

V. Wigner Distribution Function

V-A Wigner Distribution Function (WDF)

Definition 1: $W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} d\tau$

Definition 2: $W_x(t, \omega) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j\omega\tau} d\tau$

Another way for computation from the frequency domain

$$\text{Definition 1: } W_x(t, f) = \int_{-\infty}^{\infty} X(f + \eta/2) \cdot X^*(f - \eta/2) e^{j2\pi\eta t} d\eta$$

where $X(f)$ is the Fourier transform of $x(t)$

$$\text{Definition 2: } W_x(t, \omega) = \int_{-\infty}^{\infty} X(\omega + \eta/2) \cdot X^*(\omega - \eta/2) e^{j\eta t} d\eta$$

The Wigner distribution function is also called the Wigner Ville distribution.

Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 5, Prentice Hall, N.J., 1996.

Other References

- [Ref] E. P. Wigner, “On the quantum correlation for thermodynamic equilibrium,” *Phys. Rev.*, vol. 40, pp. 749-759, 1932.
- [Ref] T. A. C. M. Classen and W. F. G. Mecklenbrauker, “The Wigner distribution—A tool for time-frequency signal analysis; Part I,” *Philips J. Res.*, vol. 35, pp. 217-250, 1980.
- [Ref] F. Hlawatsch and G. F. Boudreux–Bartels, “Linear and quadratic time-frequency signal representation,” *IEEE Signal Processing Magazine*, pp. 21-67, Apr. 1992.
- [Ref] R. L. Allen and D. W. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*, Wiley-Interscience, NJ, 2004.

The operators that are related to the WDF:

(a) Signal auto-correlation function:

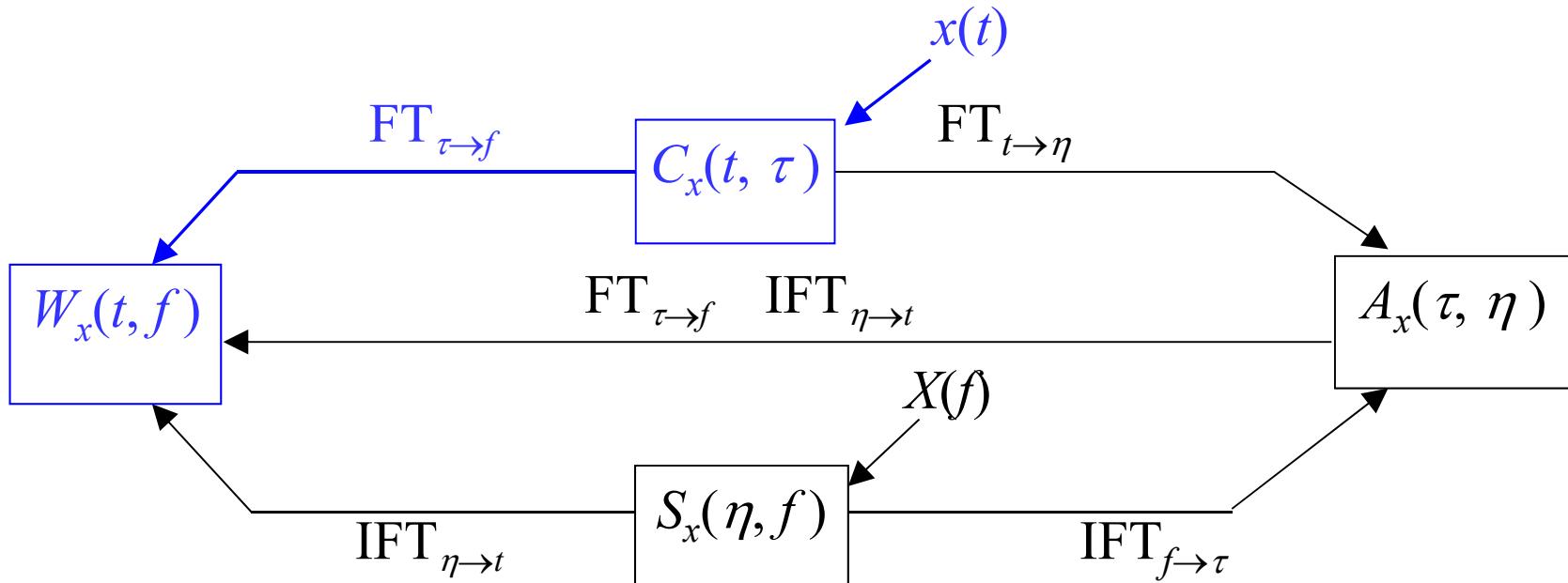
$$C_x(t, \tau) = x(t + \tau/2) \cdot x^*(t - \tau/2)$$

(b) Spectrum auto-correlation function:

$$S_x(\eta, f) = X(f + \eta/2) \cdot X^*(f - \eta/2)$$

(c) Ambiguity function (AF):

$$A_x(\tau, \eta) = \int_{-\infty}^{\infty} x(t + \tau/2) x^*(t - \tau/2) e^{-j2\pi t\eta} dt$$



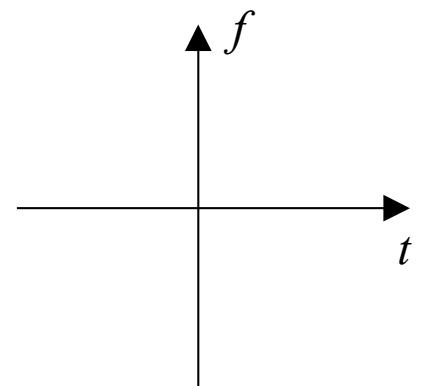
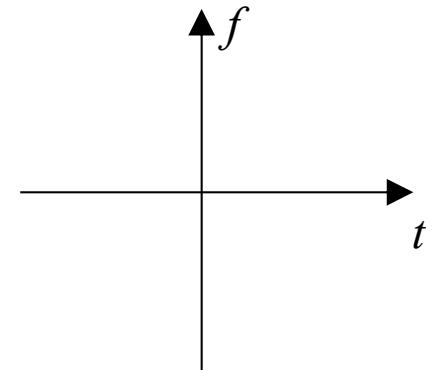
V-B Why the WDF Has Higher Clarity?

Due to signal auto-correlation function

$$(1) \text{ If } x(t) = 1$$

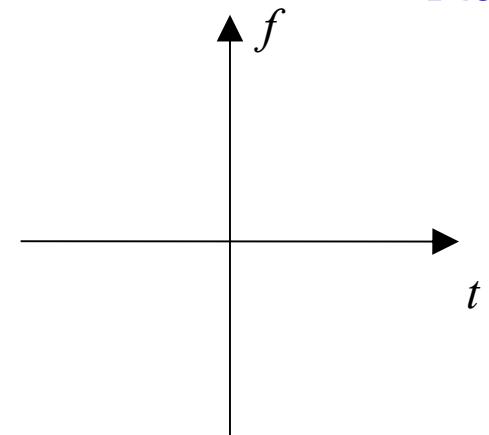
$$(2) \text{ If } x(t) = \exp(j2\pi h t)$$

$$\begin{aligned} W_x(t, f) &= \int_{-\infty}^{\infty} e^{j2\pi h(t+\tau/2)} e^{-j2\pi h(t-\tau/2)} \cdot e^{-j2\pi \tau f} d\tau \\ &= \int_{-\infty}^{\infty} e^{j2\pi h\tau} \cdot e^{-j2\pi \tau f} d\tau \\ &= \int_{-\infty}^{\infty} e^{-j2\pi \tau(f-h)} d\tau \\ &= \delta(f - h) \end{aligned}$$



