XIV. Discrete Wavelet Transform (DWT)

14.1 概念

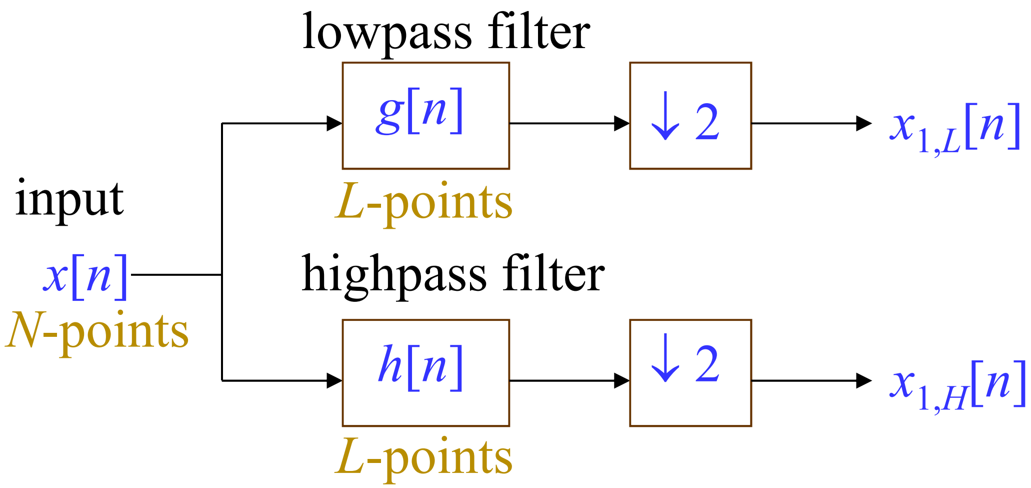
(1) discrete input to discrete output

(2) 由 continuous wavelet transform with discrete coefficients 演變而來的，
(比較 page 418)
但是大幅簡化了其中的數學

(3) 忽略了 scaling function 和 mother wavelet function 的分析
但是保留了階層式的架構

14.2 1-D Discrete Wavelet Transform (1D DWT)

simplified from page 421



$\boxed{\downarrow 2}$: downsampling by the factor of 2

$$x[n] \longrightarrow \boxed{\downarrow Q} \longrightarrow z[n] \qquad\qquad z[n] = x[Qn]$$

Input： $x[n]$ (不需算 $\chi_w(n,m)\big|_{m\to\infty}$，
直接以 $x[n]$ 作為 initial

Low pass filter $g[n]$
角色似 scaling function
(相當於 page 418 的 g_n)

High pass filter $h[n]$
角色似 wavelet function
(相當於 page 418 的 h_n)

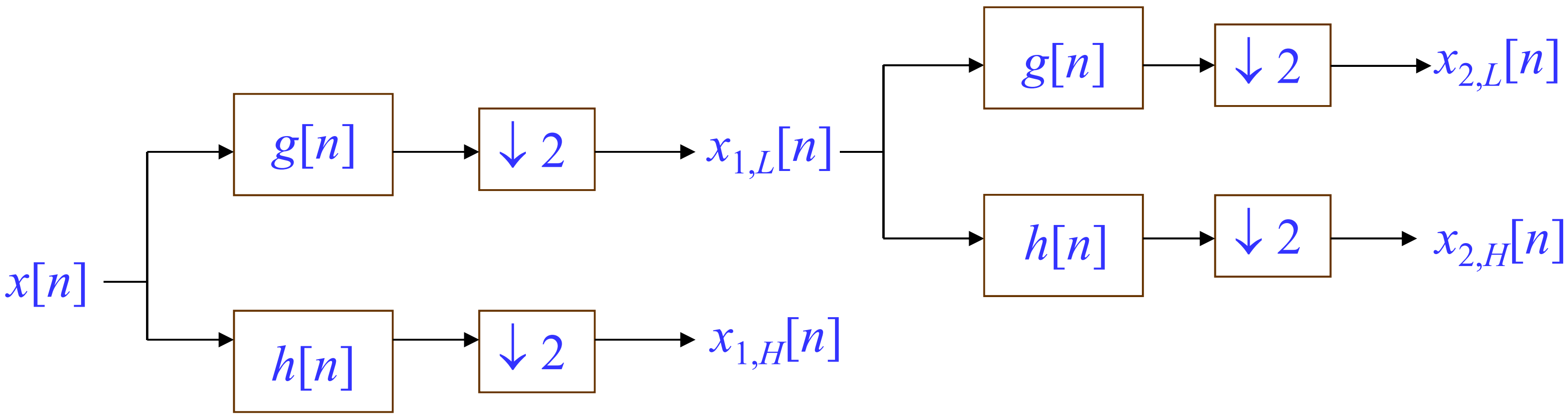
1st stage
$$x_{1,L}[n]=\sum_{k=0}^{K-1}x[2n-k]g[k]$$

$$x_{1,H}[n]=\sum_{k=0}^{K-1}x[2n-k]h[k]$$

further decomposition (from the $(a-1)^{\text{th}}$ stage to the a^{th} stage)

