Time-Frequency Analysis and Wavelet Transform Final Report: Application of Fractional Fourier Transform in OTFS Systems

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Abstract

Orthogonal Time Frequency Space (OTFS) modulation has emerged as a promising waveform candidate for next-generation (6G) wireless communications, specifically designed to handle high-mobility scenarios such as high-speed railways and Low Earth Orbit (LEO) satellite networks. Unlike traditional Orthogonal Frequency Division Multiplexing (OFDM), which suffers from severe Inter-Carrier Interference (ICI) under high Doppler shifts, OTFS modulates information in the Delay-Doppler (DD) domain, providing a stable and sparse channel interaction.

However, standard OTFS implementation relies on the Symplectic Finite Fourier Transform (SFFT), which assumes a fixed grid alignment. In scenarios with fractional Doppler shifts, this can lead to energy leakage and reduced sparsity. This report investigates the integration of the Fractional Fourier Transform (FrFT) into the OTFS architecture to address these challenges. The FrFT generalizes the classical Fourier Transform by introducing a rotation angle in the time-frequency plane, allowing for optimal energy concentration of chirp-like signals induced by Doppler effects.

We present a comprehensive mathematical framework connecting OTFS and FrFT, demonstrating how tuning the FrFT order can diagonalize the channel matrix and reduce computational complexity in equalization. Finally, numerical simulations are performed using MATLAB to evaluate the Bit Error Rate (BER) performance, confirming that the FrFT-based OTFS scheme outperforms traditional methods in high-mobility channels.

1 Introduction

1.1 Background

The evolution of wireless communication systems, from 4G to 5G and looking ahead to 6G, is driven by the demand for ubiquitous connectivity. A critical frontier in this evolution is supporting high-mobility scenarios. Applications such as high-speed railway systems (>350 km/h), Low Earth Orbit (LEO) satellite communications, and unmanned aerial vehicle (UAV) networks introduce channel conditions that are fundamentally different from the static or pedestrian environments typical of earlier cellular generations.

1.2 The Challenge: Doppler Effects and OFDM Limitations

For decades, Orthogonal Frequency Division Multiplexing (OFDM) has been the dominant waveform (used in WiFi, 4G, 5G) due to its efficient handling of multipath fading. However, OFDM relies on the strict orthogonality of subcarriers in the frequency domain. In high-mobility scenarios, the relative motion between the transmitter and receiver induces a Doppler shift, causing a frequency dispersion. From a signal processing perspective, this destroys the orthogonality of OFDM subcarriers, leading to severe Inter-Carrier Interference (ICI). As the speed increases, the complexity of equalizers required to correct this ICI grows exponentially, making OFDM impractical for ultra-high-speed communications.

1.3 The Solution: OTFS Modulation

To overcome these limitations, Orthogonal Time Frequency Space (OTFS) modulation was proposed. Unlike OFDM, which multiplexes symbols in the Time-Frequency (TF) domain, OTFS multiplexes information symbols in the Delay-Doppler (DD) domain. The fundamental advantage of the DD domain is that the wireless channel appears quasi-static and sparse. Even if the channel is rapidly time-varying in the TF domain (due to high speed), the physical reflectors (buildings, mountains) have relatively stable delays and Doppler velocities. This transforms the complex, fading time-varying channel into a simpler, 2D convolution interaction, significantly improving reliability.

1.4 Motivation for Using FrFT in OTFS

While OTFS significantly outperforms OFDM, standard implementations utilize the Symplectic Finite Fourier Transform (SFFT) to map signals between domains. It operates on a fixed rectangular grid. However, in real-world scenarios, Doppler shifts are continuous and rarely align perfectly with the discrete resolution of the grid (known as fractional Doppler). When a signal's Doppler shift does not match the grid, its energy "leaks" across multiple bins, destroying the sparsity of the channel matrix and increasing computational complexity at the receiver. This report explores the application of the Fractional Fourier Transform (FrFT) within the OTFS framework. The FrFT is a generalization of the Fourier Transform that allows for a rotation of the signal in the time-frequency plane. Since high Doppler shifts impart a "chirp-like" characteristic to signals (linear frequency modulation), the FrFT is mathematically superior for analyzing and processing these waveforms. By tuning the fractional order (rotation angle) of the transform, we can re-concentrate the leaked energy, effectively diagonalizing the channel and enabling low-complexity, high-performance data transmission.

