VI. Other Time Frequency Distributions

Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 6, Prentice Hall, N.J., 1996.

Requirements for time-frequency analysis:

(1) higher clarity $\xleftrightarrow{\text{tradeoff}}$ (2) avoid cross-term

(3) less computation time (4) good mathematical properties

VI-A Cohen's Class Distribution

VI-A-1 Ambiguity Function

$$A_x(\tau,\eta) = \int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt$$

(1) If $x(t) = \exp\left[-\alpha\pi(t-t_0)^2 + j2\pi f_0 t\right]$

$$A_x(\tau,\eta) = \int_{-\infty}^{\infty} e^{-\alpha\pi(t+\tau/2-t_0)^2 + j2\pi f_0(t+\tau/2)} e^{-\alpha\pi(t-\tau/2-t_0)^2 - j2\pi f_0(t-\tau/2)} \cdot e^{-j2\pi t\eta}\cdot dt$$

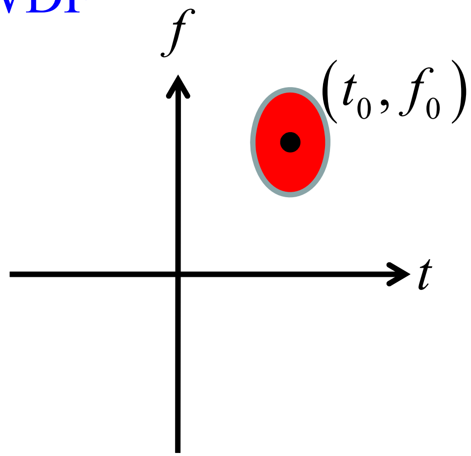
$$= \int_{-\infty}^{\infty} e^{-\alpha\pi\left[2(t-t_0)^2+\tau^2/2\right]+j2\pi f_0\tau} \cdot e^{-j2\pi t\eta}\cdot dt$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\pi\left[2t^2+\tau^2/2\right]+j2\pi f_0\tau} \cdot e^{-j2\pi t\eta} e^{-j2\pi t_0\eta}\cdot dt$$

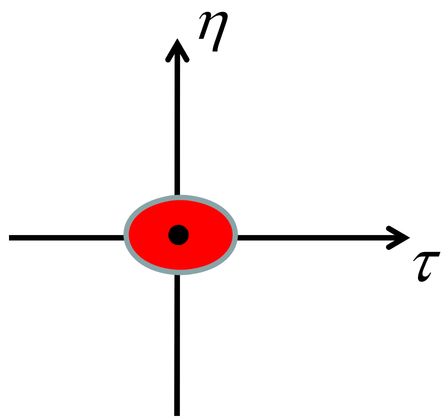
$$\boxed{A_x(\tau,\eta) = \sqrt{\frac{1}{2\alpha}}\exp\left[-\pi\left(\frac{\alpha\tau^2}{2}+\frac{\eta^2}{2\alpha}\right)\right]\exp\left[j2\pi\left(f_0\tau - t_0\eta\right)\right]}$$

WDF and AF for the signal with only 1 term

WDF



AF



(2) If $x(t)=\underbrace{\exp\left[-\alpha_1\pi(t-t_1)^2+j2\pi f_1 t\right]}_{x_1(t)}+\underbrace{\exp\left[-\alpha_2\pi(t-t_2)^2+j2\pi f_2 t\right]}_{x_2(t)}$

$$
\begin{aligned}
A_x(\tau,\eta)=&\int_{-\infty}^{\infty} x_1(t+\tau/2)\cdot x_1^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt+ &&\Leftarrow A_{x_1}(\tau,\eta)\\
&\int_{-\infty}^{\infty} x_2(t+\tau/2)\cdot x_2^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt+ &&\Leftarrow A_{x_2}(\tau,\eta)\\
&\int_{-\infty}^{\infty} x_1(t+\tau/2)\cdot x_2^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt+ &&\Leftarrow A_{x_1x_2}(\tau,\eta)\\
&\int_{-\infty}^{\infty} x_2(t+\tau/2)\cdot x_1^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt+ &&\Leftarrow A_{x_2x_1}(\tau,\eta)
\end{aligned}
$$

$$\boxed{A_x(\tau,\eta)=A_{x_1}(\tau,\eta)+A_{x_2}(\tau,\eta)+A_{x_1x_2}(\tau,\eta)+A_{x_2x_1}(\tau,\eta)}$$

$$A_{x_1}(\tau,\eta)=\sqrt{\frac{1}{2\alpha_1}}\exp\left[-\pi\left(\frac{\alpha_1\tau^2}{2}+\frac{\eta^2}{2\alpha_1}\right)\right]\exp\left[j2\pi\left(f_1\tau-t_1\eta\right)\right]$$

$$A_{x_2}(\tau,\eta)=\sqrt{\frac{1}{2\alpha_2}}\exp\left[-\pi\left(\frac{\alpha_2\tau^2}{2}+\frac{\eta^2}{2\alpha_2}\right)\right]\exp\left[j2\pi\left(f_2\tau-t_2\eta\right)\right]$$