$$x_{a,L}[n] = \sum_{k=0}^{K-1} x_{a-1,L}[2n-k]\,g[k]$$

$$x_{a,H}[n] = \sum_{k=0}^{K-1} x_{a-1,L}[2n-k]\,h[k]$$



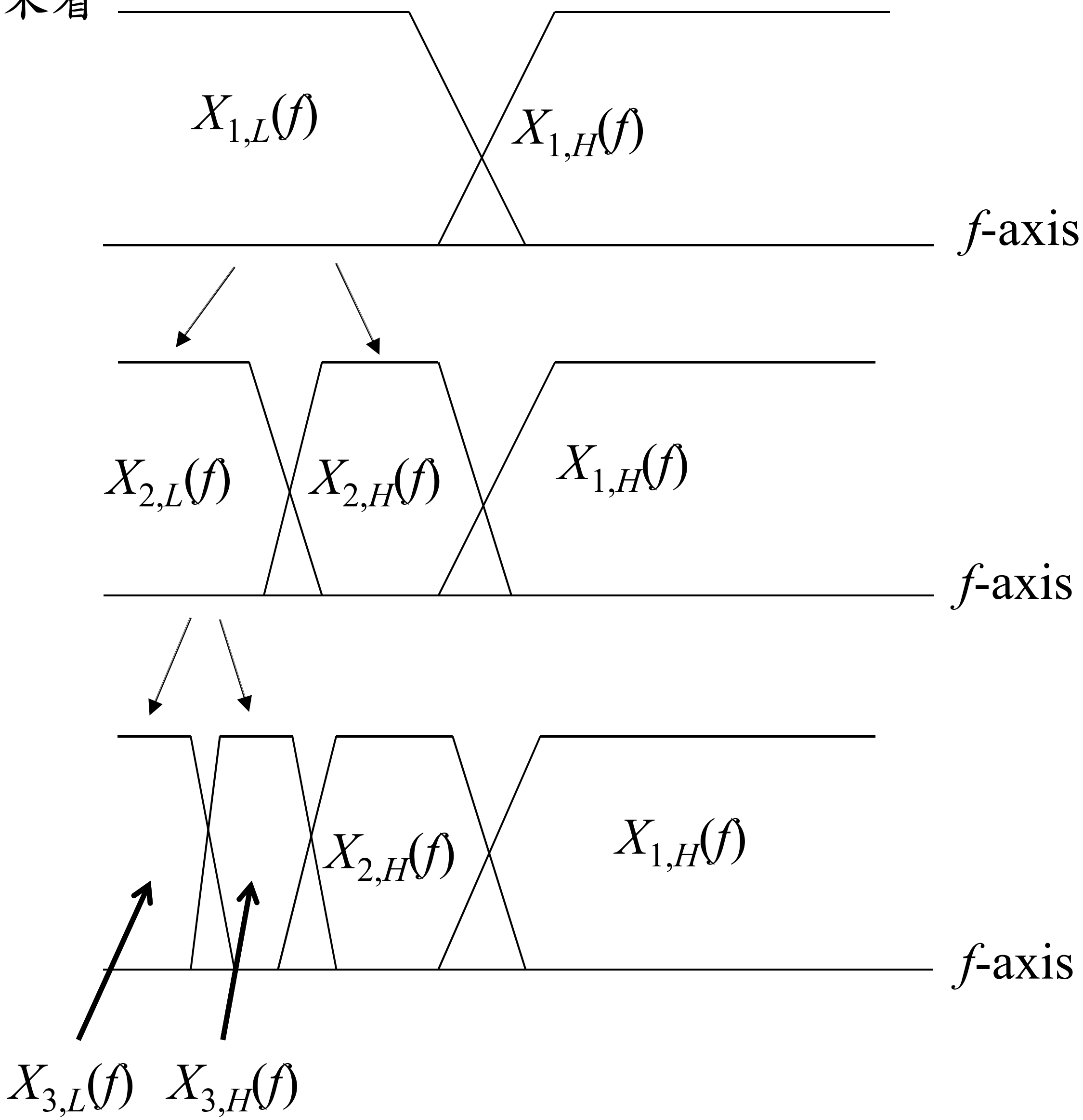
(1) 有的時候，對於 $x_{a,H}[n]$ 也再作細分

(2) 若 input 的 $x[n]$ 的 length 為 N,

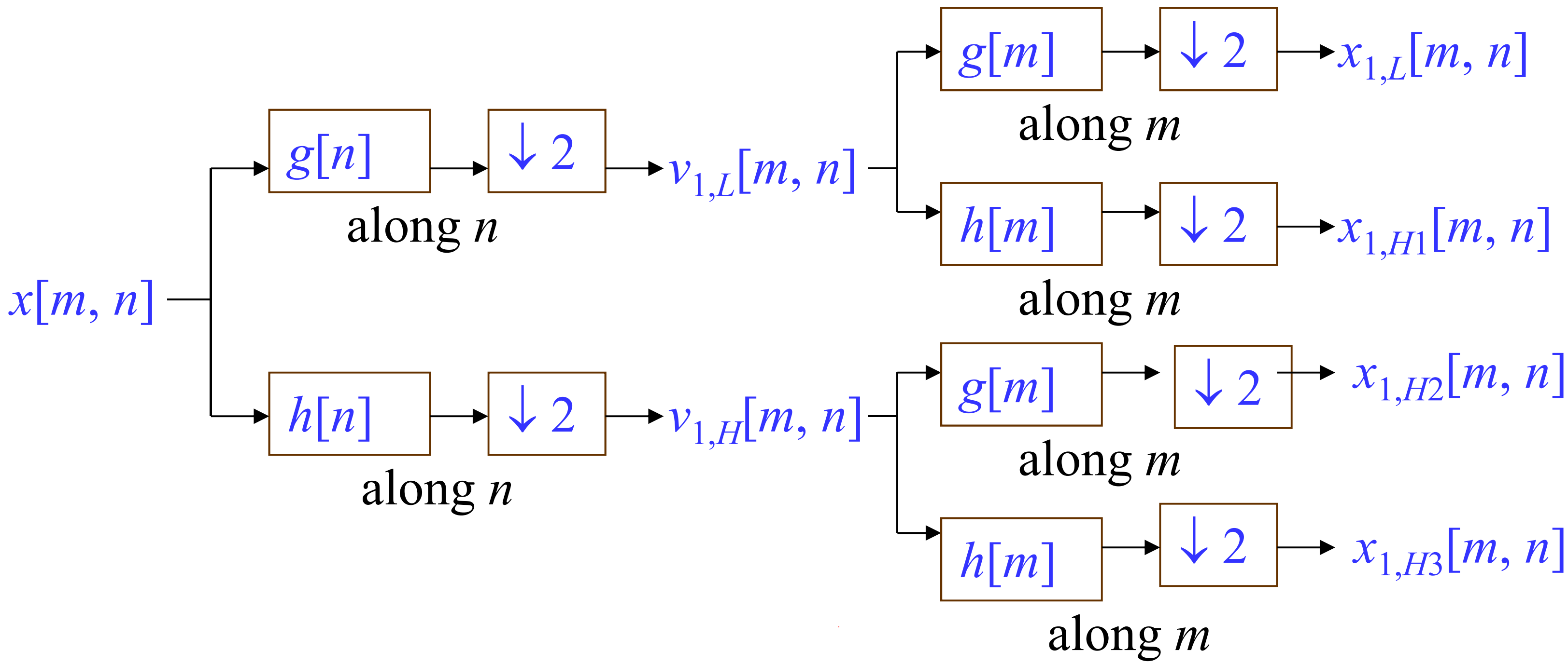
則 a^{th} stage $x_{a,L}[n]$, $x_{a,H}[n]$ 的 length 為 $N/2^a$

(3) 經過 DWT 之後，全部點數仍接近 N 點

(4) 以頻譜來看


$X_{1,L}(f)$
$X_{1,H}(f)$
f-axis
$X_{2,L}(f)$
$X_{2,H}(f)$
$X_{1,H}(f)$
f-axis
$X_{2,H}(f)$
$X_{1,H}(f)$
f-axis
$X_{3,L}(f)$
$X_{3,H}(f)$

14.3 2-D Discrete Wavelet Transform (2D DWT)



See page 40

輸入： $x[m, n]$

Low pass filter $g[n]$ High pass filter $h[n]$

- along n

$$v_{1,L}[m,n]=\sum_{k=0}^{K-1}x[m,2n-k]g[k] \qquad v_{1,H}[m,n]=\sum_{k=0}^{K-1}x[m,2n-k]h[k]$$

- along m

$$x_{1,L}[m,n]=\sum_{k=0}^{K-1}v_{1,L}[2m-k,n]g[k] \qquad x_{1,H_2}[m,n]=\sum_{k=0}^{K-1}v_{1,H}[2m-k,n]g[k]$$

$$x_{1,H_1}[m,n]=\sum_{k=0}^{K-1}v_{1,L}[2m-k,n]h[k] \qquad x_{1,H_3}[m,n]=\sum_{k=0}^{K-1}v_{1,H}[2m-k,n]h[k]$$

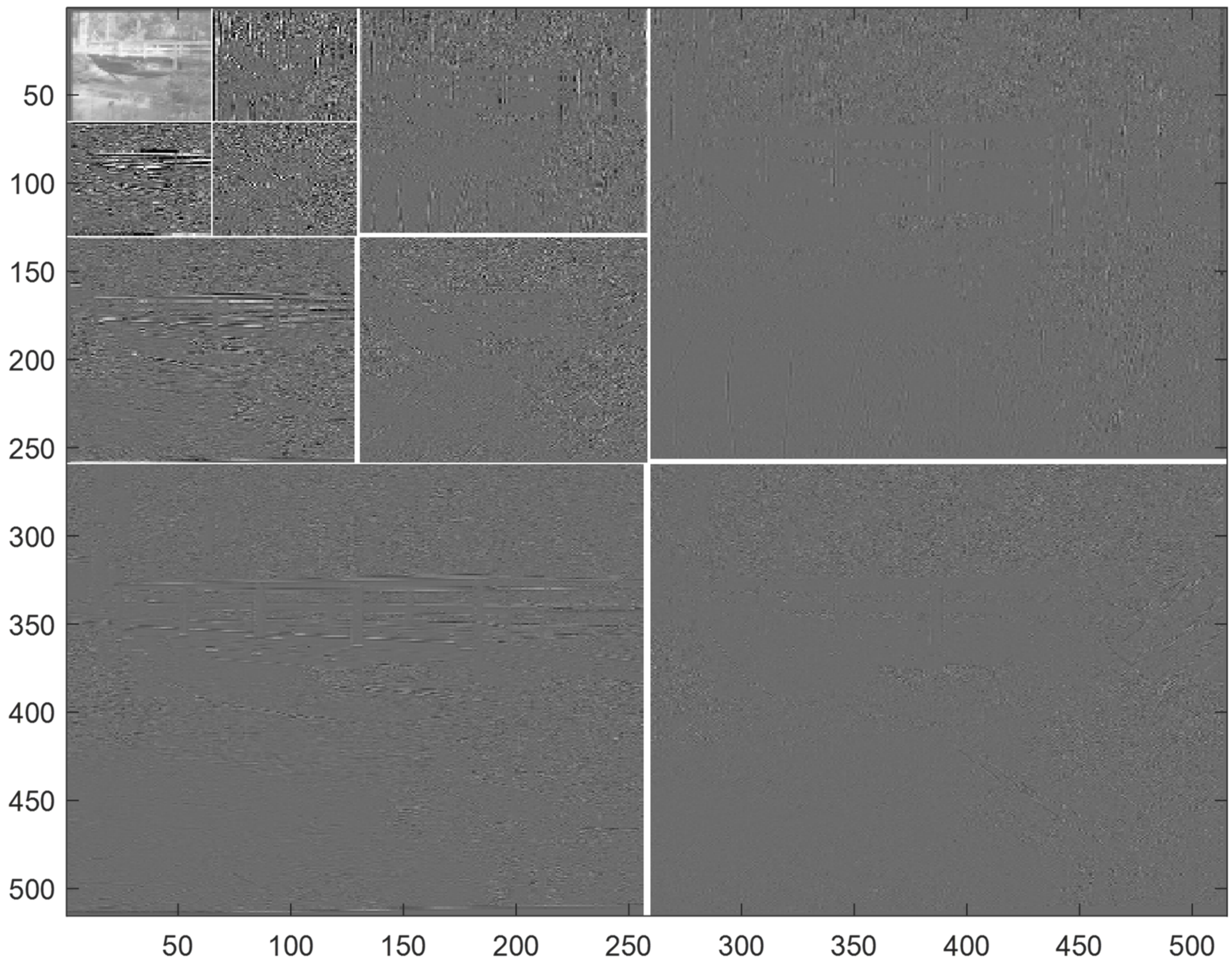
Input image:

A square.

from R. C. Gonzalez and R. E. Woods, *Digital Image Processing*, Chap. 7, 2nd edition, Prentice Hall, New Jersey, 2002.

