Homework 5 (Due: 25th Dec.)

(1) What are the vanish moments of (a) $\frac{d^{12}}{dt^{12}}e^{-\pi t^2}$; (b) the sinc wavelet; (c) the 10-point Daubechies wavelet; (d) the 18-point coiflet? (15 scores)

(a) $\psi(t) = \frac{d^p}{dt^p} e^{-\pi t^2}$, $\Psi(f) = (j2\pi f)^p e^{-\pi f^2}$

$m_0 = \int_{-\infty}^{\infty} \psi(t)\,dt = \Psi(0) = 0$ if $p>0$

$m_k = \int_{-\infty}^{\infty} t^k \psi(t)\,dt = \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k}\Psi(f)\Big|_{f=0} = 0$ if $k<p$ ⇒ $m_0, m_1, \dots, m_{p-1} = 0$ ⇒ vanish moment $=12$

(b) $\Psi(f) = \begin{cases}1, & |f|<\frac14 \\ 0, & \frac14<|f|<\frac12\end{cases}$ $\quad \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k}\Psi(f)\Big|_{f=0} = 0$ for all k ⇒ vanish moment $= \infty$

(c) According to physical meaning, a 10 point Daubechies has vanish moment 5

(d) According to physical meaning, a 18 point coiflet has vanish moment 3 #

(2) (a) What is the role of the admissibility criterion in the continuous wavelet transform? (b) What is the role of the generating function in the continuous wavelet transform with discrete coefficients? (10 scores)

(a) Admissibility criterion: $C_\psi = \int_0^\infty \frac{|\Psi(f)|^2}{|f|}\,df < \infty$, this has to be true for reversibility of continuous wavelet transform.

(b) Once generating function $G(f)$ is determined, the mother wavelet $\psi(t)$ & scaling function $\phi(t)$ are determined

$H(f) = -e^{-j2\pi f} G^*(f+\frac12)$, $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$, $\Psi(f) = H\left(\frac f2\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$ #

(3) Why the complexity of the 1-D discrete wavelet transform is $O(N)$? (10 scores)

Without secondary convolution: Complexity $= O(N\log_2 N)$

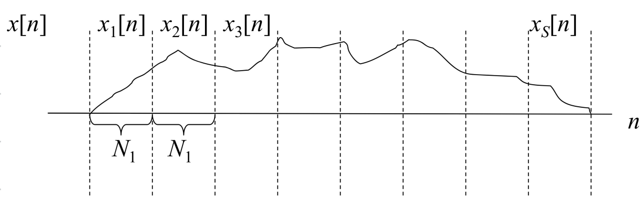
But with secondary convolution: Cut $x[n]$ into many segments each with length N_1, $N > N_1 >> L$

$x[n] = x_1[n] + \dots + x_S[n]$, $S = \frac{N}{N_1}$

$x[n]*g[n] = x_1[n]*g[n] + \dots + x_S[n]*g[n]$

$x[n]*h[n] = x_1[n]*h[n] + \dots + x_S[n]*h[n]$

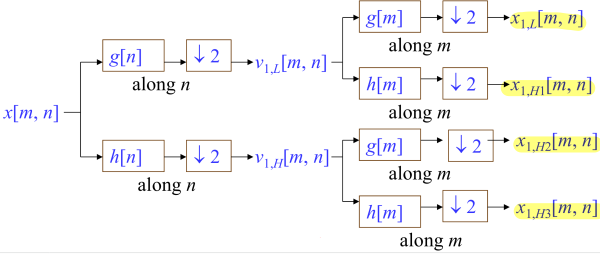
Complexity $= S\times 3(N_1+L-1)\log_2(N_1+L-1) = O\left(\frac{N}{N_1}(N_1)\log_2 N_1\right) = O(N)$



∴ By secondary convolution, we slightly improve the complexity from $O(N\log_2 N)$ to $O(N)$ #

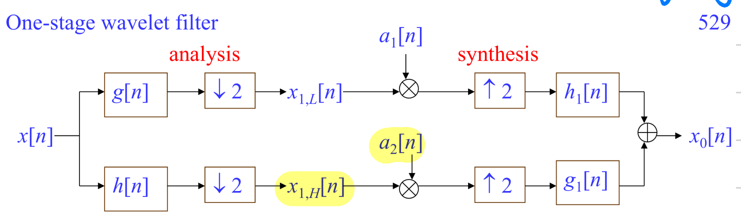
(4) Why the wavelet transform can be used for, (a) image compression, (b) pattern recognition, and (c) adaptive filter design? (15 scores)

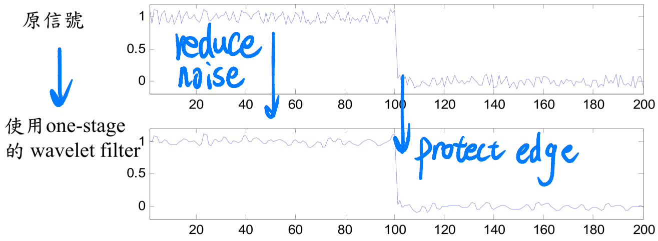
(a) For most part of an image, they have a lower freq., and high freq. only exist at edges. Thus, apply 2D-DWT (for several times if you want to), we devide $x[m,n]$ into $x_{1,L}[m,n]$, $x_{1,H_1}[m,n]$, $x_{1,H_2}[m,n]$, $x_{1,H_3}[m,n]$, which represent main part, horizontal edge, vertical edge, and corner, respectively. If we only take the main part, we can compress image by about 1/4 for one 2D-DWT.



