Towards Producing Shorter Congruence Closure Proofs in a State-of-the-art SMT Solver (Extended Abstract)

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Congruence closure

- The property of congruence states that, for any terms x, y, and any function f,

$$x = y \to f(x) = f(y)$$

- For a given set of equalities, the congruence closure is a minimal equivalence relation that satisfies them, as well as reflexivity, symmetry, transitivity and *congruence*

Congruence closure in SMT solvers

- Congruence closure algorithms are essential for solving the theory of equality and uninterpreted functions (EUF)

- From the perspective of the congruence closure algorithm, the solver will provide a set of equalities, and then make queries for whether two terms are equivalent given those equalities

Congruence closure in SMT solvers

- However, for SMT solvers, it is not sufficient to just determine whether two terms are equivalent
- We also require an *explanation*, that is, a minimal set of equations that makes the terms equivalent

Congruence closure in SMT solvers

- However, for SMT solvers, it is not sufficient to just determine whether two terms are equivalent
- We also require an *explanation*, that is, a minimal set of equations that makes the terms equivalent
- Furthermore, we might also want a structured *proof*, that uses these equalities as assumptions to derive the equivalence of the two terms

Congruence closure in SMT solvers

- Ideally we want these proofs and explanations to be small
- For proof-producing SMT solvers, a smaller proof from congruence closure will result in a smaller overall proof for the problem

Congruence closure in SMT solvers

- Ideally we want these proofs and explanations to be small
- For proof-producing SMT solvers, a smaller proof from congruence closure will result in a smaller overall proof for the problem
- A smaller explanation represents a smaller conflict clause, which will prune a larger portion of the search space

Proof-producing congruence closure

- A proof-producing congruence closure algorithm was presented by Nieuwenhuis et al. in 2005 [1]

- It is based on a union-find data structure, and contructs an equality graph to represent the equivalence relation

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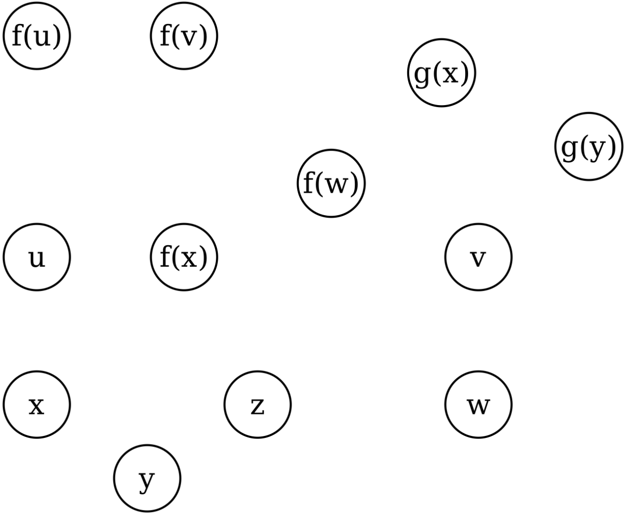
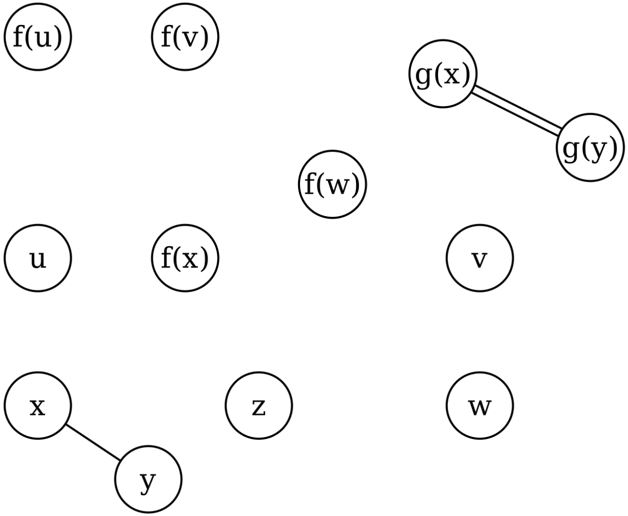
- It is based on a union-find data structure, and contructs an equality graph to represent the equivalence relation
- Then, finding the explanation for the equivalence of two terms consists in finding a path between them in the graph

Proof-producing congruence closure: Example



Proof-producing congruence closure: Example

1. $x = y$



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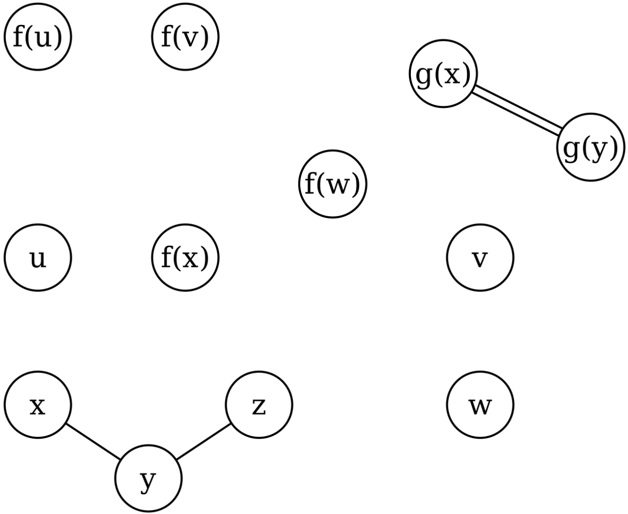