Comparing: for the case of the STFT

(3) If $x(t) = \exp(j2\pi k t^2)$



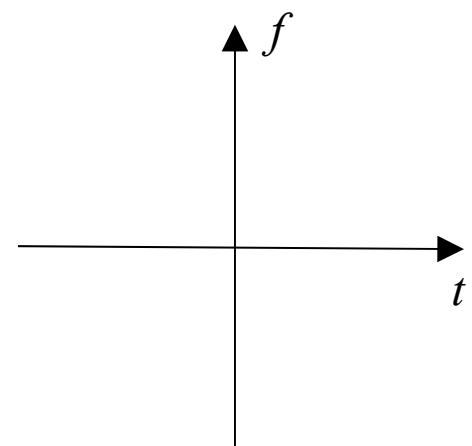
(4) If $x(t) = \delta(t)$

$$W_x(t, f) = \int_{-\infty}^{\infty} \delta(t + \tau/2) \cdot \delta(t - \tau/2) e^{-j2\pi\tau f} d\tau$$

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公式(2)

$$\begin{aligned} &= 4 \int_{-\infty}^{\infty} \delta(2t + \tau) \cdot \delta(2t - \tau) e^{-j2\pi\tau f} d\tau \\ &= 4\delta(4t) e^{j4\pi t f} = \delta(t) e^{j4\pi t f} = \delta(t) \end{aligned}$$

$\int_{-\infty}^{\infty} \delta(\tau - \tau_0) \cdot y(\tau) d\tau$ Page 139 Page 139 Page 139
 $= y(\tau_0)$ 公式(4) 公式(2) 公式(5), $t_0 = 0$



V-C The WDF is not a Linear Distribution

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

If $h(t) = \alpha g(t) + \beta s(t)$

Linear:

If $x_1 \rightarrow y_1, x_2 \rightarrow y_2$
then $ax_1 + bx_2 \rightarrow ay_1 + by_2$

$$\begin{aligned} W_h(t, f) &= \int_{-\infty}^{\infty} h(t + \tau/2) \cdot h^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau \\ &= \int_{-\infty}^{\infty} [\alpha g(t + \tau/2) + \beta s(t + \tau/2)] [\alpha^* g^*(t - \tau/2) + \beta^* s^*(t - \tau/2)] e^{-j2\pi\tau f} d\tau \\ &= \int_{-\infty}^{\infty} [|\alpha|^2 g(t + \tau/2)g^*(t - \tau/2) + |\beta|^2 s(t + \tau/2)s^*(t - \tau/2) \\ &\quad + \alpha\beta^* g(t + \tau/2)s^*(t - \tau/2) + \alpha^*\beta g^*(t - \tau/2)s(t + \tau/2)] e^{-j2\pi\tau f} d\tau \\ &= |\alpha|^2 W_g(t, f) + |\beta|^2 W_s(t, f) \\ &\quad + \int_{-\infty}^{\infty} [\alpha\beta^* g(t + \tau/2)s^*(t - \tau/2) + \alpha^*\beta g^*(t - \tau/2)s(t + \tau/2)] e^{-j2\pi\tau f} d\tau \end{aligned}$$

