XIII. Continuous WT with Discrete Coefficients

13-A Definition

The parameters a and b are not chosen arbitrarily.

For example,
$$a = n2^{-m} \quad \text{and} \quad b = 2^{-m}.$$

$$X_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$$

$$n \in Z, \quad n \in (-\infty, \infty)$$
$$m \in Z, \quad m \in (-\infty, \infty)$$

註：某些文獻把這個式子稱作是 discrete wavelet transform，實際上仍然是
　　continuous wavelet transform 的特例

- Main reason for constrain a and b to be $n2^{-m}$ and 2^{-m} :

Easy to implementation

$X_w(n, m)$ can be computed from $X_w(n, m-1)$ by digital convolution.

13-B Inverse Wavelet Transform

Inverse Wavelet Transform

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi\left(2^m t - n\right) X_w(n,m)$$

Suppose that $\quad Y_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$

$$Y_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \psi\left(2^{m_1} t - n_1\right) X_w(n_1, m_1) \psi(2^m t - n)dt$$

$$= 2^{-m/2} \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \left(2^m \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right)\psi(2^m t - n)dt\right) X_w(n_1, m_1)$$

If $\quad \boxed{\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1} t - n_1\right)\psi(2^m t - n)dt = \delta\left(m - m_1\right)\delta(n - n_1)}$

then $\quad Y_w(n,m) = 2^{-m/2} \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \delta\left(m - m_1\right)\delta(n - n_1) X_w(n_1, m_1)$
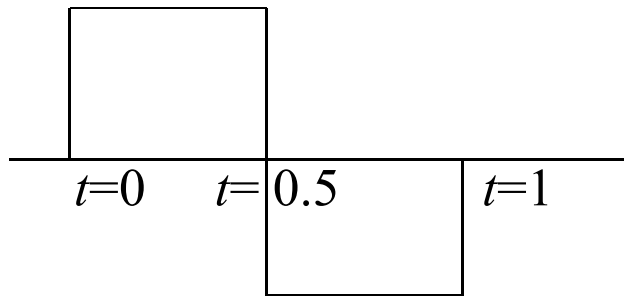
$$= 2^{-m/2} 2^{m/2} X_w(n,m) = X_w(n,m)$$

Constraint for reversibility:

$$2^m \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$
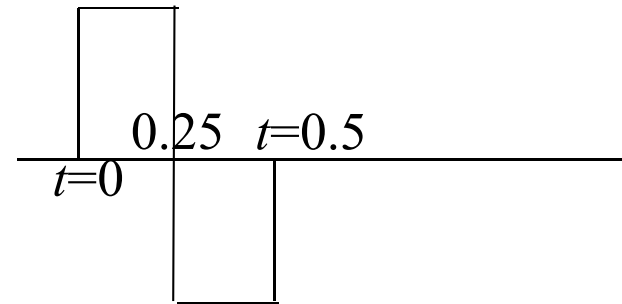
$\psi(t)$ mother wavelet
     (wavelet function)

$\psi(2t)$

The Haar wavelet satisfies

$$2^m \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$

Without the loss of generalization, suppose that $m_1 \geq m$. Set

$$t_1 = 2^m t - n \qquad dt_1 = 2^m dt$$

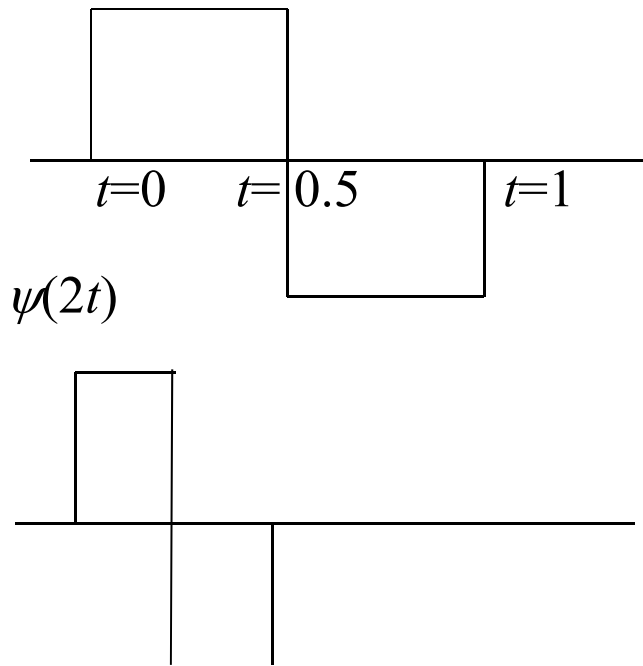$$2^{m_1} t - n_1 = 2^{m_1 - m} t_1 + 2^{m_1 - m} n - n_1$$

$$2^m \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \int_{-\infty}^{\infty} \psi\left(2^{m_1 - m} t_1 + 2^{m_1 - m} n - n_1\right) \psi\left(t_1\right) dt_1$$

Therefore, we only have to prove that

$$\int_{-\infty}^{\infty} \psi\left(2^m t - n\right) \psi\left(t\right) dt = \delta\left(m\right) \delta\left(n\right)$$

for $m \geq 0$.

$\psi(t)$ mother wavelet
(wavelet function)

