INDIAN INSTITUTE OF TECHNOLOGY GANDHINAGAR

Project Report

Numerical Analysis of Quasi-One-Dimensional Nozzle Flow

Submitted by

Aditya Prasad - 22110018
Bhavik Patel - 22110047
Jinil Patel - 22110184
Pranav Patil - 22110199
Sumeet Sawale - 22110234
Indian Institute of Technology
Gandhinagar, Gujrat

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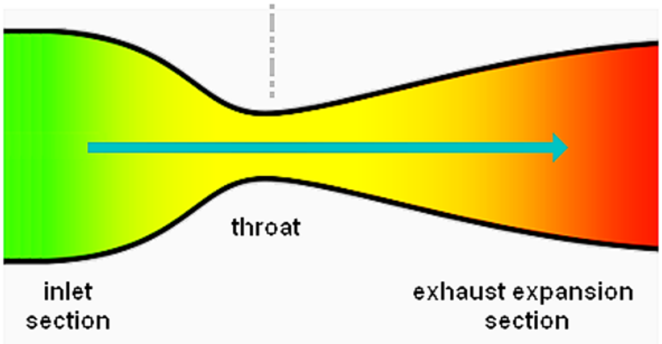
1 Problem Description

Fluid flow through nozzles is a fundamental concept in fluid dynamics and engineering. It plays a crucial role in various applications, such as aerospace engineering (rocket propulsion and aircraft design), automotive engineering (combustion engines), and industrial processes (spray nozzles and chemical reactors). Understanding the behaviour of fluid flow through a convergent-divergent nozzle is of particular importance due to its widespread applications.



The Convergent-Divergent nozzle is also known as Super-Sonic Nozzle. The most important property or the application of the Convergent-Divergent Nozzle is to get supersonic flow at the output.

At the inlet of the nozzle, the flow comes from a reservoir with a very low velocity, relatively it is so low that it is considered around 0. Now the flow enters the nozzle and travels through the first section of the nozzle which is the convergent section. The Cross-section area gradually decreases along the flow direction. The pressure of the fluid decreases with the decrease in the area as Velocity increases. With the Gradual decrease in the cross-section area, the area reaches a critical area where the Velocity becomes sonic, that is at the throat section. At the throat, the velocity of the fluid reaches the local speed of sound, and after that, the velocity of the fluid increases with the increase in area. After the throat, there is a divergent section where the cross-sectional area gradually increases toward the direction of the flow and the velocity of the fluid goes Super-Sonic.

2 Physical Model

This project will consider the steady, isentropic flow through a convergent-divergent nozzle. The flow at the inlet to the nozzle from a reservoir where the pressure and temperature are denoted by p_0, and T_0, respectively. The cross-sectional area of the reservoir is large (theoretically, ∞), and hence the velocity is very small (V_0). Thus, p_0 and T_0 are the stagnation values, or total pressure and total temperature, respectively. The flow expands isentropically to supersonic speeds at the nozzle exit, where the exit pressure, temperature, velocity, and Mach number are denoted by p_e, T_e, V_e and M_e, respectively.

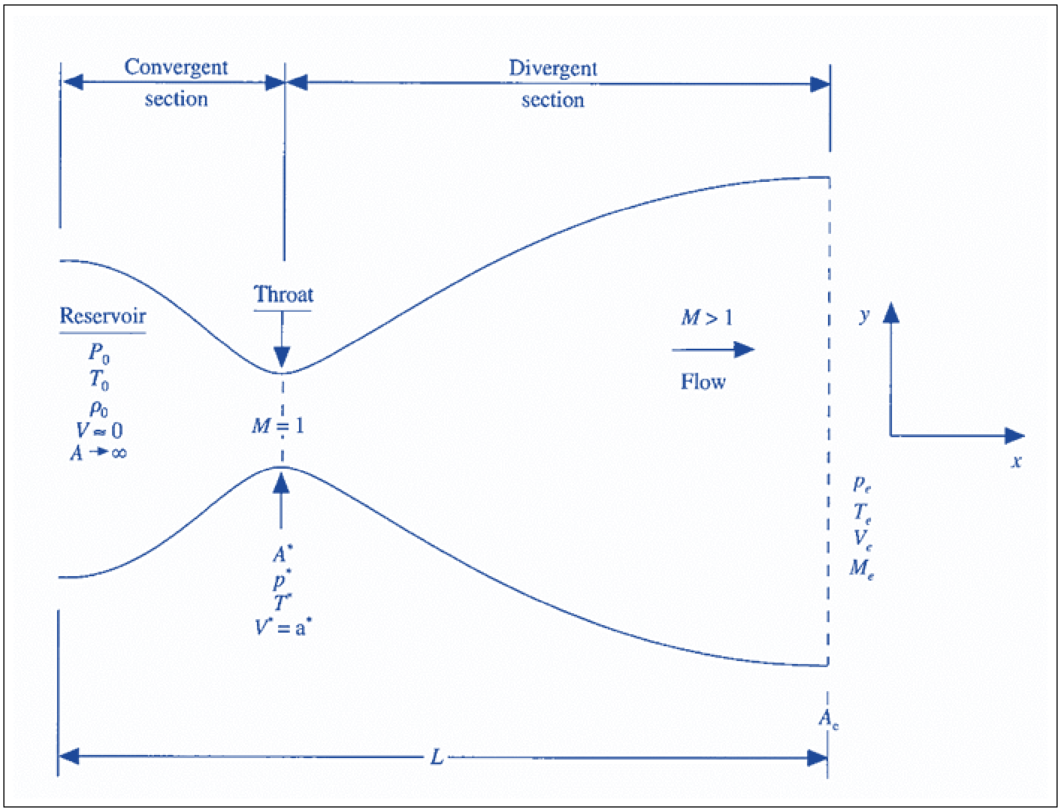


Figure 1: Schematic for subsonic-supersonic isentropic nozzle flow.

The flow is locally subsonic in the convergent section of the nozzle, sonic at the throat (minimum area), and supersonic at the divergent section. The sonic flow ($M = 1$) at the throat means that the local velocity at this location is equal to the local speed of the sound. Using an asterisk to denote sonic flow values, we have at the throat $V = V^* = a^*$. Similarly, the sonic flow values of the pressure and temperature are denoted by p^* and T^*, respectively. The area of the sonic throat is denoted by A*. We assume that at a section of cross-sectional area A, the flow properties are uniform across that section.Even though in reality the flow field is two-dimensional, we make the assumption that the flow properties vary only with x. Such flow is defined as quasi-one-dimensional flow.