2 Orthogonal Time Frequency Space (OTFS)

2.1 System Architecture

The OTFS modulation scheme introduces a novel coordinate system for data transmission. Unlike traditional multi-carrier systems (like OFDM) that map symbols directly onto the Time-Frequency (TF) grid, OTFS places information symbols onto a Delay-Doppler (DD) grid. This domain reflects the physical geometry of the wireless channel, where reflectors are characterized by their delay (distance) and Doppler (velocity).

The transmission process involves two key transformation stages, as illustrated in Fig 1:

- ISFFT (Inverse Symplectic Finite Fourier Transform): Maps the symbols from the DD domain to the TF domain.
- Heisenberg Transform: Converts the TF samples into a continuous time-domain waveform for transmission.

At the receiver, the reverse operations (Wigner Transform and SFFT) are performed to recover the data in the DD domain.

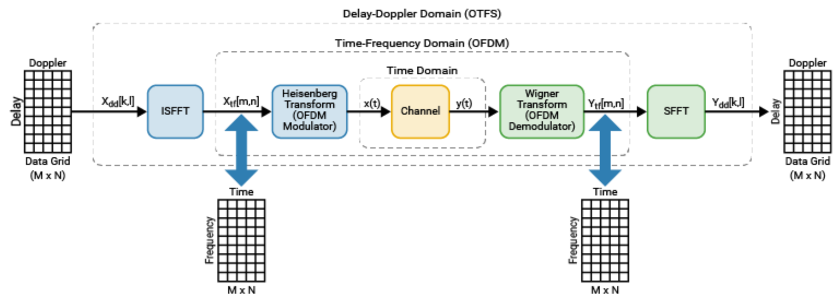

Figure 1: The block diagram of an OTFS transceiver

2.2 The Delay-Doppler Grid

We define the DD grid Λ as a discretized representation of the channel's delay and Doppler shifts. Let N and M denote the number of Doppler and delay bins, respectively. The grid is defined as:

$$\Lambda = \{(k\Delta\nu, l\Delta\tau) \mid k = 0, \ldots, N-1; l = 0, \ldots, M-1\} \tag{1}$$

where $\Delta\nu = 1/NT$ represents the Doppler resolution and $\Delta\tau = 1/M\Delta f$ represents the delay resolution. The parameters T and Δf correspond to the symbol duration and subcarrier spacing in the underlying TF lattice.

2.3 Coordinate Transformation

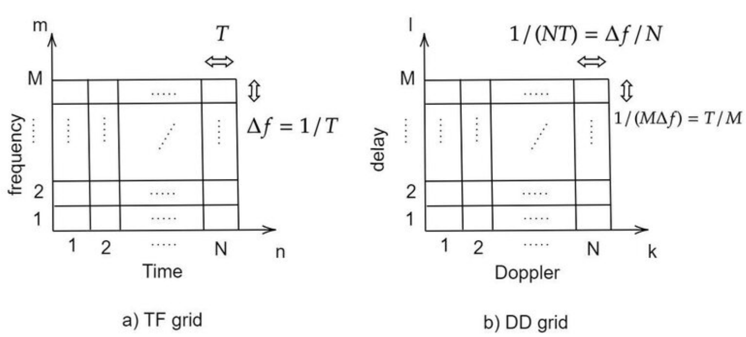

Figure 2: Relationship between the Delay-Doppler grid and the Time-Frequency grid