$\phi(t)$ scaling function

$\psi(2t)$

$\phi(2t)$ $2\phi(2t) = \phi(t) + \psi(t)$

$\phi(t) = \phi(2t) + \phi(2t{-}1)$

$\psi(t) = \phi(2t) - \phi(2t{-}1)$

$$\psi(2^m t - n)$$

$$\phi(2^m t - n)$$

$t=n2^{-m} \quad t=(n+.5)2^{-m} \quad t=(n+1)2^{-m}$

$t=n2^{-m} \qquad\qquad t=(n+1)2^{-m}$
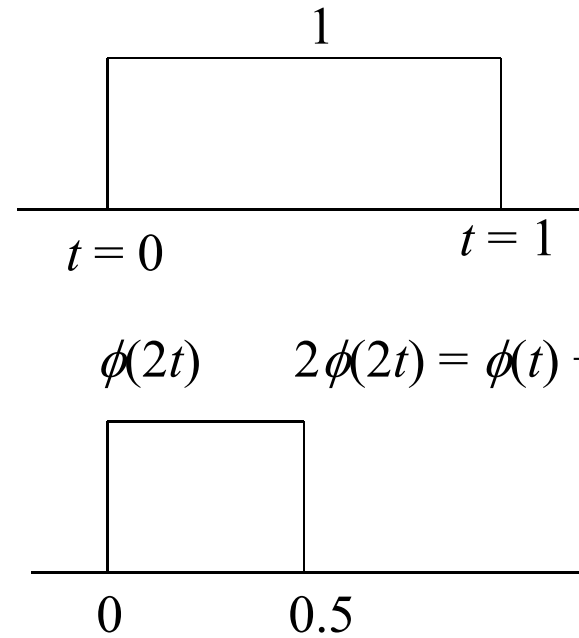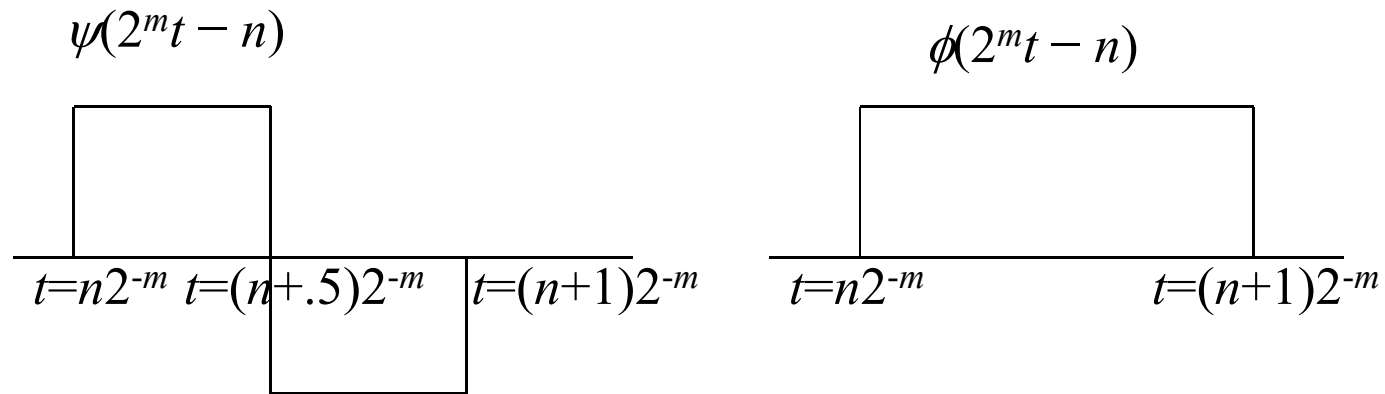
- Advantages of Haar wavelet

  (1) Simple

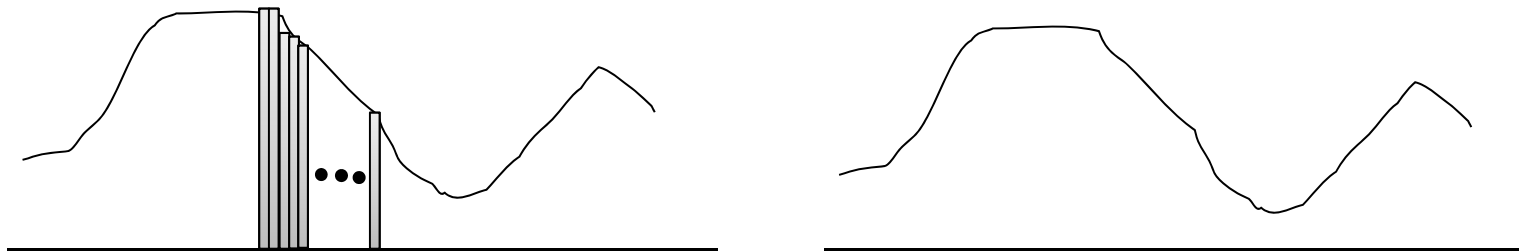 (2) Fast algorithm

  (3) Orthogonal →reversible

  (4) Compact, real, odd

- Disadvantages of Haar wavelet

  vanishing moment =

(1) Any function can be expressed by a linear combination of $\phi(t)$, $\phi(2t)$, $\phi(4t)$, $\phi(8t)$, $\phi(16t)$, ..., and their shifting.

(2) 任何平均為 0 的 function 都可以由 $\psi(t)$, $\psi(2t)$, $\psi(4t)$, $\psi(8t)$, $\psi(16t)$, ...
所組成

換句話說....... 任何 function 都可以由 constant, $\psi(t)$, $\psi(2t)$, $\psi(4t)$, $\psi(8t)$, $\psi(16t)$, ... 所組成

(4) 不同寬度 (也就是不同 m) 的 wavelet / scaling functions 之間會有一個關係

$$\phi(t) = \phi(2t) + \phi(2t - 1)$$

$$\phi(t - n) = \phi(2t - 2n) + \phi(2t - 2n - 1)$$

$$\phi(2^m t - n) = \phi(2^{m+1}t - 2n) + \phi(2^{m+1}t - 2n - 1)$$

$$\psi(t) = \phi(2t) - \phi(2t - 1)$$

$$\psi(t - n) = \phi(2t - 2n) - \phi(2t - 2n - 1)$$

$$\psi(2^m t - n) = \phi(2^{m+1}t - 2n) - \phi(2^{m+1}t - 2n - 1)$$
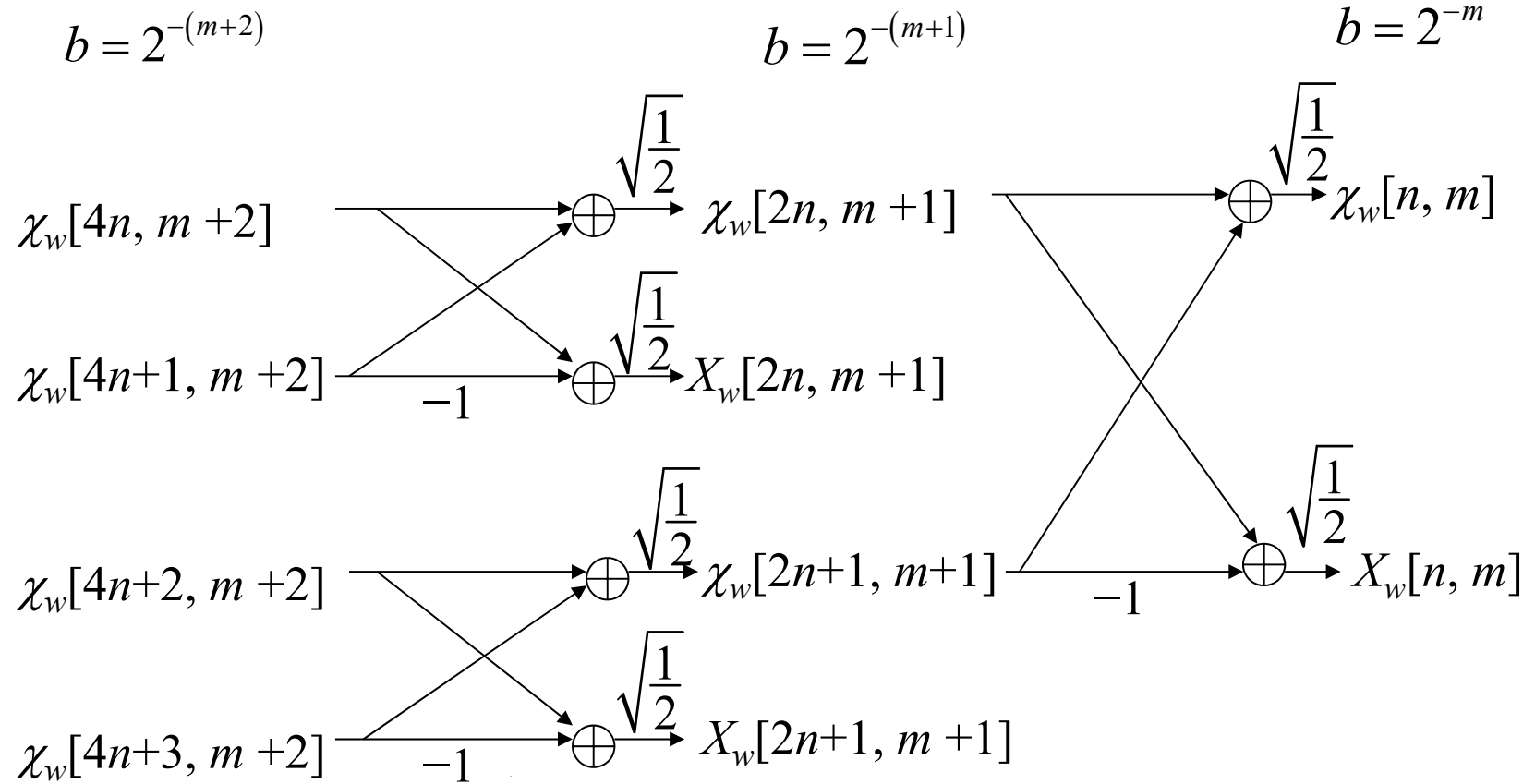
(5) 可以用 $m+1$ 的 coefficients 來算 m 的 coefficients