原圖：Bridge

原圖：Bridge


50
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- compression & noise removing

保留 $x_{1,L}[m, n]$，捨棄其他部分

- (directional) edge detection

保留 $x_{1,H1}[m, n]$
或保留 $x_{1,H2}[m, n]$
捨棄其他部分

- $x_{1,H3}[m, n]$ 當中所包含的資訊較少

corner detection?

14.4 Complexity of the DWT

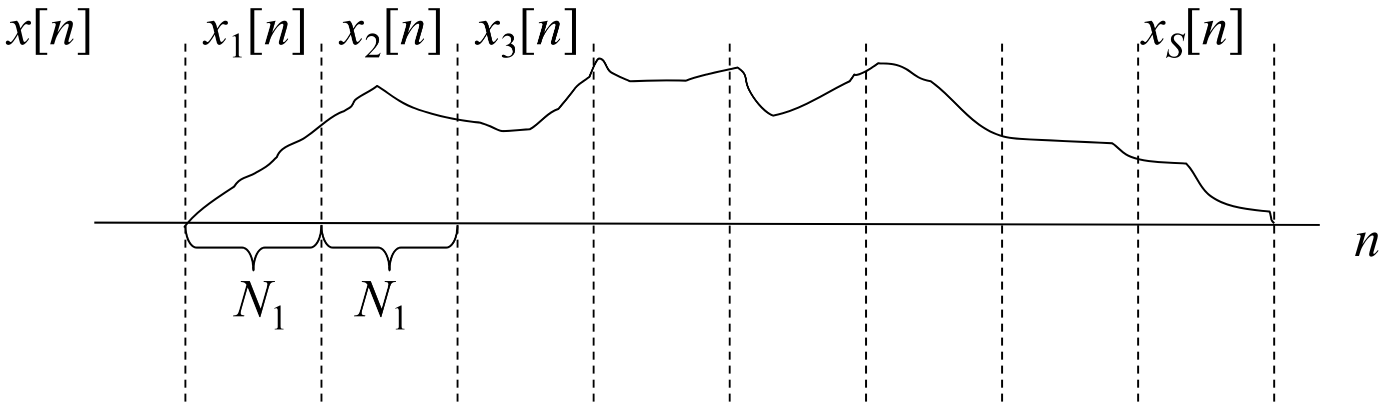
$$x[n] * y[n], \quad \text{length}(x[n]) = N, \quad \text{length}(y[n]) = L,$$

$$\underbrace{IDFT_{N+L-1}}_{(N+L-1)\text{-point inverse discrete Fourier transform (IDFT)}}\left[\underbrace{DFT_{N+L-1}\left(x[n]\right)}_{(N+L-1)\text{-point discrete Fourier transform (DFT)}} DFT_{N+L-1}\left(y[n]\right)\right]$$

(1) Complexity of the 1-D DWT (without sectioned convolution)

$$3(N+L-1)\log_2(N+L-1) \approx 3N\log_2 N$$

(2) 當 $N >>> L$ 時，使用 "sectioned convolution" 的技巧



將 $x[n]$ 切成很多段，每段長度為 N_1 $(N > N_1 >> L)$

總共有 $S = N / N_1$ 段

$$x[n] = x_1[n] + x_2[n] + \cdots\cdots + x_S[n]$$

$$x[n] * g[n] = x_1[n] * g[n] + x_2[n] * g[n] + \cdots\cdots + x_S[n] * g[n]$$

$$x[n] * h[n] = x_1[n] * h[n] + x_2[n] * h[n] + \cdots\cdots + x_S[n] * h[n]$$

complexity:

$$
\begin{aligned}
3S(N_1+L-1)\log_2(N_1+L-1) &\approx 3SN_1\log_2(N_1+L-1) \\
&= 3N\log_2(N_1+L-1) \\
&\approx 3N\log_2 N_1
\end{aligned}
$$

- 重要概念：

The complexity of the 1-D DWT is **linear with *N***

$$O(N)$$

when $N >>> L$

(3) Multiple stages 的情形下

- 若 $x_{a,H}[n]$ 不再分解

Complexity 近似於: $$\left(N+\frac{N}{2}+\frac{N}{4}+\frac{N}{8}+\cdots\cdots+2\right)\log_2 N_1$$
$$=(2N-2)\log_2 N_1 \approx 2N\log_2 N_1$$

- 若 $x_{a,H}[n]$ 也細分

Complexity 近似於:

$$\left(N+2\frac{N}{2}+4\frac{N}{4}+8\frac{N}{8}+\cdots\cdots+\frac{N}{2}\cdot 2\right)\log_2 N_1$$
$$=(N\log_2 N)\log_2 N_1$$

(和 DFT 相近)

(4) Complexity of the 2-D DWT on page 465 (without sectioned convolution)

$$3M(N+L-1)\log_2(N+L-1)+3(N+L-1)(M+L-1)\log_2(M+L-1)$$

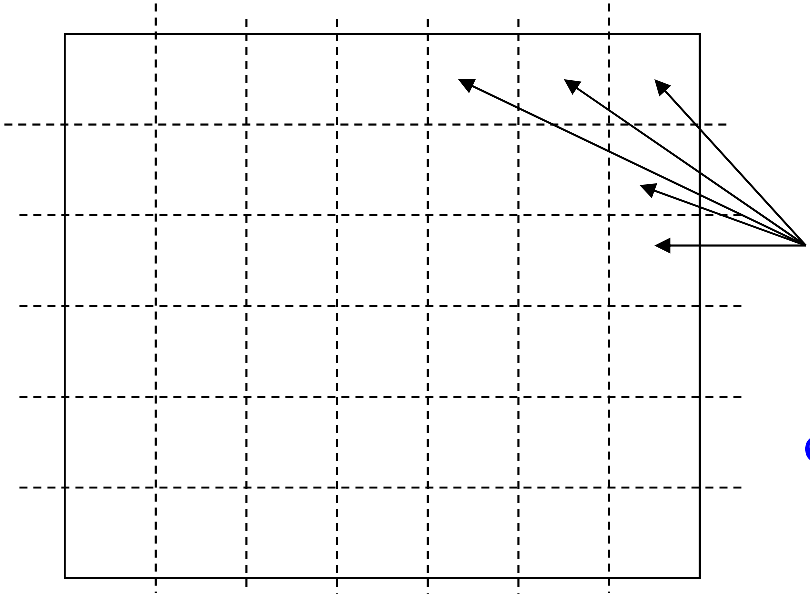
The first part needs M 1-D DWTs and the input for each 1-D DWT has N points

The second part needs $N+L-1$ 1-D DWTs and the input for each 1-D DWT has M points

$$\begin{aligned}\text{complexity} &\approx 3MN\log_2 N+3MN\log_2 M\\ &=3MN(\log_2 N+\log_2 M)\\ &=3MN\log_2(MN)\end{aligned}$$

(5) Complexity of the 2-D DWT (with Sectioned Convolution)

Image



The original size: $M \times N$

The size of each part: $M_1 \times N_1$

$$\text{complexity} \approx \left(\frac{MN}{M_1 N_1}\right) 3 M_1 N_1 \log_2(M_1 N_1)$$
$$= 3MN \log_2(M_1 N_1)$$

● 重要概念：

If the method of the sectioned convolution is applied,

the complexity of the 2-D DWT is **linear with *MN*.**

$$O(MN)$$

(6) Multiple stages, two dimension

$x[m, n]$ 的 size 為 $M \times N$

- 若 $x_{a,H1}[n], x_{a,H2}[n], x_{a,H3}[n]$ 不細分，只細分 $x_{a,L}[n]$

total complexity

$$\left(MN + \frac{MN}{4} + \frac{MN}{16} + \cdots\cdots \right) \log_2(M_1 N_1) \approx \frac{4}{3} MN \log_2(M_1 N_1)$$

- 若 $x_{a,H1}[n], x_{a,H2}[n], x_{a,H3}[n]$ 也細分

total complexity

$$\left(MN + 4\frac{M}{2}\frac{N}{2} + 16\frac{M}{4}\frac{N}{4} + \cdots\cdots \right) \log_2(M_1 N_1)$$
$$= \left[MN \log_2\left(\min(M, N)\right) \right] \log_2(M_1 N_1)$$

14.5 Many Operations Also Have Linear Complexities

- 事實上，不只 wavelet 有 linear complexity

  當 input 和 filter 長度或大小相差懸殊時

  1-D convolution 的 complexity 是 linear with N.

  2-D convolution 的 complexity 是 linear with MN.