3 Assumptions

- **One-Dimensionality:** Even though the flow field is not constant at a cross-section, the flow field is assumed to vary only in one direction (e.g., along the x-axis), making it quasi-one-dimensional. This simplification allows the problem to be treated as a function of a single spatial variable.
- **Steady State:** The analysis assumes that the flow parameters do not change with time, i.e., it's in a steady-state condition. Unsteady effects are not considered.
- **Isentropic Flow:** The flow is assumed to be isentropic, i.e. it is adiabatic and reversible. This assumption implies that there is no heat transfer or friction loss within the flow.
- **No Viscous Effects:** Viscous effects within the fluid, such as boundary layer development or shear stresses, are neglected. The flow is considered inviscid(zero viscosity).
- **Perfect Gas:** The fluid is treated as an ideal gas, with constant specific heat ratios (γ) with temperature. This assumption simplifies the equations of state.
- **Negligible Body forces:** The influence of body forces (e.g., magnetic or electromagnetic forces) on the flow is considered negligible unless specifically relevant to the problem.
- **No Chemical Reactions:** Chemical reactions within the fluid are not considered.
- **Quasi-One-Dimensional Nozzle:** The geometry of the nozzle is assumed to be quasi-one-dimensional, which means that variations in the cross-sectional area occur primarily in one direction (typically along the nozzle length).
- **Inflow and Outflow Assumptions:** Specific boundary conditions are applied at the inflow and outflow boundaries, as discussed earlier, considering the interaction of flow characteristics with boundaries.
- **Conservation Laws:** Conservation laws for mass, momentum, and energy are applied to the flow field, forming the basis of the governing equations.

4 Governing Equations

4.1 Ideal Gas Equation

$$p = \rho RT \tag{1.1}$$

Taking its derivative with respect to position (x).

$$\frac{\partial \rho}{\partial x} = R\left(\rho\frac{\partial T}{\partial x} + T\frac{\partial \rho}{\partial x}\right) \tag{1.2}$$

4.2 Equation of Continuity

$$\frac{\partial(pA)}{\partial t} + \rho A\frac{\partial V}{\partial x} + \rho V\frac{\partial A}{\partial x} + VA\frac{\partial \rho}{\partial x} = 0 \tag{2}$$

4.3 Conservation of Momentum

$$\rho\frac{\partial V}{\partial t} + \rho V\frac{\partial V}{\partial x} + \frac{\partial \rho}{\partial x} = 0 \tag{3.1}$$

Substituting Eq. (1.2) in Eq (3.1) we get,

$$\rho\frac{\partial V}{\partial t} + \rho V\frac{\partial V}{\partial x} = -R\left(\rho\frac{\partial T}{\partial x} + T\frac{\partial \rho}{\partial x}\right) \tag{3.2}$$

4.4 Conservation of Energy

$$\rho C_V\frac{\partial T}{\partial t} + \rho V C_V A\frac{\partial T}{\partial x} = -\rho RT\left[\frac{\partial V}{\partial x} + V\frac{\partial(lnA)}{\partial x}\right] \tag{4}$$

5 Non-Dimenstionalization

We will convert flow-field variables in terms of nondimensional variables, where the flow variables are referenced to their reservoir values.

The reservoir temperature and density are denoted by T_0 and ρ_0, respectively. So, we define dimensionless temperature and density as

$$T' = \frac{T}{T_0}$$

$$\rho' = \frac{\rho}{\rho_0}$$

Let L be the total length of the nozzle; we define dimensionless length as

$$x' = \frac{x}{L}$$

Denoting the speed of the sound in the reservoir as a_0,where

$$a_0 = \sqrt{\gamma R T_0}$$

we define a dimensionless velocity as,

$$V' = \frac{V}{a_0}$$

and dimensionless time as

$$t' = \frac{t}{L/a_0}$$

Finally, we ratio the local area A to the sonic throat area A^* and define dimensionless area as,

$$A' = \frac{A}{A^*}$$

5.1 Governing Equations in the form of non-dimensional variables

Returning the Eq.(2) and introducing non-dimensional variables, we have

$$\frac{\partial \rho' A'}{\partial t'}\left(\frac{\rho_0 A^*}{L/a_0}\right)+\rho' A'\frac{\partial V'}{\partial x'}\left(\frac{\rho_0 A^* a_0}{L}\right)+\rho' V'\frac{\partial A'}{\partial x'}\left(\frac{\rho_0 A^* a_0}{L}\right)+V' A'\frac{\partial \rho'}{\partial x'}\left(\frac{\rho_0 A^* a_0}{L}\right) = 0 \tag{5.1}$$

As we know, area A' is the function of x only, and it does not vary with time.

$$\therefore \frac{\partial \rho' A'}{\partial t'} = A' \frac{\partial \rho'}{\partial t'} \tag{5.2}$$

Using Eq.(5.2), we will get the non-dimensional form of the equation of continuity as

$$Continuity : \boxed{\frac{\partial \rho'}{\partial t'} = -\rho' \frac{\partial V'}{\partial x'} - \rho' V' \frac{\partial (lnA')}{\partial x'} - V' \frac{\partial \rho'}{\partial x'}} \tag{5.3}$$

Now, returning to Eq.(3.2), substituting non-dimensional variables, we have the equation of conservation of momentum as

$$\rho' \frac{\partial V'}{\partial t'} \left(\frac{\rho_0 a_0}{L/a_0} \right) + \rho' V' \frac{\partial V'}{\partial x'} \left(\frac{\rho_0 a_0^2}{L} \right) = -R \left(\rho' \frac{\partial T'}{\partial x'} + T' \frac{\partial \rho'}{\partial x'} \right) \left(\frac{\rho_0 T_0}{L} \right) \tag{6.1}$$

Note that,

$$\frac{RT_0}{a_0^2} = \frac{\gamma R T_0}{\gamma a_0^2} = \frac{a_0^2}{\gamma a_0^2} = \frac{1}{\gamma}$$

Hence, Eq.(6.1) becomes,

$$Momentum : \boxed{\frac{\partial V'}{\partial t'} = -V' \frac{\partial V'}{\partial x'} - \frac{1}{\gamma} \left(\frac{\partial t'}{\partial x'} + \frac{T'}{\rho'} \frac{\partial \rho'}{\partial x'} \right)} \tag{6.2}$$

Now, introducing non-dimensional variables to Eq.(4),

$$\rho' C_V \frac{\partial T'}{\partial t'} \left(\frac{\rho_0 T_0}{L/a_0} \right) + \rho' V' C_V \frac{\partial T'}{\partial x'} \left(\frac{\rho_0 a_0 T_0}{L} \right) = -\rho' R T' \left[\frac{\partial V'}{\partial x'} + V' \frac{\partial (lnA')}{\partial x'} \right] \times \left(\frac{\rho_0 T_0 a_0}{L} \right) \tag{7.1}$$