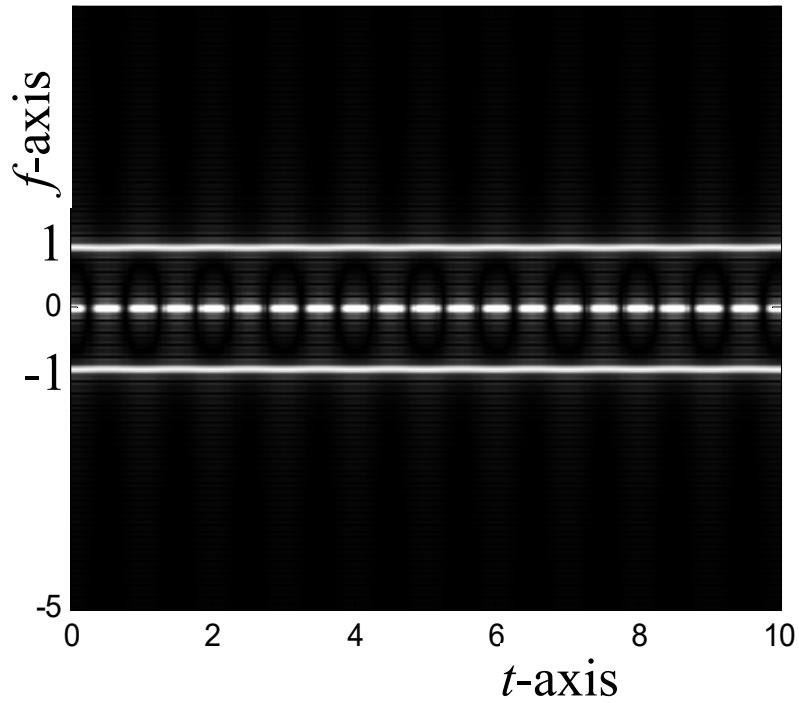
 cross terms

V-D Examples of the WDF

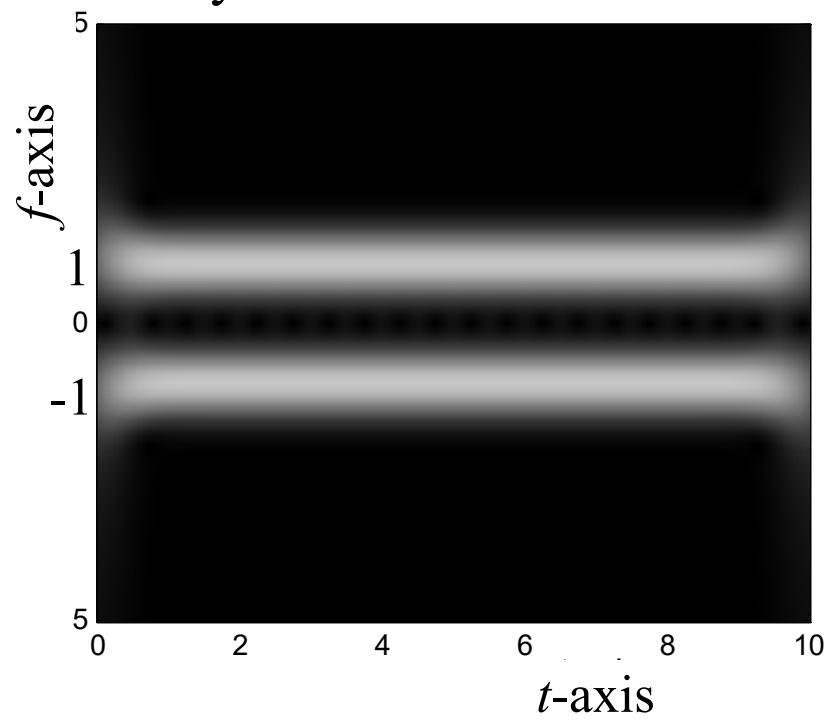
Simulations

$$x(t) = \cos(2\pi t) = 0.5[\exp(j2\pi t) + \exp(-j2\pi t)]$$

by the WDF



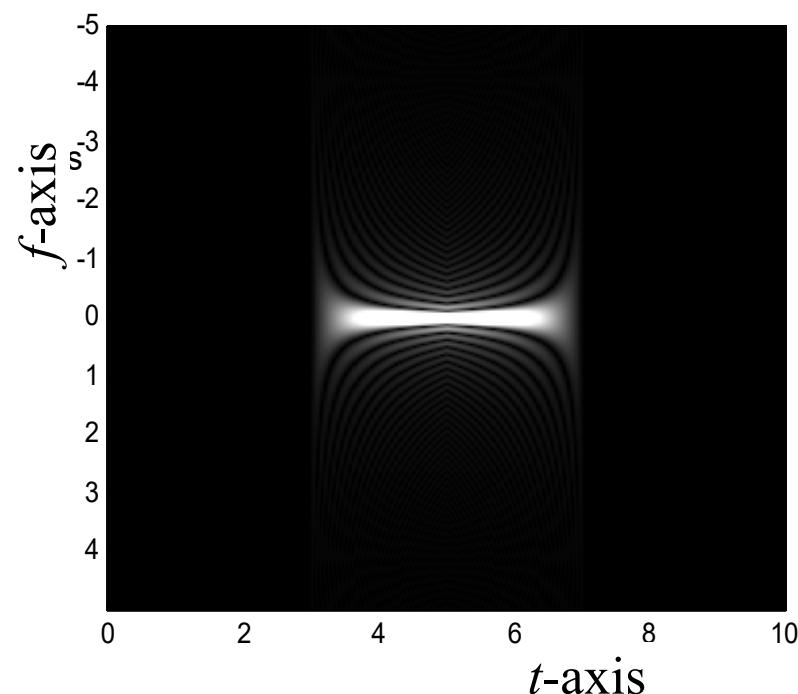
by the Gabor transform



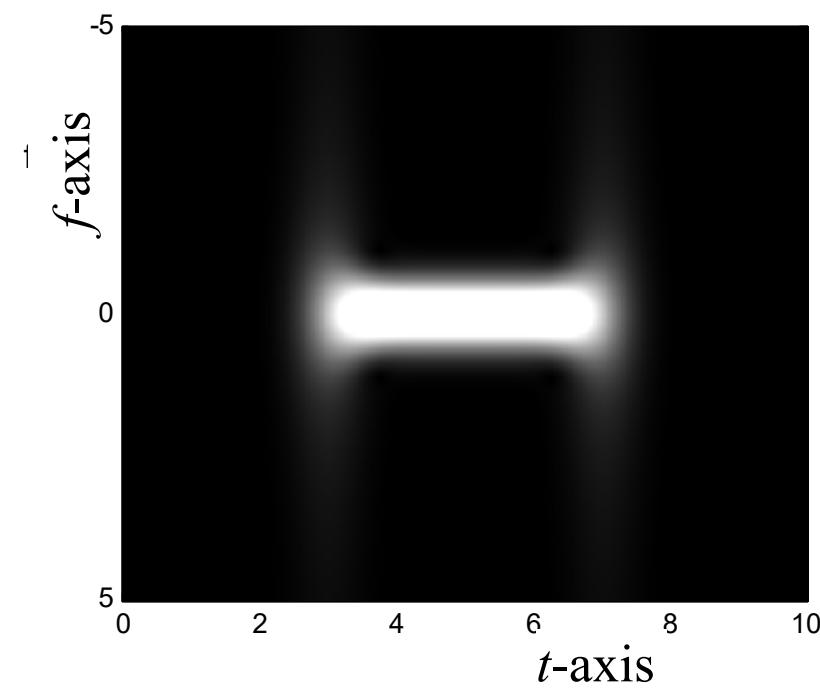
$$x(t) = \Pi((t-5)/4)$$

Π : rectangular function

by the WDF

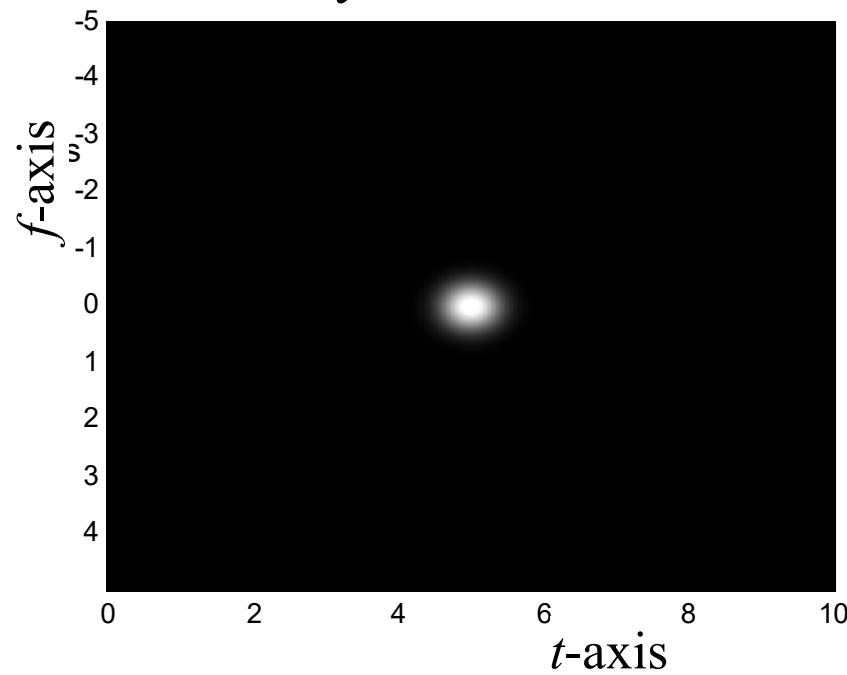


by the Gabor transform

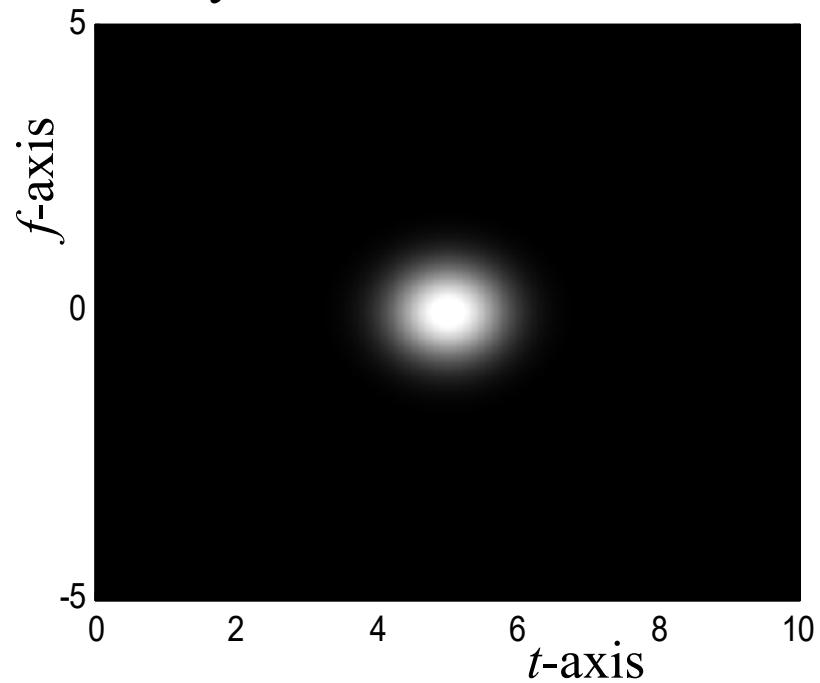


$$x(t) = \exp[-\pi(t-5)^2]$$

by the WDF



by the Gabor transform



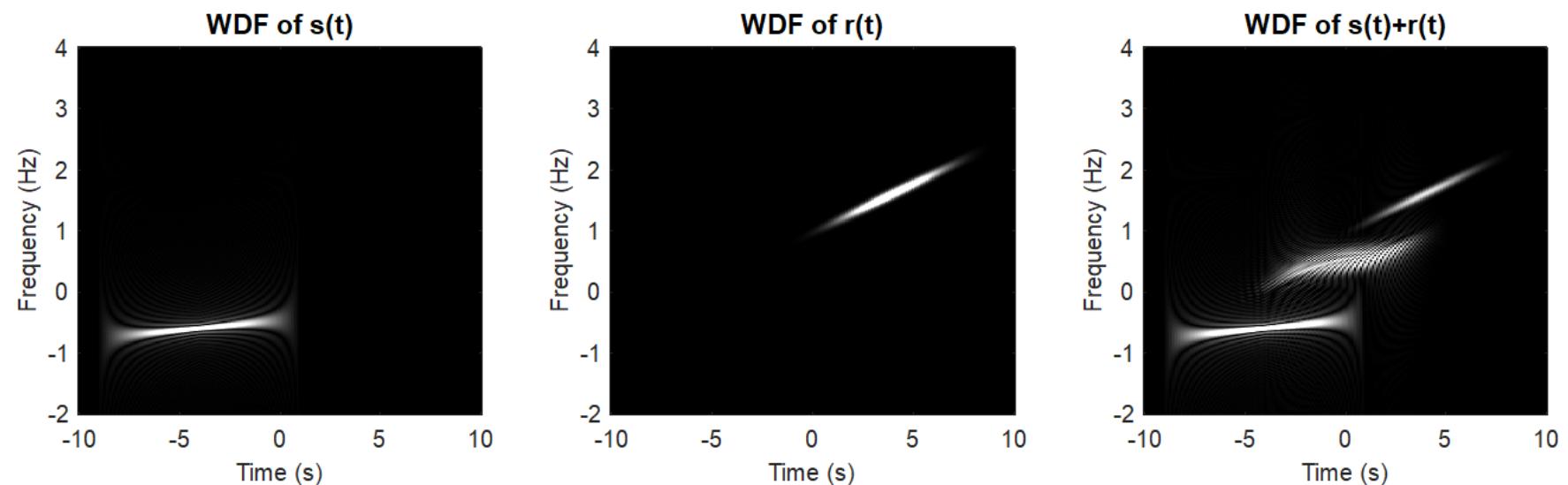
Gaussian function: $e^{-\pi t^2} \xrightarrow{FT} e^{-\pi f^2}$

Gaussian function's T-F area is minimal.

$$s(t) = \exp\left(jt^2/10 - j3t\right) \quad \text{for } -9 \leq t \leq 1, s(t) = 0 \text{ otherwise,}$$