若 $\quad \chi_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t)\phi(2^m t - n)dt$

$$\chi_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t)\phi(2^{m+1}t - 2n)dt + 2^{m/2} \int_{-\infty}^{\infty} x(t)\phi(2^{m+1}t - 2n - 1)dt$$

$$= \sqrt{\frac{1}{2}} \left(\chi_w(2n, m+1) + \chi_w(2n+1, m+1) \right)$$

$$X_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$$

$$X_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t)\phi(2^{m+1}t - 2n)dt - 2^{m/2} \int_{-\infty}^{\infty} x(t)\phi(2^{m+1}t - 2n - 1)dt$$

$$= \sqrt{\frac{1}{2}} \left(\chi_w(2n, m+1) - \chi_w(2n+1, m+1) \right)$$

layer:

$$b = 2^{-(m+2)} \qquad b = 2^{-(m+1)} \qquad b = 2^{-m}$$

$\chi_w[4n, m+2]$

$\chi_w[4n+1, m+2]$ $\quad -1$

$\chi_w[4n+2, m+2]$

$\chi_w[4n+3, m+2]$ $\quad -1$

$\sqrt{\dfrac{1}{2}} \; \chi_w[2n, m+1]$

$\sqrt{\dfrac{1}{2}} \; X_w[2n, m+1]$

$\sqrt{\dfrac{1}{2}} \; \chi_w[2n+1, m+1]$ $\quad -1$

$\sqrt{\dfrac{1}{2}} \; X_w[2n+1, m+1]$

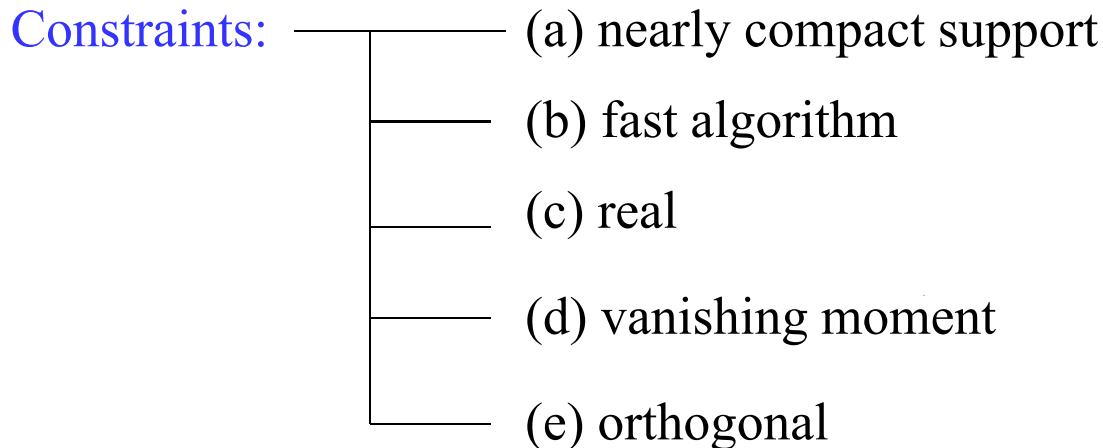$\sqrt{\dfrac{1}{2}} \; \chi_w[n, m]$

$\sqrt{\dfrac{1}{2}} \; X_w[n, m]$

structure of multiresolution analysis (MRA)

13-D   General Methods to Define the Mother Wavelet and the Scaling Function

Constraints:
- (a) nearly compact support
- (b) fast algorithm
- (c) real
- (d) vanishing moment
- (e) orthogonal

和 continuous wavelet transform 比較：

(1) compact support 放寬為 "nearly compact support"

(2) 沒有 even, odd symmetric 的限制

(3) 由於只要是 complete and orthogonal, 必定可以 reconstruction

所以不需要 admissibility criterion 的限制

(4) 多了對 fast algorithm 的要求

13-E Fast Algorithm Constraints

Higher and lower resolutions 的 recursive relation 的一般化

$$\phi(t) = 2\sum_k g_k \phi(2t-k)$$ 稱作 dilation equation

$$\psi(t) = 2\sum_k h_k \phi(2t-k)$$

$\psi(t)$: mother wavelet, $\phi(t)$: scaling function

這些關係式成立，才有fast algorithms

$$\boxed{\phi(t)=2\sum_k g_k\phi(2t-k)} \qquad\qquad \boxed{\psi(t)=2\sum_k h_k\phi(2t-k)}$$

If $\quad \chi_w(n,m)=2^{m/2}\int_{-\infty}^{\infty}x(t)\phi(2^m t-n)dt$

then $\quad \chi_w(n,m)=\sum_k 2^{\frac{m}{2}+1}\int_{-\infty}^{\infty}x(t)g_k\phi(2^{m+1}t-2n-k)dt$

$$=2^{\frac{1}{2}}\sum_k g_k\chi_w(2n+k,m+1)$$

If $\quad X_w(n,m)=2^{m/2}\int_{-\infty}^{\infty}x(t)\psi(2^m t-n)dt$

then $\quad X_w(n,m)=\sum_k 2^{\frac{m}{2}+1}\int_{-\infty}^{\infty}x(t)h_k\phi(2^{m+1}t-2n-k)dt$

