**Homework 3 (Due: Nov. 13th)**

(1) (a) In what condition the polynomial WDF cannot remove the cross term (write two conditions)? (b) In what condition Cohen's class distribution cannot remove the cross term (write two conditions)? (10 scores)

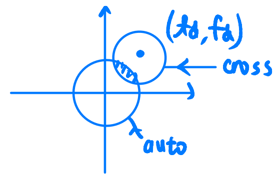
(a) Polynomial WDF $= W_x(t,f) = \int_{-\infty}^{\infty}\left[\prod_{l=1}^{q/2} x(t+d_l\tau)\, x^*(t+d_{-l}\tau)\right] e^{-j2\pi f\tau}\, d\tau$

① The order of phase is larger than $\frac{q}{2}+1$

② There exist more than one component (cross term between components can't be avoid)

(b) Cohen's class distribution $= C_x(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_x(\tau,\eta)\,\Phi(\tau,\eta)\, e^{j2\pi(\eta t - \tau f)}\, d\eta\, d\tau$

$, A_x(\tau,\eta) = \int_{-\infty}^{\infty} x(t+\frac{\tau}{2})\, x^*(t-\frac{\tau}{2})\, e^{-j2\pi t\eta}\, dt$




① Two signal components overlap / lie very close in time & freq. → cross-terms falls near the auto term

② The order of phase $\geq 4$ (instantaneous freq. order $\geq 3$)

(2) (a) Compared to the original STFT, what is the advantage of the S transform?
(b) Which of the following function is most suitable to be the window function of the S transform? Why?

(i) $w(t) = |f|^3 \exp\left[-\pi t^2 f^6\right]$ (ii) $w(t) = \left|(1+0.2\sqrt{|f|})\right| \exp\left[-\pi t^2 (1+0.2\sqrt{|f|})^2\right]$

(iii) $w(t) = |2+\cos(f)|$ (15 scores)

(a) S transform =

| freq. | window | time resolution | freq. resolution |
|---|---|---|---|
| High | small | High | Low |
| Low | large | Low | High |

→ High time resolution at high freq., High freq. resolution at low freq.

Application = vocal signal = Humans hear different pitch according to freq. ratio, not difference → freq. resolution has to be high at low freq. Also, time resolution is allowed to be lower at low freq.

(b) A suitable S transform $= w(t,f) = |f| e^{-\pi t^2 f^2}$ (a Gaussian form)

(i) yes, Let $f' = f^3$ → Gaussian ✓

(ii) yes, Let $f' = 1+0.2\sqrt{|f|}$ → Gaussian ✓

(iii) no, not even a Gaussian /#

Compare (i) & (ii) = at $f=0$, (i) has greatest freq. resolution ($\because$ inf window width)

However (ii) still has finite window width ($\because$ we added a constant) → freq. resolution worse than (i)

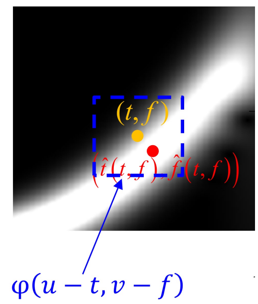
at low freq. Therefore, (i) is the most suitable. /#

(3) Why (a) the generalized spectrogram and (b) reassignment can make time-frequency distribution more concentrated? (10 scores)

(a). Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)G^*_{x,w_2}(t,f)$

Choose small $w_1$ → high time resolution
Chose large $w_2$ → high freq. resolution
} high resolution in both time / freq. domain.

(b). Reassignment method finds a "center of gravity" $(\hat{t}(t,f), \hat{f}(t,f))$ in a patch of




time-freq. plot with
$$\hat{t}(t,f) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\cdot\varphi(u-t,v-f)\cdot X(u,v)\,du\,dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u-t,v-f)\cdot X(u,v)\,du\,dv}$$
$$\hat{f}(t,f) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v\cdot\varphi(u-t,v-f)\cdot X(u,v)\,du\,dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u-t,v-f)\cdot X(u,v)\,du\,dv}$$