$$r(t) = \exp\left(jt^2/2 + j6t\right) \exp\left[-(t-4)^2/10\right]$$

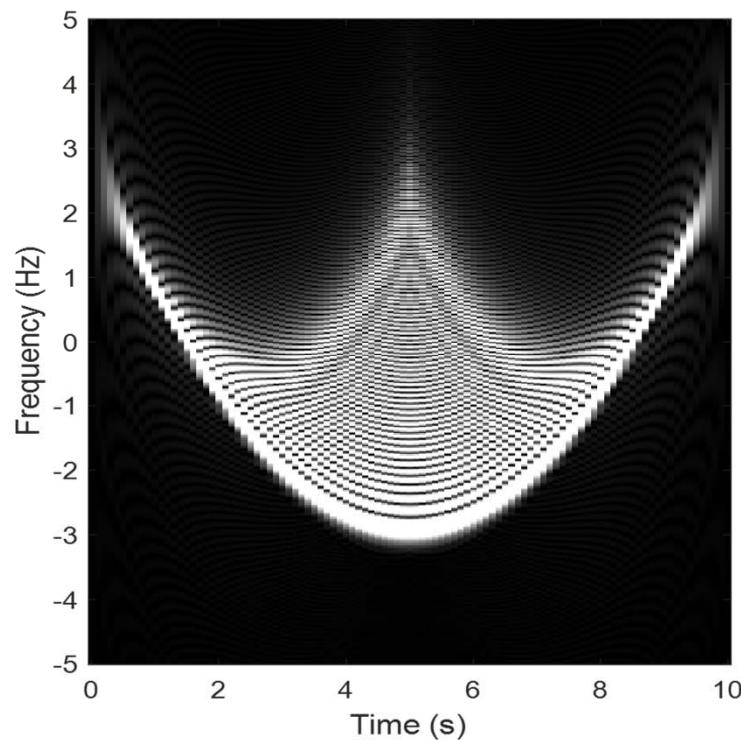
$$f(t) = s(t) + r(t)$$



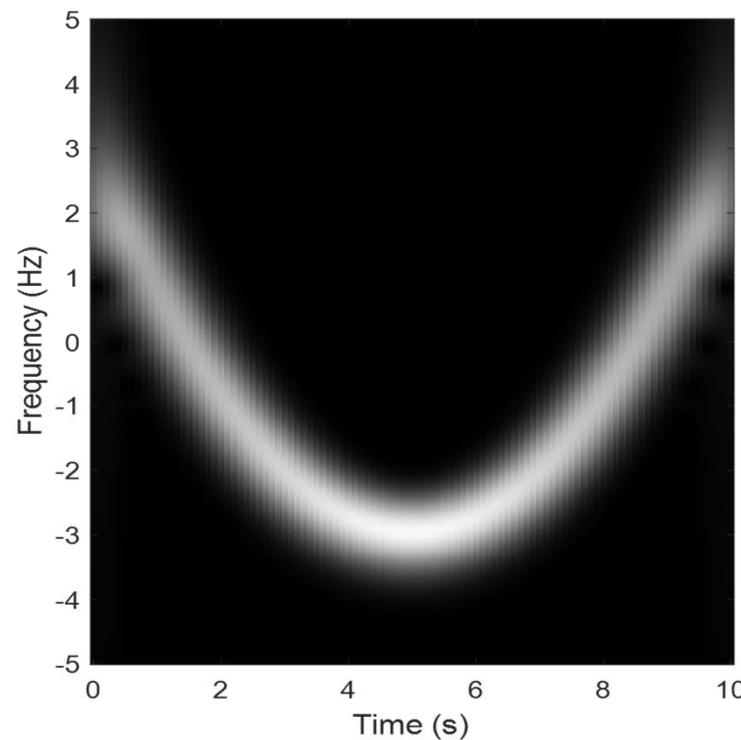
橫軸: t -axis, 縱軸: f -axis

$$x(t) = \exp\left(\frac{j}{2}(t-5)^3 - j6\pi t\right)$$

by the WDF



by the Gabor transform



V-E Digital Implementation of the WDF

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau ,$$

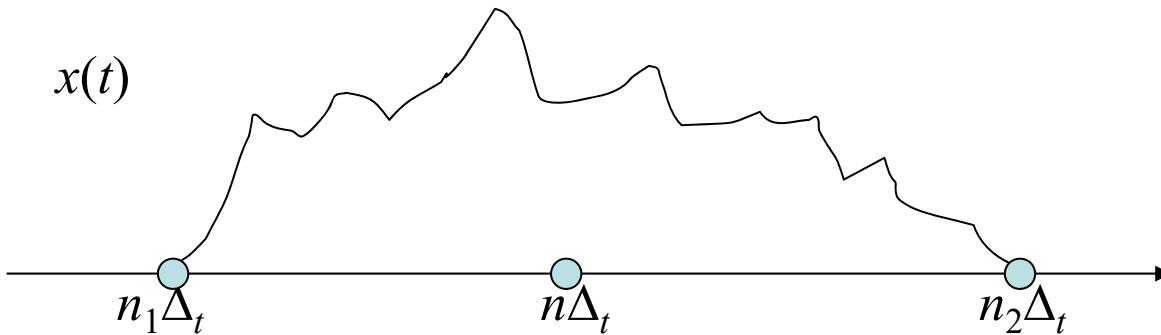
$$W_x(t, f) = 2 \int_{-\infty}^{\infty} x(t + \tau') \cdot x^*(t - \tau') e^{-j4\pi\tau' f} \cdot d\tau' \quad (\text{using } \tau' = \tau/2)$$

Sampling: $t = n\Delta_t$, $f = m\Delta_f$, $\tau' = p\Delta_t$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-\infty}^{\infty} x((n+p)\Delta_t) x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t \Delta_f) \Delta_t$$

When $x(t)$ is not a time-limited signal, it is hard to implement.