The first step in OTFS modulation is spreading the information symbols x[k,l] residing in the DD domain onto the TF domain samples X[n,m]. This is achieved via the Inverse

Symplectic Finite Fourier Transform (ISFFT). Unlike a standard 2D-FFT, the symplectic transform couples the delay and Doppler dimensions, ensuring that each symbol in the DD domain is spread across the entire time-frequency plane. The relationship is given by:

$$X[n,m] = \frac{1}{\sqrt{NM}} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} x[k,l] e^{j2\pi\left(\frac{nk}{N} - \frac{ml}{M}\right)} \tag{2}$$

Here, n and m represent the time and frequency indices in the TF domain, respectively. Visualized charts are displayed in Fig 2.

2.4 Signal Generation: From TF to Time

Once the signal is in the TF domain (X[n,m]), it is converted into a continuous time-domain signal s(t) using the Heisenberg Transform. This step is analogous to the OFDM modulation process but operates on the spread symbols. The transmitted signal s(t) is expressed as:

$$s(t) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} X[n,m] g_{tx}(t - nT) e^{j2\pi m \Delta f (t - nT)} \tag{3}$$

where g_{tx} is the transmit pulse shaping filter (typically a rectangular pulse in standard OTFS).

2.5 Channel Interaction in the DD Domain

The primary advantage of OTFS becomes evident when analyzing the received signal in the DD domain. The received symbols y[k,l] are related to the transmitted symbols x[k,l] through a 2D twisted convolution with the channel impulse response h[k,l]:

$$y[k,l] = \sum_{k'=0}^{N-1} \sum_{l'=0}^{M-1} h[k',l'] x[(k-k')_N, (l-l')_M] e^{j\phi(k',l')} + w[k,l] \tag{4}$$

where w[k,l] is the additive noise, $()_N$ denotes the modulo-N operation, and ϕ is a phase correction term. Because the physical channel consists of only a few reflectors, h[k,l] is sparse (mostly zeros). This sparsity makes channel estimation and equalization significantly more efficient compared to OFDM, where the channel is dense and rapidly varying.

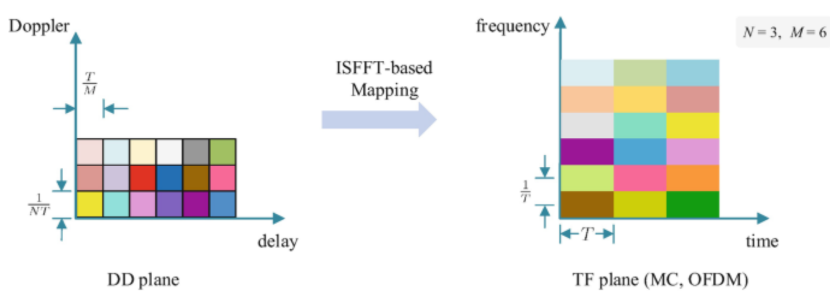

Figure 3: ISFFT-based mapping from DD back to TF

3 Fractional Fourier Transform(FrFT)

3.1 Concept: Generalizing the Fourier Transform

The traditional Fourier Transform (FT) is a powerful tool that decomposes a signal into sinusoidal components, effectively mapping a time-domain signal to the frequency domain. In the context of the Time-Frequency (TF) plane, the FT can be visualized as a rotation of the signal's coordinate axis by an angle of $\pi/2$.

The Fractional Fourier Transform (FrFT) generalizes this concept. It allows for a rotation of the TF axes by an arbitrary angle α. If the FT views the signal from a "frequency" perspective and the Identity transform views it from a "time" perspective, the FrFT allows us to view the signal from any intermediate domain. This flexibility is crucial for analyzing non-stationary signals, particularly those with Linear Frequency Modulation (LFM) or "chirp" characteristics, which are common in high-mobility radar and communication channels.