In Eq.(7.1), the factor R/C_V is given by

$$\frac{R}{C_V} = \frac{R}{R/(\gamma - 1)} = \gamma - 1$$

Hence, Eq.(7.1) becomes

$$Energy : \boxed{\frac{\partial T'}{\partial t'} = -V' \frac{\partial T'}{\partial x'} - (\gamma - 1) T' \left(\frac{\partial V'}{\partial x'} + V' \frac{\partial (lnA')}{\partial x'} \right)} \tag{7.2}$$

6 Numerical Solution

6.1 Grid Discretization

To implement a finite-difference solution, we divide the Nozzle length into a number of discrete grid points along the X-direction, as shown in the following. We have assumed Quasi 1-D Nozzle Flow, which means the Flow Variables values at the center of a nozzle cross-section will be the same and uniform at all cross-section points, whether the point is at the center or along the wall. This assumption of Quasi-1D and inviscid flow reduces the complexity of the grid generation, as we can study the flow by creating a grid at the center/axis of the nozzle, rather than the complex profile of the Nozzle wall.

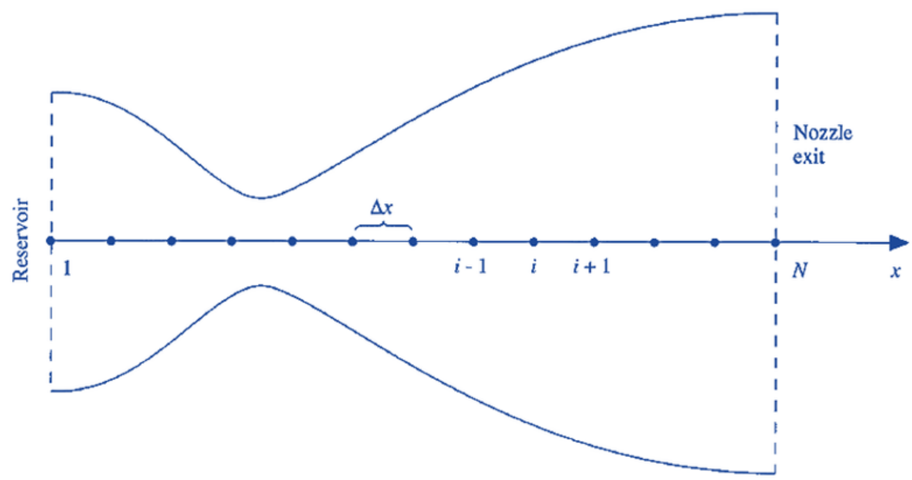


Figure 2: Grid point distribution along the nozzle.

The Grid points are evenly distributed along the x-axis, with Δx or dx denoting the spacing between the grid points.

6.2 MacCormack Technique

The MacCormack's technique is a predictor-corrector method. In the time-marching approach. we know the flow-field variables at the time t, and we use the definite difference equations to solve explicitly for the variables at time $t + \Delta t$.

The MacCormack method is as follows,

a) Predator Step

- Discretizing the governing equation using the forward difference method. We set the spatial derivatives as forward differences and calculate the time derivative terms.
- From the time derivatives, updating the values of primitive variables.

b) Corrector Step

- Discretizing the equations using the backward difference and from the predicted values of the flow variables from the predictor step.
- Computing the average of the time derivatives of governing equations solved by the predictor and corrector method.
- Updating the primitive variable terms and obtaining the final corrected values, by the average time derivatives.

In our scenario, we're dealing with three equations that describe the variations of variables V, T, and ρ with respect to both time and space. By maintaining a fixed spatial discretization, we can express these equations in terms of changes over time, denoted as Δt.

$$\boxed{\rho_i^{t+\Delta t} = \rho_i^t + \left(\frac{\partial \rho}{\partial t}\right)_{av} \Delta t} \tag{8.1}$$

In the above equation the average time derivative is given as,

$$\left(\frac{\partial \rho}{\partial t}\right)_{av} = 0.5 \left[\left(\frac{\partial \rho}{\partial t}\right)_i^t + \left(\frac{\overline{\partial \rho}}{\partial t}\right)_i^{t+\Delta t} \right] \tag{8.2}$$

where,

$$\left(\frac{\partial \rho}{\partial t}\right)_i^t = -\rho_i^t \frac{V_{i+1}^t - V_i^t}{\Delta x} - \rho_i^t V_i^t \frac{lnA_{i+1} - lnA_i}{\Delta x} - V_i \frac{\rho_{i+1}^t - \rho_i^t}{\Delta x} \tag{8.3}$$

And,

$$\begin{aligned} \left(\frac{\overline{\partial \rho}}{\partial t}\right)_i^{t+\Delta t} = & -\overline{\rho}_i^{t+\Delta t} \frac{\overline{V}_i^{t+\Delta t} - \overline{V}_{i-1}^{t+\Delta t}}{\Delta x} - \overline{\rho}_i^{t+\Delta t} \overline{V}_i^{t+\Delta t} \frac{lnA_i - lnA_i - 1}{\Delta x} \\ & - \overline{V}_i^{t+\Delta t} \frac{\overline{\rho}_i^{t+\Delta t} - \overline{\rho}_{i-1}^{t+\Delta t}}{\Delta x} \end{aligned} \tag{8.4}$$

Similarly, as ρ we can express V and T in terms of changes over time,

$$\boxed{V_i^{t+\Delta t} = V_i^t + \left(\frac{\partial V}{\partial t}\right)_{av} \Delta t} \tag{9.1}$$

Here,

$$\left(\frac{\partial V}{\partial t}\right)_{av} = 0.5\left[\left(\frac{\partial V}{\partial t}\right)_i^t + \left(\frac{\overline{\partial V}}{\partial t}\right)_i^{t+\Delta t}\right] \tag{9.2}$$

where,

$$\left(\frac{\partial V}{\partial t}\right)_i^t = -V_i^t \frac{V_{i+1}^t - V_i^t}{\Delta x} - \frac{1}{\gamma}\left(\frac{T_{i+1}^t - T_i^t}{\Delta x} + \frac{T_i^t}{\rho_i^t}\frac{\rho_{i+1}^t - \rho_i^t}{\Delta x}\right) \tag{9.3}$$

And,

$$\begin{aligned}\left(\frac{\overline{\partial V}}{\partial t}\right)_i^{t+\Delta t} &= -\overline{V}_i^{t+\Delta t}\frac{\overline{V}_i^{t+\Delta t} - \overline{V}_{i-1}^{t+\Delta t}}{\Delta x} \\ &\quad - \frac{1}{\gamma} \times \left(\frac{\overline{T}_i^{t+\Delta t} - \overline{T}_{i-1}^{t+\Delta t}}{\Delta x} + \frac{\overline{T}_i^{t+\Delta t}}{\overline{\rho}_i^{t+\Delta t}}\frac{\overline{\rho}_i^{t+\Delta t} - \overline{\rho}_{i-1}^{t+\Delta t}}{\Delta x}\right)\end{aligned} \tag{9.4}$$

