

III. Gabor Transform

III-A Definition

Standard Definition:

$$G_x(t, f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau) d\tau$$

Alternative Definitions:

$$G_{x,1}(t, f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f(\tau-\frac{t}{2})} x(\tau) d\tau \quad \text{with rotation property}$$

$$G_{x,2}(t, f) = \sqrt[4]{2} \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau) d\tau \quad \text{normalization}$$

$$G_{x,3}(t, \omega) = \int_{-\infty}^{\infty} e^{-(\tau-t)^2/2} e^{-j\omega\tau} x(\tau) d\tau$$

$$G_{x,4}(t, \omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(\tau-t)^2}{2}} e^{-j\omega(\tau-\frac{t}{2})} x(\tau) d\tau$$

Main Reference

- S. Qian and D. Chen, [Sections 3-2 ~ 3-6](#) in *Joint Time-Frequency Analysis: Methods and Applications*, Prentice-Hall, 1996.

Other References

- D. Gabor, “Theory of communication”, *J. Inst. Elec. Eng.*, vol. 93, pp. 429-457, Nov. 1946. (最早提出 Gabor transform)
- M. J. Bastiaans, “Gabor’s expansion of a signal into Gaussian elementary signals,” *Proc. IEEE*, vol. 68, pp. 594-598, 1980.
- R. L. Allen and D. W. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*, Wiley- Interscience.
- S. C. Pei and J. J. Ding, “Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing,” *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

Note :

許多文獻把 Gabor transform 直接就稱作 short-time Fourier transform (STFT)，實際上，Gabor transform 是 STFT 當中的一個 special case.

III-B Approximation of the Gabor Transform

Although the range of integration is from $-\infty$ to ∞ , due to the fact that

$$e^{-\pi a^2} < 0.00001 \quad \text{when } |a| > 1.9143$$

$$e^{-a^2/2} < 0.00001 \quad \text{when } |a| > 4.7985$$

the Gabor transform can be simplified as:

$$G_x(t, f) \approx \int_{t-1.9143}^{t+1.9143} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau) d\tau$$

$$G_{x,3}(t, \omega) = \sqrt{\frac{1}{2\pi}} \int_{t-4.7985}^{t+4.7985} e^{-\frac{(\tau-t)^2}{2}} e^{-j\omega(\tau-\frac{t}{2})} x(\tau) d\tau$$

III-C Why Do We Choose the Gaussian Function as a Mask

(1) Among all functions, the Gaussian function has the advantage that the area in time-frequency distribution is **minimal**.

(和其他的 STFT 相比，比較能夠同時讓 time-domain 和 frequency domain 擁有較好的清晰度)

$w(t)$ 太寬 → time domain 的解析度較差

$w(t)$ 太窄 → $W(f) = FT[w(t)]$ 太寬 → frequency domain 的解析度較差

(2) Special relation between the Gaussian function and the rectangular function

(Note): 由於 Gaussian function 是 FT 的 eigenfunction，因此 Gabor transform 在 time domain 和 frequency domain 的性質將互相對稱

$$\int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j2\pi f t} dt = e^{-\pi f^2}$$

$$\int_{-\infty}^{\infty} e^{-t^2/2} e^{-j\omega t} dt = e^{-f^2/2}$$

according to $\int_{-\infty}^{\infty} e^{-(at^2+bt)} dt = \sqrt{\pi/a} \cdot e^{b^2/4a}$

M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., 2009.

Gaussian function is also an eigenmode in optics, radar system, and other electromagnetic wave systems.

(will be illustrated in the 8th week)

Uncertainty Principle (Heisenberg, 1927)

For a signal $x(t)$, if $\sqrt{t} x(t) = 0$ when $|t| \rightarrow \infty$, then

$$\sigma_t \sigma_f \geq 1/4\pi$$

where $\sigma_t^2 = \int (t - \mu_t)^2 P_x(t) dt$ $\sigma_f^2 = \int (f - \mu_f)^2 P_X(f) df,$

$$\mu_t = \int t P_x(t) dt,$$

$$\mu_f = \int f P_X(f) df$$

$$P_x(t) = \frac{|x(t)|^2}{\int |x(t)|^2 dt},$$

$$P_X(f) = \frac{|X(f)|^2}{\int |X(f)|^2 df},$$

(Proof of Henseinberg's uncertainty principle):

From simplification, we consider the case where $\mu_t = \mu_f = 0$

Then, use Parseval's theorem

$$\sigma_t^2 \sigma_f^2 = \frac{1}{4\pi^2} \frac{\int t^2 |x(t)|^2 dt}{\int |x(t)|^2 dt} \frac{\int |x'(t)|^2 dt}{\int |x(t)|^2 dt}$$

$$\int |x(t)|^2 dt = \int |X(f)|^2 df \quad \text{if } X(f) = FT[x(t)]$$

From Schwarz's inequality $\langle x(t), x(t) \rangle \langle y(t), y(t) \rangle \geq |\langle x(t), y(t) \rangle|^2$