Then, we shift all the distribution toward $(\hat{t}(t,f), \hat{f}(t,f))$ with $\hat{X}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t_1,f_1)\,\delta(t-\hat{t}(t_1,f_1))\,\delta(f-\hat{f}(t_1,f_1))\,dt_1 df$

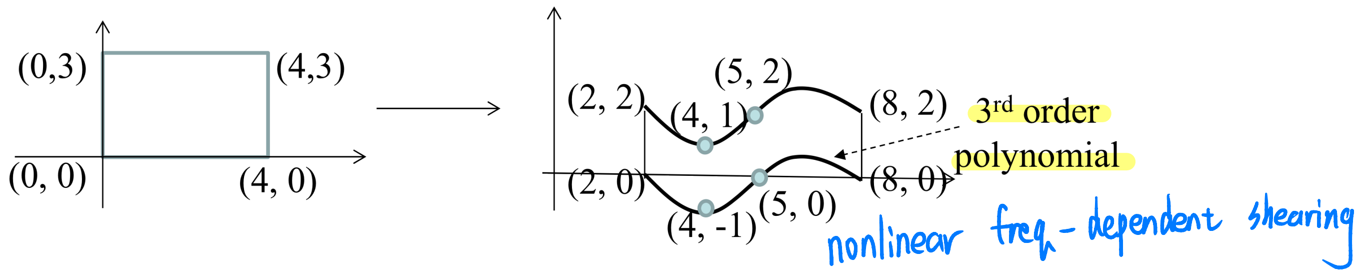
⇒ The distribution is more concentrated. #

(4) Give an example of the physical system that the input and the output have the relation of (a) chirp multiplication and (b) chirp convolution. (10 scores)

(a). EM Wave propagating through spherical lens ⇒ LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix}$

(b) EM Wave through air : Fresnel transform ⇒ LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$ #

(5) Suppose that the WDF of a signal is as the left figure. How do we change its WDF into the right figure? (10 scores)




① Dilation, freq. interval $[0,3] \to [0,2]$ $W(t,f) \to W(\frac{2}{3}t, \frac{3}{2}f)$
time interval $[0,4] \to [0,6]$

② Shift, time interval $[0,6] \to [2,8]$ $W(\frac{2}{3}t, \frac{3}{2}f) \to W(\frac{2}{3}(t-2), \frac{3}{2}f)$

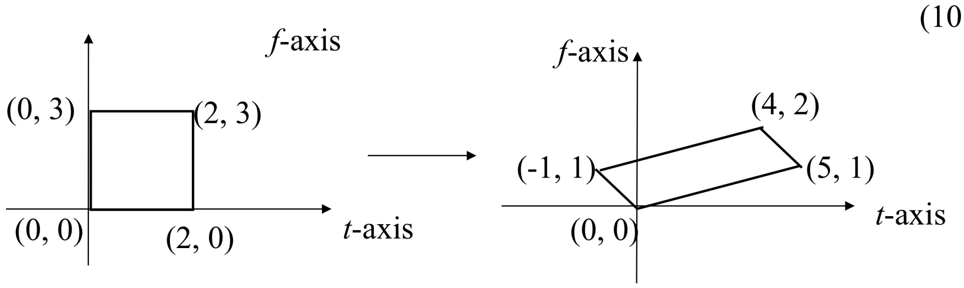
③ Shearin : $f \to f - k(t)$, with $\begin{cases} k(4) = -1 \\ k(2) = k(8) = k(5) = 0 \end{cases}$ ⇒ $k(t) = a(t-8)(t-5)(t-2)$

$a \times -4 \times -1 \times 2 = -1 \to a = -\frac{1}{8}$ ⇒ $k(t) = -\frac{1}{8}(t-8)(t-5)(t-2)$

∴ $W(\frac{2}{3}(t-2), \frac{3}{2}f) \to W(\frac{2}{3}(t-2), \frac{3}{2}(f-k(t)))$, $k(t) = -\frac{1}{8}(t-8)(t-5)(t-2)$ #

(6) Suppose that the time-frequency distribution of $x(t)$ is as the left figure. How do we change the time-frequency distribution into the right figure using the LCT?

