II. Short-time Fourier Transform

II-A Definition

Short-time Fourier transform (STFT)

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

Alternative definition

$$X(t,\omega)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j\omega\tau}d\tau$$

參考資料

[1] S. Qian and D. Chen, Section 3-1 in *Joint Time-Frequency Analysis: Methods and Applications*, Prentice-Hall, 1996.

[2] S. H. Nawab and T. F. Quatieri, "Short time Fourier transform," in *Advanced Topics in Signal Processing*, pp. 289-337, Prentice Hall, 1987.

STFT $X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$

$$X(t,\omega)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j\omega\tau}d\tau$$

Inverse of the STFT: To recover $x(t)$,

$$x(t)=w^{-1}(t_1-t)\int_{-\infty}^{\infty} X(t_1,f)e^{j2\pi f t}df$$

where $w(t_1-t)\neq 0$.

For the alternative definition, the inverse transform is:

$$x(t)=\frac{1}{2\pi}w^{-1}(t_1-t)\int_{-\infty}^{\infty} X(t_1,\omega)e^{j\omega t}d\omega$$

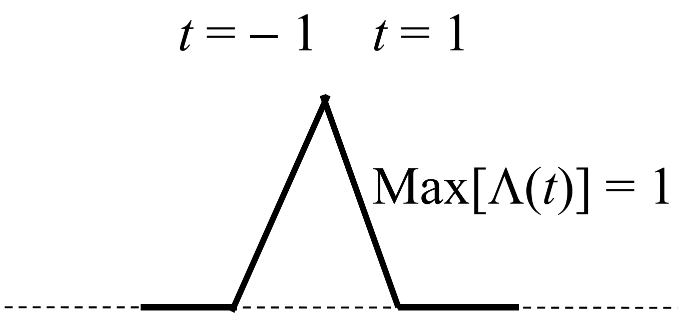
The mask function $w(t)$ always has the property of

(a) even: $w(t) = w(-t)$, (通常要求這個條件要滿足)

(b) $\max(w(t)) = w(0)$, $w(t_1) \geq w(t_2)$ if $|t_2| > |t_1|$

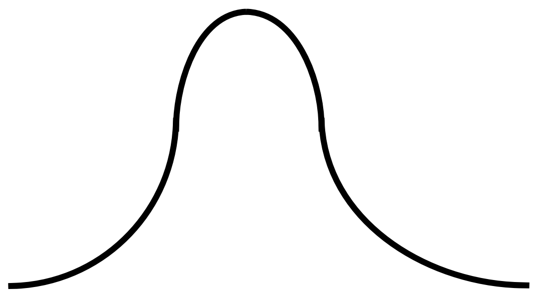
(c) $w(t) \approx 0$ when $|t|$ is large

$w(t) = \Lambda(t)$ (triangular function)



$w(t) = \exp(-a|t|^b)$

(hyper-Laplacian function)

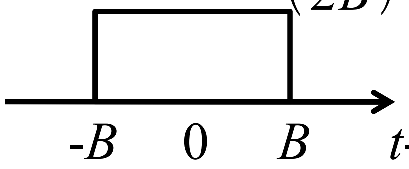


II-B Rec-STFT

Rectangular mask STFT (rec-STFT)

$$X(t,f)=\int_{t-B}^{t+B} x(\tau)e^{-j2\pi f\tau}d\tau$$

$$w(t)=rect\left(\frac{t}{2B}\right)$$



Inverse of the rec-STFT

$$x(t)=\int_{-\infty}^{\infty} X(t_1,f)e^{j2\pi f t}df$$

where $t - B < t_1 < t + B$

The simplest form of the STFT

Other types of the STFT may require more computation time than the rec-STFT.