Similarly, for T,

$$\boxed{T_i^{t+\Delta t} = T_i^t + \left(\frac{\partial T}{\partial t}\right)_{av} \Delta t} \tag{10.1}$$

Here,

$$\left(\frac{\partial T}{\partial t}\right)_{av} = 0.5\left[\left(\frac{\partial T}{\partial t}\right)_i^t + \left(\frac{\overline{\partial T}}{\partial t}\right)_i^{t+\Delta t}\right] \tag{10.2}$$

where,

$$\left(\frac{\partial T}{\partial t}\right)_i^t = -V_i^t \frac{T_{i+1}^t - T_i^t}{\Delta x} - (\gamma - 1)T_i^t\left(\frac{V_{i+1}^t - V_i^t}{\Delta x} + V_i^t \frac{lnA_{i+1} - lnA_i}{\Delta x}\right) \tag{10.3}$$

And,

$$\left(\frac{\overline{\partial T}}{\partial t}\right)_i^{t+\Delta t} = -\overline{V}_i^{t+\Delta t}\frac{\overline{T}_i^{t+\Delta t} - \overline{T}_{i-1}^{t+\Delta t}}{\Delta x} - (\gamma - 1) \times \overline{T}_i^{t+\Delta t}\left(\frac{\overline{V}_i^{t+\Delta t} - \overline{V}_{i-1}^{t+\Delta t}}{\Delta x} + \overline{V}_i^{t+\Delta t}\frac{lnA_i - lnA_{i-1}}{\Delta x}\right) \tag{10.4}$$

In the above equations $\overline{\rho}_i^{t+\Delta t}$, $\overline{V}_{i-1}^{t+\Delta t}$, and $\overline{T}_i^{t+\Delta t}$ are given by

$$\overline{\rho}_i^{t+\Delta t} = \rho_i^t + \left(\frac{\partial \rho}{\partial t}\right)_i^t \Delta t \tag{11.1}$$

$$\overline{V}_i^{t+\Delta t} = V_i^t + \left(\frac{\partial V}{\partial t}\right)_i^t \Delta t \tag{11.2}$$

$$\overline{T}_i^{t+\Delta t} = \rho_i^t + \left(\frac{\partial T}{\partial t}\right)_i^t \Delta t \tag{11.3}$$

6.3 Time Step Calculation

In numerical analysis, selecting an appropriate time step is crucial for obtaining accurate and stable results when solving differential equations.The stability of a numerical scheme is a fundamental consideration. It dictates the maximum allowable time step to prevent the solution from diverging.To achieve a desired level of accuracy, one must balance the trade-off between time step size and computational cost.

We can represent the round-off error in the following form:

$$\epsilon(x) = \sum_m A_m e^{ik_m x} \tag{12.1}$$

Equation (12.1) gives the spatial variation at a given time for an assessment of numerical stability, we are interested in the variation with time. Therefore, we extend Eq. (12.1) by assuming the amplitude A, is a function of time.

$$\epsilon(x,t) = \sum_m A_m(t) e^{ik_m x} \tag{12.2}$$

Moreover, it is reasonable to assume an exponential variation with time; errors tend to grow or diminish exponentially with time. Therefore, we write

$$\epsilon(x,t) = \sum_m e^{at} e^{ik_m x} \tag{12.3}$$

where a is a constant (which may take on different values for different m's). Equation (12.3) represents a final, reasonable form for the variation of round-off error in both space and time.

Let us deal with just one term of the series and write

$$\epsilon(x,t) = e^{at} e^{ik_m x} \tag{12.4}$$

The stability characteristics can be studied using just this form for ϵ with no loss in generality.

The exact form of the stability criterion depends on the form of the difference equation. For example, let us briefly examine the stability characteristics of another simple equation, this time a hyperbolic equation (we are considering hyperbolic equation as our model employs hyperbolic equations). Consider the first-order wave equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{12.5}$$

Let us replace the spatial derivative with a central difference

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \tag{12.6}$$

If we replace the time derivative with a simple forward difference, then the resulting difference equation representing Eq. (12.5) would be

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -c\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \tag{12.7}$$

let us replace the time derivative with a first-order difference, where $u(t)$ is represented by an average value between grid points $i+1$ and $i-1$, i.e.,

$$u(t) = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) \tag{12.8}$$

Then,

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - \frac{1}{2}(u_{i+1}^n + u_{i-1}^n)}{\Delta t} \tag{12.9}$$

Substituting Eqs. (12.6) and (12.9) into (12.5), we have

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - c\frac{\Delta t}{\Delta x}\frac{u_{i+1}^n - u_{i-1}^n}{2} \tag{12.10}$$

As derived above, assume error of the form

$$\epsilon(x,t) = e^{at}e^{ikt}$$

Substitute this form in eqn (12.10) to find the amplification factor i.e. e^{at}

$$e^{a(t+\Delta t)}e^{ikx} = \frac{e^{at}e^{ik(x+\Delta x)} + e^{at}e^{ik(x-\Delta x)}}{2} - C\left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2}\right)$$

$$\Rightarrow e^{a\Delta t} = \frac{1}{2}\left[(1-c)e^{ik\Delta x} + (1+c)e^{-ik\Delta x}\right]$$

On simplifying we get,

$$\Rightarrow e^{a+\Delta t} = cos(k\Delta x) - iCsin(k\Delta x)$$

For stability $|e^{\alpha\Delta t}| \leq 1$,

$$C = c\frac{\Delta t}{\Delta x} \leq 1$$

where, C is called Courant number. This equation says that, $\Delta t \leq \frac{\Delta x}{c}$ for numerical solution to be stable. Applying stability constraint on the given system we have,

$$\Delta t = C\frac{\Delta x}{a+V}$$

where, V is a local flow velocity at a point int the flow and a is the local speed of sound. Note that we have modified the denominator according to our model.

7 Boundary Condition

To analyze the boundary conditions, we consider the presence of two extremely weak Mach waves at the boundary points, both propagating at the speed of sound ($a_0 = \sqrt{\gamma R T_0}$). These waves are labeled as the 'Left-Running Characteristic' and the 'Right-Running Characteristic,' moving towards the left and right directions, respectively.