(b) Following (a), edge is an important feature when doing pattern recognition and wavelet transform can find edge at different scale and location. It also save computation time, we can do recognition to image after DWT, if the probability of existence of the target is high, we search on the image before DWT for a finer resolution. This reduces the time spent on searching unnecessary details.

(c) Sometimes we want the filter to remove high-freq. noise, but we also remove edge at the same time, which is not desired. By wavelet transform, $X_{1,H}[n]$ contains both edge & noise signal, with different amplitude. If we let $a_2[n] = \begin{cases} 0, & \text{amplitude} < \text{threshold (noise)} \\ 1, & \text{amplitude} < \text{threshold (edge signal)} \end{cases}$, we can keep the edge signal and remove noise. //#





(5) (a) What is the advantage of the symlet compared to the Daubechies wavelet? (b) What is the advantage of the curvelet compared to the original 2D wavelet? (10 scores)

(a). The energy of symlet is more concentrated in the middle, this allows the image to have smaller shift after 2D-DWT.

(b). Curvelet do 2D-DWT along a rotation of x-axis & y-axis. The performance will be better if edges of the image have certain direction. //#

(6) Suppose that the wavelet filter $g[n]$ is an orthonormal filter. (a) What is the value of a if $g[0] = 1/2$, $g[1] = a$, and $g[n] = 0$ otherwise? (b) What are the constraints of a, b, and c if if $g[0] = 1/2$, $g[1] = a$, $g[2] = b$, $g[3] = c$, and $g[n] = 0$ otherwise? (10 scores)

(a) $G(z) = a + bz^{-1}$, $G_1(z) = G(z^{-1}) = a + bz$, $G(z)G_1(z) = a^2 + b^2 + abz + abz^{-1}$, $G_1(z)G_1(z^{-1}) + G_1(-z)G_1(-z^{-1}) = G(z)G_1(z) + G(-z)G_1(-z)$

$= 2(a^2+b^2) = 2 \to a^2+b^2 = 1 \to g[1] = a = \pm\sqrt{1-\frac{1}{2^2}} = \pm\frac{\sqrt{3}}{2}$

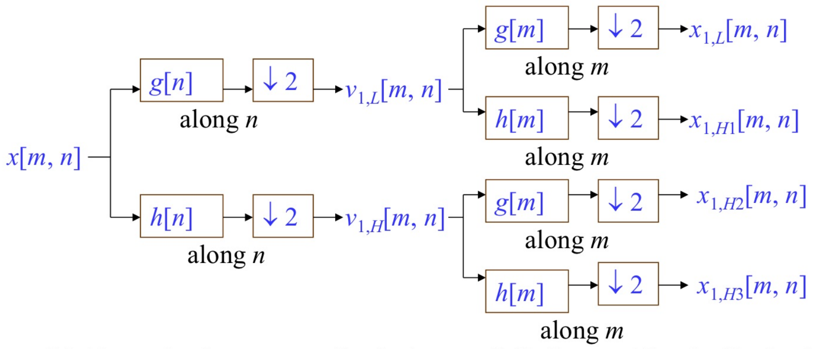
(b) $\sum_n g[n-2k]g[n] = \delta[k],\ k \in \mathbb{Z}$

$k=0 \to \frac{1}{4} + a^2 + b^2 + c^2 = 1$

$k=1 \to g[0]g[2] + g[1]g[3] = \frac{1}{2}b + ac = 1$ $\to \begin{cases} a^2+b^2+c^2 = \frac{3}{4} \\ \frac{1}{2}b + ac = 1 \end{cases}$ //#

(7) (a) Write a Matlab or Python code for the following 2-D discrete 10-point Daubechies wavelet.

[x1L, x1H1, x1H2, x1H3] = wavedbc10(x)



(b) Also write the program for the inverse 2-D discrete 10-point Daubechies wavelet transform.

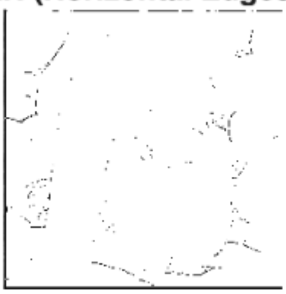
x = iwavedbc10(x1L, x1H1, x1H2, x1H3)

The code should be handed out by NTUCool. (30 scores)

(a).

LL (Approximation)

LH (Horizontal Edges)



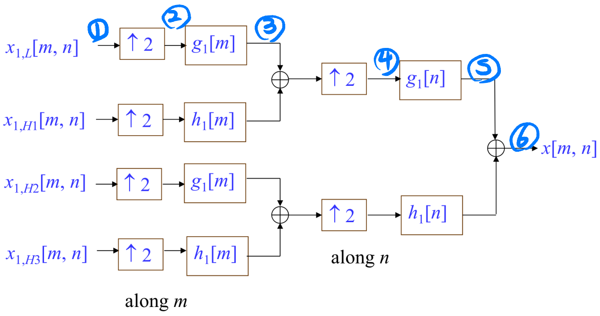
HL (Vertical Edges)

HH (Diagonal Edges)

(b). iwavedbc10:

原图:

Extra (学号尾数4) = P.481 What is the difference of number of data points between output of inverse 2D-DWT and original signal?



Stage	# of point
1	$\frac{(M+L-1)(N+L-1)}{4}$
2	$\frac{(M+L-1)(N+L-1)}{2}$
3	$\frac{(M+2L-2)(N+L-1)}{2}$
4	$(M+2L-2)(N+L-1)$
5 = 6	$(M+2L-2)(N+2L-2)$

Difference $= (M+2L-2)(N+2L-2) - MN$

$= 2L(M+N) - 2(M+N) + 4L^2 - 8L + 4$

$= 2(L-1)(M+N+2L-2)$ points #