  (和傳統 $N\log_2 N$, $MN\log_2(MN)$ 的觀念不同)

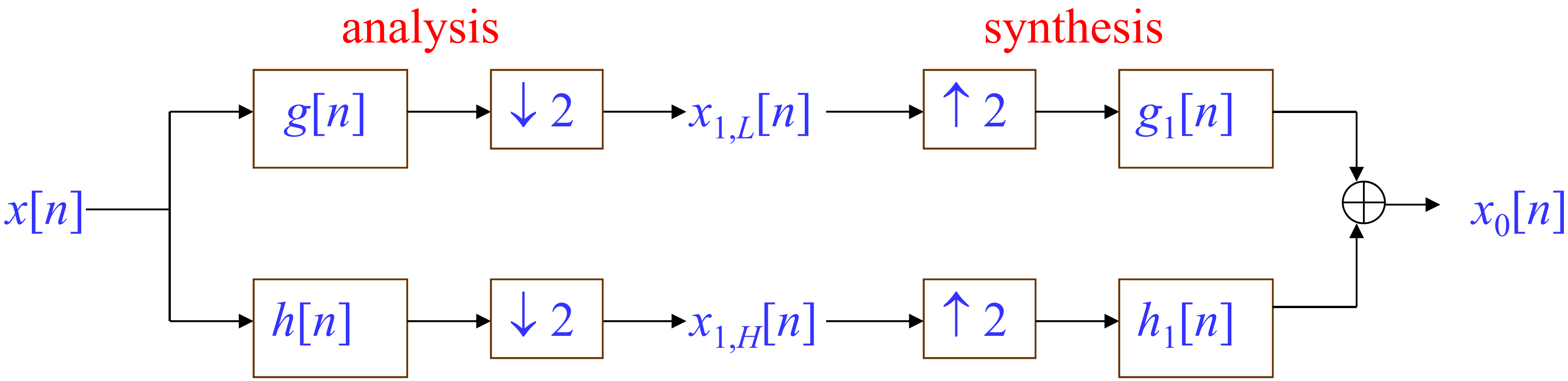
很重要的概念

- Note：DCT 的 complexity 也是 linear with MN

(divided into 8×8 blocks)

complexity : $\dfrac{MN}{64}(8\times 8\log_2 8 + 8\times 8\log_2 8) = MN\log_2 64$

14.6 Reconstruction



$g_1[n], h_1[n]$ 要滿足什麼條件，才可以使得 $x_0[n] = x[n]$？

$\boxed{\uparrow 2}$: upsampling by the factor of 2

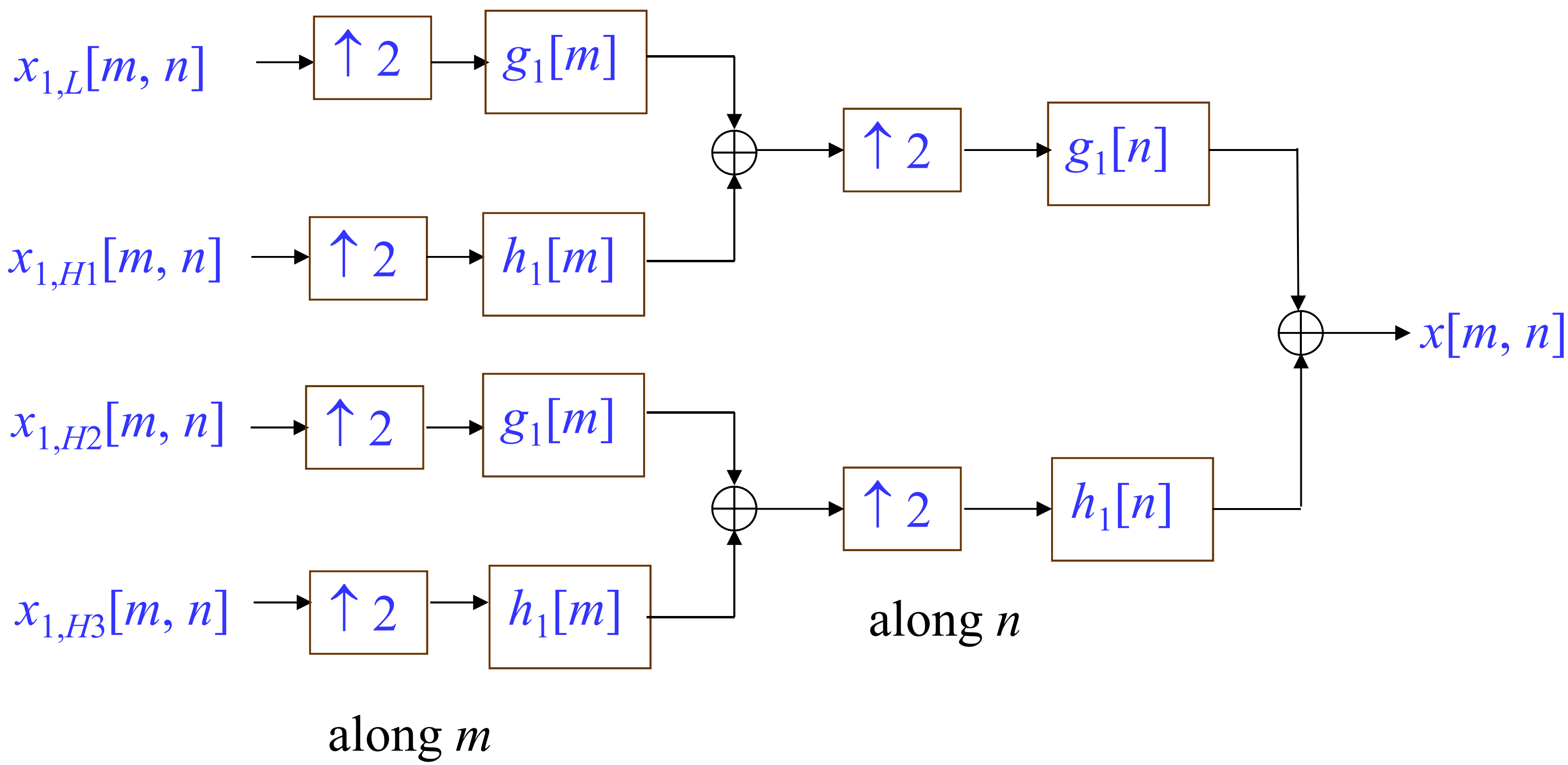
$a[n] \rightarrow \boxed{\uparrow Q} \rightarrow b[n]$

$$b[Qn] = a[n]$$

$$b[Qn + r] = 0 \quad \text{for } r = 1, 2, Q\text{–}1$$

the analysis part of the 2D DWT: page 465

the synthesis part of the 2D DWT



Reconstruction Problem

用 Z transform 來分析 $X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$

- If $a[n]=b[2n]$, $\xrightarrow{\text{Z transform}}$ $A(z)=\frac{1}{2}\left[B\left(z^{1/2}\right)+B\left(-z^{1/2}\right)\right]$

  $\downarrow 2$ (downsampling)

(Proof):

$$B\left(z^{1/2}\right)+B\left(-z^{1/2}\right)=\sum_{n=-\infty}^{\infty} b[n]z^{-n/2}+\sum_{n=-\infty}^{\infty}(-1)^n b[n]z^{-n/2}$$

$$=\sum_{n=-\infty}^{\infty}\left(1+(-1)^n\right)b[n]z^{-n/2}=2\sum_{n_1=-\infty}^{\infty} b[2n_1]z^{-n_1}=2\sum_{n_1=-\infty}^{\infty} a[n_1]z^{-n_1}=2A(z)$$

- If $a[2n]=b[n]$, $\Longrightarrow$ $A(z)=B\left(z^2\right)$

  $a[2n+1]=0$

  $\uparrow 2$ (upsampling)

$$X_{1,L}(z)=\frac{1}{2}\left[X\left(z^{1/2}\right)G\left(z^{1/2}\right)+X\left(-z^{1/2}\right)G\left(-z^{1/2}\right)\right]$$

$$X_{1,H}(z)=\frac{1}{2}\left[X\left(z^{1/2}\right)H\left(z^{1/2}\right)+X\left(-z^{1/2}\right)H\left(-z^{1/2}\right)\right]$$

$$\begin{aligned}X_o(z)&=\frac{1}{2}\left[X(z)G(z)+X(-z)G(-z)\right]G_1(z)\\&\quad+\frac{1}{2}\left[X(z)H(z)+X(-z)H(-z)\right]H_1(z)\\&=\frac{1}{2}\left[G(z)G_1(z)+H(z)H_1(z)\right]X(z)\\&\quad+\frac{1}{2}\left[G(-z)G_1(z)+H(-z)H_1(z)\right]X(-z)\end{aligned}$$

