Homework 2 (Due: 16th Oct.)

(1) Which of the following functions satisfy the lower bound of the uncertainty principle? Why? (a) $\exp(-t^2-2t-3)$; (b) $3\exp(-t^2-j2t)$; (c) $t\exp(-\pi t^2)$; (d) $\exp(-\pi t^2) * \exp(-2\pi t^2)$ where * means convolution? (15 scores)

The function that satisfy the lower bound is scaling, time/freq. shift of Gaussian ($e^{-\pi t^2}$ or $e^{-t^2/2}$)

(a). $\exp(-(t+1)^2-2)$ = scaling & time shift → satisfy ✓

(b). $3\exp(-j2t)\exp(-t^2)$ = scaling & freq. shift → satisfy ✓

(c). $t\exp(-\pi t^2)$ = not a pure Gaussian → not satisfy ✗

(d). $FT\{\exp(\pi t^2) * \exp(-2\pi t^2)\} = FT\{\exp(\pi t^2)\} \times FT\{\exp(-2\pi t^2)\}$ = Guassian × Gaussian = Gaussian $\xrightarrow{IFT}$ Gaussian
→ satisfy ✓

(2) For which of the following functions the WDFs may have the cross term problem? Why? (a) $\exp(j\pi t^2)$; (b) $\sin(-\pi t^2)$; (c) $\exp(j\pi t^4)$; (d) $\exp(-\pi t^4)$; (e) A typical speech signal. (15 scores)

(a) single complex term with order = 2 → no cross term

(b) $\sin(-\pi t^2) = -\sin(\pi t^2) = \frac{j}{2}(e^{j\pi t^2} - e^{-j\pi t^2})$, which is sum of 2 chirp → cross term

(c) single term, but order = 4 → cross term

(d) it is a single & real component → no cross term

(e) it is real & multi-component → cross term

(3) Compare the 4 methods to implement the STFT in terms of (a) complexity and (b) constraints. (c) Which methods can also be used for implementing the WDF? (15 scores)

	(a) Complexity	(b) Constraits
Direct Implementation	$O(TFQ)$	$\Delta t < \frac{1}{2(\Omega x + \Omega w)}$
FFT-based method	$O(TN\log_2 N)$	$\Delta t < \frac{1}{2(\Omega x + \Omega w)}$, $\Delta t \Delta f = \frac{1}{N}$, $N \geq 2Q+1$
FFT-based method with recursive formula	$O(FT)$	$\Delta t < \frac{1}{2(\Omega x + \Omega w)}$, $\Delta t \Delta f = \frac{1}{N}$, $N \geq 2Q+1$, rec function only
Chirp z-transform method	$O(TN\log_2 N)$	$\Delta t < \frac{1}{2(\Omega x + \Omega w)}$

(c). Direct Implementation, FFT-based method, Chirp z-transform method / #
(Note that the length of signal has to be finite.)

(4) Calculate the Wigner distribution function (WDF) of $\exp(-\sigma\pi t^2)$.
Hint: Using the fact that the FT of $\exp(-\pi t^2)$ is $\exp(-\pi f^2)$. (10 scores)

$$W(t,f)=\int_{-\infty}^{\infty}\exp\left(-\sigma\pi\left(t+\frac{\tau}{2}\right)^2\right)\exp\left(-\sigma\pi\left(t-\frac{\tau}{2}\right)^2\right)\exp(-j2\pi f\tau)\,d\tau$$

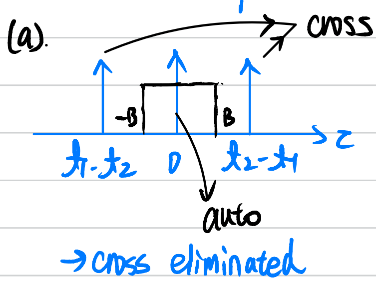
$$=\int_{-\infty}^{\infty}\exp\left(-\sigma\pi t^2-\sigma\pi t\tau-\frac{\sigma\pi\tau^2}{4}\right)\exp\left(-\sigma\pi t^2+\sigma\pi t\tau-\frac{\sigma\pi\tau^2}{4}\right)\exp(-j2\pi f\tau)\,d\tau$$

$$=\int_{-\infty}^{\infty}\exp\left(-\pi\sigma\left(2t^2+\frac{\tau^2}{2}\right)\right)\exp(-j2\pi f\tau)\,d\tau=e^{-2\pi\sigma t^2}\int_{-\infty}^{\infty}\exp\left(-\pi\sigma\frac{\tau^2}{2}\right)\exp(-j2\pi f\tau)\,d\tau$$