$$\begin{aligned}
& \int t^2 |x(t)|^2 dt \int |x'(t)|^2 dt \geq \left(\left| \int tx^*(t) \frac{d}{dt} x(t) dt \right|^2 + \left| \int tx(t) \frac{d}{dt} x^*(t) dt \right|^2 \right) / 2 \\
& \geq \left| \int \left(tx^*(t) \frac{d}{dt} x(t) + tx(t) \frac{d}{dt} x^*(t) \right) dt \right|^2 / 4 \quad (\text{using } |a+b|^2 + |a-b|^2 \geq 2|a|^2) \\
& = \left| \int t \frac{d}{dt} [x(t)x^*(t)] dt \right|^2 / 4 = \left| tx(t)x^*(t) \Big|_{-\infty}^{\infty} - \int x^*(t)x(t) dt \right|^2 / 4 \\
& = \left[\left| tx(t)x^*(t) \right|_{t \rightarrow \infty} - \left| tx(t)x^*(t) \right|_{t \rightarrow -\infty} \right] - \left| \int x^*(t)x(t) dt \right|^2 / 4 \\
& = \left| \int |x(t)|^2 dt \right|^2 / 4
\end{aligned}$$

$$\sigma_t^2 \sigma_f^2 \geq \frac{1}{16\pi^2} \implies \sigma_t \sigma_f \geq \frac{1}{4\pi}$$

For Gaussian function

$$x(t) = e^{-\pi t^2} \quad X(f) = e^{-\pi f^2}$$

$$\sigma_t^2 = \int (t - \mu_t)^2 P_x(t) dt = \frac{\int (t - \mu_t)^2 |x(t)|^2 dt}{\int |x(t)|^2 dt} = \frac{\int t^2 |x(t)|^2 dt}{\int |x(t)|^2 dt}$$

Since $\mu_t = 0$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} e^{-2\pi t^2} dt = ?$$

use $\int_{-\infty}^{\infty} e^{-(at^2+bt)} dt = \sqrt{\pi/a} \cdot e^{b^2/4a}$

$$\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt = \int_{-\infty}^{\infty} t^2 e^{-2\pi t^2} dt = 2 \int_0^{\infty} t^2 e^{-2\pi t^2} dt = 2 \frac{\Gamma(3/2)}{2(2\pi)^{3/2}} = \frac{1}{2^{5/2}\pi}$$

use $\int_0^{\infty} t^m e^{-at^2} dt = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$

$$\Gamma(1/2) = \sqrt{\pi} \quad \Gamma(n+1) = n\Gamma(n), \quad \Gamma(3/2) = \sqrt{\pi}/2$$

$$\sigma_t^2 = \frac{\int t^2 |x(t)|^2 dt}{\int |x(t)|^2 dt} = \frac{1}{4\pi}, \quad \sigma_t = \sqrt{\frac{1}{4\pi}}$$

