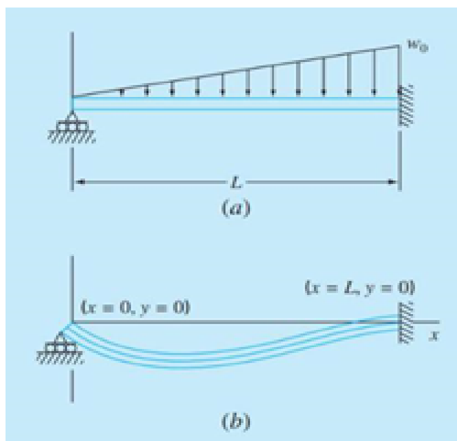
Problem 3

The Figure below shows a uniform beam subject to a linearly increasing distributed load. The equation for the resulting elastic curve is (see figure below)

$$y = \frac{w_0}{120EIL}(-x^5 + 2L^2x^3 - L^4x)$$

Determine the point of maximum deflection. Then substitute this value into the above equation to determine the value of the maximum deflection. Use the following parameter values in your computation: $L = 600\ cm$, $E = 50,000 \frac{kN}{cm^2}$, $I = 30,000\ cm^4$, $w_0 = 2.5 \frac{kN}{cm}$. Use any open method for root-finding.



Description of Problem:

The problem involves analyzing a uniform beam subjected to a linearly increasing distributed load. The equation for the elastic curve of the beam's deflection is given as

$$y = \frac{w_0}{120EIL}(-x^5 + 2L^2x^3 - L^4x)$$

We are required to find the point of maximum deflection by determining the root of the equation where the first derivative of the deflection equation is equal to zero. After finding the point of maximum deflection, you will substitute its value into the equation to determine the value of the maximum deflection.

Elastic Curve Equation:

The elastic curve equation describes the deflection of a beam subjected to loads, accounting for material properties and geometry. It reveals how the beam bends along its length, which is crucial for structural analysis and design. The equation is given by:

$$\boxed{y = \frac{w_0}{120EIL}(-x^5 + 2L^2x^3 - L^4x)}$$

Here:

- w_0 is the load per unit length.
- E is Young's modulus of the material.
- L is the length of the beam.
- I represents the area moment of inertia of the beam.

To Find: We need to determine the point of maximum deflection and the value of maximum deflection by using any open method.

Given: $L = 600\ cm, E = 50,000\ \frac{kN}{cm^2}, I = 30,000\ cm^4, w_0 = 2.5\ \frac{kN}{cm}$.

Method Used: *Newton Raphson Method*

Procedure:

To determine the point of maximum deflection and its corresponding value using the Newton-Raphson method, you need to find the root of the equation representing the derivative of the given elastic curve equation. The maximum deflection occurs at the point where the derivative is equal to zero.

Step 1: Derive the Equation for Derivative

Calculate the derivative of the given elastic curve equation with respect to x:

$$y' = \frac{dy}{dx} = \frac{w_0}{120EIL}(-5x^4 + 6L^2x^2 - L^4)$$

Let the derivative of y with respect to x i.e. y' be a function $f(x)$.

$$f(x) = y' = \frac{w_0}{120EIL}(-5x^4 + 6L^2x^2 - L^4)$$

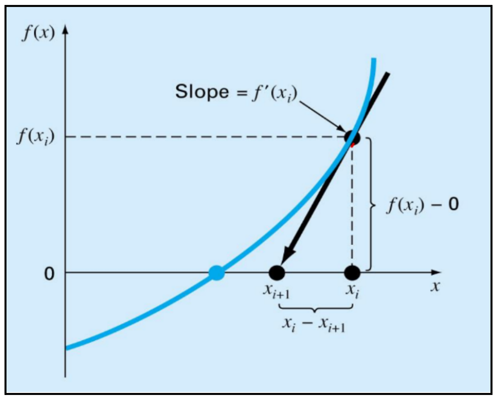
Step 2: Formulate the Equation to Find the Maximum Deflection Point

The root corresponds to the point where the maximum deflection occurs. We need to solve $y' = 0$ or $f(x) = 0$ for x.

Step 3: Implement the Newton-Raphson Method.

The Newton-Raphson method is a numerical technique for finding the roots of a real-valued function. It's an iterative approach that refines an initial guess by approximating the function with its tangent line and finding where the tangent line intersects the x-axis.