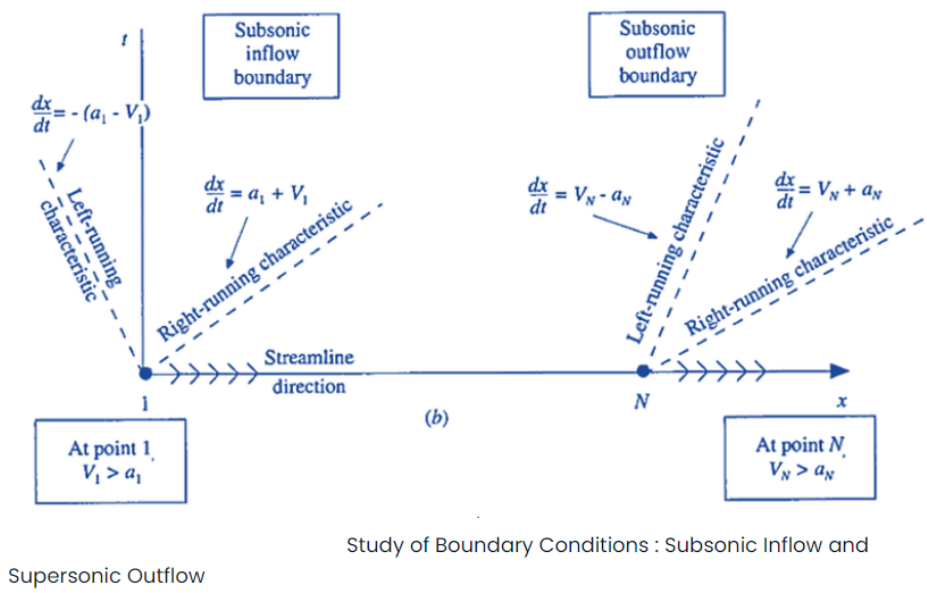


Figure 3: Grid point distribution along the nozzle.

Subsonic Inflow Boundary Condition:
Examining the figure, we observe that at the subsonic inlet boundary, the left-running characteristic moves towards the left, while the incoming flow from the inlet moves towards the right with a subsonic speed. The relative velocity of the left characteristic is given by -(a1 - V1), and for the right-running characteristic, it is (a1 + V1). Importantly, the left-running characteristic propagates outside the nozzle region, beyond the domain of influence, while the right-running characteristic remains inside the domain.

Consequently, at the subsonic inflow boundary, we must specify the values of two fixed flow-field variables that remain independent of time. In contrast, one other variable, typically the velocity, is allowed to vary with time. Estimating the velocity at the inlet boundary can be achieved through linear extrapolation based on velocities obtained at inner nodes during each time step.

Supersonic Outflow Boundary Condition:
Applying the same concept to the outflow boundary, we note that the fluid flow moves towards the right direction, similar to the inflow condition. However, now the flow speed is supersonic. Thus, the relative velocity for the left characteristic is $V_N - a_N$, and for the right characteristic, it is $V_N + a_N$, both primarily influenced by the fluid flow's speed. Consequently, both characteristics propagate towards the right direction, out of the domain.

Hence, at the supersonic outflow boundary, where both characteristics and the fluid stream exit the domain, there is no need to specify values for flow-field variables. Instead, all variables, including density, temperature, and velocity, are allowed to evolve naturally with time.

The Boundary conditions for Quasi 1-D inviscid, unsteady, Isentropic Subsonic-Supersonic Nozzle Flow -

Subsonic Boundary Inflow (M<1, Point=1)	Supersonic Boundary Outflow (M>1, Point=N)
$V_1 = 2V_2 - V_3$ $\rho_1 = 1$ $T_1 = 1$	$\rho'_N = 2\rho'_{N-1} - \rho'_{N-2}$ $V'_N = 2V'_{N-1} - V'_{N-2}$ $T'_N = 2T'_{N-1} - T'_{N-2}$

8 Initial Conditions

The nozzle shape, $A = A(x)$, is specified and held fixed, independent of time. For the case the parabolic area distribution given by

$$A = 1 + 2.2(x - 1.5)^2 \qquad 0 \leq x \leq 3$$

Note that $x = 1.5$ is the throat of the nozzle, that the convergent section occurs for $x < 1.5$ and that the divergent section occurs for $x > 1.5$. This nozzle shape is as shown in the following figure,

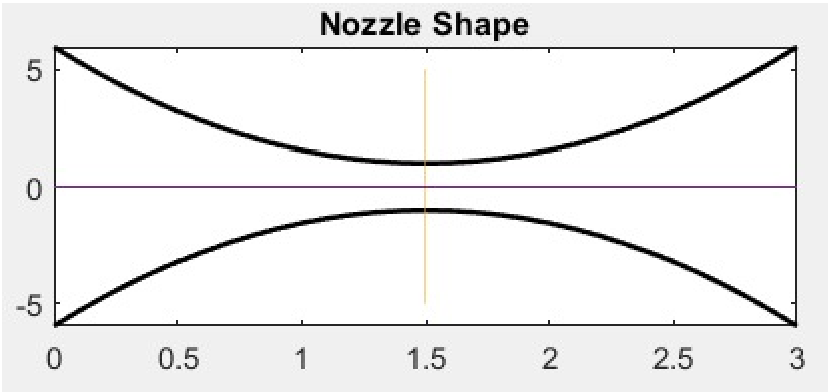


Figure 4: Non-dimensional distance along nozzle.

To start the simulation, we have to stipulate the initial conditions for the ρ, T and V as a function of x, at time $t = 0$. There are two reasons why the choosing of initial conditions should be done carefully.

- The closer the initial conditions are to the final Steady-State result, the faster the time-marching procedure will converge, and hence, computer execution time will be shorter.
- If the initial conditions are too far away from the reality, the inital time wise gradients(time derivatives) can become huge at earlt timesteps. These large gradients at the early part of time-stepping can cause system to go unstable.

