IV. Implementation

IV-A Method 1: Direct Implementation

以 STFT 為例

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

Converting into the Discrete Form

$$t=n\Delta_t,\quad f=m\Delta_f,\quad \tau=p\Delta_t$$

$$X(n\Delta_t,m\Delta_f)=\sum_{p=-\infty}^{\infty} w((n-p)\Delta_t)x(p\Delta_t)e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t$$

Suppose that $w(t)\cong 0$ for $|t|>B$, $B/\Delta_t=Q$

$$\boxed{X(n\Delta_t,m\Delta_f)=\sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)x(p\Delta_t)e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t}$$

Problem：對 scaled Gabor transform 而言，$Q=?$

• **Constraint for Δ_t** (The only constraint for the direct implementation method)

To avoid the aliasing effect,

$\Delta_t < 1/2\Omega$, Ω is the bandwidth of ?

There is no constraint for Δ_f when using the direct implementation method.

Four Implementation Methods

(1) Direct implementation

Complexity:

假設 t-axis 有 T 個 sampling points, f-axis 有 F 個 sampling points

(2) FFT-based method

Complexity:

(3) FFT-based method with recursive formula

Complexity:

(4) Chirp-Z transform method

Complexity:

(A) Direct Implementation

Advantage： simple, flexible

Disadvantage ： higher complexity

(B) DFT-Based Method

Advantage ： lower complexity

Disadvantage ： with some constraints

(C) Recursive Method

Advantage ：

Disadvantage ：

(D) Chirp Z Transform

Advantage ：

Disadvantage ：

IV-B Method 2: FFT-Based Method

Constraints：(ii) $\Delta_t\Delta_f = 1/N$,

(iii) $N = 1/(\Delta_t\Delta_f) \geqq 2Q+1$: ($\Delta_t\Delta_f$是整數的倒數)

Standard form of the DFT $\quad Y[m] = \sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi mn}{N}}$

$$\boxed{X(n\Delta_t, m\Delta_f) = \sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)\, x(p\Delta_t)\, e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t}$$

$\downarrow \; \Delta_t\Delta_f = 1/N$

$$\boxed{X(n\Delta_t, m\Delta_f) = \sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)\, x(p\Delta_t)\, e^{-j\frac{2\pi pm}{N}}\Delta_t}$$

$\downarrow \; q = p-(n-Q) \rightarrow p = (n-Q)+q$

$$\boxed{X(n\Delta_t, m\Delta_f) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{2Q} w((Q-q)\Delta_t)\, x((q+n-Q)\Delta_t)\, e^{-j\frac{2\pi qm}{N}}}$$

Note that the input of the N-point FFT should have N points (others are set to zero).

$$\boxed{X(n\Delta_t, m\Delta_f) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}}} \quad , \quad q = p-(n-Q) \rightarrow p = (n-Q)+q$$

where $x_1(q) = w((Q-q)\Delta_t)\, x((n-Q+q)\Delta_t)$ for $0 \le q \le 2Q,$

$x_1(q) = 0$ for $2Q < q < N.$

$n\text{-}Q \le n\text{-}Q+q \le n+Q$

$Q \ge Q - q \ge -Q$

$$\big\downarrow\; k = q\text{-}Q$$

$$\boxed{X(n\Delta_t, m\Delta_f) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}}\, DFT(x_1(q))}$$

where $x_1(q) = w(k\Delta_t)\, x((n+k)\Delta_t)$ for $0 \le q \le 2Q,$ $\quad -Q \le k \le Q \quad (k = q\text{-}Q)$

$x_1(q) = 0$ for $2Q < q < N.$

(Suppose that $w(t) = w(-t)$)