When $\alpha_1 = \alpha_2$

$$\boxed{\begin{aligned} A_{x_1x_2}(\tau,\eta) &= \sqrt{\frac{1}{2\alpha_\mu}}\exp\left[-\pi\left(\alpha_\mu\frac{(\tau-t_d)^2}{2}+\frac{(\eta-f_d)^2}{2\alpha_\mu}\right)\right] \\ &\times\exp\left[j2\pi(f_\mu\tau-t_\mu\eta+f_d t_\mu)\right] \end{aligned}}$$

$$t_\mu=(t_1+t_2)/2\,,\quad f_\mu=(f_1+f_2)/2\,,\quad \alpha_\mu=(\alpha_1+\alpha_2)/2\,,$$

$$t_d=t_1-t_2\,,\quad f_d=f_1-f_2\,,\quad \alpha_d=\alpha_1-\alpha_2$$

$$\boxed{A_{x_2x_1}(\tau,\eta)=A^*_{x_1x_2}(-\tau,-\eta)}$$

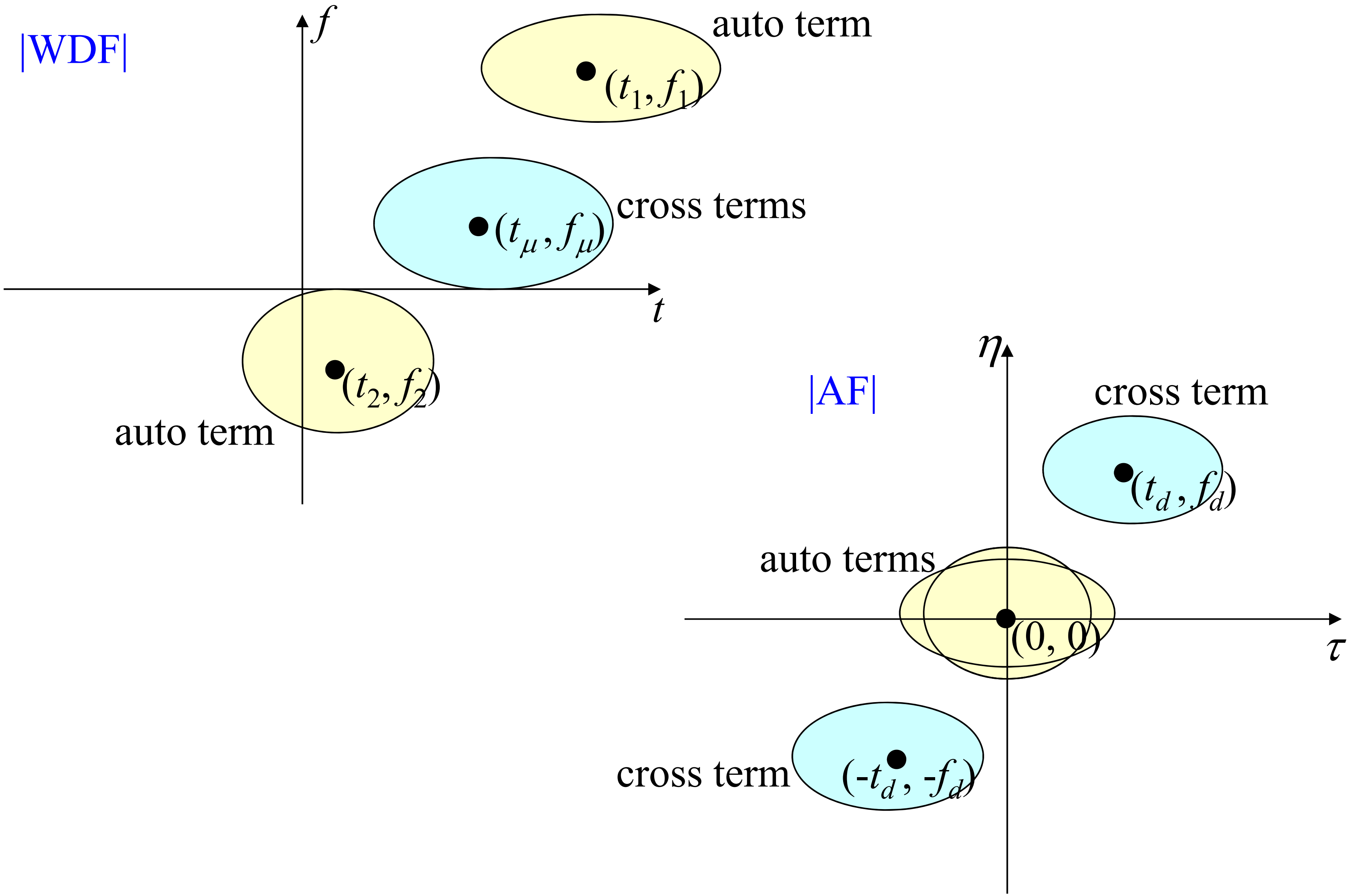
When $\alpha_1 \neq \alpha_2$

$$A_{x_1x_2}(\tau,\eta)=\sqrt{\frac{1}{2\alpha_\mu}}\exp\left[-\pi\frac{[(\eta-f_d)+j(\alpha_1t_1+\alpha_2t_2)-j\alpha_d\tau/2]^2}{2\alpha_\mu}\right]$$

$$\exp\left[-\pi\left(\alpha_1(t_1-\frac{\tau}{2})^2+\alpha_2(t_2+\frac{\tau}{2})^2\right)\right]\exp\left[j2\pi f_\mu\tau\right]$$

$$A_{x_2x_1}(\tau,\eta)=A^*_{x_1x_2}(-\tau,-\eta)$$

WDF and AF for the signal with 2 terms



For the ambiguity function

The auto term is always near to the origin

The cross-term is always far from the origin

VI-A-2 Definition of Cohen's Class Distribution

The Cohen's Class distribution is a further generalization of the Wigner distribution function

$$C_x(t,f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_x(\tau,\eta)\Phi(\tau,\eta)\exp\left(j2\pi(\eta t-\tau f)\right)d\eta\, d\tau$$

where $A_x(\tau,\eta)=\int_{-\infty}^{\infty}x(t+\tau/2)\cdot x^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt$

is the ambiguity function (AF).

$\Phi(\eta,\ \tau) = 1 \rightarrow$ WDF

$$C_x(t,f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(u+\tau/2)x^*(u-\tau/2)\phi(t-u,\tau)du\ e^{-j2\pi f\tau}d\tau$$

where $\phi(t,\tau)=\int_{-\infty}^{\infty}\Phi(\tau,\eta)\exp(j2\pi\eta t)d\eta$

How does the Cohen's class distribution avoid the cross term?

Chose $\Phi(\tau, \eta)$ low pass function.