In the current case, we know that T decreases and V increases as the flow expands through the nozzle. Therefore, we have taken the initial condition that qualitatively beahaves in same fashion. The initial conditions with the linear variations of the flow-feild variables as the function of x is as follows,

$$\left.\begin{aligned}\rho &= 1 - 0.3146x\\ T &= 1 - 0.2314x\\ V &= (0.1 + 1.09)T^{1/2}\end{aligned}\right\} intial\ conditions\ at\ t = 0$$

8.1 Calculating time step

As discussed in section(6.3), we will be using the Courant number to calculate the Time Step Size for a stable solution. Inorder to avoid the solution to blow-up, the time-step size should be calculated at every timestep loop from the Courant number. For the our problem we will consider C = 0.5 for the optimum solution.

$$\Delta t = dt = min\left(C.\frac{dx}{a+V}\right)$$

Non-Dimensional velocity of sound,$a' = \frac{a}{a_0}$,

Non-Dimensional Velocity of fluid, $V' = \frac{V}{a_0}$

Now,

$$a' = \frac{a}{a_0} = \frac{\sqrt{\gamma RT}}{\sqrt{\gamma RT_0}} = \sqrt{\frac{T}{T_0}} = (T')^{\frac{1}{2}}$$

$$\Rightarrow dt = min\left(C.\frac{dx}{(T)'^{\frac{1}{2}} + V'}\right)$$

9 Algorithm Walkthrough:

1. Grid Discretization:
    - The nozzle is divided into discrete grid points along the x-axis, with a fixed spacing (Δx).
    - The Quasi 1-D assumption simplifies the grid generation by focusing on the axis of the nozzle.
2. Nozzle Shape:
    - The nozzle shape is defined as a function of x, representing the area variation along the nozzle.
    - A parabolic nozzle shape is considered.
3. MacCormack Technique:
    - The MacCormack method is used for solving the unsteady compressible flow equations.
4. Predictor Step:
    - Forward difference discretization is used to calculate spatial derivatives.
    - Time derivatives of flow variables (density, velocity, temperature) are calculated.
    - Temporary values of flow variables are updated using these derivatives.
5. Corrector Step:
    - Backward difference discretization is used.
    - Time derivatives are recalculated based on the updated predictor values.
    - Corrected values of flow variables are obtained by averaging the time derivatives from the predictor and corrector steps.
6. Time Step Calculation:
    - A stable time step (t) is determined using the Courant number less than 1 (CFL condition) to ensure numerical stability.
7. Boundary Conditions:

- Subsonic Inflow Boundary Condition: The left-running characteristic moves outside the domain, while the incoming subsonic flow is linearly extrapolated for the velocity.
- Supersonic Outflow Boundary Condition: Both characteristics move out of the domain, and no fixed values are specified; all variables evolve naturally with time.

8. Initial Conditions:

   - Initial conditions for density, velocity, temperature, and pressure are set.
   - These conditions are specified at the beginning.

9. Results:

   - The code performs a time loop, iterating through time steps to calculate and update the flow variables.
   - The results are plotted for nozzle shape, pressure, density, temperature, and velocity profiles.
   - Non-dimensional mass flow rates are also calculated and plotted at different time steps

10 Simulation

Through our simulation, we gained valuable insights into the behavior of water flow within a convergent-divergent nozzle. The visual representation of individual water trajectories provided a comprehensive view of the fluid dynamics at play.

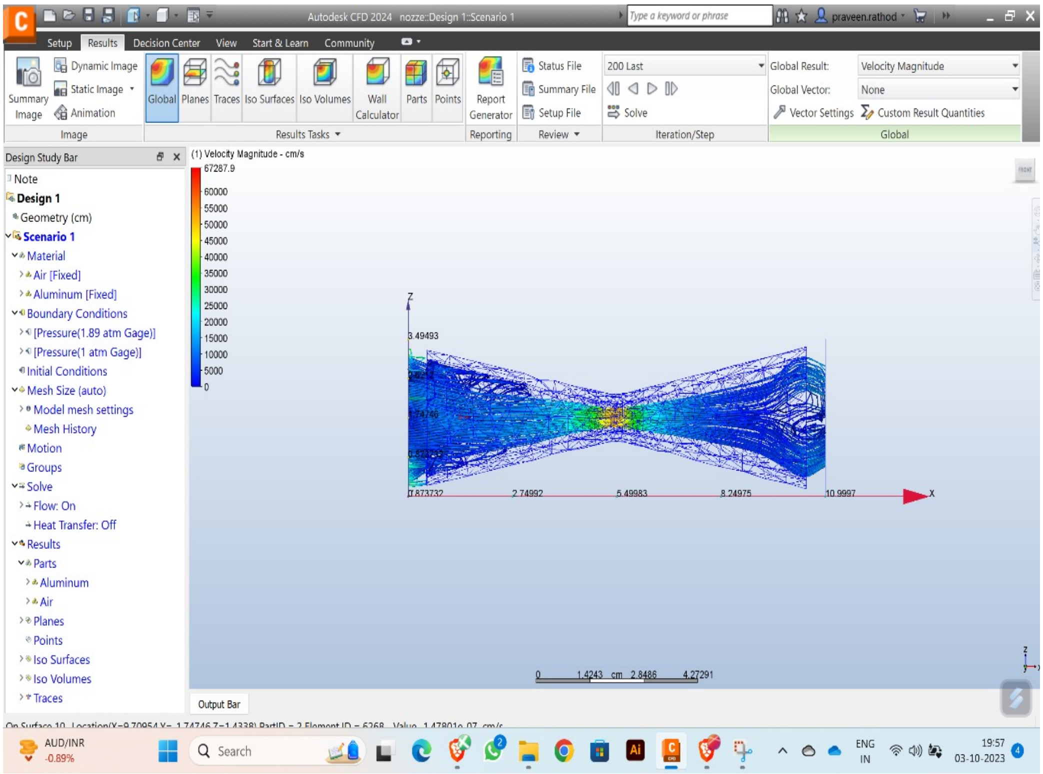


Figure 5: Simulation of path of different particles.

