COMP 330 Winter 2021
Assignment 1

Belle Pan 260839939

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Solution 1.
Given a finite alphabet Σ and let $\emptyset \neq L \subseteq \Sigma^*$. We define the following relation R on words from Σ^*: $\forall x, y \in \Sigma^*, xRy$ if $\forall z \in \Sigma^*, xz \in L$ iff $yz \in L$. Prove that this is an equivalence relation.

To prove this is an equivalence relation, we must check if it fulfills the three properties of an equivalence relation: reflexitivity, symmetry, and transitivity.

1. Reflexitivity: $\forall x \in \Sigma^*$, xRx
   - In this case, we see that $\forall z \in \Sigma^*$, $xz \in L$ iff $xz \in L$.
   - This case holds (i.e. is true) because $xz \in L$ is on both sides of the iff statement.
2. Symmetry: $\forall x, y \in \Sigma^*$, $xRy \implies yRx$
   - In this case, we see that $\forall z \in \Sigma^*$, $xz \in L$ iff $yz \in L$ implies that $\forall z \in \Sigma^*$, $yz \in L$ iff $xz \in L$.
   - This is clearly true, as the iff statement may be reversed.
3. Transitivity: $\forall x, y, z \in \Sigma^*$,
   1) If $\forall w \in \Sigma^*$, $xw \in L$ iff $yw \in L$
   2) And $\forall w \in \Sigma^*$, $yw \in L$ iff $zw \in L$
   3) Then $\forall w \in \Sigma^*$, $xw \in L$ iff $zw \in L$
   - First, supposing $w \in \Sigma^*$ and supposing $xw \in L$, we know that $yw \in L$ by 1).
   - Knowing that $yw \in L$, we also gather that $zw \in L$ by 2). We also know that if $zw \in L$, then $yw \in L$ (the iff statement implies both ways!).
   - Hence, we can conclude using 3) that if $zw \in L$ then $xw \in L$.

All properties hold; therefore, R is an equivalence relation.

Solution 2.
Consider, pairs of natural numbers $\langle m, n\rangle$ where $m, n \in N$. We order them by the relation $m, n\rangle \sqsubseteq \langle m', n'\rangle$ if $m < m'$ or $(m = m')$^$n \leq n'$, where $\leq$ is the usual numerical order. Prove that the relation $\sqsubseteq$ is a partial order.

To prove that $\sqsubseteq$ is a partial order, we must check if it fulfills the three properties of a partial order: reflexitivity, antisymmetry, and transitivity.

1. Reflexitivity: $\forall x \in \Sigma^*$, xRx
   - $\langle m, n\rangle$ compared with itself yields $m = m$, and we also see that $n \leq n$.
   - Therefore, $\langle m, n\rangle \sqsubseteq \langle m, n\rangle$ (clearly reflexive).
2. Antisymmetry: $\forall x, y \in \Sigma^*$, if xRy and $yRx \implies x = y$
   - Suppose that $\langle m_1, n_1\rangle \sqsubseteq \langle m_2, n_2\rangle$ and $\langle m_2, n_2\rangle \sqsubseteq \langle m_1, n_1\rangle$ both hold.
   - If $m_1 > m_2$, then this contradicts the first statement in which $\langle m_1, n_1\rangle \sqsubseteq \langle m_2, n_2\rangle$ holds.
   - Similarly, if $m_1 < m_2$, then this contradicts the statement in which $\langle m_2, n_2\rangle \sqsubseteq \langle m_1, n_1\rangle$ holds.
   - Therefore, m_1 must be equal to m_2 if both statements hold.
   - Now, we know that these statements must also be true: $n_1 \leq n_2$, and $n_2 \leq n_1$.

- The only way this is possible is if $n_1 = n_2$.
- Thus, we can conclude that $\langle m_1, n_1 \rangle$ and $\langle m_2, n_2 \rangle$ are equal.