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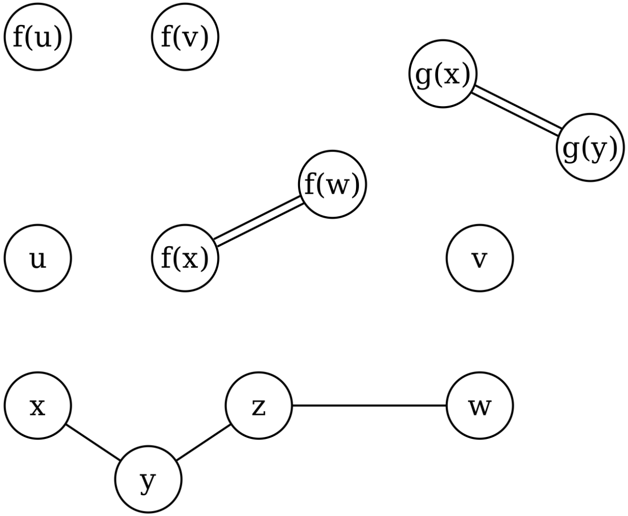

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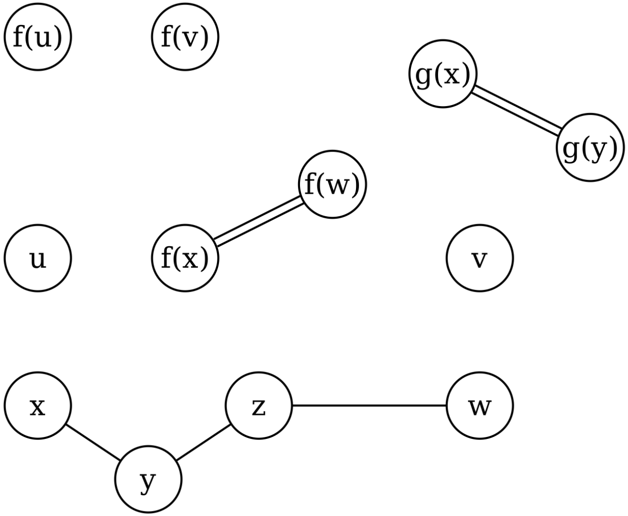

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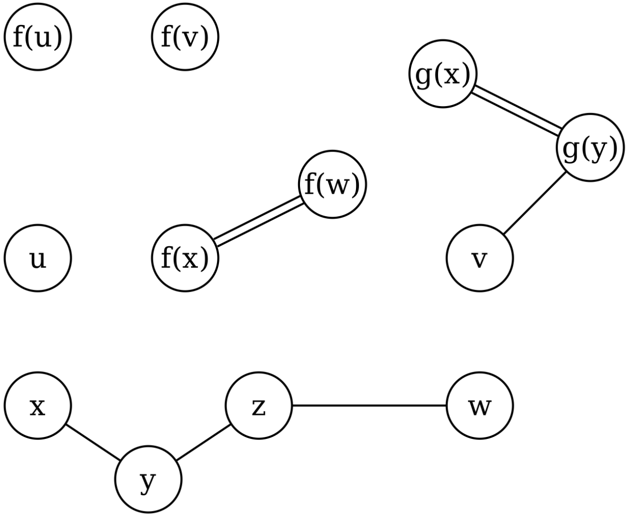

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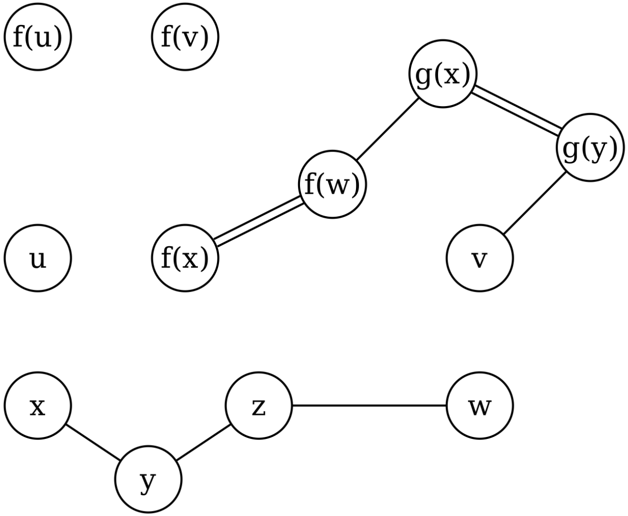

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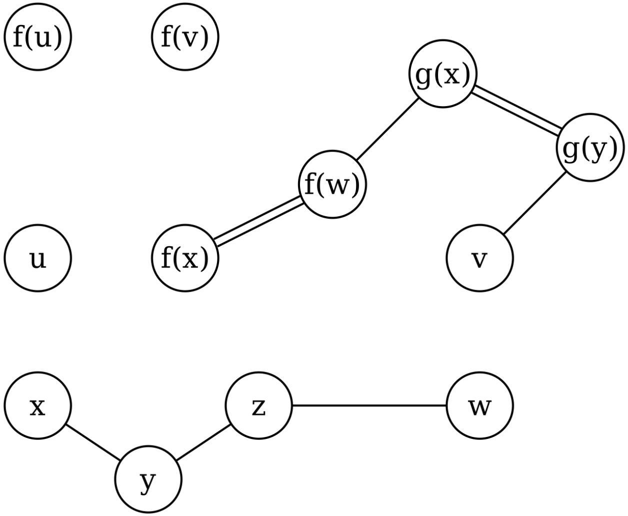

