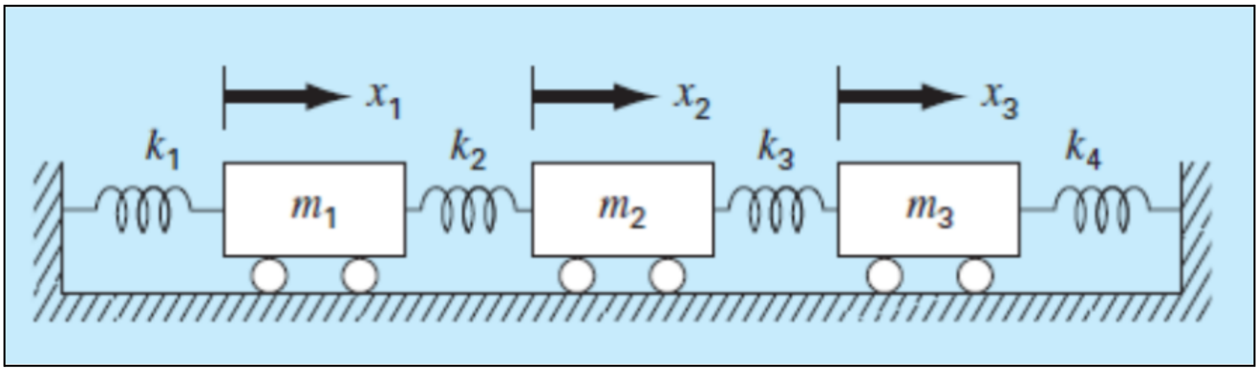
Problem 5

Consider the three-mass four-spring system shown below. Determine the equations of motion from $\sum F_x = ma$ for each mass using its free-body diagram:



where k_1 = k_4 = 10 N/m, k_2 = k_3 = 30 N/m, and m_1=m_2=m_3= 2 kg. Write the three equations in matrix form,

0= [Acceleration vector] + [k/m matrix] [displacement vector x]

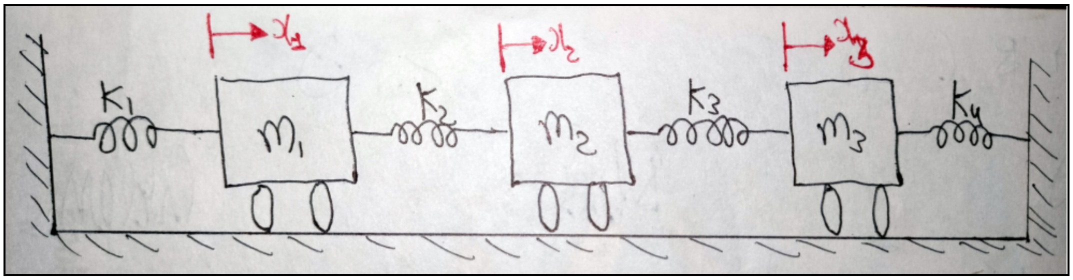
At a specific time when $a_1 =- 0.4\ m/s^2$,$a_2 = a_3 = 0$, this forms a tridiagonal matrix. Solve for the displacement of each mass using the Tridiagonal Matrix Algorithm (TDMA).

Description of Problem:

The problem involves analyzing a mass-spring system consisting of three masses connected by four springs. The springs have different spring constants (k_1, k_2, k_3, k_4), and the masses are identical (m_1,m_2,m_3). The task is to determine the equations of motion for each mass using Newton's second law ($\sum F_x = ma$) and the free-body diagrams. These equations can then be expressed in matrix form, where the acceleration vector is related to the displacement vector using a matrix equation. Additionally, at a specific time when one of the accelerations is given, the matrix equation simplifies into a tridiagonal matrix system. The Tridiagonal Matrix Algorithm (TDMA) will be employed to solve for the displacements of each mass.

Procedure:

- For each mass, construct the free-body diagram to identify the forces acting on it (spring forces and inertial forces). Apply Newton's second law ($\sum F_x$ = ma) to write down the equations of motion for each mass.