Suppose that $x(t) = 0$ for $t < n_1\Delta_t$ and $t > n_2\Delta_t$



$$x((n+p)\Delta_t)x^*((n-p)\Delta_t) \neq 0 \quad \text{only when } n+p \in [n_1, n_2] \text{ and } n-p \in [n_1, n_2]$$

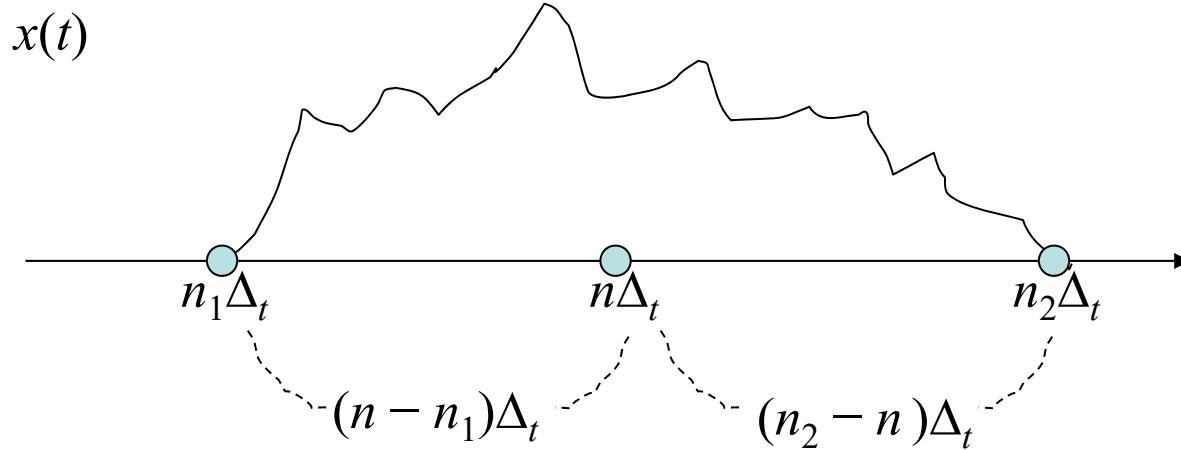
• p 的範圍的問題 (當 n 固定時)

$$n_1 \leq n + p \leq n_2 \longrightarrow n_1 - n \leq p \leq n_2 - n$$

$$n_1 \leq n - p \leq n_2 \longrightarrow n_1 - n \leq -p \leq n_2 - n, \quad n - n_2 \leq p \leq n - n_1$$

$$\max(n_1 - n, n - n_2) \leq p \leq \min(n_2 - n, n - n_1)$$

$$-\min(n_2 - n, n - n_1) \leq p \leq \min(n_2 - n, n - n_1)$$



$$-\min(n_2 - n, n - n_1) \leq p \leq \min(n_2 - n, n - n_1)$$

$\quad \quad \quad Q = \min(n_2 - n, n - n_1).$

$(n_2 - n)\Delta_t, (n - n_1)\Delta_t:$ 離兩個邊界的距離

注意：當 $n > n_2$ 或 $n < n_1$ 時，

由於 $Q < 0$ ，將沒有 p 能滿足上面的不等式

此時 $W_x(n\Delta_t, m\Delta_f) = 0$

If $x(t) = 0$ for $t < n_1\Delta_t$ and $t > n_2\Delta_t$

$$W_x(n\Delta_t, m\Delta_f) = \underset{T\text{點}}{2} \sum_{p=-Q}^Q \underset{F\text{點}}{x((n+p)\Delta_t)} x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t\Delta_f) \Delta_t$$

$$Q = \min(n_2 - n, n - n_1). \quad (\text{varies with } n)$$

$$p \in [-Q, Q], \quad n \in [n_1, n_2],$$

possible for implementation

Method 1: Direct Implementation (brute force method)

唯一的限制條件？

Method 2: Using the DFT

3 大限制條件

When $\Delta_t \Delta_f = \frac{1}{2N}$ and $N \geq 2\text{Max}(Q)+1 = 2(n_2-n_1)/2+1 = n_2-n_1+1 = T$

$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\Delta_t \sum_{p=-Q}^Q x((n+p)\Delta_t)x^*((n-p)\Delta_t)e^{-j\frac{2\pi mp}{N}}}$$

$$q = p+Q \rightarrow p = q - Q$$

$$W_x(n\Delta_t, m\Delta_f) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{2Q} x((n+q-Q)\Delta_t)x^*((n-q+Q)\Delta_t)e^{-j\frac{2\pi mq}{N}}$$

$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{N-1} c_1(q) e^{-j\frac{2\pi mq}{N}}}$$

$$Q = \min(n_2-n, n-n_1), \\ n \in [n_1, n_2],$$

$$c_1(q) = x((n+q-Q)\Delta_t)x^*((n-q+Q)\Delta_t) \quad \text{for } 0 \leq q \leq 2Q$$

$$\text{i.e., } c_1(Q+k) = x((n+k)\Delta_t)x^*((n-k)\Delta_t) \quad \text{for } -Q \leq k \leq Q \quad (k = q-Q)$$

$$c_1(q) = 0 \quad \text{for } 2Q+1 \leq q \leq N-1$$

假設 $t = n_0\Delta_t, (n_0+1)\Delta_t, (n_0+2)\Delta_t, \dots, n_1\Delta_t$

$f = m_0\Delta_f, (m_0+1)\Delta_f, (m_0+2)\Delta_f, \dots, m_1\Delta_f$

Step 1: Calculate n_0, n_1, m_0, m_1, N

Step 2: $n = n_0$

Step 3: Determine Q

Step 4: Determine $c_1(q)$

Step 5: $C_1(m) = \text{FFT}[c_1(q)]$

Step 6: Convert $C_1(m)$ into $C(n\Delta_t, m\Delta_f)$

Step 7: Set $n = n+1$ and return to Step 3 until $n = n_1$.

Method 3: Using the Chirp Z Transform

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-Q}^Q x((n+p)\Delta_t) x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t\Delta_f) \Delta_t$$

$$W_x(n\Delta_t, m\Delta_f) = 2\Delta_t e^{-j2\pi m^2 \Delta_t \Delta_f} \sum_{p=-Q}^Q x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j2\pi p^2 \Delta_t \Delta_f} e^{j2\pi(p-m)^2 \Delta_t \Delta_f}$$

Step 1 $x_1(n, p) = x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j2\pi p^2 \Delta_t \Delta_f}$

Step 2 $X_2[n, m] = \sum_{p=-Q}^Q x_1[n, p] c[m-p] \quad c[m] = e^{j2\pi m^2 \Delta_t \Delta_f}$

Step 3 $X(n\Delta_t, m\Delta_f) = 2\Delta_t e^{-j2\pi m^2 \Delta_t \Delta_f} X_2[n, m]$

Q : What is the complexity of Method 1?