Perfect reconstruction: $X_o(z)=X(z)$

Perfect reconstruction: $X_o(z) = X(z)$

條件： $\begin{cases} G(z)G_1(z) + H(z)H_1(z) = 2 \\ G(-z)G_1(z) + H(-z)H_1(z) = 0 \end{cases}$

$$\begin{bmatrix} G(z) & H(z) \\ G(-z) & H(-z) \end{bmatrix} \begin{bmatrix} G_1(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G_1(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} G(z) & H(z) \\ G(-z) & H(-z) \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G_1(z) \\ H_1(z) \end{bmatrix} = \frac{1}{\det(\mathbf{H}_m(z))} \begin{bmatrix} H(-z) & -H(z) \\ -G(-z) & G(z) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

where $\mathbf{H}_m(z) = \begin{bmatrix} G(z) & H(z) \\ G(-z) & H(-z) \end{bmatrix}$

$$\boxed{\begin{bmatrix} G_1(z) \\ H_1(z) \end{bmatrix} = \frac{2}{\det\left(\mathbf{H}_m(z)\right)} \begin{bmatrix} H(-z) \\ -G(-z) \end{bmatrix}}$$

where $\boxed{\det\left(\mathbf{H}_m(z)\right) = G(z)H(-z) - H(z)G(-z)}$

14.7 Reconstruction 的等效條件

$$\begin{bmatrix} G_1(z) \\ H_1(z) \end{bmatrix} = \frac{2}{\det\left(\mathbf{H}_m(z)\right)} \begin{bmatrix} H(-z) \\ -G(-z) \end{bmatrix}$$

if and only if

$$\boxed{\sum_p g[p]g_1[2n-p] = \delta[n]}$$

$$\boxed{\sum_p h[p]h_1[2n-p] = \delta[n]}$$

$$\boxed{\sum_p g[p]h_1[2n-p] = 0}$$

$$\boxed{\sum_p g_1[p]h[2n-p] = 0}$$

這四個條件被稱作

biorthogonal conditions

(Proof)

Note: (a) $\det\left(\mathbf{H}_m(-z)\right) = -\det\left(\mathbf{H}_m(z)\right)$

(b) 令 $P(z) = G(z)G_1(z) = \dfrac{2G(z)H(-z)}{\det\left(\mathbf{H}_m(z)\right)}$

$$P(-z) = \frac{2G(-z)H(z)}{\det\left(\mathbf{H}_m(-z)\right)} = H(z)\frac{-2G(-z)}{\det\left(\mathbf{H}_m(z)\right)} = H(z)H_1(z)$$

Therefore,

$$H(z)H_1(z) = P(-z) = G(-z)G_1(-z)$$

From $G(z)G_1(z) + H(z)H_1(z) = 2$

$$G(z)G_1(z) + G(-z)G_1(-z) = 2$$

$$\Big\downarrow \text{ inverse } Z \text{ transform}$$

$$\sum_p g[p]g_1[n-p] + (-1)^n \sum_p g[p]g_1[n-p] = 2\delta[n]$$

$$\sum_p g[p]g_1[n-p]+(-1)^n\sum_p g[p]g_1[n-p]=2\delta[n]$$

$$\downarrow$$

$$\boxed{\sum_p g[p]g_1[2n-p]=\delta[n]}$$ ← orthogonality 條件 1

(c) Similarly, substitute $G(z)G_1(z)=H(-z)H_1(-z)$

into $G(z)G_1(z)+H(z)H_1(z)=2$

$$H(-z)H_1(-z)+H(z)H_1(z)=2$$

↓ after the process the same as that of the above

$$\boxed{\sum_p h[p]h_1[2n-p]=\delta[n]}$$ ← orthogonality 條件 2

(d) Since $$G(z)H_1(z)+G(-z)H_1(-z)$$

$$=-G(z)\frac{G(-z)}{\det(\mathbf{H}_m(z))}-G(-z)\frac{G(z)}{\det(\mathbf{H}_m(-z))}$$

$$=-\frac{G(z)G(-z)}{\det(\mathbf{H}_m(z))}+\frac{G(-z)G(z)}{\det(\mathbf{H}_m(z))}=0$$

$$\sum_p g[p]h_1[n-p]+(-1)^n\sum_p g[p]h_1[n-p]=0$$

$$\boxed{\sum_p g[p]h_1[2n-p]=0}$$ ← orthogonality 條件 3

(e) 同理 $$G_1(z)H(z)+G_1(-z)H(-z)=0$$

$$\boxed{\sum_p g_1[p]h[2n-p]=0}$$ ← orthogonality 條件 4

14.8 DWT 設計上的條件

- Reconstruction
- Finite length　　為了 implementation 速度的考量

$$\boxed{g[n] \neq 0 \ \text{ only when } -L \le n \le L}$$

$$\boxed{h[n] \neq 0 \ \text{ only when } -L \le n \le L}$$

$h_1[n]$, $g_1[n]$?

令 $\boxed{\det\left(\mathbf{H}_m(z)\right) = \alpha z^k}$ 　則根據 page 484,

$$G_1(z) = 2\alpha^{-1} z^{-k} H(-z) \qquad H_1(z) = -2\alpha^{-1} z^{-k} G(-z)$$

複習: $x[n-k] \xrightarrow{Z\ transform} z^{-k} X(z)$

$$g_1[n] = 2\alpha^{-1}(-1)^{n-k} h[n-k] \qquad h_1[n] = -2\alpha^{-1}(-1)^{n-k} g[n-k]$$

- 因為 $\det\left(\mathbf{H}_m(z)\right) = G(z)H(-z) - H(z)G(-z)$

  $\det\left(\mathbf{H}_m(z)\right) = -\det\left(\mathbf{H}_m(-z)\right)$

  k 必需為 odd

- Lowpass-highpass pair

14.9 整理：DWT 的四大條件

(1) $$\begin{bmatrix} G_1(z) \\ H_1(z) \end{bmatrix} = \frac{2}{\det\left(\mathbf{H}_m(z)\right)} \begin{bmatrix} H(-z) \\ -G(-z) \end{bmatrix}$$ (for reconstruction)

$$\det\left(\mathbf{H}_m(z)\right) = G(z)H(-z) - H(z)G(-z)$$

(2) $h[n] \neq 0$ only when $0 \leq n \leq L-1$

$g[n] \neq 0$ only when $0 \leq n \leq L-1$

($h[n]$, $g[n]$ have finite lengths)

(3) $\boxed{\det\left(\mathbf{H}_m(z)\right) = \alpha z^k}$ k 必需為 odd

($h_1[n]$, $g_1[n]$ have finite lengths)

(4) $h[n]$ 為 highpass filter

$g[n]$ 為 lowpass filter

(lowpass and highpass pair)

第三個條件較難達成，是設計的核心

14.10 Two Types of Perfect Reconstruction Filters

(1) QMF (quadrature mirror filter)

$G(z)$ satisfy $\boxed{G^2(z) - G^2(-z) = 2z^k}$ $\underline{k \text{ is odd}}$

$g[n]$ has finite length

$H(z) = G(-z)$ $\quad h[n] = (-1)^n g[n]$

$G_1(z) = G(z)z^{-k}$ $\quad g_1[n] = g[n-k]$

$H_1(z) = -G(-z)z^{-k}$ $\quad h_1[n] = (-1)^{n-k+1} g[n-k]$

$$\det(\mathbf{H}_m(z)) = G(z)H(-z) - H(z)G(-z) = ?$$

(2) Orthonormal

$G(z)$ satisfy $\boxed{G(z)G(z^{-1})+G(-z)G(-z^{-1})=2}$

$g[n]$ has finite length

$H(z)=-z^{k}G(-z^{-1})$ <u>k is odd</u> $h[n]=(-1)^{n}g[-n-k]$

$G_1(z)=G(z^{-1})$ $g_1[n]=g[-n]$

$H_1(z)=-z^{-k}G(-z)=H(z^{-1})$ $h_1[n]=h[-n]$

$$\det\left(\mathbf{H}_m(z)\right)=G(z)H(-z)-H(z)G(-z)$$
$$=G(z)z^{k}G(z^{-1})+G(-z)z^{k}G(-z^{-1})=2z^{k}$$

大部分的 wavelet 屬於 orthonormal wavelet

For the orthonormal wavelet

$$\sum_{n=0}^{N-\tau-1} g[n]g[n+\tau]=0$$

$$\sum_{n=0}^{N-\tau-1} h[n]h[n+\tau]=0$$

for $\tau = 2, 4, \ldots, N-2$

(orthonormal to the shift versions of themselves)

It can be proved by pages 486 and 494.

(Note): 文獻上，有時會出現另一種 perfect reconstruction filter, 稱作 CQF (conjugate quadrature filter)

然而，CQF 本質上和 orthonormal filter 相同

14.11 Several Types of Discrete Wavelets

- discrete Haar wavelet **(最簡單的)**

$g[-1] = g[0] = 1$ $\quad$ $g[n] = 0$ otherwise

$h[-1] = -1, \quad h[0] = 1$ $\quad$ $h[n] = 0$ otherwise

$g_1[0] = g_1[1] = 1$ $\quad$ $g_1[n] = 0$ otherwise

$h_1[0] = 1, \quad h_1[1] = -1$ $\quad$ $h_1[n] = 0$ otherwise

是一種 orthonormal filter

- discrete Daubechies wavelet (8-point case)

$$g[n] = [-0.0106 \quad 0.0329 \quad 0.0308 \quad -0.1870 \quad -0.0280 \quad 0.6309 \quad 0.7148 \quad 0.2304]$$

$n = 0 \sim 7$ $\qquad g[n] = 0$ otherwise

$$h[n] = [0.2304 \quad -0.7148 \quad 0.6309 \quad 0.0280 \quad -0.1870 \quad -0.0308 \quad 0.0329 \quad 0.0106]$$

$n = 0 \sim 7$ $\qquad h[n] = 0$ otherwise

$$g_1[n] = [0.2304 \quad 0.7148 \quad 0.6309 \quad -0.0280 \quad -0.1870 \quad 0.0308 \quad 0.0329 \quad -0.0106]$$