3. Transitivity: $\forall x, y, z \in \Sigma^*$,
   1) If $\forall w \in \Sigma^*$, $xw \in L$ iff $yw \in L$
   2) And $\forall w \in \Sigma^*$, $yw \in L$ iff $zw \in L$
   3) Then $\forall w \in \Sigma^*$, $xw \in L$ iff $zw \in L$

   - In this case, we must show that $\langle m_1, n_1 \rangle \sqsubseteq \langle m_3, n_3 \rangle$ given that $\langle m_1, n_1 \rangle \sqsubseteq \langle m_2, n_2 \rangle$ and $\langle m_2, n_2 \rangle \sqsubseteq \langle m_3, n_3 \rangle$.
   - This leads us to the following cases:
     - If $m_1 < m_2$ and $m_2 < m_3$, then $m_1 < m_3 \implies \langle m_1, n_1 \rangle \sqsubseteq \langle m_3, n_3 \rangle$.
     - If $m_1 < m_2$ and $m_2 = m_3$, then $m_1 < m_3 \implies \langle m_1, n_1 \rangle \sqsubseteq \langle m_3, n_3 \rangle$.
     - If $m_1 = m_2$ and $m_2 = m_3$, then $m_1 < m_3 \implies \langle m_1, n_1 \rangle \sqsubseteq \langle m_3, n_3 \rangle$.
     - If $m_1 = m_2$, $m_2 = m_3$, $n_1 \leq n_2$, and $n_2 \leq n_3$, then we must have $m_1 = m_3$ and $n_1 \leq n_3$. As both equality and $\leq$ are transitive, this implies $\langle m_1, n_1 \rangle \sqsubseteq \langle m_3, n_3 \rangle$.

All properties hold; therefore, $\sqsubseteq$ is a partial order.

Solution 3.
Give deterministic finite automata accepting the following languages over the alphabet $\{0, 1\}$.

1. **The set of all words ending in 00.**
2. **The set of all words ending in 00 or 11.**
3. **The set of all words such that the second last element is a 1.**

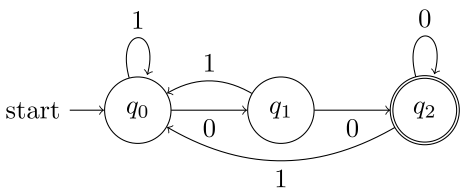

Figure 1: Deterministic finite automata accepting the set of all words ending in 00.

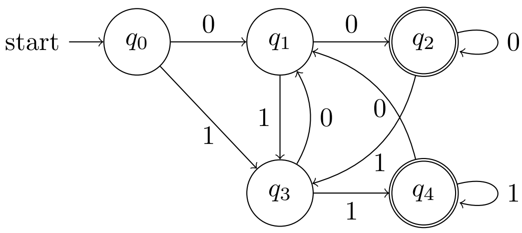

Figure 2: Deterministic finite automata accepting the set of all words ending in 00 or 11.

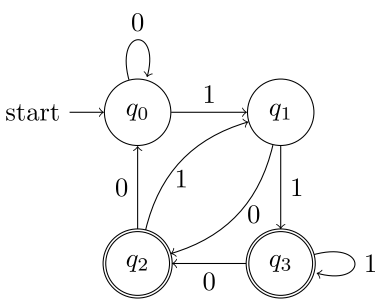

Figure 3: Deterministic finite automata accepting the set of all words such that the second last element is a 1.

Solution 4.
Suppose that L is a language accepted by a DFA (i.e. a regular language). Show that the following language is also regular: lefthalf$(L) := \{w_1 | \exists w_2 \in^* $ such that $w_1 w_2 \in L$ and $|w_1| = |w_2|\}$.

Let L be defined as (Q, q_0, δ, F). Let us then define a new NFA denoted L' as (Q', Q'_0, Δ, F'), where

- $Q' = Q \times Q$, i.e. the states in L' are pairs of states from the machine of L
- $Q'_0 = \{(q_0, f) | f \in F\}$
- $\Delta((s,t), a) = \{(\delta(s,a), t') | t = \delta(t', b)$ for some $b \in \Sigma\}$
- $F' = \{(s,s) | s \in Q\}$

We may note a couple of characteristics of L':

- Given a word w, the states in L' keep track of how the machine for L processes w via the first element in the pair; the second element of the pair keeps track of all the possible paths possible via processing w letter by letter by traversing through the states of the machine for L starting from an accept state.
- If L' finishes processing w (i.e. it reaches the end of w) and ends at the state (q_n, q_n), the machine for L will have also ended in the state q_n after processing w.