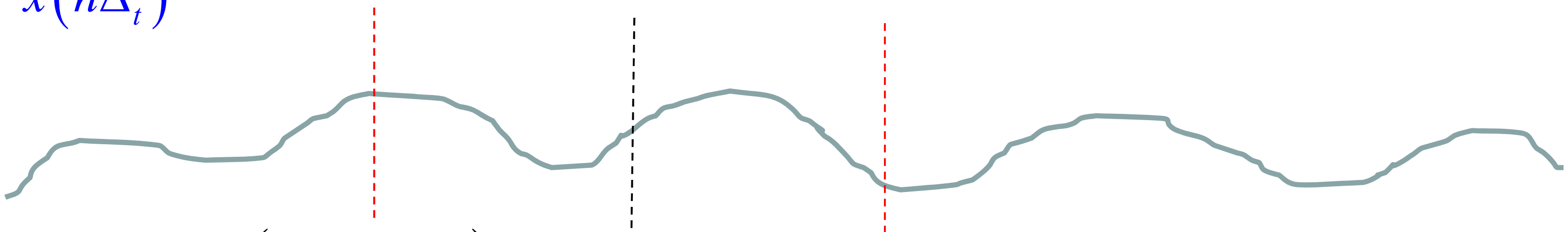
$$\boxed{X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} DFT\left(x_1\left(q\right)\right)}$$ for each n

where $x_1\left(q\right) = w\left(k\Delta_t\right)x\left((n+k)\Delta_t\right)$ for $-Q \le k \le Q$ $(k = q-Q)$

$x_1\left(q\right) = 0$ for $2Q < q < N$. (Suppose that $w(t) = w(-t)$)

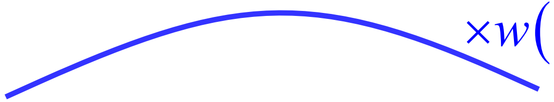
$x\left(n\Delta_t\right)$

center



$x\left((n-Q)\Delta_t\right)$ $x\left(n\Delta_t\right)$ $x\left((n+Q)\Delta_t\right)$

take a segment of the signal



$\times w\left(k\Delta_t\right)$ $-Q \le k \le Q$

multiplied by the window

$x_1\left(q\right)$: $w\left(k\Delta_t\right)x\left((n+k)\Delta_t\right)$ padding N-2Q-1 zeros

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} DFT\left(x_1\left(q\right)\right)$$

$$\boxed{X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} DFT\left(x_1\left(q\right)\right)} \qquad \text{for each } n$$

注意：

(1) 可以用 Matlab 或 Python 的 fft 指令來計算 $\sum_{q=0}^{N-1} x_1\left(q\right) e^{-j\frac{2\pi qm}{N}} = X_1\left(m\right)$

(2) 對每一個固定的 n，都要計算一次下方的式子

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{N-1} x_1\left(q\right) e^{-j\frac{2\pi qm}{N}} = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} X_1\left(m\right)$$

(fixed n)

(3) Complexity = ?

假設 $t = n_0\Delta_t,\ (n_0+1)\,\Delta_t,\ (n_0+2)\,\Delta_t,\ \cdots\cdots,\ (n_0+T\text{-}1)\Delta_t$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ (m_0+F\text{-}1)\Delta_f$

Step 1: Calculate n_0, m_0, T, F, N, Q

Step 2: $n = n_0$

Step 3: Determine $x_1(q)$

Step 4: $X_1(m) = \text{FFT}[x_1(q)]$

Step 5: Convert $X_1(m)$ into $X(n\Delta_t, m\Delta_f)$

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$m = f/\,\Delta_f$

$m_1 = \text{mod}(m, N)$

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} X_1\left(m_1\right)$$

$$X_1[m] = \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}} \qquad X_1[m] = X_1[m+N]$$

Step 6: Set $n = n+1$ and return to Step 3 until $n = n_0+T$-1.

IV-C Method 3: Recursive Method

- **A very fast way for implementing the rec-STFT**

(n 和 $n-1$ 有recursive的關係)

$$\boxed{X\left(n\Delta_t, m\Delta_f\right) = \sum_{p=n-Q}^{n+Q} x\left(p\Delta_t\right) e^{-j\frac{2\pi pm}{N}} \Delta_t}$$

$$X\left((n-1)\Delta_t, m\Delta_f\right) =$$

(1) Calculate $X(\min(n)\Delta_t, m\Delta_f)$ by the N-point FFT

$$\boxed{X\left(n_0\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n_0)m}{N}} \sum_{q=0}^{N-1} x_1\left(q\right) e^{-j\frac{2\pi qm}{N}}}, \quad n_0 = \min(n),$$

$$x_1\left(q\right) = x\left((n-Q+q)\Delta_t\right) \quad \text{for } q \le 2Q, \qquad x_1(q) = 0 \quad \text{for } q > 2Q$$