$$=2^{\frac{1}{2}}\sum_k h_k\chi_w(2n+k,m+1)$$

(Step 1) convolution

$$\tilde{\chi}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{g}_k \chi_w(n-k, m+1) \qquad \tilde{g}_k = g_{-k}$$
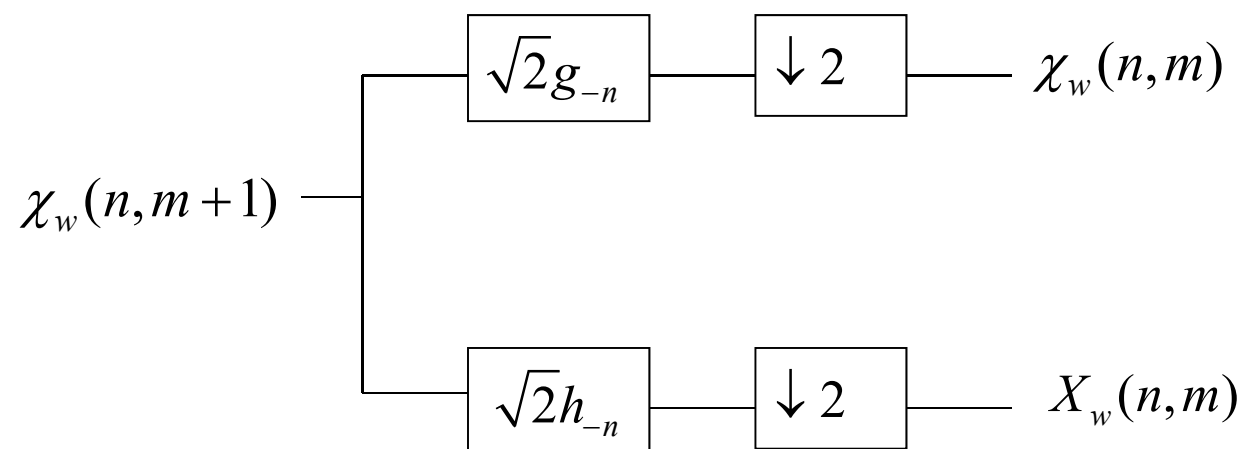
$$\tilde{X}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{h}_k \chi_w(n-k, m+1) \qquad \tilde{h}_k = h_{-k}$$

(Step 2) down sampling

$$\chi_w(n, m) = \tilde{\chi}_w(2n)$$

$$X_w(n, m) = \tilde{X}_w(2n)$$

$$\chi_w(n, m+1) \quad
\begin{cases}
\boxed{\sqrt{2}\,g_{-n}} - \boxed{\downarrow 2} - \chi_w(n,m) \\
\\
\boxed{\sqrt{2}\,h_{-n}} - \boxed{\downarrow 2} - X_w(n,m)
\end{cases}$$

m 越大，越屬於細節

- To satisfy $\phi(t) = 2\sum_k g_k \phi(2t-k)$,

$$\phi(t/2) = 2\sum_k g_k \phi(t-k) = 2\sum_k g_k \delta(t-k) * \phi(t)$$

FT $\downarrow$ FT $\downarrow$

where $\Phi(f) = FT[\phi(t)] = \int_{-\infty}^{\infty} \phi(t) e^{-j2\pi f t} dt$

$$2\Phi(2f) = 2G(f)\Phi(f)$$

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

$$G(f) = FT\left[\sum_k g_k \delta(t-k)\right]$$

$$= \sum_k g_k \int_{-\infty}^{\infty} \delta(t-k) e^{-j2\pi f t} dt$$

$$= \sum_k g_k e^{-j2\pi f k}$$

$\Phi(f)$ 是 $\phi(t)$ 的 continuous Fourier transform

$G(f)$ 是 $\{g_k\}$ 的 discrete time Fourier transform

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right) \qquad \Phi\left(\frac{f}{2}\right) = G\left(\frac{f}{4}\right)\Phi\left(\frac{f}{4}\right)$$

$$\Phi(f) = G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)\Phi\left(\frac{f}{4}\right) = G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)G\left(\frac{f}{8}\right)\Phi\left(\frac{f}{8}\right) = \cdots\cdots$$

$$\Phi(f) = \Phi\left(\frac{f}{2^{\infty}}\right)\prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right) = \Phi(0)\prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$$

↑

連乘

$$\Phi(0) = \int_{-\infty}^{\infty} \phi(t)\,dt \qquad (可以藉由 \text{ normalization}, 讓 \Phi(0) = 1)$$

$$\boxed{\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)} \qquad 若 G(f) \text{ 決定了,則 } \Phi(f) \text{ 可以被算出來}$$

$G(f)$: 被稱作 generating function

constraint 1

● 同理

$$\psi(t) = 2\sum_k h_k \phi(2t - k)$$

$$\Psi(f) = \int_{-\infty}^{\infty} \psi(t) e^{-j2\pi f t} dt$$

$$\psi(t/2) = 2\sum_k h_k \phi(t - k)$$

$$\Psi(f) = H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

$$H(f) = \sum_k h_k e^{-j2\pi f k}$$

$$\boxed{\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)}$$

constraint 2

● 另外，由於

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

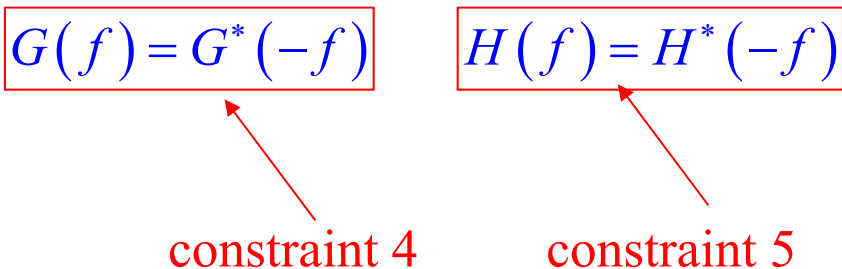$$\Phi(0) = G(0)\Phi(0) \qquad (f = 0 \text{ 代入})$$

$$\boxed{G(0) = 1}$$ 必需滿足

constraint 3

13-F Real Coefficient Constraints

Since
$$\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right) \qquad \Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$

If $\boxed{G(f) = G^*(-f)}$ $\boxed{H(f) = H^*(-f)}$ are satisfied,

constraint 4   constraint 5

then $\Phi(f) = \Phi^*(-f),\ \Psi(f) = \Psi^*(-f),$ and $\phi(t),\ \psi(t)$ are real.

Note: If these constraints are satisfied, $g_k,\ h_k$ on page 418 are also real.

13-G Vanishing Moment Constraint

If $\psi(t)$ has p vanishing moments,

$$\int_{-\infty}^{\infty} t^k \psi(t)\, dt = 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1$$

Since $FT\left[t^k \psi(t)\right] = \left(\dfrac{j}{2\pi}\right)^k \dfrac{d^k}{df^k} \Psi(f)$

$$\int_{-\infty}^{\infty} x(t)\, dt = X(0) \quad \text{if} \quad X(f) = FT(x(t))$$

$$\int_{-\infty}^{\infty} t^k \psi(t)\, dt = 0 \implies FT\left[t^k \psi(t)\right]\Big|_{f=0} = \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k} \Psi(f)\Big|_{f=0} = 0$$

Therefore, $\dfrac{d^k}{df^k} \Psi(f)\Big|_{f=0} = 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1$

$$\frac{d^k}{df^k} \Psi(f)\bigg|_{f=0} = 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1$$

Since $\quad \Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$

$$\frac{d^k}{df^k} \Psi(f) = \sum_{n=0}^{k} \binom{k}{n} \frac{d^n}{df^n} H\left(\frac{f}{2}\right) \frac{d^{k-n}}{df^{k-n}} \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$

$$= \sum_{n=0}^{k} \binom{k}{n} \frac{1}{2^n} \frac{d^n}{df^n} H(f) \frac{d^{k-n}}{df^{k-n}} \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$

if $\quad \boxed{\frac{d^k}{df^k} H(f)\bigg|_{f=0} = 0} \quad$ for $k = 0, 1, 2, \ldots, p-1$ is satisfied,

constraint 6

then $\quad \dfrac{d^k}{df^k} \Psi(f)\bigg|_{f=0} = 0 \quad$ for $k = 0, 1, 2, \ldots, p-1$ are satisfied

and the wavelet function has p vanishing moments.

