Estimating a relative change using a log-transformation of the outcome

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December 19, 2018

1 Result

Let's denote by Y the outcome and by G a group variable G (binary variable). We are interested in the relative change in Y between the groups. We decide to model the group effect on the log scale:

$$\log(Y) = Z = \alpha + \beta G + \varepsilon \text{ where } \varepsilon \sim \mathcal{N}\left(0, \sigma^2\right)$$

We claim that:

$$\frac{\mathbb{E}\left[Y|G=1\right] - \mathbb{E}\left[Y|G=0\right]}{\mathbb{E}\left[Y|G=0\right]} = e^{\beta} - 1$$

2 Proof

2.1 Re-writting the model as a multiplicative model

We can re-write the model as:

$$Y = e^{\alpha+\beta G} e^{\varepsilon} \text{ where } \varepsilon \sim \mathcal{N}\left(0, \sigma^2\right)$$

So for $g \in \{1, 2\}$:

$$\mathbb{E}\left[Y|G=g\right] = e^{\alpha+\beta g}\mathbb{E}\left[e^{\varepsilon}\right]$$

Then:

$$\begin{aligned}\frac{\mathbb{E}\left[Y|G=1\right] - \mathbb{E}\left[Y|G=0\right]}{\mathbb{E}\left[Y|G=0\right]} &= \frac{e^{\alpha+\beta}\mathbb{E}\left[e^{\varepsilon}\right] - e^{\alpha}\mathbb{E}\left[e^{\varepsilon}\right]}{e^{\alpha}\mathbb{E}\left[e^{\varepsilon}\right]} \\ &= \frac{e^{\alpha+\beta} - e^{\alpha}}{e^{\alpha}} = e^{\beta} - 1\end{aligned}$$

2.2 Using a Taylor expansion

Using a second order Taylor expansion of $\exp(Z)$ around $\mu(G) = \alpha + \beta G$ and assuming that the first moments of Z are finite and the remaining moments are neglectable regarding the factorial of the moment order (i.e. $\forall i \geq 1$, $\frac{1}{i!}\mathbb{E}\left[\varepsilon^i\right] < +\infty$ and $\sum_{i=1}^{\infty} \frac{1}{i!}\mathbb{E}\left[\varepsilon^i\right] < +\infty$), we get:

$$
\begin{aligned}
Y = e^Z &= e^\mu + \sum_{i=1}^{\infty} \frac{1}{i!}(Z-\mu)^i \frac{\partial^i e^\mu}{(\partial\mu)^i} \\
&= e^{\alpha+\beta G} + \sum_{i=1}^{\infty} \frac{1}{i!}(Z-\alpha-\beta G)^i e^{\alpha+\beta G} \\
\mathbb{E}\left[Y|G=g\right] &= e^{\alpha+\beta G} + \sum_{i=1}^{\infty} \frac{1}{i!}\mathbb{E}\left[(Z-\alpha-\beta g)^i\right] e^{\alpha+\beta G} \\
&= e^{\alpha+\beta G}\left(1 + \sum_{i=1}^{\infty} \frac{1}{i!}\mathbb{E}\left[\varepsilon^i\right]\right)
\end{aligned}
$$

where we used that the distribution of ε is independent of g. [Optional] ε follows a zero-mean normal distribution, so the uneven moments are 0:

$$
\mathbb{E}\left[Y|G=g\right] = e^{\alpha+\beta G}\left(1 + \sum_{i=1}^{\infty} \frac{1}{2i!}\mathbb{E}\left[\varepsilon^{2i}\right]\right)
$$

We can now express our parameter of interest:

$$
\begin{aligned}
\Delta_G &= \frac{\mathbb{E}\left[Y|G=1\right] - \mathbb{E}\left[Y|G=0\right]}{\mathbb{E}\left[Y|G=0\right]} = \frac{\mathbb{E}\left[Y|G=1\right]}{\mathbb{E}\left[Y|G=0\right]} - 1 \\
&= \frac{e^{\alpha+\beta}\left(1 + \sum_{i=1}^{\infty} \frac{1}{2i!}\mathbb{E}\left[\varepsilon^{2i}\right]\right)}{e^{\alpha}\left(1 + \sum_{i=1}^{\infty} \frac{1}{2i!}\mathbb{E}\left[\varepsilon^{2i}\right]\right)} - 1 \\
&= e^\beta - 1
\end{aligned}
$$