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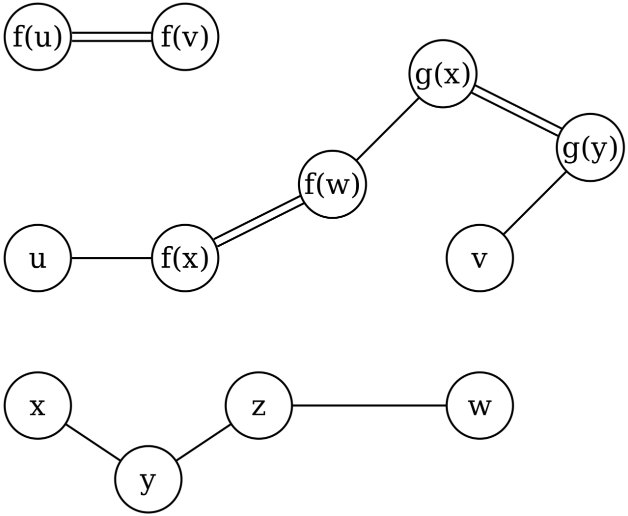

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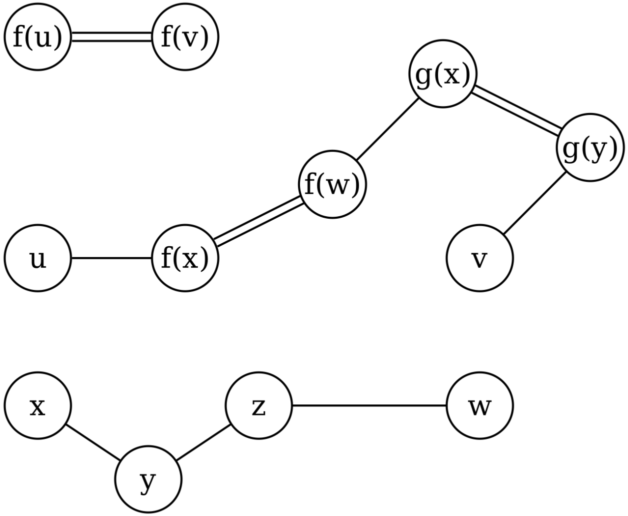

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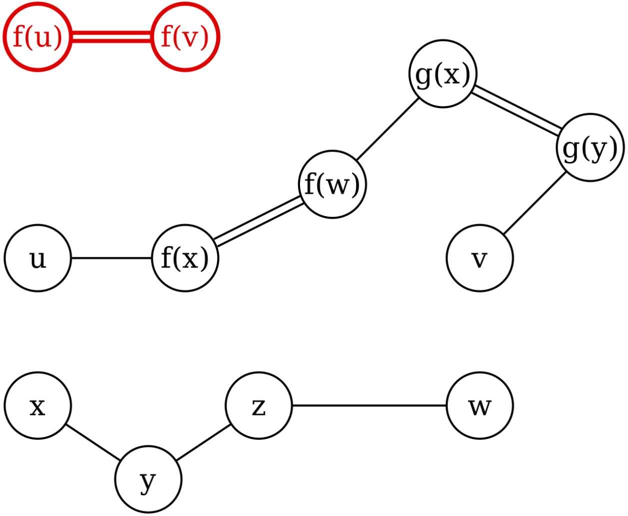

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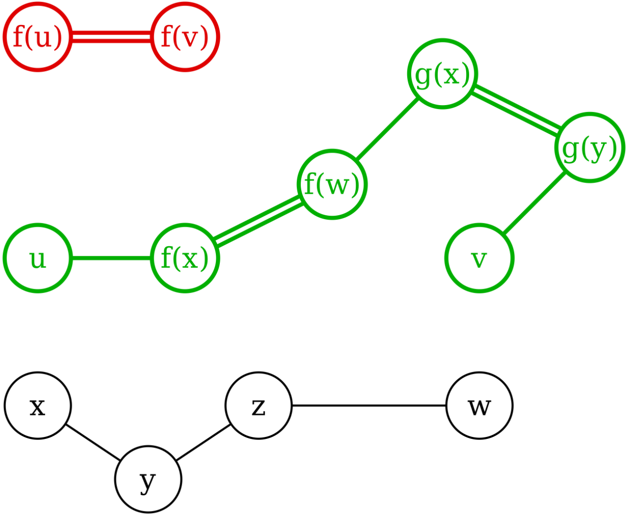

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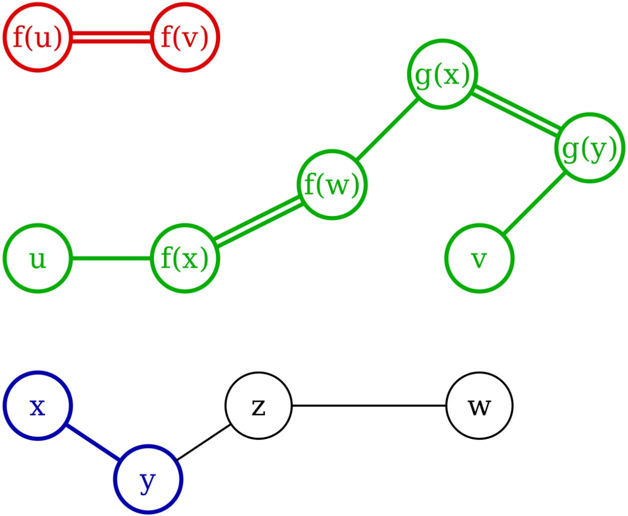