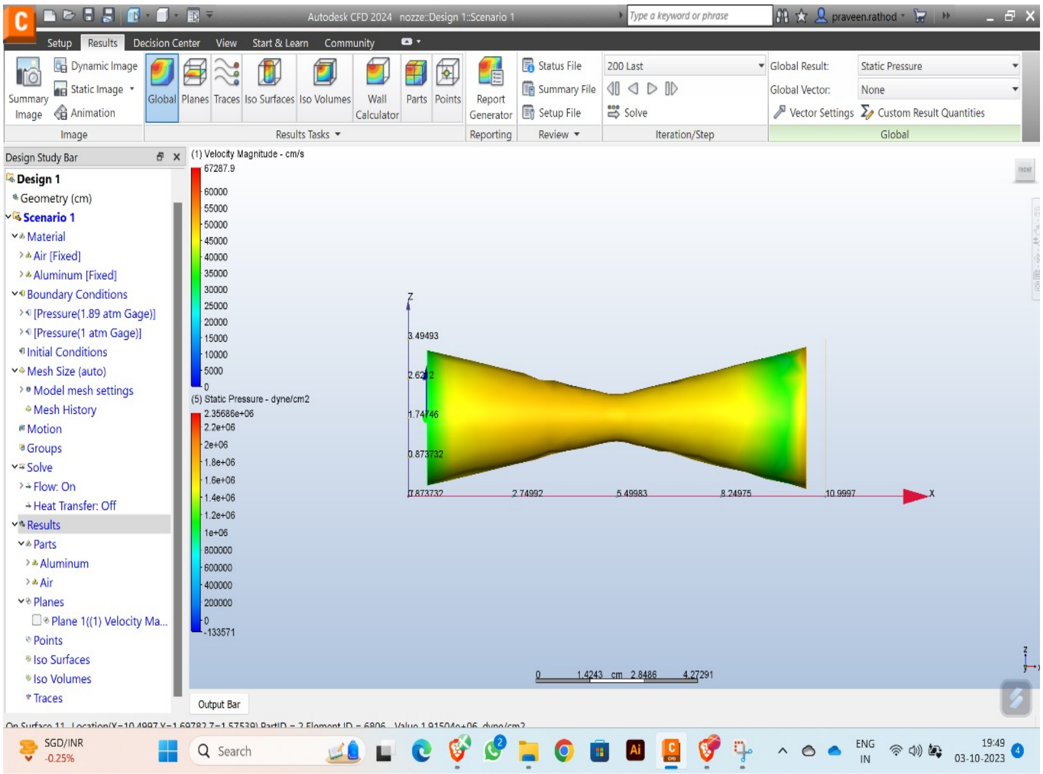


Figure 6: Simulation of pressure

11 Result and Discussion

Our analysis reveals a direct correlation between the number of time steps employed in the numerical solution and the level of convergence achieved. As expected, a higher number of time steps lead to a more accurate and stable solution. This finding highlights the importance of carefully selecting an appropriate time step to ensure accurate and meaningful numerical simulations.

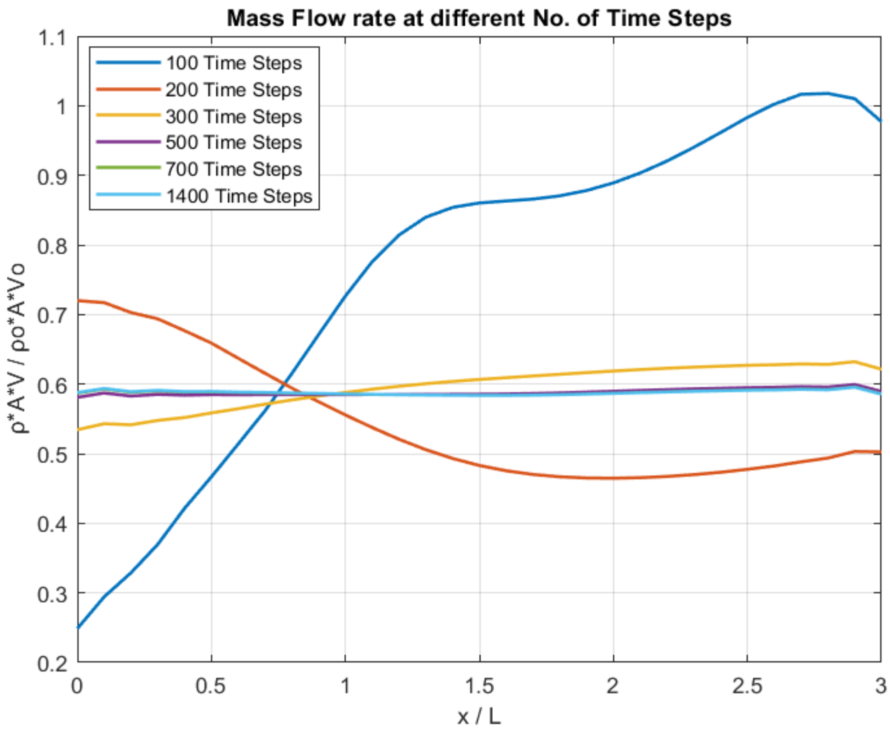


Figure 7: Mass flow rate at different time steps.

The four graphs below depict the variation in pressure ratio (P/Po), temperature ratio (T/To) , velocity ratio (V/Vo), density ratio (ρ/ρ_0) across the spatial domain normalized by the characteristic length (x/L) at the final time step. This visualization provides a crucial snapshot of the various distributions and their relative values at this critical juncture.