$\Phi(\tau, \eta) \approx 1$ for small $|\eta|, |\tau|$

$\Phi(\tau, \eta) \approx 0$ for large $|\eta|, |\tau|$

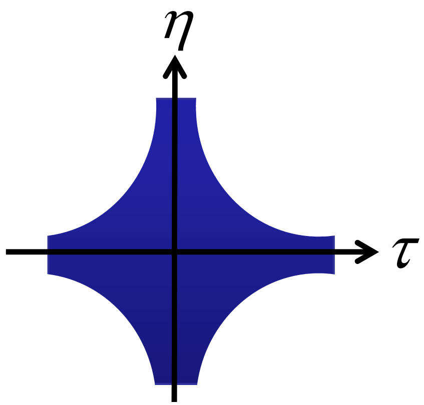
[Ref] L. Cohen, "Generalized phase-space distribution functions," *J. Math. Phys.*, vol. 7, pp. 781-806, 1966.

[Ref] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, New York, 1995.

VI-A-3 Several Types of Cohen's Class Distribution

Choi-Williams Distribution (One of the Cohen's class distribution)

$$\Phi(\tau,\eta) = \exp\left[-\alpha(\eta\tau)^2\right]$$

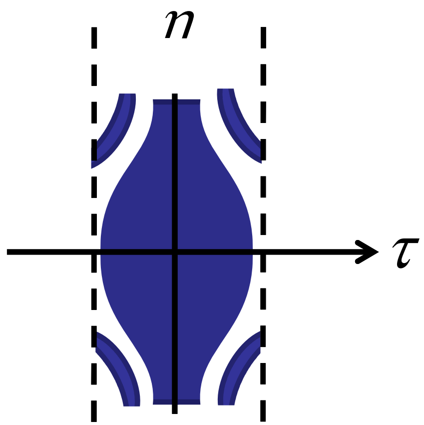


[Ref] H. Choi and W. J. Williams, "Improved time-frequency representation of multicomponent signals using exponential kernels," *IEEE. Trans. Acoustics, Speech, Signal Processing*, vol. 37, no. 6, pp. 862-871, June 1989.

Cone-Shape Distribution (One of the Cohen's class distribution)

$$\phi(t,\tau) = \frac{1}{|\tau|}\exp\left(-2\pi\alpha\tau^2\right)\Pi\left(\frac{t}{\tau}\right)$$

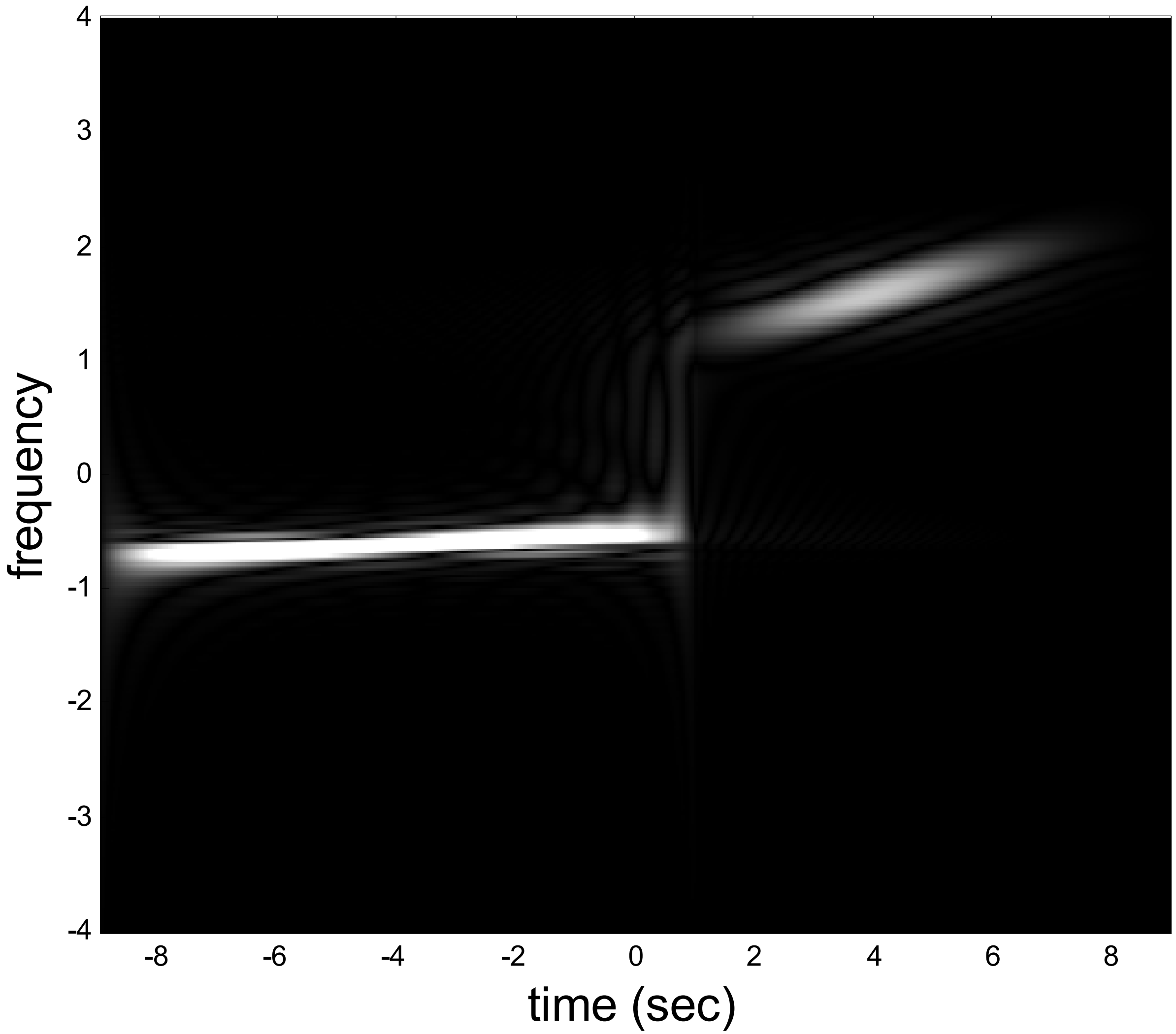
$$\Phi(\tau,\eta) = \sin c(\eta\tau)\exp\left(-2\pi\alpha\tau^2\right)$$



[Ref] Y. Zhao, L. E. Atlas, and R. J. Marks, "The use of cone-shape kernels for generalized time-frequency representations of nonstationary signals," *IEEE Trans. Acoustics, Speech, Signal Processing*, vol. 38, no. 7, pp. 1084-1091, July 1990.