Q : What is the complexity of Method 2?

Q : What is the complexity of Method 3?

The computation time of the WDF is more than those of the rec-STFT and the Gabor transform.

V-F Properties of the WDF

(1) Projection property	$ x(t) ^2 = \int_{-\infty}^{\infty} W_x(t, f) df \quad X(f) ^2 = \int_{-\infty}^{\infty} W_x(t, f) dt$
(2) Energy preservation property	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t, f) dt df = \int_{-\infty}^{\infty} x(t) ^2 dt = \int_{-\infty}^{\infty} X(f) ^2 df$
(3) Recovery property	$\int_{-\infty}^{\infty} W_x(t/2, f) e^{j2\pi f t} df = x(t) \cdot x^*(0)$ $x^*(0)$ 已知 $\int_{-\infty}^{\infty} W_x(t, f/2) e^{-j2\pi f t} dt = X(f) \cdot X^*(0)$
(4) Mean condition frequency and mean condition time	If $x(t) = x(t) \cdot e^{j2\pi \phi(t)}$, $X(f) = X(f) \cdot e^{j2\pi \Psi(f)}$ then $\phi'(t) = x(t) ^{-2} \cdot \int_{-\infty}^{\infty} f \cdot W_x(t, f) \cdot df$ $-\Psi'(f) = X(f) ^{-2} \int_{-\infty}^{\infty} t \cdot W_x(t, f) \cdot dt$
(5) Moment properties	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t^n W_x(t, f) dt df = \int_{-\infty}^{\infty} t^n x(t) ^2 dt$, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^n W_x(t, f) dt df = \int_{-\infty}^{\infty} f^n X(f) ^2 df$

(6) $W_x(t, f)$ is bound to be real	$\overline{W_x(t, f)} = W_x(t, f)$
(7) Region properties	If $x(t) = 0$ for $t > t_2$ then $W_x(t, f) = 0$ for $t > t_2$ If $x(t) = 0$ for $t < t_1$ then $W_x(t, f) = 0$ for $t < t_1$
(8) Multiplication theory	If $y(t) = x(t)h(t)$, then $W_y(t, f) = \int_{-\infty}^{\infty} W_x(t, \rho)W_h(t, f - \rho) \cdot d\rho$
(9) Convolution theory	If $y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau) d\tau$, then $W_y(t, f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(t - \rho, f) \cdot d\rho$
(10) Correlation theory	If $y(t) = \int_{-\infty}^{\infty} x(t + \tau)h^*(\tau) d\tau$, then $W_y(t, f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(-t + \rho, f) \cdot d\rho$

(11) Time-shifting property	If $y(t) = x(t - t_0)$, then $W_y(t, f) = W_x(t - t_0, f)$
(12) Modulation property	If $y(t) = \exp(j2\pi f_0 t)x(t)$, then $W_y(t, f) = W_x(t, f - f_0)$
(13) Constant multiplication property	If $y(t) = cx(t)$, then $W_y(t, f) = c ^2 W_x(t, f)$
(14) Conjugation property	If $y(t) = x^*(t)$, then $W_y(t, f) = W_x(t, -f)$
(15) Scaling property	If $y(t) = x(ct)$, then $W_y(t, f) = \frac{1}{ c } W_x(ct, \frac{1}{c}f)$

The STFT (including the rec-STFT, the Gabor transform) does not have real region, multiplication, convolution, and correlation properties.

- Why the WDF is always real?

What are the advantages and disadvantages it causes?

- Try to prove of the projection and recovery properties

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

- Proof of the region properties

If $x(t) = 0$ for $t < t_0$,

$$x(t + \tau/2) = 0 \quad \text{for } \tau < (t_0 - t)/2 = -(t - t_0)/2,$$

$$x(t - \tau/2) = 0 \quad \text{for } \tau > (t - t_0)/2,$$

Therefore, if $t - t_0 < 0$, the nonzero regions of $x(t + \tau/2)$ and $x(t - \tau/2)$ does not overlap and $x(t + \tau/2) x^*(t - \tau/2) = 0$ for all τ .

The importance of the region property

- Extra Property:

(16) The relation between the WDF and the spectrogram:

Suppose that $x(t)$ is the input function, $w(t)$ is the window function of the STFT, $X(t, f)$ is the STFT of $x(t)$, and $W_x(t, f)$ and $W_w(t, f)$ are the WDFs of $x(t)$ and $w(t)$, respectively, then

$$|X(t, f)|^2 = W_x(t, f) * W_w(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t-u, f-v) W_w(u, v) du dv$$

$$\begin{aligned} (\text{Proof}): \quad & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t-u, f-v) W_w(u, v) du dv \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t-u+\tau/2) x^*(t-u-\tau/2) e^{-j2\pi(f-v)\tau} d\tau \\ & \quad \int_{-\infty}^{\infty} w\left(u + \frac{\eta}{2}\right) w\left(u - \frac{\eta}{2}\right) e^{-j2\pi v \eta} d\eta du dv \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\left(t-u+\frac{\tau}{2}\right) x^*\left(t-u-\frac{\tau}{2}\right) w\left(u + \frac{\eta}{2}\right) w\left(u - \frac{\eta}{2}\right) e^{-j2\pi f \tau} \\ & \quad \int_{-\infty}^{\infty} e^{-j2\pi v(\eta-\tau)} dv d\tau d\eta du \end{aligned} \quad (Cont.)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\left(t-u+\frac{\tau}{2}\right) x^*\left(t-u-\frac{\tau}{2}\right) w\left(u+\frac{\eta}{2}\right) w\left(u-\frac{\eta}{2}\right) e^{-j2\pi f\tau} \\ \delta(\eta-\tau) d\tau d\eta du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\left(t-u+\frac{\tau}{2}\right) x^*\left(t-u-\frac{\tau}{2}\right) w\left(u+\frac{\tau}{2}\right) w\left(u-\frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau du$$

$$\text{Set } \tau_1 = t - u + \frac{\tau}{2}, \quad \tau_2 = t - u - \frac{\tau}{2}$$

$$d\tau_1 d\tau_2 = \left| \det \begin{bmatrix} \partial \tau_1 / \partial \tau & \partial \tau_1 / \partial u \\ \partial \tau_2 / \partial \tau & \partial \tau_2 / \partial u \end{bmatrix} \right| d\tau du = d\tau du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau_1) x^*(\tau_2) w(t-\tau_2) w(t-\tau_1) e^{-j2\pi f(\tau_1-\tau_2)} d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty} x(\tau_1) w(t-\tau_1) e^{-j2\pi f\tau_1} d\tau_1 \int_{-\infty}^{\infty} x^*(\tau_2) w(t-\tau_2) e^{j2\pi f\tau_2} d\tau_2$$

$$= X(t, f) X^*(t, f)$$

$$= |X(t, f)|^2$$

V-G Advantages and Disadvantages of the WDF

Advantages: clarity

many good properties

suitable for analyzing the random process

Disadvantages: cross-term problem

not suitable for $\exp(jt^n)$, $n \neq 0, 1, 2$

more time for computation, especial for the signal with long time duration

not one-to-one

V-H Windowed Wigner Distribution Function

When $x(t)$ is not time-limited, its WDF is hard for implementation

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

↓
with mask

$$W_x(t, f) = \int_{-\infty}^{\infty} w(\tau) x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

$w(\tau)$ is real and time-limited

The windowed WDF is also called the pseudo Wigner-Ville distribution.