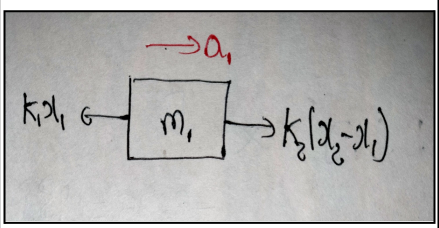
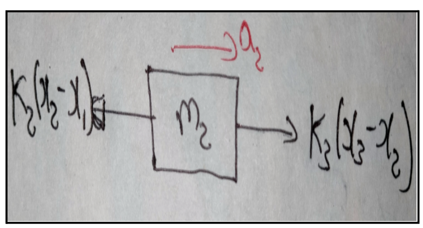
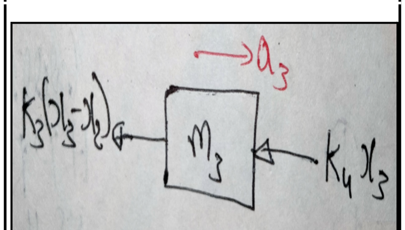


- Spring 1 is extended by x_1

  Spring 2 is extended by $(x_2 - x_1)$

  Spring 3 is extended by $(x_3 - x_2)$

  Spring 4 is compressed by x_3

Body 1	Body 2	Body 3
		
$m_1a_1 = k_2(x_2 - x_1) - k_1x_1$	$m_2a_2 = k_3(x_3 - x_2) - k_2(x_2 - x_1)$	$m_3a_3 =- k_4x_3 - k_3(x_3 - x_2$
$(\frac{k_2}{m_1})x_2 - \frac{(k_1+k_2)}{m_1}x_1 = a_1$	$k_2x_2 - (k_2 + k_3)x_2 + k_3x_3 =$	$k_3x_2 - (k_4 + k_3)x_3 = 0$

- Convert the equations of motion into a matrix equation by rearranging terms and form the acceleration vector, displacement vector, and the matrix of spring constants divided by masses (k/m matrix).

$$AX = B$$

$$\begin{pmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -(k_3+k_4) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -(k_3+k_4) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -0.4 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -20 & 15 & 0 \\ 30 & -60 & 30 \\ 0 & 30 & -40 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -0.4 \\ 0 \\ 0 \end{pmatrix}$$

Tridiagonal Systems:

$$\begin{bmatrix} f_1 & g_1 & & & & & \\ e_2 & f_2 & g_2 & & & & \\ & e_3 & f_3 & g_3 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & e_{n-1} & f_{n-1} & g_{n-1} \\ & & & & & e_n & f_n \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \cdot \\ \cdot \\ \cdot \\ r_{n-1} \\ r_n \end{Bmatrix}$$

Algorithm:

1. **Decomposition**

```
DOFOR k = 2, n
  e_k = e_k / f_k-1
  f_k = f_k - e_k · g_k-1
END DO
```

2. **Forward Substitution**

```
DOFOR k = 2, n
  r_k = r_k - e_k · r_k-1
END DO
```

3. **Back Substitution**

```
x_n = r_n / f_n
DOFOR k = n -1, 1, -1
  x_k = (r_k - g_k · x_k+1)/f_k
END DO
```

- After decomposition of the TDMA matrix the new system we get is,

$$\begin{pmatrix} -20 & 15 & 0 \\ -1.5 & -37.5 & 30 \\ 0 & -0.8 & -16 \end{pmatrix}$$

- After forward substitution($LD = B$) is implemented, we get,

$$\begin{pmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & -0.8 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} -0.4 \\ 0 \\ 0 \end{pmatrix}$$

$d_1 = -0.4$

$d_2 = -0.6$

$d_3 = -0.48$

Thus, the right-sided vector is modified to,

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} -0.4 \\ -0.6 \\ -0.48 \end{pmatrix}$$

- After backward substitution ($UX = D$) we get,

$$\begin{pmatrix} -20 & 15 & 0 \\ 0 & -37.5 & 30 \\ 0 & 0 & -16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -0.4 \\ -0.6 \\ -0.48 \end{pmatrix}$$

The Final answers are given below

- ➢ $x_1 = 0.05\, m$
- ➢ $x_2 = 0.04\, m$
- ➢ $x_3 = 0.03\, m$