COMP 330 Winter 2021
Assignment 2

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Solution 1.
Give regular expressions for the following languages over $\{a, b\}$:

1. $\{w|w$ **contains an even number of occurrences of** $a\}$
2. $\{w|w$ **contains an odd number of occurrences of** $b\}$
3. $\{w|$ **does not contain the substring** $ab\}$
4. $\{w|w$ **does not contain the substring** $aba\}$

Below are the corresponding regular expressions:

1. $(b^*ab^*ab^*)^* + b^*$
2. $(a^*ba^*)(ba^*ba^*)^*$
3. b^*a^*
4. $b^* + b^*aa^*(bbb^*aa^*)^*b^*$

Solution 2.
Suppose you have a DFA $M = (S, \Sigma, s_0, \delta, F)$. Consider two sitinct states s_1, s_2, i.e. $s_1 \neq s_2$. Suppose further that for all $a \in \Sigma$ $\delta(s_1, a) = \delta(s_2, a)$. Show that for any nonempty word w over Σ we have $\delta^*(s_1, a) = \delta^*(s_2, a)$.

Using induction on the length of the word w:

- Base case: $|w| = 1$ (length of the word w is 1)
  - We must show that for all $a \in \Sigma, \delta^*(s_1, a) = \delta^*(s_2, a)$.
  - This is already given in the question, $\forall a \in \Sigma$ $\delta(s_1, a) = \delta(s_2, a)$.
  - Therefore, there is no other verification processes needed.
- Inductive Hypothesis: $\forall w \in \Sigma^*$, $|w| < n \implies \delta^*(s_1, w) = \delta^*(s_2, w)$.
  - Consider a word wa where a is any letter $\in \Sigma$ and $|w| = n - 1$.
  - This yields : $\delta^*(s_1, wa)$
  - By the definition of δ^*, $\delta^*(s_1, wa) = \delta(\delta^*(s_1, w), a)$.
  - Then, by the inductive hypothesis, $\delta^*(s_1, wa) = \delta(\delta^*(s_1, w), a) = \delta(\delta^*(s_2, w), a)$
  - And finally, by the definition of δ^*, $\delta^*(s_1, wa) = \delta(\delta^*(s_1, w), a) = \delta(\delta^*(s_2, w), a) = \delta^*(s_2, wa)$

The inductive hypothesis holds and thus shows that for any nonempty word w over Σ we have $\delta^*(s_1, a) = \delta^*(s_2, a)$.

Solution 3. Show that the following languages are not regular by using the pumping lemma:

1. $\{a^n b^m a^{n+m} | n, m \geq 0\}$
2. $\{x | x = x^R, x \in \Sigma^*\}$ **where x^R means x reversed (palindromes).**

Below are the corresponding proofs using the demon - angel strategy of showing the pumping lemma:

1. (a) The demon picks a number p.
   (b) The angel picks the word $a^p ba^{(p+1}$.
   (c) The demon is forced to pick a y value that is made of exclusively a's due to the constraints $|xy| \leq p$ and $|y| > 0$. Suppose that $y = a^k$.
   (d) The angel picks $i = 3$ such that the new word is $a^{p+2k} ba^{p+1}$. This word is not in the language as $1 \neq 2k$.
   (e) Thus, by the pumping lemma, this language is indeed not regular.
2. (a) The demon picks a number p.
   (b) The angel picks the word $a^p bba^p$.
   (c) The demon is forced to pick x, y, z values such that for any $i \geq 0$, $xy'z$ remains a palindrome; thus, $x = a^j$ for some $j \in \{0, ..., p-1\}$, $y = a^k$ for some $k \in \{1, ..., p\}$, and $z = a^l bba^p$ for some $l \in \{0, ..., p-1\}$.
   (d) The angel picks $i = 2$ such that the new word is $a^{p+k} bba^p$. As $k > 0$, this word is not a palindrome and is thus not in the language.
   (e) Therefore, by the pumping lemma, this language is indeed not regular.

Solution 4. Show that the following languages are not regular by using the pumping lemma:

1. $\{a \in \{a, b, c\}^* || x| \text{ is a square}\}$**, where** $|x|$ **means the length of** x**.**

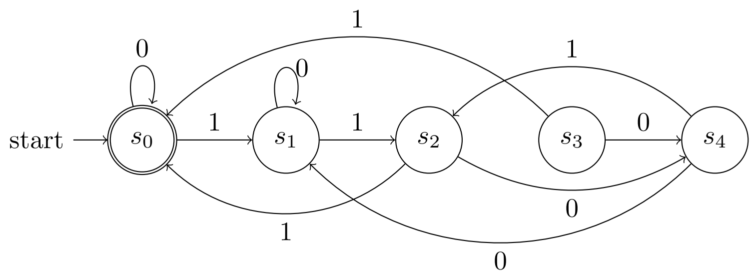
2. $\{a^{2n} b^n\}$

Below are the corresponding proofs using the demon - angel strategy of showing the pumping lemma:

1. (a) The demon picks a number p.
   (b) The angel picks the word $a^{p^2} b^{3p^2}$. $|a^{p^2} b^{3p^2}| = 4p^2$.
   (c) The demon is forced to pick a y value that is made of exclusively a's due to the constraints $|xy| \leq p$ and $|y| > 0$. Let $|y| = k$ and $0 < k \leq p$.
   (d) The angel picks $i = 2$ such that the new word is $a^{p^2+k} b^{3p^2}$ and $|a^{p^2+k} b^{3p^2}| = 4p^2 + k$. As $k \leq p < 4p+1$, we know that $4p^2 + q < 4p^2 + 4p + 1 = (2p+1)^2$; $4p^2 + q$ cannot be a square number as it is strictly less than the square after $4p^2$ and thus is not in the language.
   (e) Therefore, by the pumping lemma, this language is indeed not regular.
2. (a) The demon picks a number p.
   (b) The angel picks the word $a^{2p} b^p$.
   (c) The demon is forced to pick a y value that is made of exclusively a's due to the constraints $|xy| \leq p$ and $|y| > 0$. Let $|y| = k$ and $k > 0$.
   (d) The angel picks $i = 0$ such that the new word is $a^{2p-k} b^p$. As $2p - k \neq 2p$, this word is not in the language.
   (e) Thus, by the pumping lemma, this language is indeed not regular.

Solution 5. We are using the alphabet $\{0, 1\}$. We have a DFA with 5 states, $S = \{s_0, s_1, s_2, s_3, s_4\}$. The start state is s_0 and the only accepting state is also s_0. The transitions are given by the formula $\delta(s_i, a) = s_j$ where $j = i^2 + a \bmod 5$. Draw the table showing which pairs of states are in-equivalent and then construct the minimal automaton. Remember to remove useless states right from the start, before you draw the table.

Let us draw the automaton described:



We notice that s_3 is an unreachable state; this means that we may remove it immediately, before the table is drawn.

States	s_0	s_1	s_2	s_4
s_0	Self	0	0	0
s_1		Self	0	**1**
s_2			Self	0
s_4				Self

Table 1: The table achieved by running the algorithm.

There is only one pair of states that are equivalent by the algorithm : s_1 and s_4. We may now create the following minimized DFA: