COMP 330 Winter 2021
Assignment 3

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Solution 1.
Are the following statements true or false? Prove your answer in each case, but the proof need only be a simple example or a couple of lines of explanation. We have some fixed alphabet Σ with at least two letters. In the following A and B stand for languages, i.e. subsets of Σ^*

- **If A is regular and $A \subseteq B$ then B must be regular.**
- **If A is regular and AB are both regular then B must be regular.**
- **If $\{A_i | i \in\}$ is an infinite family of regular sets then $\bigcup_{i=1}^{\infty} A_i$ is regular.**
- **If A is not regular it cannot have a regular subset.**

The following are the true/false value with the reasoning for each respective sub-question:

- False. We know that A is regular because all finite sets are regular; however, there is no information stating that B is a finite set! Therefore, we cannot simply say that B is regular.
- False. If A is the finite set Σ^* and B is irregular, AB is still regular as $AB = \Sigma^*$. Therefore, B is not always regular in this case.
- False. $\bigcup_{i=1}^{\infty} A_i$ is an infinite set and is therefore not regular.
- False. All finite sets are regular, so a finite subset of A must also be regular.

Solution 2.
Show that the following language is not regular using the pumping lemma : $\{a^n b^{2n} | n > 0\}$.

Below is a proof using the demon - angel strategy of showing the pumping lemma:

1. The demon picks a number p.
2. The angel picks the word $a^p ba^{2p}$.
3. The demon is forced to pick a y value that is made of exclusively a's due to the constraints $|xy| \leq p$ and $|y| > 0$. Suppose that $y = a^k$ such that $0 < k \leq p$.
4. The angel chooses $i = 2$ such that the new word becomes $a^{p+k} ba^{2p}$. This word is not in the language as $p + k \neq 2 \times p$.

Thus, by the pumping lemma, $\{a^n b^{2n} | n > 0\}$ is indeed not a regular language.

Solution 3.
Show that the language $F = \{a^i b^j c^k | i, j, k \geq 0$ and if $i = 1$ then $j = k\}$ is not regular. Show, however, that it satisfies the statement of the pumping lemma as shown in class, i.e. there is a p such that all three conditions for the pumping lemma are met. Explain why this does not contradict the pumping lemma.

We start with showing that F is not regular:

- First, we know that we cannot use the pumping lemma, as it is clear by the question that F satisfies the pumping lemma.

- We must use the closure properties of regular languages to prove that F is not regular, particularly the property that the intersect ($\cap$) of two regular languages produces a regular language.
- We define a language $F' = \{ab^jc^k | j, k \geq 0\}$ such that $F \cap F' = \{ab^nc^n \ |n \geq 0\}$. This is clearly NOT a regular language, as a DFA cannot count the number of occurrences of b and c and ensure that they are equal.
- Therefore, F is NOT a regular language!

Next, we must show that F satisfies the pumping lemma:

- It is important to note that the pumping lemma states that all regular languages are pump-able, but it does NOT state that a pump-able language is regular (i.e. all regular languages $\implies$ pump-able, and the converse is not true). Thus, even if F satisfies the pumping lemma, it does not necessarily mean that F is regular!
- As we will be proving that F satisfies the pumping lemma, we do not use the negation method that is often used to prove that a language is not regular, i.e. in this case the angel will begin the game and choose the p value.
  - The angel chooses $p = 2$.
  - The demon chooses $w = a^mb^jc^k$.
    * When $m = 0$, $w = b^jc^k$.
      · The angel chooses $y = b$ if the word begins with b, OR the angel chooses $y = c$ if the word begins with c; these follow the restraints of $|xy| \leq p$ and $|y| \leq p$ when $x = \epsilon$.
      · The demon chooses an i value, such that the new word is either $b^ib^{j-1}c^k$ if $y = b$, OR c^ic^{k-1} if $y = c$. These two words are clearly in F, thus F satisfies the pumping lemma when $m = 0$.
    * When $m = 1$, $w = ab^jc^k$.
      · The angel chooses $y = a$; this follows the restraints of $|xy| \leq p$ and $|y| \leq p$ when $x = \epsilon$.
      · The demon chooses an i value such that the new word is $a^ib^jc^k$; this is clearly in F, and thus F satisfies the pumping lemma when $m = 1$.
    * When $m = 2$, $w = a^2b^jc^k$.
      · The angel chooses $y = a^2$; this follows the restraints of $|xy| \leq p$ and $|y| \leq p$ when $x = \epsilon$. We choose $y = a^2$ because words in the form of ab^jc^k may not necessarily be in F.
      · The demon chooses an i value such that the new word is $a^{2i}b^jc^k$; this is clearly in F, and thus F satisfies the pumping lemma when $m = 2$.
    * When $m > 2$, $w = a^mb^jc^k$.
      · The angel chooses $y = a$; this follows the restraints of $|xy| \leq p$ and $|y| \leq p$ when $x = \epsilon$.
      · The demon chooses an i value such that the new word is $a^ia^{m-1}b^jc^k$; this is clearly in F as $m - 1 > 1$, and thus F satisfies the pumping lemma when $m > 2$.
- Therefore, given the above proof, F satisfies the pumping lemma. F, despite being a non-regular language, satisfies the pumping lemma because the lemma only states that all regular languages are pump-able, NOT that all pump-able languages are regular!

Solution 4.
Let D be the language of words w over the alphabet $\{a, b\}$ such that w has an even number of a's and an odd number of b's and does not contain the substring ab. By this last statement, it is meant that one can never have an a followed by a b.

1. **Give a DFA with only five states, including any dead states, that recognizes D.**
2. **Give a regular expression for this language.**

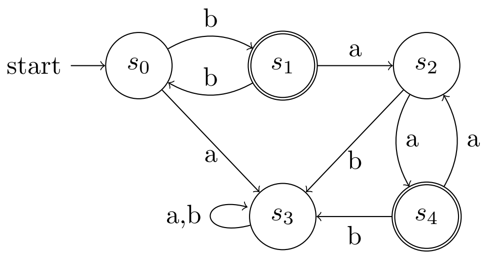

Figure 1: DFA accepting the set of all words in D.

The regular expression for D is $b(bb)^*(aa)^*$.

Solution 5.
Consider the language $L = \{a^n b^m | n \neq m\}$; as seen in class, this is not regular. Recall the definition of the equivalence $\equiv_L$ which we used in the proof of the Myhill-Nerode theorem; we used the notation R_L in the notes but it means the same thing as $\equiv_L$. Since this language is not regular $\equiv_L$ cannot have finitely many equivalence classes. Exhibit explicitly, infinitely many distinct equivalence classes of $\equiv_L$.

- We must first show that for two strings $x \in L$ and $y \in L$ such that $x \not\equiv_L y$, there exists a string z such that $xz \notin L$ and $yz \in L$ or vice versa.
  - Consider the two strings a^j and a^k: we first claim that $a^j \not\equiv_L a^k$ where $j \neq k$.
  - Then, we find a string z such that $a^j z \notin L$ and $a^k z \in L$.
  - Now, we state that $z = b^j$, which yields us $a^j b^j$, which is clearly $\notin L$, and $a^k b^j$ which is clearly $\in L$.
- Then, we need to show that there are infinitely many distinct equivalence classes.
  - There is clearly an infinite number of equivalence classes as the equivalence classes of a^k are different for every value of k, and there are infinitely many values of k.
- Therefore, there are an infinite number of equivalence classes of $\equiv_L$.