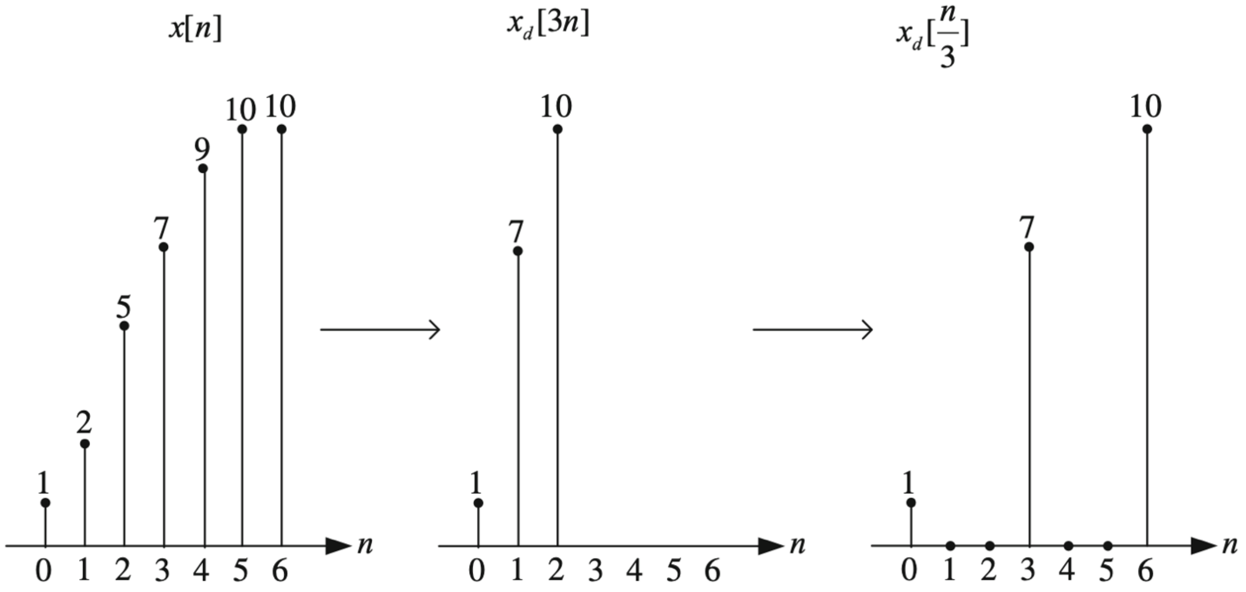


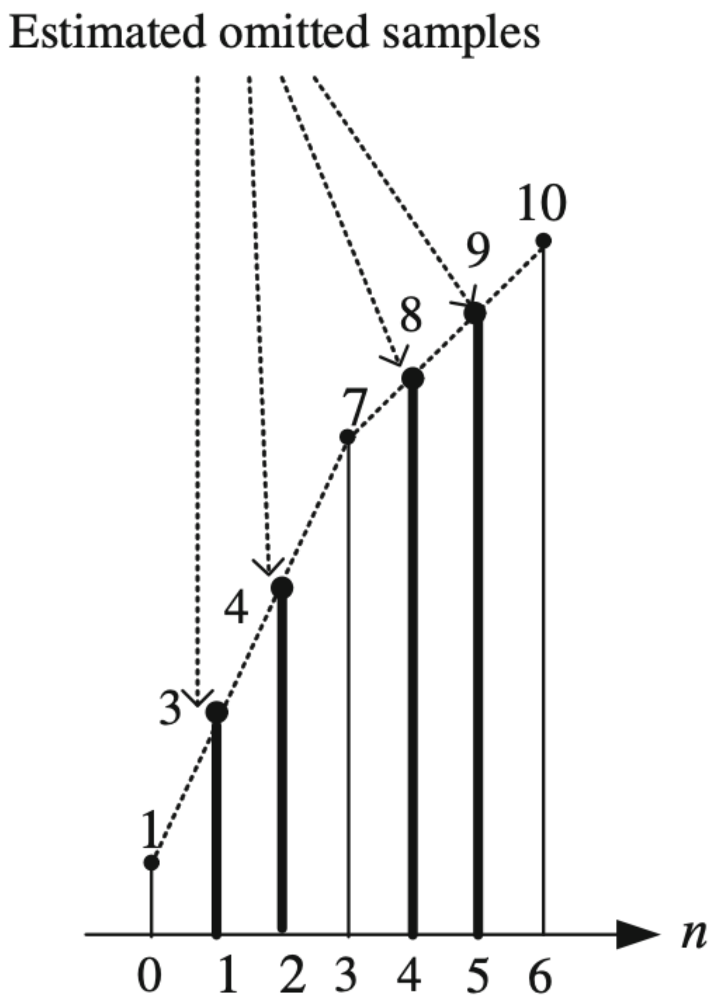
Interpolation – the Naïve Method

$$x[n] = [\underbrace{1}_{n=0}\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10]$$

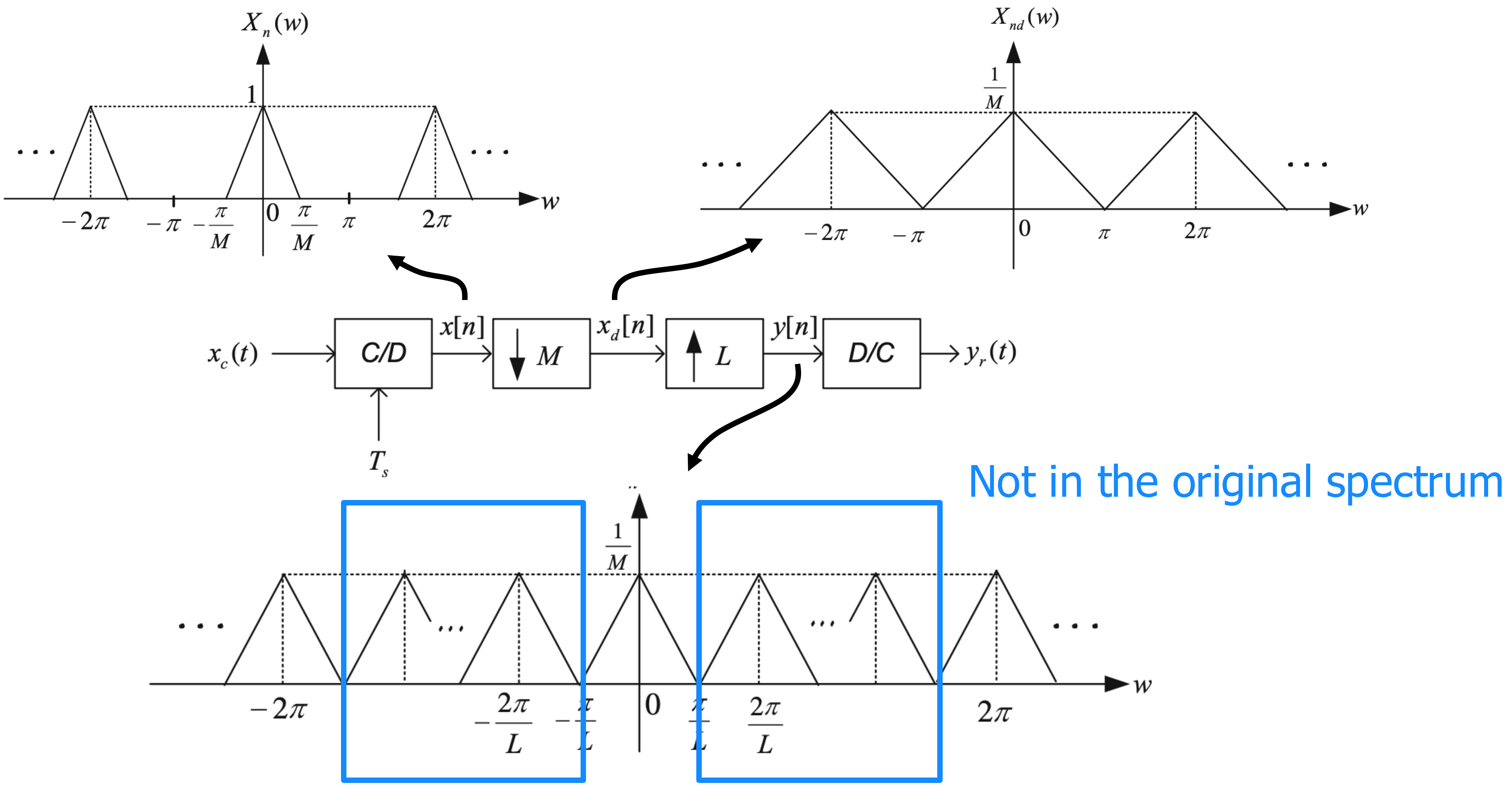
$$y[n] = [\underbrace{1}_{n=0}\ 0\ 3\ 0\ 5\ 0\ 7\ 0\ 9\ 0]$$

Replace 0's by the estimated values of the omitted samples



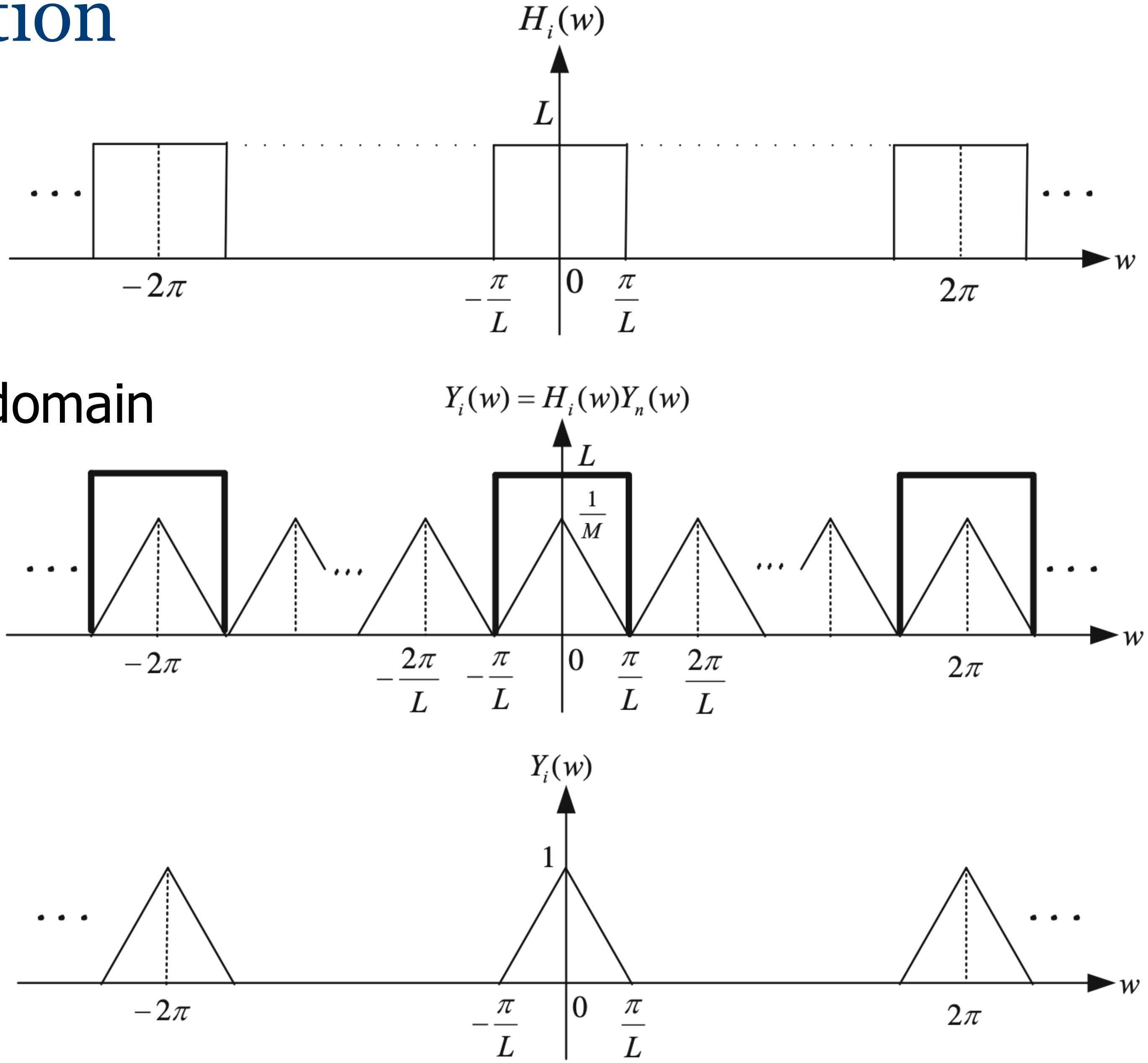


From Frequency Domain Perspective

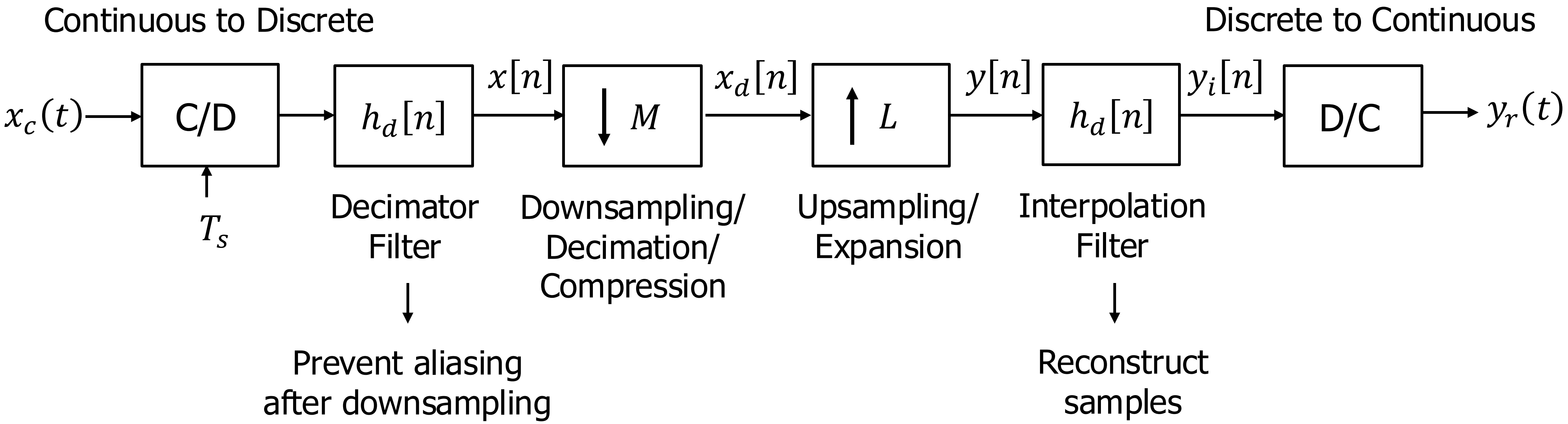


Signal Reconstruction

Filter from the frequency domain



Summary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Sampling Rate Conversion by Non-integer Factors

<u>Example</u>

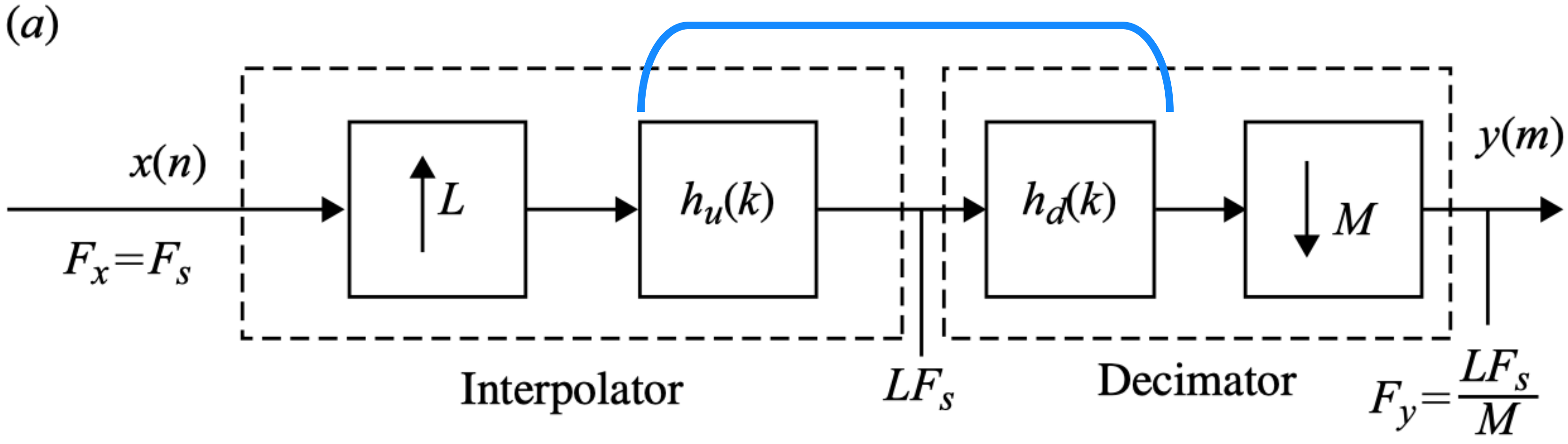
Compact disk (CD) at a rate of 44.1 kHz
↓
Digital audio tape at 48 kHz

$\Longrightarrow$ Upsample by a factor of $\frac{48}{44.1} = \frac{160}{147}$

Sampling rate conversion by a rational number $\frac{L}{M}$

- First interpolate by a factor of L
- Then Decimate by a factor of M

Do we need both filters?



References

- Madhow, Upamanyu. *Introduction to communication systems*. Cambridge University Press, 2014. [Unofficial version on UC Santa Barbara Website]
  - Chap 6
- Joachim Speidel. *Introduction to digital communications*. Springer Nature, 2021. [NTU Library Link]
  - Chap. 3
- Andreas F. Molisch, *Wireless communications*. Vol. 34. John Wiley & Sons, 2012. [NTU Library Link]
  - Chap. 12

Orhan Gazi

Associate Professor

Çankaya University, Turkey

Reference

Orhan Gazi. *Understanding digital signal processing*. Springer Singapore, 2018. [NTU Library link]

Chapter 2: Multirate Signal Processing

S. Palani. *Principles of Digital Signal Processing*. Springer Cham, 2022. [NTU Library link]

Chap. 6: Multi-rate Digital Signal Processing

Filtering

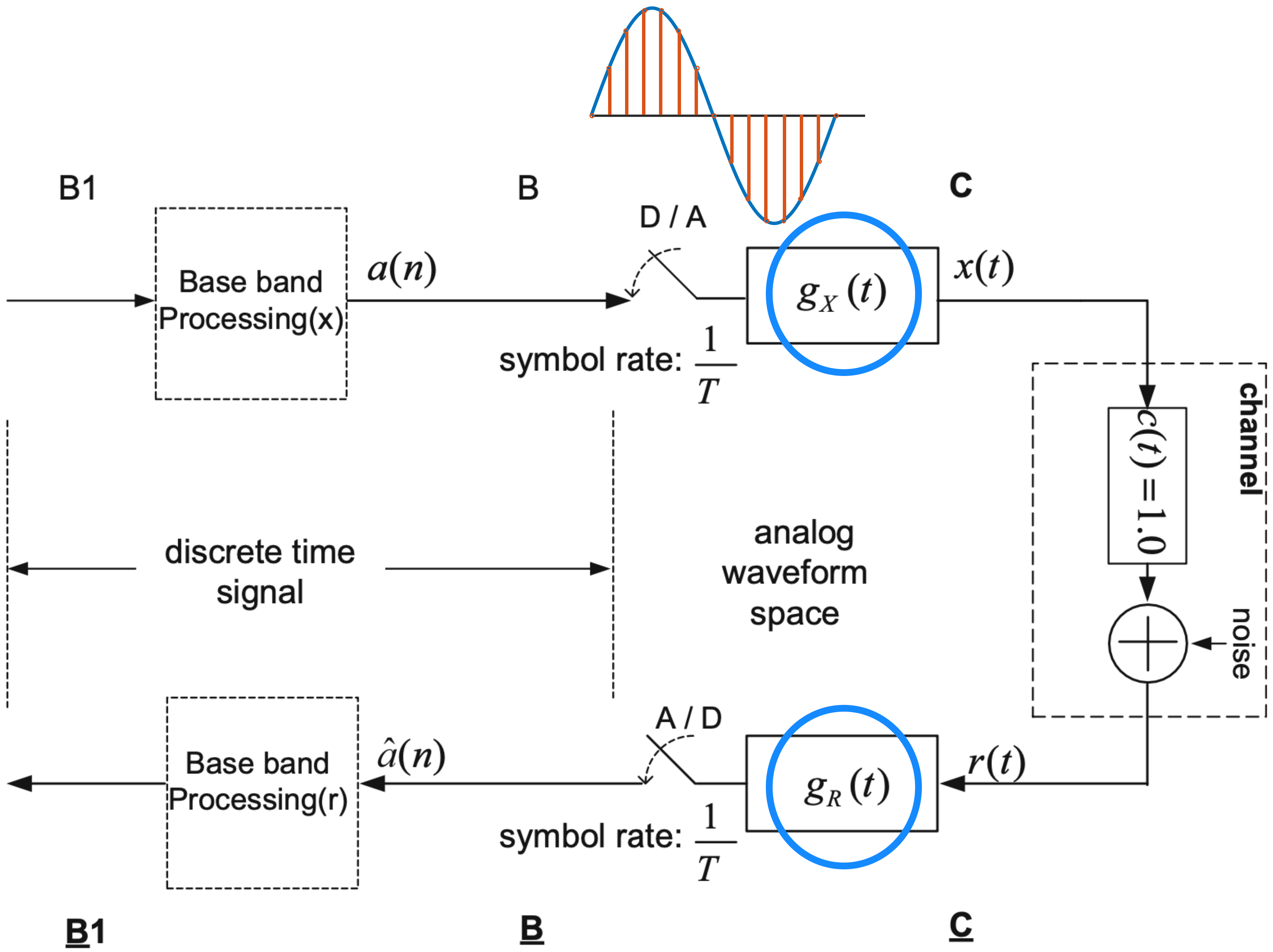
Why Do We Need Filters

Spectral mask compliance (pulse shaping)

Noise/interference rejection

Channel equalization

How to select the transmit and receive filter pair?

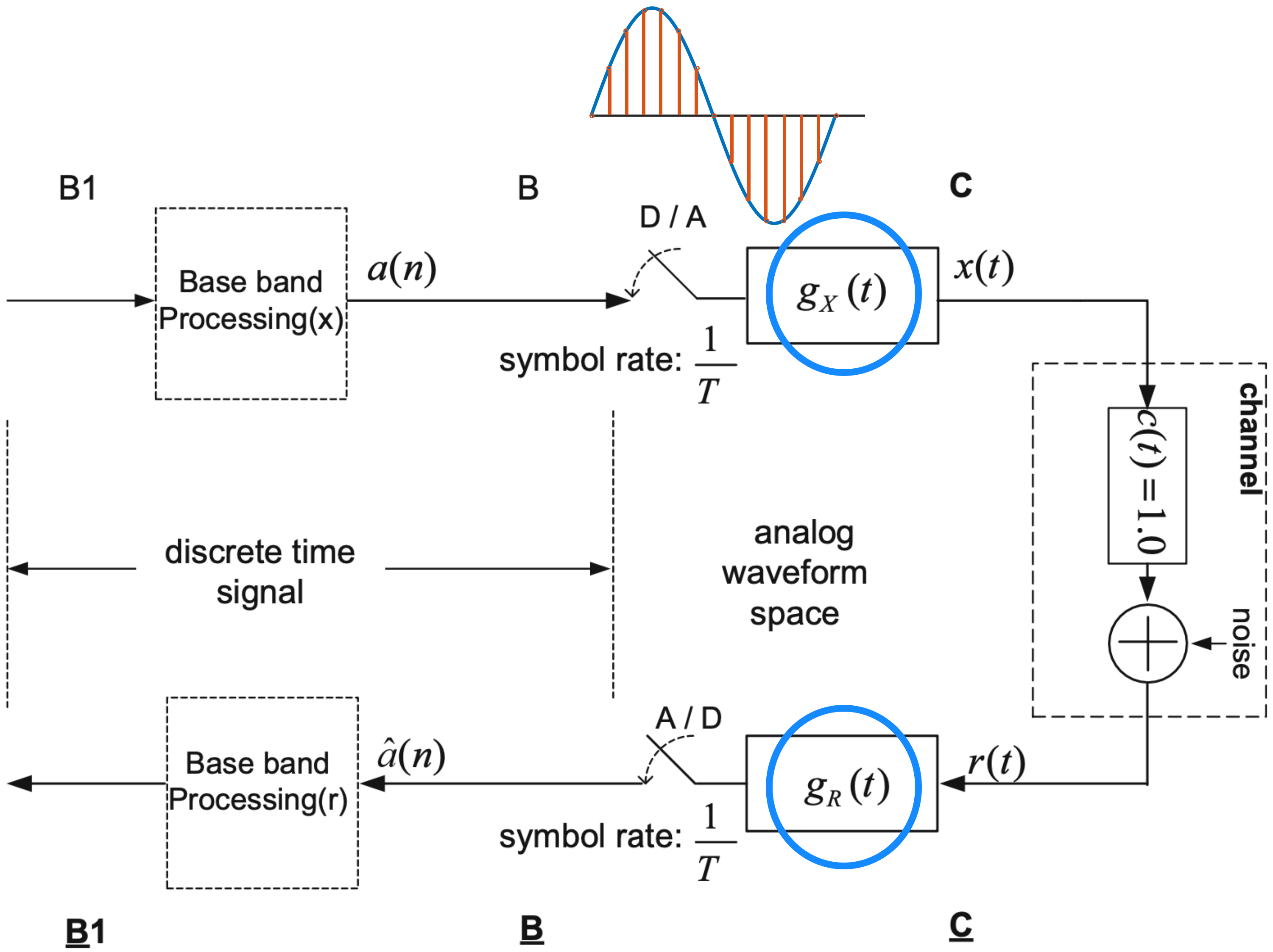


Band Limited

Select g_X and g_R

Achieve good SNR	Avoid inter-symbol interference

How to select the transmit and receive filter pair?



Band Limited

Select g_X and g_R

Achieve good SNR	Avoid inter-symbol interference

Maximize SNR: A Pair of Matched Filters

$$\boxed{g_R(t) = g_X^*(-t)}$$

Time Domain

OR

$$\boxed{G_R(f) = G_X^*(f)}$$

Frequency Domain

Why do matched filters maximize SNR?

Schwarz Inequality

$$\left|\int f(x)g(x)dx\right|^2 \leq \int |f(x)|^2 dx \int |g(x)|^2 dx$$

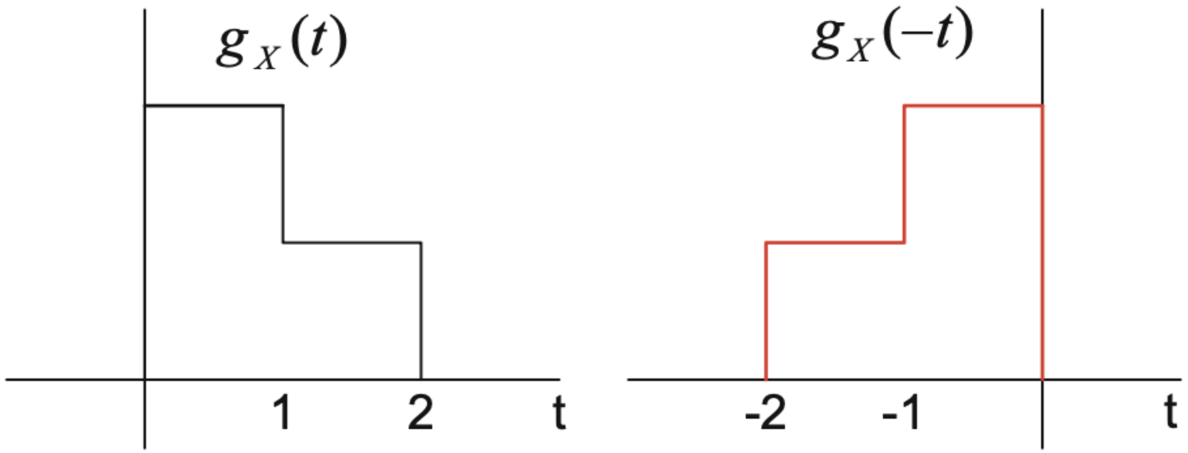
Equality holds if and only if
$f(x) = kg^*(x)$ with $k = \text{constant}$

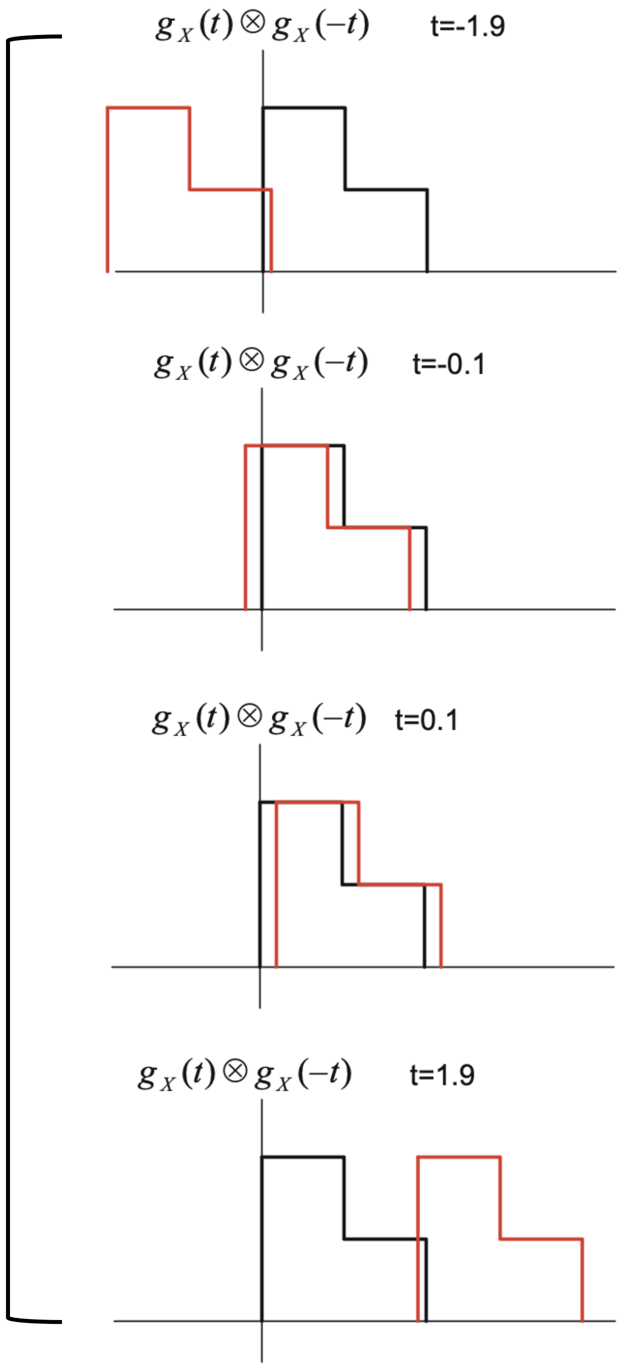
End-to-end frequency response

$$\begin{aligned} H(f) &= G_X(f)G_R(f) \\ &= G_X(f)G_X^*(-f) \\ &= |G_X(f)|^2 \end{aligned}$$

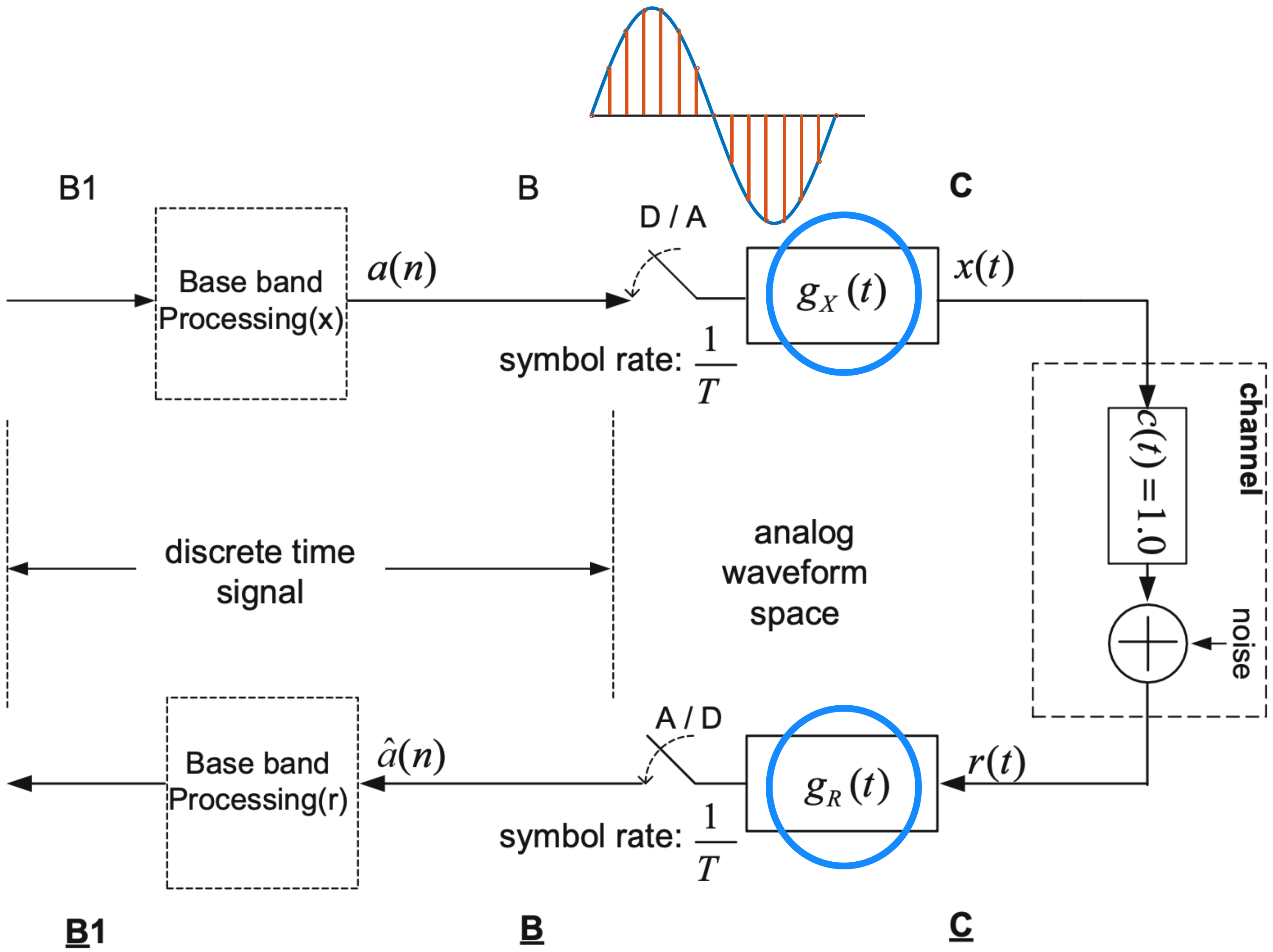
Example Matched Filter

Convolution of $g_X(t) \otimes g_X(-t)$





How to select the transmit and receive filter pair?



Band Limited

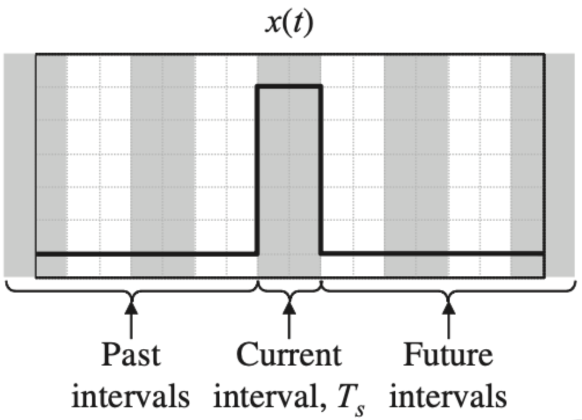
Select g_X and g_R

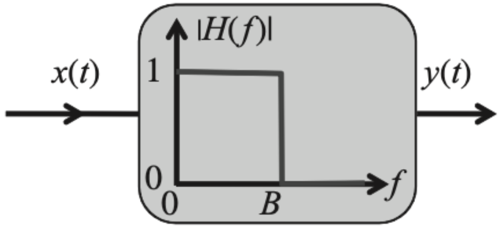
Achieve good SNR

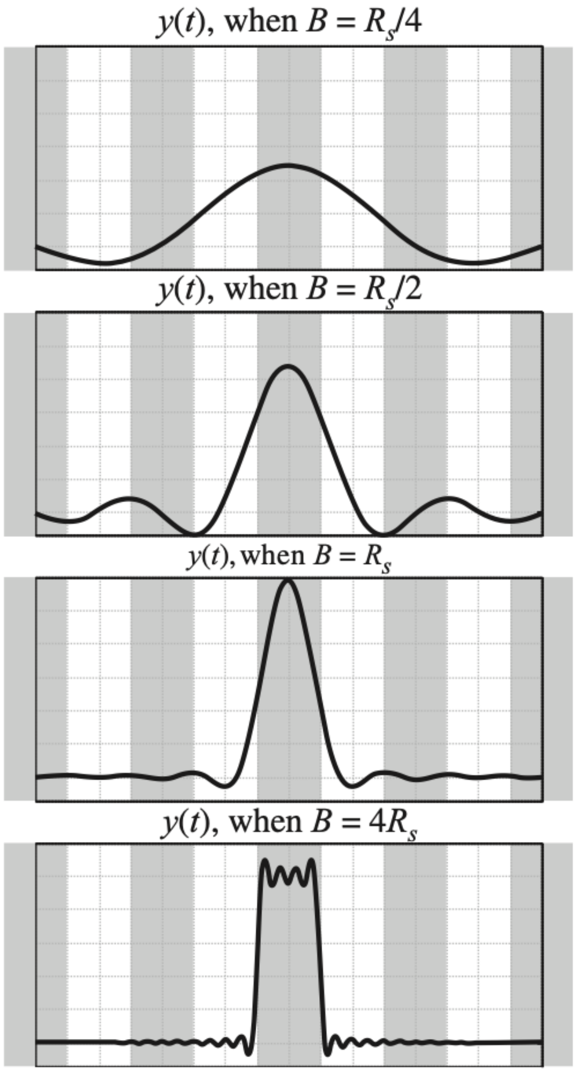
Avoid inter-symbol interference

Matched Filter

Inter-Symbol Interference (ISI)