Cone-Shape distribution for the example on pages 97, 150

$(\alpha = 1)$



Distributions	$\Phi(\tau, \eta)$
Wigner	1
Cohen (circular)	$\Phi(\tau, \eta) = 1$ for $\sqrt{\eta^2+\tau^2} < r$ $\Phi(\tau, \eta) = 0$ otherwise
Cohen (rectangular)	$\Phi(\tau, \eta) = 1$ for $Max(\lvert\eta\rvert, \lvert\tau\rvert) < T$ $\Phi(\tau, \eta) = 0$ otherwise
Choi-Williams	$\exp\left[-\alpha(\eta\tau)^2\right]$
Cone-Shape	$\sin c(\eta\tau)\exp\left(-2\pi\alpha\tau^2\right)$
Page	$\exp\left(j\pi\eta\lvert\tau\rvert\right)$
Levin (Margenau-Hill)	$\cos(\pi\eta\tau)$
Born-Jordan	$\sin c(\eta\tau)$

註：感謝 2007年修課的王文阜同學

VI-A-4 Advantages and Disadvantages of Cohen's Class Distributions

The Cohen's class distribution may avoid the cross term and has higher clarity.

However, it requires more computation time and lacks of well mathematical properties.

Moreover, there is a tradeoff between the quality of the auto term and the ability of removing the cross terms.

VI-A-5 Implementation for the Cohen's Class Distribution

$$C_x(t,f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_x(\tau,\eta)\Phi(\tau,\eta)\exp\left(j2\pi(\eta t-\tau f)\right)d\eta d\tau$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\left(u+\frac{\tau}{2}\right)x^*\left(u-\frac{\tau}{2}\right)\cdot\Phi(\tau,\eta)e^{-j2\pi u\eta+j2\pi(\eta t-\tau f)}du\,d\eta\,d\tau$$

簡化法 1：不是所有的 $A_x(\eta, \tau)$ 的值都需要算出

If $\Phi(\tau,\eta)=0$ for $|\eta| > B$ or $|\tau| > C$

$$C_x(t,f)=\int_{-C}^{C}\int_{-B}^{B}\int_{-\infty}^{\infty}x\left(u+\frac{\tau}{2}\right)x^*\left(u-\frac{\tau}{2}\right)\cdot\Phi(\tau,\eta)e^{-j2\pi u\eta+j2\pi(\eta t-\tau f)}du\,d\eta\,d\tau$$

簡化法 2：注意，η 這個參數和input 及output 都無關

$$C_x(t,f)=\int_{-C}^{C}\int_{-\infty}^{\infty}x\left(u+\frac{\tau}{2}\right)x^*\left(u-\frac{\tau}{2}\right)\cdot\left[\int_{-B}^{B}\Phi(\tau,\eta)e^{j2\pi\eta(t-u)}d\eta\right]e^{-j2\pi\tau f}du\,d\tau$$

$$=\int_{-C}^{C}\int_{-\infty}^{\infty}x\left(u+\frac{\tau}{2}\right)x^*\left(u-\frac{\tau}{2}\right)\cdot\Psi(\tau,t-u)e^{-j2\pi\tau f}du\,d\tau$$

$$\Psi(\tau,t)=\int_{-B}^{B}\Phi(\tau,\eta)e^{j2\pi\eta t}d\eta$$

由於 $\Psi(\tau,t)$ 和 input 無關，可事先算出，所以只剩 2 個積分式

VI-B Modified Wigner Distribution Function

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2) \cdot x^*(t-\tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} X(f+\eta/2) \cdot X^*(f-\eta/2) e^{j2\pi t\eta} \cdot d\eta$$

where $X(f) = FT[x(t)]$

Modified Form I

$$W_x(t,f) = \int_{-B}^{B} w(\tau) x(t+\tau/2) \cdot x^*(t-\tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

Modified Form II

$$W_x(t,f) = \int_{-B}^{B} w(\eta) X(f+\eta/2) \cdot X^*(f-\eta/2) e^{j2\pi t\eta} \cdot d\eta$$

Modified Form III (Pseudo *L*-Wigner Distribution)

$$W_x(t,f)=\int_{-\infty}^{\infty} w(\tau)\,x^L\left(t+\frac{\tau}{2L}\right)\cdot\overline{x^L\left(t-\frac{\tau}{2L}\right)}e^{-j2\pi\tau f}\cdot d\tau$$

增加 L 可以減少 cross term 的影響 (但是不會完全消除)

[Ref] L. J. Stankovic, S. Stankovic, and E. Fakultet, “An analysis of instantaneous frequency representation using time frequency distributions-generalized Wigner distribution,” *IEEE Trans. on Signal Processing,* pp. 549-552, vol. 43, no. 2, Feb. 1995

P.S.: 感謝2006年修課的林政豪同學

Modified Form IV (Polynomial Wigner Distribution Function)

$$W_x(t,f)=\int_{-\infty}^{\infty}\left[\prod_{l=1}^{q/2}x(t+d_l\tau)x^*(t-d_{-l}\tau)\right]e^{-j2\pi\tau f}d\tau$$

When $q = 2$ and $d_1 = d_{-1} = 0.5$, it becomes the original Wigner distribution function.

It can avoid the cross term when the order of phase of the exponential function is no larger than $q/2 + 1$.