同理， $\sigma_f = \sqrt{\frac{1}{4\pi}}$

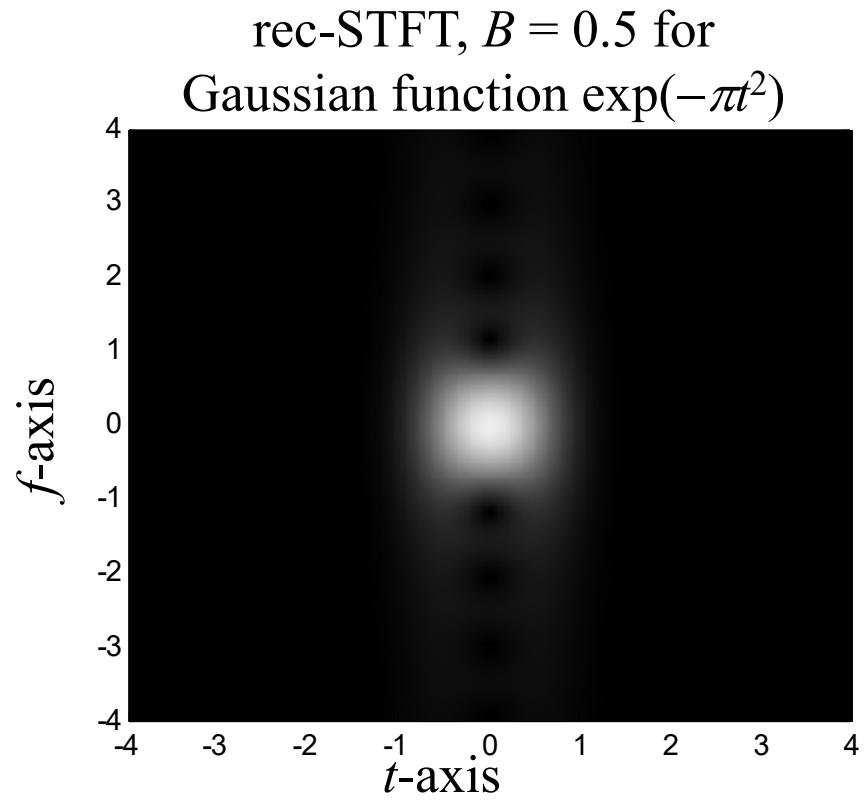
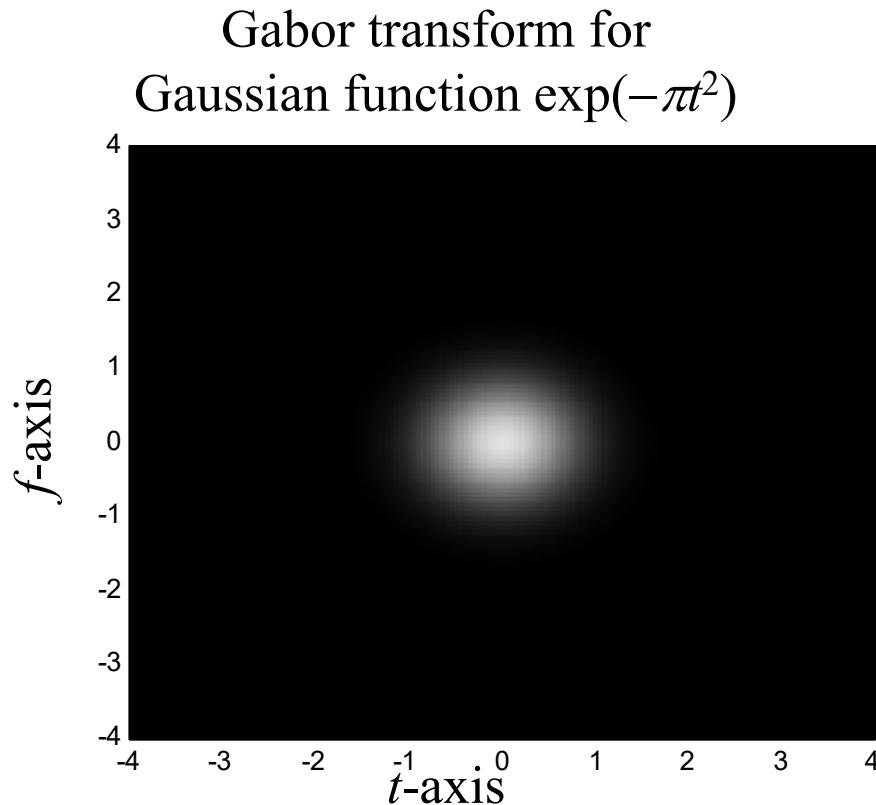
所以對 Gaussian function 而言，