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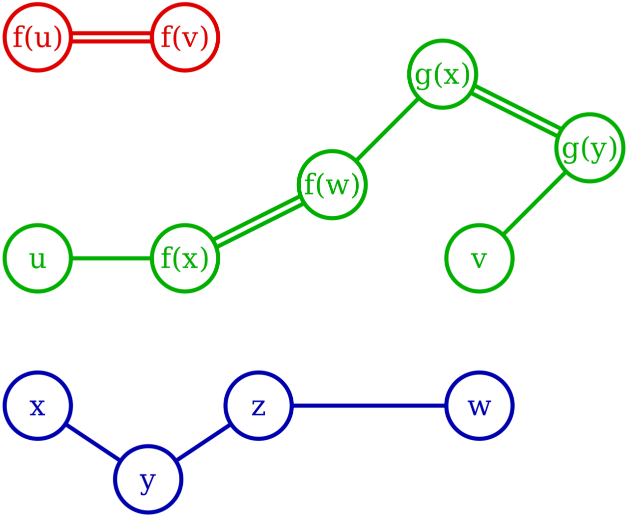

Keeping redundant equalities

- You may have noticed that we discard equalities between terms we already know to be equivalent

- However, keeping these equalities might allow us to find shorter paths between terms, and thus shorter proofs

Keeping redundant equalities

- Flatt et al. at FMCAD'22 [2] presented two proof producing congruence closure algorithms that make use of redundant equalities to find smaller proofs (called TreeOpt and Greedy)

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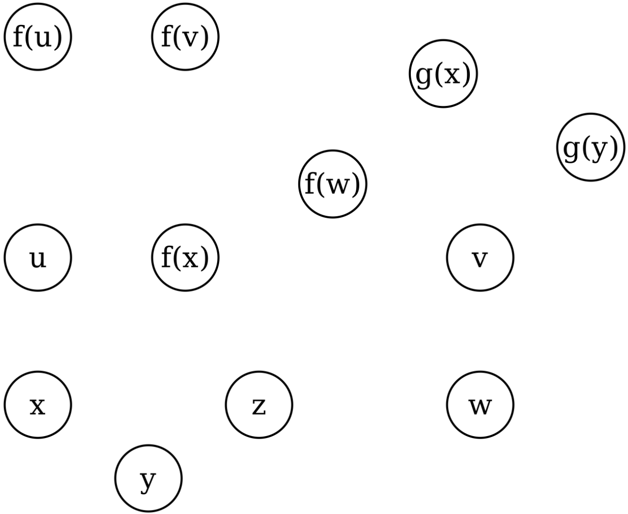
- Flatt et al. at FMCAD'22 [2] presented two proof producing congruence closure algorithms that make use of redundant equalities to find smaller proofs (called TreeOpt and Greedy)
- In the original work, they were implemented in an equality saturation tool
- Here, we present our effort to implement these algorithms in cvc5 [3], a state-of-the-art SMT solver

Keeping redundant equalities: Example



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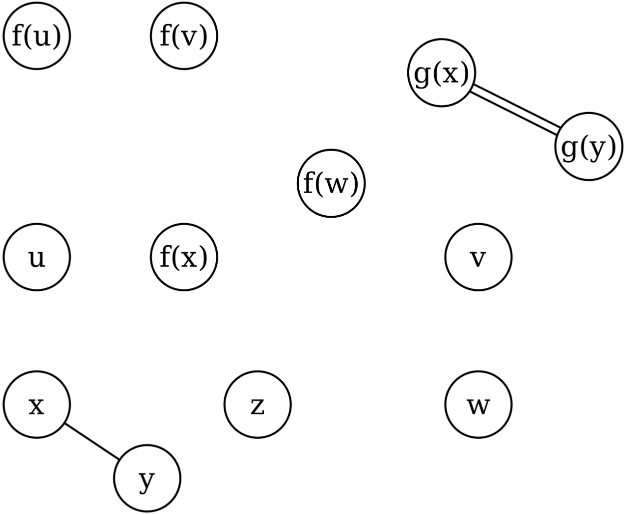

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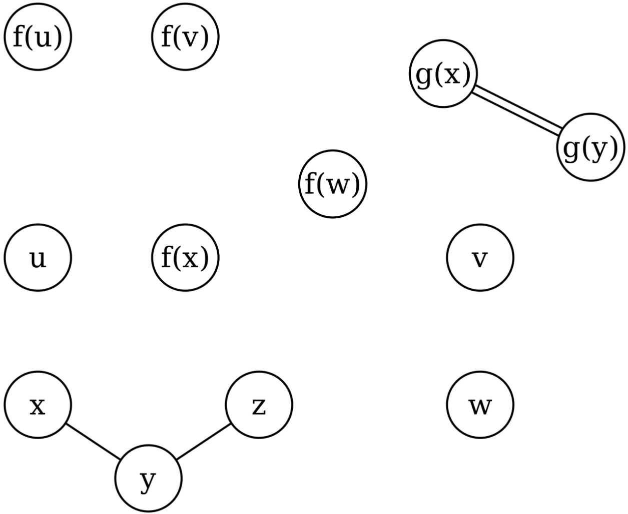

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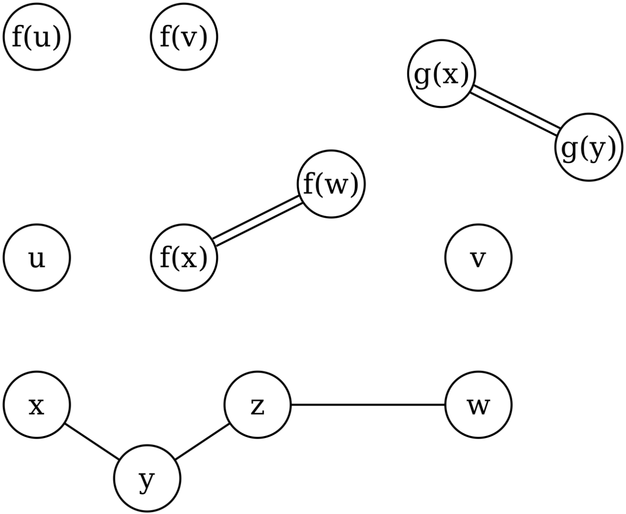

