COMP 330 Winter 2021
Assignment 3
Due Date: 18^{th} February 2021

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There are **5** questions for credit and some extra questions at the end. There are three questions which are a accessible to everyone but perhaps harder than the usual questions. There is one for your spiritual growth. All the regular questions are excellent practice for the mid-term. The extra questions will not help you prepare for the mid-term. Please submit the homework through myCourses by 5pm on the due date.

Question 1[20 points] Are the following statements true or false? Prove your answer in each case, but the proof need only be a simple example or a couple of lines of explanation. We have some fixed alphabet Σ with at least two letters. In the following A and B stand for languages, *i.e.* subsets of Σ^*.

- If A is regular and $A \subseteq B$ then B must be regular. [3]
- If A and AB are both regular then B must be regular. [7]
- If $\{A_i | i \in \mathbb{N}\}$ is an infinite family of regular sets then $\bigcup_{i=1}^{\infty} A_i$ is regular. [5]
- If A is not regular it cannot have a regular subset. [5]

Question 2[20 points]
Show that the following language is not regular using the pumping lemma.

$$\{a^n b^{2n} | n > 0\}$$

Question 3[20 points] Show that the language

$$F = \{a^i b^j c^k | i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$$

is not regular. Show, however, that it satisfies the statement of the pumping lemma as I proved it in class, i.e. there is a p such that all three conditions for the pumping lemma are met. Explain why this does not contradict the pumping lemma.

Question 4[20 points] Let D be the language of words w over the alphabet $\{a, b\}$, such that w has an even number of a's and an odd number of b's and does not contain the substring ab. By this last statement I mean that one can never have an a followed by a b.

1. Give a DFA with *only five* states, including any dead states, that recognizes D.
2. Give a regular expression for this language.

Question 5[20 points] Consider the language $L = \{a^n b^m | n \neq m\}$; as we have seen this is not regular. Recall the definition of the equivalence $\equiv_L$ which we used in the proof of the Myhill-Nerode theorem; we used the notation R_L in the notes but it means the same thing as $\equiv_L$. Since this language is not regular $\equiv_L$ cannot have finitely many equivalence classes. Exhibit explicitly, infinitely many distinct equivalence classes of $\equiv_L$.

Please turn over for the spiritual growth question.

Extra Question 1[0 points]
If L is a language over an alphabet with strictly more than one letter we define $CYC(L) = \{uv | u, v \in \Sigma^*, vu \in L\}$. Show that if L is regular then $CYC(L)$ is also regular; [12]. Give an example of a *non-regular* language such that $CYC(L)$ is regular.

Extra Question 2[0 points] In assignment 1 we had a question that asked you to prove that if a language is regular then the lefthalf of the language is also regular. Similarly, if I define the *middle thirds* of a regular language by

$$\mathsf{mid}(L) = \{y \in \Sigma^* | \exists x, z \in \Sigma^* \text{ s.t. } xyz \in L \text{ and } |x| = |y| = |z|\}$$

then $\mathsf{mid}(L)$ is also regular. *I am not asking you to prove this; it is too easy after you have done left-half.* What if I delete the "middle" and keep the outer portions? More precisely define,

$$\mathsf{outer}(L) = \{xz | \exists y \in \Sigma^*, xyz \in L, \text{ and } |x| = |y| = |z|\}$$

then is it true that $\mathsf{outer}(L)$ is regular if L is regular? Give a proof if your answer is "yes" and a counter-example, with a proof that it is not regular, if your answer is "no."

Extra Question 3[0 points] Consider regular expressions as an algebraic structure with operations of $\cdot$ and $+$ and constants of $\emptyset$ and ε. Now consider equations of the form

$$X = A \cdot X + B$$

where A and B are regular expressions. Show that this always has a solution given by $A^* \cdot B$. Show,in addition, that if A does not contain the empty word this is the unique solution.

Spiritual Growth Question[0 points] Consider a probabilistic variant of a finite automaton. Come up with a formalization of what this might mean. Suppose that you have a reasonable definition and now you define acceptance to mean that your word causes the machine to reach an accept state with probability at least $\frac{2}{3}$. Show that such automata can recognize non-regular languages.