(2) Applying the recursive formula to calculate $X(n\Delta_t, m\Delta_f)$,

$$n = n_0 + 1 \sim \max(n)$$

$$\boxed{\begin{aligned} X\big(\underbrace{n\Delta_t}_{T\text{點}}, \underbrace{m\Delta_f}_{F\text{點}}\big) = {} & X\left((n-1)\Delta_t, m\Delta_f\right) - x\left((n-Q-1)\Delta_t\right) e^{-j2\pi(n-Q-1)m/N} \Delta_t \\ & + x\left((n+Q)\Delta_t\right) e^{-j2\pi(n+Q)m/N} \Delta_t \end{aligned}}$$

IV-D Method 4: Chirp Z Transform

$$\exp\left(-j2\pi\, pm\Delta_t\Delta_f\right) = \exp\left(-j\pi\, p^2\Delta_t\Delta_f\right)\exp\left(j\pi\,(p-m)^2\,\Delta_t\Delta_f\right)\exp\left(-j\pi m^2\Delta_t\Delta_f\right)$$

For the STFT

$$X\left(n\Delta_t, m\Delta_f\right) = \sum_{p=n-Q}^{n+Q} w\left((n-p)\Delta_t\right)x\left(p\Delta_t\right)e^{-j2\pi\, pm\Delta_t\Delta_f}\,\Delta_t$$

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{-j\pi\, m^2\Delta_t\Delta_f} \sum_{p=n-Q}^{n+Q} w\left((n-p)\Delta_t\right)x\left(p\Delta_t\right)e^{-j\pi\, p^2\Delta_t\Delta_f}\, e^{j\pi\,(p-m)^2\,\Delta_t\Delta_f}$$

Step 1 multiplication

Step 2 convolution

Step 3 multiplication

For every fixed n

Step 1 $x_1[p] = w((n-p)\Delta_t)\, x(p\Delta_t)\, e^{-j\pi p^2 \Delta_t \Delta_f}$ $\qquad n\text{-}Q \le p \le n+Q$

Step 2 $X_2[n,m] = \sum_{p=n-Q}^{n+Q} x_1[p]\, c[m-p]$ $\qquad c[m] = e^{j\pi m^2 \Delta_t \Delta_f}$

Step 3 $X(n\Delta_t, m\Delta_f) = \Delta_t\, e^{-j\pi m^2 \Delta_t \Delta_f} X_2[n,m]$

Step 2 在計算上，需要用到 linear convolution 的技巧

Question: <u>Step 2 要用多少點的 DFT</u>?

• Illustration for the Question on Page 125

$$y[n] = \sum_k x[n-k]h[k]$$

- Case 1

When length($x[n]$) = N, length($h[n]$) = K, N and K are finite,

$\longrightarrow$ length($y[n]$) = $N+K-1$,

Using the ($N+K-1$)-point DFTs (學信號處理的人一定要知道的常識)

- Case 2

$x[n]$ has finite length but $h[n]$ has infinite length ????

$$y[n]=\sum_k x[n-k]h[k]$$

- Case 2

$x[n]$ has finite length but $h[n]$ has infinite length

$x[n]$ 的範圍為 $n \in [n_1, n_2]$，範圍大小為 $N = n_2 - n_1 + 1$

$h[n]$ 無限長

$y[n]=\sum_k x[n-k]h[k]$ 　　　$y[n]$ 每一點都有值 (範圍無限大)

但我們只想求出 $y[n]$ 的其中一段

希望算出的 $y[n]$ 的範圍為 $n \in [m_1, m_2]$，範圍大小為 $M = m_2 - m_1 + 1$

$h[n]$ 的範圍？

要用多少點的 FFT？

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[n-k]h[k]$$

改寫成 $$y[n]=x[n]*h[n]=\sum_{s=n_1}^{n_2}x[s]h[n-s]$$

$$y[n]=x[n_1]h[n-n_1]+x[n_1+1]h[n-n_1-1]+x[n_1+2]h[n-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[n-n_2]$$