Pulse duration T_s

Symbol Rate $R_s = \frac{1}{T_s}$

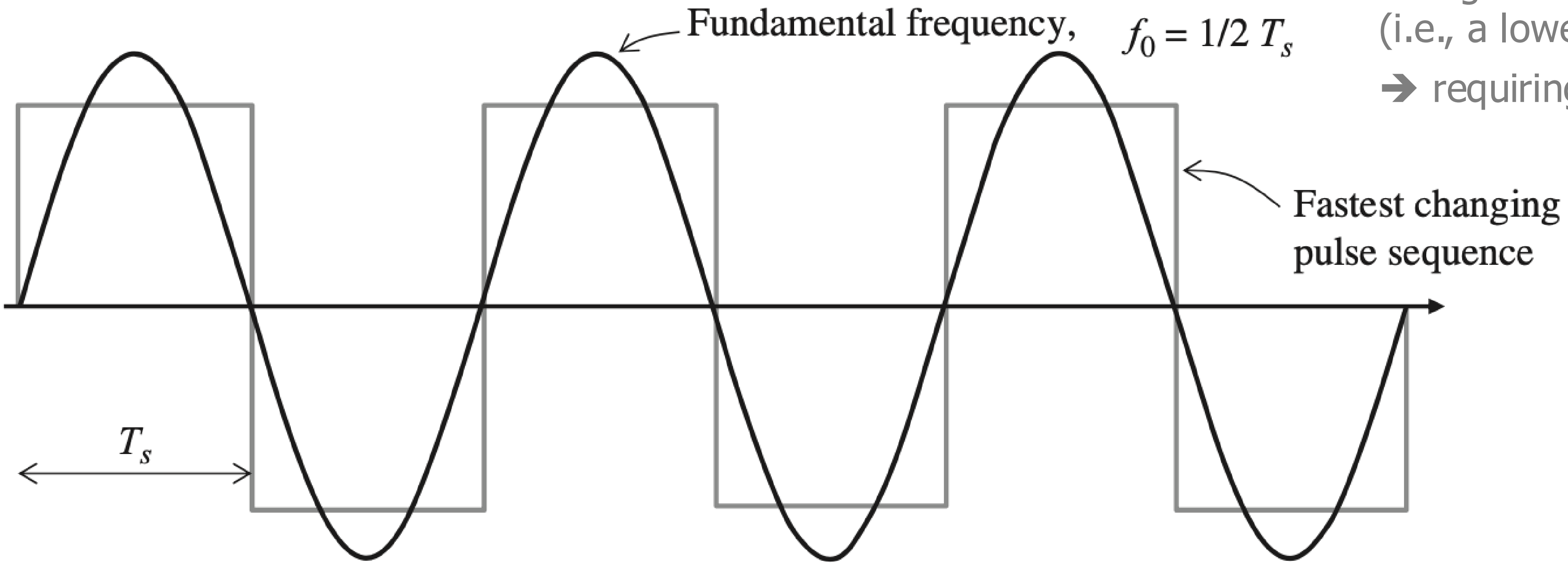
Bandwidth B

Band Limited

Larger B

Less ISI,
but inefficient

Bandwidth and Symbol Rate



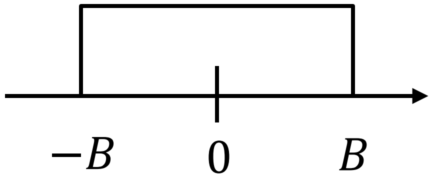
All other pulse sequences will change slower (i.e., a lower fundamental freq)

➔ requiring less bandwidth

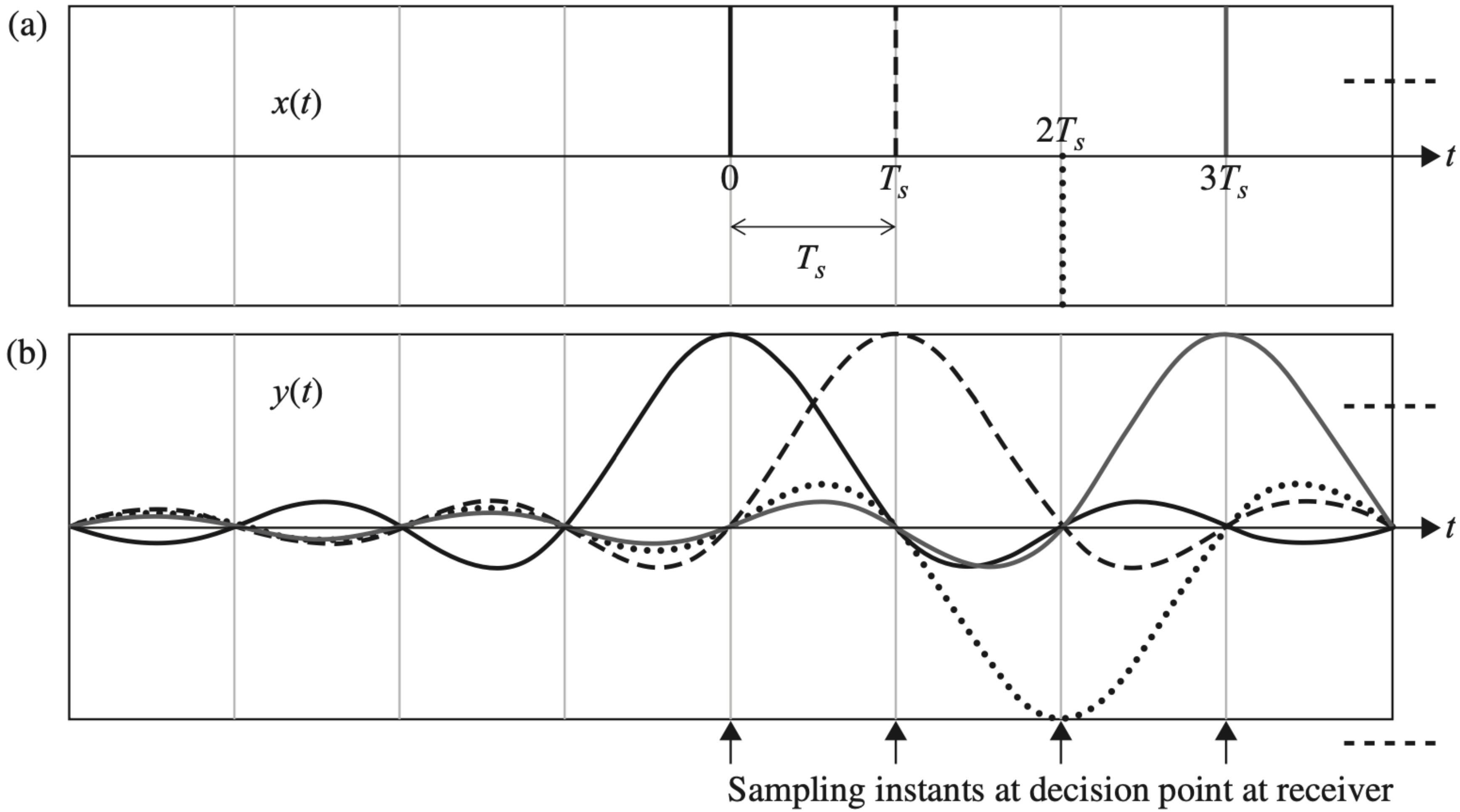
Channel bandwidth B must be at least wide enough to pass the fundamental frequency f_0 of the fastest-changing sequence ⇨

With bandwidth B

Max Symbol Rate $R_{s\,max} = 2B$



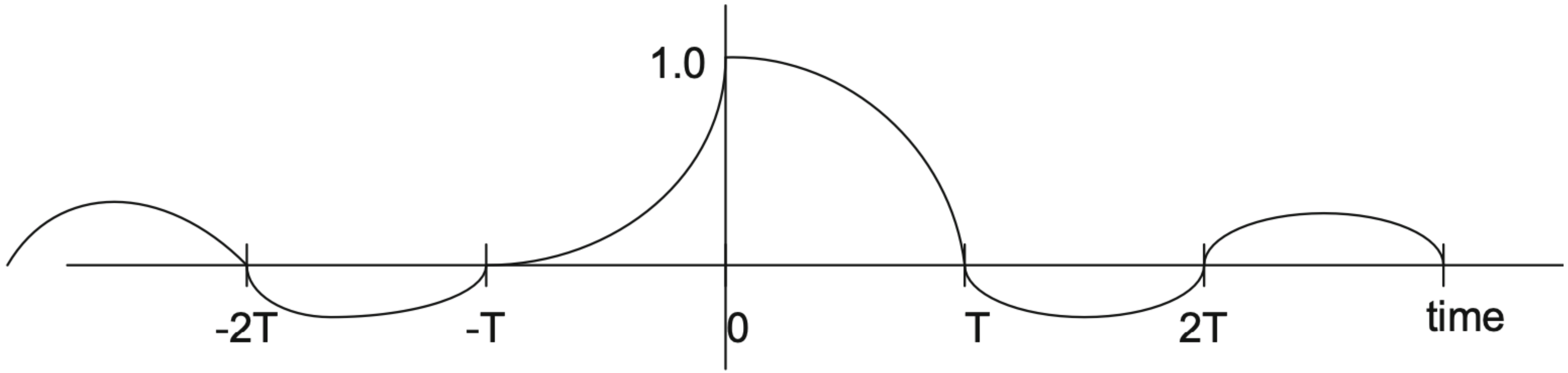
Nyquist Criterion for Zero ISI



All other symbols happen to be zero

Nyquist Criterion for Zero ISI

$$h(mT_s - nT_s) = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}$$



Does This Example Satisfy The Nyquist Criterion?