$$\sigma_t \sigma_f = \frac{1}{4\pi}$$

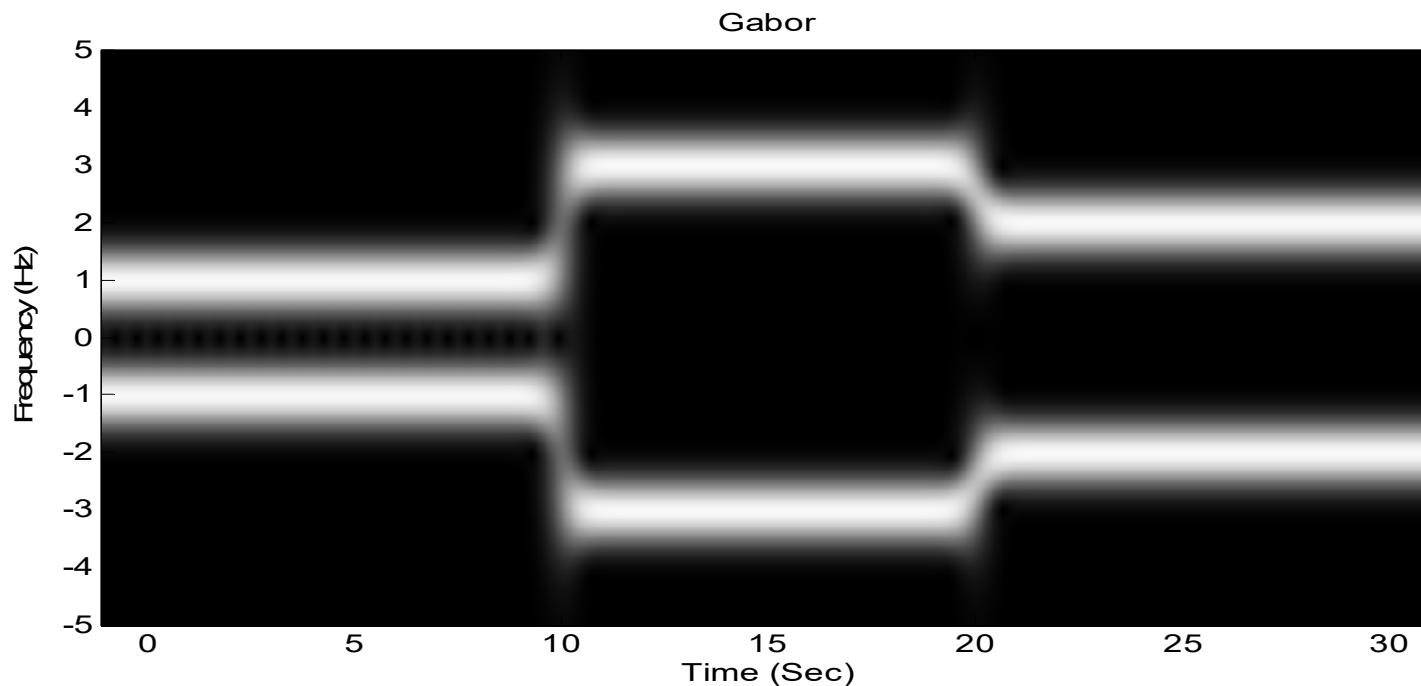
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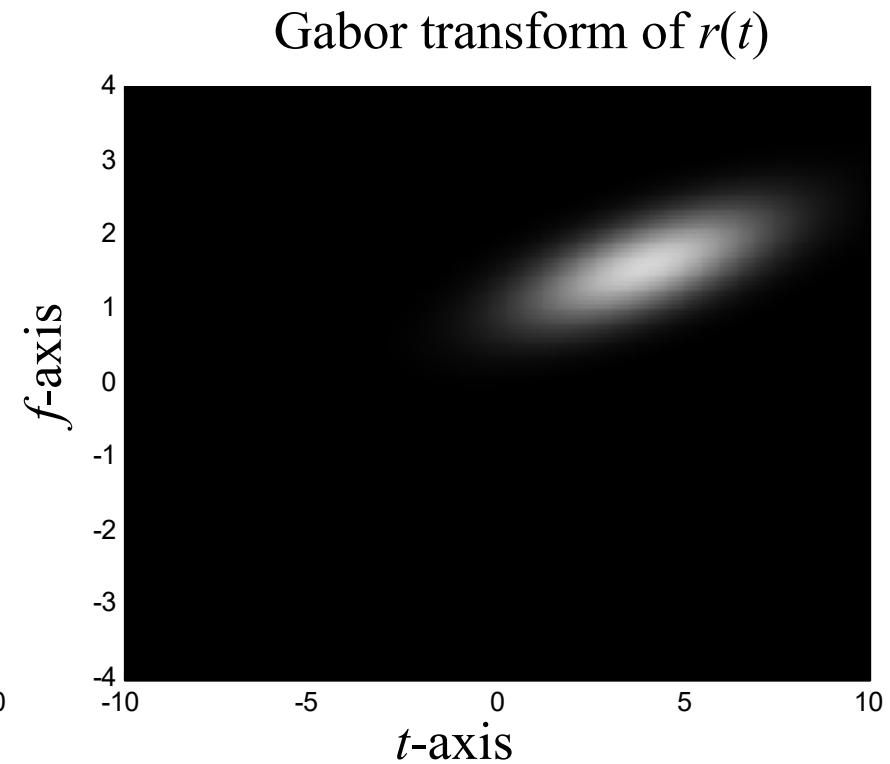
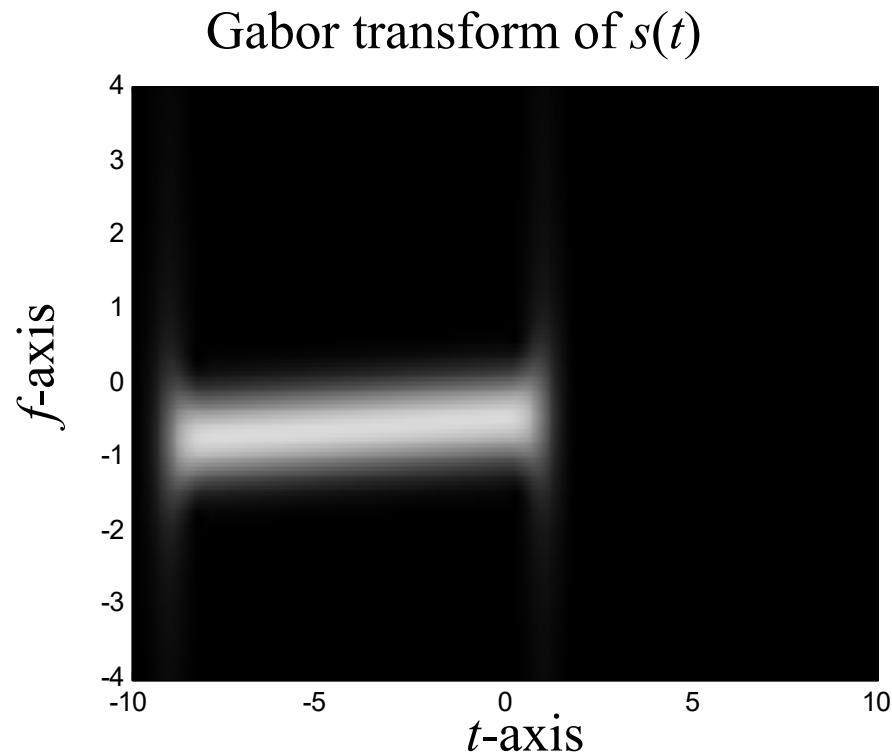
[工具書] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., 2009.