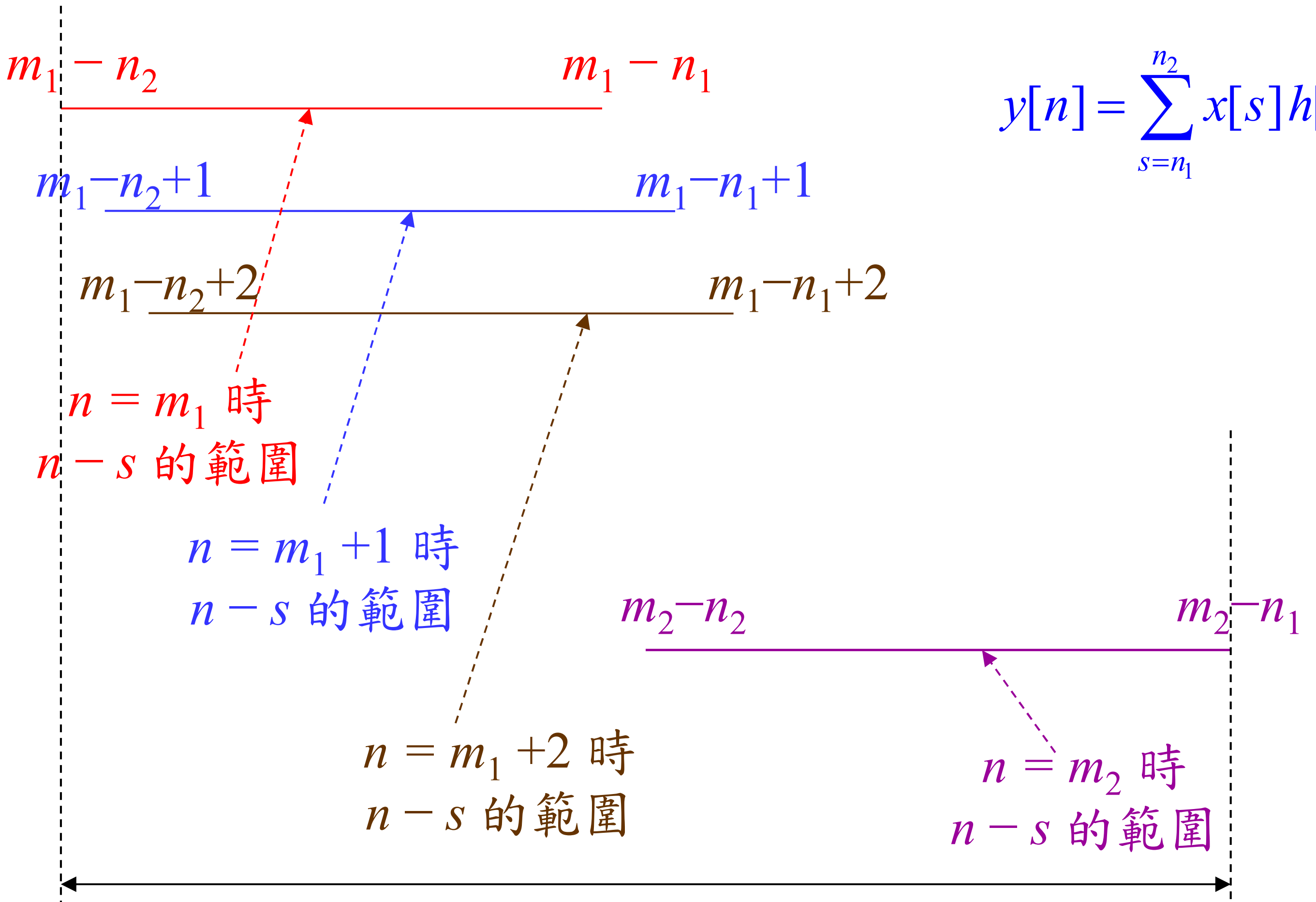
當 $n=m_1$

$$y[m_1]=x[n_1]h[m_1-n_1]+x[n_1+1]h[m_1-n_1-1]+x[n_1+2]h[m_1-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[m_1-n_2]$$