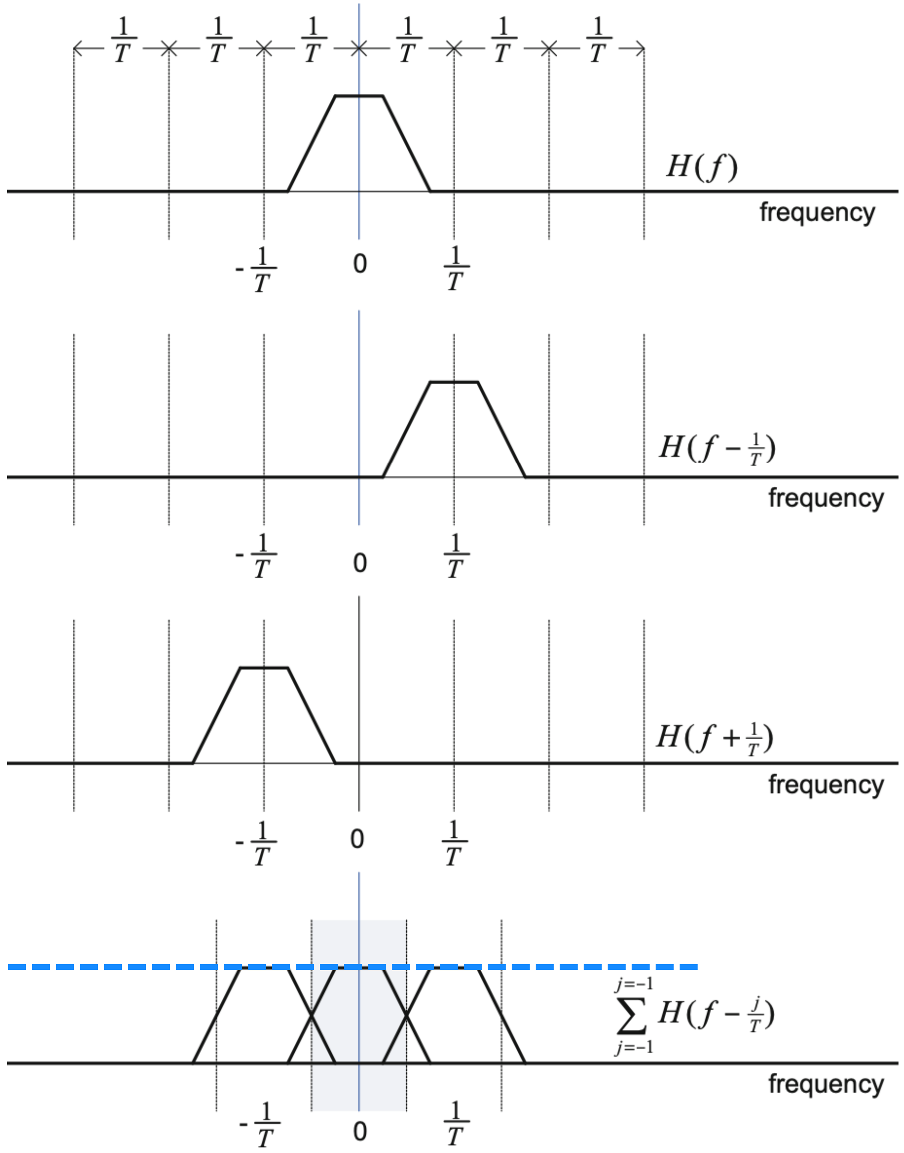
Yes!

Frequency Domain Expression of Nyquist Criterion

$$\sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T}\right) = T$$

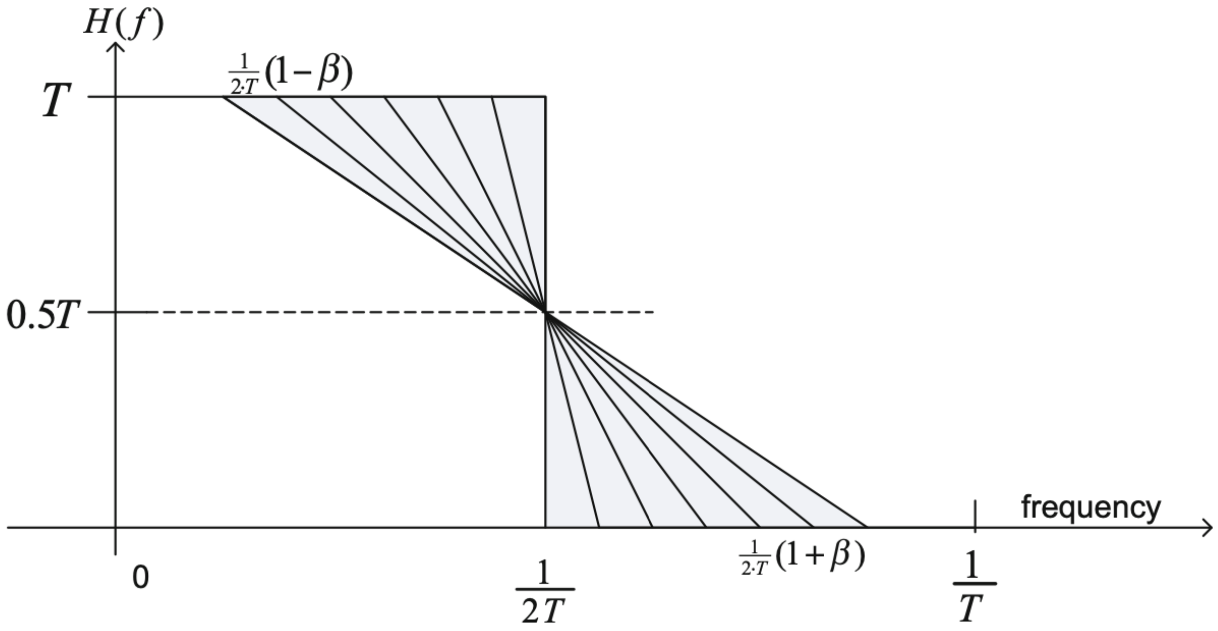
The sum of all periodic repetitions of H is a constant

A lowpass satisfying this condition is also called a Nyquist lowpass

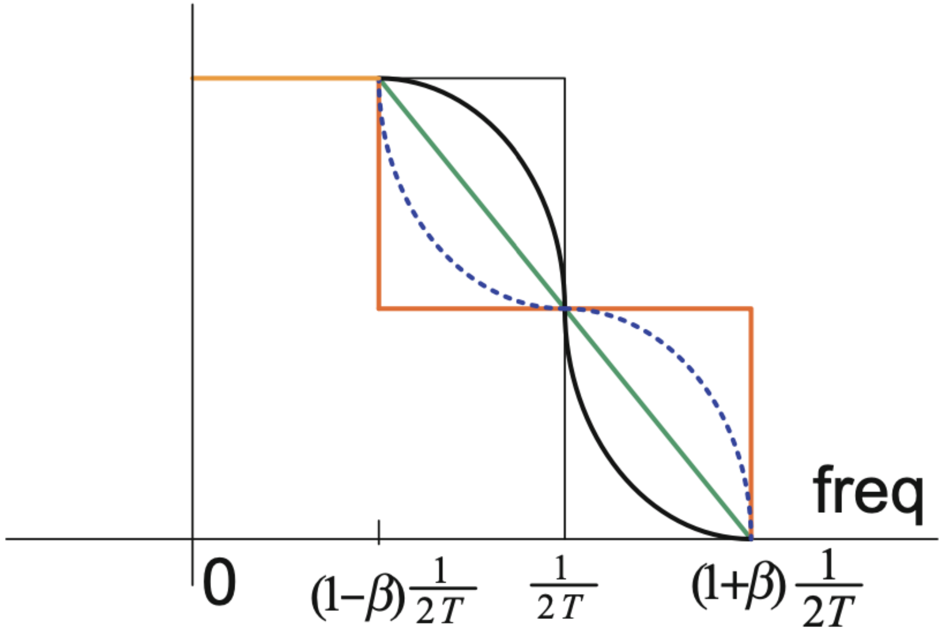


邑恆：在 Digital communications by John G. Proakis 第五版中的 605 頁有提供完整的推導。這本書在 Lab 331 中有，應該可以提供給同學做參考。

Band Edge Vestigial Symmetry (for Nyquist Criterion)



β: excess bandwidth parameter



Which of these satisfies Nyquist criterion?

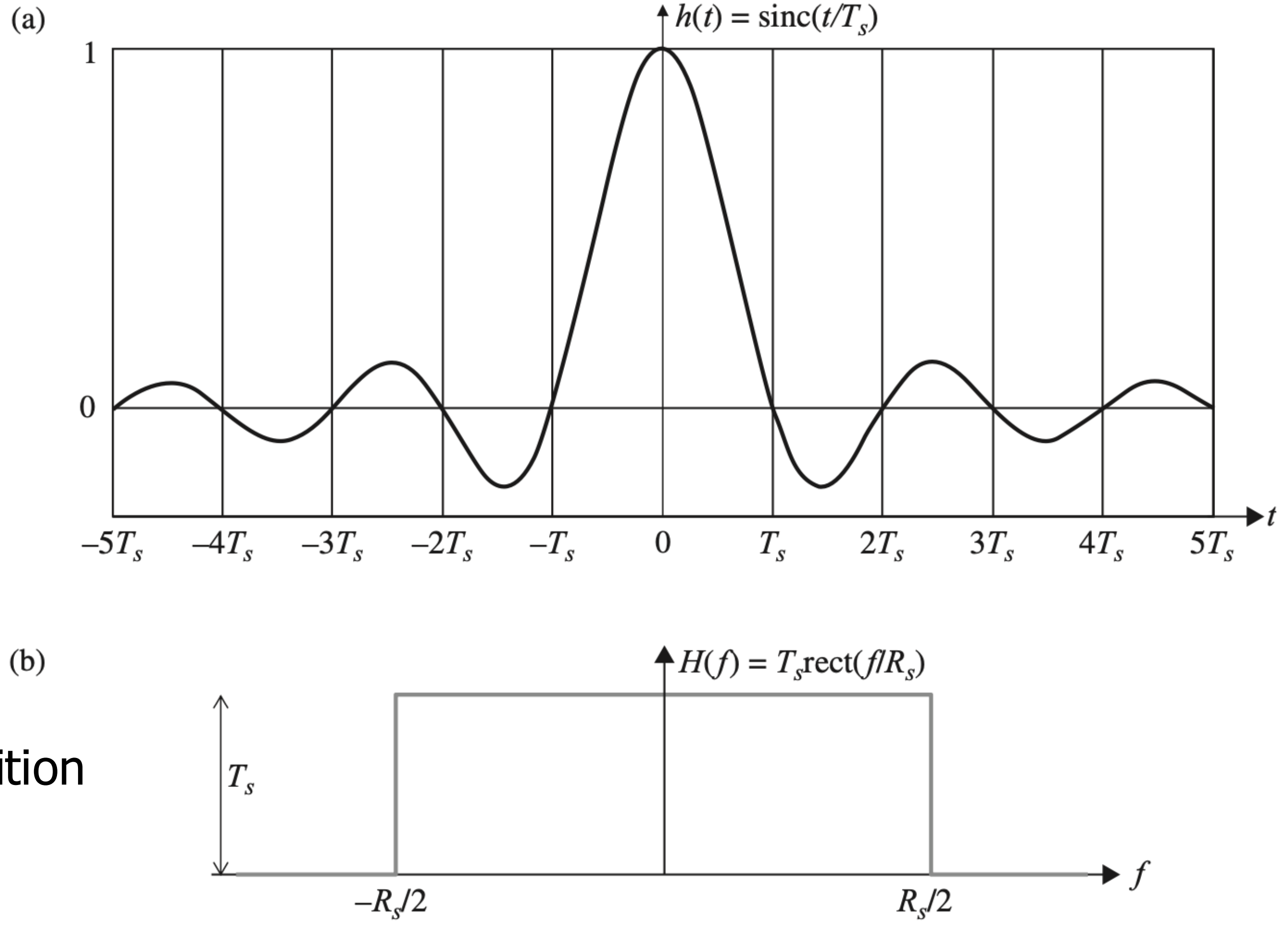
$$\frac{1}{2T} = B \longrightarrow \frac{1}{2T}(1+\beta) = B$$

All of them!