III-D Simulations



$x(t) = \cos(2\pi t)$ when $t < 10$,
 $x(t) = \cos(6\pi t)$ when $10 \leq t < 20$,
 $x(t) = \cos(4\pi t)$ when $t \geq 20$

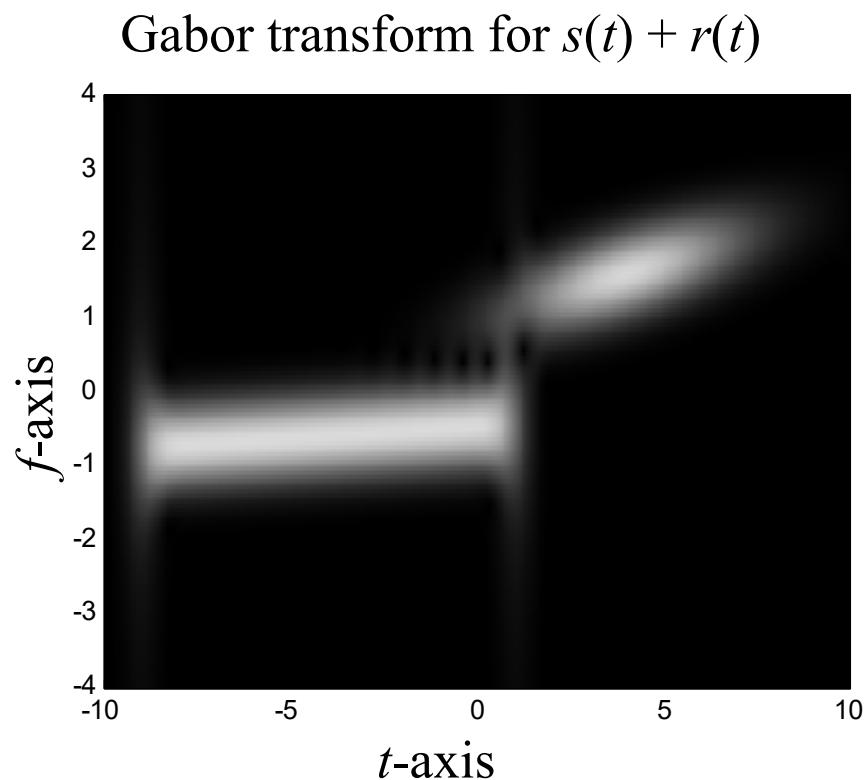




$$s(t) = \exp(jt^2/10 - j3t) \text{ for } -9 \leq t \leq 1,$$

$$s(t) = 0 \text{ otherwise,}$$

$$r(t) = \exp(jt^2/2 + jt) \exp[-(t-4)^2/10]$$



III-E Properties of Gabor Transforms

$$G_x(t, f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau} e^{-\pi(\tau-t)^2} x(\tau) d\tau$$

(1) Integration property

When $k \neq 0$, $\int_{-\infty}^{\infty} G_x(t, f) e^{j2\pi kt f} df = e^{-\pi(k-1)^2 t^2} x(kt)$

When $k = 0$, $\int_{-\infty}^{\infty} G_x(t, f) df = e^{-\pi t^2} x(0)$

When $k = 1$, $\int_{-\infty}^{\infty} G_x(t, f) e^{j2\pi t f} df = x(t)$ (recovery property)

(2) Shifting property

If $y(t) = x(t - t_0)$, then $G_y(t, f) = G_x(t - t_0, f) e^{-j2\pi f t_0}$.

(3) Modulation property

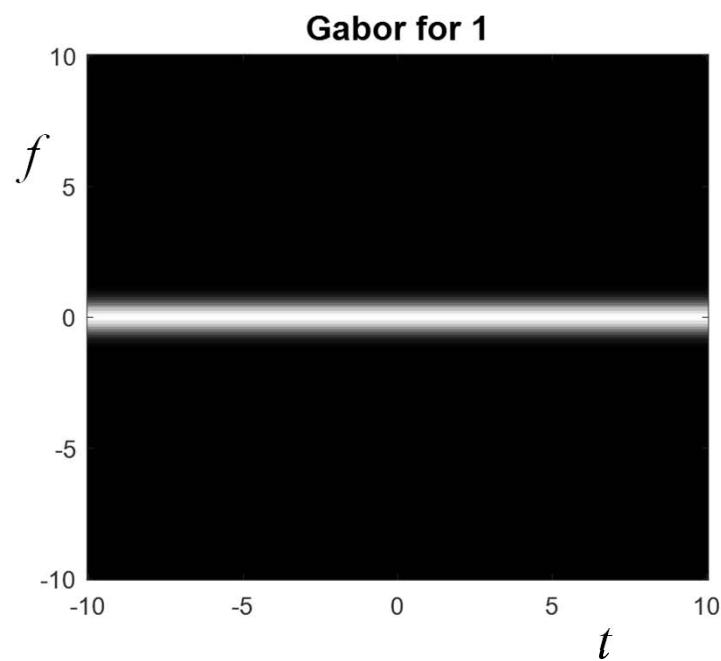
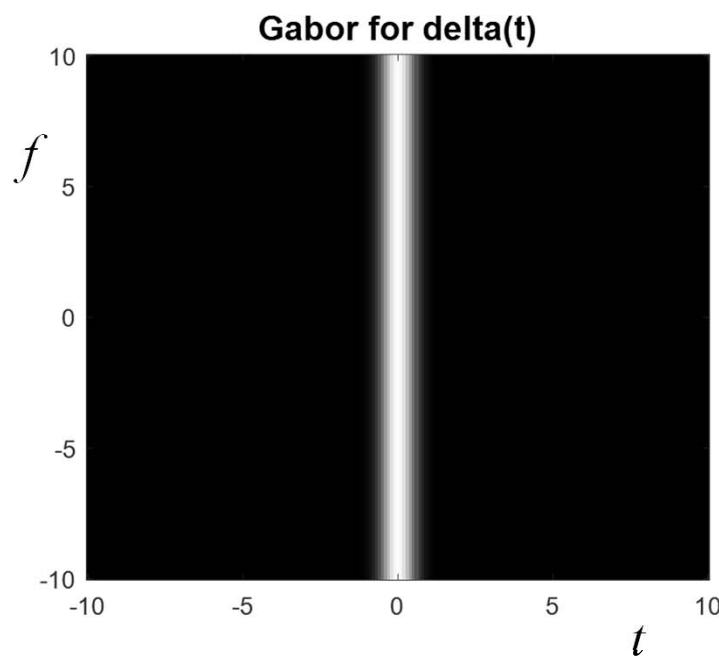
If $y(t) = x(t) \exp(j2\pi f_0 t)$, then $G_y(t, f) = G_x(t, f - f_0)$