當 $n=m_2$

$$y[m_2]=x[n_1]h[m_2-n_1]+x[n_1+1]h[m_2-n_1-1]+x[n_1+2]h[m_2-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[m_2-n_2]$$

$$y[n]=\sum_{s=n_1}^{n_2} x[s]h[n-s]$$



有用到的 $h[k]$ 的範圍：$k\in[m_1-n_2,\, m_2-n_1]$

所以，有用到的 $h[k]$ 的範圍是 $k \in [m_1 - n_2, m_2 - n_1]$

範圍大小為 $m_2 - n_1 - m_1 + n_2 + 1 = N + M - 1$

FFT implementation for Case 2

$x_1[n] = x[n + n_1]$ for $n = 0, 1, 2, \ldots, N-1$

$x_1[n] = 0$ for $n = N, N+1, N+2, \ldots\ldots, L-1$ $L = N + M - 1$

$h_1[n] = h[n + m_1 - n_2]$ for $n = 0, 1, 2, \ldots, L-1$

$$y_1[n] = IFFT_L\left(FFT_L\{x_1[n]\}\, FFT_L\{h_1[n]\}\right)$$

$y[n] = y_1[n - m_1 + N - 1]$ for $n = m_1, m_1+1, m_1+2, \ldots, m_2$

IV-E Unbalanced Sampling for STFT and WDF

將 pages 115 and 119 的方法作修正

$$\boxed{X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau}$$

$$\boxed{X(n\Delta_t, m\Delta_f)=\sum_{p=nS-Q}^{nS+Q} w((nS-p)\Delta_\tau)x(p\Delta_\tau)e^{-j2\pi pm\Delta_\tau\Delta_f}\Delta_\tau}$$

where $t = n\Delta_t$, $f = m\Delta_f$, $\tau = p\Delta_\tau$, $B = Q\Delta_\tau$ (假設 $w(t) \cong 0$ for $|t| > B$),

$S = \Delta_t / \Delta_\tau$ $\Delta_t \neq \Delta_\tau$

註：Δ_τ (sampling interval for the input signal)

Δ_t (sampling interval for the output t-axis) can be different.

However, it is better that $S = \Delta_t / \Delta_\tau$ is an integer.

When (1) $\Delta_\tau\Delta_f = 1/N$, (2) $N = 1/(\Delta_\tau\Delta_f) > 2Q + 1$: ($\Delta_\tau\Delta_f$只要是整數的倒數即可)

(3) $\Delta_\tau < 1/2\Omega$, Ω is the bandwidth of $w(\tau - t)x(\tau)$

i.e., $|FT\{w(\tau - t)x(\tau)\}| = |X(t, f)| \approx 0$ when $|f| > \Omega$

$$\boxed{X(n\Delta_t, m\Delta_f) = \sum_{p=nS-Q}^{nS+Q} w((nS-p)\Delta_\tau)x(p\Delta_\tau)e^{-j\frac{2\pi pm}{N}}\Delta_\tau}$$

令 $q = p - (nS-Q) \rightarrow p = (nS-Q) + q$

$$\boxed{X(n\Delta_t, m\Delta_f) = \Delta_\tau e^{j\frac{2\pi(Q-nS)m}{N}} \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}}}$$

$x_1(q) = w((Q-q)\Delta_\tau)x((nS-Q+q)\Delta_\tau)$ for $0 \le q \le 2Q$,

$x_1(q) = 0$ for $2Q < q < N$.

If $w(t) = w(-t)$)

$x_1(q) = w(k\Delta_\tau)x((nS+k)\Delta_\tau)$ for $0 \le q \le 2Q$, $k = q-Q$, $-Q \le k \le Q$

$x_1(q) = 0$ for $2Q < q < N$.