13-H Orthogonality Constraints

- orthogonality constraint:

$$\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$

$\psi(t)$: wavelet function

If the above equality is satisfied,

forward wavelet transform:

$$X_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

inverse wavelet transform:

$$x(t) = C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi\left(2^m t - n\right) X_w(m,n)$$

(much easier for inverse) $\qquad C = $ mean of $x(t)$

(證明於後頁)

If $\quad x(t) = C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi\left(2^m t - n\right) X_w(m,n)$

and $\quad \int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta(m - m_1)\delta(n - n_1),$

then $\quad 2^{m/2} \int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$

$$= 2^{m/2} \int_{-\infty}^{\infty} \left[C + \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \psi\left(2^{m_1} t - n_1\right) X_w(m_1, n_1) \right] \psi(2^m t - n)dt$$

$$= 2^{m/2} \int_{-\infty}^{\infty} C\psi(2^m t - n)dt + 2^{m/2} \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right) \psi(2^m t - n)dt X_w(m_1, n_1)$$

$$= 0 + \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \delta(m_1 - m)\delta(n_1 - n) X_w(m_1, n_1)$$

$$= X_w(m, n)$$

due to $\int_{-\infty}^{\infty} \psi(t)dt = 0$

Therefore, $\quad 2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt \quad$ is the inverse operation of

$$C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2}\psi\left(2^m t - n\right) X_w\left(m,n\right) \qquad \#$$

※ 要滿足

$$\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$

之前，需要滿足以下三個條件

(1) $\int_{-\infty}^{\infty} \psi\left(t - n_1\right) \psi\left(t - n\right) dt = \delta\left(n_1 - n\right)$ for mother wavelet

這個條件若滿足，$\int_{-\infty}^{\infty} 2^m \psi\left(2^m t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(n - n_1\right)$

對所有的 m 皆成立

(2) $\int_{-\infty}^{\infty} \phi\left(t - n_1\right) \phi\left(t - n\right) dt = \delta\left(n_1 - n\right)$ for scaling function

嚴格來說，這並不是必要條件，但是可以簡化 第 (3) 個條件的計算

(3) $\displaystyle\int_{-\infty}^{\infty} \psi(t-n_1)\psi(2^{-k}t-n)\,dt = 0$ for any n, n_1 if $k > 0$

若 (1) 和 (3) 的條件滿足，則

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1}t-n_1)\psi(2^m t-n)\,dt = \delta(m-m_1)\delta(n-n_1)$$

也將滿足

(Proof): Set $t_1 = 2^m t, \quad dt_1 = 2^m dt$

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1}t-n_1)\psi(2^m t-n)\,dt = \int_{-\infty}^{\infty} \psi(2^{m_1-m}t_1-n_1)\psi(t_1-n)\,dt_1$$

If (3) is satisfied,

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1}t-n_1)\psi(2^m t-n)\,dt = 0 \quad \text{when } m \neq m_1$$

In the case where $m = m_1$, if (1) is satisfied, then

$$\int_{-\infty}^{\infty} 2^m \psi(2^m t-n_1)\psi(2^m t-n)\,dt = \int_{-\infty}^{\infty} \psi(t_1-n_1)\psi(t_1-n)\,dt_1 = \delta(n_1-n)$$

#

- 由 Page 431 的條件 (1)

$$\int_{-\infty}^{\infty} \psi(t-n_1)\psi(t-n)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi n_1 f}\Psi(f)e^{j2\pi n f}\Psi^*(f)\,df$$

Parseval's theorem

$$\int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \int_{-\infty}^{\infty} X(f)Y^*(f)\,df$$

$$= \int_{-\infty}^{\infty} e^{j2\pi(n-n_1)f}\Psi(f)\Psi^*(f)\,df$$

$$= \sum_{p=-\infty}^{\infty} \int_{0}^{1} e^{j2\pi(n-n_1)(f'+p)}\Psi(f'+p)\Psi^*(f'+p)\,df'$$

$$e^{j2\pi(n-n_1)(f'+p)} = e^{j2\pi(n-n_1)f'}$$

$$= \int_{0}^{1} e^{j2\pi(n-n_1)f'} \sum_{p=-\infty}^{\infty} |\Psi(f'+p)|^2 \, df' = \delta(n-n_1) \qquad \text{if } p \text{ is an integer}$$

Therefore,

$$\int_{0}^{1} e^{-j2\pi n_2 f'} \sum_{p=-\infty}^{\infty} |\Psi(f'+p)|^2 \, df = \delta(-n_2) = \delta(n_2)$$

$$\boxed{\sum_{p=-\infty}^{\infty} |\Psi(f'+p)|^2 = 1} \qquad \text{for all } f' \text{ should be satisfied}$$

- 同理，由 Page 431 的條件 (2)

$$\int_{-\infty}^{\infty} \phi(t-n_1)\phi(t-n)dt = \delta(n_1-n) \qquad \text{for scaling function}$$

推導過程類似 page 433

$$\boxed{\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1}$$ for all f should be satisfied

衍生的條件：將 $\Psi(f) = H\left(\dfrac{f}{2}\right)\Phi\left(\dfrac{f}{2}\right)$ 代入 $\displaystyle\sum_{p=-\infty}^{\infty} |\Psi(f+p)|^2 = 1$

$$\sum_{p=-\infty}^{\infty} |H\left(\frac{f}{2} + \frac{p}{2}\right)\Phi\left(\frac{f}{2} + \frac{p}{2}\right)|^2 = 1$$

$$\sum_{q=-\infty}^{\infty} |H\left(\frac{f}{2} + q\right)\Phi\left(\frac{f}{2} + q\right)|^2 + \sum_{q=-\infty}^{\infty} |H\left(\frac{f}{2} + q + \frac{1}{2}\right)\Phi\left(\frac{f}{2} + q + \frac{1}{2}\right)|^2 = 1$$