II-C Properties of the Rec-STFT

(1) Integration (recovery):

(a) $\int_{-\infty}^{\infty} X(t,f) e^{j2\pi f v} df = x(v)$ when $v - B < t < v + B$,

$= 0$ otherwise

(b)
$$\int_{-\infty}^{\infty} X(t,f) df = \int_{t-B}^{t+B} x(\tau) \int_{-\infty}^{\infty} e^{-j2\pi f \tau} df \, d\tau$$
$$= \int_{t-B}^{t+B} x(\tau) \delta(\tau) d\tau$$
$$= \begin{cases} x(0) & \text{when } t - B < 0 < t + B, \quad -B < t < B \\ 0 & \text{otherwise} \end{cases}$$

(2) Shifting property (橫的方向移動)

$$\int_{t-B}^{t+B} x(\tau-\tau_0)e^{-j2\pi f\tau}d\tau = X(t-\tau_0, f)e^{-j2\pi f\tau_0}$$

(3) Modulation property (縱的方向移動)

$$\int_{t-B}^{t+B} [x(\tau)e^{j2\pi f_0\tau}]e^{-j2\pi f\tau}d\tau = X(t, f-f_0)$$

(4) Special inputs:

(1) When $x(t) = \delta(t)$,

$$X(t,f) = 1 \text{ when } -B < t < B, \quad X(t,f) = 0 \text{ otherwise}$$

(2) When $x(t) = 1$

$$X(t,f) = 2B\,\mathrm{sinc}(2Bf)\,e^{-j2\pi f t}$$

思考： ***B*** 值的大小，對解析度的影響是什麼？

(5) Linearity property

If $h(t) = \alpha\, x(t) + \beta\, y(t)$ and $H(t, f)$, $X(t, f)$ and $Y(t, f)$ are their rec-STFTs, then

$$H(t, f) = \alpha\, X(t, f) + \beta\, Y(t, f).$$

(6) Energy sum property (Parseval's theorem)

$$\int_{-\infty}^{\infty} \left|X(t,f)\right|^2 df = \int_{t-B}^{t+B} \left|x(\tau)\right|^2 d\tau$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|X(t,f)\right|^2 df\, dt = 2B\int_{-\infty}^{\infty} \left|x(\tau)\right|^2 d\tau$$

Comparison: for the original Fourier transform

$$\int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|X(f)\right|^2 df$$

(7) Generalized Parseval's theorem

$$\int_{-\infty}^{\infty} X(t,f)\, Y^*(t,f)\, df = \int_{t-B}^{t+B} x(\tau)\, y^*(\tau)\, d\tau$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t,f)\, Y^*(t,f)\, df\, dt = 2B\int_{-\infty}^{\infty} x(\tau)\, y^*(\tau)\, d\tau$$

思考:

(1) 哪些性質 Fourier transform 也有？

(2) 其他型態的 STFT 是否有類似的性質？

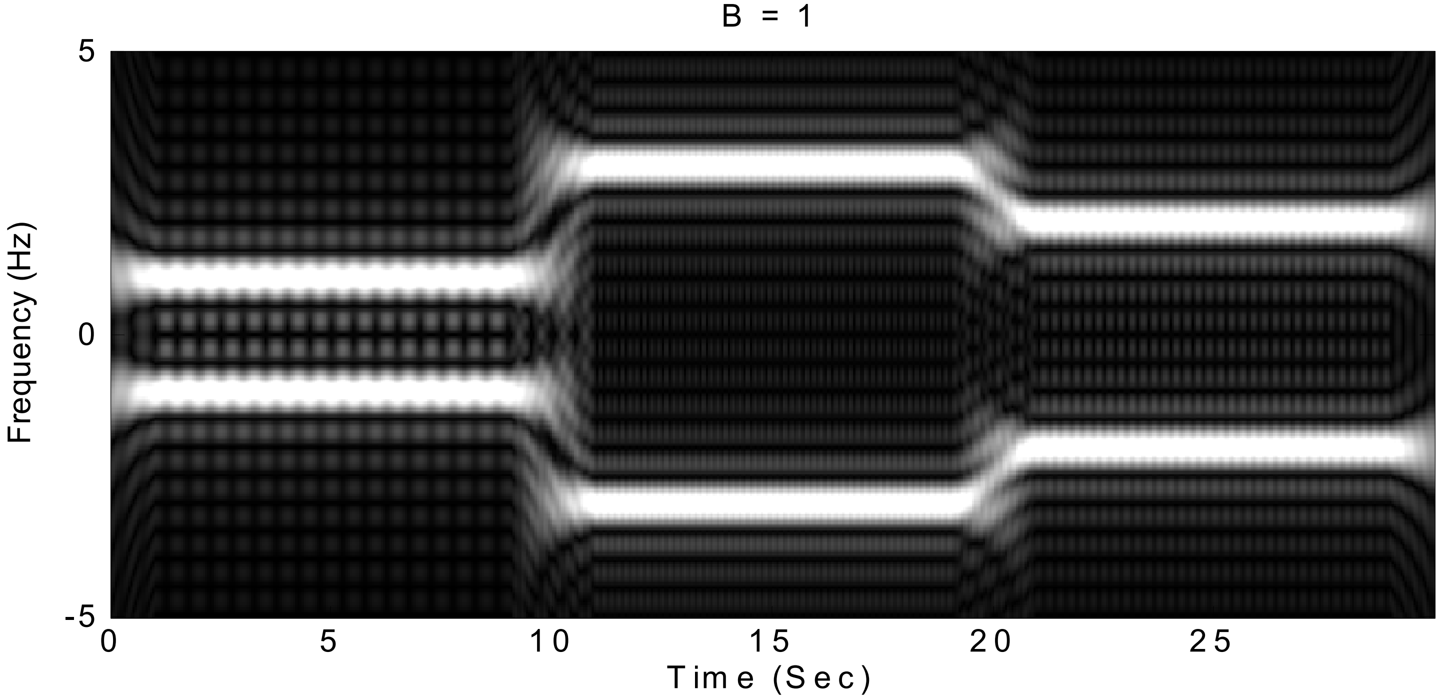
Shifting
$$\int_{-\infty}^{\infty} w(t-\tau)x(\tau-\tau_0)e^{-j2\pi f\tau}d\tau$$
$$=\int_{-\infty}^{\infty} w(t-\tau-\tau_0)x(\tau)e^{-j2\pi f\tau}e^{-j2\pi f\tau_0}d\tau$$
$$=X(t-\tau_0,f)e^{-j2\pi f\tau_0}$$