假設 $t = c_0\Delta_t,\ (c_0+1)\,\Delta_t,\ (c_0+2)\,\Delta_t,\ \cdots\cdots,\ (c_0+C-1)\,\Delta_t$

$= c_0S\Delta_\tau,\ (c_0S+S)\,\Delta_\tau,\ (c_0S+2S)\,\Delta_\tau,\ \cdots\cdots,\ [c_0S+(C-1)S]\,\Delta_\tau$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ (m_0+F-1)\,\Delta_f$

$\tau = n_0\,\Delta_\tau,\ (n_0+1)\,\Delta_\tau,\ (n_0+2)\,\Delta_\tau,\ \cdots\cdots,\ (n_0+T-1)\,\Delta_\tau,$ $S = \Delta_t/\,\Delta_\tau$

Step 1: Calculate c_0, m_0, n_0, C, F, T, N, Q

Step 2: $n = c_0$

Step 3: Determine $x_1(q)$

Step 4: $X_1(m) = \text{FFT}[x_1(q)]$

Step 5: Convert $X_1(m)$ into $X(n\Delta_t, m\Delta_f)$

Step 6: Set $n = n+1$ and return to Step 3 until $n = c_0 + C - 1$.

Complexity = ?

IV-F Non-Uniform Δ_t

(A) 先用較大的 Δ_t

(B) 如果發現 $\left|X\left(n\Delta_t, m\Delta_f\right)\right|$ 和 $\left|X\left((n+1)\Delta_t, m\Delta_f\right)\right|$ 之間有很大的差異

則在 $n\Delta_t$, $(n+1)\,\Delta_t$ 之間選用較小的 sampling interval Δ_{t1}

($\Delta_\tau < \Delta_{t1} < \Delta_t$, Δ_t / Δ_{t1} 和 $\Delta_{t1} / \Delta_\tau$ 皆為整數)

再用 page 132 的方法算出

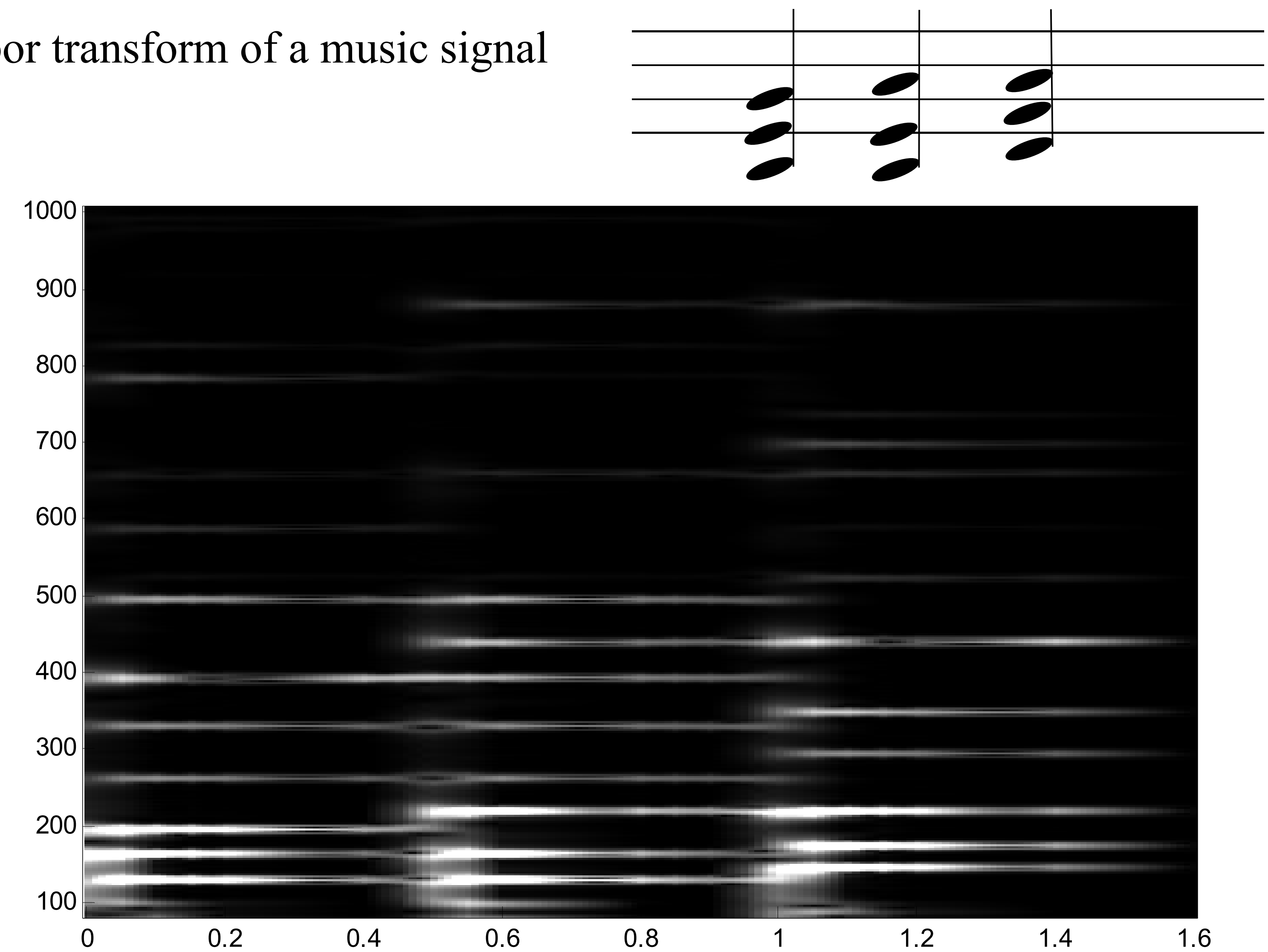
$$X\left(n\Delta_t + \Delta_{t1}, m\Delta_f\right), \quad X\left(n\Delta_t + 2\Delta_{t1}, m\Delta_f\right), \quad \cdots\cdots, \quad X\left((n+1)\Delta_t - \Delta_{t1}, m\Delta_f\right)$$