$n = -7 \sim 0$ $\qquad g_1[n] = 0$ otherwise

$$h_1[n] = [0.0106 \quad 0.0329 \quad -0.0308 \quad -0.1870 \quad 0.0280 \quad 0.6309 \quad -0.7148 \quad 0.2304]$$

$n = -7 \sim 0$ $\qquad h_1[n] = 0$ otherwise

https://wavelets.pybytes.com/

- discrete Daubechies wavelet (4-point case)

$$g[n] = [-0.1294 \quad 0.2241 \quad 0.8365 \quad 0.4830]$$

- discrete Daubechies wavelet (6-point case)

$$g[n] = [0.0352 \quad -0.0854 \quad -0.1350 \quad 0.4599 \quad 0.8069 \quad 0.3327]$$

- discrete Daubechies wavelet (10-point case)

$$g[n] = [0.0033 \quad -0.0126 \quad -0.0062 \quad 0.0776 \quad -0.0322 \quad -0.2423$$
$$0.1384 \quad 0.7243 \quad 0.6038 \quad 0.1601]$$

- discrete Daubechies wavelet (12-point case)

$$g[n] = [-0.0011 \quad 0.0048 \quad 0.0006 \quad -0.0316 \quad 0.0275 \quad 0.0975$$
$$-0.1298 \quad -0.2263 \quad 0.3153 \quad 0.7511 \quad 0.4946 \quad 0.1115]$$

symlet (6-point case)

$$g[n] = [0.0352 \quad -0.0854 \quad -0.1350 \quad 0.4599 \quad 0.8069 \quad 0.3327]$$

symlet (8-point case)

$$g[n] = [-0.0757 \quad -0.0296 \quad 0.4976 \quad 0.8037 \quad 0.2978 \quad -0.0992 \\ -0.0126 \quad 0.0322]$$

symlet (10-point case)

$$g[n] = [0.0273 \quad 0.0295 \quad -0.0391 \quad 0.1993 \quad 0.7234 \quad 0.6339 \\ 0.0166 \quad -0.1753 \quad -0.0211 \quad 0.0195]$$

Daubechies wavelets and symlets are defined for <u>N is a multiple of 2</u>

https://wavelets.pybytes.com/

coiflet (6-point case)

$$g[n] = [-0.0157 \quad -0.0727 \quad 0.3849 \quad 0.8526 \quad 0.3379 \quad -0.0727]$$

coiflet (12-point case)

$$g[n] = [0.0232 \quad -0.0586 \quad -0.0953 \quad 0.5460 \quad 1.1494 \quad 0.5897$$
$$-0.1082 \quad -0.0841 \quad 0.0335 \quad 0.0079 \quad -0.0026 \quad -0.0010]$$

Coiflets are defined for <u>N is a multiple of 6</u>

The Daubechies wavelet, the symlet, and the coiflet are all orthonormal filters.

https://wavelets.pybytes.com/

The Daubechies wavelet, the symlet, and the coiflet are all derived from the "continuous wavelet with discrete coefficients" case.

Physical meanings:

- Daubechies wavelet

  The ? point Daubechies wavelet has the vanishing moment of p.

- Symlet

  The vanishing moment is the same as that of the Daubechies wavelet, but the filter is more symmetric.

- Coiflet

  The ? point coiflet has the vanishing moment of p.

  The scaling function also has the vanishing moment.

$$\int_{-\infty}^{\infty} \phi(t)\,dt \neq 0 \qquad \int_{-\infty}^{\infty} t^k \phi(t)\,dt = 0 \quad \text{for } 1 \leqq k \leqq p$$

14.12 產生 Discrete Daubechies Wavelet 的流程

Step 1 $P(y)=\sum_{k=0}^{p-1} C_k^{p-1+k} y^k$

Q: 如何用 Matlab 寫出 C_n^m

(When p = 2, $P(y) = 2y + 1$)

Step 2 $P_1(z)=P\left(\frac{2-z-z^{-1}}{4}\right)$

Hint: $\left((2-z-z^{-1})/4\right)^k$ 在 Matlab 當中，可以用 [-.25, .5, -.25] 自己和自己 convolution k-1 次算出來

(When p = 2, $P_1(z) = 2 - 0.5z - 0.5z^{-1}$)

Step 3 算出 $z^kP_1(z)$ 的根 (i.e., $z^kP_1(z) = 0$ 的地方)

Q: 在 Matlab 當中應該用什麼指令

(When p = 2, roots = 3.7321, 0.2679)

Step 4　算出

$$P_2(z) = (z - z_1)(z - z_2)\cdots(z - z_{p-1})$$

$z_1, z_2, \ldots, z_{p\text{-}1}$ 為 $z^k P_1(z)$ 當中，絕對值小於 1 的 roots

Step 5　算出

$$G_0(z) = (1 + z)^p P_2(z)$$

$$g_0[n] = Z^{-1}\{G_0(z)\}$$

注意：Z transform 的定義為 $G_0(z) = \sum_n g_0[n] z^{-n}$

所以 coefficients 要做 reverse

(When p = 2, $g_0[n]$ = [1　1.7321　0.4641　-0.2679])

n = -3 ~ 0

Step 6 Normalization

$$g_1[n] = \frac{g_0[n]}{\|g_0\|}$$

(When $p = 2$, $g_1[n]$ = [0.4830 0.8365 0.2241 -0.1294])

n = -3 ~ 0

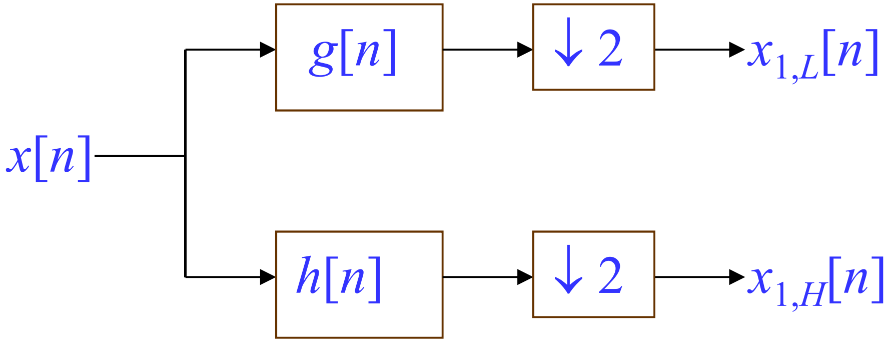
Step 7 Time reverse

$$g[n] = g_1[-n] \qquad\qquad h[n] = (-1)^n g[2p-1-n]$$

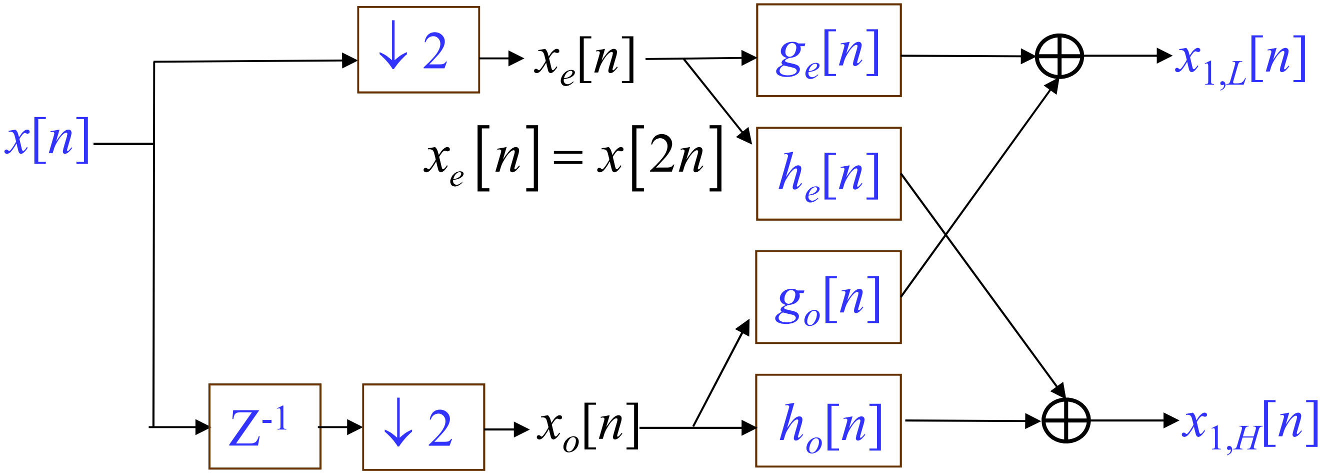
Then, the (2p)-point discrete Daubechies wavelet transform can be obtained

