Uniform and Isotropic Distribution

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1 Coordinate system

To simulate a uniform and isotropic distribution at detector in space, we use a sphere of radius R_{max} to cover all detectors. R_{max} is the maximum value of impact parameter (R_p) and should be greater than the maximum distance from coordinate center, which could be the center of detector or any specific points. This sphere is called "detector sensitive region (DSR) from now on.

The requirement of "uniform" means uniform probability of impact points on any plane slicing through the DSR. In a 3-D coordinate, this condition can be parameterized as

$$\frac{dN}{dv} = \text{const} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Where dN is number of events simulated at a volume of dv.

The requirement of "isotropic" means uniform probability of unit vectors of velocity of any events.

$$\frac{dN}{d\Omega} = \text{const} \quad \ldots \text{(2)},$$

where $d\Omega$ is the solid angle.

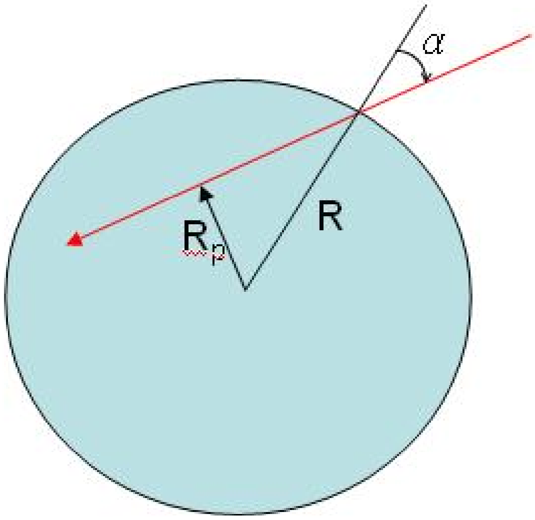

Figure 1: Definition of detector sensitive region (DSR) and impact parameter (R_p).

2 Correct Algorithm

This section describes a typical procedure to **"uniform and isotropic"** distribution, which I used for a long time. I will use spherical coordinate system (r, θ, ϕ) primary and Cartesian coordinate system (x, y, z) for coordinate transformation. Four random probability, P_i, i=1,4, $0 \le P_i \le 1$, are used to generate uniform and isotropic distribution in this Monte-Carlo simulation.

2.1 Isotropic distribution

The first step is sampling velocity vector from all phase space. Assuming all sky coverage, zenith angle covers from 0 to π and azimuth angle covers from 0 to 2π. The unit solid angle $d\Omega = \sin\theta d\theta d\phi$ or equivalent to $d(\cos\theta)d\phi$.

$$\begin{aligned} \cos\theta &= 2P_1 - 1 \\ \phi &= P_2 \times 2\pi \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)},$$

Notice the cos θ is sampled form -1 to 1, this algorithm cover 4π sr. This method guarantees even distribution in a projection of cosθ vs. ϕ plot or equal-area projection such as Mollweide projection and Aitoff projection, both are widely used in geosciences and astronomy.

A common error of this sampling is sampling θ by $P_1 \times \pi$. In this way, it appears to be even density in a plot of θ vs. ϕ. However, this is not an equal area projection. This mistakes results in over-sampling (higher probability density) at higher latitude or θ near 0 or π.

When you need to focus on events from certain direction, you may specify maximum and minimum value of cosθ and ϕ, then simulate by

$$\begin{aligned} \cos\theta &= \cos\theta_{\min} + P_1 \times (\cos\theta_{\max} - \cos\theta_{\min}) \\ \phi &= \phi_{\min} + P_2 \times (\phi_{\max} - \phi_{\min}) \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4),$$

An alternative method is sampling of dθ and sinθdϕ. It requires two stages of simulations. First equal probability in sampling of θ

$$\theta = \theta_{\min} + P_1 \times (\theta_{\max} - \theta_{\min}), \quad \frac{dN}{d\theta} = c \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The second stage of sampling of ϕ requires to scale to probability by a factor of sinθ.

$$\phi = \phi_{\min} + P_2 \times (\phi_{\max} - \phi_{\min}), \quad \frac{dN}{d\phi} = c\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

This method is not convenient in a event-by-event simulation, it is used in a grid-like simulation, which separate phase space in many pixels of angular size (dθ, dϕ).

After sampling of one direction (*θ, ϕ*), the velocity unit vector can be expressed in (1*, θ, ϕ*) or transform to Cartesian coordinate system

$$\vec{V} = (\sin\theta\cos\phi, \quad \sin\theta\sin\phi, \quad \cos\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

2.2 Uniform distribution

The second step is sampling of an "impact point" at some plane. To have even probability density, we select a plane perpendicular to the trajectory of particle. The cross-section of DSR and this normal plane become a circle. The informality becomes a simple 2-dimensional distribution on this circle. We define the intersection point of event track and this circle as "Minimum Impact Parameter Point (MIPP)" and expressed it's location by a polar coordinate (R_p, ω)

$$\begin{aligned} R_p &= R_{\max} \times \sqrt{P_3} \\ \omega &= 2\pi \times P_4 \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

The first equation comes from uniform assumption, which requires number of events per unit area to be the same.

$$\frac{\text{Area}}{\text{Prob}} = \frac{\pi R_{\max}^2}{1} = \frac{\pi r^2}{P} \longrightarrow r = R_{\max}\sqrt{P}$$

The angle ω is a phase angle on the circle.

Then a rotational matrix transforms MIPP vector back to the center of DSR. This rotation rotate default (DSR) coordinate system so the particle trajectory becomes new Z axis and plane of (R_{p}, ω) coincide with new X-Y plane. First rotate around Z-axis by azimuth angle ϕ **counterclockwise**. Then rotate around new Y' axis by zenith angle θ **clockwise**. Those two rotational matrixes are:

$$T_1 = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

The combined matrix $T=T_2T_1$ will transform the particle velocity to Z" axis of new coordinate, i.e. $V'' = (0,0,1)$.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

When MIPP are determined to be (R_{p}, ω) in the new coordinate is

$$\vec{R}_p'' = \left(R_p \cos\omega, \quad R_p \sin\omega, \quad 0 \right)$$

To transform it back to the default coordinate, just use inverse matrix T^{-1}.

$$\vec{R}_p = T^{-1}\vec{R}_p'' = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} R_p\cos\omega \\ R_p\sin\omega \\ 0 \end{pmatrix} \quad \text{..........................(9)}$$

Rp vector is the location of MIPP relative to coordinate center. Then use this vector (eq. 9) and velocity vector (eq. 7) to calculate the particle trajectory.