(4) Special inputs:

(a) When $x(\tau) = \delta(\tau)$, $G_x(t, f) = e^{-\pi t^2}$

(b) When $x(\tau) = 1$, $G_x(t, f) = e^{-j2\pi f t} e^{-\pi f^2}$

(symmetric for the time and frequency domains)



(5) Power decayed property

- If $x(t) = 0$ for $t > t_0$, then

$$\int_{-\infty}^{\infty} |G_x(t, f)|^2 df < e^{-2\pi(t-t_0)^2} \int_{-\infty}^{\infty} |G_x(t_0, f)|^2 df$$

i.e., $\underset{(\text{fix } t, \text{ vary } f)}{\text{average of}} |G_x(t, f)|^2 < e^{-2\pi(t-t_0)^2} \times \underset{(\text{fix } t_0, \text{ vary } f)}{\text{average of}} |G_x(t_0, f)|^2$ for $t > t_0$.

(Proof):

$$G_x(t, f) = \int_{-\infty}^{t_0} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau) d\tau \quad G_x(t_0, f) = \int_{-\infty}^{t_0} e^{-\pi(\tau-t_0)^2} e^{-j2\pi f\tau} x(\tau) d\tau$$

$$\text{Since } (\tau - t)^2 > (\tau - t_0)^2 + (t_0 - t)^2 \quad e^{-\pi(t-\tau)^2} < e^{-\pi(t-t_0)^2} e^{-\pi(t_0-\tau)^2}$$

$$G_x(t, f) < e^{-\pi(t-t_0)^2} G_x(t_0, f)$$

- If $X(f) = FT[x(t)] = 0$ for $f > f_0$, then

$\underset{(\text{fix } f, \text{ vary } t)}{\text{average of}} |G_x(t, f)|^2 < e^{-2\pi(f-f_0)^2} \times \underset{(\text{fix } f_0, \text{ vary } t)}{\text{average of}} |G_x(t, f_0)|^2$ for $f > f_0$.

(6) Linearity property

If $z(\tau) = \alpha x(\tau) + \beta y(\tau)$ and $G_z(t, f)$, $G_x(t, f)$ and $G_y(t, f)$ are their Gabor transforms, then

$$G_z(t, f) = \alpha G_x(t, f) + \beta G_y(t, f)$$

(7) Power integration property:

$$\int_{-\infty}^{\infty} |G_x(t, f)|^2 df = \int_{-\infty}^{\infty} e^{-2\pi(\tau-t)^2} |x(\tau)|^2 d\tau \approx \int_{u-1.9143}^{u+1.9143} e^{-2\pi(\tau-u)^2} |x(\tau)|^2 d\tau$$

(8) Energy sum property

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_x(t, f) G_y^*(t, f) df dt = \int_{-\infty}^{\infty} x(\tau) y^*(\tau) d\tau$$

where $G_x(t, f)$ and $G_y(t, f)$ are the Gabor transforms of $x(\tau)$ and $y(\tau)$, respectively.