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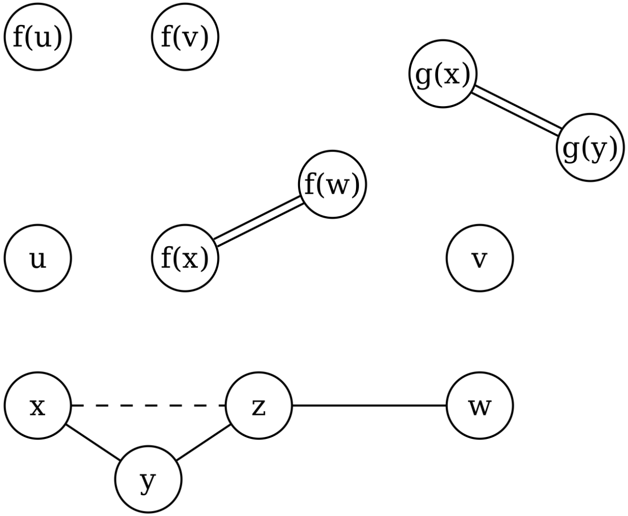

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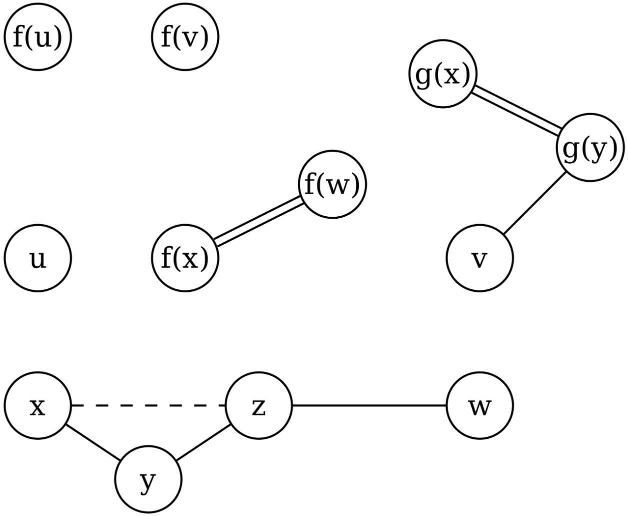

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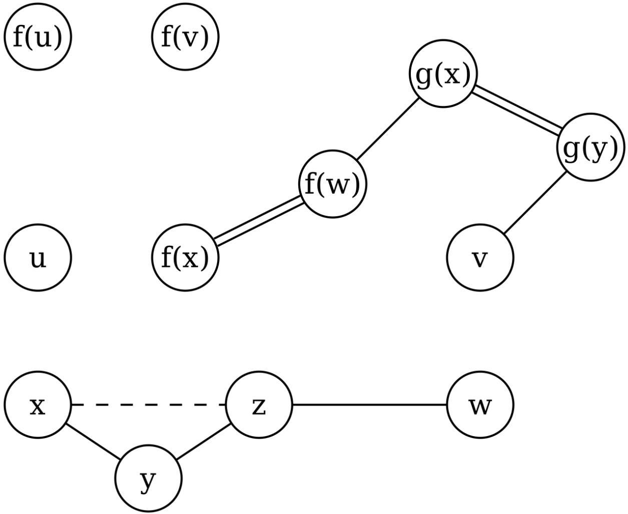

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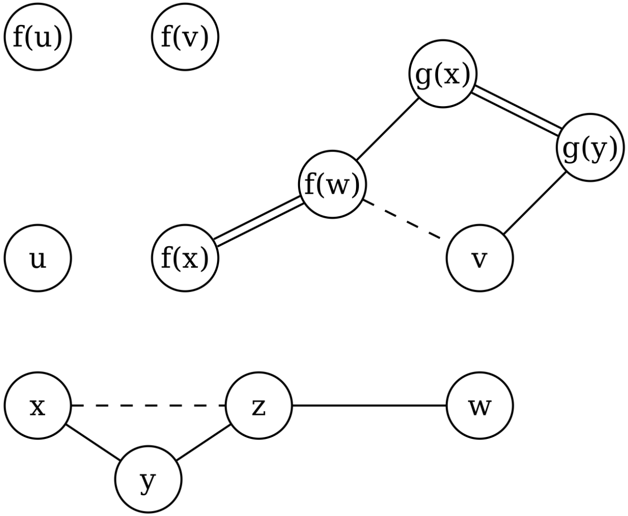

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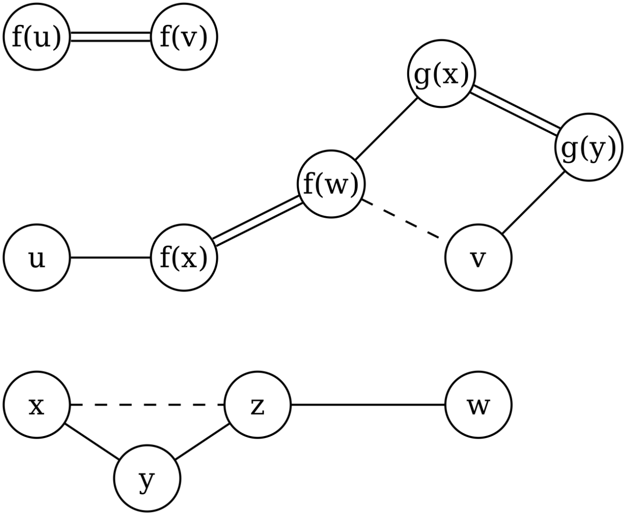