14.13 2x2 Structure Form and the Lifting Scheme

The analysis part



can be changed into the following 2x2 structure



$$x_e[n] = x[2n]$$

$$x_o[n] = x[2n-1]$$

where

$g_e[n] = g[2n]$

$g_o[n] = g[2n+1]$

$h_e[n] = h[2n]$

$h_o[n] = h[2n+1]$

Z^{-1} means delayed by 1

(Proof): From page 461,

$$x_{1,L}[n] = \sum_{k=0}^{K-1} x[2n-k]\,g[k]$$

$$x_{1,L}[n] = \sum_{k=0}^{K/2-1} x[2n-2k]\,g[2k] + \sum_{k=0}^{K/2-1} x[2n-2k-1]\,g[2k+1]$$

$$= \sum_{k=0}^{K/2-1} x_e[n-k]\,g_e[k] + \sum_{k=0}^{K/2-1} x_o[n-k]\,g_o[k]$$

where

$$x_e[n] = x[2n], \qquad x_o[n] = x[2n-1]$$

$$x[n] \longrightarrow \boxed{Z^{-1}} \longrightarrow \boxed{\downarrow 2} \longrightarrow x[2n\text{-}1]$$

Similarly,

$$x_{1,H}[n] = \sum_{k=0}^{K/2-1} x_e[n-k]\,h_e[k] + \sum_{k=0}^{K/2-1} x_o[n-k]\,h_o[k]$$

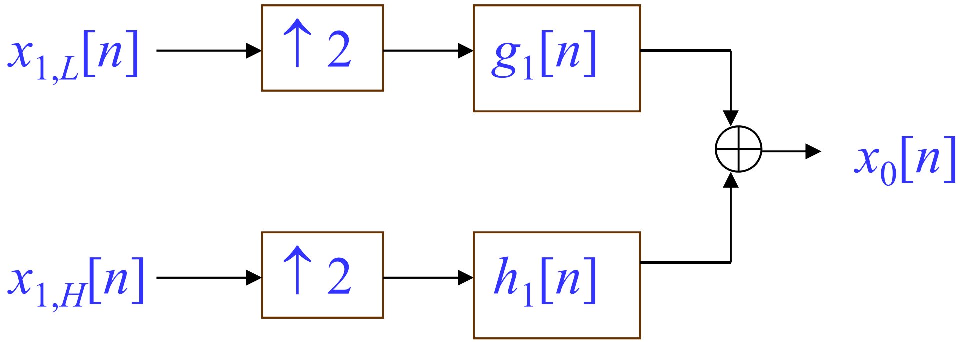
Original Structure:

Two Convolutions of an N-length input and an L-length filter

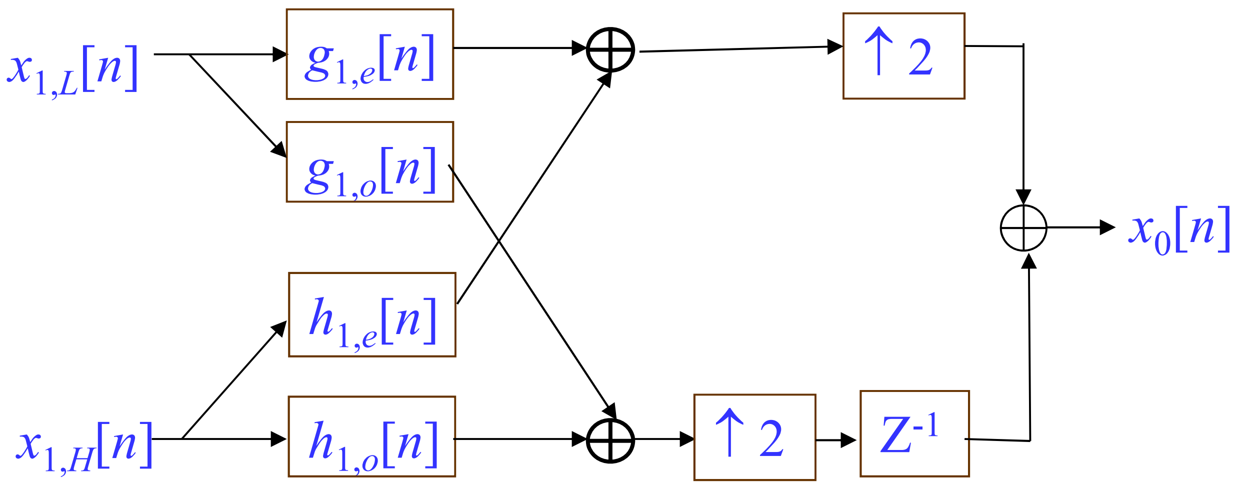
New Structure:

Four Convolutions of an $(N/2)$-length input and an $(L/2)$-length filter, which is more efficient. (Why?)

Similarly, the synthesis part



can be changed into the following 2x2 structure



where

$$g_{1,e}[n] = g_1[2n]$$

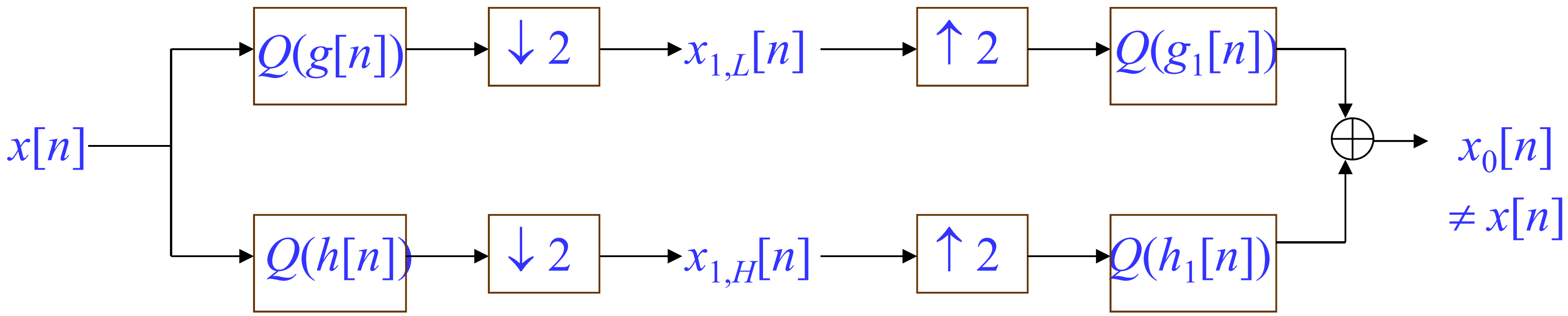
$$g_{1,o}[n] = g_1[2n+1]$$

$$h_{1,e}[n] = h_1[2n]$$

$$h_{1,o}[n] = h_1[2n+1]$$

14.14 Lifting Scheme

After performing quantization, the DWT may not be perfectly reversible



$Q(\ )$ means quantization (rounding, flooring, ceiling ……)

Lifting Scheme:

Reversible After Quantization

From page 505

$$\begin{bmatrix} G_e(z) & G_o(z) \\ H_e(z) & H_o(z) \end{bmatrix}\begin{bmatrix} X_e(z) \\ X_o(z) \end{bmatrix} = \begin{bmatrix} X_{1,L}(z) \\ X_{1,H}(z) \end{bmatrix}$$

Since

$$G_e(z) = \left[G\left(z^{1/2}\right) + G\left(-z^{1/2}\right)\right]/2 \qquad G_o(z) = z^{1/2}\left[G\left(z^{1/2}\right) - G\left(-z^{1/2}\right)\right]/2$$

$$H_e(z) = \left[H\left(z^{1/2}\right) + H\left(-z^{1/2}\right)\right]/2 \qquad H_o(z) = z^{1/2}\left[H\left(z^{1/2}\right) - H\left(-z^{1/2}\right)\right]/2$$

$$\det\left(\begin{bmatrix} G_e(z) & G_o(z) \\ H_e(z) & H_o(z) \end{bmatrix}\right) = z^{\frac{1}{2}}\left(G\left(-z^{\frac{1}{2}}\right)H\left(z^{\frac{1}{2}}\right) - G\left(z^{\frac{1}{2}}\right)H\left(-z^{\frac{1}{2}}\right)\right)/2$$

from page 492, one set that, if α = -1 and k = -2m-1,

$$\det\left(\mathbf{H}_m(z)\right) = G(z)H(-z) - H(z)G(-z) = -z^{-2m-1}$$

then

$$\det\left(\begin{bmatrix} G_e(z) & G_o(z) \\ H_e(z) & H_o(z) \end{bmatrix}\right) = z^{-m}/2$$