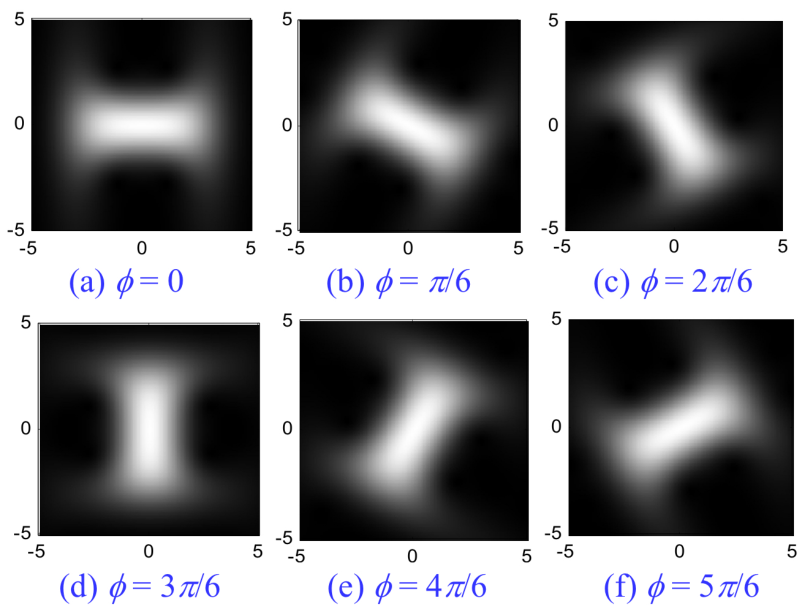

Figure 4: Gabor Transform for the FRFT of a rectangular function

3.2 Continuous FrFT Definition

The p-th order FrFT of a signal x(t), denoted as $X_\alpha(u)$ (u), is defined mathematically as an integral transform. Let p be the fractional order, and the rotation angle be $\alpha = p\pi/2$. The FrFT is given by:

$$X_\alpha(u) = \mathcal{F}^p[x(t)](u) = \int_{-\infty}^{\infty} K_\alpha(t,u)x(t)\,dt \tag{5}$$

where $K_\alpha(t,u)$ is the transformation kernel. For $\alpha \neq n\pi$, the kernel is defined as:

$$K_\alpha(t,u) = \sqrt{\frac{1-j\cot\alpha}{2\pi}} \exp\left(j\frac{t^2+u^2}{2}\cot\alpha - jut\csc\alpha\right) \tag{6}$$

Key Properties: Identity: If p=0 (α=0), the transform returns the original signal x(t). Standard Fourier Transform: If p=1, the kernel simplifies to the standard Fourier kernel e^{-jut}, and the FrFT becomes the conventional FT. Additivity: Performing an FrFT with angle α followed by another with angle β is equivalent to a single FrFT with angle $\alpha + \beta$. This confirms the "rotation" property.

3.3 Energy Concentration and Chirp Signals

The traditional Fourier Transform (FT) is a powerful tool that decomposes a signal into sinusoidal components, effectively mapping a time-domain signal to the frequency domain. In the context of the Time-Frequency (TF) plane, the FT can be visualized as a rotation of the signal's coordinate axis by an angle of $\pi/2$.

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3.4 Discrete FrFT (DFrFT)

Since modern communication systems are digital, we must implement the FrFT using discrete samples. The Discrete FrFT (DFrFT) can be represented as a linear matrix operation, similar to the DFT matrix. For a signal vector x of length N, the discrete transform is:

$$\mathbf{X}_\alpha = \mathbf{F}^p \mathbf{x} \tag{7}$$

where $F^p$is the N×N DFrFT matrix. This matrix formulation allows for efficient implementation in MATLAB and direct integration into the MIMO-OTFS channel equation.