Advantages: (1) reduce the computation time

(2) may reduce the cross term problem

Disadvantages:

$$W_x(t, f) = 2 \int_{-\infty}^{\infty} w(2\tau') x(t + \tau') \cdot x^*(t - \tau') e^{-j4\pi\tau' f} \cdot d\tau'$$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-\infty}^{\infty} w(2p\Delta_t) x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j4\pi mp\Delta_t \Delta_f} \Delta_t$$

Suppose that $w(t) = 0$ for $|t| > B$

$$w(2p\Delta_t) = 0 \quad \text{for } p < -Q \text{ and } p > Q$$

$$Q = \frac{B}{2\Delta_t}$$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-Q}^{Q} w(2p\Delta_t) x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j4\pi mp\Delta_t \Delta_f} \Delta_t$$

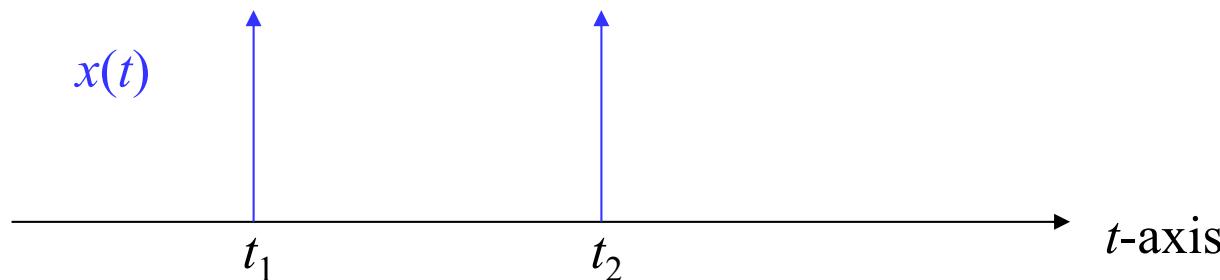
當然，乘上 mask 之後，有一些數學性質將會消失

(B) Why the cross term problem can be avoided ?

$$W_x(t, f) = \int_{-\infty}^{\infty} w(\tau) x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

$w(\tau)$ is real

Viewing the case where $x(t) = \delta(t - t_1) + \delta(t - t_2)$

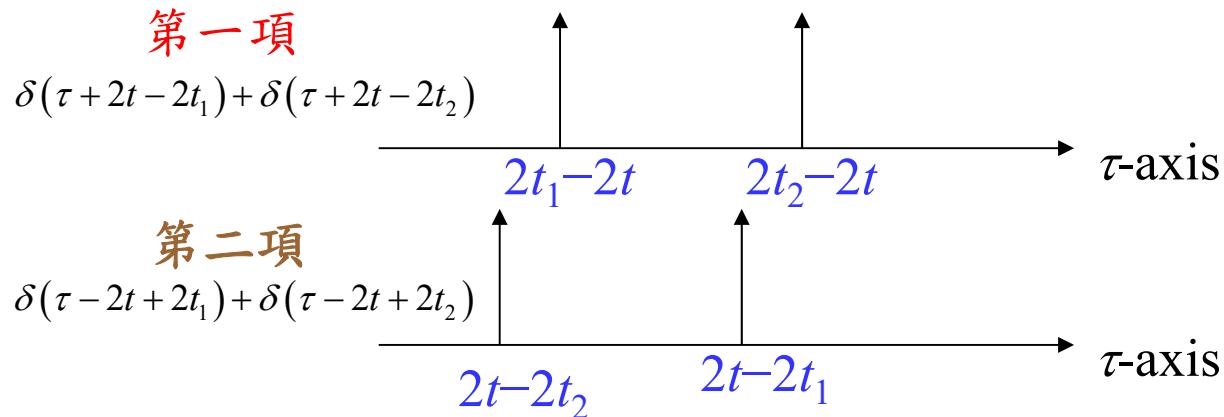


理想情形： $W_x(t, f) = 0$ for $t \neq t_1, t_2$

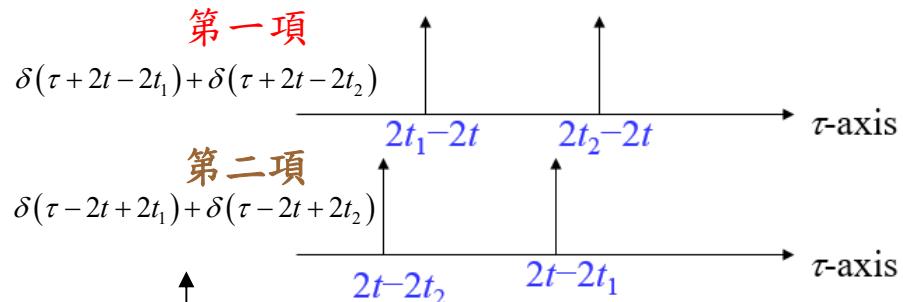
然而，當 mask function $w(\tau) = 1$ 時 (也就是沒有使用 mask function)

$$x(t) = \delta(t - t_1) + \delta(t - t_2)$$

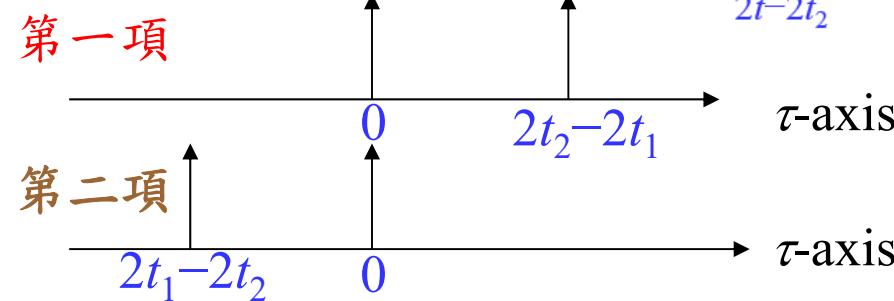
$$\begin{aligned} W_x(t, f) &= \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi\tau f} \cdot d\tau \\ &= \int_{-\infty}^{\infty} \left[\delta\left(t + \frac{\tau}{2} - t_1\right) + \delta\left(t + \frac{\tau}{2} - t_2\right) \right] \left[\delta\left(t - \frac{\tau}{2} - t_1\right) + \delta\left(t - \frac{\tau}{2} - t_2\right) \right] e^{-j2\pi\tau f} \cdot d\tau \\ &\stackrel{\text{from page 139, property 2}}{=} 4 \int_{-\infty}^{\infty} \underbrace{[\delta(\tau + 2t - 2t_1) + \delta(\tau + 2t - 2t_2)]}_{\text{第一項}} \underbrace{[\delta(\tau - 2t + 2t_1) + \delta(\tau - 2t + 2t_2)]}_{\text{第二項}} e^{-j2\pi\tau f} \cdot d\tau \end{aligned}$$