Then $\begin{bmatrix} G_e(z) & G_o(z) \\ H_e(z) & H_o(z) \end{bmatrix}$ can be decomposed into

$$\boxed{\begin{bmatrix} G_e(z) & G_o(z) \\ H_e(z) & H_o(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z^{-m}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L_1(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & L_2(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L_3(z) & 1 \end{bmatrix}}$$

where

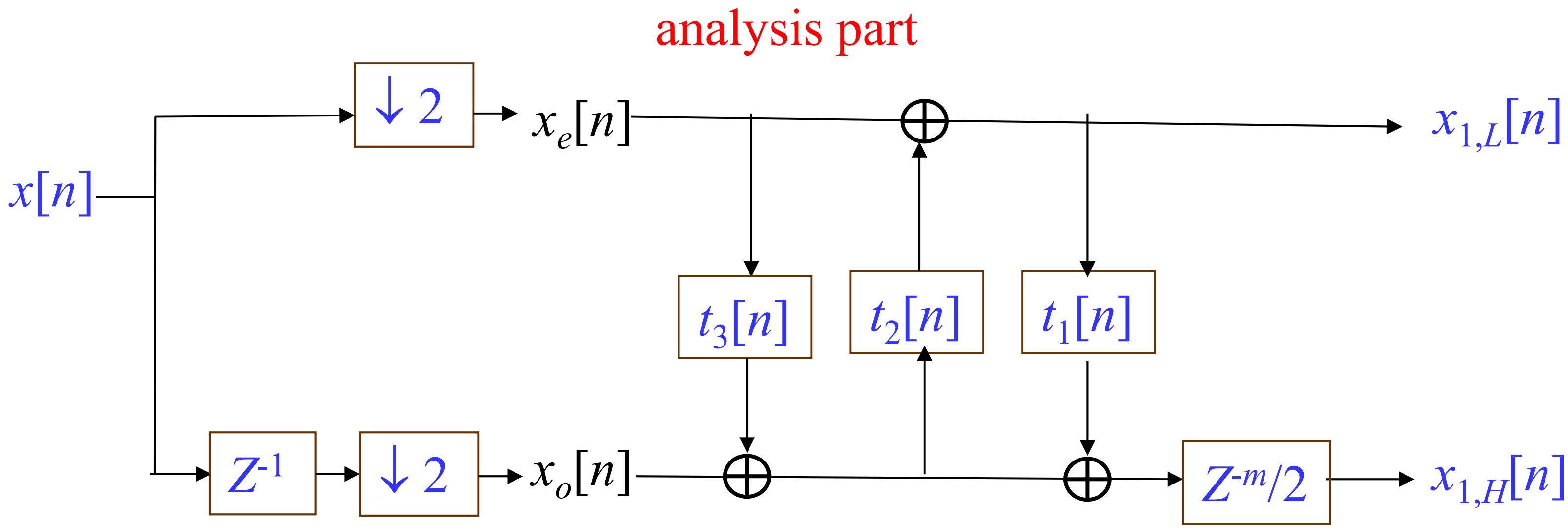
$$L_1(z) = \frac{2z^m H_o(z) - 1}{G_o(z)} \qquad L_2(z) = G_o(z) \qquad L_3(z) = \frac{G_e(z) - 1}{G_o(z)}$$

Then the DWT can be approximated by

$$\boxed{\begin{bmatrix} 1 & 0 \\ 0 & z^{-m}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ T_1(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & T_2(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ T_3(z) & 1 \end{bmatrix} \begin{bmatrix} X_e(z) \\ X_o(z) \end{bmatrix} = \begin{bmatrix} X_{1,L}(z) \\ X_{1,H}(z) \end{bmatrix}}$$

where $T_1(z) \cong L_1(z)$, $T_2(z) \cong L_2(z)$, $T_3(z) \cong L_3(z)$

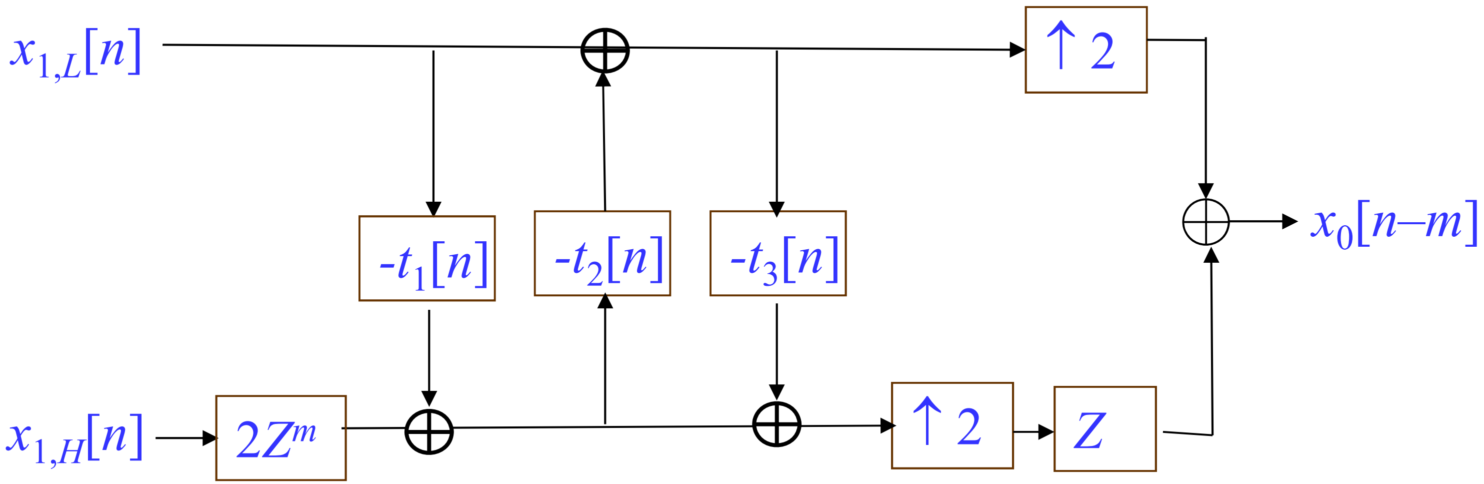
Lifting Scheme



The Z transforms of $t_1[n]$, $t_2[n]$, and $t_3[n]$ are $T_1(z)$, $T_2(z)$, and $T_3(z)$, respectively.

Lifting Scheme

synthesis part



If one perform quantization for $t_1[n]$, $t_2[n]$, and $t_3[n]$, then the discrete wavelet transform is still reversible.

$$\begin{bmatrix} 1 & 0 \\ L_1(z) & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -L_1(z) & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ Q\left(L_1(z)\right) & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -Q\left(L_1(z)\right) & 1 \end{bmatrix}$$

W. Sweldens, "The lifting scheme: a construction of second generation wavelets," *Applied Comput. Harmon. Anal.*, vol. 3, no. 2, pp. 186-200, 1996.

I. Daubechies and W. Sweldens, "Factoring wavelet transforms into lifting steps," *J. Fourier Anal. Applicat.*, vol. 4, pp. 246-269. 1998.

附錄十四 誤差計算的標準

若原來的信號是 $x[m, n]$，要計算 $y[m, n]$ 和 $x[m, n]$ 之間的誤差，有下列幾種常見的標準

(1) maximal error

$$Max\left(\left|y[m,n]-x[m,n]\right|\right)$$

(2) square error

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|y[m,n]-x[m,n]\right|^2$$

(3) error norm (i.e., Euclidean distance)

$$\sqrt{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|y[m,n]-x[m,n]\right|^2}$$

(4) mean square error (MSE)，信號處理和影像處理常用

$$\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|y[m,n]-x[m,n]\right|^2$$

(5) root mean square error (RMSE)

$$\sqrt{\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|y[m,n]-x[m,n]\right|^2}$$

(6) normalized mean square error (NMSE)

$$\frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|y[m,n]-x[m,n]\right|^2}{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|x[m,n]\right|^2}$$

(7) normalized root mean square error (NRMSE)，
信號處理和影像處理常用

$$\sqrt{\frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|y[m,n]-x[m,n]\right|^2}{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|x[m,n]\right|^2}}$$

(8) signal to noise ratio (SNR)，信號處理常用

$$10\log_{10}\left(\frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|x[m,n]\right|^2}{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|y[m,n]-x[m,n]\right|^2}\right)$$

(9) peak signal to noise ratio (PSNR)，影像處理常用

$$10\log_{10}\left(\frac{X_{Max}^2}{\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|y[m,n]-x[m,n]\right|^2}\right)$$

X_{Max}: the maximal possible value of $x[m, n]$

In image processing, $X_{Max} = 255$

for color image: $$10\log_{10}\left(\frac{X_{Max}^2}{\frac{1}{3MN}\sum_{R,G,B}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|y_{color}[m,n]-x_{color}[m,n]\right|^2}\right)$$

color = R, G, or B

(10) structural dissimilarity (DSSIM)

有鑑於 MSE 和 PSNR 無法完全反應人類視覺上所感受的誤差，在 2004 年被提出來的新的誤差測量方法

$$DSSIM(x,y) = 1 - SSIM(x,y)$$

$$SSIM(x,y) = \frac{\left(2\mu_x\mu_y + c_1 L\right)}{\left(\mu_x^2 + \mu_y^2 + c_1 L\right)} \frac{\left(2\sigma_{xy} + c_2 L\right)}{\left(\sigma_x^2 + \sigma_y^2 + c_2 L\right)}$$

μ_x, μ_y: means of x and y σ_x^2, σ_y^2: variances of x and y

$\sigma_x\sigma_y$: covariance of x and y c_1, c_2: adjustable constants

L: the maximal possible value of x − the minimal possible value of x

Z. Wang, A. C. Bovik, H. R. Sheikh, and E. P. Simoncelli, “Image quality assessment: From error visibility to structural similarity,” *IEEE Trans. Image Processing*, vol. 13, no. 4, pp. 600−612, Apr. 2004.