However, the cross term between two components cannot be removed.

[Ref] B. Boashash and P. O'Shea, "Polynomial Wigner-Ville distributions & their relationship to time-varying higher order spectra," *IEEE Trans. Signal Processing*, vol. 42, pp. 216–220, Jan. 1994.

[Ref] J. J. Ding, S. C. Pei, and Y. F. Chang, "Generalized polynomial Wigner spectrogram for high-resolution time-frequency analysis," *APSIPA ASC*, Kaohsiung, Taiwan, Oct. 2013.

d_l should be chosen properly such that

$$\prod_{l=1}^{q/2} x(t+d_l\tau)x^*(t-d_{-l}\tau) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} na_n t^{n-1}\tau \right)$$

when $\quad x(t) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} a_n t^n \right)$

then

$$W_x(t,f) = \int_{-\infty}^{\infty} \exp\left(-j2\pi(f - \sum_{n=1}^{q/2+1} na_n t^{n-1})\tau \right) d\tau \cong \delta\left(f - \sum_{n=1}^{q/2+1} na_n t^{n-1} \right)$$

(from page 139(1))

page 139(3)

$$\prod_{l=1}^{q/2} x(t+d_l\tau)x^*(t-d_{-l}\tau) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} n a_n t^{n-1}\tau \right)$$

$$x(t) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} a_n t^n \right)$$

when $q = 2$ $\quad x(t) = \exp\left(j2\pi(a_1 t + a_2 t^2) \right)$

$$x(t+d_1\tau)x^*(t-d_{-1}\tau) = \exp\left(j2\pi(a_1 + 2a_2 t)\tau \right)$$

$$a_2\left(t+d_1\tau\right)^2 + a_1\left(t+d_1\tau\right) - a_2\left(t-d_{-1}\tau\right)^2 - a_1\left(t-d_{-1}\tau\right) = 2a_2 t\tau + a_1\tau$$

$$2a_2(d_1+d_{-1})t\tau + a_2(d_1^2 - d_{-1}^2)\tau^2 + a_1(d_1+d_{-1})\tau = 2a_2 t\tau + a_1\tau$$

$\Rightarrow$ $d_1 + d_{-1} = 1 \qquad d_1 - d_{-1} = 0$

$\Rightarrow$ $d_1 = d_{-1} = 1/2$

When $q = 4$, $x(t) = \exp\left(j2\pi(a_1 t + a_2 t^2 + a_3 t^3)\right)$

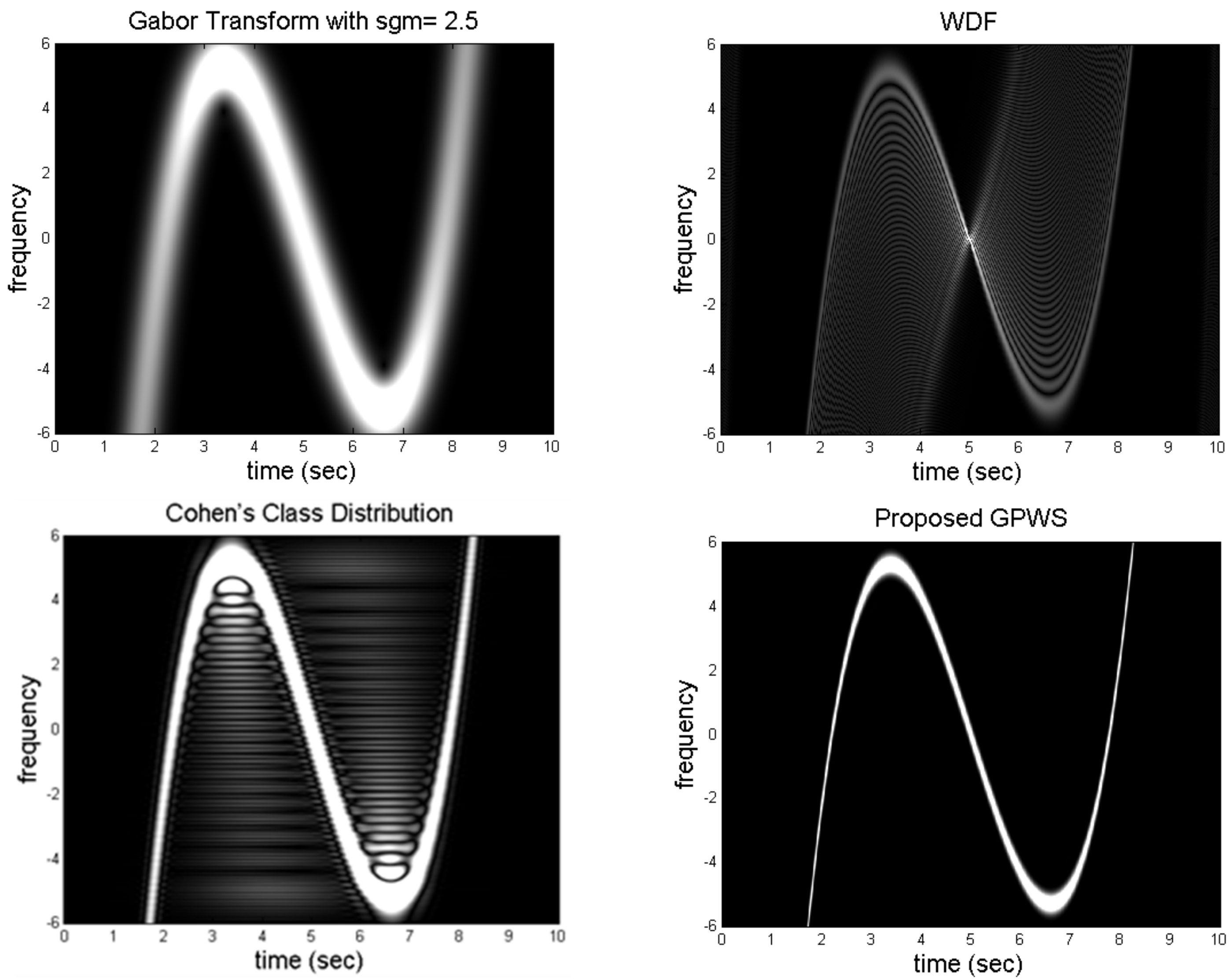
$$\prod_{l=1}^{2} x(t + d_l \tau) x^*(t - d_{-l}\tau) = \exp\left(j2\pi \sum_{n=1}^{3} n a_n t^{n-1} \tau \right)$$