Deriving the Newton Raphson method to find approximate root x_{i+1}:



1^{st} order Talyor series approximation,

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

We used ' $\approx$ ' as we want x_{i+1} to be approximate root i.e. $f(x_{i+1}) \approx 0$.

This gives,

$$0 \approx f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

Or

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

This is the Newton Raphson formula where x_{i+1} is the point where the tangent to $y = f(x)$ at $(x_i, f(x_i))$ cuts the X-axis.

Algorithm for Newton Raphson method:

1. Choose an initial guess x_0 for the root.
2. Initialize a counter variable $iter$ to keep track of iteration number.

3. Repeat the following steps until convergence ($|f(x_i) < \varepsilon|$) or until a maximum number of iterations is reached:
   a. Calculate the function value and its derivative at the current guess: $f(x_i)$ and $f'(x_i)$.
   b. Compute the next approximation using the Newton-Raphson formula
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$
   c. Increment the iteration counter: i = i + 1.
4. If the iteration limit is reached and convergence is not achieved, consider the method as failed.
5. The final approximation, x_{i+1}, is the estimated root of the function.

❖ The success of the Newton-Raphson method depends on the choice of initial guess and the properties of the function being analyzed. If the initial guess is too far from the actual root or if the function has multiple roots, the method might fail to converge. It's also important to consider cases where the derivative approaches zero, as this can lead to numerical instability.

Solution:

➢ So to solve the problem we first need to define the values of stopping criteria for our program.

➢ Let us take the max number of iterations to 100 and the minimum value of approximation error ε_a that we want to achieve to be 0.005%.

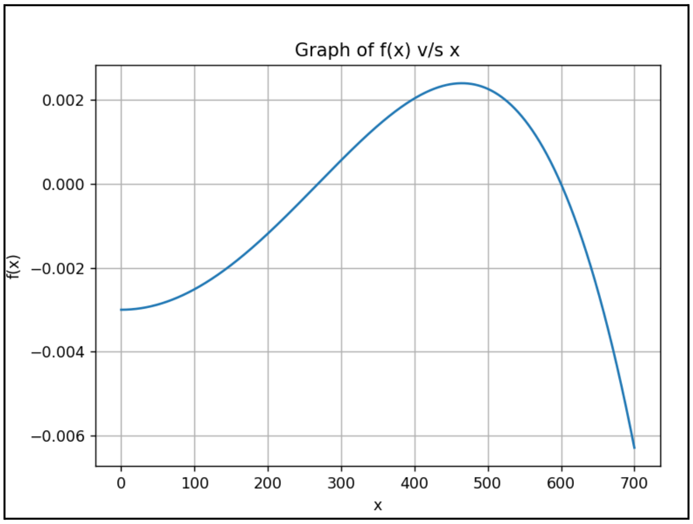
$$itermax = 100 \; and \; \varepsilon_s = 0.005$$

➢ We have the equation $f(x) = y' = \frac{w_0}{120EIL}(-5x^4 + 6L^2x^2 - L^4)$

Also, we require $f'(x)$ in newtons method. Therefore,

$$f'(x) = \frac{w_0}{120EIL}(-20x^3 + 12L^2x)$$

- From the graph of $f(x)$ we can infer that there are two roots, one near $x = 200$ and other near $x = 550$.



- So let us make the first guess to be $x_0 = 200$.

Iteration	x_i+1	ea(%)	et(%)
1	200.000000	-	25.464401
2	272.727273	26.666667	1.639454
3	268.326967	1.639904	0.000444
4	268.328157	0.000444	0.000000

The root we got is $x = 268.328157\ cm$.

- Let the second guess be $x = 550$.

Iteration	x_i+1	ea(%)	et(%)
1	550.000000	-	8.333333e+00
2	619.646611	11.239731	3.274435e+00
3	601.690663	2.984249	2.817771e-01
4	600.014119	0.279417	2.353224e-03
5	600.000001	0.002353	1.661129e-07
6	600.000000	0.0	0.000000e+00

The root we got this time is $x = 600\ cm$.

Substituting the value of the roots in the original equation $y = \frac{w_0}{120EIL}(-x^5 + 2L^2x^3 - L^4x)$.

$$y_1 = -0.51282\ cm \text{ when } x = 268.328157\ cm$$

$$\text{And, } y_2 = 0 \text{ when } x = 600\ cm.$$

Hence, point of maximum deflection is $x = 268.3281\ cm$ and the value of maximum deflection is $|y|_{max} = 0.51282\ cm$.