III-F Scaled Gabor Transforms

$$G_x(t, f) = \sqrt[4]{\sigma} \int_{-\infty}^{\infty} e^{-\sigma\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau) d\tau$$

 (finite interval form)

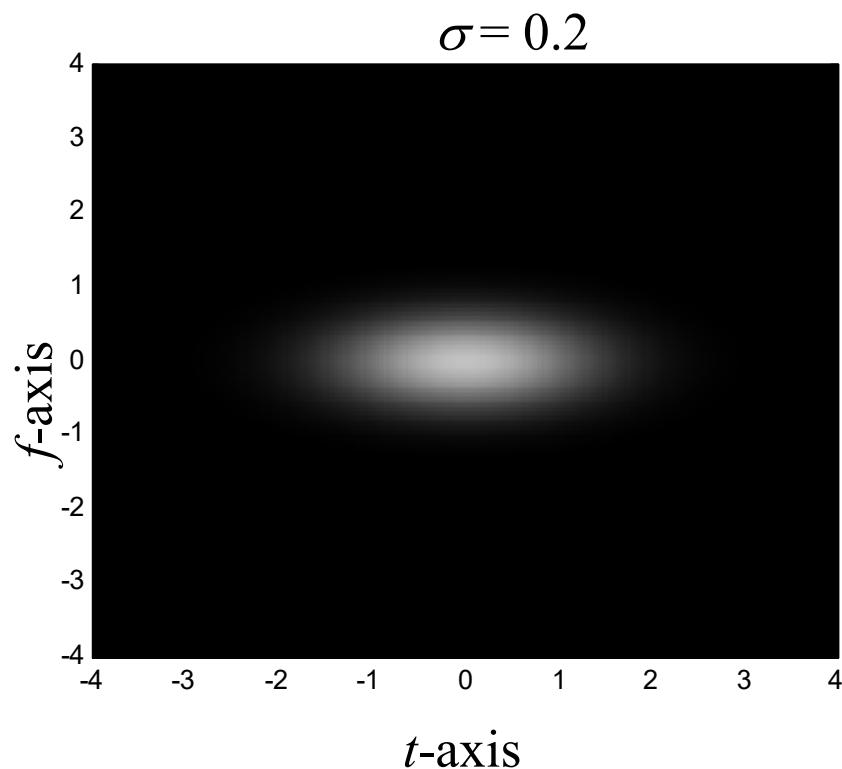
larger σ : higher resolution in the time domain

lower resolution in the frequency domain

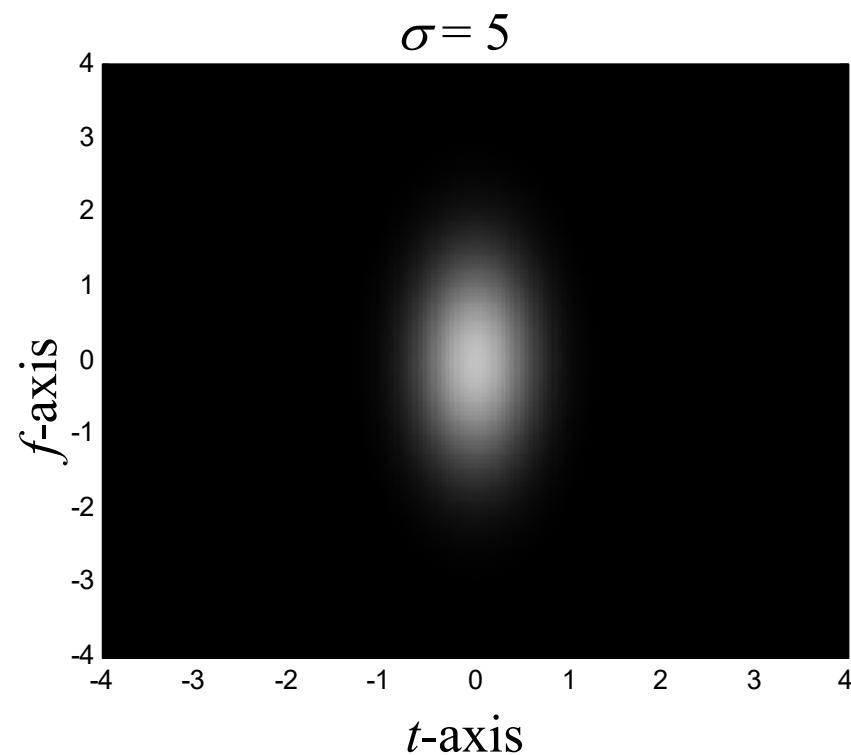
smaller σ : higher resolution in the frequency domain

lower resolution in the time domain

Gabor transform for
Gaussian function $\exp(-\pi t^2)$



Gabor transform for
Gaussian function $\exp(-\pi t^2)$



處理對 time resolution 相對上比 frequency resolution 敏感的信號

- (1) Using the generalized Gabor transform with larger σ
- (2) Using other time unit instead of second

例如，原本 t (單位 : sec) f (單位 : Hz)

對聲音信號可以改成

t (單位 : 0.1 sec) f (單位 : 10 Hz)

III-G Gabor Transforms with Adaptive Window Width

For a signal,

when the instantaneous frequency varies fast \rightarrow larger σ

when instantaneous frequency varies slowly \rightarrow smaller σ

$$G_x(t, f) = \sqrt{\sigma(t)} \int_{-\infty}^{\infty} e^{-\sigma(t)\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau) d\tau$$

$\sigma(t)$ is a function of t

S. C. Pei and S. G. Huang, “STFT with adaptive window width based on the chirp rate,” *IEEE Trans. Signal Processing*, vol. 60, issue 8, pp. 4065-4080, 2012.