因為 h_k 是 discrete sequence, $H(f)$ 是 h_k 的 discrete-time Fourier transform

$$H(f) = H(f+1) = H(f+2) = \cdots\cdots$$

$$|H\left(\frac{f}{2}\right)|^2 \sum_{q=-\infty}^{\infty} |\Phi\left(\frac{f}{2} + q\right)|^2 + |H\left(\frac{f}{2} + \frac{1}{2}\right)|^2 \sum_{q=-\infty}^{\infty} |\Phi\left(\frac{f}{2} + q + \frac{1}{2}\right)|^2 = 1$$

$$\mid H\left(\frac{f}{2}\right)\mid^2 \sum_{q=-\infty}^{\infty} \mid \Phi\left(\frac{f}{2}+q\right)\mid^2 + \mid H\left(\frac{f}{2}+\frac{1}{2}\right)\mid^2 \sum_{q=-\infty}^{\infty} \mid \Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\mid^2 = 1$$

因為 $$\sum_{p=-\infty}^{\infty} \mid \Phi\left(f+p\right)\mid^2 = 1 \quad \text{for all } f$$

(page 433 的條件)

$$\mid H\left(\frac{f}{2}\right)\mid^2 + \mid H\left(\frac{f}{2}+\frac{1}{2}\right)\mid^2 = 1$$

$$\mid H(f)\mid^2 + \mid H\left(f+\frac{1}{2}\right)\mid^2 = 1$$

constraint 7

同理，將 $\Phi(f) = G\left(\dfrac{f}{2}\right)\Phi\left(\dfrac{f}{2}\right)$ 代入 $\displaystyle\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1$

(page 433)

經過運算可得

$$|G(f)|^2 + |G\left(f+\dfrac{1}{2}\right)|^2 = 1$$

constraint 8

- Page 432 條件 (3) 的處理

由於

$\psi\left(2^{-k}t-n\right)$　　是 $\phi\left(2^{-k+1}t-n_1\right)$ 的 linear combination　　$\psi(t)=2\sum_k h_k\phi(2t-k)$

$\phi\left(2^{-k+1}t-n_1\right)$ 是 $\phi\left(2^{-k+2}t-n_2\right)$ 的 linear combination　　$\phi(t)=2\sum_k g_k\phi(2t-k)$

$\phi\left(2^{-k+2}t-n_2\right)$　是 $\phi\left(2^{-k+3}t-n_3\right)$ 的 linear combination

$$\vdots$$

$$\vdots$$

$\phi\left(2^{-1}t-n_{k-1}\right)$　是　$\phi\left(t-n_k\right)$　　的 linear combination

所以

$\psi\left(2^{-k}t-n\right)$　必定可以表示成　$\phi\left(t-n_k\right)$　的 linear combination

$$\psi\left(2^{-k}t-n\right)=\sum_{n_k}b_{n_k}\phi\left(t-n_k\right)$$

$$\psi\left(2^{-k}t-n\right)=\sum_{n_k}b_{n_k}\phi\left(t-n_k\right)$$

所以，若 $\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\phi\left(t-n_k\right)dt=0$ for any n_1, n_k 可以滿足

則 $\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\psi\left(2^{-k}t-n\right)dt=0$ for any n_1, n_k 必定能夠成立

Page 432 條件 (3) 可改寫成

$$\boxed{\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\phi\left(t-n_k\right)dt=0}$$

$$\int_{-\infty}^{\infty}\psi(t)\phi(t-\tau)dt=0 \quad (\text{將 } t-n_1 \text{ 變成 } t, \quad \tau=n_k-n_1)$$

$$\int_{-\infty}^{\infty}\Psi(f)\Phi^*(f)e^{j2\pi\tau f}\,df=0 \quad \text{(from Parseval's theorem)}$$

$$\int_{-\infty}^{\infty} \Psi(f)\Phi^*(f)e^{j2\pi\tau f}\, df = 0$$

Since $\quad \Psi(f) = H\left(\dfrac{f}{2}\right)\Phi\left(\dfrac{f}{2}\right) \qquad \Phi(f) = G\left(\dfrac{f}{2}\right)\Phi\left(\dfrac{f}{2}\right)$

$$\int_{-\infty}^{\infty} H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)\left|\Phi\left(\frac{f}{2}\right)\right|^2 e^{j2\pi\tau f}\, df = 0$$

$$\sum_{p=-\infty}^{\infty}\int_{0}^{1} H\left(\frac{f+p}{2}\right)G^*\left(\frac{f+p}{2}\right)\left|\Phi\left(\frac{f+p}{2}\right)\right|^2 e^{j2\pi\tau(f+p)}\, df = 0$$

$$e^{j2\pi\tau(f+p)} = e^{j2\pi\tau f} \quad \text{(since from page 439,}$$
$$\tau = n_k - n_1 \text{ is an integer)}$$

$$\sum_{q=-\infty}^{\infty}\int_{0}^{1} H\left(\frac{f}{2}+q\right)G^*\left(\frac{f}{2}+q\right)\left|\Phi\left(\frac{f}{2}+q\right)\right|^2 e^{j2\pi\tau f}\, df$$

$$+\sum_{q=-\infty}^{\infty}\int_{0}^{1} H\left(\frac{f}{2}+q+\frac{1}{2}\right)G^*\left(\frac{f}{2}+q+\frac{1}{2}\right)\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 e^{j2\pi\tau f}\, df = 0$$

Since $\quad H(f) = H(f+1) = H(f+2) = \cdots\cdots$

$$G(f) = G(f+1) = G(f+2) = \cdots\cdots$$

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)\int_0^1 \sum_{q=-\infty}^{\infty} \left|\Phi\left(\frac{f}{2}+q\right)\right|^2 e^{j2\pi\tau f}\, df$$

$$+H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right)\int_0^1 \sum_{q=-\infty}^{\infty} \left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 e^{j2\pi\tau f}\, df = 0$$

Since $\quad \displaystyle\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1 \quad$ for all f $\quad$ (page 433)

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right) + H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right) = 0$$

$$\boxed{H(f)G^*(f) + H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right) = 0}$$

constraint 9

整理： 設計 mother wavelet 和 scaling function 的九大條件
(皆由 page 417 的 constraints 衍生而來)

(1) $\Phi(f) = \prod_{q=1}^{\infty} G\left(\dfrac{f}{2^q}\right)$ for fast algorithm , page 423

(2) $\Psi(f) = H\left(\dfrac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\dfrac{f}{2^q}\right)$ for fast algorithm , page 424

(3) $G(0) = 1$ for fast algorithm , page 424

(4) $H(f) = H^*(-f)$ for real , page 425

(5) $G(f) = G^*(-f)$ for real , page 425

(6) $\left.\dfrac{d^k}{df^k} H(f)\right|_{f=0} = 0$ for p vanishing moments , page 427

for $k = 0, 1, \ldots, p\text{-}1$

(7) $\quad |H(f)|^2 + |H\left(f + \frac{1}{2}\right)|^2 = 1 \quad$ for orthogonal , page 436

(8) $\quad |G(f)|^2 + |G\left(f + \frac{1}{2}\right)|^2 = 1 \quad$ for orthogonal , page 437

(9) $\quad H(f)G^*(f) + H\left(f + \frac{1}{2}\right)G^*\left(f + \frac{1}{2}\right) = 0 \quad$ for orthogonal , page 441

$G(f)$
$H(f)$ are the discrete-time Fourier transform of $\{g_k\}$ $\{h_k\}$ on page 418.