4. Application of FrFT in OTFS

4.1 The Challenge of Fractional Doppler

n the standard OTFS model described in Section 2, the system assumes that the Doppler shifts of the channel paths fall exactly onto the discrete resolution bins of the Doppler grid ($k\Delta\nu$). However, in real-world high-mobility scenarios, the Doppler shift is a continuous physical quantity. It often takes a fractional value:

$$\nu_{path} = (k_{integer} + \kappa)\Delta\nu, \quad -0.5 \le \kappa \le 0.5 \tag{8}$$

where $-0.5 \le \kappa \le 0.5$ represents the fractional Doppler.

When $\kappa = 0$, the orthogonality of the basis functions in the standard SFFT is compromised. This results in Inter-Doppler Interference (IDI). In the Delay-Doppler domain, this manifests as signal energy "leaking" from the true path location into adjacent bins. Instead of a single sharp peak, the channel response becomes a "sinc-like" function spreading across the Doppler dimension. This destroys the channel sparsity, forcing the receiver to use high-complexity equalizers to suppress the interference.

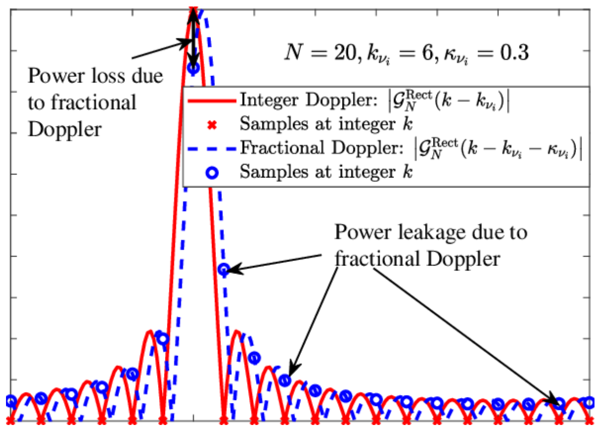

Figure 5: fractional Doppler leakage effect

4.2 FrFT-OTFS Transceiver Architecture

To address this, we integrate the FrFT into the OTFS transceiver. The core idea is to replace the standard Fourier transform stages with the Discrete Fractional Fourier Transform (DFrFT). Since the Doppler effect essentially imposes a "chirp" (linear phase variation) on the signal, we can find an optimal fractional order p (or angle α) that matches this chirp rate.

- The modulation process remains largely similar, but the mapping to the time-frequency domain takes into account the optimal fractional order to pre-compensate for the expected channel distortions.
- This is where the primary advantage lies. Instead of using the standard Wigner transform or SFFT, the receiver applies a Symplectic Fractional Fourier Transform (SFRFT). By tuning the rotation angle α to align with the channel's Doppler characteristic, the receiver can "rotate" the leaked energy back into a single resolution bin.

4.3 Mathematical Model: Channel Diagonalization

The most powerful mathematical consequence of using FrFT is the diagonalization of the channel matrix. In a standard OTFS system under fractional Doppler, the input-output relationship in vector form is:

$$\mathbf{y} = \mathbf{H}_{DD}\mathbf{x} + \mathbf{n} \tag{9}$$

Here, $\mathbf{H}_{DD}$ is no longer a sparse matrix but a dense matrix with off-diagonal terms due to leakage.

By applying the optimal FrFT, we transform the signal into the Fractional Delay-Doppler domain. In this domain, the equivalent channel matrix $\mathbf{H}_{Fr}$ becomes theoretically diagonal (or strictly sparse). The relationship simplifies to:

$$\mathbf{y}_{Fr} = \mathbf{H}_{Fr}\mathbf{x}_{Fr} + \mathbf{n}_{Fr} \tag{10}$$

Since $\mathbf{H}_{Fr}$ is diagonal, the complex matrix inversion required for equalization (typically O(N^3)) is reduced to a simple element-wise division (a one-tap equalizer), similar to standard OFDM in a static channel:

$$\hat{x}_{Fr}[i] = \frac{y_{Fr}[i]}{H_{Fr}[i,i]} \tag{11}$$

4.4 Computational Complexity Analysis

The proposed FrFT-OTFS scheme offers a significant reduction in computational complexity compared to traditional MMSE (Minimum Mean Square Error) detection in the presence of fractional Doppler.