Looking at the second element of the pair of states in L', we observe that it took $|w|$ transitions from an accept state to reach q_n.

- This leads us to the understanding that there must be a path in the machine of L that starts at q_n and ends in an accept state.
- Therefore, there must exist another word w' of which, when read after w by the machine of L, would reach an accept state. I.e. If the machine reads ww', reading w would take us from the start state to q_n and reading w' after it would take us from q_n to an accept state.

This concludes that $ww' \in L$ and w would only be accepted by L' if $w \in$ lefthalf(L). Hence, lefthalf(L) must be a regular language.

Solution 5.

1. **Give a deterministic finite automaton accepting the following language over the alphabet $\{0,1\}$: The set of all words containing 100 or 110.**
2. **Show that any DFA for recognizing this language must have at least 5 states.**

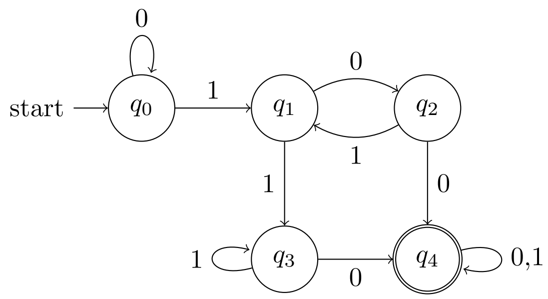

Figure 4: Deterministic finite automata accepting the set of all words containing 100 or 110 over the alphabet $\{0,1\}$.

This DFA may receive five different types of words and end up in five states:

1. ε, which ends up in a state we denote s_0
2. 1, which leads to a state we denote s_1 from the start state
3. 10, which leads to a state we denote s_2 from the start state
4. 11, which leads to a state we denote s_3 from the start state
5. 110, which leads to a state we denote s_4 from the start state

We check to see if any of the states stated above are redundant (i.e. if two of the above words leads to the same state):

- s_4 must be different than all the other states, as the DFA accepts 110 and none of the other words listed above without some form of modification.
- If $s_0 = s_1$, this means that ε and 1 must reach the same state; however, if this were the case, $\varepsilon \cdot 10$ and $1 \cdot 10$ should both be accepted by the DFA. 10 is not accepted by the DFA, and thus s_0 and s_1 must be distinct.
- If $s_0 = s_2$, this means that ε and 10 must reach the same state; however, if this were the case, $\varepsilon \cdot 0$ and $10 \cdot 0$ should both be accepted by the DFA. 0 is not accepted by the DFA, and thus s_0 and s_2 must be distinct.
- If $s_0 = s_3$, this means that ε and 11 must reach the same state; however, if this were the case, $\varepsilon \cdot 0$ and $11 \cdot 0$ should both be accepted by the DFA. 0 is not accepted by the DFA, and thus s_0 and s_3 must be distinct.
- If $s_1 = s_2$, this means that 1 and 10 must reach the same state; however, if this were the case, $1 \cdot 0$ and $10 \cdot 0$ should both be accepted by the DFA. 10 is not accepted by the DFA, and thus s_1 and s_2 must be distinct.
- If $s_1 = s_3$, this means that 1 and 11 must reach the same state; however, if this were the case, $1 \cdot 0$ and $11 \cdot 0$ should both be accepted by the DFA. 0 is not accepted by the DFA, and thus s_1 and s_3 must be distinct.
- If $s_2 = s_3$, this means that 10 and 11 must reach the same state; however, if this were the case, $10 \cdot 10$ and $11 \cdot 10$ should both be accepted by the DFA. 1010 is not accepted by the DFA, and thus s_2 and s_3 must be distinct.

Therefore, there must be five distinct states in this DFA for it to function properly.