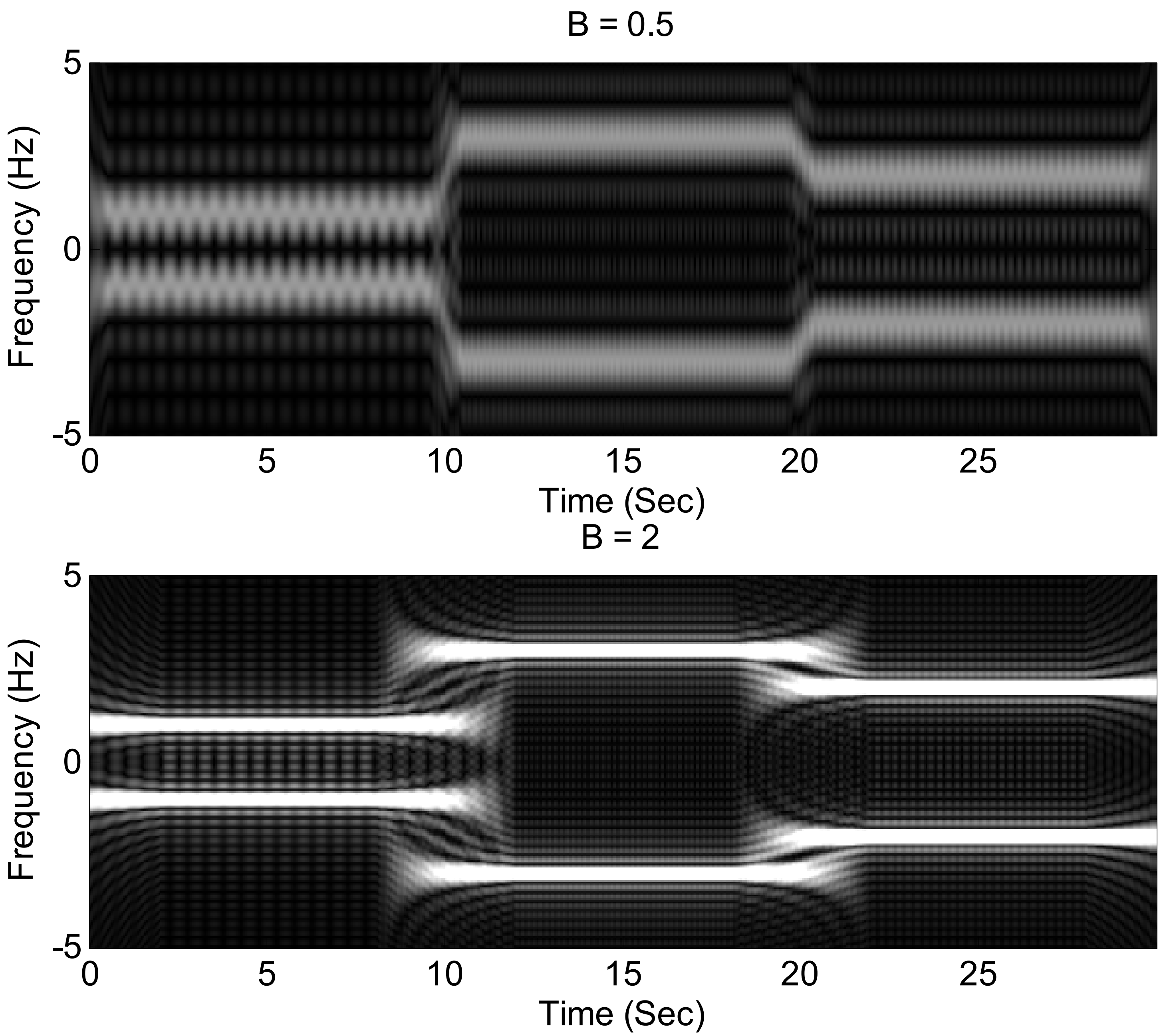
Modulation
$$\int_{-\infty}^{\infty} w(t-\tau)[x(\tau)e^{j2\pi f_0\tau}]e^{-j2\pi f\tau}d\tau = X(t,f-f_0)$$