(10 scores)



$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3b \\ 3d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow b = -\frac{1}{3},\ d = \frac{1}{3}$ $\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5/2 & -1/3 \\ 1/2 & 1/3 \end{bmatrix}$ #

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a \\ 2c \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow a = \frac{5}{2},\ c = \frac{1}{2}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2a+3b \\ 2c+3d \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ ✓, $ad - bc = \frac{5}{6} + \frac{1}{6} = 1$ ✓

(7) Write a Matlab or Python program for the scaled Gabor transform (unbalanced form).

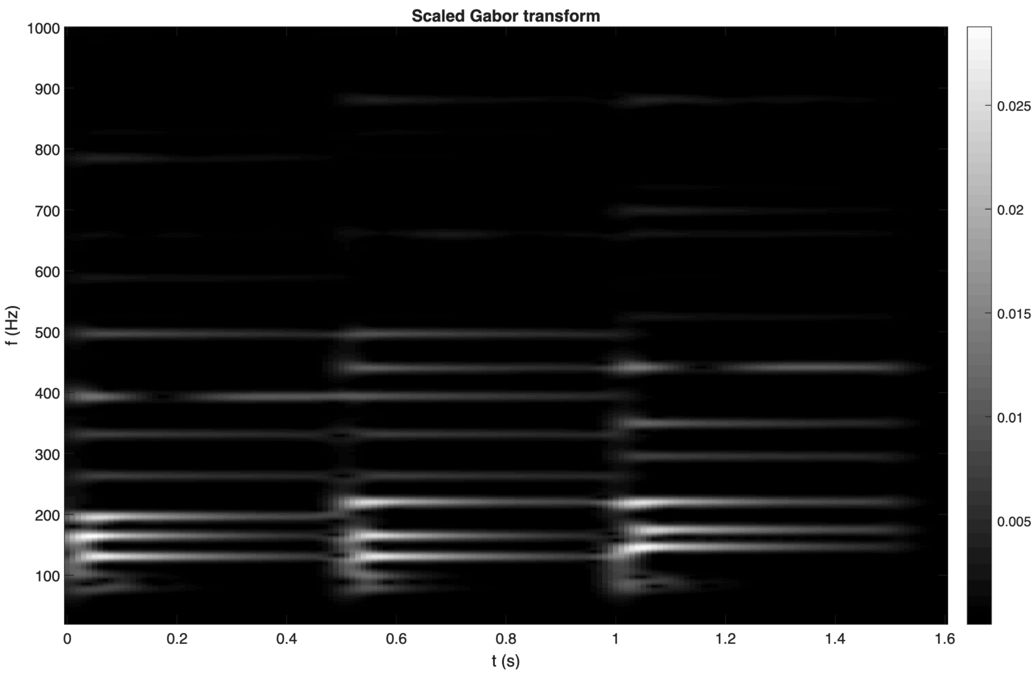
y = Gabor(x, tau, t, f, sgm) (35 scores)

*x*: input, *tau*: samples on *t*-axis for the input, *t*: samples on *t*-axis for the output
*f*: samples on *f*-axis, *sgm*: scaling parameter, *y*: output

(i) The code should be handed out by NTU Cool, (ii) Choose an input x (Use *.wav), plot the output y, (iii) Use tic and toc to show the running time, (iv) The running time for the following example should be within 1.5 seconds.

```
[a1, fs] = audioread('Chord.wav');
x=a1(:,1).‘;   % only extract the first channel
tau = (?  Please think how to determine tau);
dt = 0.01;          df= 1;      sgm= 200;
t= 0:dt:max(tau);     f= 20:df:1000;
tic
y= Gabor (x, tau, t, f, sgm);
toc
```

Choose $\Delta\tau = 1/fs = 1/44100$, with Running time: 0.0807 s (Running Chord.wav)




Extra (尾數 4) Week 7 20251016_1 14:10

Cohen class distribution have high resolution and can avoid cross-term, but it's computation time is N times (for N-sample signal) of Wigner distribution without any simplification.

($\approx O(N^3)$ vs $O(N^2)$, or $O(N^2\lg N)$ vs $O(N\lg N)$ if FFT is used) #