$$x(t + d_1\tau)x^*(t - d_{-1}\tau)x(t + d_2\tau)x^*(t - d_{-2}\tau) = \exp\left(j2\pi \sum_{n=1}^{3} n a_n t^{n-1} \tau \right)$$

$$\begin{aligned}
&a_3\left(t + d_1\tau\right)^3 + a_2\left(t + d_1\tau\right)^2 + a_1\left(t + d_1\tau\right) \\
&+a_3\left(t + d_2\tau\right)^3 + a_2\left(t + d_2\tau\right)^2 + a_1\left(t + d_2\tau\right) \\
&-a_3\left(t - d_{-1}\tau\right)^3 - a_2\left(t - d_{-1}\tau\right)^2 - a_1\left(t - d_{-1}\tau\right) \\
&-a_3\left(t - d_{-2}\tau\right)^3 - a_2\left(t - d_{-2}\tau\right)^2 - a_1\left(t - d_{-2}\tau\right) \\
&= 3a_3 t^2 \tau + 2a_2 t\tau + a_1\tau
\end{aligned}$$

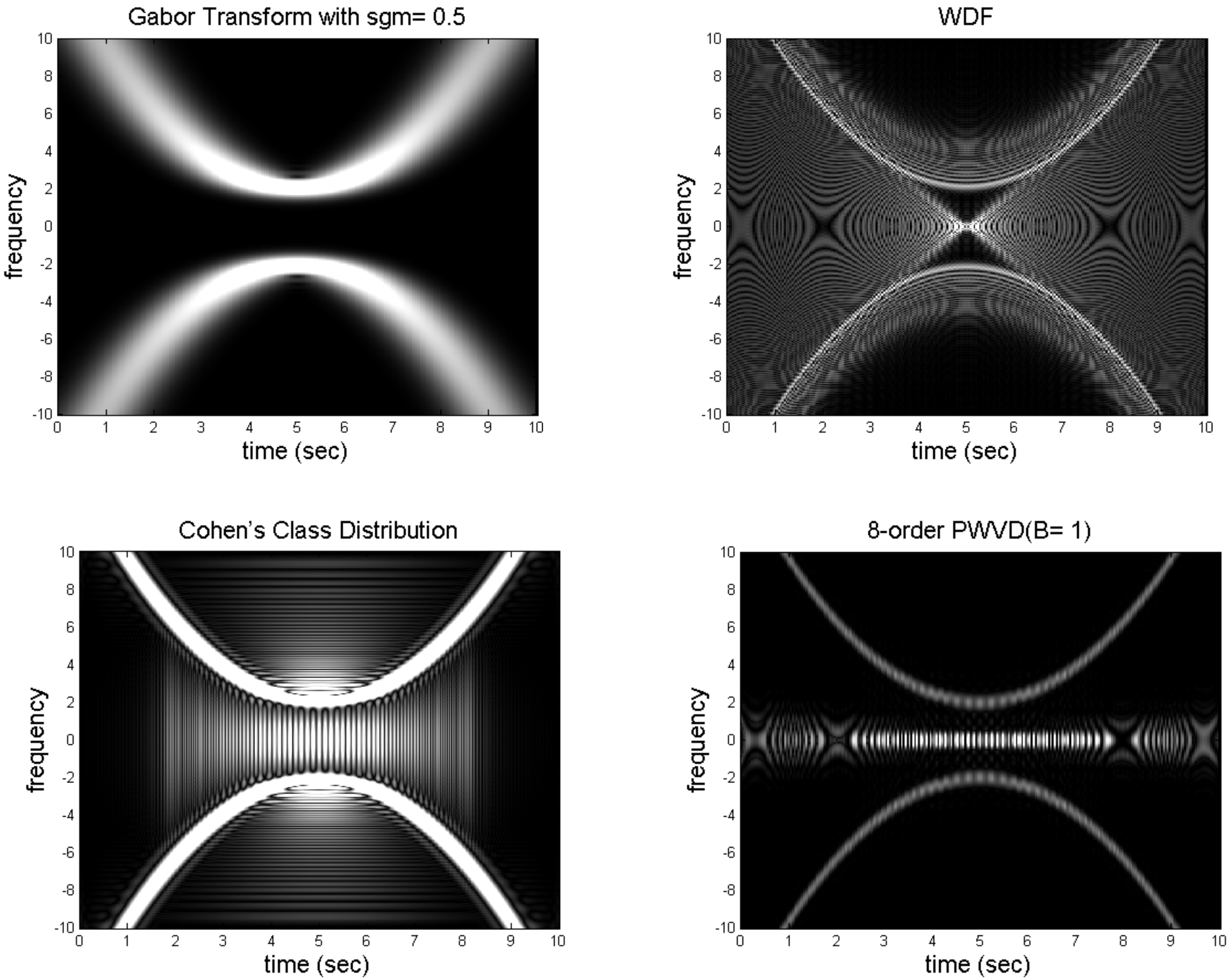
$$\Rightarrow \begin{cases} d_1 + d_2 + d_{-1} + d_{-2} = 1 \\ d_1^2 + d_2^2 - d_{-1}^2 - d_{-2}^2 = 0 \\ d_1^3 + d_2^3 + d_{-1}^3 + d_{-2}^3 = 0 \end{cases}$$

$$x(t) = \exp(j(t-5)^4 - j5\pi(t-5)^2)$$



$q = ?$

$$x(t) = 2\cos((t-5)^3 + 4\pi t)$$



VI-C Gabor-Wigner Transform

[Ref] S. C. Pei and J. J. Ding, "Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing," *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