Example: $x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$





B = 0.5
Frequency (Hz)
5
0
-5
0
5
10
15
20
25
Time (Sec)
B = 2
Frequency (Hz)
5
0
-5
0
5
10
15
20
25
Time (Sec)

II-D Advantage and Disadvantage

• Compared with the Fourier transform:

All the time-frequency analysis methods has the advantage of:

The instantaneous frequency can be observed.

All the time-frequency analysis methods has the disadvantage of:

Higher complexity for computation

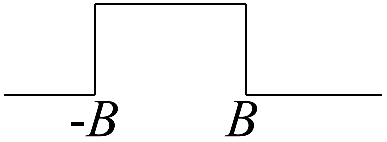
- Compared with other types of time-frequency analysis:

The rec-STFT has an advantage of the least computation time for digital implementation

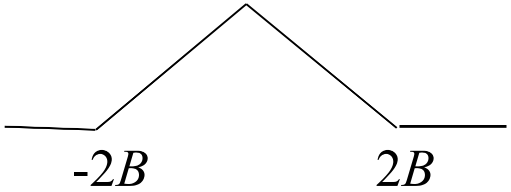
but its performance is worse than other types of time-frequency analysis.

II-E STFT with Other Windows

(1) Rectangle



(2) Triangle



(3) Hanning

$$w(t) = \begin{cases} 0.5 + 0.5\cos(\pi t / B) & \text{when } |t| \le B \\ 0 & \text{otherwise} \end{cases}$$