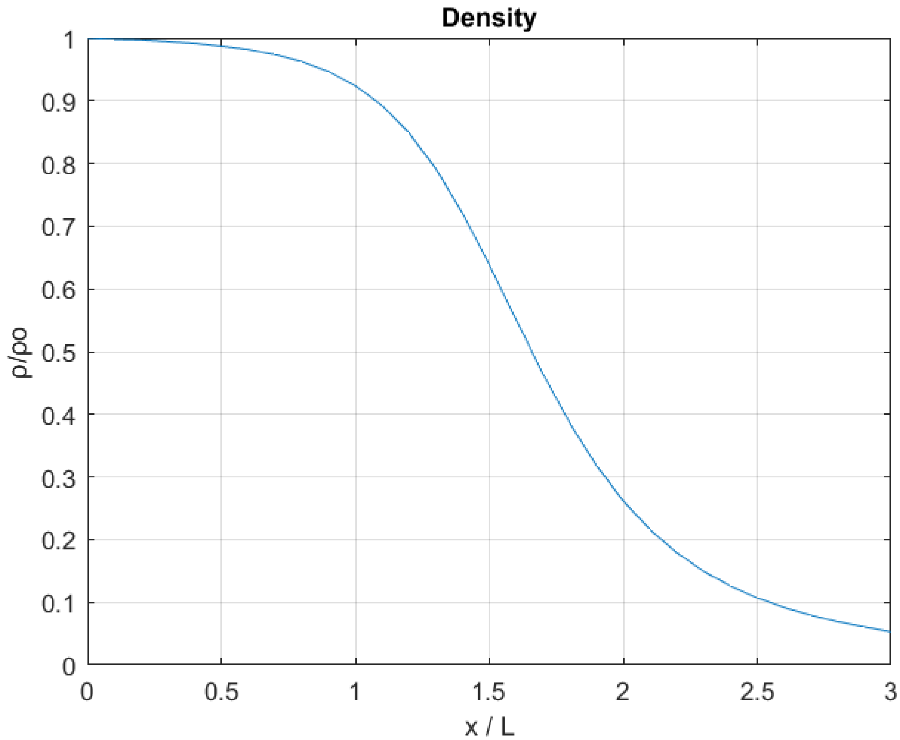


Figure 8: Variation of density with distance

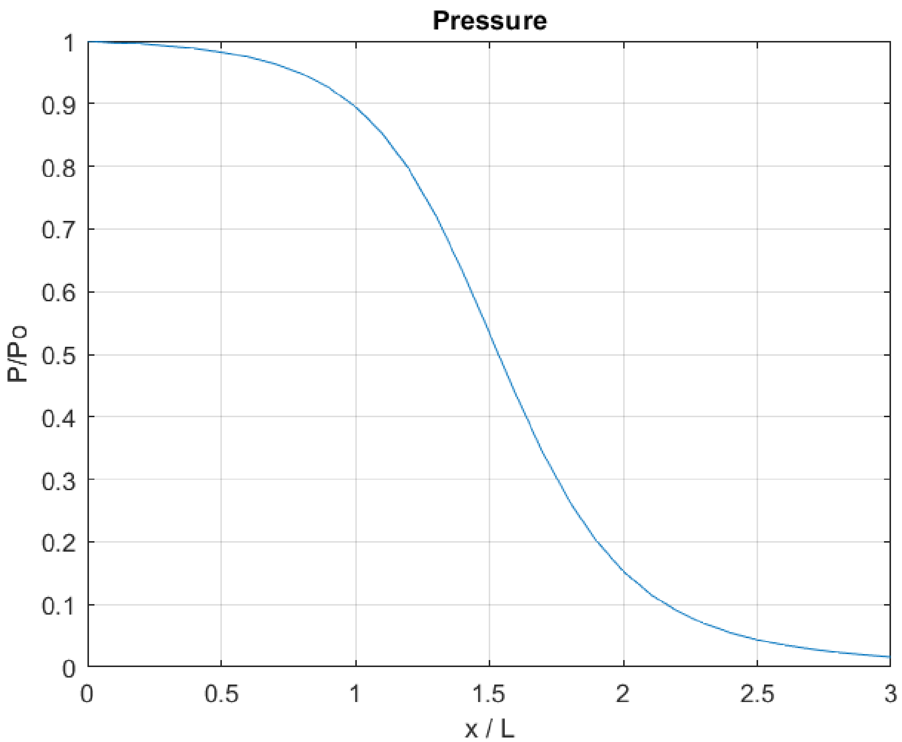


Figure 9: Variation of pressure with distance

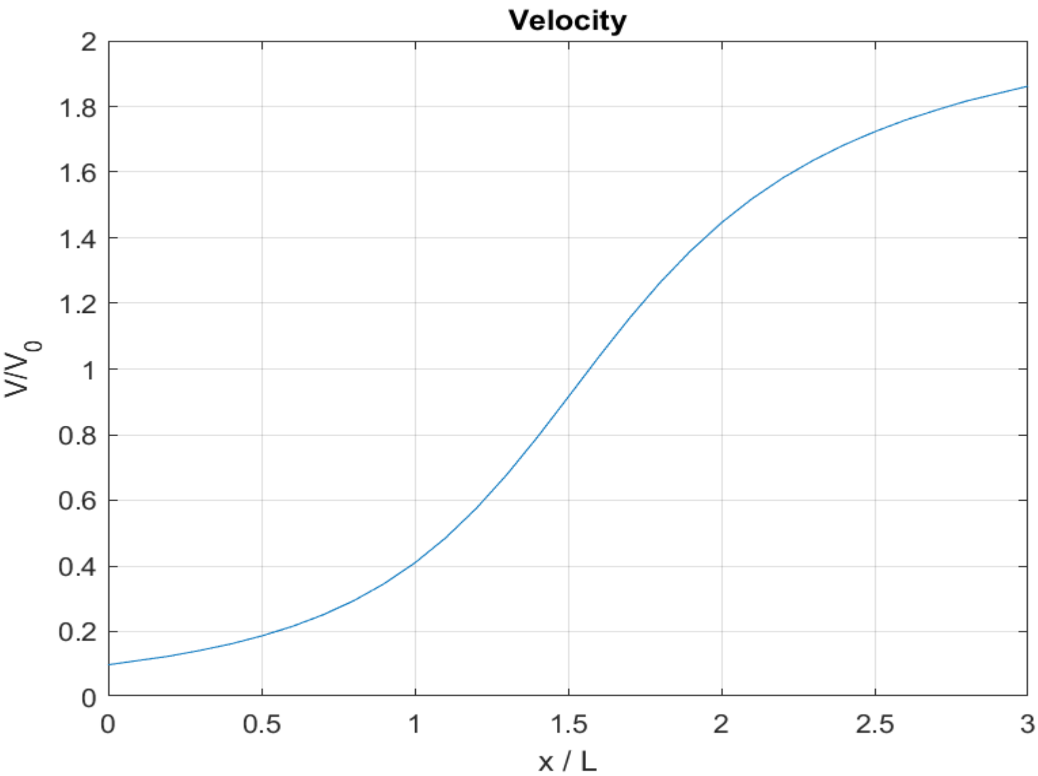


Figure 10: Variation of velocity with distance

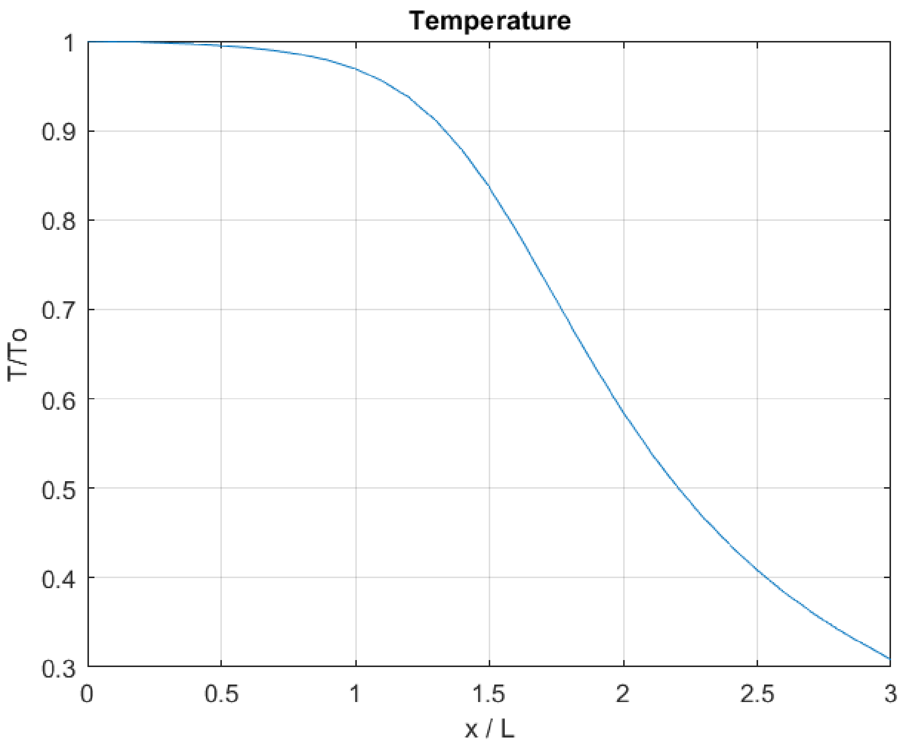


Figure 11: Variation of temperature with distance