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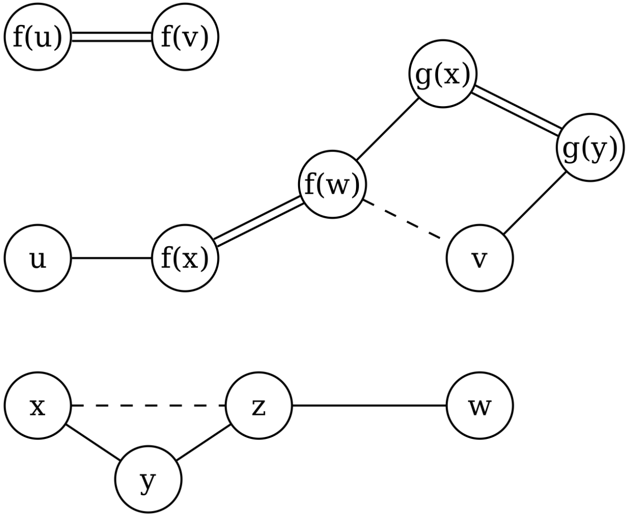

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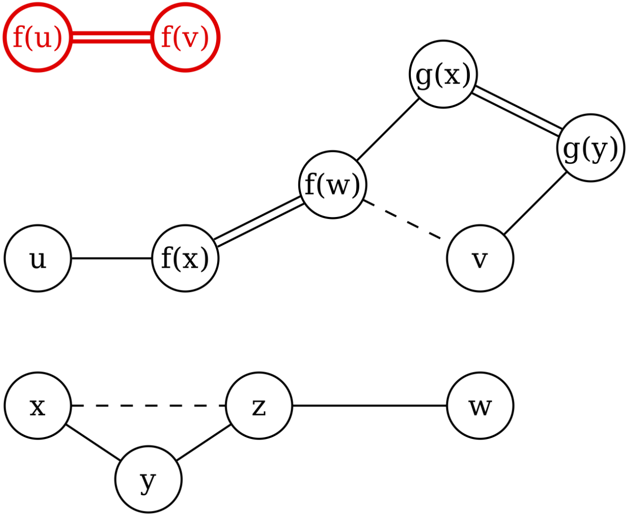

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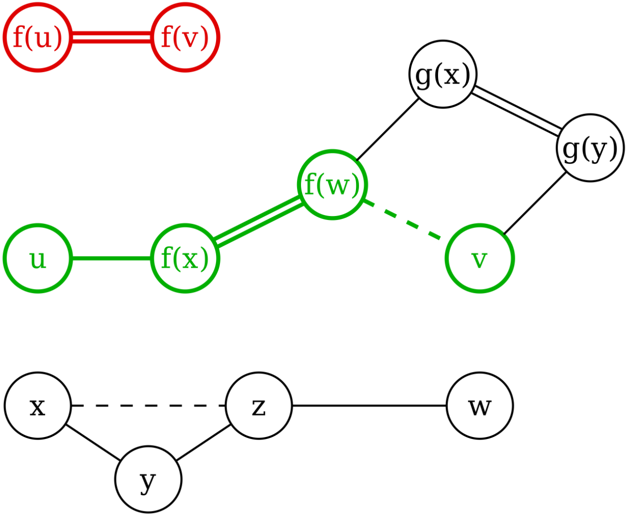

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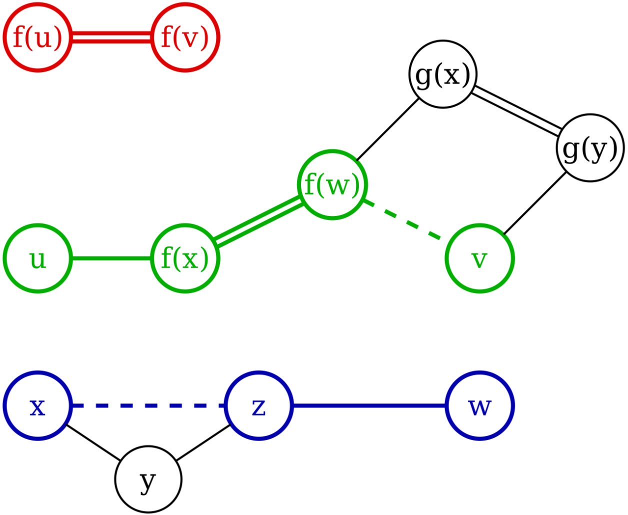

A side note: tree-size vs DAG-size

- Finding the minimal proof for the equivalence of two terms is an NP-hard problem [4]
- However, this problem becomes easier if we don't allow the reuse of proof steps

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- Finding the minimal proof for the equivalence of two terms is an NP-hard problem [4]
- However, this problem becomes easier if we don't allow the reuse of proof steps
- We call this the *tree size* of the proof, in contrast to the *DAG size*

TreeOpt algorithm

- Optimal algorithm (with regards to proof tree size)

- Computes the weight of each congruence edge by finding the size of the explanation of its justification, until a fixed point

- When asked to explain the equivalence between two terms, simply finds the shortest path between them considering these weights

TreeOpt algorithm

```
function compute_weights():
    let weights = {}
    for edge in edges:
        if not edge.is_congruence_edge():
            weights[edge] = 1

    until fixed point:
        for edge in congruence_edges:
            let (j1, j2) = edge.justification()
            weights[edge] = find_shortest_path(j1, j2, weights).size()

    return weights
```