- Simplification

Let

$$\left|H(f)\right| = \left|G(f + 1/2)\right|$$

$$G(f) = \sum_k g_k e^{-j2\pi fk}, \quad H(f) = \sum_k h_k e^{-j2\pi fk}$$

$$G(f) = G(f+1), \quad H(f) = H(f+1)$$

Low frequency: around $f = 0$

High frequency: around $f = \pm 1/2$

Specially, if we set that

$$h_k = (-1)^k g_{1-k} \qquad\qquad H(f) = -e^{-j2\pi f} G^*(f+1/2)$$

when the following constraints are satisfied:

$$|G(f)|^2 + |G(f+\tfrac{1}{2})|^2 = 1$$

$$G(f) = G^*(-f) \qquad\qquad \text{(條件 (5), (8) 滿足)}$$

then

$$|H(f)|^2 + |H(f+\tfrac{1}{2})|^2 = |G(f+\tfrac{1}{2})|^2 + |G(f)|^2 = 1$$

$$H(f)G^*(f) + H(f+\tfrac{1}{2})G^*(f+\tfrac{1}{2})$$

$$= -e^{-j2\pi f} G^*(f+\tfrac{1}{2})G^*(f) - e^{-j2\pi(f+\frac{1}{2})} G^*(f) G^*(f+\tfrac{1}{2})$$

$$= -e^{-j2\pi f} G^*(f+\tfrac{1}{2})G^*(f) + e^{-j2\pi f} G^*(f) G^*(f+\tfrac{1}{2}) = 0$$

$$H^*(-f) = -e^{-j2\pi f} G(-f+1/2) = -e^{-j2\pi f} G^*(f-1/2) = H(f)$$

條件 (4), (7), (9) 也將滿足

整理：設計 mother wavelet 和 scaling function 的幾個要求 (簡化版)

(1) $\quad \Phi(f) = \prod\limits_{q=1}^{\infty} G\left(\dfrac{f}{2^q}\right)$ for fast algorithm

(2) $\quad \Psi(f) = H\left(\dfrac{f}{2}\right) \prod\limits_{q=2}^{\infty} G\left(\dfrac{f}{2^q}\right)$ for fast algorithm

(3) $\quad G(0) = 1$ for fast algorithm

(4) $\quad G(f) = G^*(-f)$ for real

(5) $\quad \left. \dfrac{d^k}{df^k} H(f) \right|_{f=0} = 0$ for p vanishing moments

$\qquad\qquad$ for $k = 0, 1, \ldots, p\text{-}1$

(6) $\quad |G(f)|^2 + |G\left(f + \dfrac{1}{2}\right)|^2 = 1$ for orthogonal

(7) $\quad H(f) = -e^{-j2\pi f} G^*(f + 1/2)$

設計時，只要 $G(f)$ $(0 \leq f \leq 1/4)$ 決定了，mother wavelet 和 scaling function 皆可決定

$G(f)$: 被稱作 generating function

Design Process (設計流程):

(Step 1): 給定 $G(f)$ $(0 \leq f \leq 1/4)$，滿足以下的條件

(a) $G(0) = 1$

(b) $\left. \dfrac{d^k}{df^k} G(f) \right|_{f=\frac{1}{2}} = 0$ for $k = 0, 1, 2, \ldots, p\text{-}1$

(Step 2) 由 $G(f) = G^*(-f)$ 　　　　　決定 $G(f)$ (-1/4 $\leq f <$ 0)

(Step 3) 由 $|G(f)|^2 + |G(f + \frac{1}{2})|^2 = 1$ 　決定 $G(f)$ (1/4 $< f <$ 1/2)

　　　　　　　　　　　　　　　　　　　　(-1/2 $< f <$ -1/4)

　　再根據 $G(f) = G(f+1)$，決定所有的 $G(f)$ 值

(Step 4) 由 $H(f) = -e^{-j2\pi f} G^*(f + 1/2)$ 　決定 $H(f)$

(Step 5) 由 $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$

　　　　　$\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$ 　決定 $\Phi(f),\ \Psi(f)$

註： (1) 當 Step 1 的兩個條件滿足，由於 $|G(f)|^2 + |G(f+1/2)|^2 = 1$

$$\left. \frac{d^k}{df^k} G(f) \right|_{f=1/2} = 0 \quad \text{for } k = 0, 1, 2, \ldots, p\text{-}1$$

又由於 $\quad H(f) = -e^{-j2\pi f} G^*(f+1/2)$

$$\left. \frac{d^k}{df^k} H(f) \right|_{f=0} = 0 \quad \text{for } k = 0, 1, 2, \ldots, p\text{-}1$$

(2) $\quad |G(f)|^2 + |G(f+1/2)|^2 = 1 \qquad |G(f)|^2 = |G(-f)|^2$

所以當 *G(f)* (0 ≤ *f* ≤ 1/4) 給定，|*G(f)*| 有唯一解

(3) 對於離散信號而言，*G(f)* = *G(f*+1)
有意義的頻率範圍為 -1/2 < *f* < 1/2

$$G(f) = \sum_k g_k e^{-j2\pi f k}$$

13-K Several Continuous Wavelets with Discrete Coefficients

(1) Haar Wavelet

$$g[0] = 1, \ g[1] = 1 \qquad G(f) = 1 + \exp(-j2\pi f)$$

$$h[0] = 1, \ h[1] = -1 \qquad H(f) = 1 - \exp(-j2\pi f)$$

或

$$g[0] = 1/2, \ g[1] = 1/2 \qquad G(f) = \left[1 + \exp(-j2\pi f)\right] / 2$$

$$h[0] = 1/2, \ h[1] = -1/2 \qquad H(f) = \left[1 - \exp(-j2\pi f)\right] / 2$$

vanishing moment $= ?$

(2) Sinc Wavelet

$$G(f) = 1 \quad \text{for } |f| \le 1/4$$

$$G(f) = 0 \quad \text{otherwise}$$

vanishing moment $= ?$

(3) 4-point Daubechies Wavelet

$$g_k : \left[\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8} \right]$$

vanishing moment $= ?$

vanishing moment VS the number of coefficients

[Daubechies Wavelet]:

It can be viewed as a generalization of the Haar wavelet.

(Haar wavelet = 2-point Daubechies wavelet).

The $2p$-point Daubechies wavelet has the vanish moment of p.

[Ref]: Ingrid Daubechies: *Ten Lectures on Wavelets*, SIAM 1992.

[Ref]: "Daubechies wavelets", Encyclopedia of Mathematics, EMS Press, 2001, https://encyclopediaofmath.org/index.php?title=Daubechies_wavelets.