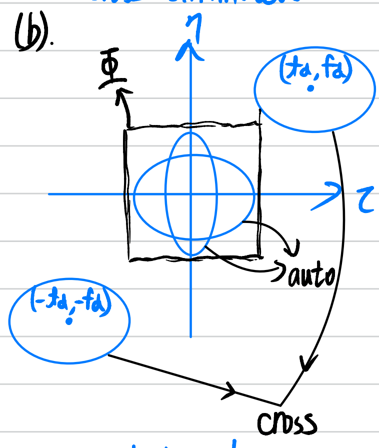
$$=e^{-2\pi\sigma t^2}\sqrt{\frac{2}{\sigma}}\,e^{-\frac{2}{\sigma}\pi f^2}=\sqrt{\frac{2}{\sigma}}\exp\left(-2\pi\sigma\left(t^2+\left(\frac{f}{\sigma}\right)^2\right)\right)$$ #

(5) Why (a) windowed Wigner distribution function and (b) Cohen's class distribution can avoid the cross term problem in some cases? (10 scores)

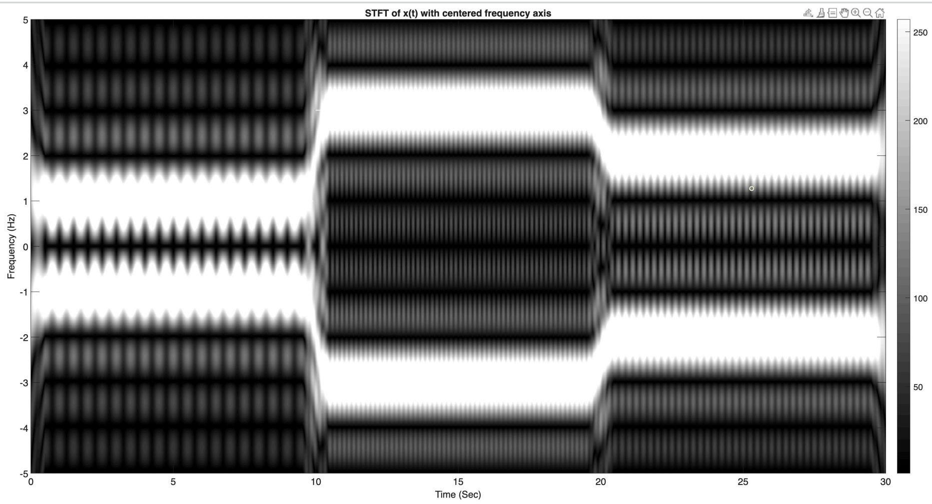
(a). Consider the example $x(t)=\delta(t-t_1)+\delta(t-t_2)$, we found out that the cross term appears at $t=\frac{t_1+t_2}{2}$ $(\tau=\pm(t_2-t_1))$, and the auto term appears at $t=t_1$ or $t=t_2$ $(\tau=0)$. So if we apply a lowpass window $w(\tau)=0$ for $|\tau|>B$, where $B<t_2-t_1$, we can avoid cross term problem.



(b). Consider the example $x(t)=\exp(-\alpha_1\pi(t-t_1)^2+j2\pi f_1 t)+\exp(K_2\pi(t-t_2)^2+j2\pi f_2 t)$, the auto term is independent of t_1, f_1, t_2, f_2 and is a Gaussiaun with max at $(\tau,\eta)=(0,0)$. The cross term, however, is a Gaussiaun with max centered at $(\tau,\eta)=\pm(t_d,f_d)$, where $t_d=t_2-t_1$, $f_d=f_2-f_1$. Applying a mask with $\Phi(\tau,\eta)=0$ for small $|\tau|,|\eta|$ and $\Phi(\tau,\eta)=1$ for large $|\tau|,|\eta|$, the cross term problem can be avoided.



(6) Write a code for the rectangular STFT. (35 scores)
(the window is $w(t)=1$ if $|t|<B$, $w(t)=0$ otherwise).
y = recSTFT(x, t, f, B)
x: input, t: samples on t-axis, f: samples on f-axis, y: output
(i) 要交本題的程式碼 (*.m 檔或 *.py檔，可用 Matlab 或 Python寫)，
(ii) 自己選一個 input x, 用你們的程式將 output y 算出來並畫出來
(iiii) 計算程式的 computation time
(iv) 不可以用 direct implementation 的方法
(v) The code should be handed out by NTUcool.



FFT-based method
Computation time ≒ 0.046238 sec #

Bonus (尾投4). : $\phi(t)=\frac{t^2}{10}-3t \Rightarrow$ instanteous frequency $=\frac{\phi'(t)}{2\pi}=\frac{\frac{1}{5}t-3}{2\pi}=\frac{t-15}{10\pi}$ Hz for $-9\le t\le 1$ #