TreeOpt algorithm

```
function get_explanation(start, end, weights=compute_weights()):
    let explanation = []
    let path = find_shortest_path(start, end, weights)
    for edge in path:
        if edge.is_congruence_edge():
            let (j1, j2) = edge.justification()
            explanation += get_explanation(j1, j2, weights)
        else:
            explanation += edge
    return explanation
```

GREEDY algorithm

- First, estimates the proof size of each congruence edge

- To explain the equivalence between two terms, finds the sortest path between them using these estimates as edge weights, but recurses when it encounters a congruence edge

GREEDY algorithm

- First, estimates the proof size of each congruence edge
- To explain the equivalence between two terms, finds the sortest path between them using these estimates as edge weights, but recurses when it encounters a congruence edge
- However, we don't recurse forever; we only go until a certain depth, determined by a *fuel* parameter

Greedy algorithm

```
function compute_weights():
    let weights = {}
    for edge in edges:
        if edge.is_congruence_edge():
            weights[l, r] = unoptimized_explanation(l, r).size()
        else:
            weights[l, r] = 1

    return weights
```

GREEDY algorithm

```
function get_explanation(start, end, weights=compute_weights(), fuel):
    if fuel == 0:
        return unoptimized_explanation(start, end)

    let explanation = []
    let path = find_shortest_path(start, end, weights)
    for edge in path:
        if edge.is_congruence_edge():
            let (j1, j2) = edge.justification()
            explanation += get_explanation(j1, j2, fuel - 1)
        else:
            explanation += edge

    return explanation
```

Dealing with backtracking

- Modern SMT solvers work by trying many possible (partial) solutions, and backtracking when a solution is determined to be invalid

- So, a congruence closure algorithm must be able to efficiently revert to a previous state when backtracking

Dealing with backtracking

- In our case, this is done by carefully recording the steps the congruence closure engine did, and undoing them when backtracking

- This includes clearing any cache that became invalid because of the backtrack (e.g., the TreeOpt and Greedy edge weights)

Avoiding circular explanations

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- This is because, after a congruence edge is added, the path between the terms of its justification will not change anymore

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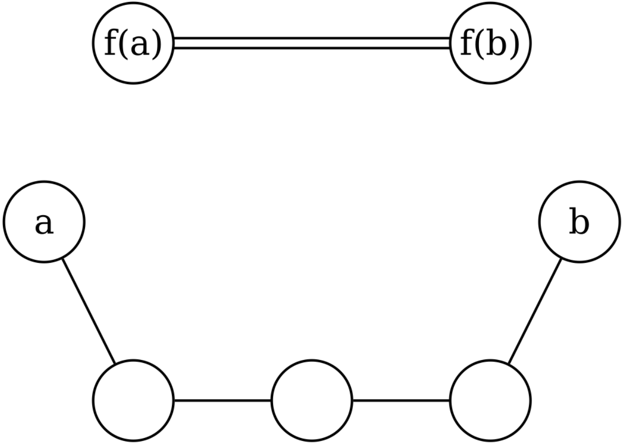
- When we were discarding redundant equalities, there could never be a circular explanation
- This is because, after a congruence edge is added, the path between the terms of its justification will not change anymore
- Now that we keep redundant edges, this is no longer true

Avoiding circular explanations: Example



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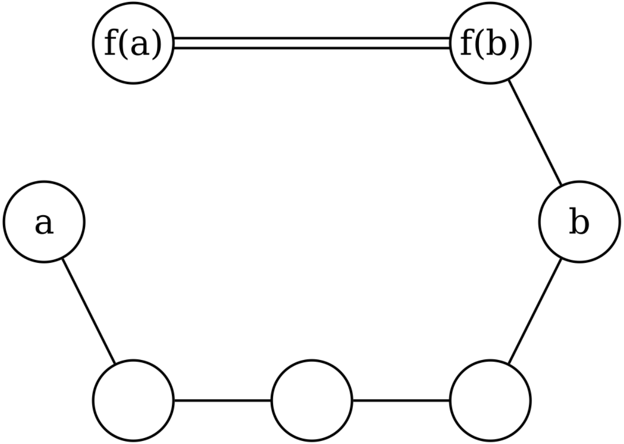

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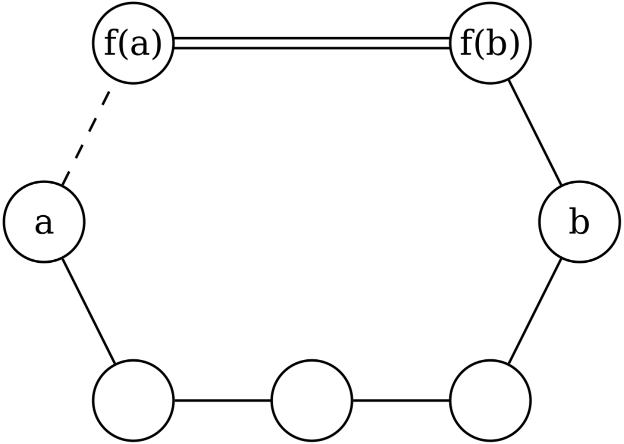


Avoiding circular explanations

- To prevent this problem, we store in each edge the *level* in which it was added