Advantages:

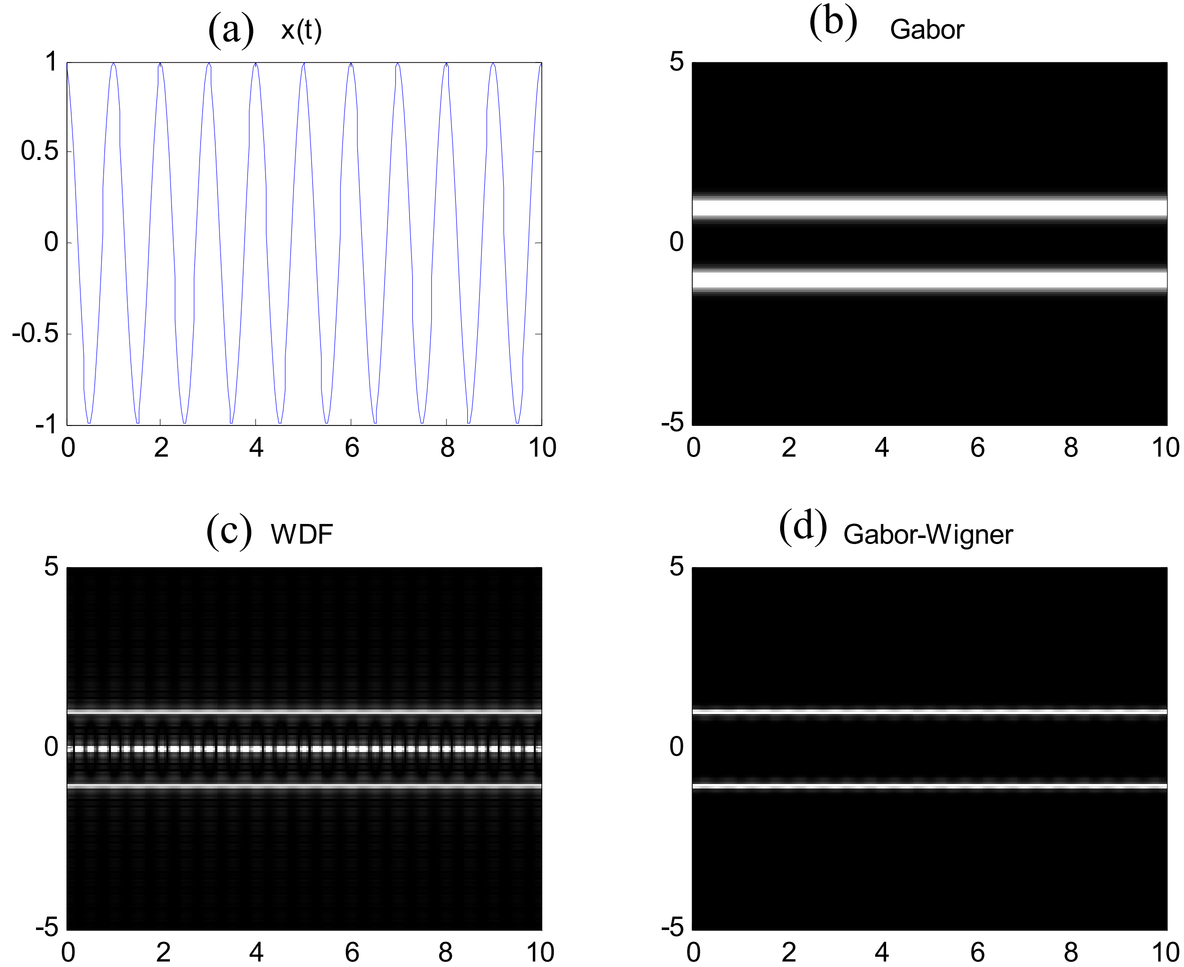
combine the advantage of the WDF and the Gabor transform