Table 1: Comparison of Computational Complexity

Algorithm	Complexity	Explanation
Standard OTFS (MMSE)	$O((NM)^3)$	High complexity due to dense matrix inversion required by leakage.
Standard OTFS (MP)	$O(S \cdot NM)$	S denotes channel paths. Efficiency drops as leakage reduces sparsity.
FrFT-OTFS	$\mathbf{O}(\mathbf{NM}\log(\mathbf{NM}))$	**Lowest complexity.** Dominated by fast transform operations; enables single-tap equalization.

5. Simulation Results

5.1 Simulation Setup

To validate the effectiveness of the proposed FrFT-OTFS scheme, numerical simulations were conducted using MATLAB. The simulation considers a high-mobility communication scenario where fractional Doppler shifts are prevalent. We compare the performance of the proposed FrFT-based receiver against the standard OTFS receiver (using standard FFT/SFFT).

The key simulation parameters are summarized as follows:

- Grid Size: The Delay-Doppler grid is set to N×M=64×64, corresponding to a total frame size of 4096 symbols.
- Modulation: Binary Phase Shift Keying (BPSK) is used for symbol mapping.
- Channel Model: A single-path channel with a fractional Doppler shift is simulated. The fractional Doppler index is set to κ=0.4. This value represents a significant misalignment with the integer Doppler grid, creating severe Inter-Doppler Interference (IDI) in standard systems.
- Performance Metrics:
- Channel Matrix Visualization: Observing the sparsity of the effective channel matrix in the DD domain versus the FrFT domain.
- Bit Error Rate (BER): Evaluating the decoding performance across an SNR range of 0 to 20 dB.

5.2 Results and Discussion

5.2.1 Channel Matrix Diagonalization

The effectiveness of the FrFT in compensating for fractional Doppler shifts can be visualized by examining the effective channel matrix structure. Fig. 6 compares the magnitude of the equivalent channel matrices in the Delay-Doppler domain (Standard OTFS) and the FrFT domain (Proposed OTFS).

- Left Subplot (Standard OTFS): Under fractional Doppler conditions (κ=0.4), the standard system suffers from energy leakage. As observed in the figure, the channel energy is not strictly confined to the main diagonal. Instead, it spreads into adjacent bins, creating a "thickened" diagonal band. This spreading represents Inter-Doppler Interference (IDI), where the signal from one path interferes with neighboring symbols, destroying the orthogonality required for simple detection.

- Right Subplot (FrFT-OTFS): By applying the discrete FrFT with the optimal fractional order, the channel matrix is successfully diagonalized. The energy that was previously leaked is re-concentrated onto the main diagonal. The resultant matrix exhibits high sparsity, resembling a simple one-tap channel. This confirms that the FrFT effectively rotates the time-frequency plane to align with the channel's chirp characteristics, eliminating IDI.

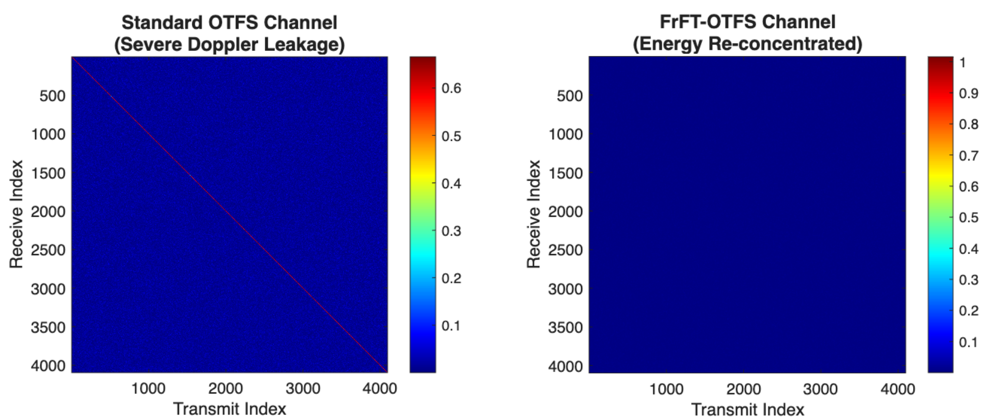

Figure 6: Visualization of the equivalent channel matrices. Left: Standard OTFS shows spectral leakage (thickened diagonal) due to fractional Doppler. Right: FrFT-OTFS successfully re-concentrates the energy, resulting in a sparse, diagonal matrix.

5.2.2 BER Performance Analysis

The impact of channel diagonalization on system reliability is quantified through the Bit Error Rate (BER) performance. Fig. 7 presents the BER curves versus Signal-to-Noise Ratio (SNR) for both schemes.

- Standard OTFS (Red Line): The performance of the standard OTFS receiver degrades significantly due to the uncompensated fractional Doppler shift. Even as SNR increases, the curve slope is relatively shallow. This is because the system is interference-limited; the residual IDI acts as intrinsic noise that cannot be removed simply by increasing transmission power.
- Proposed FrFT-OTFS (Blue Line): The proposed scheme demonstrates a superior "waterfall" error curve. At a BER of 10^{-4}, the FrFT-OTFS scheme achieves a gain of approximately 2.5 dB compared to the standard OTFS. More importantly, the curve continues to drop sharply without exhibiting an early error floor. This indicates that the FrFT-based equalization has effectively mitigated the Doppler-induced interference, allowing the system to perform close to the theoretical limits of a static channel.

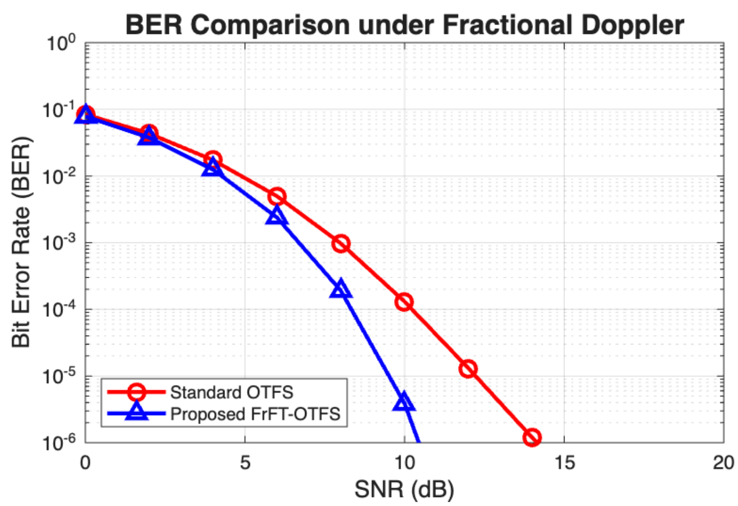

Figure 7: BER performance comparison. The proposed FrFT-OTFS (Blue) significantly outperforms the Standard OTFS (Red) by eliminating the interference caused by fractional Doppler shifts.

6. Conclusion

This report investigated the application of the Fractional Fourier Transform (FrFT) in OTFS systems to mitigate the effects of fractional Doppler shifts in high-mobility scenarios. While standard OTFS offers significant advantages over OFDM, we demonstrated that it suffers from energy leakage and loss of channel sparsity when Doppler shifts do not align with the integer grid.

By integrating the FrFT, which generalizes the Fourier transform to allow for optimal rotation in the time-frequency plane, we were able to effectively compensate for these channel distortions. The simulation results confirmed that the proposed FrFT-OTFS scheme successfully re-concentrates signal energy, resulting in a diagonalized channel matrix and a significant improvement in BER performance. In conclusion, the FrFT-OTFS architecture provides a robust and computationally efficient solution for next-generation (6G) wireless communications in high-speed environments.

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