

Estimating a relative change using a log-transformation of the outcome

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1 Theory

Let's denote by Y the outcome and by G a group variable G (binary variable). We are interested in the relative change in Y between the groups. We decide to model the group effect on the log scale:

$$\log(Y) = Z = \alpha + \beta G + \varepsilon \text{ where } \varepsilon \sim \mathcal{N}(0, \sigma^2)$$

Using a second order Taylor expansion of $\exp(Z)$ around $\mu(G) = \alpha + \beta G$ and assuming that the first moments of Z are finite and the remaining moments are neglectable regarding the factorial of the moment order (i.e. $\forall i \geq 1, \frac{1}{i!} \mathbb{E}[\varepsilon^i] < +\infty$ and $\sum_{i=1}^{\infty} \frac{1}{i!} \mathbb{E}[\varepsilon^i] < +\infty$), we get:

$$\begin{aligned} Y &= e^Z = e^\mu + \sum_{i=1}^{\infty} \frac{1}{i!} (Z - \mu)^i \frac{\partial^i e^\mu}{(\partial \mu)^i} \\ &= e^{\alpha + \beta G} + \sum_{i=1}^{\infty} \frac{1}{i!} (Z - \alpha - \beta G)^i e^{\alpha + \beta G} \\ \mathbb{E}[Y|G = g] &= e^{\alpha + \beta G} + \sum_{i=1}^{\infty} \frac{1}{i!} \mathbb{E}[(Z - \alpha - \beta g)^i] e^{\alpha + \beta G} \\ &= e^{\alpha + \beta G} \left(1 + \sum_{i=1}^{\infty} \frac{1}{i!} \mathbb{E}[\varepsilon^i] \right) \end{aligned}$$

where we used that the distribution of ε is independent of g . [Optional] ε follows a zero-mean normal distribution, so the uneven moments are 0:

$$\mathbb{E}[Y|G = g] = e^{\alpha + \beta G} \left(1 + \sum_{i=1}^{\infty} \frac{1}{2i!} \mathbb{E}[\varepsilon^{2i}] \right)$$

We can now express our parameter of interest:

$$\begin{aligned}
\Delta_G &= \frac{\mathbb{E}[Y|G=1] - \mathbb{E}[Y|G=0]}{\mathbb{E}[Y|G=0]} = \frac{\mathbb{E}[Y|G=1]}{\mathbb{E}[Y|G=0]} - 1 \\
&= \frac{e^{\alpha+\beta} \left(1 + \sum_{i=1}^{\infty} \frac{1}{2i!} \mathbb{E}[\varepsilon^{2i}]\right)}{e^{\alpha} \left(1 + \sum_{i=1}^{\infty} \frac{1}{2i!} \mathbb{E}[\varepsilon^{2i}]\right)} - 1 \\
&= e^{\beta} - 1
\end{aligned}$$