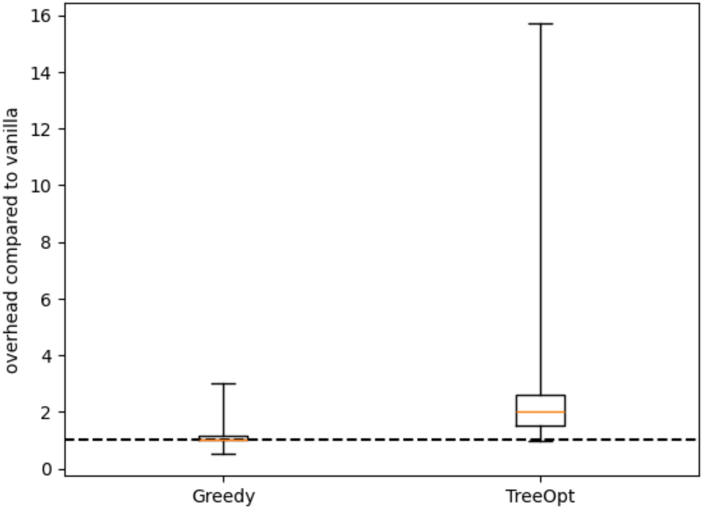
- Then, when explaining a congruence edge that was added in level n, we can only use edges whose level is no greater than n

Evaluation

- To evaluate our implementation, we ran cvc5 on all the 7502 problems from the QF_UF logic from the SMT-LIB benchmarks library
- We ran the solver using each of the two new algorithms, as well as the original algorithm present in cvc5 (from here on referred to as VANILLA)

Results

- Average runtime overhead was 1.18x for GREEDY, 2.68x for TREEOPT



Results

- The new algorithms had no meaningful overall impact in the final proof size
- On average the proofs from GREEDY and TREEOPT were respectively 0.2% and 0.1% larger that the VANILLA proofs
- They were 80% smaller than the VANILLA proofs in the best case, and 70% bigger in the worst case

Results: local impact

- The final proof returned by the solver will include things other than congruence closure steps, and can be affected by many variables that are hard to control

- To measure the impact more directly, we also recorded the size of the proof returned by each call to `get_explanation`

Results: local impact

- In 83% of cases, the proofs returned by the three algorithms had the same size

- On average, the “local” proofs from the two new algorithms are 2% smaller than the VANILLA proofs

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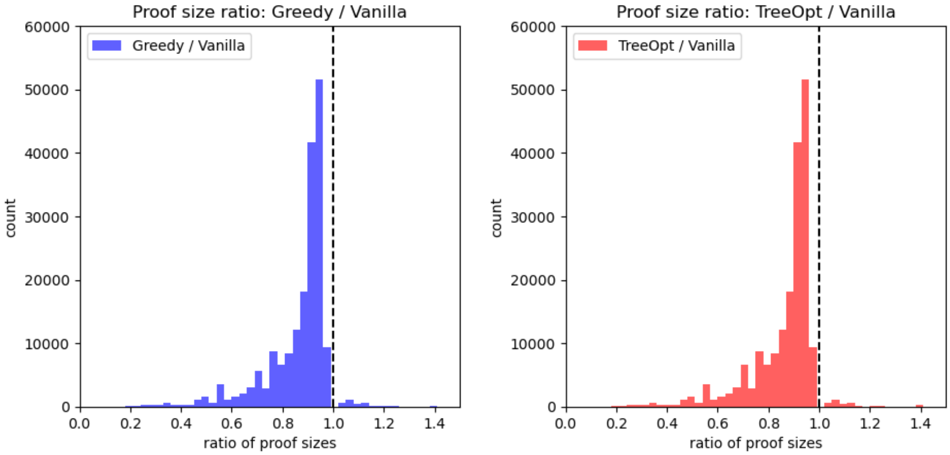
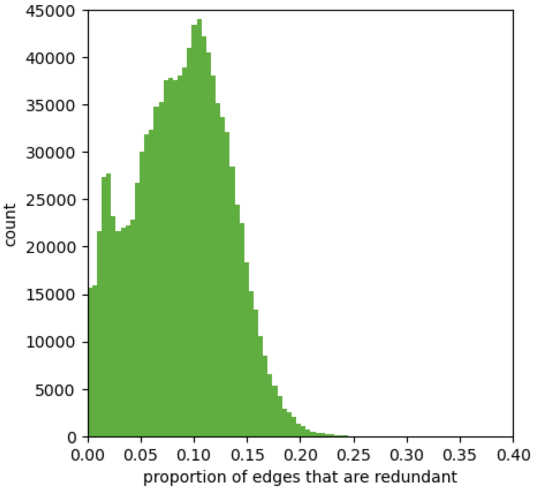
- In 83% of cases, the proofs returned by the three algorithms had the same size
- On average, the “local” proofs from the two new algorithms are 2% smaller than the VANILLA proofs
- If we exclude identical proofs, they are on average 14% smaller

Results: local impact



Results: extra edges

- We also measured the number of redundant equalities added
- On average, 8.7% of the equality graph edges were redundant, and in the most extreme case, 44.63%



Conclusion

- We were able to show that congruence closure algorithms that make use of redundant equalities can be efficiently implemented in a state-of-the-art SMT solver, with reasonable overhead
- However, they did not present a meaningful reduction in the final proof size
- Furthermore, even when looking at the local impact, the reduction in proof size is modest

Conclusion

- We were able to show that congruence closure algorithms that make use of redundant equalities can be efficiently implemented in a state-of-the-art SMT solver, with reasonable overhead

- However, they did not present a meaningful reduction in the final proof size

- Furthermore, even when looking at the local impact, the reduction in proof size is modest

- It is soon to say whether this or similar techniques will be practical for use in modern SMT solvers

[1] Robert Nieuwenhuis and Albert Oliveras. "Proof-Producing Congruence Closure". In: *Term Rewriting and Applications*. Ed. by Jürgen Giesl. Berlin, Heidelberg: Springer Berlin Heidelberg, 2005, pp. 453–468. ISBN: 978-3-540-32033-3.

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