附錄五：Matlab 寫程式的原則以及部分常用的指令

- (1) 迴圈能避免就儘量避免
- (2) 儘可能使用 Matrix 及 Vector operation
- (3) 能夠不在迴圈內做的運算，則移到迴圈外
- (4) 寫一部分即測試，不要全部寫完再測試 (縮小範圍比較容易 debug)
- (5) 先測試簡單的例子，成功後再測試複雜的例子

註：作業 Matlab Program (or Python program) 鼓勵各位同學儘量用精簡而快速的方式寫。Program 執行速度越快，分數就越高。

一些重要的 Matlab 指令

(1) `function`: 放在第一行，可以將整個程式函式化

(2) `tic, toc`: 計算時間

`tic` 為開始計時，`toc` 為顯示時間

(3) `find`: 找尋一個 vector 當中不等於 0 的 entry 的位置

範例： $\text{find}([1\ 0\ 0\ 1]) = [1, 4]$

$\text{find}(\text{abs}([-5:5]) \leq 2) = [4, 5, 6, 7, 8]$

(因為 $\text{abs}([-5:5]) \leq 2 = [0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0]$)

(4) `'`: Hermitian (transpose + conjugation) , `.'`: transpose

(5) `imread`: 讀圖

(註：較老的 Matlab 版本 `imread` 要和 `double` 並用

```
A=double(imread('Lena.bmp'));
```

(6) `image`: 將圖顯示出來，

(i) 顯示灰階圖

`image(A)` % A has the size of $M \times N \times 1$

`colormap(gray(256))`

(ii) 顯示彩色圖，整數的情形

`image(A)` % A has the size of $M \times N \times 3$ and the entries are integer

(iii) 顯示彩色圖，非整數的情形

`image(A)` % A has the size of $M \times N \times 3$ and the entries are non-integer

(7) `imshow`, `imagesc`: 也可用來顯示圖

(8) `imwrite`: 製做圖檔

(9) `aviread`: 讀取 video 檔，限副檔名為 avi

(10) `VideoReader`: 讀取 video 檔

(11) `VideoWriter`: 製作 video 檔

- (12) `xlsread` 或 `readmatrix` 或 `readcell`: 由 Excel 檔讀取資料
- (13) `xlswrite`: 將資料寫成 Excel 檔
- (14) `dlmread`: 讀取 *.txt 或其他類型檔案的資料
- (15) `dlmwrite`: 將資料寫成 *.txt 或其他類型檔案

附錄六：寫 Python 版本程式可能會用到的重要指令

建議必安裝模組

pip install numpy

pip install scipy

pip install opencv-python

pip install matplotlib

(1) 定義函式：使用def

(2) 計算時間

```
import time
```

```
start_time = time.time() #獲取當前時間
```

```
end_time = time.time()
```

```
total_time = end_time - start_time #計算時間差來得到總執行時間
```

感謝2021年擔任助教的蔡昌廷同學

(3) 讀取圖檔、顯示圖檔、輸出圖檔

(方法一)

```
import cv2  
  
image = cv2.imread(file_name) #預設color channel為BGR  
cv2.imshow('test', image)  
# 若 image 的值非整數，要改成 cv2.imshow('test', image/255)  
cv2.waitKey(0)  
cv2.destroyAllWindows()  
cv2.imwrite(file_name, image) #需將color channel轉為BGR
```

(方法二)

```
import matplotlib.pyplot as plt  
  
image = plt.imread(file_name) #預設color channel為RGB  
plt.imshow(image)  
# 若 image 的值非整數，要改成 plt.imshow(image/255)  
plt.show()  
plt.imsave(file_name, image) #需將color channel轉為RGB
```

(4) 尋找array中滿足特定條件的值的位置

(相當於 Matlab 的 find 指令)

```
import numpy as np
```

```
a = np.array([0, 1, 2, 3, 4, 5])
```

```
index = np.where(a > 3) # 回傳array([4, 5])
```

```
print(index)
```

```
array([4, 5], dtype=int64),)
```

```
index[0][0]
```

```
4
```

```
index[0][1]
```

```
5
```

```
A1= np.array([[1,3,6],[2,4,5]])  
index = np.where(A1 > 3)  
print(index)
```

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 5 \end{bmatrix}$$

(array([0, 1, 1], dtype=int64), array([2, 1, 2], dtype=int64))

(代表滿足 $\mathbf{A}_1 > 3$ 的點的位置座標為 [0, 2], [1, 1], [1, 2]

```
[index[0][0], index[1][0]]
```

[0, 2]

```
[index[0][1], index[1][1]]
```

[1, 1]

```
[index[0][2], index[1][2]]
```

[1, 2]

(5) Hermitian、transpose

```
import numpy as np  
result = np.conj(matrix.T)    # Hermitian  
result = matrix.T    # transpose
```

(6) 在 Python 當中讀取 Matlab 當中的 mat 檔

```
data = scipy.io.loadmat('***.mat')  
y = np.array(data['y'])    # 假設 y 是 ***.mat 當中儲存的資料
```