(C) 以此類推，如果 $\left|X\left(n\Delta_t + k\Delta_{t1}, m\Delta_f\right)\right|$, $\left|X\left(n\Delta_t + (k+1)\Delta_{t1}, m\Delta_f\right)\right|$

的差距還是太大，則再選用更小的 sampling interval Δ_{t2}

($\Delta\tau < \Delta t_2 < \Delta t_1$, $\Delta t_1 / \Delta t_2$ 和 $\Delta t_2 / \Delta\tau$ 皆為整數)

Gabor transform of a music signal



$\Delta_\tau = 1/44100$ (總共有 44100×1.6077 sec $+ 1 = 70902$ 點)

(A) Choose $\Delta_t = \Delta_\tau$

running time = out of memory (2008年) 15.262140 sec (2022年)

(B) Choose $\Delta_t = 0.01 = 441\Delta_\tau$ (1.6/0.01 + 1 = 161 points)

running time = 1.0940 sec (2008年) 0.041053 sec (2022年)

(C) Choose the non-uniform sampling points on the t-axis as

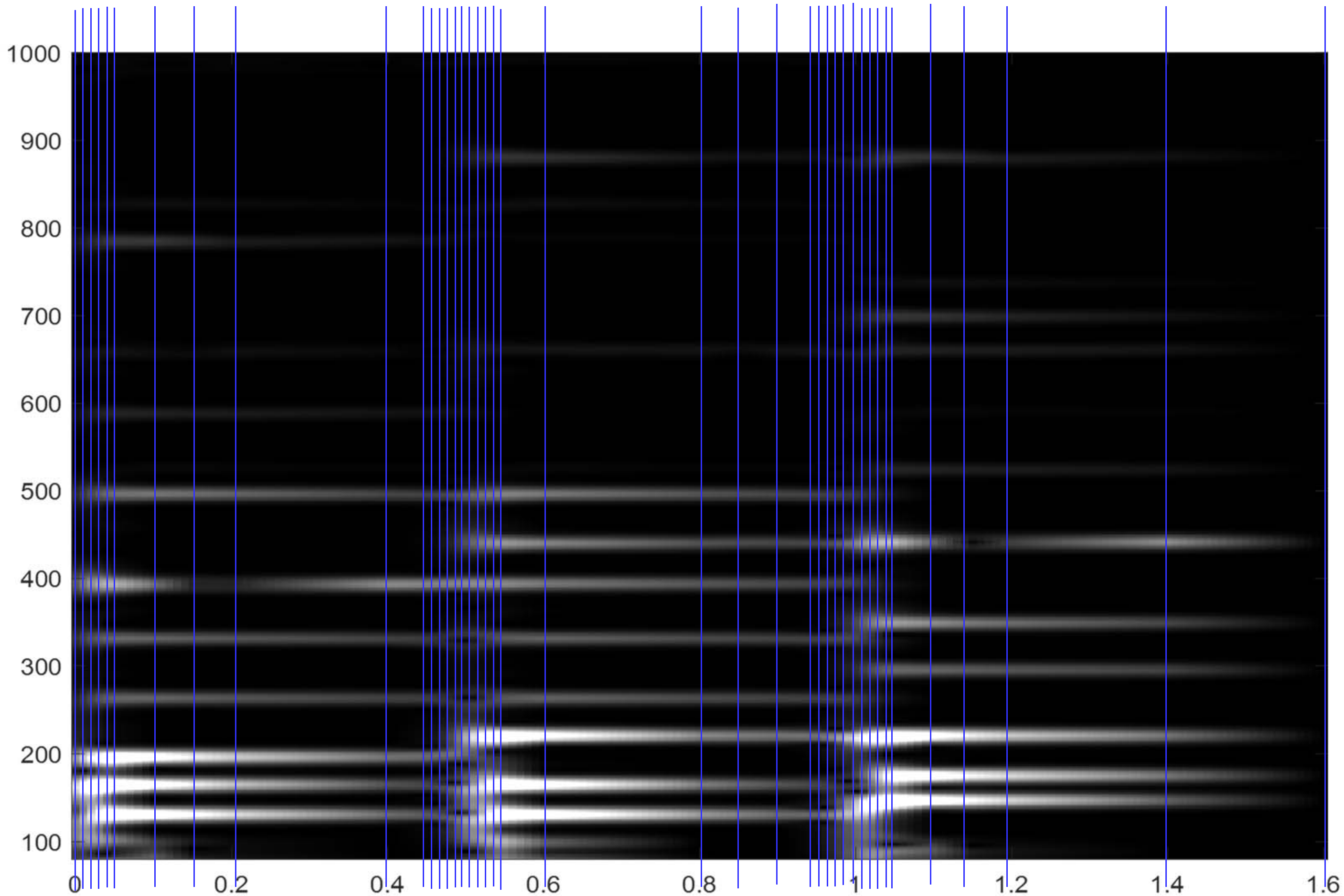