Nyquist Filtering – The Ideal Case

$$h(t) = \mathrm{sinc}(t/T_s)$$

$$H(f) = T_s \mathrm{rect}(fT_s) = \frac{1}{R_s}\mathrm{rect}\left(\frac{f}{R_s}\right)$$



Practical problems

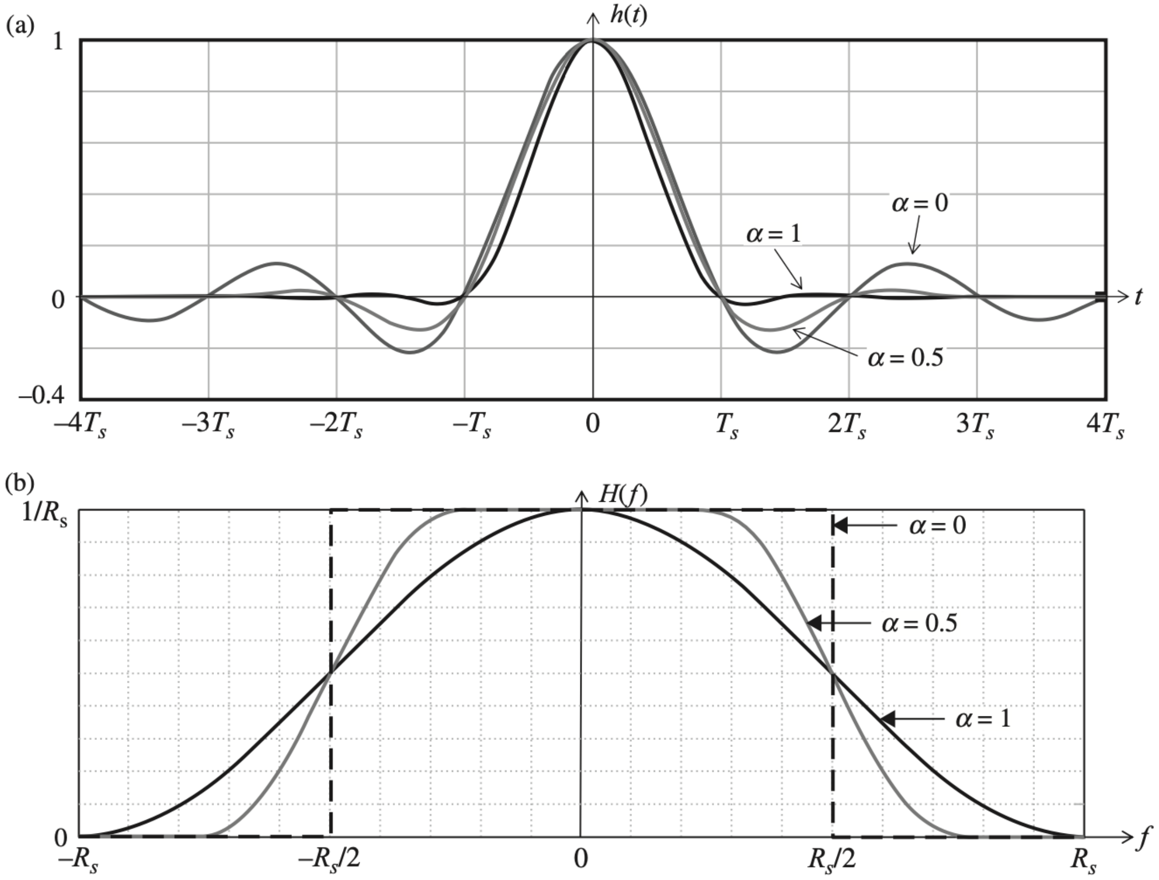
Hard to implement sharp cut-off frequency transition
(future inputs contribute to current output)

envelope of the sinc pulse decays very slowly
➔ imposes a very stringent requirement on timing accuracy

Raised Cosine Filtering

$$H(f) = \frac{1}{R_s} \times \begin{cases} 1, & |f| \le f_1 \\ \frac{1}{2}\left[1 + \cos\left(\pi \frac{|f| - f_1}{f_2 - f_1}\right)\right], & f_1 \le |f| \le f_2 \\ 0, & |f| \ge f_2 \end{cases}$$

$$f_1 = (1-\alpha)R_s/2;\ f_2 = (1+\alpha)R_s/2;\ 0 \le \alpha \le 1$$



α: roll-off factor

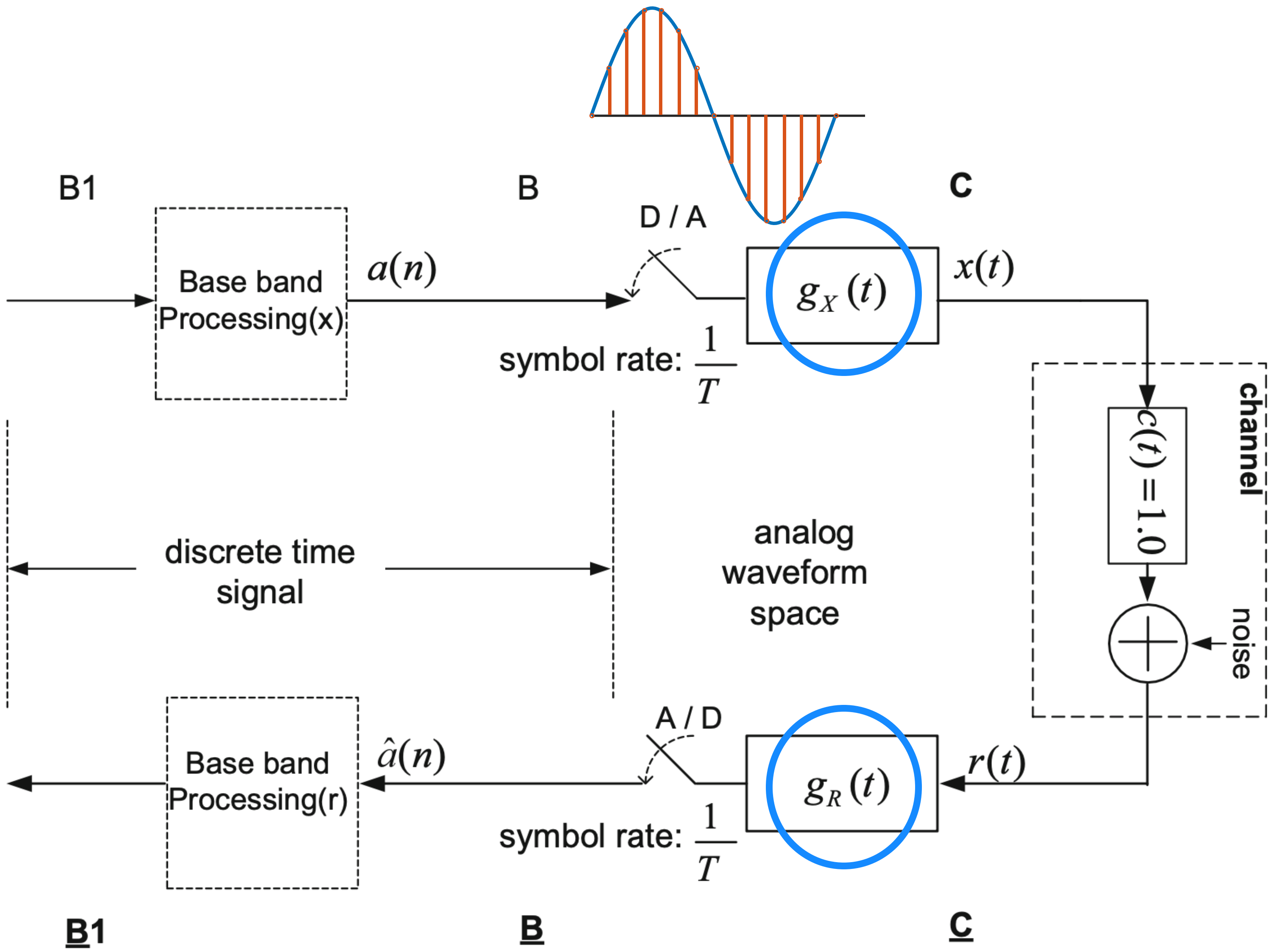
gradual roll-off makes the raised cosine filter characteristic easier to approximate than the ideal Nyquist filter

$\alpha = 0$: Nyquist Filtering

Does it satisfy the Nyquist Criterion?

Yes!

How to select the transmit and receive filter pair?