Ingrid Daubechies
https://en.wikipedia.org/wiki/Ingrid_Daubechies

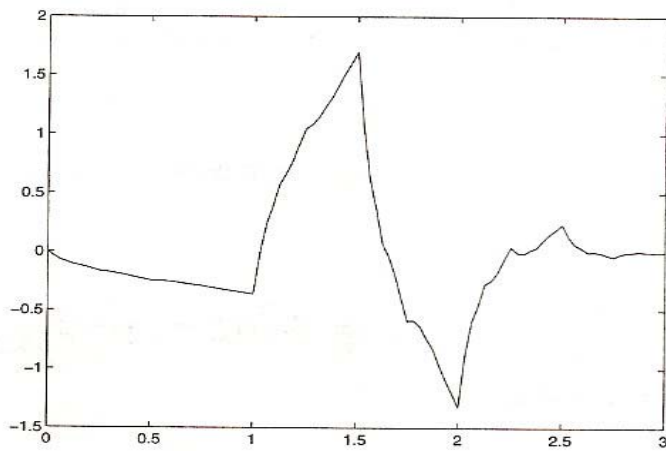From: S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Prentice Hall, N.J., 1996.
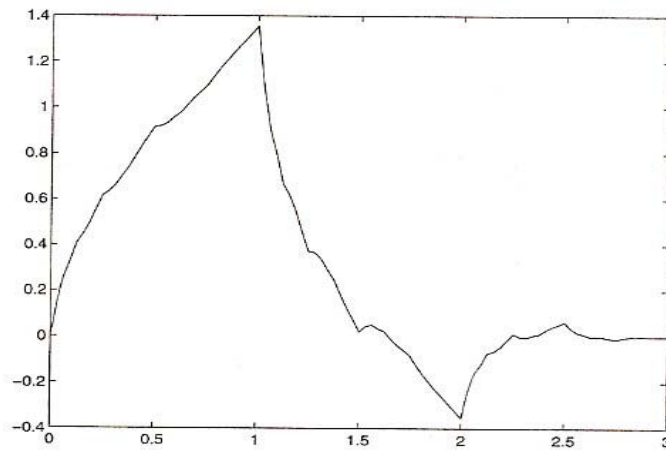
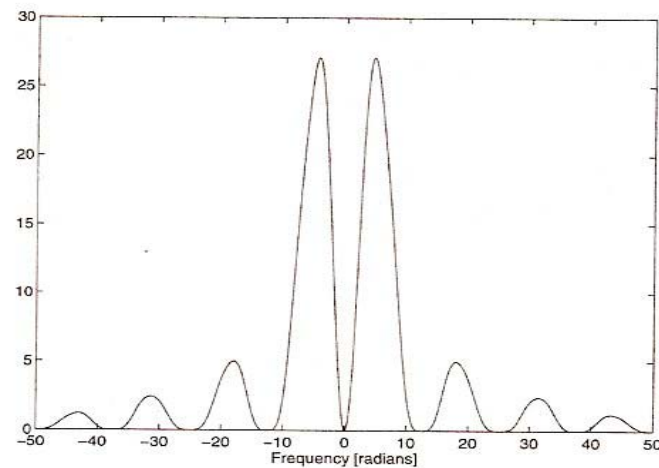(a) Scaling function ϕ(t)
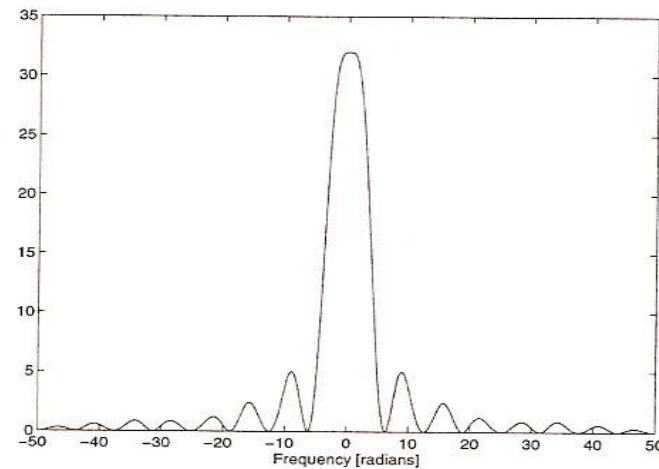
(b) |Φ(ω)|

(c) Daubechies wavelet ψ(t)

(d) |Ψ(ω)|

● Advantages:

(1) Fast algorithm for MRA

(2) Non-uniform frequency analysis

$$\psi\left(2^m t - n\right) \xrightarrow{\text{FT}} 2^{-m} e^{-j2\pi n 2^{-m} f} \Psi\left(2^{-m} f\right)$$

(3) Orthogonal

• Disadvantages:

(a) 無限多項連乘

(b) problem of initial

$$\chi_w(n,m), \quad X_w(n,m) \quad 皆由 \quad \chi_w(n,m+1) \quad 算出$$

$$\chi_w(n,m)\big|_{m\to\infty} \quad 如何算$$

(c) 難以保證 compact support

(d) 仍然太複雜

附錄十三 幾種常見的影像壓縮格式

(1) JPEG: 使用 discrete cosine transform (DCT) 和 8×8 blocks

是當前最常用的壓縮格式 (副檔名為 *.jpg 的圖檔都是用 JPEG 來壓縮)

可將圖檔資料量壓縮至原來的 1/8 (對灰階影像而言) 或 1/16 (對彩色影像而言)

(2) JPEG2000: 使用 discrete wavelet transform (DWT)

壓縮率是 JPEG 的 5 倍左右

(3) JPEG-LS: 是一種 lossless compression

壓縮率較低，但是可以完全重建原來的影像

(4) JPEG2000-LS: 是 JPEF2000 的 lossless compression 版本

(5) JBIG: 針對 bi-level image (非黑即白的影像) 設計的壓縮格式

(6) DjVu：適用於掃描的文件，使用漸進載入與算術編碼等技術，lossy

(7) WebP： 由 Google 開發，善用 Huffman code，Walsh transform ，字典
編碼，以及顏色索引變換的編碼技術，方法較複雜但壓縮率
也較高，open license，適用於 lossy 和 lossless 的編碼

(8) AVIF：由開放媒體聯盟（AOMedia）開發，使用預測技術，非對稱
DCT，算術編碼，可適性分割的編碼技術，open license，適
用於 lossy 和 lossless 的編碼

(9) JPEG XR (又稱 HD Photo)： 使用 Integer DCT，適用於 lossy 和 lossless
的編碼，在 lossy compression 的情形下壓縮率可和 JPEG 2000
差不多

(10) OpenEXR: 適合於高動態範圍的影像編碼，支援 16-bit 浮點數、32-bit
浮點數和32-bit 整數的 pixel values，有 lossy 和 lossless 的版本

(11) GIF: 使用 LZW (Lempel–Ziv–Welch) algorithm (類似字典的建構)

　　　　　適合卡通圖案和動畫製作，lossless

(12) MNG: 和 GIF 相似，適合動畫製作，有 lossy 和 lossless 的版本

(13) PNG: 使用 LZ77 algorithm (類似字典的建構，並使用 sliding window)

　　　　　適合於包含文字以及高對比度成份的影像，lossless

(14) APNG: PNG 的改良版本，有更小的資料量， lossless

(15) TIFF: 使用標籤，最初是為圖形的印刷和掃描而設計的，lossless