t = 0, 0.01, 0.02, 0.03, 0.04, <u>0.05</u>, <u>0.1</u>, <u>0.15</u>, 0.2, 0.4, <u>0.45</u>, 046, 0.47, 0.48, 0.49, <u>0.5</u>, 0.51, 0.52, 0.53, 0.54, <u>0.55</u>, 0.6, 0.8, <u>0.85</u>, <u>0.9</u>, <u>0.95</u>, 0.96, 0.97, 0.98, 0.99, 1, 1.01, 1.02, 1.03, 1.04, <u>1.05</u>, <u>1.1</u>, <u>1.15</u>, 1.2, 1.4, 1.6

(41 points)

intervals: 0.2 → 0.05 → 0.01

running time = 0.2970 sec (2008年) 0.010594 sec (2022年)

with adaptive output sampling intervals



附錄七 和 Dirac Delta Function 相關的常用公式

(1) $\int_{-\infty}^{\infty} e^{-j2\pi t f} dt = \delta(f)$

(2) $\delta(t) = |a|\delta(at)$ (scaling property)

(3) $\int_{-\infty}^{\infty} e^{-j2\pi t g(f)} dt = \delta(g(f)) = \sum_{n} |g'(f_n)|^{-1} \delta(f - f_n)$

where f_n are the zeros of $g(f)$

(4) $\int_{-\infty}^{\infty} \delta(t-t_0) y(t,\cdots\cdots) dt = y(t_0,\cdots\cdots)$ (sifting property I)

(5) $\delta(t-t_0) y(t,\cdots\cdots) = \delta(t-t_0) y(t_0,\cdots\cdots)$ (sifting property II)