Band Limited

Select g_X and g_R

Achieve good SNR	Avoid inter-symbol interference
Matched Filter	**Nyquist criterion**

Square Root Raised Cosine (RRC) Filter

End-to-end frequency response = Raised Cosine Filter

$$H(f) = G_X(f)G_R(f) = G_X(f)G_X^*(-f) = |G_X(f)|^2$$
$$= Raised\ Cosine\ Filter$$

$$|G_X(f)| = |G_R(f)| = \sqrt{H(f)} \equiv \sqrt{Raised\ Cosine\ Filter\ Gain\ Response}$$
$$\equiv |H_{RRC}(f)|$$

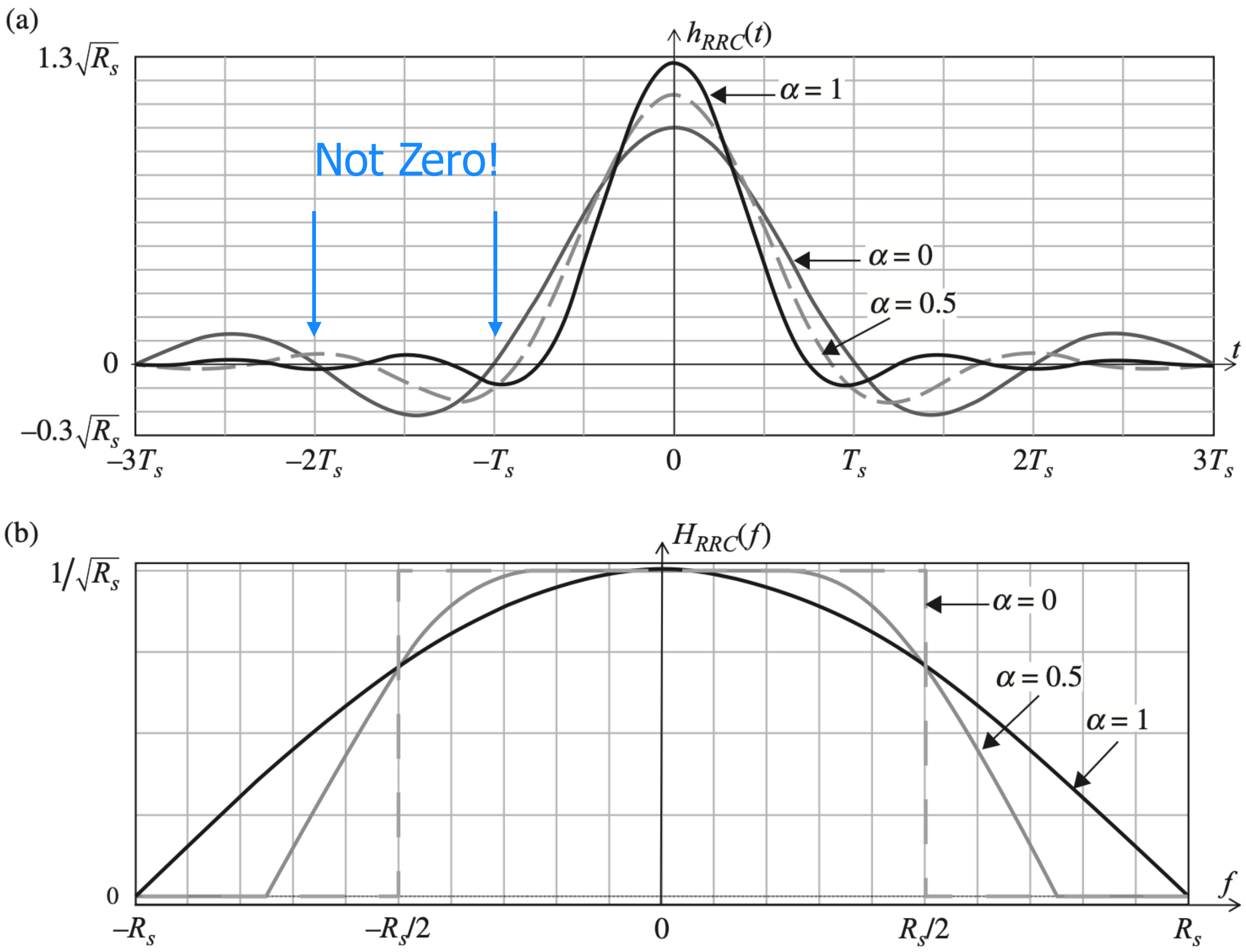
$$H_{RRC}(f) = \frac{1}{\sqrt{R_s}} \times \begin{cases} 1, & |f| \le f_1 \\ \cos\left(\frac{\pi}{2}\frac{|f|-f_1}{f_2-f_1}\right), & f_1 \le |f| \le f_2 \\ 0, & |f| \ge f_2 \end{cases}$$

$$f_1 = (1-\alpha)R_s/2;\ \ f_2 = (1+\alpha)R_s/2;\ \ 0 \le \alpha \le 1$$

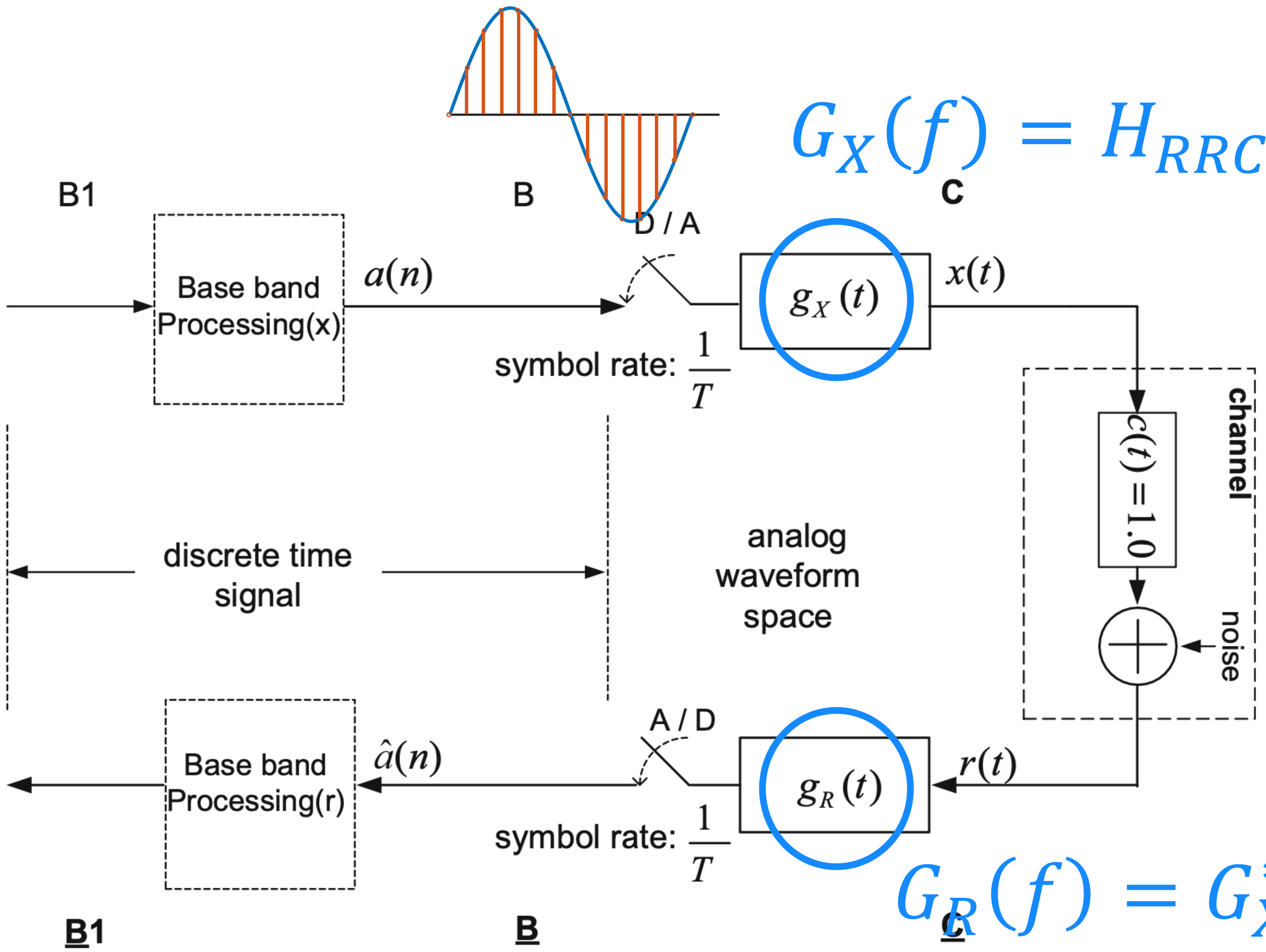
Square Root Raised Cosine (RRC) Filter

A single RRC filter cannot eliminate ISI although it has the same bandwidth as its raised cosine counterpart

RRC filters must be used in pairs



How to select the transmit and receive filter pair?



$$G_X(f) = H_{RRC}(f)$$

Band Limited

Select g_X and g_R

$$G_R(f) = G_X^*(f)$$

Achieve good SNR → **Matched Filter**

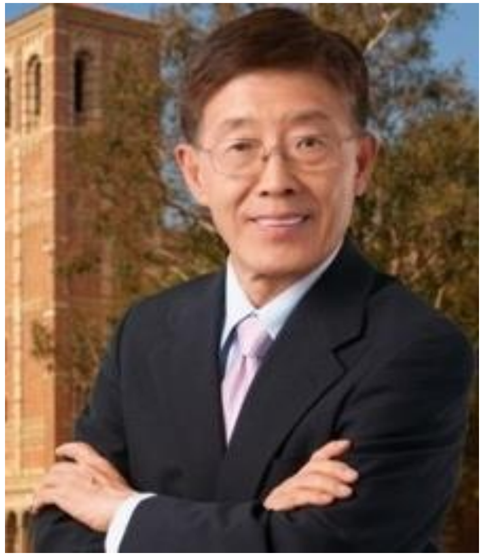
Avoid inter-symbol interference → **Nyquist criterion**

Filtering

Ifiok Otung

Professor

University of Bradford, UK



Sung-Moon Michael Yang

Principal Consulting Engineer

Baycore Wireless, USA

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Chapter 8: Transmission through band limited AWGN channels

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Chap 3: Matched Filter & Nyquist Pulse