Breakpoint model

1 No covariate, no plateau

Consider:

- a response variable Y

- a explanatory variable X taking values in the interval $[x_{\min}; x_{\max}]$.

Solving $a+bx_{\min} = -1$ and $a+bx_{\max} = -1$ leads to $b = \frac{2}{x_{\max}-x_{\min}}$ and $a = -\frac{x_{\max}+x_{\min}}{x_{\max}-x_{\min}}$. Therefore the transformation $f(.) := \tanh^{-1}(a + b.)$ moves from the constrained to the unconstrained scaled. We will thus consider the following broken line model for the mean:

$$
\begin{aligned}
\mathbb{E}\left[Y\right] &= \alpha + \beta_1 X + \beta_2 (X - \psi)_+ \\
&= \alpha + \beta_1 X + \beta_2 (X - \tanh^{-1}(a + b\phi))_+
\end{aligned}
$$

We have that at the true breakpoint value ψ_0 and for $X \neq \psi_0$

$$
(X - \psi)_+ = f(\psi) = f(\psi_0) + (\psi - \psi_0)f'(\psi_0) = (X - \psi_0)_+ - (\psi - \psi_0)\mathbb{1}_{X > \psi_0}
$$

Now with transformation:

$$
(X - \tanh^{-1}(a + b\phi))_+ = f(\phi) = f(\phi_0) + (\phi - \phi_0)f'(\phi_0) = (X - \psi_0)_+ - (\phi - \phi_0)\frac{b\mathbb{1}_{X > \psi_0}}{1 - \psi_0^2}
$$