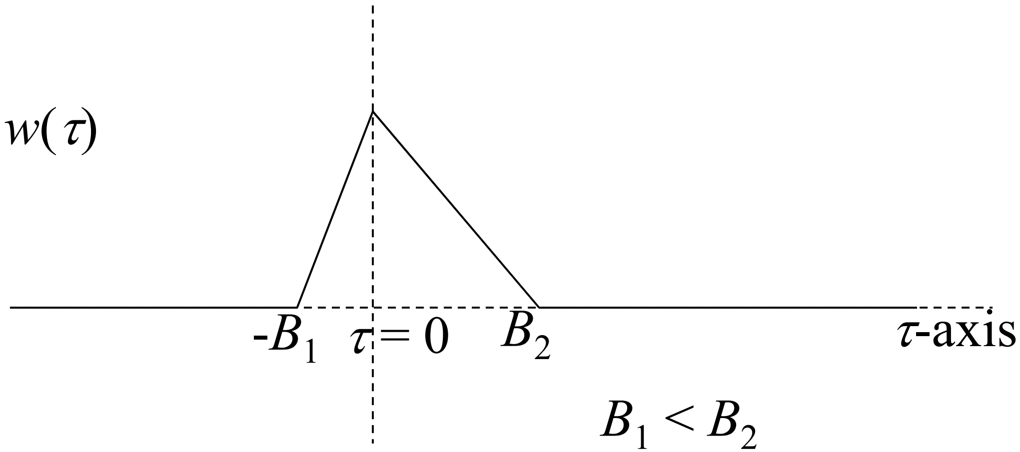
(4) Hamming

$$w(t) = \begin{cases} 0.54 + 0.46\cos(\pi t / B) & \text{when } |t| \le B \\ 0 & \text{otherwise} \end{cases}$$

(5) Gaussian

$$w(t) = \exp\left(-\pi \sigma t^2\right)$$

(6) Asymmetric window



$w(t-\tau) = ?$

應用： seismic wave analysis, collision detection

(The applications that require real-time processing)

動腦思考：

(1) Are there other ways to choose the mask of the STFT?

(2) Which mask is better?

沒有一定的答案

II-F Spectrogram

STFT 的絕對值平方，被稱作 Spectrogram

$$SP_x(t,f)=\left|X(t,f)\right|^2=\left|\int_{-\infty}^{\infty} w(t-\tau)e^{-j2\pi f\tau}x(\tau)\,d\tau\right|^2$$

比較： spectrum 為 Fourier transform 的絕對值平方

文獻上，spectrogram 這個名詞出現的頻率多於 STFT

但實際上， spectrogram 和 STFT 的本質是相同的

附錄三：使用 Matlab 將時頻分析結果 Show 出來

可採行兩種方式：

(1) 使用 mesh 指令畫出立體圖

(但結果不一定清楚，且執行時間較久)

(2) 將 amplitude 變為 gray-level，用顯示灰階圖的方法將結果表現出來

假設 y 是時頻分析計算的結果

image(abs(y)/max(max(abs(y)))*C) % C 是一個常數，我習慣選 C=400

或 image(t, f, abs(y)/max(max(abs(y)))*C)

colormap(gray(256)) % 變成 gray-level 的圖

set(gca,‘Ydir’,‘normal’) % 若沒這一行, y-axis 的方向是倒過來的

```
set(gca,'Fontsize',12)        % 改變橫縱軸數值的 font sizes
xlabel('Time (Sec)','Fontsize',12)          % x-axis
ylabel('Frequency (Hz)','Fontsize',12)      % y-axis
title('STFT of x(t)','Fontsize',12)         % title
```

計算程式執行時間的指令：

tic (這指令如同按下碼錶)

toc (show 出碼錶按下後已經執行了多少時間)

註：通常程式執行第一次時，由於要做程式的編譯，所得出的執行時間會比較長

程式執行第二次以後所得出的執行時間，是較為正確的結果

附錄四：使用 Python 將時頻分析的圖畫出來

事前安裝模組

```
pip install numpy
pip install matplotlib
```

假設y為時頻分析結果(應為二維的矩陣數列)，將 y 以灰階方式畫出來

```
import numpy as np
import matplotlib.pyplot as plt
C = 400
y = np.abs(y) / np.max(np.abs(y)) * C
plt.imshow(y, cmap='gray', origin='lower')
# 加上 origin='lower' 避免上下相反
plt.xlabel('Time (Sec)')
plt.ylabel('Frequency (Hz)')
plt.show()
```

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若要加上座標軸數值(在plt.show()之前加上以下程式碼)

```
x_label = ['0', '10', '20', '30']   # 橫軸座標值
y_label = ['-5', '0', '5']              # 縱軸座標值
plt.xticks(np.arange(0, x_max, step=int(x_max/(len(x_label)-1)), x_label)
plt.yticks(np.arange(0, y_max, step=int(y_max/(len(y_label)-1)), y_label)
```

Reference :
https://matplotlib.org/stable/api/_as_gen/matplotlib.pyplot.xticks.html

計算時間

```
import time
start_time = time.time()   #獲取當前時間
end_time = time.time()
total_time = end_time - start_time   #計算時間差來得到總執行時間
```