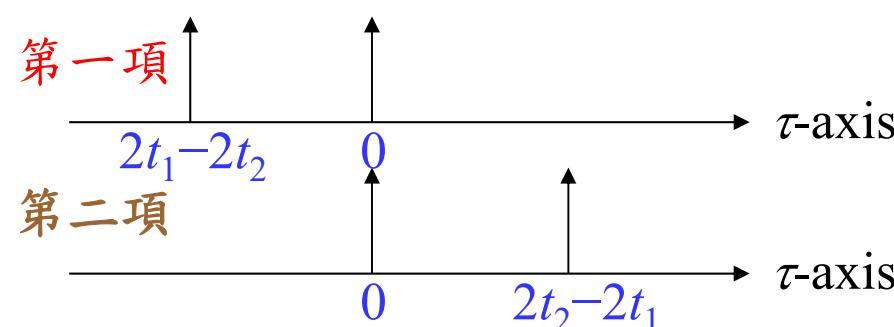
3種情形 $W_x(t, f) \neq 0$



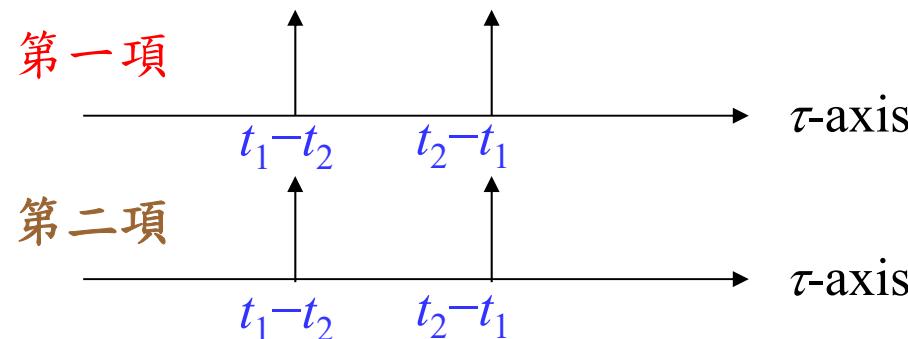
(1) If $t = t_1$



(2) If $t = t_2$



(3) If $t = (t_1 + t_2)/2$

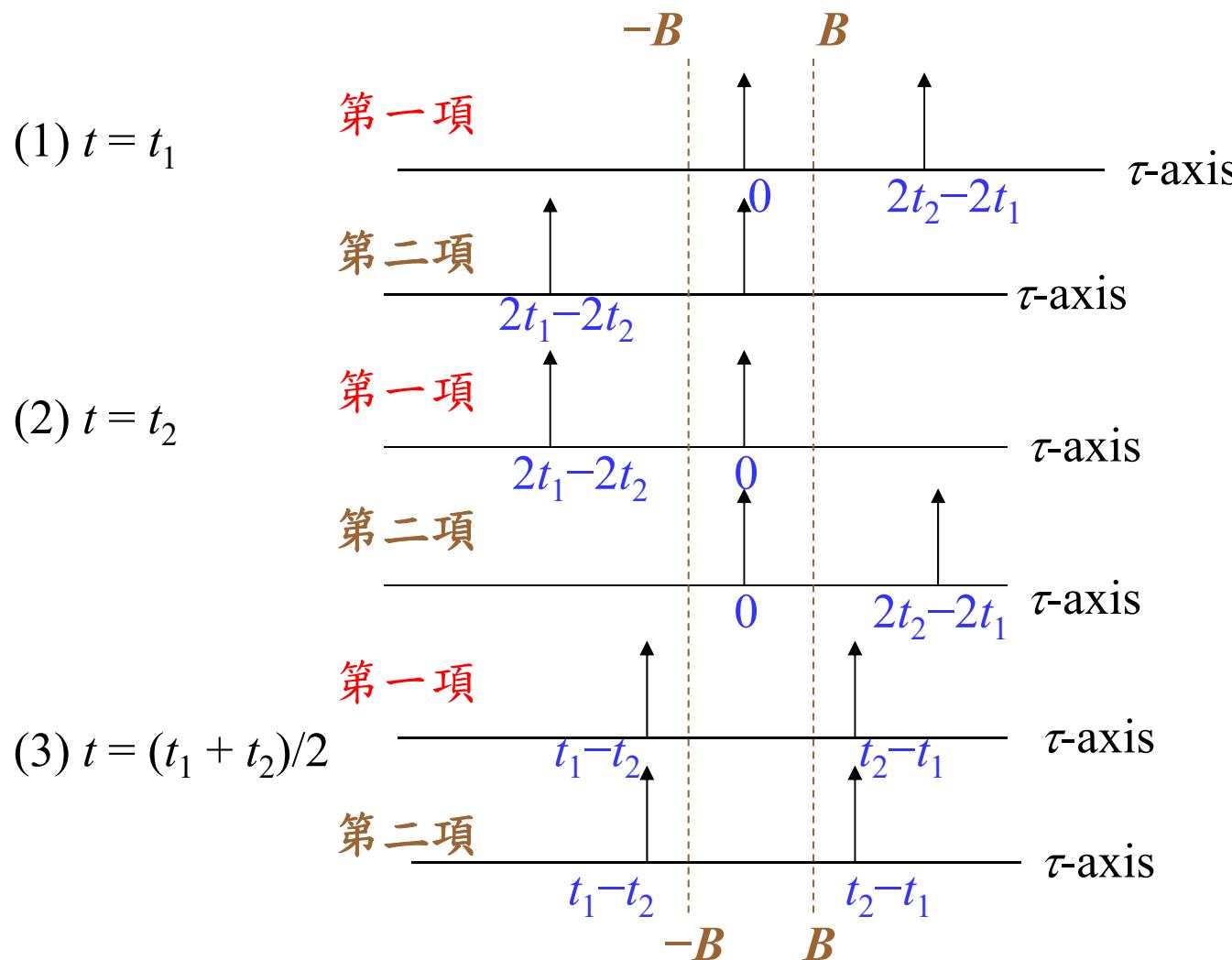


With mask function

$$\begin{aligned}
 W_x(t, f) &= \int_{-\infty}^{\infty} w(\tau) x(t + \tau/2) x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau \\
 &= \int_{-\infty}^{\infty} w(\tau) [\delta(\tau + 2t - 2t_1) + \delta(\tau + 2t - 2t_2)] \\
 &\quad \times [\delta(\tau - 2t + 2t_1) + \delta(\tau - 2t + 2t_2)] e^{-j2\pi\tau f} \cdot d\tau
 \end{aligned}$$

Suppose that $w(\tau) = 0$ for $|\tau| > B$, B is positive.

If $B < t_2 - t_1$



附錄八：研究所學習新知識把握的要點

- (1) Concepts: 這個方法的核心概念、基本精神是什麼
- (2) Comparison: 這方法和其他方法之間，有什麼相同的地方？
有什麼相異的地方
- (3) Advantages: 這方法的優點是什麼
(3-1) Why? 造成這些優點的原因是什麼
- (4) Disadvantages: 這方法的缺點是什麼
(4-1) Why? 造成這些缺點的原因是什麼
- (5) Applications: 這個方法要用來處理什麼問題，有什麼應用
- (6) Innovations: 這方法有什麼可以改進的地方
或是可以推廣到什麼地方

看過一篇論文或一個章節之後，若能夠回答(1)-(5)的問題，就代表你已經學通了這個方法

如果你的目標是發明創造出新的方法，可試著回答(3-1), (4-1), 和(6)的問題

每個領域每個月至少都有100篇以上的新論文，閱讀能力再厲害的人，也不可能都像大學讀書一樣篇篇都逐字讀過，所以，要選讀哪幾篇論文，哪些要詳讀，哪些可以只抓重點，這種擇要的能力，是研究所學生們應該練習的。