advantage of the WDF → higher clarity

advantage of the Gabor transform → no cross-term

$$D_x(t,f) = G_x^2(t,f)W_x(t,f)$$

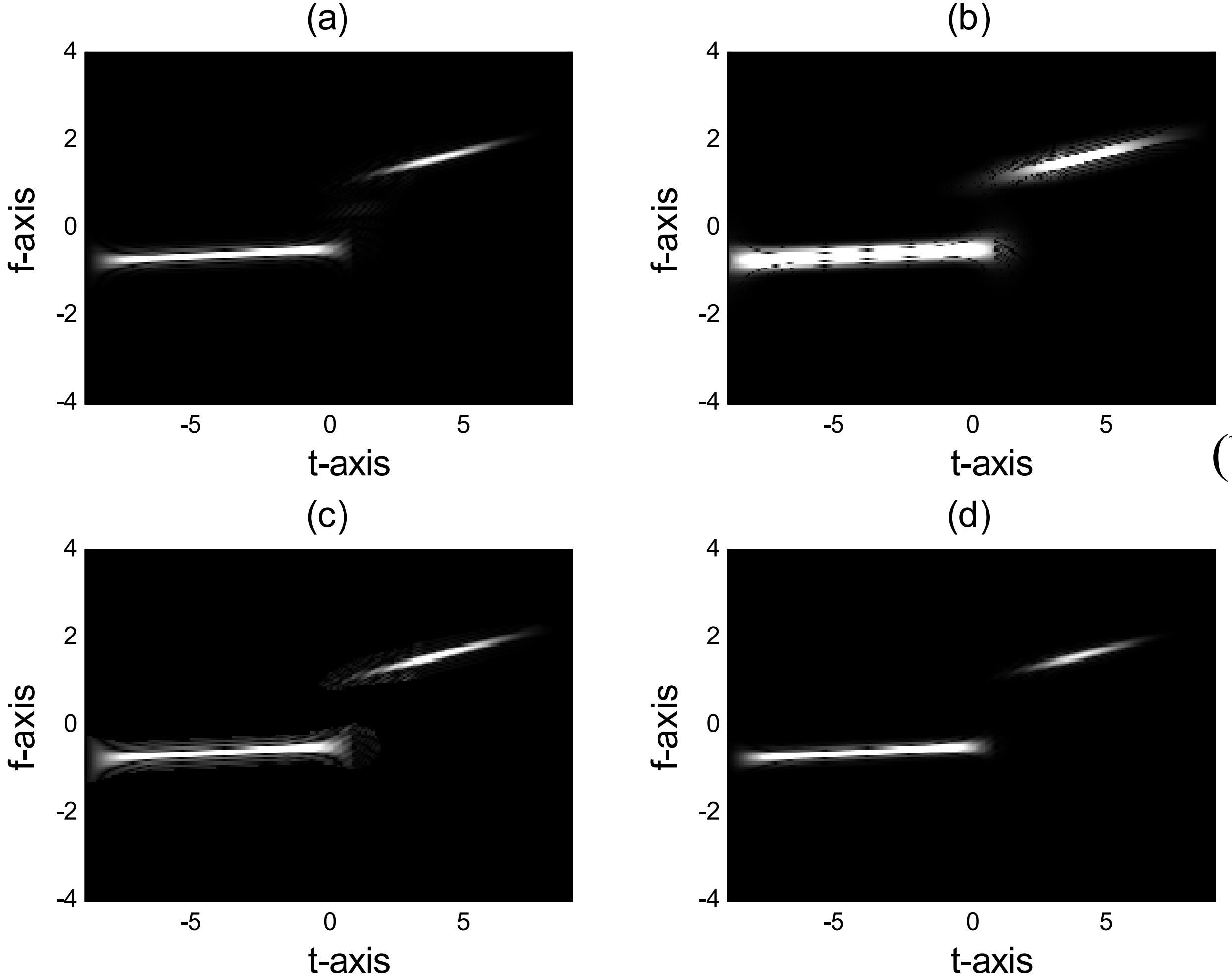
$$x(t) = \cos(2\pi t)$$



(a) $D_x\left(t,f\right)=G_x(t,f)W_x(t,f)$ (b) $D_x\left(t,f\right)=\min\left(|G_x(t,f)|^2,|W_x(t,f)|\right)$

(c) $D_x\left(t,f\right)=W_x(t,f)\times\left\{|G_x(t,f)|>0.25\right\}$

(d) $D_x\left(t,f\right)=G_x^{\alpha}(t,f)W_x^{\beta}(t,f),\quad \alpha=2.6,\quad \beta=0.7$



(b)、(c) are real

思考：

(1) Which type of the Gabor-Wigner transform is better?

(2) Can we further generalize the results?

Implementation of the Gabor-Wigner Transform ：簡化技巧

(1) When $G_x(t,f) \approx 0$, $D_x\left(t,f\right) = G_x^{\alpha}(t,f)W_x^{\beta}(t,f) \approx 0$

先算 $G_x(t,f)$

$W_x(t,f)$ 只需算 $G_x(t,f)$ 不近似於 0 的地方

(2) When $x(t)$ is real，對 Gabor transform 而言

$X(f) = X^*(-f)$ if $x(t)$ is real, where $X(f) = FT[x(t)]$

附錄九： Properties of the Fourier Transform

$$X(f) = FT[x(t)] = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi f t)\,dt$$

(1) Recovery (inverse Fourier transform)	$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi f t)\,df$
(2) Integration	$x(0) = \int_{-\infty}^{\infty} X(f)\,df$
(3) Modulation	$FT\left[x(t)e^{j2\pi f_0 t}\right] = X(f - f_0)$
(4) Time Shifting	$FT[x(t - t_0)] = X(f)e^{-j2\pi f t_0}$
(5) Scaling	$FT[x(at)] = \frac{1}{\lvert a \rvert} X\left(\frac{f}{a}\right)$
(6) Time Reverse	$FT[x(-t)] = X(-f)$
(7) Real Output	If $x(t) = x^*(-t)$, then $X(f)$ is real.

(8) Real / Imaginary Input	If $x(t)$ is real, then $X(f) = X^*(-f)$; If $x(t)$ is pure imaginary, then $X(f) = -X^*(-f)$
(9) Even / Odd Input	If $x(t) = x(-t)$, then $X(f) = X(-f)$; If $x(t) = -x(-t)$, then $X(f) = -X(-f)$;
(10) Conjugation	$FT\left[x^*(t)\right] = X^*(-f)$
(11) Differentiation	$FT\left[x'(t)\right] = j2\pi f\, X(f)$
(12) Multiplication by t	$FT\left[tx(t)\right] = \frac{j}{2\pi} X'(f)$
(13) Division by t	$FT\left[\frac{x(t)}{t}\right] = -j2\pi \int_{-\infty}^{f} X(\mu)\, d\mu$
(14) Parseval's Theorem (Energy Preservation)	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$
(15) Generalized Parseval's Theorem	$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$

(16) Linearity	$FT\left[ax(t)+by(t)\right]=aX(f)+bY(f)$
(17) Convolution	If $z(t)=x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau$, then $Z(f)=X(f)Y(f)$
(18) Multiplication	If $z(t)=x(t)y(t)$, then $Z(f)=X(f)*Y(f)=\int_{-\infty}^{\infty}X(\mu)Y(f-\mu)d\mu$
(19) Correlation	If $z(t)=\int_{-\infty}^{\infty}x(\tau)y^*(\tau-t)d\tau$, then $Z(f)=X(f)Y^*(f)$
(20) Two Times of Fourier Transforms	$FT\left\{FT\left[x(t)\right]\right\}=x(-t)$
(21) Four Times of Fourier Transforms	$FT\left[FT\left(FT\left\{FT\left[x(t)\right]\right\}\right)\right]=x(t)$